1 Simple Metropolis: Normal Precision – Gamma.

We know that the prior $\pi(\theta) \propto \theta^{\alpha_0-1}e^{-\beta_0\theta}$, where $\alpha_0 = \frac{1}{2}$ and $\beta_0 = 1$. We also know that the likelihood $f(x|\theta) \propto \theta^{\frac{1}{2}}e^{-\frac{1}{2}\theta x^2}$. Then the posterior

$$\pi(\theta|x) \propto \pi(\theta)f(x|\theta) = \theta^{\alpha_0-\frac{1}{2}}e^{-\theta\left(\beta_0+\frac{1}{2}x^2\right)} = e^{-\theta\left(1+\frac{1}{2}x^2\right)}.$$

We first derive the acceptance ratio for the Metropolis algorithm.

$$\begin{aligned}\rho(\theta_n, \theta') &= \min\left\{1, \frac{\pi(\theta')}{\pi(\theta_n)}\frac{q(\theta_n|\theta')}{q(\theta'|\theta_n)}\right\}\\ &= \min\left\{1, \frac{e^{-\theta'\left(1+\frac{1}{2}x^2\right)}}{e^{-\theta_n\left(1+\frac{1}{2}x^2\right)}}\frac{q(\theta_n|\theta')}{q(\theta'|\theta_n)}\right\} = \min\left\{1, \frac{e^{-3\theta'}}{e^{-3\theta_n}}\frac{q(\theta_n|\theta')}{q(\theta'|\theta_n)}\right\},\end{aligned}$$

where $q(\theta_n|\theta')$ and $q(\theta'|\theta_n)$ are determined based on the proposal distribution ($\mathcal{G}a(\alpha, \beta)$).

We then apply the Metropolis algorithm to approximate the posterior distribution with the following steps.

- **Step 1**. Start with an arbitrary θ_0.
- **Step 2**. At stage n, generate proposal θ' from $\mathcal{G}a(\alpha, \beta)$ for the chosen α and β.
- **Step 3**. Set

  $\theta_{n+1} = \theta'$ with probability $\rho(\theta_n, \theta')$; and

  $\theta_{n+1} = \theta_n$ with probability $1 - \rho(\theta_n, \theta')$.
- **Step 4**. Increase n by 1 and go to **Step 2**.

We consider the following two examples of choosing α and β (any reasonable choices of α and β are accepted).

- For independent Metropolis, we can choose $\alpha = 1$ and $\beta = 3$. Then the acceptance ratio can be found as

  $$\rho(\theta_n, \theta') = \min\left\{1, \frac{e^{-3\theta'}}{e^{-3\theta_n}}\frac{e^{-3\theta_n}}{e^{-3\theta'}}\right\}.$$

  Notice that we get the acceptance ratio equals 1 for all generated θ'. This is due to the fact that our proposal distribution is identical to the posterior distribution.

- Or if we want to use the information of θ_n, we may consider setting $\alpha = 1$ and $\beta = \frac{1}{\theta_n}$. This means that

$$[\theta_n|\theta'] \sim \mathcal{G}a(1, \frac{1}{\theta'}), \text{ and } [\theta'|\theta_n] \sim \mathcal{G}a(1, \frac{1}{\theta_n}).$$

The acceptance ratio can be found as

$$\rho(\theta_n, \theta') = \min\left\{1, \frac{e^{-3\theta'}}{e^{-3\theta_n}} \frac{e^{-\frac{\theta_n}{\theta'}}}{e^{-\frac{\theta'}{\theta_n}}}\right\}.$$

We implement the Metropolis algorithm with the two proposal distributions. The posterior distribution with the proposal distribution discussed in (a) is shown in Figure 1, and the posterior distribution with the proposal distribution discussed in (b) is shown in Figure 2.

