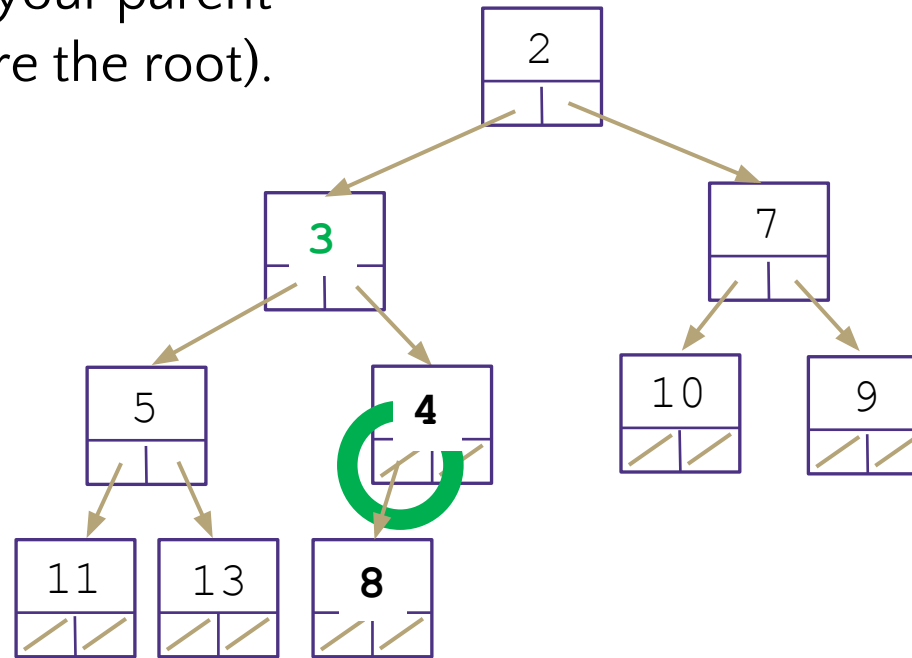# Lecture 14: Heap Array Implementation

CSE 373 Data Structures and Algorithms

Implementing add()

add() Algorithm:
1. Insert a node on the bottom level that ensure no gaps
2. Fix heap invariant by percolate **UP**

i.e. swap with parent, until your parent
is smaller than you (or you're the root).

Worst case runtime is similar to `removeMin` and `percolateDown` – might have to do log(n) swaps, so the
worst-case runtime is Theta(log(n))

Practice: Building a minHeap

Construct a Min Binary Heap by adding the following values in this order:

- 5, 10, 15, 20, 7, 2

2

7   **5**

percolateUp!

20   **10**   **15**

percolateUp!

percolateUp!

add() Algorithm:

1. Insert a node on the bottom level that ensure no gaps
2. Fix heap invariant by percolate **UP**

i.e. swap with parent, until your parent is smaller than you (or you're the root).
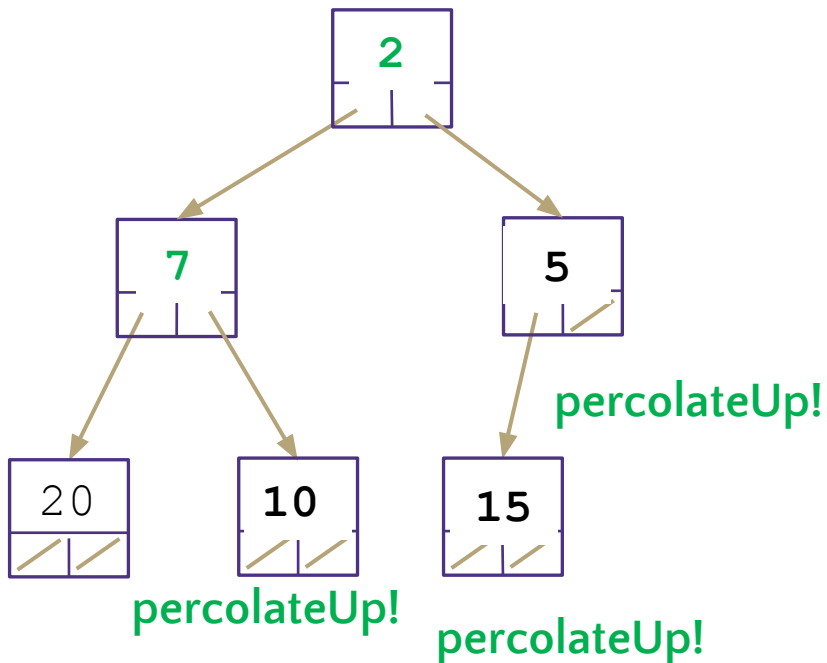
Min Binary Heap Invariants
1. **Binary Tree** – each node has at most 2 children
2. **Min Heap** – each node's children are larger than itself
3. **Level Complete** – new nodes are added from left to right completely filling each level before creating a new one

minHeap runtimes

removeMin():

