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Jobs, Strikes, and Wars: Probability Models for Duration

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A class of stochastic models with behaviorally meaningful parameters and manpower planning implications is presented. These models are used to reanalyze some previously published data on the durations of wars, strikes, and jobs. Job durations are found to be qualitatively different from strike durations. The concepts developed in this paper also allow researchers to "eyeball" and plot their data better even if a formal statistical analysis is not performed.

INTRODUCTION

The past decade has seen a number of published articles dealing with job durations, turnover rates, and career paths. Some of these articles have used explicit probabilistic models, e.g., March and March (1977, 1978) on the tenure of school superintendents and Vroom and MacCrimmon (1968) on a stochastic model for managerial careers. Other more descriptive research includes Beehr and Gupta's (1978) work on the progression of withdrawal leading to turnover, Krant's (1975) use of job attitudes to predict turnover, and Johnson and Graen's (1973) study examining role ambiguity, role conflict, and turnover. Additional studies with statistical analyses of job duration can be found in Bartholomew (1959, 1973), Hedberg (1961), and Silcock (1954).

The duration of strikes has received considerable study, e.g., Lancaster (1972) and Horvath (1968). [The duration of wars originally collected by Richardson (1945, 1960a, b) is also discussed and analyzed by Horvath (1968).] In summary, the important phenomena of job and strike durations have received some theoretical attention and rather extensive descriptive statistical analysis.

The purpose of this paper is to present what we feel are more appropriate probability models for analyzing duration times in general and job and strike data in particular. The parameters of these models have clear behavioral interpretations; hence we are not engaged in *ad hoc* curve fitting. As a result the reader should obtain some new insights into the col-

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lection and interpretation of duration data. A description of a typical study will illustrate the points that we wish to make.

Job duration statistics usually contain one, or at most a few, duration times from many different individuals. In virtually every study that has been done these aggregated data show the following characteristic of inertia. For a particular job group (e.g., MBAs in their first job since receiving the degree), assume that the average stay with that company is 2 years. Now given that an individual has stayed 2 years, we expect this person to stay an *additional* 3 years. Given that an individual has stayed 4 years, we expect an additional 6 years of service. In summary, the longer an individual stays in that company, the longer the expected additional time in that same company. The question is can this aggregate phenomenon of inertia be assigned to the individuals being studied? In many cases the answer is "no."

In this paper we develop mathematical models and statistical estimation procedures that will enable us to separate the spurious inertia effect due to aggregating individuals with differing average duration times from the true time-dependent quitting rates at the individual level. Most previous work has not allowed the separation of individual-level phenomena from spurious aggregation effects.

The policy implications of this study are important. Two different companies (or divisions within a company) can both show strong inertia as judged from their aggregate job duration statistics, and in addition each may have the same average duration. However, at the first company the individual quitting rate can be remaining *constant* over time, while at the second company the individual quitting rate can be actually *increasing* over time. If both companies wish to decrease turnover, the suggested remedies can be quite different. The first company should try to obtain better initial selection since there is nothing in the work environment that is causing individuals to quit more readily as the time in the job increases. The second company may also try better initial selection, but there is also something in the work environment that needs improving since individuals' quitting rates are increasing as they stay with the company longer.

The remainder of the paper will present the necessary methodology, which will then be applied to some rather rich sets of data on the duration of jobs, strikes, and wars. One of our more interesting results is that job durations differ in a fundamental manner from the durations of strikes and wars. Without the models developed in this paper this difference would not have been detected—at least not as clearly.

THE CONCEPT OF INERTIA

Average Additional Duration

There are three basically similar descriptive measures of inertia that are used in the social sciences. One of these is what the mathematical reliability theorists call the "mean residual lifetime." This very appropriate name is self-explanatory. That is, given that an item has lasted t units of time, how much longer is it expected to last? Therefore defining S as the random variable which is the additional duration time, the mean residual duration is the conditional expectation

$$E[S \mid t] = \text{mean additional duration.}$$

That is, $E[S \mid t]$ is merely the expected value of the additional duration, given that the total duration is at least t units of time. Of course $E[S \mid t = 0]$ is simply the unconditional average duration.

At this point it would be helpful to give a specific empirical example. Horvath (1968) collected data for 3317 United States strikes that were settled in 1961. The average length of these strikes was 25 days. However, the longer a strike was in process, the longer it was expected to continue, as the following mean residual durations show.

t	$E[S \mid t]$
0	25
10	35
20	43
50	53
100	69
150	90

This illustrates the aggregate level inertia phenomenon that appears in most social science durations data. The average length of a strike is only 25 days, but given that a strike has gone on for 100 days, it is expected to last an additional 69 days. From a purely descriptive point of view $E[S \mid t]$ is clearly a monotonically increasing function of t. Does this empirical fact imply that at the individual company level the longer a strike goes on, the longer it is expected to continue? The answer is "no," but the justification for this answer must be postponed until the formal model building begins.

A more fundamental question is raised. Is $E[S \mid t]$ even a valid descriptive measure of aggregate level inertia? For Horvath-type data the answer is "yes, up to a point." Horvath collected data on all strikes *settled* (not started) in 1961. Hence every strike had an exact duration. Clearly all of the various conditional expectations could be calculated. Nevertheless any data set has a single largest value. In the Horvath data this value is 550 days. (Actually he only reported two strikes with durations from 501 to 600 days, which we estimate as 550.) For these data the expected additional duration, given that a strike has gone on 549 days, is only 1 day. Obviously the empirical $E[S \mid t]$ will eventually start decreasing as a function of t because of the upper limit on the observed duration values. Therefore for large values of t an empirically discovered lack of inertia (i.e., $E[S \mid t]$ starting to fall as t increases) may be "real" or it may just be an artifact of the finiteness of the data.

A more damaging argument against the use of $E[S \mid t]$ is that for most duration sets it cannot be calculated. For most job duration studies a cohort of people is observed for some fixed period of time—say 5 years. Often as many as half the people do not change jobs. The upper duration limit is then the somewhat vague "greater than 5 years" value. $E[S \mid t]$ can be calculated if some average value is assigned to this group. However, there is usually no compelling argument for any particular value and $E[S \mid t]$ can be very sensitive to the value that is picked. One measure of inertia that does not depend on the value assigned to the last group is the *median* additional duration.

Median Additional Duration

The median of a distribution is the point at which half of the values are below and half above. The median additional (residual) duration is then analogous to the mean additional duration, with the median playing the role of the mean. This concept of median residual duration has two advantages with respect to the mean residual duration. First, the median is a more stable statistic than the mean for the highly skewed distributions encountered in most duration studies. Second, the median can be calculated even with censored (truncated) data where the last group is those with greater than or equal to some upper value duration.

The mathematics of the median residual duration is given in the Appendix (Sections A.1 and A.2). The histograms necessary to calculate this statistic are also presented (except for the very extensive Horvath strike data histogram). We will calculate the median residual duration for the Richardson war data (see Table 1), however, most of the discussion in the paper will center around the parameter estimates of the specific probability duration models to be discussed shortly. Letting $M[S \mid t]$ be the median additional duration given a duration of at least t units of time, the Richardson war data yield:

t (years)	$M[S \mid t]$
0	0.96
1	1.95
2	1.95
3	2.22
4	2.60

The sample sizes are too small after $t = 4$ to give meaningful numbers. However, the *mean* additional duration could not have been calculated, and we do see (at least in the aggregate) the inertia phenomenon in the war durations data.

