Question 1: Maxwell

(a) We let $\boldsymbol{y} = (y_1, \ldots, y_n)$ and find the likelihood as

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \sqrt{\frac{2}{\pi}} \theta^{3/2} y_i^2 \exp\left\{-\theta y_i^2/2\right\} \\
&= \left(\sqrt{\frac{2}{\pi}}\right)^n \theta^{\frac{3}{2}n} \exp\left\{-\frac{\theta}{2}\sum_{i=1}^{n} y_i^2\right\} \\
&\propto \theta^{3n/2} \exp\left\{-\frac{\theta}{2}\sum_{i=1}^{n} y_i^2\right\}
\end{aligned}
$$

Then the posterior distribution is

$$
\begin{aligned}
\pi(\theta \mid \boldsymbol{y}) &\propto \theta^{\frac{3}{2}n} \exp\left\{-\frac{\theta}{2}\sum_{i=1}^{n} y_i^2\right\} \cdot \exp\{-\lambda\theta\} = \theta^{\frac{3}{2}n} \exp\left\{-\left(\frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda\right)\theta\right\} \\
&\sim \mathcal{G}amma\left(\frac{3}{2}n + 1, \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda\right).
\end{aligned}
$$

Hence, the posterior is $\mathcal{G}amma(\alpha, \beta)$ where $\alpha = \frac{3}{2}n + 1$ and $\beta = \frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda$, which means that it depends on data via $\sum_{i=1}^{n} y_i^2$.

(b) Since $\sum_{i=1}^{3} y_i^2 = (1.4)^2 + (3.1)^2 + (2.5)^2 = 17.82$ and the mean of $\mathcal{G}amma(\alpha, \beta)$ is $\frac{\alpha}{\beta}$ we calculate the Bayes estimator as

$$
\widehat{\theta}_B = E[\theta \mid \boldsymbol{y}] = \frac{\frac{3}{2}n + 1}{\frac{1}{2}\sum_{i=1}^{n} y_i^2 + \lambda} = \frac{\frac{3}{2}(3) + 1}{\frac{1}{2}(17.82) + \frac{1}{2}} = 0.5845.
$$

The MLE for θ is $\widehat{\theta}_{MLE} = \frac{3n}{\sum_{i=1}^{n} y_i^2} = \frac{(3)(3)}{17.82} = 0.5051$, and the prior mean is $E[\theta] = 1/\lambda = 2$. Based on our results, the relationship among Bayes estimator, MLE and prior mean is

MLE(0.5051) $<$ Bayes estimator (0.5845) $<$ prior mean (2)

(c) We use the following Matlab code:

```
gaminv(0.025,4.5+1,1 /(17.82 / 2+0.5)) % 0.2027
gaminv(0.975,4.5+1,1 /(17.82 / 2+0.5)) % 1.1647
```

The 95% equitailed credible set for θ to be $[0.2027, 1.1647]$.

(d) As $EY = 2\sqrt{\frac{2}{\pi\theta}}$ and $\pi(\theta \mid x) \sim \mathcal{G}amma(5.5, 9.41)$, we find the predictive value for a single future observation as

$$\begin{aligned}
\widehat{y}_{n+1} &= \int_0^\infty 2\sqrt{\frac{2}{\pi\theta}}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta = \int_0^\infty 2\sqrt{\frac{2}{\pi\theta}}\frac{9.41^{5.5}}{\Gamma(5.5)}\theta^{4.5}e^{-9.41\theta}d\theta \\
&\approx 2.2445
\end{aligned}$$

Question 2: Jeremy Mixture

(a) Marginal Distribution:

$$\begin{aligned}
m(x) &= \int f(x \mid \theta)\pi(\theta)d\theta \\
&= \int f(x \mid \theta)\left[\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta)\right]d\theta \\
&= \epsilon\int f(x \mid \theta)\pi_1(\theta)d\theta + (1-\epsilon)\int f(x \mid \theta)\pi_2(\theta)d\theta \\
&= \epsilon m_1(x) + (1-\epsilon)m_2(x)
\end{aligned}$$