- remove root node
- find last node in tree and swap to top level
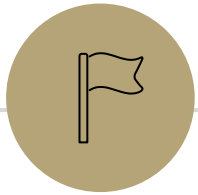- percolate down to fix heap invariant

add()

- insert new node into next available spot
- percolate up to fix heap invariant

Finding the last node/next available spot is the hard part.
You can do it in $\Theta(\log n)$ time on complete trees, with some extra class variants
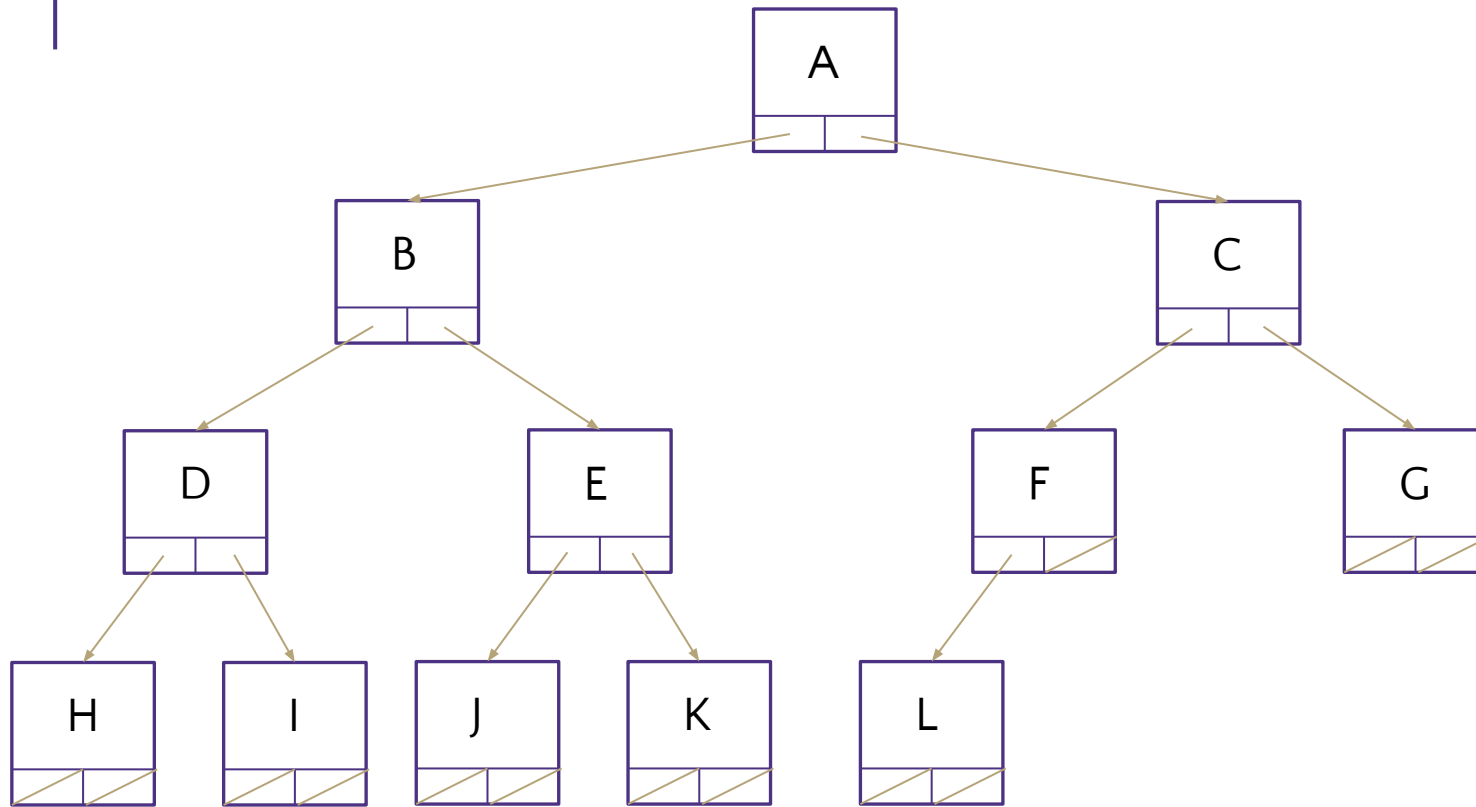
But it's NOT fun

And there's a much better way (that we'll talk about Wednesday)!

Heap Array Implementation
More Priority Queue Operations

Implement Heaps with an array

We map our binary-tree representation of a heap into an array implementation where you fill in the array in level-order from left to right.

<mark>The array implementation of a heap is what people actually implement, but the tree drawing is how to think of it conceptually.</mark> Everything we've discussed about the tree representation still is true!