We now move on to another empirical statistic, the time-dependent quitting rate, which also can be used to show inertia at the aggregate level. This statistic is not affected by a censored upper category.

Time-Dependent Quitting Rates

Again using the Horvath strike data, we can calculate the "quitting" (actually "settlement") rate each day. Of the 3317 total strikes, the numbers that were settled on Days 1 through 6 were:

t	Number settled	$u(t)$
1	388	.117 = 388/3317
2	317	.108 = 317/2929
3	214	.092 = 241/2612
4	154	.067 = 154/2371
5	175	.080 = 175/2212
6	117	.057 = 117/2037

Let $u(t)$ be the proportion of strikes that were on at the start of Day t and were settled on Day t. This concept $u(t)$ is called the hazard function (for obvious reasons) in the reliability literature. (In the Appendix, Section A.4, a better way to calculate the quitting rate function is given.) Except for a slight "blip" at Day 5, the empirical $u(t)$ is monotonically decreasing with time. That is, the longer a strike has been on, the less likely a settlement is to occur in the next interval of time.

One additional empirical quitting rate example will be helpful. March and March (1978), in their study of school superintendents, state (but do not give the actual data):

> Before age 58, both the promotion rate and the exit rate show a similar pattern as a function of tenure (controlling for age and number of superintendencies). The rates are relatively low the first year, rise to a peak about the second, third or fourth year and decline thereafter. The exit rates appear to peak a year or two later than the promotion rate.

As we will see later in this paper, this nonmonotonic quitting and promotion rate phenomenon found by March and March has very strong implications for the type of stochastic model that is generating the durations data. The monotonically decreasing aggregate quitting rate found in the Horvath strike data implies a much different behavioral assumption at the individual level than does the starting low, reaching a peak, and then declining quitting rate found by March and March.

In both job duration and strike data inertia is found when the aggregate data are analyzed. However, at the individual level this inertia phenomenon appears to be "real" with strikes (and also wars), but it is spurious in job durations. For job durations it is much more likely that at the individual level the quitting rate actually increases with time. We now move on to the model building and statistical methods that allow us to make such conclusions.

INDIVIDUAL-LEVEL JOB DURATIONS

Random Quitting

The simplest possible stochastic model for job duration would be that of a constant quitting rate. That is, each day (or week or month) the individual has a probability p of quitting. This probability is independent of previous time in the job. Given this assumption the number of days (or weeks or months) that an individual stays on the job is a geometric random variable with a mean of $1/p$. Since the quitting rate is independent of t, the time in the job, then the mean additional duration $E[S \mid t]$ is also independent of t. That is,

$$E[S \mid t] = 1/p, \qquad \text{for all values of } t.$$

In model building it is usually more convenient to work with continuous time. The continuous analog to the discrete geometric distribution is the exponential, with a constant quitting rate λ. Hence T, the random variable, has the following probability density function (p.d.f.), cumulative distribution function (C.D.F.), and quitting rate, respectively:

$$f(t;\lambda) = \lambda e^{-\lambda t}, \qquad t > 0. \qquad \lambda > 0. \tag{1}$$

$$F(t;\lambda) = 1 - e^{-\lambda t}. \qquad t > 0. \tag{2}$$

$$u(t;\lambda) = f(t;\lambda)/(1 - F(t;\lambda)) = \lambda, \qquad t > 0. \tag{3}$$

The mean duration and mean residual duration are

$$E[T] = 1/\lambda.$$

$$E[S \mid t] = 1/\lambda, \qquad \text{for all values of } t.$$

Clearly the exponential distribution is the appropriate model for random (meaning time-independent) quitting.

Time-Dependent Quitting

When modeling duration times the quitting rate function is a more natural starting point than either the p.d.f. or the C.D.F. (Mathematically it doesn't matter which is specified first since any one of the three concepts uniquely determines the other two. See Section A.3 of the Appendix for the formulas.) The logical generalization of the exponential that allows for a time varying quitting rate would be to let $u(t)$ vary as a power of t. Namely,

$$u(t;\lambda,\beta) = \lambda\beta t^{\beta-1}, \qquad t > 0. \tag{4}$$

Notice that when $\beta = 1$, (4) reduces to (3) and we again have the exponential. When $\beta < 1$ the quitting rate decreases with time (i.e., there is inertia at the individual level). When $\beta > 1$, $u(t)$ increases with time. That is, the longer an individual is on the job, the more likely he or she is to quit in any given succeeding interval of time. Hence β is a very parsimonious index of inertia—or the lack thereof.

The quitting rate (4) defines the Weibull distribution which has been used extensively in the reliability literature. The p.d.f. and C.D.F. of this distribution are, respectively:

$$f(t;\lambda,\beta) = (\lambda\beta)t^{\beta-1}e^{-\lambda t^{\beta}}, \qquad t > 0. \ \lambda,\beta > 0. \tag{5}$$
$$F(t;\lambda,\beta) = 1 - e^{-\lambda t^{\beta}}, \qquad t > 0. \tag{6}$$

In further discussions we will refer to λ as the scale parameter and β as the shape parameter. We now move on to the crux of our model building effort where λ, the scale parameter, is allowed to vary across the individuals in the population. The resulting aggregate probability distributions of duration times can have properties that are very different from the properties of the individual duration times.

MODELING POPULATION HETEROGENEITY

Mixtures of Exponentials

Suppose that each person's duration time is distributed exponentially—but with different means. Suppose further that the average duration for the population as a whole is 2 years. Therefore an individual picked at random has an expected duration of 2 years. However, if we pick an individual at random and observe that in (say) 4 years he or she has not changed jobs, then the chances are that we have picked someone with an average duration longer than the population overall average of 2 years. Since at the individual level the mean additional duration is the original mean duration, the mean additional duration for this individual with no change in 4 years will be longer than 2 years. In a Bayesian sense we have updated our knowledge of this individual's λ (scale) parameter by observing no change in the 4-year period of observation. At the aggregate level $E[S\,|\,t]$ increases with t, showing apparent inertia. The best way to illustrate these points is to develop a particular (and mathematically tractable) mixture of exponentials model.

Assume that each individual has an exponential duration with parameter λ (i.e., a mean of $1/\lambda$). That is,

$$f(t;\lambda) = \lambda e^{-\lambda t}, \qquad t > 0.$$

Now let λ be distributed according to some p.d.f. $g(\lambda)$ across the population of individuals. In other words $g(\lambda)$ captures the heterogeneity of the quitting rates in the population. The aggregate distribution of quitting rates is then each $f(t;\lambda)$ weighted by the likelihood of that particular value of λ occurring. Formally, the aggregate duration p.d.f. $h(t)$ is

$$h(t) = \int_0^\infty f(t;\lambda)\, g(\lambda) d\lambda, \qquad t > 0. \tag{7}$$

In the language of probability mixture models $g(\lambda)$ is called the mixing distribution and $h(t)$ is called the mixture.

