Concrete models and lower bounds

In this lecture, we will examine some simple, concrete models of computation, each with a precise definition of what counts as a step, and try to get tight upper and lower bounds for a number of problems. Specific models and problems examined in this lecture include:

- The number of comparisons needed to find the largest and second-largest item in an array,
- The number of comparisons needed to sort an array,
- The number of swaps needed to sort an array,
- The number of queries on a graph to determine whether it is connected.

Objectives of this lecture

In this lecture, we want to:

- Understand some concrete models of computation, with several examples (the comparison model, the exchange model, the query-edge model)
- Understand the definition of a *lower bound* in a specific model
- See some examples of how to prove lower bounds in specific models, particularly for sorting and selection problems

Recommended study resources

- CLRS, *Introduction to Algorithms*, Chapter 8.1, Lower bounds for sorting
- DPV, *Algorithms*, Chapter 2.3, Mergesort (Page 59)

1 Terminology: Upper Bounds and Lower Bounds

In this lecture, we will look at (worst-case) upper and lower bounds for a number of problems in several different concrete models. Each model will specify exactly what operations may be performed on the input, and how much they cost. Each model will have some operations that cost a certain amount (like performing a comparison, or swapping a pair of elements), some that are free, and some that are not allowed at all.

Definition: Upper bound

By an *upper bound* of U_n for some problem and some length n, we mean that *there exists an algorithm A that for every input x of length n costs at most U_n*.

A lower bound for some problem and some length n, is obtained by the negation of an upper bound for that n. It says that some upper bound is not possible (for that value of n). If we take the above statement (in italics) and negate it, we get the following. *for every algorithm A there exists an input x of length n such that A costs more than U_n on input x*. Rephrasing:

Definition: Lower bound

By a *lower bound* of L_n for some problem and some length n, we mean that *for any algorithm A there exists an input x of length n on which A costs at least L_n steps.*

These were definitions for a single value of n. Now a function $f : \mathbb{N} \to \mathbb{R}$ is an upper bound for a problem if $f(n)$ is an upper bound for this problem for every $n \in \mathbb{N}$. And a function $g(\cdot)$ is an lower bound for a problem if $g(n)$ is a lower bound for this problem for every n.

The reason for this terminology is that if we think of our goal as being to understand the "true complexity" of each problem, measured in terms of the best possible worst-case guarantee achievable by any algorithm, then an upper bound of $f(n)$ and lower bound of $g(n)$ means that the true complexity is somewhere between $g(n)$ and $f(n)$.

Finally, what is the *cost* of an algorithm? As we said before, that depends on the particular model of computation we're using. We will consider different models below, and show each has their own upper and lower bounds.

One natural model for examining problems like sorting and selection is the comparison model from last lecture, which we recall as follows.

Definition: Comparison Model

In the *comparison model*, we have an input consisting of n elements in some initial order. An algorithm may compare two elements (asking is $a_i < a_j$?) at a cost of 1. Moving the items, copying them, swapping them, etc., is *free*. No other operations on the items are allowed (using them as indices, adding them, hashing them, etc).

2 Selection in the comparison model

2.1 Finding the maximum of n elements

How many comparisons are necessary and sufficient to find the maximum of n elements, in the comparison model of computation?

Claim: Upper bound on select-max in the comparison model

$n-1$ comparisons are sufficient to find the maximum of n elements.

Proof. Just scan left to right, keeping track of the largest element so far. This makes at most $n-1$ comparisons. □

Now, let's try for a lower bound. One simple lower bound is that we have to look at all the elements (else the one not looked at may be larger than all the ones we look at). But looking at all n elements could be done using $n/2$ comparisons, so this is not tight. In fact, we can give a better lower bound:

Claim: Lower bound on select-max in the comparison model

$n-1$ comparisons are necessary for any deterministic algorithm in the worst-case to find the maximum of n elements.

Proof. The key claim is that every item that is not the maximum must lose at least one comparison (by lose, we mean it is compared to another element and is the lesser of the two). Why is this true? Suppose there were two elements a_i and a_j and neither lost a comparison. Suppose without loss of generality that $a_i > a_j$. If the algorithm outputs a_j it is incorrect. Otherwise, if it outputs a_i then we could construct another input that is the same except that a_j is now the maximum (we don't change the relative order of any other elements). On this new input, none of the results of any comparisons change since a_j never lost any comparisons in the first place, so the algorithm, being deterministic, must output the same answer. However, the algorithm is now incorrect. Therefore there must be $n-1$ elements that lose a comparison, and since only one element loses per comparison, a correct algorithm must perform $n-1$ comparisons. □

Since the upper and lower bounds are equal, the bound of $n-1$ is *tight*.