Fill array in **level-order** from left to right

0	1	2	3	4	5	6	7	8	9	10	11	12	13
A	B	C	D	E	F	G	H	I	J	K	L		

Implement Heaps with an array
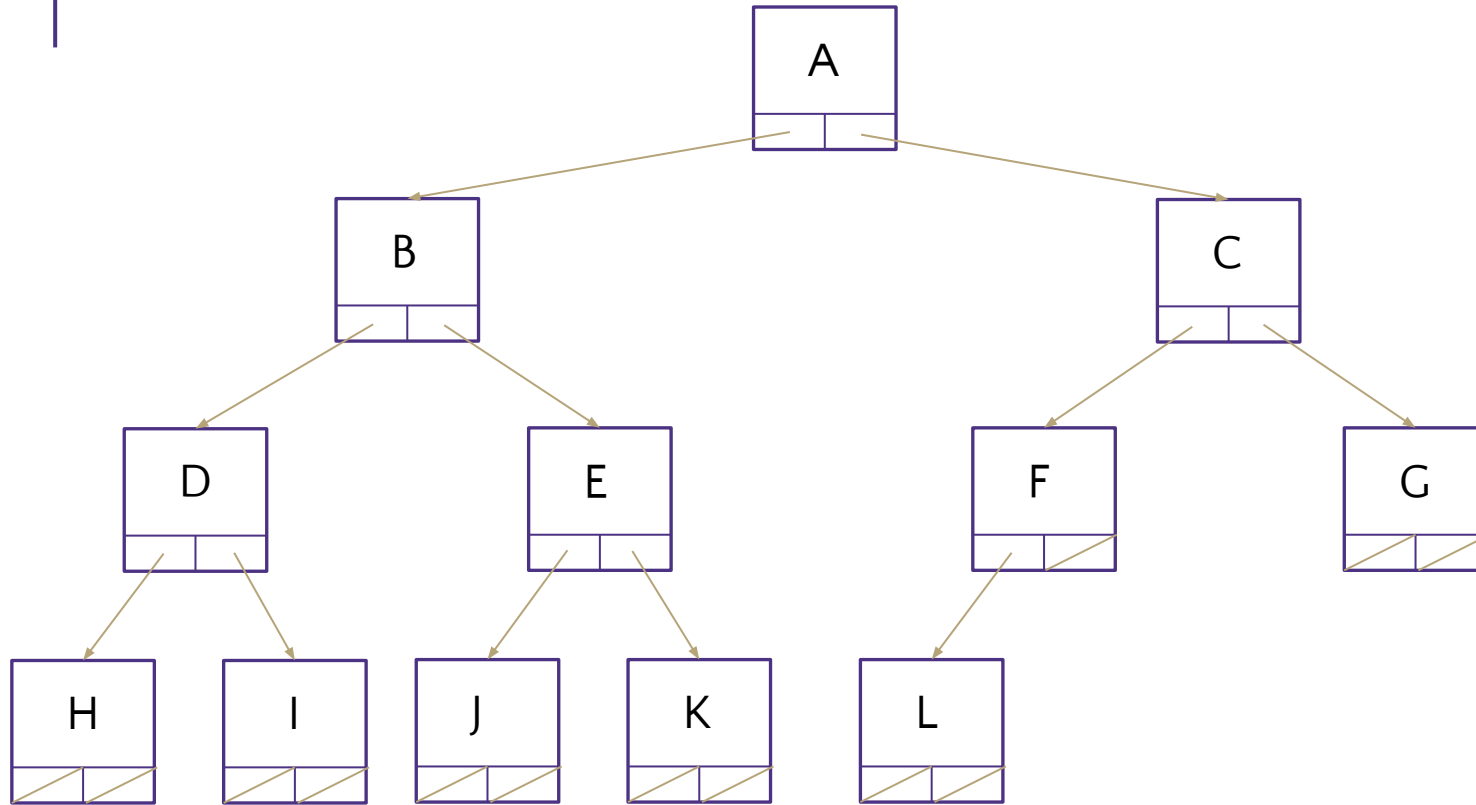
Fill array in **level-order** from left to right

	0	1	2	3	4	5	6	7	8	9	10	11	12	13
	A	B	C	D	E	F	G	H	I	J	K	L		

How do we find the minimum node?

$$peekMin() = arr[0]$$

How do we find the last node?

$$lastNode() = arr[size - 1]$$

How do we find the next open space?

$$openSpace() = arr[size]$$

How do we find a node's left child?

$$leftChild(i) = 2i + 1$$
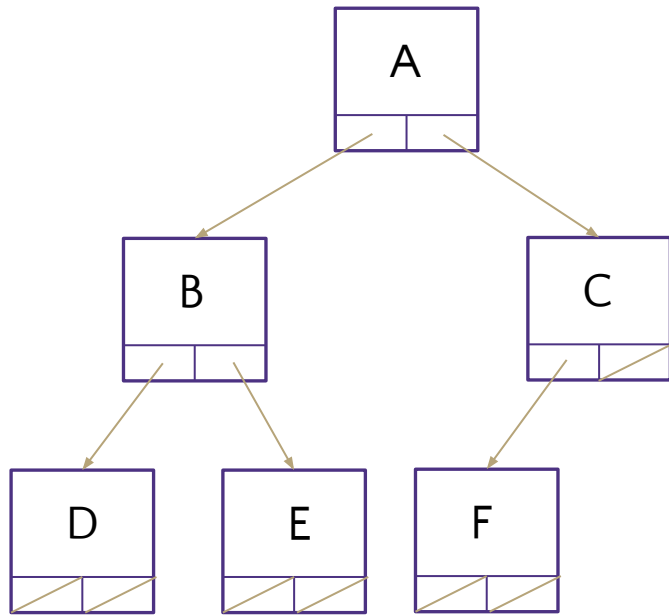
How do we find a node's right child?

