Part : 2 [Set Theory & Algebra]

→ Power set $P(A)$ is set of all possible subset of A.

→ $|P(A)| = 2^{|A|}$

→ Symmetric difference (XOR): $A \Delta B = A \oplus B = (A-B) \cup (B-A)$

→ Idempotent law: $A \cap A = A \cup A = A$

→ Absorption law: $A \cup (A \cap B) = A$

→ $A \cap P(A) = \phi$, $P(A) \cap P(P(A)) = \{\phi\}$

→ Sets are commutative and associative over $(\cup, \cap, \oplus)$ but distributive over $(\cup, \cap)$ only.

→ No of finite relation possible over cartesian set $= 2^{|m \times n|}$

→ Reflexive: Comparison among same item of set (x, x).

Smallest reflexive $|R_n| = n$, Largest reflexive $|R_n| = n^2$

No. of reflexive relation possible $= 2^{n^2-n}$

No. of irreflexive relⁿ possible $= 2^{n^2} - 2^{n^2-n}$

If R_1 and R_2 are reflexive then $(R_1 \cup R_2)$ & $(R_1 \cap R_2)$ both are reflexive but if R_1 and R_2 are Irreflexive then $(R_1 \cup R_2)$, $(R_1 \cap R_2)$ & $(R_1 - R_2)$ all are irreflexive.

→ Reflexive: all the diagonal elements are necessary.
Irrefelxive: even a single diagonal element doesn't required.

→ There is no relation which is both Reflexive & Irreflexive.

No. of Irreflexive relation $= 2^{n^2} - 2^{n^2-n}$, Smallest $= \{\}, \phi$

largest $= n^2 - n$

→ Symmetric: If (aRb) then (bRa)

Smallest relation $= \{\}$ Largest relⁿ $= n^2$

No of symmetric relations $= 2^n \cdot 2^{\frac{n^2-n}{2}} = 2^{\frac{n(n+1)}{2}}$

$|\text{Symmetric} \cap \text{Reflexive}| = 2^{\frac{n^2-n}{2}} = |\text{Sym.} \cap \text{Irr}|$

Antisymmetric (As): If (xRy) and (yRx) then $x = y$.

$|As| = n + \frac{n^2-n}{2}$

Smallest $= n$, Largest $= |A \times A| = n^2$

→ Symmetric relations are closed under $(\cup, \cap, -)$

$n(S \cap I) = 2^{\frac{n^2-n}{2}}$

$n(A_s) = 2^n \, 3^{n^2-n/2}$

$n(A_s \cap S) = 2^n$

$n(R \cap A_s) = 3^{n^2-n/2}$

$n(IR \cap A_s) = 3^{n^2-n/2}$

Asymmetric: Stricter version of Anti-symmetric.

$|Asy| = 3^{\frac{n^2-n}{2}}$, Smallest = { }, Largest = $\frac{n^2-n}{2}$

→ Reflexive relⁿ can't be ~~antisymmetric~~ Asymmetric bcoz of no diagonal element.

→ Every asymmetric relation is irreflexive and anti-symmetric.

→ It is closed under subset, intersection and set difference operation.

Transitivity: If (xRy) & (yRz) then (xRz)

→ It is closed under '∩' but not '∪'.

Equivalence Relⁿ: Closed under '∩' but not '∪'.

→ Reflexive, Symmetric & Transitive together.

Partial Order Relation: Reflexive, transitive, Antisymmetric together

Totally Ordered Set: Linear order set or chain. It means relation exist b/w every pair of elements

→ Representation of poset : Hasse diagram.

→ Don't need to show reflexive (loop) or transitive relation in Hasse diagram.

Maximal element → No relation above elements

Minimal element → No relation with below elements

Maximum → Maximum in all the maximal.

Minimum → Minimum in all the minimal.

Upper bound | Lower bound : In a given hasse digram either go only up or only down but not both up-down.

GLB or LUB : GLB : (Meet, ∩), Infimum LUB (Join, ∪, Supremum)

Join Semi Lattice : It consist LUB for each pair of vertices.

Meet Semi Lattice : ——— GLB ———.

Lattice → If hasse diagram is both Join semi Lattice and Meet semi Lattice.

Properties → follows commutative, associative, but not distributive

$a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$
$a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$

If lattice follow these called distributed lattice.

Sub lattice : It has same LUB and GLB as parent lattice.

→ If 'I' is Upper bound & 'O' is lower bound, then

LUB (a and I) = $a \vee I = I$, LUB $(a, 0) = a \vee 0 = a$

GLB $(a, I) = a \wedge I = a$, GLB $(a, 0) = 0$

→ Bounded lattice is a finite lattice because it has both the upper bound and a lower bound.