2.2 Finding the second-largest of n elements

How many comparisons are necessary (lower bound) and sufficient (upper bound) to find the second largest of n elements? Again, let us assume that all elements are distinct.

Claim: Lower bound on select-second-max in the comparison model

$n-1$ comparisons are needed in the worst-case to find the second-largest of n elements.

Proof. The same argument used in the lower bound for finding the maximum still holds. □

Let us now work on finding an upper bound. Here is a simple one to start with.

Claim: Upper bound #1 on select-second-max in the comparison model

$2n-3$ comparisons are sufficient to find the second-largest of n elements.

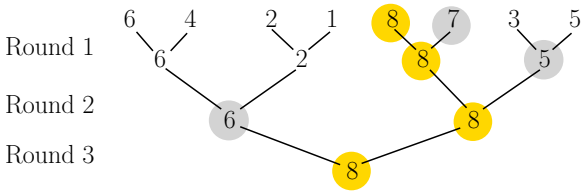
Proof. Just find the largest using $n-1$ comparisons, and then the largest of the remainder using $n-2$ comparisons, for a total of $2n-3$ comparisons. □

We now have a gap: $n-1$ versus $2n-3$. It is not a huge gap: both are $\Theta(n)$, but remember today's theme is tight bounds. So, which do you think is closer to the truth? It turns out, we can reduce the upper bound quite a bit:

Claim: Upper bound #2 on select-second-max in the comparison model

$n+\lg n-2$ comparisons are sufficient to find the second-largest of n elements.

Proof. As a first step, let's find the maximum element using $n-1$ comparisons, but in a tennis-tournament or playoff structure. That is, we group elements into pairs, finding the maximum in each pair, and recurse on the maxima. E.g.,



Now, given just what we know from comparisons so far, what can we say about possible locations for the second-highest number (i.e., the second-best player)? The answer is that the second-best must have been directly compared to the best, and lost.[1] This means there are only $\lg n$ possibilities for the second-highest number, and we can find the maximum of them making only $\lg(n)-1$ more comparisons. □

At this point, we have a lower bound of $n-1$ and an upper bound of $n+\lg(n)-2$, so they are nearly tight. It turns out that, in fact, the lower bound can be improved to exactly meet the upper bound, but the proof is rather complicated so we won't do it for now.

[1] Apparently the first person to have pointed this out was Charles Dodgson (better known as Lewis Carroll!), writing about the proper way to award prizes in lawn tennis tournaments.

2.3 An alternate technique: decision trees

Our lower bound arguments so far have been based on an *adversary* technique. We argued that if an algorithm makes too few comparisons, then we can concoct an input such that it will produce the wrong answer. There are many techniques that can be used to prove lower bounds. Another powerful one are *decision trees*.

A decision tree is a binary tree that represents the behavior of a specific algorithm based on the outcomes of each comparison it makes. Specifically, each internal node corresponds to a comparison such that the left subtree corresponds to the outcome of the comparison being true and the right subtree corresponds to it being false. At a leaf node the algorithm performs no more comparisons and thus is finished and produces an output.

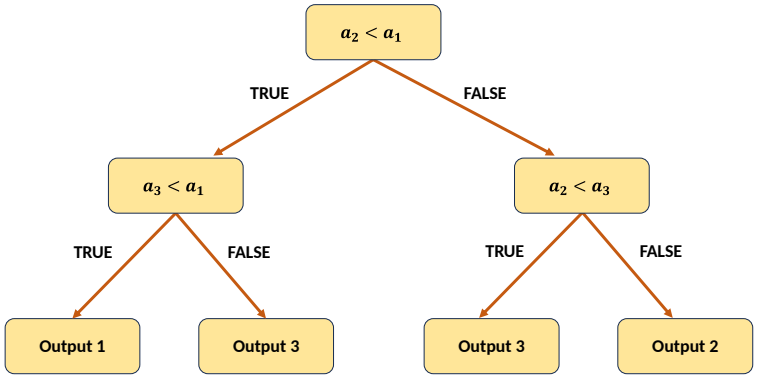
Remark: Decision trees are for particular algorithms

It is very important to remember that a decision tree encodes **a specific algorithm**. Different algorithms will have different decision trees. The decision tree does not however depend on the input to the algorithm, it encodes its behavior on any possible input. In some sense, you can think of the decision tree as a flow chart that tells you exactly what the algorithm does based on the results of the comparisons.