When the mixing distribution $g(\lambda)$ is a gamma distribution,

$$g(\lambda;r,\alpha) = \frac{\alpha}{\Gamma(r)}(\alpha\lambda)^{r-1} e^{-\alpha\lambda}, \qquad \lambda > 0, \qquad r,\alpha > 0, \tag{8}$$

the heterogeneity of the quitting rate λ can take on a variety of J shapes when $r < 1$ and unimodal shapes when $r > 1$. Notice that when $r = 1$ the gamma becomes an exponential distribution. The only distributions of quitting rates that cannot be captured in spirit by the gamma are ones that are strongly bimodal. That is, many individuals have low rates and many have high rates but very few fall in the middle. However, a different class of mixture models will be used for that situation.

The mean and variance of the gamma distribution are

$$\begin{aligned} E[\lambda] &= r/\alpha, \\ \operatorname{Var}[\lambda] &= r/\alpha^2. \end{aligned}$$

Notice that if we define the heterogeneity of λ across the population by the coefficient of variation

$$k = SD(\lambda)/E[\lambda],$$

where $SD(\lambda)$ equals the standard deviation of λ, we have a particularly parsimonious measure, namely,

$$k = r^{-1/2}. \tag{9}$$

When the exponentially distributed individual durations $f(t;\lambda)$ are mixed by a gamma mixing distribution $g(\lambda)$, the resulting mixture is

$$h(t;r,\alpha) = \frac{r}{\alpha}\left(\frac{\alpha}{\alpha + t}\right)^{r+1}, \qquad t > 0. \tag{10}$$

This distribution is called the Pareto distribution [see Johnson & Kotz (1970)] or, equivalently, a Type XI distribution in the Pearson system. The C.D.F., mean additional duration, and quitting rate function for this Pareto distribution are derived in the Appendix and are, respectively:

$$H(t;r,\alpha) = 1 - \left(\frac{\alpha}{\alpha + t}\right)^r. \tag{11}$$

$$E[S\,|\,t] \;= \frac{\alpha}{r-1} + \left(\frac{1}{r-1}\right)t, \qquad \text{if } r > 1. \tag{12}$$

$$\bar{u}(t;r,\alpha) \;= \frac{r}{\alpha + t}\,. \tag{13}$$

Notice that the mean residual duration $E[S\,|\,t]$ is a linearly increasing function of t and that $\bar{u}(t)$ is a monotonically decreasing function of t. Morrison (1978) shows that the gamma is the unique mixing distribution that gives a linearly increasing $E[S\,|\,t]$. Recall that both an increasing $E[S\,|\,t]$ and a decreasing $\bar{u}(t)$ are characteristics of inertia at the aggregate level. However, since each individual has an exponentially distributed duration (and hence by definition has no inertia), the apparent inertia shown by equations (12) and (13) is totally due to the heterogeneity of the individual quitting rate λ across the population. In this model all of the observed inertia at the aggregate level is a spurious effect due to heterogeneity. We now move on to a more general probability mixture model of durations that allows for time varying quitting rates at the individual level.

Mixtures of Weibulls

As mentioned earlier the Weibull distribution is the natural generalization of the exponential. Each individual's duration time p.d.f. is defined by (5) with a time-dependent quitting rate (4), i.e.,

$$u(t;\lambda,\beta) = \lambda\beta t^{\beta-1}, \qquad t > 0.$$

We could let both λ and β vary across the population and characterize this heterogeneity by a bivariate mixing distribution $g(\lambda,\beta)$. However, this becomes mathematically burdensome and probably overly complicated. If we allow the scale parameter λ to vary, we allow for differing mean durations. Clearly this form of heterogeneity is mandatory. However, if we fix β we are not being all that restrictive. We are merely forcing the time-dependent quitting rates to be of the same form for all individuals. For example, if we set $\beta = 2$, then all individuals have a linearly increasing quitting rate but the slope of this linear function will vary as λ varies.

The main model of this paper will keep the shape parameter β fixed for all individuals and have a gamma distribution on the scale parameter λ. The resulting aggregate mixture distribution $h(t)$ will be

$$h(t;r,\alpha,\beta) = \int_0^\infty f(t;\lambda,\beta)\, g(\lambda;r,\alpha)\,d\lambda, \qquad t > 0,$$

with $f(t;\lambda,\beta)$ defined by (5) and $g(\lambda;r,\alpha)$ defined by (8). This yields the following p.d.f., C.D.F., and aggregate quitting rate functions, respectively.

(Again these are derived in the Appendix, Section A.2. The mean additional duration is very complex and not presented here.)

$$h(t;r,\alpha,\beta) = \frac{r\beta\alpha^r t^{\beta-1}}{(\alpha + t^\beta)^{r+1}}, \qquad t > 0. \tag{14}$$

$$H(t;r,\alpha,\beta) = 1 - \frac{\alpha^r}{(\alpha + t^\beta)^r}, \qquad t > 0. \tag{15}$$

$$\bar{u}(t;r,\alpha,\beta) = \frac{r\beta t^{\beta-1}}{\alpha + t^\beta}, \qquad t > 0. \tag{16}$$

Some additional properties of this gamma mixture of Weibulls can be found in Dubey (1968).

The aggregate quitting rate $\bar{u}(t;r,\alpha,\beta)$ yields the most insight for analyzing empirical data. When $\beta < 1$ (i.e., each individual's quitting rate decreases with time—there is inertia at the individual level), $\bar{u}(t;r,\alpha,\beta)$ also decreases monotonically with time. Now notice what happens when $\beta > 1$ (i.e., each individual's quitting rate increases with time—we could call this "negative inertia"). When $t = 0$, $\bar{u}(t;r,\alpha,\beta)$ equals zero since the numerator is zero. When $t \to \infty$, $\bar{u}(t;r,\alpha,\beta)$ also goes to zero since the t^β term in the denominator dominates the $t^{\beta-1}$ term in the numerator. As shown in Section A.2 of the Appendix, $\bar{u}(t;r,\alpha,\beta)$ starts at zero, monotonically increases to a point

$$t^* = \left[\alpha(\beta - 1)\right]^{1/\beta},$$

and then monotonically decreases to zero. Notice that this is the qualitative phenomenon described by March and March (1978) which we discussed earlier. Also notice that for $t > t^*$ the aggregate duration distribution shows inertia, while not a single individual has inertia. A naive interpretation of the aggregate quitting rate would be totally misleading.

A Bayesian interpretation of $\bar{u}(t;r,\alpha,\beta)$ is instructive. For small values of t (i.e., we have observed the individual for a very short period of time) we receive very little added information on this individual's mean duration. Therefore since each individual has an increasing quitting rate (recall that all individuals have the same $\beta > 1$), the aggregate quitting rate also increases. However, when $t > t^*$ the Bayesian updating on the individual's mean duration (we have observed this individual for a long time with no job change, therefore chances are that we are observing an individual with a long mean duration) overtakes the fact that each individual has an increasing quitting rate.

Interpretation of the Parameters

The gamma mixture of Weibull distributions that we have presented is a three-parameter distribution. The scale parameter α of the gamma mixing distribution $g(\lambda;r,\alpha)$ has no behavioral significance. It merely reflects the units by which we measure time (e.g., days, weeks, months, etc.). However, the numerical values of the two remaining parameters, β and r, are not affected by units of time and each has a rather profound, yet simple behavioral interpretation. The shape parameter β of the individual-level Weibull distribution of durations $f(t;\lambda,\beta)$ measures the degree (or lack thereof) of inertia. When $\beta > 1$ the individual quitting rates actually increase with time. Hence even though the aggregate duration distribution $h(t;r,\alpha,\beta)$ shows inertia, there is in fact no true inertia. In summary, β is an index of true inertia at the individual level.

