ISyE6669 Homework 7
Fall 2021

1 Week 7

1. **Geometry of LP**

(a) Let P be a triangle in a plane with three extreme points $A = [0,0]^\top, B = [2,0]^\top, C = [1,2]^\top$. Let $x = [1, 2/3]^\top$ be a point in the triangle P. Express x as a convex combination of the three extreme points A, B, C. You should write $x = \lambda_1 A + \lambda_2 B + \lambda_3 C$, where $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and all nonnegative. You need to give the numerical values of $\lambda_1, \lambda_2, \lambda_3$.

(b) Consider the following polyhedron P defined as

$$P = \{(x, y) \in \mathbb{R}^2 : x + y \geq 2,\ x - y \leq 4,\ y \geq 1,\ x \leq 5\}.$$

Find all the extreme points and extreme rays of P.

(c) Let B be the unit box in $\mathbb{R}^3$, i.e.

$$B = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 1\}.$$

You are allowed to cut the box B with one plane. What is the maximum number of extreme points you can get from the cut? That is, Find a plane H such that its intersection with B has the maximum possible number of extreme points. You need to give an example of such a plane, and write down the coordinates of all the extreme points.

Solution:

(a) We solve the following system of equations:

$$x_1 = \lambda_1 x_{1,A} + \lambda_2 x_{1,B} + \lambda_3 x_{1,C} \tag{1}$$
$$x_2 = \lambda_1 x_{2,A} + \lambda_2 x_{2,B} + \lambda_3 x_{2,C} \tag{2}$$
$$1 = \lambda_1 + \lambda_2 + \lambda_3 \qquad \lambda_i \geq 0, i = \{1, 2, 3\} \tag{3}$$

(2) with $x_{2,A} = x_{2,B} = 0$ yields $x_2 = \lambda_3 x_{2,C} \Rightarrow \lambda_3 = \frac{x_2}{x_{2,C}} = \frac{2/3}{2} = 1/3$.
(1) with $x_{1,A} = 0$ yields $x_1 = \lambda_2 x_{1,B} + \lambda_3 x_{1,C} \Rightarrow \lambda_2 = \frac{x_1 - \lambda_3 x_{1,C}}{x_{1,B}} = \frac{1-(1/3)(1)}{2} = 1/3$.
Finally, (3) yields $\lambda_1 = 1 - \lambda_2 - \lambda_3 = 1 - (1/3) - (1/3) = 1/3$.
Thus $x = (1/3)A + (1/3)B + (1/3)C$.

An alternative way to find the coefficients is as follows. First, note that the first coordinate of $x = [1, 2/3]^\top$ is 1, which is the middle point of A and B, so we can first form $y = (1/2)A + (1/2)B = [1, 0]^\top$. Then, it is easy to see that x is on the line segment connecting y and C. In fact, since the second coordinate of x, which is $2/3$, can be computed as $2/3 = 2/3 \times 0 + 1/3 \times 2$, we get $x = (2/3)y + (1/3)C$. Combining with the formula for y, we get $x = 2/3((1/2)A + (1/2)B) + (1/3)C = (1/3)A + (1/3)B + (1/3)C$. This derivation is quite intuitive and it embodies an important principle that "convex combination of convex combination is a convex combination."

(b) We generate the extreme points by finding basic solutions of $n = 2$ linearly independent active constraints and verifying their feasibility:

$$
\begin{aligned}
&x + y = 2, x - y \leq 4 : [3, -1]^\top \text{ violates constraint } y \geq 1 \\
&x + y = 2, y = 1 && \Rightarrow x^1 = [1, 1]^\top \\
&x + y = 2, x = 5 : [5, -3]^\top \text{ violates constraint } y \geq 1 \\
&x - y = 4, y = 1 && \Rightarrow x^2 = [5, 1]^\top \\
&x - y = 4, x = 5 : \text{ same as } x^2 \\
&y = 1, x = 5 : \text{ same as } x^2
\end{aligned}
$$

Thus the extreme points are $x^1 = [1, 1]^\top, x^2 = [5, 1]^\top$.