It turns out that decision trees can be a useful tool for analyzing lower bounds. Keeping in mind that a decision tree always represents a particular algorithm, to prove a lower bound, we must argue some property about the structure of *any possible decision tree* for the problem (if we make an argument about a specific decision tree, that is just like arguing about a specific algorithm, which does not help us derive a lower bound for the problem).

Since we are interested in the worst-case number of comparisons, we should observe that the number of comparisons performed by the algorithm on a particular input is the *depth* of the leaf node corresponding to that output. Therefore the worst-case cost (number of comparisons) of the algorithm corresponds exactly to the *longest root-to-leaf path*, i.e., the height of the tree. Therefore, if we can successfully argue about the height of any possible decision tree for a problem, we have an argument for a lower bound!

Here is a decision tree for some arbitrary algorithm that solves the select-max problem.



You can follow it just like a flowchart to determine for any input what index the algorithm will output! We can also use it to argue about lower bounds.

Proof of select-max lower bound via decision trees. At the root node of any decision tree for the select-max problem there are n possible outputs (positions $1 \dots n$). For each comparison, exactly one element loses, and hence the set of possible outputs at each node is one fewer than at its parent node. Therefore all of the leaves of this decision tree have depth $n-1$ and hence $n-1$ comparisons are required to determine the maximum element. □

3 Sorting in the comparison model

For the problem of *sorting* in the comparison model, the input is an array $a = [a_1, a_2, \dots, a_n]$, and the output is a permutation of the input $\pi(a) = [a_{\pi(1)}, a_{\pi(2)}, \dots, a_{\pi(n)}]$ in which the elements are in increasing order.

Remark: Correctly defining the "output" of an algorithm

A surprisingly subtle aspect of proving lower bounds, and the source of many buggy or incorrect lower bound proofs is the seemingly simple step of defining what the *output* of the algorithm is supposed to be.

Remember, importantly, that the comparison model has no concept of "values" of the input elements. The *only* thing that an algorithms knows about them are the results of the comparisons. Therefore when a comparison-model sorting algorithm produces an output, it doesn't know the values of the elements, it only knows what order it rearranged them into – so its output can only be described as a permutation of the input elements.

Many of our lower bound proofs will be combinatoric in nature, where we will count the number of *required outputs that an algorithm could need to produce.* From the point of view of a comparison-based algorithm, the two inputs $[5,8,2]$ and $[2,3,1]$ *are exactly the same,* because their elements are in the same relative permuted order, and the actions taken by a deterministic sorting algorithm on them would therefore be 100% identical.

When thinking about comparison-model lower bound proofs, be sure to keep this important distinction in mind – values do not matter at all because the algorithm does not know them. It can only deduce/know information about relative order!

Theorem 1: Lower bound for sorting in the comparison model

Any deterministic comparison-based sorting algorithm must perform at least $\lg(n!)$ comparisons to sort n distinct elements in the worst case.[a]

[a] As is common in CS, we will use "lg" to mean "$\log_2$".

In other words, for any deterministic comparison-based sorting algorithm $\mathscr{A}$, for all $n \geq 2$ there exists an input I of size n such that $\mathscr{A}$ makes at least $\lg(n!) = \Omega(n \log n)$ comparisons to sort I.

To prove this theorem, we cannot assume the sorting algorithm is going to necessarily choose a pivot as in Quicksort, or split the input as in Mergesort — we need to somehow analyze *any possible* (comparison-based) algorithm that might exist. This is a difficult task, and its not at all obvious how to even begin to do something like this! We now present the proof, which uses a very nice *information-theoretic* argument. (This proof is deceptively short: it's worth thinking through each line and each assertion.)

Proof of Theorem 1. First remember that we are dealing with *deterministic algorithms* here. Since the algorithm is deterministic, the first comparison it makes is always the same. Depending on the result of that comparison, the algorithm could take different actions, however, critically, **the result of all the previous comparisons always determines which comparison will be made next.** Therefore for any given input to the algorithm, we could write down the sequence of results of the comparisons (e.g., True, False, True, True, False, ...) and this sequence would entirely describe the behavior and hence the output of the algorithm on that input.

Now, in the comparison model, since values are unimportant and only order matters, there are $n!$ different possible input sequences that the algorithm needs to be capable of sorting correctly, one for each possible permutation of the elements. Furthermore, **every input permutation has a unique output permutation that correctly sorts it**. So, for a comparison-based sorting algorithm to be correct, it needs to be able to produce $n!$ different possible output permutations, because if there is an output it can not produce, then there is an input which it can not sort.