$$rightChild(i) = 2i + 2$$

How do we find a node's parent?

$$parent(i) = \frac{(i - 1)}{2}$$

Heap Implementation Runtimes

Implementation	add	removeMin	Peek
Array-based heap	worst: Θ(log n) in-practice: Θ(1)	worst: Θ(log n) in-practice: Θ(log n)	Θ(1)

We've matched the **asymptotic worst-case** behavior of AVL trees.

But we're actually doing better!

- The constant factors for array accesses are better.
- The tree can be a constant factor shorter because of stricter height invariants.
- In-practice case for add is really good.
- A heap is MUCH simpler to implement.

Are heaps always better? AVL vs Heaps

- The really amazing things about heaps over AVL implementations are the constant factors (e.g. 1.2n instead of 2n) and the sweet sweet Theta(1) in-practice `add` time.

- The really amazing things about AVL implementations over heaps is that AVL trees are absolutely sorted, and they guarantee worst-case be able to find (contains/get) in Theta(log(n)) time.

- If heaps have to implement methods like contains/get/ (more generally: finding a particular value inside the data structure) – it pretty much just has to loop through the array and incur a worst case Theta(n) runtime.

- Heaps are stuck at Theta(n) runtime and we can't do anything more clever.... aha, just kidding.. unless...?

Project 3

Build a heap! Alongside hash maps, heaps are one of the most useful data structures to know – and pop up many more times this quarter!

- You'll also get practice using multiple data structures together to implement an ADT!
- Directly apply the invariants we've talked so much about in lecture! Even has an invariant checker to verify this (a *great* defensive programming technique!)

<table>
<tr><th colspan="1">MIN PRIORITY QUEUE ADT</th></tr>
</table>

State

Set of comparable values (*ordered based on "priority"*)

Behavior

add(value) – add a new element to the collection

removeMin() – returns the element with the <u>smallest</u> priority, removes it from the collection

peekMin() – find, but do not remove the element with the <u>smallest</u> priority

changePriority(item, priority) – update the priority of an element

contains(item) – check if an element exists in the priority queue

Project 3 Tips

- Project 3 adds changePriority and contains to the PriorityQueue ADT, which aren't efficient on a heap alone

- **You should utilize an extra data structure for changePriority!**

  - Doesn't affect *correctness* of PQ, just runtime. Please use a built-in Java collection instead of implementing your own (although you could in theory).

- changePriority Implementation Strategy:

  - implement without regards to efficiency (without the extra data structure) at first

  - analyze your code's runtime and figure out which parts are inefficient

  - reflect on the data structures we've learned and see how any of them could be useful in improving the slow parts in your code

MIN PRIORITY QUEUE ADT

State

Set of comparable values (*ordered based on "priority"*)

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peekMin() – find, but do not remove the element with the <u>smallest</u> priority

changePriority(item, priority) *– update the priority of an element*

contains(item) *– check if an element exists in the priority queue*

Heap Array Implementation
More Priority Queue Operations

More Operations

Min Priority Queue ADT

state
  Set of comparable values
  – Ordered based on "priority"

behavior

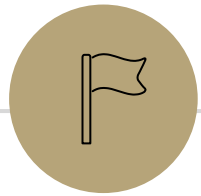 **add(value)** – add a new element to the collection

  removeMin() – returns the element with the <u>smallest</u> priority, removes it from the collection

  peekMin() – find, but do not remove the element with the smallest <u>priority</u>

We'll use priority queues for lots of things later in the quarter.

Let's add them to our ADT now.

Some of these will be **asymptotically** faster for a heap than an AVL tree!

BuildHeap(elements e_1, ..., $e_\square$)

Given **n** elements, create a heap containing exactly those **n** elements.

Even More Operations

BuildHeap(elements e_1, ..., $e_\square$)

Given **n** elements, create a heap containing exactly those **n** elements.

Try 1: Just call insert **n** times.

- Worst case running time?
- n calls, each worst case $\Theta(\log n)$. So it's $\Theta(n \log n)$ right?
- That proof isn't valid. There's no guarantee that we're getting worst case every time!
- Proof is right if we just want an $O()$ bound
- But it's not clear if it's tight

BuildHeap Running Time

Let's try again for a Theta bound.

The problem last time was making sure we always hit the worst case.

If we insert the elements in decreasing order **we will!**
- Every node will have to percolate all the way up to the root.

So we really have n $\Theta(\log n)$ operations. QED.

There's still a bug with this proof!

BuildHeap Running Time (again)

Let's try once more.