→ for Infinite lattice we can't find LB, UB.

→ for any set UB is Universal set and LB is ϕ.

Complement of element in lattice : $a \vee a^c = I$, $a \wedge a^c = 0$

Pair of element with same GLB and LUB called as complement.

In distributive lattice every element have atmost 1 complement

In complemented lattice every element have atleast 1 complement

In boolean algebra every element have only 1 complement.

Groups :

Algebric structure : Each set is closed under some operations.

Semi Group : If algebric structure is associative. $\{(Q^*, +)\}$

Monoid : Semi-group with identity element.

for union operation ϕ works as identity element.

Groups : Monoid with inverse element.

[$(Q^*, *)$ is a group but $(Q, +)$ doesn't. Similarly $(P(A), \cup)$ is also not a group].

Abelian Group : It is also called commutative group. Inverse of 'e' is 'e' itself. Unique Inverse & Identity element.

for strings identity element $e = \epsilon$.

→ Identity element : $a * e = a$, Inverse element : $a * a^{-1} = e$

$*$ → It can be any operation.

finite Group : Group with finite no. of elements

→ In modulo 'm' operations range of number should be $(0 \rightarrow m-1)$.

→ Order of an element 'a' is 'n' if $a^n = e$ for smallest value of 'n'.

→ Order of identity element is always 1.

→ Order of a, a^{-1} are same and order of element always divides order of group.

<u>Subgroup</u>: e and G are trivial subgroup. All other are proper subgroup.

→ If 'H' is a subgroup of 'G' then $O(G)/O(H) = k$
If it divides then only we will check for subgroup further. After applying divisor rule verify subgroup should be closed under group's operation.

→ Subgroups are closed under '$\cap$' but not under '$\cup$'.

→ Every subgroup of abelian group is also abelian group.

<u>Cyclic group</u>: If every element of group can be written in form of (a^n) then $a \rightarrow$ generating element.

Number of generator = $\phi(n)$ means no. of elements $< n$ which are relatively prime to n.

→ <u>Properties</u>: If $n = pq$ then $\phi(n) = \phi(p) \cdot \phi(q)$ where, p, q are distinct & prime.

② $\phi(p) = p - 1$ for $p \rightarrow$ prime.

③ $\phi(p^n) = p^n - p^{n-1}$

→ If $(G, *)$ is a cyclic group with generator a then,

① a^{-1} is also a generator

② $O(a) =' O(G)$

→ Every cyclic group is abelian, every group of prime order is cyclic.

→ If $|X| = m$, $|Y| = n$ then no. of fn possible = n^m.

→ One-One fn : $|Y| \geq |X|$: no. of fn = ${}^nP_m$

If fn is bijective $|Y| = |X|$ and it is one-one and onto then, no. of one-one = no. of onto = $n!$

If $|X| \neq |Y|$ then only onto possible. So
no. of onto fn $= n^m - {}^nC_1 (n-1)^m + {}^nC_2 (n-2)^m - \dots (-1)^{n-1} \, {}^nC_{n-1} (1)^m$

→ Two irrational no. x & y can give $(x+y)$ as rational.

→ $f(A \cup B) = f(A) \cup f(B)$, $f(A \cap B) \neq f(A) \cap f(B)$
$f(A \cap B) \subseteq f(A) \cap f(B)$ equals when both are one-one.

→ ~~Relation with empty set is symmetric and transitive but not reflexive~~

→ $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$, $\phi \in P(A)$, $\phi \subseteq P(A)$

→ If $h = f \circ g$ then if $h \Rightarrow$ onto then $f \Rightarrow$ onto.

→ N^k is countable but k^N is not countable.

→ Complemented lattice is proper subset of bounded lattice.

→ Set of all rational no. is countable.

→ A finite lattice is always bounded because it has both LUB and GLB (due to lattice).

→ If a relation is defined on power set of some set then,
$a \cap a = \phi$ for $[a = \phi]$

→ Empty relation over finite set is not reflexive but symmetric & transitive.

→ Empty relation over empty set is an equivalence reln.

→ Every finite lattice has a least elements and max. elements

→ Every poset doesn't have a least or greatest element
(Ex: Bipartite Graph).

→ There is just a single relation which is both equivalence and partial order relation. → (Identity reln)

→ No. of integer between $(1 \to n)$ divisible by 'k' is given by
$\lfloor \frac{n}{k} \rfloor$ → floor fn.

