ISyE 6669 HW 3

1. Consider the following linear optimization problem

$$\begin{aligned} \min \quad & x + y \\ \text{s.t.} \quad & x + y = 1, \\ & x \leq 0,\ y \leq 0. \end{aligned}$$

   Does this problem have an optimal solution? Is this problem feasible? Explain your answer.

2. Consider the following optimization problem

$$\begin{aligned} \min \quad & (x \cdot \sin(x))^2 \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned}$$

   (a) Find all the global minimum solutions. Explain how you find them. Hint: there may be multiple ones.

   (b) Is there any local minimum solution that is not a global minimum solution?

   (c) Is the objective function $f(x) = (x \cdot \sin(x))^2$ a convex function on $\mathbb{R}$?

3. Consider the following optimization problem

$$\begin{aligned} \min \quad & \frac{1}{x} \\ \text{s.t.} \quad & x \geq 0. \end{aligned}$$

   Does this problem have an optimal solution? Why?

4. Consider the following problem

$$\begin{aligned} \min \quad & x + f(x) \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

   where the function $f(x)$ is defined as

$$f(x) = \begin{cases} 0, & -1 < x < 1 \\ 1, & x = 1 \\ 2, & x = -1 \\ +\infty, & x > 1 \text{ or } x < -1 \end{cases}.$$

(a) Is the objective function a convex function defined on $\mathbb{R}$? Explain your answer by checking the criterion of convexity.

(b) Find an optimal solution, or explain why there is no optimal solution.

5. For each of the statements below, state whether it is true or false. Justify your answer.

(a) Consider the optimization problem

$$\min f(\boldsymbol{x}) \text{ s.t. } g(\boldsymbol{x}) \geq 0.$$

Suppose the current optimal objective value is v. Now, if I change the right-hand-side of the constraint from 0 to 1 and resolve the problem, the new optimal objective value will be less than or equal to v.

(b) Consider the following optimization problem:

$$\min f(\boldsymbol{x})^4 \text{ s.t. } \boldsymbol{x} \in X$$

where $f(\boldsymbol{x})$ is a nonconvex function and X is a non-empty set. Suppose at a feasible solution $\boldsymbol{x}^* \in X$, $f(\boldsymbol{x}^*) = 0$, then $\boldsymbol{x}^*$ must be a global optimal solution.

(c) Consider the following optimization problem

$$\begin{aligned} (P) \quad \max \quad & f(\boldsymbol{x}) \\ \text{s.t.} \quad & g_i(\boldsymbol{x}) \geq b_i, \quad \forall i \in I. \end{aligned}$$

Suppose the optimal objective value of (P) is v_P. Then, the Lagrangian dual of (P) is given by

$$(D) \quad \min\{\mathcal{L}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{1}$$

where $\mathcal{L}(\boldsymbol{\lambda}) = \max_{\boldsymbol{x}}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i(g_i(x) - b_i)\}$. Furthermore, suppose the optimal objective value of (D) is v_D, then $v_P \leq v_D$.