The posterior is as follows:

$$\begin{aligned}
\pi(\theta \mid x) &= \frac{f(x \mid \theta)\pi(\theta)}{m(x)} \\
&= \frac{f(x \mid \theta)\left[\epsilon\pi_1(\theta) + (1-\epsilon)\pi_2(\theta)\right]}{m(x)} \\
&= \frac{\epsilon f(x \mid \theta)\pi_1(\theta)}{m(x)} + \frac{(1-\epsilon)f(x \mid \theta)\pi_2(\theta)}{m(x)} \\
&= \frac{\epsilon m_1(x)f(x \mid \theta)\pi_1(\theta)}{m(x)m_1(x)} + \frac{(1-\epsilon)m_2(x)f(x \mid \theta)\pi_2(\theta)}{m(x)m_2(x)} \\
&= \frac{\epsilon m_1(x)}{m(x)}\left(\frac{f(x \mid \theta)\pi_1(\theta)}{m_1(x)}\right) + \frac{\left[\epsilon m_1(x) + (1-\epsilon)m_2(x) - \epsilon m_1(x)\right]f(x \mid \theta)\pi_2(\theta)}{m(x)m_2(x)} \\
&= \frac{\epsilon m_1(x)}{m(x)}\left(\frac{f(x \mid \theta)\pi_1(\theta)}{m_1(x)}\right) + \frac{\left[m(x) - \epsilon m_1(x)\right]f(x \mid \theta)\pi_2(\theta)}{m(x)m_2(x)} \\
&= \frac{\epsilon m_1(x)}{m(x)}\left(\frac{f(x \mid \theta)\pi_1(\theta)}{m_1(x)}\right) + \left(\frac{m(x) - \epsilon m_1(x)}{m(x)}\right)\frac{f(x \mid \theta)\pi_2(\theta)}{m_2(x)} \\
&= \epsilon'\pi_1(\theta \mid x) + (1-\epsilon')\,\pi_2(\theta \mid x)
\end{aligned}$$

where

$$\epsilon' = \frac{\epsilon m_1(x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)}$$

(b) We first find the posterior distribution given the likelihood $X \mid \theta \sim \mathcal{N}(\theta, \sigma^2)$ when σ^2 is fixed, and the prior $\theta \sim \mathcal{N}(\theta_0, \sigma_0^2)$. We have

$$\begin{aligned}
\pi(\theta \mid x) \propto f(x \mid \theta)\pi(\theta) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\theta)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{(\theta-\theta_0)^2}{2\sigma_0^2}} \\
&= \frac{1}{2\pi\sigma\sigma_0} \exp\left\{ -\frac{(x-\theta)^2\sigma_0^2 + (\theta-\theta_0)^2\sigma^2}{2\sigma^2\sigma_0^2} \right\} \\
&= \frac{1}{2\pi\sigma\sigma_0} \exp\left\{ -\frac{\theta^2 - \frac{2\left(\sigma_0^2 x + \sigma^2\theta_0\right)}{\sigma_0^2+\sigma^2}\theta + \frac{\sigma_0^2 x^2 + \sigma^2\theta_0^2}{\sigma_0^2+\sigma^2}}{\frac{2\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2}} \right\} \\
&\propto \mathcal{N}\left(\frac{\sigma_0^2 x + \sigma^2\theta_0}{\sigma_0^2+\sigma^2}, \frac{\sigma^2\sigma_0^2}{\sigma_0^2+\sigma^2} \right)
\end{aligned}$$