Combinatorics

Generalized Pigeonhole: If n-objects are placed into k-boxes then atleast one box contains $\lceil N/k \rceil$ objects.

$\rightarrow$ ${}^nC_r = {}^nC_{n-r}$, ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

$\rightarrow (x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + \cdots + {}^nC_n y^n$

General term: $\boxed{{}^nC_r \, x^{n-r} y^r}$

$\rightarrow \sum_{k=0}^{n} {}^nC_k = 2^n$, $\sum_{k=0}^{n} (-1)^k \, {}^nC_k = 0$

Soln of linear homogeneous recurrence relation.

1. If eqn has same roots as 'a' then,

$$a_n = \alpha_1 a^n + \alpha_2 n a^n + \alpha_3 n^2 a^n + \cdots$$

2. If eqn has different roots then,

$$a_n = \alpha_1 a^n + \alpha_2 b^n + \alpha_3 c^n + \cdots$$

If $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k = \sum_{k=0}^{\infty} a_k x^k$

Useful generating fn

$(1+x)^n = {}^nC_0 x^0 + {}^nC_1 x^1 + \cdots + {}^nC_n x^n = \sum_{k=0}^{n} \frac{{}^nC_k x^k}{a_k}$

$(1+ax)^n = {}^nC_0 a^0 x^0 + {}^nC_1 ax + \cdots + {}^nC_n a^n x^n = \sum_{k=0}^{n} \frac{{}^nC_k a^k x^k}{a_k}$

$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \cdots x^n = \sum_{k=0}^{n} x^k$

$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{k=0}^{\infty} x^k$

$\frac{1}{1-ax} = 1 + ax + a^2x^2 + \cdots = \sum_{k=0}^{\infty} a^k x^k$

$\frac{1}{1-x^r} = 1 + x^r + x^{2r} + \cdots = \sum_{k=0}^{\infty} x^{rk}$

$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots = \sum_{k=0}^{\infty} (k+1) x^k$

$\frac{1}{(1+x)^n} = 1 - {}^nC_1 x + {}^{n+1}C_2 x^2 + \cdots = \sum_{k=0}^{\infty} {}^{n+k-1}C_k (-1)^k x^k$

$\frac{1}{(1-x)^n} = 1 + {}^nC_1 x + {}^{n+1}C_2 x^2 + \cdots = \sum_{k=0}^{\infty} {}^{n+k-1}C_k x^k$ (36)

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$

(1) $G(x)$ for $\{1\} = \frac{1}{1-x}$ (2) $G(x)$ for $n = \frac{x}{(1-x)^2}$

→ ${}^nC_r = \frac{n!}{r!(n-r)!}$, ${}^{-n}C_r = \frac{(-1)^r (n+r-1)!}{r!(n-1)!} = \boxed{{}^{n+r-1}C_r}$

→ No. of dearrangement $\boxed{D_n = \sum_{i=2}^{n} \frac{(-1)^i n!}{i!}}$ → Nothing can be placed in right box.

→ Generating fn for fibonacci series $= \frac{1}{1-x-x^2}$

→ No. of substrings formed from a length of string 'n' is –
$\boxed{\frac{n(n+1)}{2}+1}$

→ No. of ways to divide 'K' similar things among 'n' different persons/things
are $= \boxed{{}^{n+k-1}C_K}$

→ No. of moves required on travelling (i, j) to (l, m)
$= \frac{(\text{Total no. of blocks to travel})!}{(\text{Total UP})!\ (\text{Total Right})!}$

→ If their are 'n' objects of 'k' types such that all items occurs with some frequency then,
No. of ways of arranging them $= \frac{n!}{(f_1!\, f_2! \cdots f_K!) \times (k!)}$ ← if values in f are all differ

Here, $f_1 + f_2 + \cdots + f_K = n$

→ ${}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 \cdots + n \cdot {}^nC_n = \sum_{k=1}^{n} k \cdot {}^nC_k = n \cdot 2^{n-1}$

Counting Principles

① **Product Rule:** If there are n task that can be done in $n_1, n_2, \ldots n_n$ ways then total no. of ways to do the procedure is $n_1 \times n_2 \times \ldots \times n_n$.

② **Sum Rule:** If task can be done either in n_1 ways or n_2 ways then total no of ways $= n_1 + n_2$.

③ **No of f^n:** If |domain| = |m| and |codomain| = |n| then no. of fn possible $= n^m$

for one-one functions:- $n \geq m$ and No. of fn = 'k',

$$k = n(n-1)(n-2)\ldots\ldots(n-k+1)$$

Ex → A password is 6–8 characters long where each character is an uppercase letter or a digit. Each password must contain atleast one digit, No. of password is ____.

