Question 1

An automobile piston manufacturing company has two plants in North America. Plant A produces pistons of which about 2% are defective, while plant B produces about 10% defective pistons. An automobile manufacturer receives 60% of their pistons from plant A and 40% from plant B. They inspect the pistons before assembly. Answer the following questions:

1. Postulate Beta prior distributions for the defective rate of each plant: for plant A, $p \sim Beta(\alpha_0, \beta_0)$, and for plant B: $p \sim Beta(\alpha_1, \beta_1)$. Assume $\alpha_0 + \beta_0 = \alpha_1 + \beta_1 = 100$.

   Our best option is setting the mean of each prior to the estimated defective rate for each plant. For the beta distribution, $E[X] = \frac{\alpha}{\alpha+\beta}$. Obeying the constraint that the sum of the parameters must equal 100 gives us a prior of $Beta(2, 98)$ for plant A and $Beta(10, 90)$ for plant B.

2. During inspection of 200 pistons by the automobile manufacturer, 15 pistons are found to be defective. Find the posterior distribution of the overall defective rate.

   A binomial likelihood is the natural choice for the data we have available. Then it's a matter of how to combine the likelihood with the two priors. Since we don't have a breakdown of the defective products by factory, we can't give separate likelihoods for each prior from part 1. So we need a mixture prior on p:

   Where y is the overall defective rate:

$$
\begin{aligned}
Y|p &\sim Bin(n, p) \\
p &\sim .6Beta(\alpha_0, \beta_0) + .4Beta(\alpha_1, \beta_1)
\end{aligned}
$$

   First, we need the posterior. We're going to leave constants out for now.

$$
\begin{aligned}
P(p|y) &\propto p^y(1-p)^{n-y}\left(.6p^{\alpha_0-1}(1-p)^{\beta_0-1} + .4p^{\alpha_1-1}(1-p)^{\beta_1-1}\right) \\
&\propto p^y(1-p)^{n-y}\left(.6p^1(1-p)^{97} + .4p^9(1-p)^{89}\right) \\
&\propto .6p^{y+1}(1-p)^{n-y+97} + .4p^{y+9}(1-p)^{n-y+89}
\end{aligned}
$$

   Recognizing the Beta mixture:

$$
P(p|y) = \epsilon' Beta(y+2, n-y+98) + (1-\epsilon')Beta(y+10, n-y+90)
$$

   Notice that the posterior weights are still unknown. To find them, we can use the relationship explored in homework 3 question 2:

$$\pi_i(\theta \mid x) = \frac{f(x \mid \theta)\pi_i(\theta)}{m_i(x)}, \quad m_i(x) = \int_\Theta f(x \mid \theta)\pi_i(\theta)d\theta, \quad i = 1, 2, \quad \text{and}$$

$$\epsilon' = \frac{\epsilon m_1(x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)}.$$

Evaluating the marginals (B indicates the beta function):

$$\begin{aligned} m_1(x) &= \int_0^1 f(y \mid p)\pi_1(p)dp \\ &= \int_0^1 \binom{n}{y} p^y(1-p)^{n-y}\frac{p^{\alpha_0-1}(1-p)^{\beta_0-1}}{B(\alpha_0,\beta_0)}dp \\ &= \frac{\binom{n}{y}}{B(\alpha_0,\beta_0)}\int_0^1 p^y(1-p)^{n-y}p^{\alpha_0-1}(1-p)^{\beta_0-1}dp \\ &= \frac{\binom{200}{15}}{B(2,98)}\int_0^1 p^{16}(1-p)^{282}dp \approx .003869 \end{aligned}$$

$$m_2(x) = \frac{\binom{200}{15}}{B(10,90)}\int_0^1 p^{24}(1-p)^{274}dp \approx .050917$$

Depending on how you calculated the integral those approximations may vary slightly.