Saying the worst case was decreasing order was a good start.

What are the actual running times?

It's $\Theta(h)$, where h is the **current** height.
- The tree isn't height $\log n$ at the beginning.

But most nodes are inserted in the last two levels of the tree.
- For most nodes, h is $\Theta(\log n)$.

The number of operations is at least

$n/2 \cdot \Omega(\log n) = \Omega(n \log n)$.
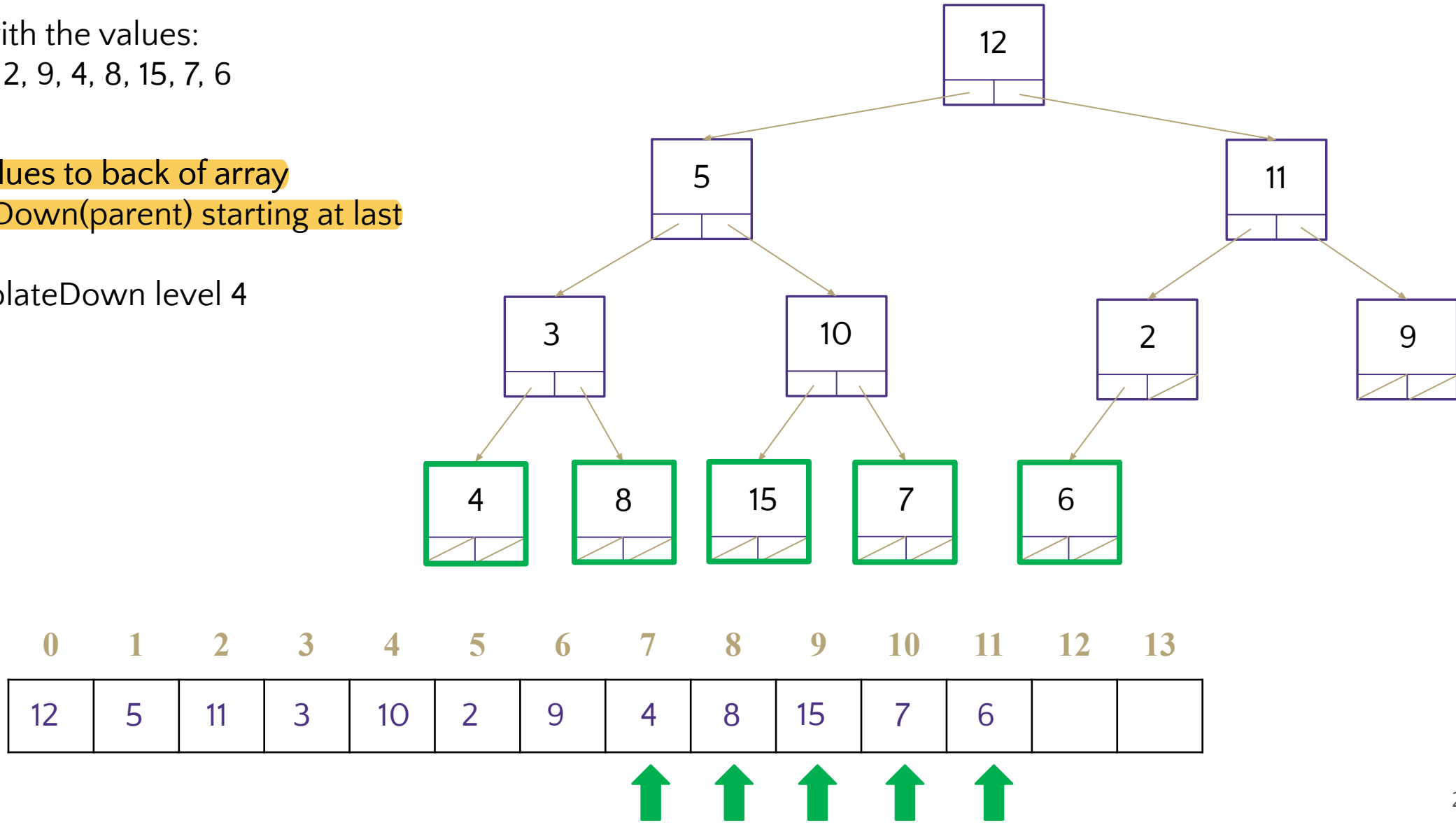
Can We Do Better?