From the graph of the feasible region in Figure 1, we can see that clearly, the two "extreme" directions that satisfy $x = x' + d$ (where x' is a convex combination of the extreme points of P) are the vectors $w^2 = [-1, 1]^\top$ and $w^1 = [0, 1]^\top$.

We can solve for the extreme rays as follows. The recession cone of P (the conic hull of the extreme rays) is $\{\mathbf{d} \in \mathbb{R}^2 \mid \mathbf{Ad} \geq \mathbf{0}\}$. The constraints defining the recession cone can therefore be written as

$$
\begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

We graph the above constraints in Figure 2 and see that the extreme rays of P are $w^2 = [-1, 1]^\top$ and $w^1 = [0, 1]^\top$ (up to scaling).

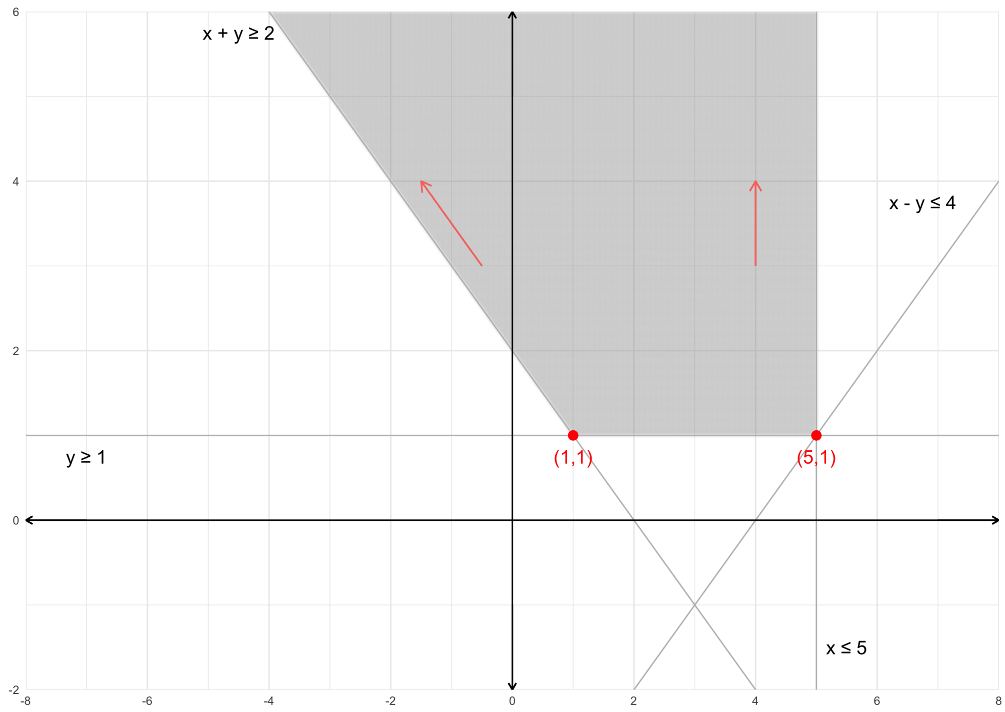

Figure 1: Shaded region represents polyhedron P. Extreme points and extreme rays shown in red.