Get the weights:

$$\begin{aligned} \epsilon' &= \frac{\epsilon m_1(x)}{\epsilon m_1(x) + (1-\epsilon)m_2(x)} \\ &= \frac{.6(.003869)}{.6(.003869) + .4(.050917)} \approx .1023 \end{aligned}$$

Which makes our posterior:

$$P(p|y) = .1023Beta(17, 283) + .8977Beta(25, 275)$$

3. Find the 95% equi-tailed credible interval for the defective rate.

   We could do either the exact calculation or use simulated samples.

   Simulating the samples by weight, then using a function like `numpy.quantile` in Python, we get an approximate answer of $[0.0459, 0.1162]$.

4. Find the 95% HPD credible interval for the overall defective rate.

   Using the same simulated samples and the `calc_hdi` function described in office hours, we find the HPD credible interval is approximately $[0.0455, 0.1157]$.

Question 2

Let $\boldsymbol{\theta} = (\theta_1, \theta_2)$. When the parameter space is multi-dimensional, we can still apply Bayes' rule / formula:

$$\pi(\boldsymbol{\theta}|x) = \frac{f(x|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{m(x)} = \frac{f(x|\boldsymbol{\theta})\pi(\boldsymbol{\theta})}{\int_\Theta f(x|\boldsymbol{\theta})\pi(\boldsymbol{\theta})d\boldsymbol{\theta}}$$

$\pi(\boldsymbol{\theta}|x) = \pi(\theta_1, \theta_2|x)$ is called the **joint posterior distribution** of θ_1 and θ_2.

To find the **marginal posterior distribution** of θ_1 given y, we integrate θ_2 out of the joint posterior distribution, i.e. $\pi(\theta_1|y) \propto \int_{\Theta_2} \pi(\theta_1, \theta_2|y)d\theta_2$

From the problem statement, we know that

$$f(y|\theta_1, \theta_2) = \frac{1}{\sqrt{2\pi}} \exp\{-0.5(y - \theta_1 - \theta_2)^2\}$$

$$\pi(\theta_1) = \frac{1}{\sqrt{2\pi}} \exp\{-0.5(\theta_1)^2\}$$

$$\pi(\theta_2) = \frac{1}{\sqrt{2\pi}} \exp\{-0.5(\theta_2)^2\}$$

We also know that $(y - \theta_1 - \theta_2)^2 = y^2 + \theta_1^2 + \theta_2^2 - 2y\theta_1 - 2y\theta_2 + 2\theta_1\theta_2$

To find the joint posterior of θ_1 and θ_2, we have the following:

$$\begin{aligned}
\pi(\theta_1, \theta_2|y) &\propto f(y|\theta_1, \theta_2) \cdot \pi_1(\theta_1, \theta_2) \\
&= f(y|\theta_1, \theta_2) \cdot \pi_1(\theta_1) \cdot \pi_2(\theta_2) \\
&\propto \exp\{-0.5(y - \theta_1 - \theta_2)^2\} \exp\{-0.5(\theta_1)^2\} \exp\{-0.5(\theta_2)^2\} \\
&= \exp\{-0.5(y^2 + \theta_1^2 + \theta_2^2 - 2y\theta_1 - 2y\theta_2 + 2\theta_1\theta_2)\} \exp\{-0.5(\theta_1)^2\} \exp\{-0.5(\theta_2)^2\} \\
&= \exp\{-0.5(y^2 + \theta_1^2 + \theta_2^2 - 2y\theta_1 - 2y\theta_2 + 2\theta_1\theta_2 + \theta_1^2 + \theta_2^2)\} \\
&= \exp\{-0.5(y^2 - 2y\theta_1 - 2y\theta_2 + 2\theta_1\theta_2 + 2\theta_1^2 + 2\theta_2^2)\}
\end{aligned}$$