- What's causing the *n* add strategy to take so long?
  - Most nodes are near the bottom, and might need to percolate all the way up

- Idea 2: Dump everything in the array, and percolate things down until the heap invariant is satisfied
  - Intuition: this could be faster!
  - The bottom two levels of the tree have $\Omega(n)$ nodes, the top two have 3 nodes
  - Maybe we can make "most of the nodes" go only a constant distance

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4

```
         12
        /    \
       5      11
      / \    /  \
     3  10  2    9
    / \ / \ /
   4  8 15 7 6
```

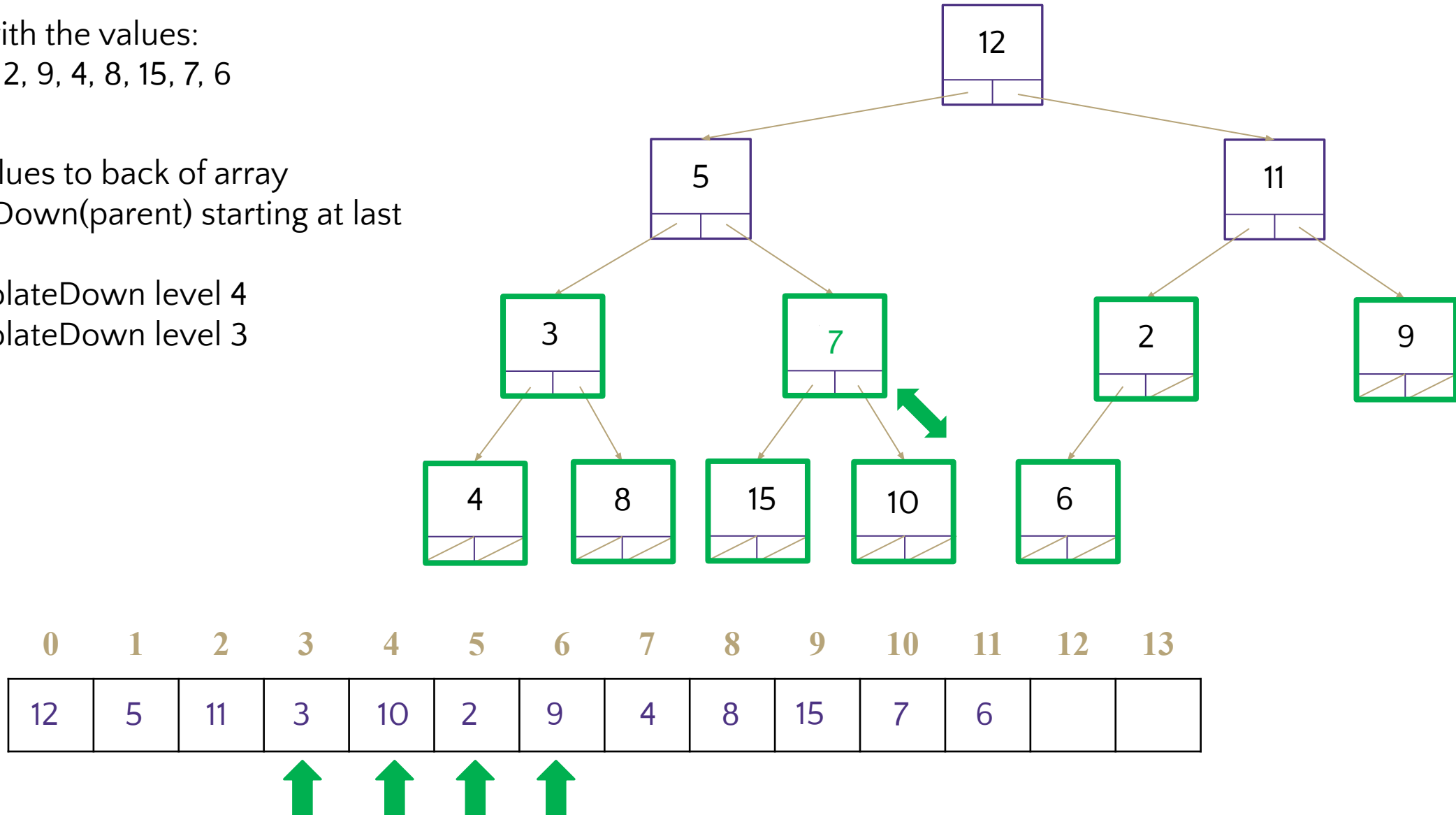| 0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	5	11	3	10	2	9	4	8	15	7	6		

