ISyE 6669 HW 1

Fall 2021

1. Consider the following maximization problem

$$\begin{aligned} \max \quad & (x-1)^2 + (y-1)^2 \\ \text{s.t.} \quad & |x| + |y| \leq 1. \end{aligned}$$

   Plot the feasible region of this problem with the feasible area shaded. Draw (in dashed lines) the contours of the objective function. Based on your drawing, find all the optimal solutions and the optimal objective value of this problem. There may be multiple optimal solutions. Find all optimal solutions.

   Solution: Figure 1 shows the feasible region and the contours of the objective. We see from the figure that the optimal solutions are $(-1, 0)$ and $(0, -1)$ with optimal objective value of 5. To find the optimal solution, we need to graph contours of the objective against the feasible region. All points on a single contour take on the same objective values. In this case we need to graph the circles $(x-1)^2 + (y-1)^2 = c$ for various scalars c. We start off by graphing $(x-1)^2 + (y-1)^2 = c$, against the feasible region, for small values of c. We gradually increase c, until the contour $(x-1)^2 + (y-1)^2 = c$ no longer intersects the feasible region. The last contour intersecting the feasible region indicates the largest objective value of any point in the feasible region. Hence, it indicates the optimal solution of this problem. The last contour intersecting the feasible region is $(x-1)^2 + (y-1)^2 = 5$ and it intersects the feasible region at $(-1, 0)$ and $(0, -1)$. Thus, the optimal solutions are $(-1, 0)$ and $(0, -1)$ with optimal objective value of 5.

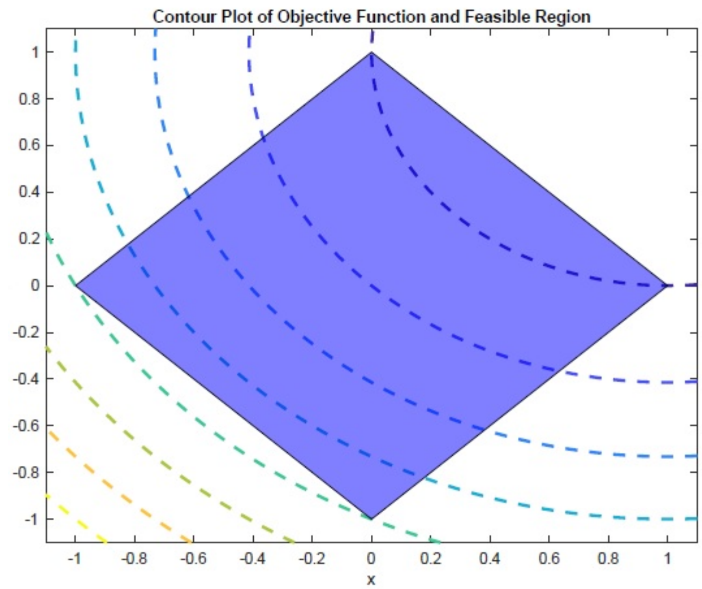

Figure 1: Feasible region and contours

2. Recall the maximum volume box from Module 1, Lesson 1. Solve the following problem using basic calculus:

$$\max\{x(1-2x)^2 :\ 0 \leq x \leq 1/2\}$$

What is the optimal solution and the optimal objective value? Draw the objective function over the feasible region.

Solution:
Taking the first and second derivatives of the objective function $f(x) = x(1-2x)^2$, we get

$$f'(x) = 12x^2 - 8x + 1, \quad f''(x) = 24x - 8.$$

Solve the equation $f'(x) = 0$, and we get solutions $x_1^* = \frac{1}{6}$, $x_2^* = \frac{1}{2}$ The solution x_2^* is clearly not a maximizer because $f(x_2^*) = 0$. Notice that $f''(x_1^*) = -4 < 0$. Using basic calculus, we know that x_1^* is a maximizer. Therefore, the optimal solution is $x_1^* = \frac{1}{6}$ and the optimal objective value is $f(x_1^*) = \frac{2}{27}$.

3. Consider the following optimization problem:

$$\begin{array}{lll} (P) & \max & x(y^2 - z^2) \\ & \text{s.t.} & |y| + z \leq 1, \\ & & x \in \{0, 1\},\ z \geq 0. \end{array}$$

Answer the questions:

(a) Is (P) a linear program, a mixed integer nonlinear program, or a mixed integer quadratic program? Choose all descriptions that apply.

(b) Write a minimization problem that is equivalent to (P).

(c) Find all the optimal solutions.

Solution:

(a) (P) is a mixed integer nonlinear program.

(b) The following is equivalent to (P)

$$\begin{aligned} \min \quad & -x(y^2 - z^2) \\ \text{s.t.} \quad & -|y| - z \geq -1, \\ & x \in \{0, 1\},\ -z \leq 0. \end{aligned}$$

(c) Substituting $|y| = w$, helps eliminate $|.|$ function and simplify the constraint. (P) is equivalent to the following problem,

$$\begin{aligned} \max \quad & x(w^2 - z^2) \\ \text{s.t.} \quad & w + z \leq 1, \\ & x \in \{0, 1\},\ w \geq 0,\ , z \geq 0. \end{aligned}$$

Since x is independent of w and z, the two factors can be maximized independently. Thus, $x = 1, w = 1, z = 0$ results in the optimal objective function. Equivalently in terms of y,

$$(x, y, z) = \{(1, 1, 0), (1, -1, 0)\}$$

are the optimal solutions.

4. Recall the portfolio optimization problem solved in Module 2, Lesson 3. Now, collect the prices of MSFT, V, and WMT from the last 24 months (data can be collected from e.g. Yahoo Finance). Use the code and data file format used in the lesson (this will be provided) to solve the exact same portfolio problem using the new data. Compare and contrast your solution to the one in the lesson.

Solution:
The output is shown below.

```
 ----------------------
MSFT: Exp ret = 0.032716, Risk = 0.063093
V: Exp ret = 0.007660, Risk = 0.084906
WMT: Exp ret = 0.012749, Risk = 0.051777
----------------------
Optimal portfolio
----------------------
x[MSFT] = 0.380516
```

```
x[V] = 0.068093
x[WMT] = 0.551391
----------------------
Exp ret = 0.020000
risk = 0.047298
----------------------
```

We use adjusted closing prices near the end of each month. Please see the attached `hw_monthly_prices.xlsx` to see the values used.

Any reasonable comparison should be awarded full points.

Data source for Microsoft: https://finance.yahoo.com/quote/MSFT/history?period1=1577750400&period2=1640908800&interval=1mo&filter=history&frequency=1mo&includeAdjustedClose=true
Data source for Visa: https://finance.yahoo.com/quote/V/history?period1=1577664000&period2=1640822400&interval=1mo&filter=history&frequency=1mo&includeAdjustedClose=true
Data source for Walmart: https://finance.yahoo.com/quote/WMT/history?period1=1577664000&period2=1640822400&interval=1mo&filter=history&frequency=1mo&includeAdjustedClose=true