The shape parameter r of the mixing distribution $g(\lambda;r,\alpha)$ determines the heterogeneity of the scale parameter λ—which in turn determines the heterogeneity in the mean duration times across individuals in the population. Equation (9) gives the coefficient of variation of λ which is the very simple function of r, namely,

$$k = r^{-1/2}.$$

Therefore if $r = 4$, $k = 1/2$, and we know that the standard deviation of λ is half as large as the mean of λ. The shape parameter r is thus an index of the heterogeneity of individual mean durations in the population.

Two-Point Mixtures of Exponentials

If the first model presented, a gamma mixture of exponentials, does not fit a set of data very well, one reason could be that the exponential distribution of durations at the individual level is a poor assumption. This possibility led to the generalization of the exponential to the Weibull which allows for time-dependent quitting rates. However, the problem could be in the mixing distribution $g(\lambda)$. We mentioned earlier that the gamma distribution cannot take on bimodal shapes. The simpliest distribution that captures the spirit of a strongly bimodal distribution is a discrete distribution with a probability mass at two points. That is, a proportion p of the population will have exponential durations with parameter λ_1 (i.e., a mean of $1/\lambda_1$) and the remaining proportion $1 - p$ will have exponential durations with parameter λ_2. More formally,

$$h(t;p,\lambda_1, \lambda_2) = p\lambda_1 e^{-\lambda_1 t} + (1 - p)\lambda_2 e^{-\lambda_2 t}, \qquad t > 0. \tag{17}$$

This model was used by Bartholomew (1959) in analyzing job durations. If we let $\lambda_2 < \lambda_1$, then the aggregate quitting rate is monotonically decreasing starting at $p\lambda_1 + (1 - p)\lambda_2$ and reaches an asymptotic value of λ_2. This is obvious since at $t = 0$ we have no additional information and hence the aggregate quitting rate is the simple weighting of λ_1 and λ_2. As t becomes very large, chances are that we are observing an individual with the longer mean duration, $1/\lambda_2$, and the smaller quitting rate, λ_2. Therefore the aggregate quitting function decreases to λ_2. Similarly the aggregate mean additional duration function starts at $p(1/\lambda_1) + (1 - p)(1/\lambda_2)$ at $t = 0$ and increases monotonically to an asymptote at the larger of the two mean durations $1/\lambda_2$.

We will now briefly describe the data sets of duration times that will be fit to the three mixture models developed in this section: the gamma mixture of exponentials, the gamma mixture of Weibulls, and the two-point mixture of exponentials.

THE DATA

Wars

Only one histogram for wars is analyzed. These data were originally collected by Richardson (1945, 1960a, b) and also presented and analyzed by Horvath (1968). These wars occurred between 1820 and 1949. There is a total of 315 war durations in yearly intervals from 0–1 to 9–10, with about 3% of the durations in the "over 10" category.

Strikes

The Horvath data contain 3317 strike durations in intervals of 1 day up to 150 days, 10-day intervals up to 200, 20-day intervals up to 300, and 100-day intervals up to 600. These were strikes settled in 1961 and recorded by the U.S. Department of Labor.

Lancaster (1972) collected data on strikes in the United Kingdom that commenced in 1965. The Ministry of Labour attempts to record all strikes except those lasting only 1 day or in companies with less than 10 workers. The 1-day strikes are not recorded since most of them are considered to be union pressure tactics designed to last a maximum of 1 day and hence not real strikes. Lancaster records the strike durations separately for the following five industries: nonelectrical engineering, 149 observations; construction, 225 observations; shipbuilding and marine engineering, 112 observations; vehicles and cycles, 103 observations; and metal manufacturing, 199 cases. We also make an aggregate histogram which includes these five industries plus 52 additional strikes from the distributive trades sector. Where possible Lancaster also classifies the strikes by three different causes: dispute over wages, dispute over employment (persons, classes), and dispute over working arrangements (rules, discipline).

Job Duration

Bartholomew (1959) collected job duration data for five British firms identified only as Firm I through Firm V with sample sizes of 1206, 629, 1161, 7628, and 968, respectively. Unfortunately we cannot say much more about these data. In no sense do we claim that these five firms are representative of all firms or any particular subset of firms or industries.

PARAMETER ESTIMATION

For the gamma mixtures of exponentials and Weibulls we used the empirical histograms and found maximum likelihood estimates through a numerical optimization (hill climbing) procedure. See Kalwani and Morrison (1977) for a more detailed discussion of this type of maximum likelihood estimation. To facilitate comparisons, a simple homogeneous Weibull was also estimated by the same method, for all histograms except the truncated Lancaster strike data. Extensive tests were done to assure the numerical stability of the resulting parameter values. Since the gamma mixture of exponentials is a special case of the gamma mixture of Weibulls, the obvious likelihood ratio test applies. This test was performed for all of the data sets. This likelihood ratio tests the null hypothesis of the gamma mixture of exponentials versus the more general alternative hypothesis of a gamma mixture of Weibulls.

The two-point mixture of exponentials model was estimated using Bartholomew's (1959) method. The theoretical C.D.F. was fit to the empirical C.D.F. at three time points, t_1, t_2 and t_3, where these times were in a ratio 1:2:6 (e.g., 3, 6, and 18 months). Since the Lancaster strike data were missing the 1-day values, and two-point mixture model could not be fit to these data.

All of the data (except for the Horvath strike data) are presented in the next section. The observed vs. expected values for all four models (gamma mixture of exponentials, gamma mixture of Weibulls, homogeneous Weibull, and two-point mixture of exponentials—where possible) along with parameter values and goodness of fit statistics are also given.

DISCUSSION

For the data on wars, a comparison of predicted with observed durations (Table 1) demonstrates that all four models fit the data well. The goodness of fit statistic used was the Pearson χ^2:

$$\chi^2 = \sum_{i=1}^{v} \frac{(O_i - E_i)^2}{E_i},$$

where O_i is the number of observed failures in period i, E_i is the number of expected failures in that period, using the model, and v is the number of periods studied. This statistic is asymptotically distributed χ^2 with $v - q - 1$ degrees of freedom, where q is the number of model parameters estimated. In all cases we cannot reject the null hypothesis that the observed durations were generated from the estimated model. The likelihood ratio test does, however, indicate that the gamma mixture of Weibulls is significantly better (at the .05 level) in its predictions than the gamma mixture of exponentials.

TABLE 1
RICHARDSON WAR DATA

	Observed	Gamma/ Weibull	Gamma/ exptl.	2-Point exptl.	Weibull
0–1	165	163.09	157.95	165.00	163.69
1–2	46	53.22	61.94	46.00	52.22
2–3	33	29.66	30.86	29.85	29.50
3–4	22	18.65	17.70	21.20	18.77
4–5	11	12.52	11.14	15.13	12.73
5–6	11	8.78	7.49	10.81	8.99
6–7	5	6.36	5.29	7.72	6.53
7–8	4	4.71	3.88	5.51	4.85
8–9	4	3.56	2.94	3.94	3.67
9–10	3	2.74	2.28	2.81	2.81
>10	11	11.71	13.54	7.03	11.25
MSE		8.04	31.58	4.83	6.68
χ^2 goodness of fit		3.24^{NS}	8.36^{NS}	5.13^{NS}	2.97^{NS}
df		7	8	7	8
Likelihood ratio (LR) χ^2 (1) = 5.21^{a}					

Parameter estimates

	r	α	β	λ	k	p	λ_1	λ_2
Gamma/Weibull	16.2	21.7	.6896		.248			
Gamma/exptl.	1.81	2.13						
2-Point exptl.						.354	3.175	.337
Weibull			.658	.733				

Note. NS, not significant at .05 level (α > .05); a, .01 < α < .05; b, α < .01. These symbols also apply to subsequent tables.