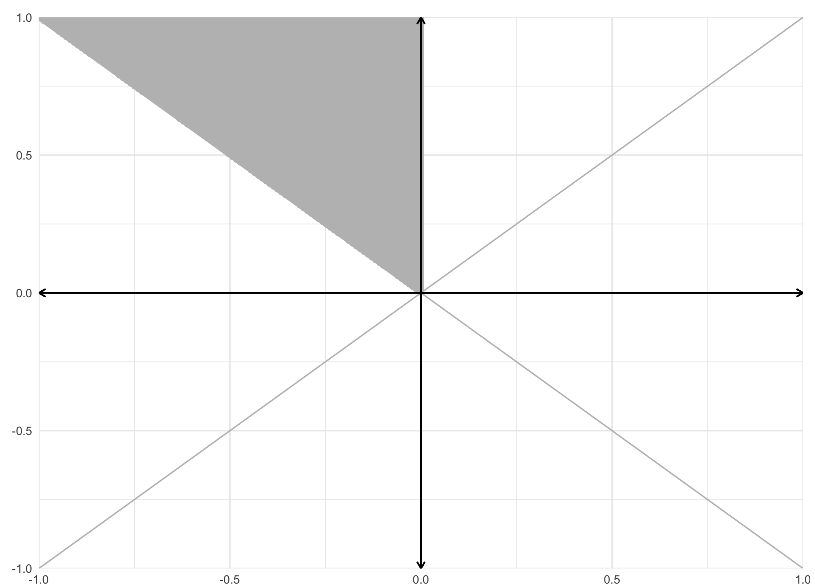

Figure 2: The recession cone of P.

(c) We wish to find a plane $H = \{a_1x + a_2y + a_3z = c\}$ such that its intersection with B yields the maximum number of extreme points. At the extreme points of $H \cap B$, there must be $n = 3$ linearly independent active constraints. All extreme points must lie on the plane H; thus, one of these active constraints must be the equation defining H and the other two linearly independent active constraints must be from B. Note that the faces of B correspond to the constraints of B and the edges of B (which are the intersections of two perpendicular faces of B) correspond to two constraints of B. So the extreme points of $H \cap B$ must lie on the edges of B. There are 12 edges of B, so which edges should we choose? Consider the following cases:

- Choose edges corresponding to two linearly independent constraints from the set $\{x = 0, y = 0, z = 0\}$. Then there are three possible extreme points: $(0, 0, c), (0, c, 0), (c, 0, 0)$.
- Choose edges corresponding to two linearly independent constraints from the set $\{x = 1, y = 1, z = 1\}$. Then there are three possible extreme points: $(1, 1, c - 2), (1, c - 2, 1), (c - 2, 1, 1)$.
- Choose edges corresponding to two linearly independent constraints such that one constraint sets a variable to 0 and the other to 1. Then there are six possible extreme points: $(0, 1, c - 1), (1, 0, c - 1), (0, c - 1, 1), (1, c - 1, 0), (c - 1, 0, 1), (c - 1, 1, 0)$.

Thus, the maximum possible number of extreme points that H can have is 6.

A slightly different argument is the following. Any two adjacent extreme points of $H \cap B$ must share two common active constraints, one of them must be H, and the other is a facet of the box. In other words, an edge of $H \cap B$ must be formed by the intersection of H with a facet of B. A facet of the box is just one of the six rectangular faces of the box. Since B only has six facets, $H \cap B$ can have at most six edges. Since $H \cap B$ is a two-dimensional polygon, the number of edges is equal to the number of extreme points. Therefore, $H \cap B$ can have at most six extreme points.

An example of such a plane is $H = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1.5\}$. The intersection of plane H and B (in Figure 3) gives extreme points $(0.5, 0, 1), (0, 0.5, 1), (1, 0, 0.5), (1, 0.5, 0)$, $(0.5, 1, 0), (0, 1, 0.5)$.

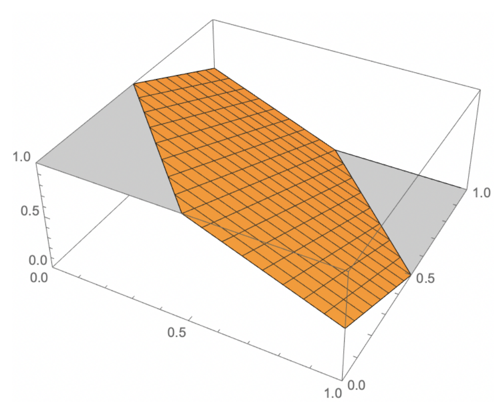

Figure 3: Shaded region represents the intersection of box B and plane H.

2. **Reformulation as linear program**

(a) Consider the following linear program:

$$\begin{aligned} \max \quad & x - y \\ \text{s.t.} \quad & x + y \geq 1, \\ & 2x - y \leq 1, \\ & x \leq 0. \end{aligned}$$

Convert the above LP to a standard form LP.