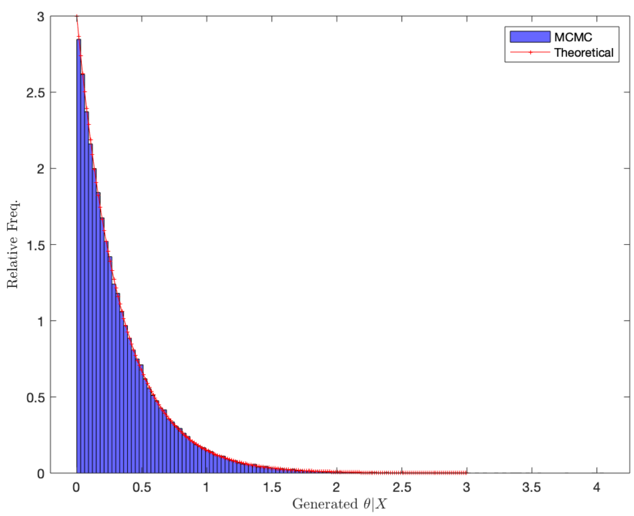

Figure 1: Histogram of generated θ

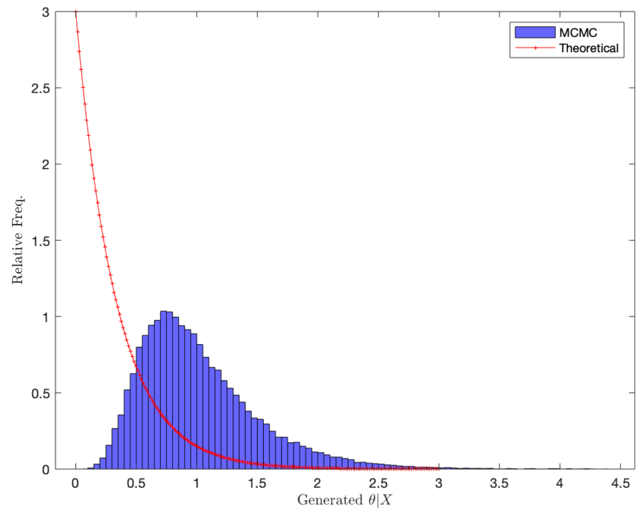

Figure 2: Histogram of generated θ

To answer the rest of the question, we stick with the first proposal. You may find the code implementation in the corresponding Python files.

Bayes estimator: `np.mean(theta_samples)=0.3320369439214918`

HPD credible set:

`calc_hdi(theta_samples, alpha=.03)=(5.549090802965444e-07, 1.163845242529721)`

Trace plot :

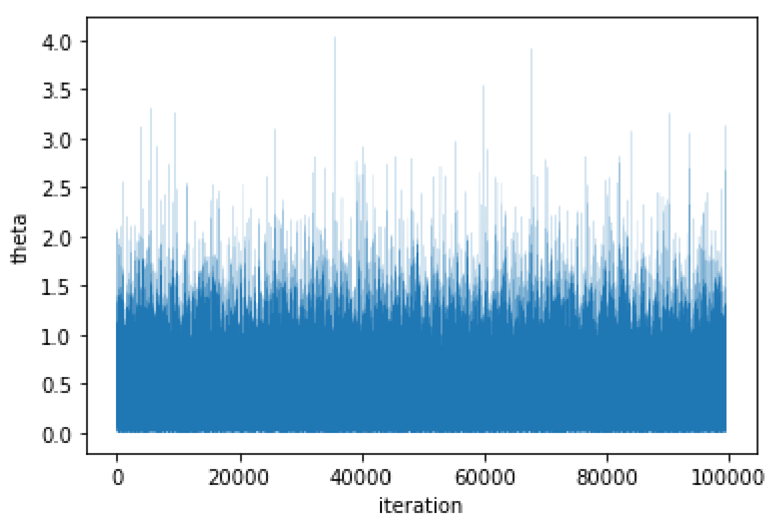

Figure 3: Trace plot of iteration versus θ

2 Normal-Cauchy by Gibbs.

1. As $\pi(\bar{y}|\theta,\sigma^2) \propto \exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right)$, $\pi(\theta|\lambda) \propto \sqrt{\frac{\lambda}{\tau^2}}\exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)$ and $\pi(\lambda) \propto \lambda^{\alpha-1}e^{-\beta\lambda}$ where $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{2}$. The product of the likelihood and the prior is proportional to

$$\exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right)\sqrt{\frac{\lambda}{\tau^2}}\exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)\lambda^{\alpha-1}\exp(-\beta\lambda).$$

