ISYE 6669 – HOMEWORK 3

ANSWER 1 –

This problem does NOT have an optimal solution. This problem is not feasible – for the constraint $x + y = 1$, if x is negative or zero, y would need to be greater than 1 or exactly 1 respectively, and that violates the third constraint $y <= 0$.

Hence, there is no feasible solution, and then of course, there is no optimal solution either.

ANSWER 2 –

(a)

To find the critical points on the function $f(x) = x^2(\sin^2 x)$, we differentiate this function to get the first derivative and equate it to zero to get critical values of x –

$2x (\sin x) (\sin x + x(\cos x)) = 0$

This gives a few possibilities –

1. $2x = 0$, hence $x = 0$
2. $\sin x = 0$, hence $x = n\pi$
3. $\sin x + x(\cos x) = 0$, or $\sin x = -x(\cos x)$, or $–(\tan x) = x$

For the first value of x, $f(x) = 0$

For the second set of values –

$F(x) = (n\pi)^2 (\sin n\pi)^2$

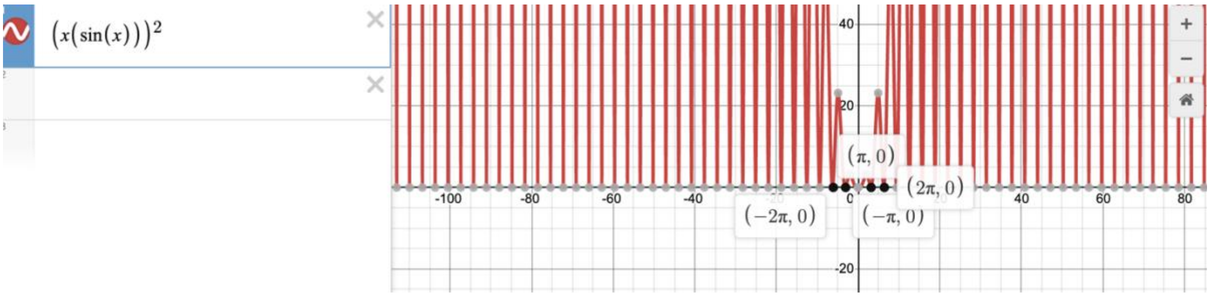
$\sin n\pi = 0$, hence the objective value $f(x) = 0$.

Do we need to look at the third possibility of x? If we look at the objective function, it is a product of squares, which means that f(x) HAS to be non-negative. The lowest value from that set is 0, which we've be able to obtain as a global optimum from the values $x = n\pi$, and the number of such solutions is infinite.

Hence, we have a global minimum of $f(x) = 0$.

(b)

The critical points derived from the first derivative cover ALL maxima and minima. From the following graph plotted we see that there are infinite local maxima –



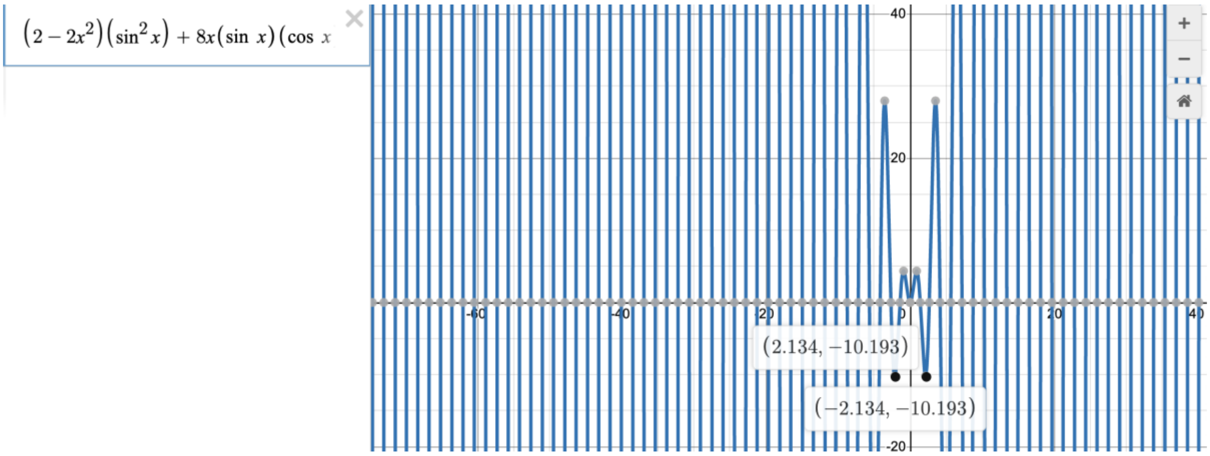
Now the cases for x from (1) and (2) cover minima which are all multiples - $n\pi$ alone, then the case (3) for x, would not cover a minimum but rather a local maximum. Hence, we can say with confidence, that there are no local minima that aren't also the global minimum.

(c)

Using the double derivative rule, we can show that f(x) is not a convex function. The second derivative of f(x) would be –

$(2 - 2x^2)(\sin^2 x) + 8x(\cos x)(\sin x) + 2x^2(\cos^2 x)$

For f(x) to be convex, the second derivative of f(x) would need to be non-negative under all cases of x. If we graphed this function –



Clearly, there are cases for x – which happen to be the local maxima values of x, to be precise – that generate negative values for the function. Hence, f(x) is NOT convex.

ANSWER 3 –

It does NOT have an optimal solution. It has infinite feasible solutions, each one better than the previous, as x increases, but the set of x is unbounded, and in this case, the improvement has no limit, hence while there are infinite feasible solutions, there are NO optimal solutions.

ANSWER 4 –

(a)

While this objective function may be discontinuous, it is still convex on a series of closed intervals, and hence it is convex on the whole.

(b)

For the case of f(x) greater than 1 or less than -1, the objective function evaluates to positive infinity. For the case of x = 1, the objective function evaluates to 2, and for the case of x = -1, the objective function evaluates to 1.

Now for the case of x from open set (-1,1), The value of x + f(x) gets smaller and approaches -1. Each feasible solution is smaller than the previous as x approaches -1, and they all beat the other solutions by the other values of f(x).

Hence, there is NO optimal solution to this objective function due to the asymptotic sequence of better and better solutions.

ANSWER 5 –

(a)

FALSE!

The set of possibilities for g(x) >= 1 is a subset of the set of possibilities g(x)>=0. That means there is a smaller set than, or an equal set of objective values as before for f(x) to take on. If the set is equal, then the *v* remains the same. If the set is smaller, *v* remains the same or gets worse since it was left out in the difference of constraints sets. This is to show that *v* won't reduce in value, it would either stay the same OR increase in value, that is, get worse.

In other words, the new *v* will be greater than or equal to the old *v*.

(b)

TRUE!

The objective function $f(x)^4$ can only take non-negative values owing the even power. And the smallest value from this set is 0. The feasible solution x^* that gives an objective value of 0 is the minimum possible value the objective function can take no matter what the constraint set is. Hence, x^* is a global optimal solution.

(c)

TRUE!

This is the dual for a maximization problem. And if for a minimization problem, the dual optimal value is less than equal the original optimal value, then for a maximization problem, owing to symmetry, the dual optimal must be greater than or equal the original optimal value, which is what is expressed by the final expression - $v_P \leq v_D$.