ISyE 6669 HW 2

Fall 2020

1. Expand the following summations:
   (For example, the answer to part (a) is $x_1 + x_2 + x_3$.)

(a)	$\sum_{i=1}^{3} x_i$	(d)	$\sum_{i=1}^{2}\sum_{j=2}^{4}(x_{ij} - y_i)$
(b)	$\sum_{t=1}^{3} \frac{x^{2t}}{t!}$	(e)	$\sum_{k=-1}^{1}(2k+1)x_{k-1}^2$
(c)	$\sum_{i=1}^{3}\sum_{j=1}^{i} x^{i+j}$	(f)	$\sum_{n=2}^{4}\sum_{m=n}^{n+2} x_n y_m$

   Note that by definition $t! = 1 \cdot 2 \cdots (t-1) \cdot t$ for integer $t \geq 1$.

   Solution:

   (a) $x_1 + x_2 + x_3$

   (b) $x^2 + \frac{x^4}{2} + \frac{x^6}{6}$

   (c) $x^2 + x^3 + x^4 + x^4 + x^5 + x^6$

   (d) $(x_{12} - y_1) + (x_{13} - y_1) + (x_{14} - y_1) + (x_{22} - y_2) + (x_{23} - y_2) + (x_{24} - y_2)$

   (e) $-x_{-2}^2 + x_{-1}^2 + 3x_0^2$

   (f) $x_2y_2 + x_2y_3 + x_2y_4 + x_3y_3 + x_3y_4 + x_3y_5 + x_4y_4 + x_4y_5 + x_4y_6$

2. Consider the following two vectors: $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and a matrix $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 3 & -1 \\ 3 & -1 & 2 \end{bmatrix}$.

   (a) Let n be the dimension of $\boldsymbol{x}$ and $\boldsymbol{y}$. What is the value of n?

   (b) Compute $2\boldsymbol{x} - \boldsymbol{y}$.

   (c) Compute the inner product $\boldsymbol{x}^\top \boldsymbol{y}$.

   (d) Compute the Euclidean norm $\|\boldsymbol{x} - \boldsymbol{y}\|_2 = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$. Also called the ℓ_2-norm.

   (e) Compute the ℓ_1-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = \sum_{i=1}^{n} |x_i - y_i|$.

   (f) Compute the ℓ_∞-norm $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max_{1 \leq i \leq n} |x_i - y_i|$.

   (g) Compute $\boldsymbol{x}^\top \boldsymbol{A} \boldsymbol{y}$.

Solution:

(a) The dimension of $\boldsymbol{x}$ and of $\boldsymbol{y}$ are 3. Thus, n is 3.

(b) $2x - y = (-1, 2, 5)^\top$

(c) $\boldsymbol{x}^\top \boldsymbol{y} = 10$

(d) $\|\boldsymbol{x} - \boldsymbol{y}\| = \sqrt{-2^2 + 0^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$

(e) $\|\boldsymbol{x} - \boldsymbol{y}\|_1 = |-2| + |0| + |2| = 4$

(f) $\|\boldsymbol{x} - \boldsymbol{y}\|_\infty = \max\{2, 0, 2\} = 2$

(g) $\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{y} = 27$

3. State whether each of the following sets is convex or not. Explain your reasoning.

(a) $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \geq 1\}$.

(b) $X = \{x \in \mathbb{R} \mid x^2 = 2\}$.

(c) $X = \{(x_1, x_2) \mid \frac{x_2}{(x_1 - 1)} \leq 1, \quad x_1 \geq 1\}$.

Solution:

(a) X is not convex. Consider $X_1 = (0, 1), X_2 = (0, -1)$ $X_1, X_2 \in X$ but the convex combination $X_3 = (\frac{1}{2} \cdot (0) + \frac{1}{2} \cdot (0), \frac{1}{2} \cdot (1) + \frac{1}{2} \cdot (-1)) = (0, 0) \notin X$.

(b) $X = \{-\sqrt{2}, \sqrt{2}\}$ and thus is not convex because $0 = \frac{1}{2} \cdot (-\sqrt{2}) + \frac{1}{2} \cdot (\sqrt{2}) \notin X$

(c) Method 1: Plot of the set X is below:

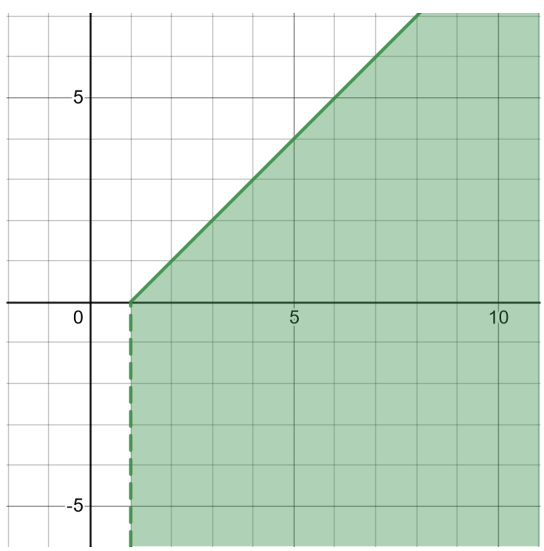

Figure 2: Set X

We can see that set X is convex.