Sol: No. of password $= P_6 + P_7 + P_8$

$P_6 = 36^6 - 26^6$, $P_7 = 36^7 - 26^7$, $P_8 = 36^8 - 26^8$.

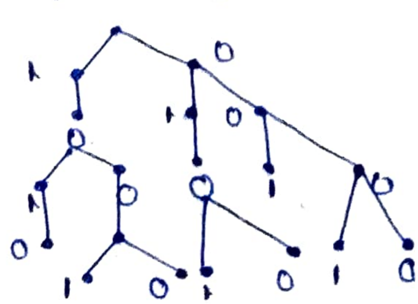
Subtraction Rule: If a task can be done in either n_1 or n_2 ways then total no. of ways possible is $n_1 + n_2 - (n_1 \cap n_2)$.

Ex → How many bit strings of length 8 either start with bit 1 or end with 00.

Ans → $2^7 + 2^6 - 2^5$.

Division Rule: (Not Important)

Tree Diagram: Bit string of length 4 dont have consecutive 1's ?



Pigeonhole: To keep $k+1$ objects in k boxes, there is atleast 1 box with two or more objects.

Collary: A function 'f' from a set with $(k+1)$ or more element to a set with 'k' elements is not one to one.

Generalized Pigeonhole: If N objects are placed into k objects boxes then atleast one box contains $\lceil N/k \rceil$ objects.

Permutations → No. of ordered arrangements

Eg → Let $S = \{1,2,3\}$ then 3,1,2 is permutation of S and 3,2 is a 2-permutation of S.

→ If n, r are positive integer such that $r \in [1, n]$ then, there are
$P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$, r-permutations of a set with n-distinct elements.

→ No. of permutation of letters ABCDEFGH contains the string ABC equals to $6!$. (Hint: Pair ABC as a single element)

Combinations: No. of r-combinations of a set with n-distinct elements is denoted by $C(n,r)$ or $\binom{n}{r}$ and called as binomial-coefficient.

$${}^{n}C_{r} = {}^{n}C_{n-r}, \quad {}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$$

Binomial Theorem:

$$(x+y)^n = {}^{n}C_0 x^n + {}^{n}C_1 x^{n-1} y + \cdots + {}^{n}C_n y^n$$

→ coefficient of $x^{12}y^{13}$ in,

① $(x+y)^{25} = {}^{25}C_{13}$

② $(2x-3y)^{25} = {}^{25}C_{13} \times 2^{12} (-3)^{13}$

$({}^{n}C_{r} x^{n-r} y^{r})$

→ If n is a non-negative integer then,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

or

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

or

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Vandermonde's Identity

$${}^{n+m}C_r = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k} \quad \text{--- (1)}$$

(or) $${}^{2n}C_n = \sum_{k=0}^{n} \binom{n}{k}^2 \text{, at } m=r=n \text{ in eq}^n \text{ (1)}$$

(or) $${}^{n+1}C_{r+1} = \sum_{k=r}^{n} \binom{k}{r}$$

Permutation with Repetetion: The no. of r-permutations of a set of n-objects with repitition allowed is n^r.

Combination with repetition: There are $C(n+r-1, r)$ r-combinations from a set with n-elements when repition of elements is allowed.

Permutation with Indistinguishable Object:

Ex: How many strings can be made by reordering the letters of word SUCCESS?

Sol^n: $\frac{7!}{3!\,2!\,1!\,1!}$

Distributing objects into boxes:

Ex: How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards.

Sol^n: $\frac{52!}{47!\,5!} \cdot \frac{47!}{42!\,5!} \cdot \frac{42!}{37!\,5!} \cdot \frac{37!}{5!\,32!} = \frac{52!}{5!\,5!\,5!\,5!\,32!}$

Recurrence Relation and Generating function

The recurrence relation $P_n = (1.11) P_{n-1}$ is a linear homogeneous recurrence relation of degree 1 and $f_n = f_{n-1} + f_{n-2}$ is linear and having degree 2.

→ $a_n = a_{n-1} + a^2_{n-2}$: not a linear.

$H_n = 2H_{n-1} + 1$: not Homogeneous

$B_n = nB_{n-1}$: no constant coefficients

→ **Solving Linear Homogeneous Recurrence Relation with Constant coefficients.**

Ex-1 $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$. find the Soln.