CSE 373 24WI

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4
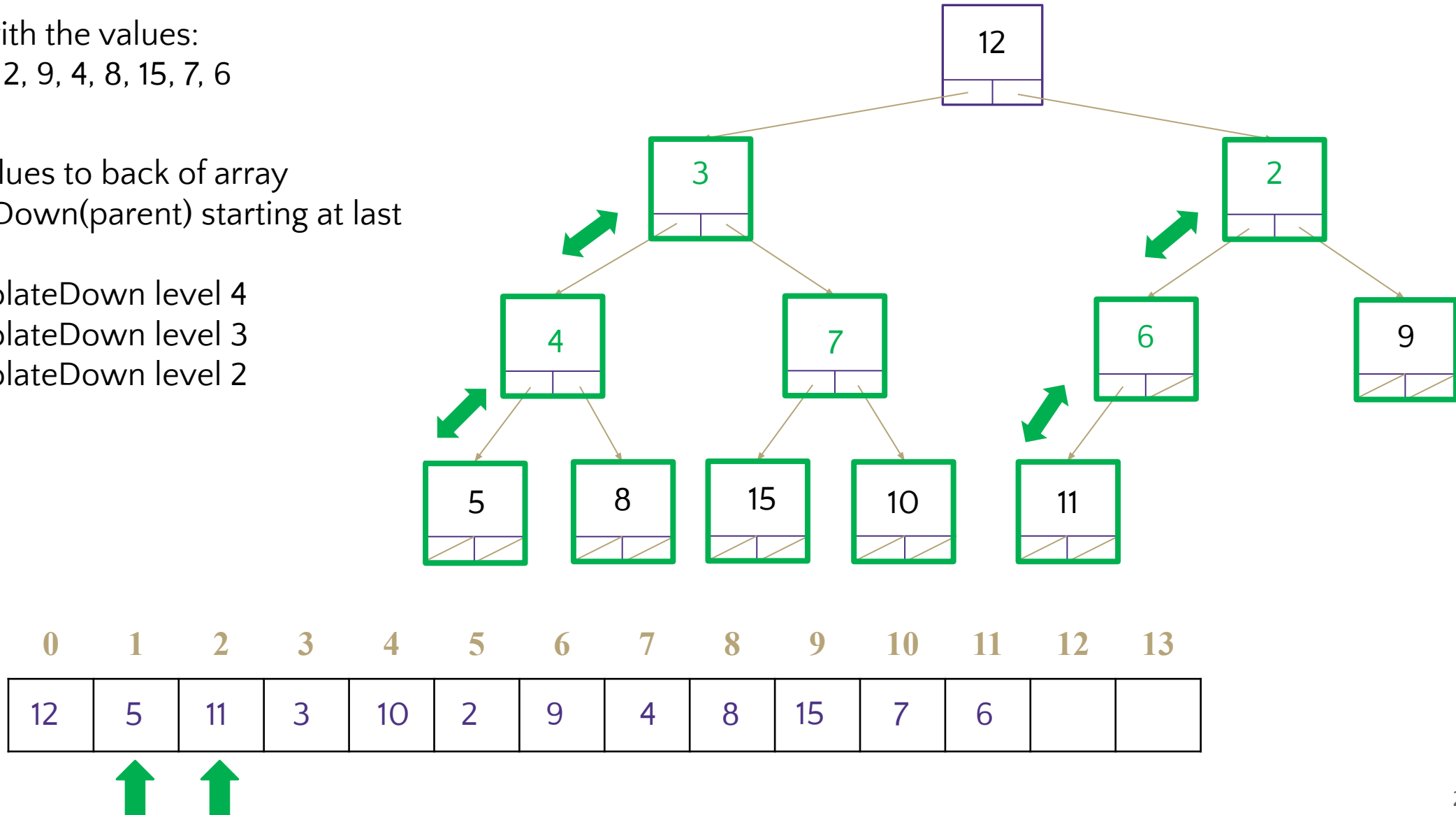   2. percolateDown level 3

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4
   2. percolateDown level 3
   3. percolateDown level 2

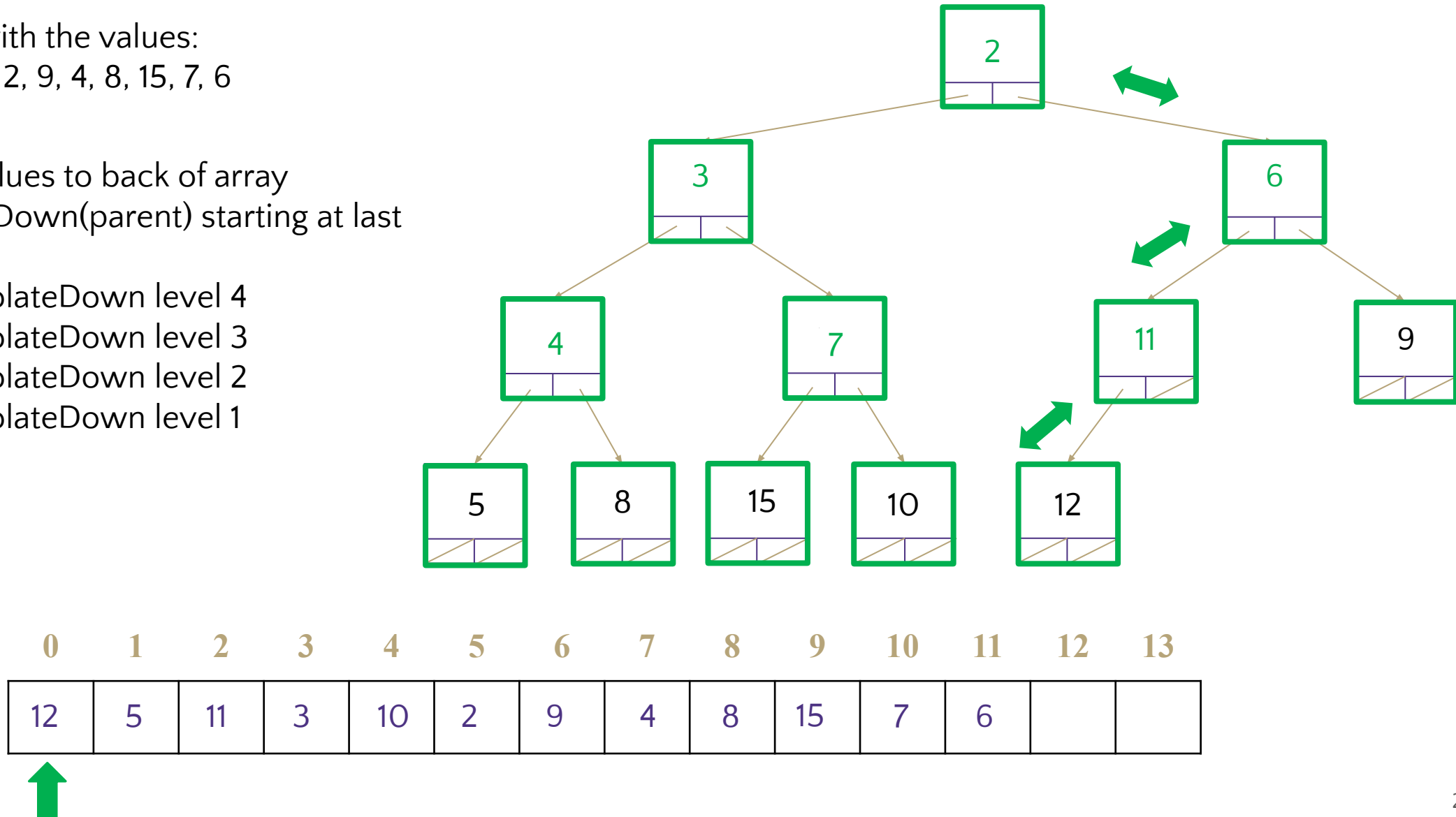| 0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	5	11	3	10	2	9	4	8	15	7	6		