   We find the full conditional for $[\theta|\bar{y},\lambda]$ as

$$\begin{aligned}
\pi(\theta|\overline{y},\lambda) &\propto \exp\left(-\frac{(\bar{y}-\theta)^2}{2\frac{\sigma^2}{n}}\right)\exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right) \\
&= \exp\left(-\frac{(n\tau^2+\lambda\sigma^2)\theta^2 - 2(\bar{y}n\tau^2+\lambda\sigma^2\mu)\theta + n\tau^2\overline{y}^2 + \lambda\sigma^2\mu^2}{2\sigma^2\tau^2}\right) \\
&= \exp\left(-\frac{\theta^2 - 2\frac{\bar{y}n\tau^2+\lambda\sigma^2\mu}{n\tau^2+\lambda\sigma^2}\theta + \frac{n\tau^2\bar{y}^2+\lambda\sigma^2\mu^2}{n\tau^2+\lambda\sigma^2}}{2\frac{\sigma^2\tau^2}{n\tau^2+\lambda\sigma^2}}\right) \\
&\propto \exp\left(-\frac{\theta^2 - 2\frac{\bar{y}n\tau^2+\lambda\sigma^2\mu}{n\tau^2+\lambda\sigma^2}\theta + \left(\frac{\bar{y}n\tau^2+\lambda\sigma^2\mu}{n\tau^2+\lambda\sigma^2}\right)^2}{2\frac{\tau^2\sigma^2/n}{\tau^2+\lambda\sigma^2/n}}\right) \\
&= \exp\left(-\frac{\left(\theta - \left(\frac{\tau^2}{\tau^2+\lambda\sigma^2/n}\bar{y} + \frac{\lambda\sigma^2/n}{\tau^2+\lambda\sigma^2/n}\mu\right)\right)^2}{2\frac{\tau^2\sigma^2/n}{\tau^2+\lambda\sigma^2/n}}\right).
\end{aligned}$$

   This means that $[\theta|\bar{y},\lambda] \sim \mathcal{N}\left(\frac{\tau^2}{\tau^2+\lambda\sigma^2/n}\overline{y} + \frac{\lambda\sigma^2/n}{\tau^2+\lambda\sigma^2/n}\mu, \frac{\tau^2\sigma^2/n}{\tau^2+\lambda\sigma^2/n}\right)$.

   We find the full conditional for $[\lambda|\bar{y},\theta]$ as

$$\begin{aligned}
\pi(\lambda|\bar{y},\theta) &\propto \sqrt{\lambda}\exp\left(-\frac{(\theta-\mu)^2}{2\frac{\tau^2}{\lambda}}\right)\lambda^{\alpha-1}\exp(-\beta\lambda) \\
&= \lambda^{\alpha-1/2}\exp\left(-\frac{\tau^2+(\theta-\mu)^2}{2\tau^2}\lambda\right) \\
&= \exp\left(-\frac{\tau^2+(\theta-\mu)^2}{2\tau^2}\lambda\right).
\end{aligned}$$

   This means that $[\lambda|\bar{y},\theta] \sim \mathcal{E}\left(-\frac{\tau^2+(\theta-\mu)^2}{2\tau^2}\lambda\right)$.

2. We implement the Gibbs sampler and obtain the desired quantities. The code for this problem is attached.

The posterior mean and variance are estimated as 105.9569192 and 8.24610674, respectively. The 94% credible set is estimated as $[100.47866057, 111.27757733]$.