The parameter estimates for the Gamma/Weibull model indicate a population relatively homogeneous in terms of mean duration (small $k = r^{-1/2}$), which explains the fairly good fit of the homogeneous Weibull model. Furthermore, individual wars seem to exhibit a decreasing settlement rate over time ($\beta < 1$). Thus, apart from the effects of heterogeneity, there appears to be "real" inertia at the individual level.

Regarding the two-point exponential, it is clear that λ_1 is fitting the low end of the observed histogram, with λ_2 accounting for the tail. The resulting goodness of fit is impressive (smallest mean squared error of any model), but the fact that both the parameter estimates and the goodness of fit statistics are highly sensitive to the choice of percentage points for estimation limits the usefulness of those estimates for individual-level inference.

Parameter estimates for the strike data (Tables 2–5) are again consistent with a relatively homogeneous population (in terms of mean strike duration). For both the United States data and the British six-industry

TABLE 2
HORVATH STRIKE DATA: PARAMETER ESTIMATES

	r	α	β	λ	k	p	λ_1	λ_2	χ^2	df	LR
Gamma/ Weibull	4.41	30.8	.756		.476				208.31[b]	160	86.28[b]
Gamma/ exponential	1.42	13.5							300.71[b]	161	
2-Point exponential						.524	.173	.026	1525.5[b]	160	
Weibull			.651	.162					252.77[b]	161	

aggregate, there is evidence of individual-level inertia ($\beta < 1$); i.e. as with wars, the likelihood of a settlement appears to decline over time.

Breaking Lancaster's British data up by industry indicates a slight variation in β, but the nonsignificant comparison of the gamma/Weibull with the gamma/exponential and the small sample sizes here indicate that the β estimates are probably not very stable.

The same limitations apply when those data are classified by cause. The

TABLE 3
LANCASTER STRIKE DATA: ALL SIX INDUSTRIES

Time (days)	Observed	Gamma/Weibull	Gamma/exptl.
1–2	203	212.81	208.06
2–3	149	135.45	137.93
3–4	100	93.80	96.37
4–5	71	68.39	70.11
5–6	49	51.70	52.68
6–7	33	40.19	40.64
7–8	29	31.93	32.05
8–9	26	25.83	25.74
9–10	23	21.22	21.00
10–11	14	17.65	17.37
11–12	12	14.85	14.54
12–13	9	12.62	12.30
13–14	11	10.82	10.51
14–15	15	9.35	9.05
15–16	6	8.13	7.85
16–17	7	7.12	6.86
17–18	6	6.27	6.03
18–19	4	5.55	5.33
19–20	4	4.94	4.73
20–25	17	18.06	17.30
25–30	16	11.18	10.72
30–35	8	7.39	7.13
35–40	8	5.14	4.99
40–50	12	6.47	6.37
>50	8	13.15	14.35

TABLE 4A
LANCASTER STRIKE DATA—BY INDUSTRY: OBSERVED VALUES

Time (days)	Nonelectrical engineering	Construction	Shipbuilding	Vehicles & cycles	Metal mfg.
1–2	41	44	27	34	43
2–3	28	33	19	19	37
3–4	18	28	19	10	21
4–5	8	23	7	8	19
5–6	9	11	4	6	11
6–7	3	12	5	5	8
7–8	7	9	3	2	8
8–9	5	6	1	3	9
9–10	3	13	2	2	3
10–15	11	16	11	6	16
15–20	4	7	6	4	4
20–25	4	6			4
25–30		6	8	4	3
30–40	8				3
40–50		11			5
>50					5

likelihood ratio statistics are still not significant (as an index of whether β is greater or less than one), but the directions are interesting. Disputes over working arrangements, rules, and discipline are the rarest of the three types, and would be expected to cause more bargaining problems (i.e., demands are likely to be more discrete, leaving less room for compromise than, for example, the percentage wage increase in a contract).

TABLE 4B
LANCASTER STRIKE DATA—BY CAUSE: OBSERVED VALUES

Time (days)	Wage disputes	Employment of persons, classes	Working arrangements (rules, discipline)
1–2	88	53	43
2–3	65	42	24
3–4	52	22	12
4–5	26	29	8
5–6	21	13	9
6–7	16	10	3
7–8	15	6	5
8–9	16	9	4
9–10		4	4
10–15	31	16	6
15–20	13	9	4
20–30	14	11	5
>30	16	17	5

TABLE 5
LANCASTER STRIKE DATA: PARAMETER ESTIMATES

		r	α	β	k	χ^2	df	LR
6-Industry aggregate	Gamma/Weibull	2.60	5.73	.847	.620	21.64^{NS}	21	.624^{NS}
	Gamma/exptl.	1.83	4.94			22.44^{NS}	22	
By industry								
Nonelectrical engineering	Gamma/Weibull	1.55	2.97	.965	.803	5.25^{NS}	9	.0076^{NS}
	Gamma/exptl.	1.46	2.92			5.25^{NS}	10	
Construction	Gamma/Weibull	1.15	6.14	1.31	.933	12.66^{NS}	10	.482^{NS}
	Gamma/exptl.	1.98	7.06			12.84^{NS}	11	
Shipbuilding	Gamma/Weibull	.982	3.49	1.33	1.01	9.00^{NS}	8	.0082^{NS}
	Gamma/exptl.	1.64	4.07			8.93^{NS}	9	
Vehicles & cycles	Gamma/Weibull	7.63	8.25	.582	.362	1.45^{NS}	8	.468^{NS}
	Gamma/exptl.	1.79	3.15			1.97^{NS}	9	
Metal mfg.	Gamma/Weibull	.724	4.12	1.63	1.18	10.72^{NS}	12	1.09^{NS}
	Gamma/exptl.	1.53	4.10			11.42^{NS}	13	
By cause								
Wage disputes	Gamma/Weibull	1.08	4.02	1.30	.962	4.83^{NS}	8	.434^{NS}
	Gamma/exptl.	1.70	4.44			5.32^{NS}	9	
Employment	Gamma/Weibull	.565	3.48	1.80	1.33	8.31^{NS}	9	1.44^{NS}
	Gamma/exptl.	1.32	3.36			10.04^{NS}	10	
Working arrangements	Gamma/Weibull	3.17	2.92	.606	.562	4.19^{NS}	9	.319^{NS}
	Gamma/exptl.	1.31	1.91			4.57^{NS}	10	

Not surprisingly, the parameter estimates here indicate true individual level inertia—as if bargaining positions tended to harden over time. Clearly, a future research project that could shed some light on this issue would be the classification of Horvath's United States strike data by primary cause, where the resulting sample sizes would permit more reliable parameter estimation.