Floyd's buildHeap algorithm

Build a tree with the values:
12, 5, 11, 3, 10, 2, 9, 4, 8, 15, 7, 6

1. Add all values to back of array
2. percolateDown(parent) starting at last index
   1. percolateDown level 4
   2. percolateDown level 3
   3. percolateDown level 2
   4. percolateDown level 1

0	1	2	3	4	5	6	7	8	9	10	11	12	13
12	5	11	3	10	2	9	4	8	15	7	6		

Is It Really Faster?

Floyd's buildHeap runs in O(n) time!

percolateDown() has worst case log n in general, but for most of these nodes, it has a much smaller worst case!

- n/2 nodes in the tree are leaves, have 0 levels to travel
- n/4 nodes have at most 1 level to travel
- n/8 nodes have at most 2 levels to travel
- etc...

$$\text{worst-case-work}(n) \approx \underbrace{\frac{n}{2} \cdot 1}_{\substack{\text{much of} \\ \text{the work}}} + \underbrace{\frac{n}{4} \cdot 2}_{\substack{\text{a little} \\ \text{less}}} + \underbrace{\frac{n}{8} \cdot 3}_{\substack{\text{a little} \\ \text{less}}} + \cdots + \underbrace{1 \cdot (\log n)}_{\substack{\text{barely} \\ \text{anything}}}$$

Intuition: Even though there are log *n* levels, each level does a smaller and smaller amount of work. Even with infinite levels, as we sum smaller and smaller values (think $1/2^i$) we converge to a constant factor of n.

- $n/2 \cdot 1 + n/4 \cdot 2 + n/8 \cdot 3 + \cdots + 1 \cdot (\log n)$

factor out n

$\text{work(n)} \approx n \left(\frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{\log n}{n} \right)$ find a pattern -> powers of 2 $\text{work(n)} \approx n \left(\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{\log n}{2^{\log n}} \right)$ Summation!

$$work(n) \approx n \sum_{i=1}^{?} \frac{i}{2^i}$$? = upper limit should give last term

We don't have a summation for this! Let's make it look more like a summation we do know.

Infinite geometric series

$$work(n) \leq n \sum_{i=1}^{\log n} \frac{\left(\frac{3}{2}\right)^i}{2^i}$$ $if -1 < x < 1 \ then \sum_{i=0}^{\infty} x^i = \frac{1}{1-x} = x$ $$work(n) \approx n \sum_{i=1}^{\log n} \frac{i}{2^i} \leq n \sum_{i=0}^{\infty} \left(\frac{3}{4}\right)^i = n * 4$$

Floyd's buildHeap runs in O(n) time!

Even More Operations

- These operations will be useful in a few weeks...
- **IncreaseKey(element,priority)** Given an element of the heap and a new, larger priority, update that object's priority.
- **DecreaseKey(element,priority)** Given an element of the heap and a new, smaller priority, update that object's priority.
- **Delete(element)** Given an element of the heap, remove that element.

- Should just be going to the right spot and percolating...
- Going to the right spot is the tricky part.
- In the programming projects, you'll use a dictionary to find an element quickly.