Solution:

$$\begin{aligned} -\min \quad & x' + y^+ - y^- \\ \text{s.t.} \quad & -x' + y^+ - y^- - s_1 = 1, \\ & -2x' - y^+ + y^- + s_2 = 1, \\ & x', y^+, y^-, s_1, s_2 \geq 0. \end{aligned}$$

(b) Consider the following nonlinear optimization problem:

$$\begin{aligned} \min \quad & |x + y| + |x - 2y| \\ \text{s.t.} \quad & \max\{x, y, x + y\} \leq 10. \end{aligned}$$

Reformulate it as a linear program.

Solution:

$$\begin{aligned} \min \quad & z_1 + z_2 \\ \text{s.t.} \quad & x \leq 10, \\ & y \leq 10, \\ & x + y \leq 10, \\ & x + y - z_1 \leq 0, \\ & -x - y - z_1 \leq 0, \\ & x - 2y - z_2 \leq 0, \\ & -x + 2y - z_2 \leq 0. \end{aligned}$$

Note that we could add $z_1 \geq 0, z_2 \geq 0$, but these two inequalities are also implied by the last four inequalities above. Can you see why?

3. **Basic Solution and Basic Feasible Solution**

(a) Consider the following linear program:

$$\begin{array}{lrcrcr} \min & -x_1 & - & 2x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 4 \\ & -2x_1 & + & x_2 & \leq & 2 \\ & 2x_1 & + & x_2 & \leq & 6 \end{array}$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

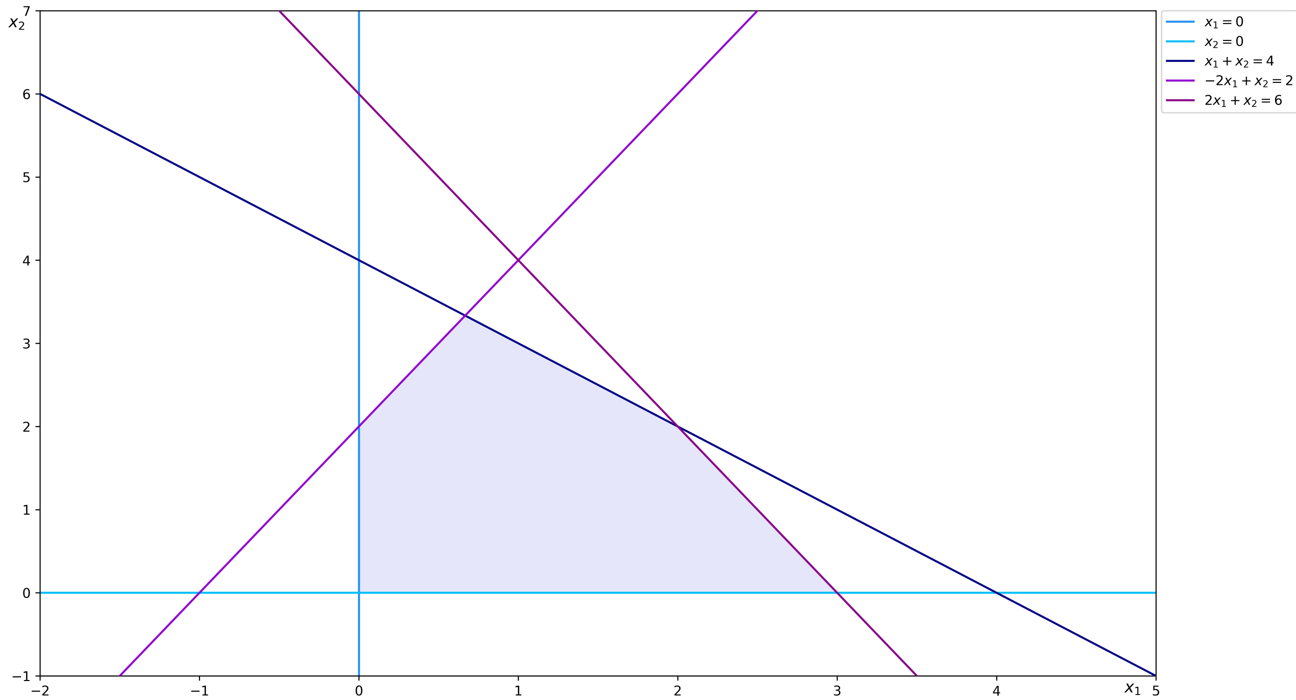

Figure 4: Feasible region

Graph the constraints of this linear program, and indicate the feasible region.