The results for the job duration data in Table 6 illustrate the general good fit of the three mixture models (with the exception of Firm IV, where overprediction of the longer durations is a problem). They also highlight the sensitivity of the χ^2 statistic to the number of observations (here, number of duration periods) examined, with statistical rejection of the models much more likely with a fine division of duration times.

The parameter estimates in Table 7 indicate a wide variation in both the individual level inertia (as measured by β) and the degree of homogeneity in employees' mean durations (the k values). For instance, workers at Firm I seem characterized by a nearly constant quitting rate over time (β close to 1). Since from equation (4) individual quitting rates increase (or

TABLE 6

BARTHOLOMEW JOB DURATION DATA: OBSERVED AND PREDICTED VALUES

Time (months)	Observed	Gamma/ Weibull	Gamma/ exptl.	2-Point exptl.	Weibull
		Firm I			
0–3	242	242.63	241.35	242.00	251.45
3^+–6	152	148.43	150.28	152.00	129.77
6^+–9	104	103.24	103.88	101.56	96.75
9^+–12	73	76.74	76.73	72.80	77.57
12^+–15	52	59.66	59.35	55.97	64.50
15^+–18	47	47.92	47.48	45.73	54.87
18^+–21	49	39.46	38.99	39.15	47.41
>21	487	487.93	487.94	496.80	483.68
MSE		22.38	21.59	27.06	111.09
χ^2 goodness of fit		3.58^{NS}	3.69^{NS}	3.05^{NS}	8.60^{NS}
df		4	5	4	5
Likelihood ratio: $\chi^2_{(1)} = .075^{NS}$					
		Firm II			
0–3	182	181.91	187.65	182.00	191.95
3^+–6	103	103.97	91.80	103.00	78.02
6^+–9	60	56.58	55.86	60.73	53.44
9^+–12	29	36.08	38.11	37.97	40.38
12^+–15	31	25.40	27.92	25.56	32.05
15^+–18	23	19.08	21.48	18.68	26.24
18^+–21	10	14.99	17.12	14.73	21.94
>21	191	190.99	189.06	186.33	184.99
MSE		16.79	40.46	21.67	135.70
χ^2 goodness of fit		5.30^{NS}	7.45^{NS}	5.92^{NS}	19.65^{b}
df		4	5	4	5
Likelihood ratio: $\chi^2_{(1)} = 2.61^{NS}$					
		Firm III			
0–3	412	411.96	416.42	412.00	421.90
3^+–6	143	135.44	124.40	143.00	104.90
6^+–9	66	67.63	68.47	62.58	67.52
9^+–12	45	43.42	45.57	37.80	50.00
12^+–15	27	31.28	33.41	29.47	39.61
15^+–18	18	24.11	26.00	26.04	32.68
>18	450	447.16	446.74	450.11	444.39
MSE		18.01	69.67	19.19	283.35
χ^2 goodness of fit		2.67^{NS}	6.64^{NS}	4.25^{NS}	25.29^{b}
df		3	4	3	4
Likelihood ratio: $\chi^2_{(1)} = 4.06^{a}$					

TABLE 6—*Continued*

Time (months)	Observed	Gamma/ Weibull	Gamma/ exptl.	2-Point exptl.	Weibull
		Firm IV			
0–1	1323	1295.96	1555.07	1336.33	1902.84
1–2	1053	1147.78	924.70	998.35	637.41
2–3	755	743.17	625.86	750.57	439.89
3–4	523	509.57	457.62	568.76	340.83
4–5	419	372.40	352.30	435.23	279.31
5–6	327	285.89	281.40	337.00	236.75
6–7	274	227.78	231.09	264.62	205.32
7–8	228	186.73	193.92	211.15	181.02
8–9	169	156.54	165.57	171.53	161.63
9–10	133	133.62	143.39	142.04	145.76
10–11	121	115.75	125.66	119.99	132.52
11–12	132	101.51	111.24	103.39	121.29
12–13	96	89.95	99.32	90.78	111.64
13–14	103	80.41	89.34	81.12	103.26
14–15	81	72.44	80.90	73.61	95.91
15–18	194	180.62	203.16	189.77	251.50
18–21	144	141.06	159.79	159.89	209.08
21–24	99	114.08	129.69	140.82	177.28
24–27	80	94.72	107.83	126.61	152.61
27–30	53	80.27	91.38	114.89	132.97
30–36	66	129.54	147.23	200.18	220.83
36–42	52	100.86	114.20	166.92	176.15
42–48	32	81.40	91.72	139.35	143.75
48–54	13	67.48	75.65	116.35	119.41
54–60	10	57.13	63.69	97.15	100.61
60–66	4	49.18	54.53	81.12	85.75
>66	1144	1012.16	951.75	410.49	762.70
MSE		2011.89	6127.79	22981.3	34286.6
χ^2 goodness of fit		298.28[b]	462.40[b]	1895.24[b]	1796.62[b]
df		23	24	23	24
Likelihood ratio: $\chi^2 = 192$[b]					
		Firm V			
0–1	80	76.90	99.28	74.39	129.06
1–2	64	76.20	60.41	63.00	40.84
2–3	51	51.70	42.58	53.37	28.92
3–4	41	37.24	32.49	45.23	23.03
4–5	38	28.53	26.04	38.36	19.38
5–6	39	22.87	21.60	32.54	16.85
6–7	27	18.95	18.37	27.63	14.98
7–8	18	16.09	15.93	23.48	13.53
8–9	16	13.93	14.02	19.96	12.36
9–10	17	12.25	12.49	17.00	11.40

TABLE 6—*Continued*

Time (months)	Observed	Gamma/ Weibull	Gamma/ exptl.	2-Point exptl.	Weibull
10–11	10	10.90	11.23	14.49	10.59
11–12	13	9.80	10.19	12.37	9.90
12–13	13	8.89	9.32	10.58	9.30
13–14	5	8.12	8.57	9.07	8.77
14–15	6	7.47	7.92	7.79	8.31
15–18	16	19.30	20.65	17.51	22.60
18–21	16	15.83	17.11	11.51	19.82
21–24	16	13.36	14.54	7.89	17.67
24–27	9	11.51	12.60	5.70	15.96
27–30	6	10.08	11.08	4.37	14.56
30–36	5	16.99	18.74	6.63	25.79
36–42	3	13.92	15.40	5.35	22.33
42–48	2	11.73	13.01	4.85	19.69
48–54	1	10.11	11.22	4.64	17.60
>54	456	445.32	443.21	450.29	434.77
MSE		50.15	76.45	14.16	274.82
χ^2 goodness of fit		64.15[b]	81.57[b]	27.30[NS]	206.31[b]
df		21	22	21	22
Likelihood ratio: $\chi^2_{(1)} = 20.32$[b]					

decrease) over time as a function of $t^{\beta-1}$, the situation $1 < \beta < 2$, as in Firms II, IV, and V, implies that individual rates increase less than linearly over time. Similarly, $\beta = 3.86$ in Firm III indicates that individual quitting rates increase sharply (more than linearly) with time.

As we have previously suggested, the implications of these differences for reducing turnover can be significant. Since all firms studied show evidence of considerable heterogeneity in individual employees' mean durations (much more than was the case for strikes or wars), Firm I may put more emphasis on initial selection procedures to identify those with longer durations. While Firm III may also follow this strategy, it should probably examine the job structure and work environment as well, for causes of the increasing quitting rates.

