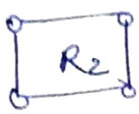
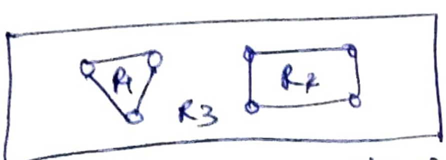
→ A planar graph with loops and parallel edges are called as a simple planar graph.

→ In a simple planar graph with atleast 2 edges, the degree of every face/region is at least 3. — ①

degree $(R_1) = 3$



$deg(R_2) = 4$



$deg(R_3) = 7$

$deg(R_1) + deg(R_2) + deg(R_3) = 14$

$3 + 4 + 7$

$3 + 3 + 3 = 9$ Using ①

$\Rightarrow 9 < 2 * 7$ $[3f \leq 2e]$ ↓ Euler's formula

$\Rightarrow 9 < 14$

Q find min. no. of edges and vertices required to form 10 faces whose min. degree is 3 in a simple connected planar graph.

→ $3(f) \leq 2e$

$\boxed{e \geq 15}$ $[f = 10]$

Using $n - e + f = 12$ [Euler's formula]

$n - 15 + 10 = 12$

$\Rightarrow n = 7$
↳ vertices

Out of 'r' faces (r-1) are bounded and 1 is unbounded

n = vertices

r = regions

Imp. Remember

$\boxed{\begin{array}{l} ① \; 3f \leq 2e \\ ② \; n - e + r = 2 \end{array}}$

four color Theorem

Euler's formula for disconnected graphs.

If G is simple planar graph with k components then, $\boxed{n - e + f = k + 1}$

four color Theorem Imp

Every planar graph is 4-colorable

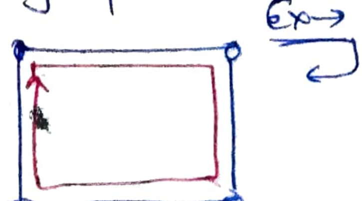
Using Greedy coloring algo. you might get more than 4 but its not possible.

Hamiltonian Cycle

If you form a cycle in a graph by passing each vertex exactly once.

→ A graph with hamiltonian cycle called as hamiltonian graph.

Ex →



→ No need to cover all the edges

→ It is not necessary that each graph is hamiltonian.

Necessary Condⁿ

① Cycle must formed

② Each vertex covered exactly once.

SC → If $G(V,E)$ is a simple graph having n vertices and for each vertex $u \in V$ we have $deg(v) \geq n/2$ then G is hamiltonian.

SC → Sufficient Condⁿ.

Edge Cuts:- [Not require for Gate]

Points to be write regarding Graph Theory:-

Kuratowski's Theorem: Graph is planar if and only if, it doesn't contain subgraph homeomorphic to K_5 or $K_{3,3}$.

→ Max. no. of possible edges in an undirected graph with 'a' vertices and 'k' components $= \frac{(a-k)*(a-k+1)}{2}$

→ K_5 is smallest non-planar graph with $a=5$ and $k=1$ So,

edges $= \frac{(5-1)(5-1+1)}{2} = 10$

Max. no. of edges in planar graph with n vertices(e) $\leq 3n-6$, $\forall n \geq 3$

Type	Edges	Multiple Edges	Loops Allowed
Simple	Undirected	×	×
Multi	"	✓	×
Pseudo	"	✓	✓
Directed Simple	Directed	×	×
Directed Multi	"	✓	✓
Mixed	Both Directed and Undirected	✓	✓

→ edge (u,v) u → v (directed)

u: Initial point, v: terminal or end

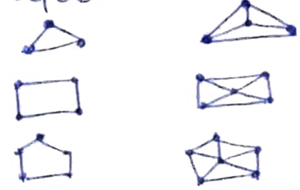
u is adjacent to v and v is adjacent from u.

n — Cycle — Wheel

3

4

5



→ Max. number of edges in an acyclic undirected graph with 'n' vertices equals to 'n-1'.

for C_n ratio of chromatic no. to diameter $= \frac{2}{n}$

→ Let G be the non-planar graph with min. possible no. of i) edges then 9 edges and 6 vertices.

(ii) vertices   10 edges and 5 vertices

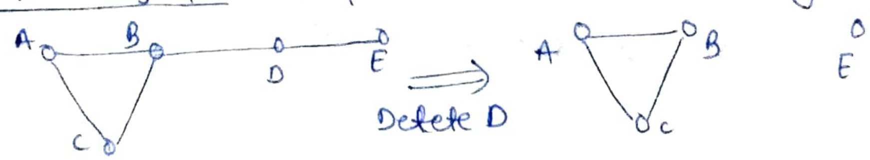
→ No. of Hamiltonian cycles in complete undirected graph = $(n-1)!/2$

Graph Theory

Graph is a triple of V, E and relation between them.
{1, 2, 3} → set of vertices (1,2) (2,3) → Order pairs.