Method 2: Notice that points for which $x_1 = 1$ do not belong to the set, because first inequality is not defined for those points. Then from $x_1 > 1$ we have $x_1 - 1 > 0$ and hence for points $(x_1, x_2) \in X$ we have $\frac{x_2}{x_1-1} \leq 1 \Leftrightarrow x_2 \leq x_1 - 1$ With that in mind we can rewrite set as:

$$X = \{(x_1, x_2) : x_2 \leq x_1 - 1,\ x_1 > 1\}$$

We will show that this set is convex using the definition. Let $a = (a_1, a_2),\ b = (b_1, b_2) \in X$. That means:

$$\begin{aligned} a_1,\ b_1 &> 1 \\ a_2 &\leq a_1 - 1 \\ b_2 &\leq b_1 - 1 \end{aligned}$$

We need to show that $c = (c_1, c_2) = \lambda a + (1-\lambda)b \in X$ i.e that

$$\begin{aligned} c_1 &> 1 \\ c_2 &\leq c_1 - 1 \end{aligned}$$

We have

$$c = (c_1, c_2) = \lambda a + (1-\lambda)b = (\lambda a_1 + (1-\lambda)b_1, \lambda a_2 + (1-\lambda)b_2)$$

From $a_1,\ b_1 > 1$ we have:

$$c_1 = \lambda a_1 + (1-\lambda)b_1 > \lambda + (1-\lambda) = 1$$

From $a_2 \leq a_1 - 1$ and $b_2 \leq b_1 - 1$ we have:

$$c_2 = \lambda a_2 + (1-\lambda)b_2 \leq \lambda(a_1 - 1) + (1-\lambda)(b_1 - 1) = \lambda a_1 + (1-\lambda)b_1 - 1$$

Hence $c \in X$, and thus set X is convex.

4. State whether the following problems are convex programs or not. Explain your reasoning.

   (a) $\min\{x_1^2 + 2x_2^2 \ :\ x_1 \leq 0,\ x_2 \leq 0\}$.

   (b) $\min\{x_1 \cdot x_2 \ :\ x_1^2 + x_2^2 \leq 1\}$.

   (c) $\min\{\sum_{i=1}^n \frac{x_i^i}{i!} \ :\ \sum_{i=1}^n x_i \geq 5\}$.

Solution:

(a) This is a convex program. The feasible region $\mathbb{R}^2_-$ is a convex set. The objective $f(x_1, x_2) = x_1^2 + 2x_2^2$ is a convex function over $\mathbb{R}^2_-$ because $\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \succ 0$.

(b) This is not a convex program. The feasible region $f(x_1, x_2) : x_1^2 + x_2^2 \leq 1$ is a ball, and thus a convex set. The objective function $x_1 \dot{x}_2$ is non-linear and $\nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and not $\succ 0$.

(c) This is a not a convex program. The feasible region $\{(x_1, ..., x_n) : \sum_{i=1}^n x_i \geq 5\}$. is a half space and thus a convex set. The objective function $f(x_1, ..., x_n) = \sum_{i=1}^n \frac{x_i^i}{i!}$ is a not a convex function because it not always a positive sum of convex functions. For example, when $n = 3$ the objective function has $\nabla^2 f = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$. This matrix does not always have non-negative eigenvalues subject to the $x_1 + x_2 + x_3 \geq 5$ constraint.

5. A quantity y is known to depend upon another quantity x. A set of n data pairs $\{y_i, x_i\}_{i=1}^n$ has been collected.

   (a) Formulate an optimization model for fitting the "best" straight line $y = a + bx$ to the data set, where best is with respect to the sum of absolute deviations. What kind of an optimization model is it?

   (b) Re-formulate the optimization model in part (a) where best is with respect to the maximum absolute deviation. What kind of an optimization model is it?

   (c) Formulate an optimization model for fitting the "best" quadratic curve $y = a + bx + cx^2$ to the data set, where best is with respect to the maximum absolute deviations. What kind of an optimization model is it ?

Solution:

(a) The optimization problem can be directly formulated as

$$\min_{\forall a, b \in \mathbb{R}} \quad \sum_{i=1}^n \mid y_i - a - bx_i \mid$$

which is an unconstrained convex nonlinear program. An equivalent formulation of the optimization problem is to introduce auxiliary variables

$t_i, i = 1, \ldots\ldots., n$ and then

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} t_i \\
\text{s.t.} \quad & t_i \geq y_i - a - bx_i, \quad \forall i = 1, \ldots, n, \\
& t_i \geq -(y_i - a - bx_i), \quad \forall i = 1, \ldots, n,
\end{aligned}$$

which is a constrained linear program.

(b) The optimization problem can be directly formulated as

$$\min_{\forall a,b \in \mathbb{R}} \max_{i=1,\ldots,n} \mid y_i - a - bx_i \mid$$

which is an unconstrained convex nonlinear program. An equivalent formulation of the optimization problem is to introduce auxiliary variables $t \in \mathbb{R}_+$ and then

$$\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & t \geq y_i - a - bx_i, \quad \forall i = 1, \ldots, n, \\
& t \geq -(y_i - a - bx_i), \quad \forall i = 1, \ldots, n,
\end{aligned}$$

which is a constrained linear program.

(c) The optimization problem can be directly formulated as

$$\min_{\forall a,b,c \in \mathbb{R}} \quad \max_{i=1,\ldots,n} \mid y_i - a - bx_i - cx_i^2 \mid$$

which is an unconstrained convex nonlinear program. An equivalent formulation of the optimization problem is to introduce auxiliary variables $t \in \mathbb{R}_+$ and then

$$\begin{aligned}
\min \quad & t \\
\text{s.t.} \quad & t \geq y_i - a - bx_i - cx_i^2, \quad \forall i = 1, \ldots, n, \\
& t \geq -(y_i - a - bx_i - cx_i^2), \quad \forall i = 1, \ldots, n,
\end{aligned}$$

which is a constrained linear program.