ISyE6669 Homework 7
Fall 2021

1 Week 7

1. **Geometry of LP**

   (a) Let P be a triangle in a plane with three extreme points $A = [0,0]^\top, B = [2,0]^\top, C = [1,2]^\top$. Let $x = [1, 2/3]^\top$ be a point in the triangle P. Express x as a convex combination of the three extreme points A, B, C. You should write $x = \lambda_1 A + \lambda_2 B + \lambda_3 C$, where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and all nonnegative. You need to give the numerical values of $\lambda_1, \lambda_2, \lambda_3$.

   (b) Consider the following polyhedron P defined as

   $$P = \{(x,y) \in \mathbb{R}^2 : x + y \geq 2,\ x - y \leq 4,\ y \geq 1,\ x \leq 5 \}.$$

   Find all the extreme points and extreme rays of P.

   (c) Let B be the unit box in $\mathbb{R}^3$, i.e.

   $$B = \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1 \}.$$

   You are allowed to cut the box B with one plane. What is the maximum number of extreme points you can get from the cut? That is, Find a plane H such that its intersection with B has the maximum possible number of extreme points. You need to give an example of such a plane, and write down the coordinates of all the extreme points.

2. **Reformulation as linear program**

   (a) Consider the following linear program:

   $$\begin{aligned} \max \quad & x - y \\ \text{s.t.} \quad & x + y \geq 1, \\ & 2x - y \leq 1, \\ & x \leq 0. \end{aligned}$$

   Convert the above LP to a standard form LP.

   (b) Consider the following nonlinear optimization problem:

   $$\begin{aligned} \min \quad & |x + y| + |x - 2y| \\ \text{s.t.} \quad & \max\{x, y, x + y\} \leq 10. \end{aligned}$$

   Reformulate it as a linear program.

3. **Basic Solution and Basic Feasible Solution**

(a) Consider the following linear program:

$$\begin{array}{lrcrcl}
\min & -x_1 & - & 2x_2 & & \\
\text{s.t.} & x_1 & + & x_2 & \leq & 4 \\
 & -2x_1 & + & x_2 & \leq & 2 \\
 & 2x_1 & + & x_2 & \leq & 6 \\
\end{array}$$
$$x_1 \geq 0, \quad x_2 \geq 0.$$

Graph the constraints of this linear program, and indicate the feasible region.

(b) Transform it into a standard form LP.

Denote $\boldsymbol{x} = [x_1, x_2, x_3, x_4, x_5]^\top$ as the vector of variables, and use the standard form notation:

$$\begin{array}{ll}
\min & \boldsymbol{c}^\top \boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
 & \boldsymbol{x} \geq \boldsymbol{0},
\end{array}$$

specify $\boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}$ for the above problem.

(c) Use the procedure discussed in lecture to find *all* basic solutions. For each basic solution, specify the basis matrix, the basic variables, the non-basic variables, and the associated cost coefficients for each part. For example, for the following basis matrix

$$\boldsymbol{B} = [\boldsymbol{A}_3, \boldsymbol{A}_4, \boldsymbol{A}_5] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\boldsymbol{A}_i$ is the i-th column of $\boldsymbol{A}$, the corresponding basic variables, non-basic variables, and cost coefficients are

$$\boldsymbol{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \boldsymbol{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{c}_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{c}_N = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

Specify all basic solutions in this way. [Hint: The nonbasic variables are always zero. So the key is to compute the basic variables. It will require inverting the basis matrix $\boldsymbol{B}$. You can use Python to compute the inverse of a matrix. Scipy has very fast linear algebra packages. You only need to invoke "from scipy import linalg" and use "linalg.inv(A)" to invert a matrix A.]

(d) Among all the basic solutions you found, which basic solutions are feasible, thus are basic feasible solutions? Which basic solutions are infeasible? Locate each basic solution on the graph you drew in part (a). [Hint: You only need to look at the (x_1, x_2) part of each basic solution.] What is the optimal solution?