→ Graph with multiple edges b/w two vertices → Multi graph.
→ Simple graph: Graph with no loops & multi edges.
→ for every undirected simple graph → ① $A = A^T$
② Principle diagonal are all zeroes. [No loop]
→ Degree of vertex: sum of all the elements in row.
→ Hand shaking lemma: $\sum_{v \in V} d(v) = 2|E(G)|$
→ Degree of loop is 2. → Number of odd d(v) are even.
→ Even graph: All vertices with even degree.
→ Number of Simple graph $= 2^{^nC_2}$
→ Isomorphism: for simple graph
i) Same no. of edges and vertices in both graphs.
ii) Degree sequence must be same for both.
iii) Both have same adjacency matrix with different order of rows can be possible.
iv) Same no. of cycles
→ In degree sequence arrange the degree of vertices in either ascending or descending order.
→ If sum of degree = odd, then no graph.
→ If sum of degree = even, then we can't say anything.
→ Havel- Hakimi Theorem = $O(n^2 \log n)$
→ Min(δ) and Max(Δ) degree: $\delta \leq \frac{2|E|}{|V|} \leq \Delta$
→ A graph is also a sub-graph of its own.
→ Induced sub-graph: Graph obtained after deleting a vertex.

Ex



→ Path: A simple graph in which two vertices are adjacent if they are consecutive in degree sequence.
→ Path is walk with edge and vertex traverse once.

→ Cycle : Graph with equal no. of vertices & edges.
Degree of each vertex is 2.

→ We can acheive path from cycle but not vice-versa.

→ Complete graph (K_n): Graph with $(n-1)$ degree of vertices and ${}^nC_2$ edges.

→ Clique: Set of pairwise adjacent vertices.

→ Independent set: Set of pairwise non-adjacent vertices.

→ Bipartite Graph: Graph whose vertices can be divide into two adjacent set.

$K_{n,m}$ → Complete with m, n vertices in both sets and $m \times n$ edges

→ A cyclic graph is bipartite if it has only even cycle.

→ Max. no. of edges in a bipartite graph = $\boxed{\frac{n^2}{4}}$

→ Regular Graph: Graph with all vertices of same degree.

→ A complete graph is $(n-1)$ regular.

→ Every cyclic graph is 2-regular.

→ No. of edges in k-regular graph with n-vertices = $(n*k)/2$

→ Hypercube (Q_k): Simple graph, Q_k : k → no of bits for vertex.

→ In Q_k, 2^k vertices, $k \cdot 2^{k-1}$ edges, $d(v) = k$.

→ We can make Q_k using two Q_{k-1}.

→ Diameter: Max. path length in the given graph.

→ Eccentricity: Max. distance of vertex from other vertexes.

→ Diameter: property of graph, Eccentricity: property of vertex.

→ Radius: Min. of eccentricity.

→ Isolated : degree(v) = 0, Pedant : degree(v) = 1

→ Walk: Contains repeated vertex but edge can't repeat.
⊢ closed, open walk

→ Trial: No repeated edges.

→ Eulerian Graph: Graph has a closed trial connecting all edges. Walk that is closed covers each vertex.

→ A graph is eulerian if it has atmost one non-trivial component and all vertices have even degree.
[Easy explaination at the end]

→ In a complete graph K_n, and all vertices are lebelled then no of cycles (distinct) of length m is

$$\boxed{\frac{{}^nC_m\,(m-1)!}{2}}$$

[No. of ways to keep 'm' people on a circular table = (m

→ Adding an edge between two components reduce the components by 1.

→ Deleting an edge increase the no. of components by 0 or 1.

→ Every graph with 'n' vertices and k-edges have atleast $n-k$ components.

→ Directed graph (digraph : It can be simple or multi graph.

→ Underlying graph : Treat an directed graph as undirected to make each vertex connected.

→ A digraph can be strongly or weakly connected. If it is weakly connected then we needed, underlying graph.

→ weakly connected $\subseteq$ strongly connected.

→ Vertex, degrees : $d^+(v)$, $d^-(v)$
out degree, in-degree

$\delta \to$ min
$\Delta \to$ max

$$\sum_{v\in V} d^+(v) = |E(G)| = \sum_{v\in V} d^-(v)$$

① $\delta^-(G) \le \frac{|E|}{|V|} \le \Delta^-(G)$ ② $\delta^+(G) \le \frac{|E|}{|V|} \le \Delta^+(G)$

→ Number of perfect matching in K_{nn} = $\boxed{n!}$ [No. of one-one f^n]

→ Number of perfect matching in K_n = $\boxed{\frac{2n!}{n!\,2^n}}$ when $2n$ = no of vertices
↳ 0 for odd no vertices, for even vertices

→ Matching : A matching in a graph is a set of non-loop edges with no shared end points.

