Homework 2: Solutions

ISyE 6420
Fall 2022

1. 2-D Density Tasks

(a) For $x > 0$, we have

$$f_X(x) = \int_x^{\infty} f(x,y)dy = \int_x^{\infty} \lambda^2 e^{-\lambda y} dy = \left[\lambda^2 \left(-\frac{1}{\lambda}\right) e^{-\lambda y}\right]\Big|_x^{\infty} = \lambda e^{-\lambda x},$$

which matches with the pdf of exponential distribution with rate parameter λ. Thus, we know that marginal distribution $f_X(x)$ is exponential $\mathcal{E}(\lambda)$.

(b) For $y > 0$, we have

$$f_Y(y) = \int_0^{y} f(x,y)dx = \int_0^{y} \lambda^2 e^{-\lambda y} dy = \left[\lambda^2 e^{-\lambda y} x\right]\big|_0^{y} = \lambda^2 y e^{-\lambda y},$$

which matches with the pdf of a gamma distribution with shape parameter 2 and rate parameter λ. Thus, we know that marginal distribution $f_Y(y)$ is Gamma $\mathcal{G}a(2,\lambda)$.

(c) For $y \geq x$, we have

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(x)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda e^{-\lambda x}} = \lambda e^{-\lambda(y-x)}, y \geq x,$$

which shows that conditional distribution $f_{Y|X}(y \mid x)$ is a shifted exponential.

(d) For $x \leq y$, we have

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}} = \frac{1}{y}, y \geq x,$$

which shows that conditional distribution $f_{X|Y}(x \mid y)$ is uniform $\mathcal{U}(0,y)$.

2 Weibull Lifetimes

(i) For a given shape parameter ν, the probability density function of X is

$$f(x \mid \theta) = \nu\theta x^{\nu-1} \exp\left\{-\theta x^{\nu}\right\}, x \geq 0,$$

and thus the likelihood is proportional to $\theta^n \exp\left\{-\theta \sum_{i=1}^{n} x_i^{\nu}\right\}$. Note that we see this because ν and x are given, and thus can be treated as constants.

As $\theta \sim \exp(\lambda)$, where $\lambda = \frac{5}{2}$, it is proportional to $\exp\{-\lambda\theta\}$. Thus, the posterior is proportional to

$$\theta^n \exp\left\{-\theta \sum_{i=1}^{n} x_i^{\nu}\right\} \exp\{-\lambda\theta\} = \theta^n \exp\left\{-\theta\left(\lambda + \sum_{i=1}^{n} x_i^{\nu}\right)\right\},$$

which is a kernel of Gamma $(n+1, \lambda + \sum_{i=1}^{n} x_i^{\nu})$ distribution. By plugging in the value of $\nu = 3, \lambda = \frac{5}{2}$ and $\sum_{i=1}^{3} X_i^3 = 99$, we obtain the posterior as $\mathcal{G}a(4, 101.5)$.

(ii) As the posterior is $\theta \mid X \sim \mathcal{G}a\left(n+1, 2 + \sum_{i=1}^{n} x_i^3\right)$, we have

$$E[\theta \mid X] = \frac{n+1}{\frac{5}{2} + \sum_{i=1}^{n} x_i^3},$$
$$\mathrm{Var}(\theta \mid X) = \frac{n+1}{\left(\frac{5}{2} + \sum_{i=1}^{n} x_i^3\right)^2}.$$

As $n = 3$ and $\sum_{i=1}^{3} X_i^3 = 99$, we know that

$$E[\theta \mid X] = \frac{n+1}{\frac{5}{2} + \sum_{i=1}^{n} x_i^3} = \frac{4}{101.5} = \frac{8}{203},$$
$$\mathrm{Var}(\theta \mid X) = \frac{n+1}{\left(\frac{5}{2} + \sum_{i=1}^{n} x_i^3\right)^2} = \frac{4}{101.5^2} = \frac{16}{41209}$$