If the algorithm makes ℓ comparisons whose results are encoded by a sequence of binary outcomes (True or False) $b_1, b_2, \ldots, b_\ell$, then since each comparison has only two possible outcomes, the algorithm can only produce 2^ℓ different outputs. Since we argued that in order to be correct the algorithm must be capable of producing $n!$ different outputs, we need

$$2^\ell \geq n! \qquad \Longrightarrow \qquad \ell \geq \lg n!,$$

which proves the theorem. □

The above is often called an "information theoretic" argument because we are in essence saying that we need at least $\lg(M) = \lg(n!)$ bits of information about the input before we can correctly decide which of M outputs we need to produce. This technique generalizes: If we have some problem with M different outputs the algorithm needs to be able to produce, then in the comparison model we have a worst-case lower bound of $\lg M$.

What does $\lg(n!)$ look like? We have:

$$\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + \ldots + \lg(1) < n\lg(n) = O(n \log n),$$

and

$$\lg(n!) = \lg(n) + \lg(n-1) + \lg(n-2) + \ldots + \lg(1) > (n/2)\lg(n/2) = \Omega(n \log n).$$

So, $\lg(n!) = \Theta(n \log n)$. However, since today's theme is tight bounds, let's be a little more precise. We can in particular use the fact that $n! \in [(n/e)^n, n^n]$ to get:

$$\begin{array}{rcccl} n\lg n - n\lg e & < & \lg(n!) & < & n\lg n \\ n\lg n - 1.443n & < & \lg(n!) & < & n\lg n. \end{array}$$

Since $1.433n$ is a low-order term, sometimes people will write this fact this as

$$\lg(n!) = (n \lg n)(1 - o(1)),$$

meaning that the ratio between $\lg(n!)$ and $n \lg n$ goes to 1 as n goes to infinity.

How Tight is this Bound? Assume n is a power of 2, can you think of an algorithm that makes at most $n \lg n$ comparisons, and so is tight in the leading term? In fact, there are several algorithms, including:

- *Binary insertion sort.* If we perform insertion-sort, using binary search to insert each new element, then the number of comparisons made is at most $\sum_{k=2}^{n} \lceil \lg k \rceil \leq n \lg n$. Note that insertion-sort spends a lot in moving items in the array to make room for each new element, and so is not especially efficient if we count movement cost as well, but it does well in terms of comparisons.
- *Mergesort.* Merging two lists of $n/2$ elements each requires at most $n-1$ comparisons. So, we get $(n-1)+2(n/2-1)+4(n/4-1)+\ldots+n/2(2-1) = n \lg n - (n-1) < n \lg n$.

3.1 An Adversary Argument

A slightly different lower bound argument comes from showing that if an algorithm makes "too few" comparisons, then an adversary can fool it into giving the incorrect answer. Here is a little example. We want to show that any deterministic sorting algorithm on 3 elements must perform at least 3 comparisons in the worst case. (This result follows from the information theoretic lower bound of $\lceil \lg 3! \rceil = 3$, but let's give a different proof.)

If the algorithm does fewer than two comparisons, some element has not been looked at, and the algorithm must be incorrect. So after the first comparison, the three elements are w the winner of the first query, l the loser, and z the other guy. If the second query is between w and z, the adversary replies $w > z$; if it is between l and z, the adversary replies $l < z$. Note that in either case, the algorithm must perform a third query to be able to sort correctly.

4 Sorting in the exchange model

Consider a shelf containing n unordered books to be arranged alphabetically. In each step, we can swap any pair of books we like. How many swaps do we need to sort all the books? Formally, we are considering the problem of *sorting* in the *exchange model*.

Definition: The Exchange Model

In the **exchange model**, an input consists of an array of n items, and the only operation allowed on the items is to swap a pair of them at a cost of 1 step. All other work is free: in particular, the items can be examined and compared to each other at no cost.

Question: how many exchanges are necessary (lower bound) and sufficient (upper bound) in the exchange model to sort an array of n items in the worst case?

Claim: Upper bound on sorting in the exchange model

$n-1$ exchanges is sufficient.

Proof. For this we just need to give an algorithm. For instance, consider the algorithm that in step 1 puts the smallest item in location 1, swapping it with whatever was originally there. Then in step 2 it swaps the second-smallest item with whatever is currently in location 2, and so on (if in step k, the kth-smallest item is already in the correct position then we just do a no-op). No step ever undoes any of the previous work, so after $n-1$ steps, the first $n-1$ items are in the correct position. This means the nth item must be in the correct position too. □

But are $n-1$ exchanges necessary in the worst-case? If n is even, and no book is in its correct location, then $n/2$ exchanges are clearly necessary to "touch" all books. But can we show a better lower bound than that?