Soln: $r^2 - r + 2 = 0 \Rightarrow r = -1, 2$

So, Soln = $a_n = \alpha_1 2^n + \alpha_2 (-1)^n$

To find α_1 and α_2 use a_0 and a_1

Ex-2 $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1, a_1 = 6$

Soln $r^2 - 6r + 9 = 0 \Rightarrow r = 3, 3$

$a_n = \alpha_1 3^n + \alpha_2 (n3^n)$

If equation have 4 same roots then

$a_n = \alpha_1 3^n + \alpha_2 n 3^n + \alpha_3 n^2 3^n + \alpha_4 n^3 3^n$

→ The generating function for the sequence $a_0, a_1, \ldots a_k$, of real no's in infinite series,

$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k = \sum_{k=0}^{\infty} a_k x^k$

Ex → The generating function for $\{a_k\}$ with $a_k = 3$ and $a_k = 2^k$ are $\sum_{k=0}^{\infty} 3x^k$ and $\sum_{k=0}^{\infty} 2^k x^k$.

Generating function for the sequence 1, 1, 1, 1, 1 ?

Ans → $1 + x + x^2 + x^3 + x^4$

→ If $a_k = C(m,k)$ then $G(x) = C(m,0) + C(m,1)x + C(m,2)x^2 + \dots + C(m,m)x^m$.

→ $f(x) = \frac{1}{(1-ax)}$ is generating f^n for $1 + ax + a^2x^2 + \dots$

→ Let 'u' be a real no and 'k' a non-negative integer, then the extended Binomial coefficient $\binom{u}{k}$ is defined as

$$\binom{u}{k} = \begin{cases} u(u-1)(u-2)\dots(u-k+1)/k! & \text{if } k > 0 \\ 1 & \text{: if } k = 0 \end{cases}$$

Ex ① $\binom{-2}{3} \Rightarrow$ $u = -2$, $k = 3$ so, $\frac{(-2)(-3)(-4)}{3!} = -4$

$$\binom{1/2}{3} = \frac{(1/2)(1/2-1)(1/2-2)}{3!} = 1/16$$

Useful Generating functions

$$(1+x)^n = 1 + C(n,1)x + C(n,2)x^2 + \dots + x^n = \sum_{k=0}^{n} \frac{C(n,k)x^k}{a_k}$$

$$(1+ax)^n = 1 + C(n,1)ax + C(n,2)a^2x^2 + \dots + a^nx^n = \sum_{k=0}^{n} \frac{C(n,k)a^kx^k}{a_k}$$

$$(1+x^r)^n = 1 + C(n,1)x^r + C(n,2)x^{2r} + \dots + x^{rn} = \sum_{k=0}^{n} C(n,k)x^{rk}$$

$$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \dots + x^n = \sum_{k=0}^{n} x^k$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{k=0}^{\infty} x^k$$

$$\frac{1}{1-ax} = 1 + ax + a^2x^2 + \dots = \sum_{k=0}^{\infty} a^kx^k$$

$$\frac{1}{1-x^r} = 1 + x^r + x^{2r} + \dots = \sum_{k=0}^{\infty} x^{rk}$$

$$\frac{1}{(1-x)^2} = 1+2x+3x^2+\cdots \sum_{k=0}^{\infty}(k+1)x^k$$

$$\frac{1}{(1+x)^n} = 1 - C(n,1)x + C(n+1,2)x^2 + \cdots = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k$$

↳ Alternate (+ and −)

$$\frac{1}{(1-x)^n} = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$$

↳ All positive.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

$$\ln(1+x) = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}x^k$$

→ $G(x)$ for $\{1\} = \frac{1}{1-x}$

$G(x)$ for $n = \frac{x}{(1-x)^2}$

If $f(x) = \sum_{k=0}^{\infty} a_k x^k$ and $g(x) = \sum_{k=0}^{\infty} b_k x^k$ then,

$f(x)+g(x) = \sum_{k=0}^{\infty}(a_k+b_k)x^k$, $f(x)g(x) = \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k} a_j b_{k-j}\right)x^k$

⇒ ${}^nC_r = \frac{n!}{r!(n-r)!}$, $\boxed{{}^{-n}C_r = (-1)^r\frac{(n+r-1)!}{r!(n-1)!} = {}^{n+r-1}C_r}$

$$(1+x)^4 = \sum_{k=0}^{\infty}\binom{4}{k}x^k$$

So, $(1+x)^{-n} = \sum_{k=0}^{\infty}\binom{-n}{k}x^k = \sum_{k=0}^{\infty}(-1)^k C(n+k-1,k)x^k.$

$(1-x)^{-n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k$

To move from co-ordinate (x_1, y_1) to (x_2, y_2)

→ No. of dearrangement $D_n = \sum_{i=2}^{n}(-1)^i\frac{n!}{i!}$