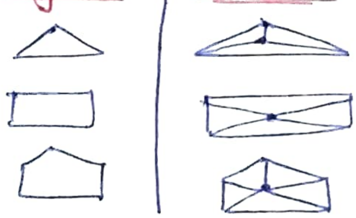
→ In saturated all vertex participated

→ No. of Hamiltonian Cycle in a complete undirected graph = $\boxed{(n-1)!/2}$

Kurtowski Theorem : Graph is planar if and only if it doesn't contain subgraph isomorphic to K_5 or K_{33}.

- Max. no. of possible edges in an undirected graph with 'a' vertices and 'k' components $= \boxed{(a-k)(a-k+1)/2}$
- K_5 is smallest non-planar with $a = 5$, $k=1$
  $\Rightarrow$ no. of edges $= 10$
- Max. no. of edges in planar-graph with 'n' vertices $\leq 3n-6$ $\forall n \geq 3$.
- Max. no. of edges in an acyclic undirected graph with n-vertices equals to '$n-1$'.
- Let G be the non-planar graph with min. no. of –
  1. edges: 9 edges and 6 vertices (K_{33})
  2. vertices: 10 edges and 5 vertices (K_5)
- In Q_K : ratio of chromatic no to diameter $= \boxed{2/K}$
- 'n' | cycle | wheel

3
4
5



- Every planar graph is 4-colorable according to 4-color theorem.
- Euler's formula.
  1. $3f \leq 2e$
  2. $n - e + r = 2$

  for cycle $6f \leq 2e$
  $4f \leq 2e$ → bipartite

- **Maximal and maximum matching** :

Maximal means you can't add more edges without disturbing its property. [Ma]

- Maximum is max among all maximal. [α']
- Vertex Cover : Min. set of vertex to cover end point of every edge.
  |vertex cover| $\geq$ matching size
- min. vertex cover [β] → NP-complete problem.
- $|Ma| \leq |\beta| \leq 2|Ma|$
- In $K_{m,n}$ $|\max Ma| = |\min \beta|$ and $\boxed{\beta = \min(m,n)}$

Independent set : Set of pair of non adjacent vertices. [α]

→ In K_{mn} = max(m,n)

Edge Cover : Set of edges to covers all the vertices. [β']

→ A perfect matching form edges cover = $|V|/2$

for each bipartite graph : $\alpha' = \beta$

with (0) isolated vertex : $\alpha = \beta'$

In a graph : $\alpha + \beta = |V|$

In a graph with 0 isolated vertex : $\alpha' + \beta' = |V|$

Vertex cut and edge cut is the set of vertex or edges to get more than one component in the graph.

→ In complete graph we need to remove all (n-1) ~~edges~~ vertex.

→ In cycle connectivity is 2, so removed vertex should be non-adjacent.

→ Connectivity of complete graph = $|V|-2$

→ Connectivity of K_{mn} = min(m,n)

→ $\delta_K \leq K$

→ Chromatic no. of $K_{mn} = 2$, odd length cycle = 3, even length cycle = 2, complete graph = n.

→ In Hamiltonian cycle, cover every vertex exactly once.

Planarity : Refer notes.

→ Max. no of edge in an acyclic undirected graph with n-vertices = n-1

→ If graph is Isomorphic to its complement called self complementry graph with edges = $\frac{(|V|)(|V|-1)}{4} = \boxed{\frac{n(n-1)}{4}}$

→ In a connected planar simple graph has e edges and 'v' vertices for $v \geq 3$ and no circuits of length 3 then $e \leq 2v-4$

→ Euler ckt / Cycle : Closed walk visits every edge present.

→ Graph contains Euler ckt called as Euler graph

→ A graph must have euler ckt if it has all vertices of even degree, but graph must be connected.

→ In planar-graph if total no. of bounded faces = n-1 then it has total 'n' faces.

→ A graph which has no circuit of odd length and atleast 1 edge is 2-chromatic.

→ Euler ckt have vertices of even degree only.

→ for a given graph with 'v' vertices 'e' edges and k-components,

$$\boxed{v-k \le e \le \frac{(v-k)(v-k+1)}{2}}$$

→ Connected component (k) can never be greater than no. of vertices.

→ No. of ways of splitting 'n' elements into two parts is $2^{n-1}-1$.

→ Max. no. of edges present in disconnected graph with 'n' vertices = $({}^{n-1}C_2)$ [Devide into (n-1) and 1]

→ A connected graph with n-vertices needs min. (n-1) edges. So we can avoid connectivity with (n-2) edges.

→ L be a lattice on a set A and operation * as (A, *) and S be a subset of A also satisfying (S, *) then A is totally ordered set.

→ To find no. of equivalence relation find B_{n+1} if no. of elements in set is "n". If $|A| = 4$ find B_3.

$$B_{n+1} = \sum_{k=0}^{n} {}^{n}C_k B_k \quad \text{and} \quad B_0 = B_1 = 1, \; B_2 = 2, \; B_3 = 5$$

→ with n-leaf node in binary tree no. of nodes with degree 2 = n-1.