Solution: See Figure 4.

(b) Transform it into a standard form LP. Denote $\boldsymbol{x} = [x_1, x_2, x_3, x_4, x_5]^\top$ as the vector of variables, and use the standard form notation:

$$\begin{aligned} \min \quad & \boldsymbol{c}^\top \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned}$$

specify $\boldsymbol{c}, \boldsymbol{A}, \boldsymbol{b}$ for the above problem.

Solution: We can rewrite the problem as:

$$\begin{array}{lrcrcrcrcrcl} \min & -x_1 & - & 2x_2 & & & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & & & = & 4 \\ & -2x_1 & + & x_2 & & & + & x_4 & & & = & 2 \\ & 2x_1 & + & x_2 & & & & & + & x_5 & = & 6 \\ & \multicolumn{11}{c}{x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0, x_5 \geq 0.} \end{array}$$

In this case, the $\mathbf{c}$, $\mathbf{A}$ and $\mathbf{b}$ are

$$\mathbf{c} = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}.$$

(c) Use the procedure discussed in lecture to find *all* basic solutions. For each basic solution, specify the basis matrix, the basic variables, the non-basic variables, and the associated cost coefficients for each part. For example, for the following basis matrix

$$\boldsymbol{B} = [\boldsymbol{A}_3, \boldsymbol{A}_4, \boldsymbol{A}_5] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $\boldsymbol{A}_i$ is the i-th column of $\boldsymbol{A}$, the corresponding basic variables, non-basic variables, and cost coefficients are

$$\boldsymbol{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \boldsymbol{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{c}_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{c}_N = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

Specify all basic solutions in this way. [Hint: The nonbasic variables are always zero. So the key is to compute the basic variables. It will require inverting the basis matrix $\boldsymbol{B}$. You can use Python to compute the inverse of a matrix. Scipy has very fast linear algebra packages. You only need to invoke "from scipy import linalg" and use "linalg.inv(A)" to invert a matrix A.]

Solution: Below are all basic solutions for this problem:

$$(b1) \quad \mathbf{B}_2 = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3] = \begin{bmatrix} 1 & 1 & 1 \\ -2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix},$$

$$\mathbf{x}_B = \mathbf{B}_2^{-1}\mathbf{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$(b2) \quad \mathbf{B}_2 = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_4] = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix},$$

$$\mathbf{x}_B = \mathbf{B}_2^{-1}\mathbf{b} = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$(b3)\quad \mathbf{B}_3 = [\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_5] = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix},$$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{10}{3} \\ \frac{5}{3} \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

$$(b4)\quad \mathbf{B}_4 = [\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_4] = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix},$$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 8 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

$$(b5)\quad \mathbf{B}_5 = [\mathbf{A}_1, \mathbf{A}_3, \mathbf{A}_5] = \begin{bmatrix} 1 & 1 & 0 \\ -2 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 8 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

$$(b6)\quad \mathbf{B}_6 = [\mathbf{A}_1, \mathbf{A}_4, \mathbf{A}_5] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

$$\mathbf{x}_B = \begin{bmatrix} x_1 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \\ -2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} -2 \\ 0 \end{bmatrix}.$$

$$(b7)\quad \mathbf{B}_7 = [\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

$$(b8)\quad \mathbf{B}_8 = [\mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_5] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

(*b*9) $\mathbf{B}_9 = [\mathbf{A}_2, \mathbf{A}_4, \mathbf{A}_5] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$,

$$\mathbf{x}_B = \begin{bmatrix} x_2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

(*b*10) $\mathbf{B}_{10} = [\mathbf{A}_3, \mathbf{A}_4, \mathbf{A}_5] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \quad \mathbf{x}_N = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{c}_N = \begin{bmatrix} -1 \\ -2 \end{bmatrix}.$$

(d) Among all the basic solutions you found, which basic solutions are feasible, thus are basic feasible solutions? Which basic solutions are infeasible? Locate each basic solution on the graph you drew in part (a). [Hint: You only need to look at the (x_1, x_2) part of each basic solution.] What is the optimal solution?