Based on the above result and $\pi_1(\theta) \sim \mathcal{N}(\theta_0 = 110, \sigma_0^2 = 60), \pi_2(\theta) \sim \mathcal{N}(\theta_0 = 100, \sigma_0^2 = 200)$, $\sigma^2 = 80$ and $x = 98$, we have

$$\begin{aligned}
\pi_1(\theta \mid x) &\sim \mathcal{N}\left(\frac{(60)(98) + (80)(110)}{60+80}, \frac{(80)(60)}{60+80} \right) = \mathcal{N}\left(\frac{734}{7}, \frac{240}{7} \right) \\
\pi_2(\theta \mid x) &\sim \mathcal{N}\left(\frac{(200)(98) + (80)(100)}{200+80}, \frac{(80)(200)}{200+80} \right) = \mathcal{N}\left(\frac{690}{7}, \frac{400}{7} \right)
\end{aligned}$$

Roshan's derivation for Marginal distribution of X:

Let $X|\theta \sim N(\theta, \sigma^2)$ and $\theta \sim N(\theta_0, \sigma_0^2)$. Then $X = \theta + \epsilon,$ where $\epsilon \sim N(0, \sigma^2)$. We have the following:

$$E[X] = \theta_0 + 0$$

$$Var[X] = Var[\theta] + Var[\epsilon] = \sigma_0^2 + \sigma^2$$

Thus, $X \sim N(\theta_0, \sigma_0^2 + \sigma^2)$. Using the provided information, we have the following for the marginal distributions:

For $N(110, 60) \rightarrow$

$$\begin{aligned}
m_1(98) &= \frac{1}{\sqrt{2\pi}\sqrt{80+60}} \cdot \exp\{-\frac{1}{2(80+60)}(98-110)^2\} \\
&= \frac{1}{\sqrt{280\pi}} \cdot \exp\{-\frac{1}{2(80+60)}(98-110)^2\} \\
&= 0.0201601890
\end{aligned}$$

For $N(100, 200) \rightarrow$

$$
\begin{aligned}
m_2(98) &= \frac{1}{\sqrt{2\pi}\sqrt{80+60}} \cdot \exp\{-\frac{1}{2(80+200)}(98-100)^2\} \\
&= \frac{1}{\sqrt{560\pi}} \cdot \exp\{-\frac{1}{2(80+200)}(98-100)^2\} \\
&= 0.02367167266471270
\end{aligned}
$$

We can then compute the ϵ' as

$$
\epsilon' = \frac{\frac{2}{3}m_1(x)}{\frac{2}{3}m_1(x) + \frac{1}{3}m_2(x)} = \frac{\frac{2}{3}(0.0202)}{\frac{2}{3}(0.0202) + \frac{1}{3}(0.0237)} = 0.6303
$$

Thus, the posterior is

$$
\pi(\theta \mid x) = 0.6303 \cdot \mathcal{N}\left(\frac{734}{7}, \frac{240}{7}\right) + 0.3697 \cdot \mathcal{N}\left(\frac{690}{7}, \frac{400}{7}\right)
$$

and the Bayes estimator for θ is

$$
\delta_B(98) = E[\theta \mid x] = (0.6303)\left(\frac{734}{7}\right) + (0.3697)\left(\frac{690}{7}\right) = 102.5333
$$

Question 3: Mendel's Experiment with Peas

Q3:

(a) To calculate the prior mean, use the mean formula for the $\mathcal{B}eta$ distribution. This is just $\frac{\alpha}{\alpha+\beta}$. The posterior distribution of p is $\mathcal{B}\text{eta}(\alpha + 787, \beta + 1064 - 787) = \mathcal{B}\text{eta}(802, 282)$. Using the same formula to calculate the posterior mean yields $\frac{802}{802+282} \approx 0.739852$

(b) Taking the CDF of the above posterior $\mathcal{B}eta$ distribution, we get: $P(p \leq 0.75) = 0.7754435$ (Done in R):

```
> pbeta(0.75,802,282)
```

(c) The 95% equitailed credible set was determined by solving the equations $\int_0^a \pi(\theta \mid x)d\theta = 0.025$ and $\int_b^1 \pi(\theta \mid x)d\theta = 0.025$ for a and b respectively. Our result is $[a, b] = [0.713334, 0.765533]$.