Thus, in contrast to the situation for war and strike durations, the apparent inertia (decreasing aggregate quitting rate) observed in job durations seems due solely to the heterogeneity in the individuals' mean durations, rather than to any "real" inertia in employees.

Two limitations in the data used previously provide an impetus for further work in this area. First, the smaller sample sizes in some cases point to a need for Monte Carlo studies, to investigate the stability of the estimates (r, α, β). This method of simulating histograms and estimating parameters is proposed since the sampling distribution of the three parameters is not known.

TABLE 7
BARTHOLOMEW JOB DURATION DATA: PARAMETER ESTIMATES

	r	α	β	λ	k	p	λ_1	λ_2
Firm I								
Gamma/Weibull	.771	8.58	9.70		1.139			
Gamma/exptl.	.705	8.06						
2-Point exptl.						.349	.202	.022
Weibull			.700	.108				
Firm II								
Gamma/Weibull	.326	2.96	1.55		1.751			
Gamma/exptl.	.692	4.49						
2-Point exptl.						.563	.211	.020
Weibull			.623	.184				
Firm III								
Gamma/Weibull	.075	.196	3.86		3.651			
Gamma/exptl.	.330	1.06						
2-Point exptl.						.463	.407	.018
Weibull			.421	.284				
Firm IV								
Gamma/Weibull	.316	1.25	1.58		1.779			
Gamma/exptl.	.605	2.18						
2-Point exptl.						.609	.313	.030
Weibull			.479	.287				
Firm V								
Gamma/Weibull	.108	.865	1.77		3.043			
Gamma/exptl.	.219	1.56						
2-Point exptl.						.491	.168	.0017
Weibull			.432	.143				

Second, in those preceding cases where $\beta > 1$ (so the aggregate quitting rate increases to a point $t^* = [\alpha(\beta - 1)]^{1/\beta}$ before decreasing to zero), the maximum point t^* occurred within the first duration interval. Therefore, a simple plot of the empirical quitting rate would not pick up that peak, and would appear consistent with the gamma mixture of Weibulls with $\beta < 1$ rather than $\beta > 1$. Model discrimination (and individual level inference) should consequently be aided when duration intervals, particularly at the lower end of the histogram, are as fine as possible.

Fortunately, the data used by March and March in calculating superintendents' quitting rates were fine enough to identify the peak in each function. This suggests that the apparent inertia in their job durations, like that of the employees in Firms I–V, is a spurious effect due to the heterogeneity in mean durations, and is not a characteristic of individuals in the organization.

Finally, we should point out that the job duration data are quite different from the strike data. Each job duration histogram was from the same firm and presumably these employees faced somewhat similar work environments. Hence, our purely descriptive models could at least detect real individual-level inertia even though we couldn't say why this inertia, or lack of inertia, existed. On the other hand, the strike histograms were from different industries in different geographic areas with unions of varying strengths and under differing state laws. A better model for predicting the duration (and even the probability) of a strike would include these important independent variables. The individual-level inertia found in the strike data might disappear if these independent variables were included. In summary, our models are not theories of duration times; rather, they are better descriptive models that separate heterogeneity in mean durations from individual-level inertia.

CONCLUSION

A class of stochastic models has been presented that enables a researcher to better collect, analyze, and interpret duration data. The models are characterized by behaviorally meaningful parameters. In addition, these models force the researcher to formally define the often vaguely stated concept of inertia. The probability mixture nature of the models helps separate the spurious effects of heterogeneity in mean duration from the individual-level time-varying quitting rates.

Even the researcher who chooses not to do the formal parameter estimation can obtain benefits from the concepts of this paper. A simple plotting of the mean or median residual lifetime (where possible) and the aggregate quitting rate function will yield insights into which of the three models seems most appropriate. At the very least we have presented better ways of "eyeballing" the data.

We have developed a framework in which most durations data can be analyzed. None of our results is a "new" mathematical finding, and two of the three models have been used on durations data for over 20 years. Nevertheless, the whole package has not been published in a form accessible to applied researchers who collect and analyze durations data.

In summary, the concept of inertia has been clarified and some better methods for quantifying this important concept have been developed. Organizations can do better manpower planning when the "real" inertia is separated from the "apparent" inertia almost always found in the aggregate duration data.

APPENDIX

A.1 Gamma Mixture of Exponentials

Using equation (7) the density function becomes

$$h(t) = \int_0^\infty \lambda e^{-\lambda t} \frac{\alpha}{\Gamma(r)} (\alpha\lambda)^{r-1} e^{-\alpha\lambda} d\lambda, \qquad t > 0; r,\alpha > 0.$$

Rearranging terms:

$$h(t) = \frac{\alpha^r (\alpha + t)^{-r-1}}{\Gamma(r)} \int_0^\infty \lambda^r (\alpha + t)^{r+1} e^{-\lambda(\alpha+t)} d\lambda,$$

$$= \frac{\alpha^r (\alpha + t)^{-r-1}}{\Gamma(r)} \Gamma(r + 1),$$

$$= \frac{r}{\alpha}\left(\frac{\alpha}{\alpha + t}\right)^{r+1}, \qquad t > 0. \qquad [\text{Note: } \Gamma(r + 1) = r\Gamma(r).]$$

The C.D.F. is

$$H(t) = \int_0^t h(x)dx = \int_0^t r\, \alpha^r (\alpha + x)^{-r-1} dx,$$

$$= 1 - \left(\frac{\alpha}{\alpha + t}\right)^r.$$

To obtain the mean residual lifetime $E[S \mid t]$, we note that this is simply the individual level value

$$E[S \mid t, \lambda] = \lambda^{-1}$$

weighted by the distribution of λ over the population. However, observation of zero failures in time t has updated (in a Bayesian sense) our distribution of this individual's λ value, to a gamma $(r + 0, \alpha + t)$ rather than the "no-information" gamma (r, α).

Thus,

$$E[S \mid t] = \int_0^\infty \lambda^{-1} \text{gamma}\,(r, \alpha + t)\, d\lambda,$$

$$= [\Gamma(r)]^{-1} \int_0^\infty \lambda^{-1}(\alpha + t)\,[\lambda(\alpha + t)]^{r-1} e^{-\lambda(\alpha+t)}\, d\lambda,$$

$$= \frac{\alpha + t}{\Gamma(r)} \Gamma(r - 1),$$

$$= \frac{\alpha + t}{r - 1} = \frac{\alpha}{r - 1} + \left(\frac{1}{r - 1}\right)t, \qquad t > 0, \qquad r > 1.$$

Note: When $r < 1$, $E[S \mid t]$ diverges to infinity.