Solution: Among solutions we found, solutions $(b2), (b3), (b4), (b8)$, and $(b10)$ are feasible thus are basic feasible solutions. They are represented as green dots in the Figure 5. Solutions $(b1), (b5), (b6), (b7)$, and $(b9)$ are infeasible. They are represented using red dots in the Figure 5. By calculating the objective value at each of the basic feasible solution we get that optimal solution is solution $(b3)$, with an optimal value of -$\frac{22}{3}$.

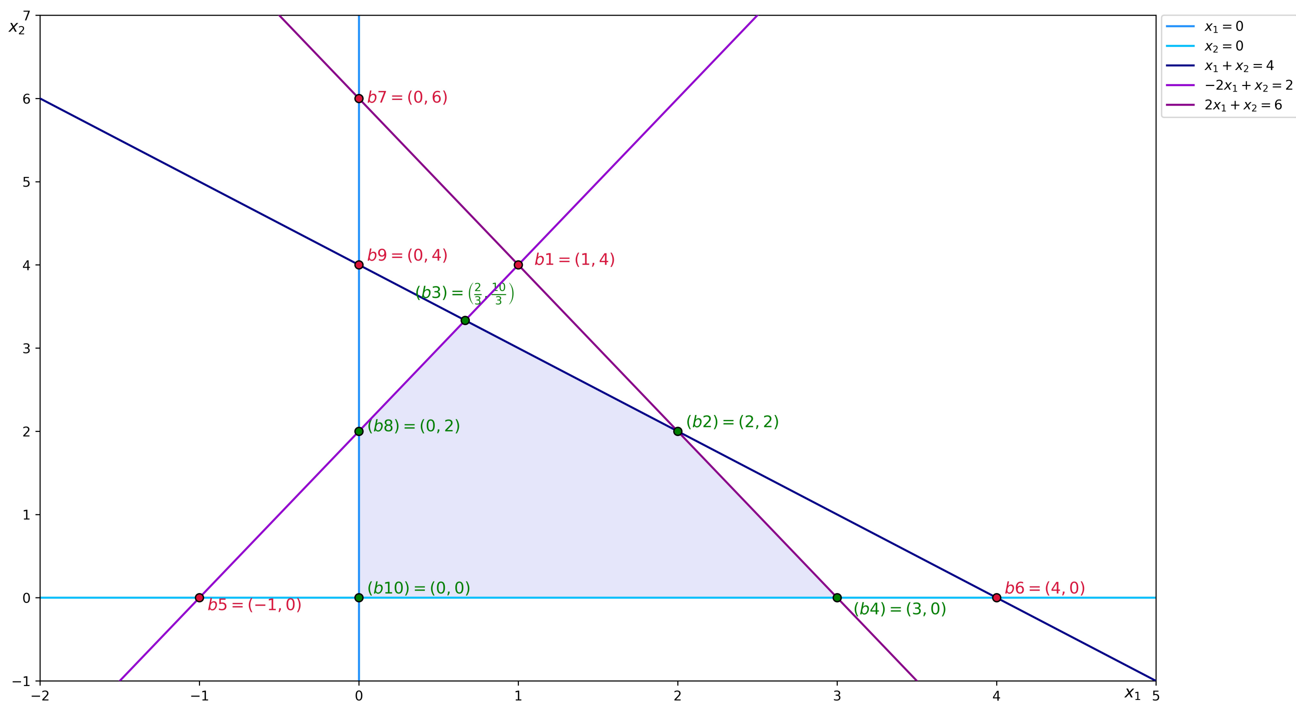

Figure 5: Basic solutions