Claim: Lower bound on sorting in the exchange model

In fact, $n-1$ exchanges are necessary, in the worst case.

Proof. Here is how we can see it. Create a graph in which a directed edge (i, j) means that that the book in location i must end up at location j. An example is given in Figure 1.

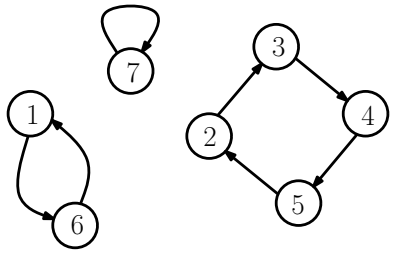


Figure 1: Graph for input `[f c d e b a g]`

This is a special kind of directed graph: it is a permutation — a set of cycles. In particular, every book points to *some* location, perhaps its own location, and every location is pointed to by exactly one book. Now consider the following points:

1. What is the effect of exchanging any two elements (books) that are in the same cycle?

   Answer: Suppose the graph had edges (i_1, j_1) and (i_2, j_2) and we swap the elements in locations i_1 and i_2. Then this causes those two edges to be replaced by edges (i_2, j_1) and (i_1, j_2) because now it is the element in location i_2 that needs to go to j_1 and the element in i_1 that needs to go to j_2. This means that if i_1 and i_2 were in the same cycle, that cycle now becomes two disjoint cycles.

2. What is the effect of exchanging any two elements that are in different cycles?

   Answer: If we swap elements i_1 and i_2 that are in different cycles, then the same argument as above shows that this merges those two cycles into one cycle.

3. How many cycles are in the final sorted array?

   Answer: The final sorted array has n cycles.

Putting the above 3 points together, suppose we begin with an array consisting of a single cycle, such as $[n, 1, 2, 3, 4, \ldots, n-1]$. Each operation at best increases the number of cycles by 1 and in the end we need to have n cycles. So, this input requires $n-1$ operations. □

5 Query models, and the evasiveness of connectivity

Optional content — Not required knowledge for the exams

To finish with something totally different, let's look at the query complexity of determining if a graph is connected. Assume we are given the adjacency matrix G for some n-node graph. That is, $G[i, j] = 1$ if there is an edge between i and j, and $G[i, j] = 0$ otherwise. We consider a model in which we can *query* any element of the matrix G in 1 step. All other computation is free. That is, imagine the graph matrix has values written on little slips of paper, face down. In one step we can turn over any slip of paper. How many slips of paper do we need to turn over to tell if G is connected?

Claim: Upper bound

$n(n-1)/2$ queries are sufficient to determine if G is connected.

Proof. This just corresponds to querying every pair (i, j). Once we have done that, we know the entire graph and can just compute for free to see if it is connected. □

Interestingly, it turns out the simple upper-bound of querying every edge is a lower bound too. Because of this, connectivity is called an "evasive" property of graphs.

Claim: Lower bound

$n(n-1)/2 = \binom{n}{2}$ queries are necessary to determine connectivity in the worst case.

Proof. We think of this as a game between two players: the *evader* and the *querier*. The querier asks a sequence of questions of the form "Is there an edge between vertices x and y?". For each question the evader must answer the question. The goal of the evader is to force the querier to ask as many questions as possible. We'll give a strategy for the evader which forces the querier to ask $\binom{n}{2}$ questions to determine if the graph is connected.

The evader's strategy will be to maintain the following invariant.

> At any point in time the edges that have been declared to exist form a forest of trees involving all the vertices of the graph. For each tree T all queries among the vertices of the T have already been asked. And for every pair of trees T and T' in the forest there exists a pair (x, y) with $x \in T$ and $y \in T'$ that has not been queried.

Note that until there is just one tree, the querier does not know if the graph is connected or not.

Initially each tree is just one vertex, and the invariant trivially holds. The evader maintains the invariant as follows. If the query is for two vertices in the same tree, then the evader just gives the answer it gave before to this query. Nothing changes and the invariant still holds.

Say the query is for vertices x and y in two different trees T_x and T_y. Then the evader determines if all the other possible edges between T_x and T_y have already been queried. If that is the case, then the evader answers 1 for "yes", otherwise it answers 0 for "no". Note that if it answers yes, (and joins the two trees) then, by induction, all the edges within the set $T_x \bigcup T_y$ have already been queried, and the invariant holds. (If it answers "no" the invariant trivially still holds.)

At any point in time before there is just one tree it is not known if the graph is connected or not. Finally when there is just one tree the graph is connected, but by that point all of the $\binom{n}{2}$ the edges have been queried. □