The quitting rate $\bar{u}(t)$ is, from (3) and the Bayesian updating above:

$$\bar{u}(t) = \int_0^\infty \lambda \text{ gamma}\,(r, \alpha + t)\, d\lambda,$$

$$= \frac{1}{\Gamma(r)(\alpha + t)} \int_0^\infty (\alpha + t)\left[\lambda(\alpha + t)\right]^r e^{-\lambda(\alpha+t)}\, d\lambda,$$

$$= \frac{\Gamma(r + 1)}{\Gamma(r)(\alpha + t)} = \frac{r}{\alpha + t}.$$

To derive the median residual lifetime $M[S \mid t]$, we use the fact that

$$M\left[S \mid t\right] = R^{-1}\left[\tfrac{1}{2}R(t)\right] - t, \tag{A1}$$

where $R(t) \equiv 1 - H(t)$ is the reliability function and $R^{-1}(x)$ is the inverse function of $R(t)$. We know that

$$R(t) = \left(\frac{\alpha}{\alpha + t}\right)^r,$$

so $R^{-1}(x) = \alpha(x^{-1/r} - 1)$. Substituting into (A1),

$$M\left[S \mid t\right] = R^{-1}\left[\tfrac{1}{2}\left(\frac{\alpha}{\alpha + t}\right)^r\right] - t,$$

$$= \alpha\left[\left[\tfrac{1}{2}\left(\frac{\alpha}{\alpha + t}\right)^r\right]^{-1/r} - 1\right] - t,$$

$$= (2^{1/r} - 1)(t + \alpha),$$

which is linear in t.

A.2 Gamma Mixture of Weibulls

Using (5), (7), and (8), the density function is

$$h(t) = \int_0^\infty (\lambda\beta)t^{\beta-1}e^{-\lambda t^\beta}\alpha\left[\Gamma(r)\right]^{-1}(\alpha\lambda)^{r-1}e^{-\alpha\lambda}\, d\lambda,$$

$$= \frac{\beta\alpha^r t^{\beta-1}(\alpha + t^\beta)^{-r-1}}{\Gamma(r)} \int_0^\infty (\alpha + t^\beta)\left[\lambda(\alpha + t^\beta)\right]^r e^{-\lambda(\alpha+t^\beta)}\, d\lambda,$$

$$= \frac{\beta\alpha^r t^{\beta-1}(\alpha + t^\beta)^{-r-1}}{\Gamma(r)}\Gamma(r + 1),$$

$$= \frac{r\,\beta\,\alpha^r\, t^{\beta-1}}{(\alpha + t^\beta)^{r+1}}.$$

The C.D.F. is

$$H(t) = r\,\alpha^r \int_0^t \beta\, x^{\beta-1}(\alpha + x^\beta)^{-r-1}\,dx,$$

$$= -\left[\frac{\alpha}{\alpha + x^\beta}\right]^r \Big|_0^t,$$

$$= 1 - \left[\frac{\alpha}{\alpha + t^\beta}\right]^r .$$

We can combine (4) with the Bayesian updating notion, to again obtain the quitting (or hazard) function. It is well known [see, e.g., Johnson & Kotz (1970, p. 250)] that if T is distributed Weibull (λ,β), then T^β is exponential (λ). Thus, observing no failure in our Weibull random variable for time t is equivalent to observing no failure in an exponential variable for time t^β. As a result, the updated distribution on λ is gamma $(r,\, \alpha + t^\beta)$, and we have:

$$\bar{u}(t) = \int_0^\infty \lambda\,\beta\, t^{\beta-1}\,\text{gamma}\,(r,\, \alpha + t^\beta)\,d\lambda,$$

$$= \frac{\beta\, t^{\beta-1}}{\Gamma(r)\,(\alpha + t^\beta)} \int_0^\infty (\alpha + t^\beta)\,\big[\lambda(\alpha + t^\beta)\big]^r\, e^{-\lambda(\alpha+t^\beta)}\,d\lambda,$$

$$= \frac{\beta\, t^{\beta-1}}{\Gamma(r)\,(\alpha + t^\beta)} .$$

Clearly, for $\beta > 1$, $\bar{u}(0) = 0$; and as $t \to \infty$, $\bar{u}(t) \to 0$ since the t^β term in the denominator dominates. To find the maximum of $\bar{u}(t)$, we differentiate and find

$$\bar{u}'(t) = r\beta t^{\beta-2}\,\big[\alpha(\beta - 1) - t^\beta\big].$$

Setting $\bar{u}'(t) = 0$ and solving for $t > 0$ yields the desired value

$$t^* = \big[\alpha(\beta - 1)\big]^{1/\beta},$$

which maximizes $\bar{u}(t)$.

The derivation of the mean residual lifetime is fairly complicated, and we simply state the result:

$$E\big[S\,|\,t\big] = \beta^{-1}\left(\frac{\alpha}{\alpha + t^\beta}\right)^{-r} \alpha^{1/\beta}\, B_{\alpha/(\alpha+t^\beta)}\,(r - \beta^{-1}, \beta^{-1}),$$

where $B_x(a,b)$ is the incomplete beta function

$$B_x(a,b) = \int_0^x u^{a-1}(1 - u)^{b-1}\,du.$$

To find the median residual lifetime we use (A1) and the results that for a gamma mixture of Weibulls,

$$1 - H(t) = R(t) = \left[\frac{\alpha}{\alpha + t^{\beta}}\right]^{r},$$

so $R^{-1}(x) = [\alpha(x^{-1/r} - 1)]^{1/\beta}$.

Substituting in (A1),

$$M[S|t] = \alpha\left[\left[\tfrac{1}{2}\left(\frac{\alpha}{\alpha + t^{\beta}}\right)^{r}\right]^{-1/r} - 1\right]^{1/\beta} - t,$$
$$= \left[2^{1/r}(\alpha + t^{\beta}) - \alpha\right]^{1/\beta} - t.$$

A.3 Relationships between $h(t)$, $H(t)$, $E[S|t]$, and $\bar{u}(t)$

To illustrate the isomorphism between the C.D.F., mean residual lifetime, and quitting rate function, we note that

$$\bar{u}(t) = \frac{h(t)}{1 - H(t)} = \frac{h(t)}{R(t)}, \tag{A2}$$

where the reliability function $R(t) \equiv 1 - H(t)$. Solving for $h(t)$ in (A2), we have

$$h(t) = \bar{u}(t)e - \int_0^t \bar{u}(x)dx. \tag{A3}$$

Thus, $\bar{u}(t)$ determines $h(t)$, and vice versa.

Similarly, one can show that

$$E[S|t] = [R(t)]^{-1}\int_0^{\infty} R(x)dx. \tag{A4}$$

Conversely,

$$1 - H(t) = R(t) = \frac{E[S|o]}{E[S|t]}\exp\left[-\int_0^t \frac{dx}{E[S|x]}\right]. \tag{A5}$$

So $H(t)$ [and consequently $h(t)$ and $\bar{u}(t)$, using the above] is isomorphic to $E[S|t]$.

Finally, using (A5) and the fact that

$$R(t) = e - \int_0^t \bar{u}(x)dx,$$

we can solve the integral equation relating the quitting rate function to the mean residual lifetime.

$$\bar{u}(t) = \frac{1 + E'\,[S\,|\,t]}{E[S\,|\,t]}, \tag{A6}$$

where $E'[S\,|\,t]$ is the derivative of $E[S\,|\,t]$ with respect to t.

A.4 The Empirical Hazard Function

In computing the empirical aggregate quitting rate function, an alternative to the naive estimate (number of failures in a given time divided by those remaining at that time) can be obtained from the relation

$$\bar{u}(t) = \frac{d}{dt}\left[-\log_e R(t)\right] = \frac{d}{dt}\left[-\log_e (1 - H(t))\right], \tag{A7}$$

which is easily verified by differentiating the above and comparing the result to (A2). Hence fitting a smooth curve to the negative of the natural logarithm of the observed reliability function may provide a more stable estimate of the aggregate hazard function. The slope (or numerical derivative) of this smooth curve will be the estimate of the hazard function.

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