3 Silver-Coated Nylon Fiber

(a) (i) Since the exponential distribution is continuous, we have $P(X \geq 5) = 1 - P(X < 5)$. The cumulative distribution function (CDF) of the Exponential Distribution is

$$F_X(x) = P(X \leq x) = 1 - e^{-\lambda x}$$

Thus, we have $P(X > 5) = 1 - (1 - e^{-(\frac{1}{4})5}) = e^{-(\frac{5}{4})} = 0.286504796860$

(ii) Using the CDF again, we have $P(X < 10) = 1 - e^{-\frac{10}{4}} = 1 - e^{-2.5} = 0.9179150$

(iii) We apply Bayes rule to solve this problem:

$$P(X > 10 \mid X > 5) = \frac{P(X > 5 \mid X > 10)P(X > 10)}{P(X > 5)} = \frac{e^{-2.5}}{e^{-(\frac{5}{4})}} = 0.28651$$

This demonstrates the memoryless property of the Exponential distribution. Note also that $P(X > 5 \mid X > 10) = 1$ because we are conditioning on $X > 10$, which means $X > 5$ is certain to happen.

(b) (i) The classical statistician would use Maximum Likelihood Estimation to estimate λ using the provided data. Students are not required to provide this derivation, but they are expected to use the MLE to obtain a numerical estimate for λ. Derivation of MLE for exponential (also covered in Office Hour 9.21.2022) is included in the appendix. The MLE for a collection of exponential data points is

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i}$$

Thus,

$$\hat{\lambda} = \frac{3}{3+5+7} = \frac{3}{15} = \frac{1}{5}$$

(ii) We apply Bayes' Rule to find the posterior. Let $T_1, \ldots, T_n$ be measurements of the time in hours between dye blockages in a factory. We choose $T \sim \mathrm{Exp}(\lambda)$ and construct the likelihood:

$$L\left(\lambda; t_1, \ldots, t_n\right) = \prod_{i=1}^{n} \lambda e^{-\lambda t_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$$

Our prior is chosen for us as $\pi(\lambda) = \frac{1}{\sqrt{\lambda}}$, $\lambda > 0$ We find the posterior as follows:

$$\begin{aligned} \pi(\lambda|\mathbf{T}) &\propto \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i} \cdot \frac{1}{\sqrt{\lambda}} \\ &= \lambda^{n-\frac{1}{2}} e^{-\lambda \sum_{i=1}^{n} t_i} \end{aligned}$$

We match terms with a Gamma distribution to find

$$\lambda \sim Gamma(n + \frac{1}{2}, \sum_{i=1}^{n} t_i)$$

Using the data provided, we have $\lambda \sim Gamma(\frac{7}{2}, 15)$. The Bayes estimator for λ is the posterior mean:

$$E[\pi(\lambda|\mathbf{T})] = \frac{7/2}{15} = \frac{7}{30}$$

Appendix: Derivation of MLE

Exponential Distribution with parameter λ:

$$f\left(t_i\right) = \lambda e^{-\lambda t_i}, \lambda > 0$$

The classical statistician would optimize for an unknown parameter given data points $t_1, \ldots, t_n$ using Maximum Likelihood Estimation. First we need the likelihood function:

$$L\left(\lambda; t_1, \ldots, t_n\right) = \prod_{i=1}^{n} \lambda e^{-\lambda t_i} = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$$

Due to the logarithm being strictly increasing, the maximum of $L()$ occurs at the same location as the maximum of $\log(L)$. So we take the logarithm first to make the algebra easier:

$$\begin{aligned}
\ln L\left(\lambda; t_1, \ldots, t_n\right) &= \ln\left(\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}\right) \\
&= n\ln(\lambda) + \ln\left(e^{-\lambda \sum_{i=1}^{n} t_i}\right) \\
&= n\ln(\lambda) - \lambda \sum_{i=1}^{n} t_i
\end{aligned}$$

Next we differentiate with respect to λ and set the result equal to 0:

$$\begin{aligned}
\frac{d}{dx} \ln L\left(\lambda; t_1, \ldots, t_n\right) &= \frac{n}{\lambda} - \sum_{i=1}^{n} t_i \\
&= 0
\end{aligned}$$

Rearranging and solving for λ yields

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} t_i}$$