Now we integrate θ_2 out of the joint posterior:

$$
\begin{aligned}
\pi(\theta_1|y) &\propto \int_{\Theta_2} \pi(\theta_1, \theta_2|y) d\theta_2 \\
&= \int_{-\infty}^{\infty} \exp\{-0.5(y^2 - 2y\theta_1 - 2y\theta_2 + 2\theta_1\theta_2 + 2\theta_1^2 + 2\theta_2^2)\} d\theta_2 \\
&= \exp\{-0.5(y^2 - 2y\theta_1 + 2\theta_1^2)\} \int_{-\infty}^{\infty} \exp\{-0.5(2)(\theta_2^2 - (y-\theta_1)\theta_2)\} d\theta_2 \\
&= \exp\{-0.5(y^2 - 2y\theta_1 + 2\theta_1^2)\} \\
&\times \int_{-\infty}^{\infty} \exp\left\{-0.5(2)(\theta_2^2 - (y-\theta_1)\theta_2 + \frac{(y-\theta_1)^2}{4} - \frac{(y-\theta_1)^2}{4})\right\} d\theta_2 \\
&= \exp\{-0.5(y^2 - 2y\theta_1 + 2\theta_1^2 - \frac{(y-\theta_1)^2}{2})\} \int_{-\infty}^{\infty} \exp\left\{-0.5(2)\left(\theta_2 - \frac{(y-\theta_1)}{2}\right)^2\right\} d\theta_2 \\
&= \exp\{-0.5(y^2 - 2y\theta_1 + 2\theta_1^2 - \frac{1}{2}y^2 + y\theta_1 - \frac{1}{2}\theta_1^2))\} \\
&= \exp\{-0.5(\frac{1}{2}y^2 - y\theta_1 + \frac{3}{2}\theta_1^2)\} \\
&\propto \exp\{-0.5(\frac{3}{2})(\theta_1^2 - \frac{2}{3}y\theta_1)\} \\
&\propto \exp\left\{-0.5\left(\frac{3}{2}\right)\left(\theta_1 - \frac{y}{3}\right)^2\right\}
\end{aligned}
$$

Since $y = 1$ was given, the marginal posterior distribution of θ_1 follows a Normal distribution with mean $\frac{1}{3}$ and variance $\frac{2}{3}$. The marginal posterior distribution of θ_2 will also follow a Normal distribution with mean $\frac{1}{3}$ and variance $\frac{2}{3}$ due to symmetry. Alternative solutions such as via Gibbs sampling were also acceptable for credit.

Question 3

We have the following: $f(y_i|\theta) = \frac{1}{(2\pi)^{1/2}} \exp\{-0.5(y_i - \theta)^2\}$

$$\pi(\theta|\tau^2) = \frac{1}{(2\pi\tau^2)^{1/2}} \exp\{-0.5\theta^2/\tau^2\}, \qquad \pi(\tau^2) = (\tau^2)^{-2} e^{-1/\tau^2}$$

Construct the joint likelihood function:

$$\begin{aligned}
L(y|\theta) &= \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}} \exp\{-0.5(y_i - \theta)^2\} \\
&= \frac{1}{(2\pi)^{n/2}} \exp\{-0.5 \sum_{i=1}^{n} (y_i - \theta)^2\} \\
&= \frac{1}{(2\pi)^{n/2}} \exp\{-0.5(\sum_{i=1}^{n} y_i^2 - 2\theta \sum_{i=1}^{n} y_i + n\theta^2)\}
\end{aligned}$$

Joint distribution:

$$\begin{aligned}
p(y, \theta, \tau^2) &= \frac{1}{(2\pi)^{n/2}} \exp\{-0.5(\sum_{i=1}^{n} y_i^2 - 2\theta \sum_{i=1}^{n} y_i + n\theta^2)\} \cdot \frac{1}{(2\pi)^{1/2}} \exp\{-0.5\theta^2/\tau^2\} \\
&\quad \cdot (\tau^2)^{-2} \cdot (\tau^2)^{-0.5} e^{-1/\tau^2} \\
&= \frac{1}{(2\pi)^{(n+1)/2}} \exp\{-0.5(\sum_{i=1}^{n} y_i^2 - 2\theta \sum_{i=1}^{n} y_i + n\theta^2 + \theta^2/\tau^2)\} \cdot (\tau^2)^{-2.5} e^{-1/\tau^2}
\end{aligned}$$

To find the conditional distribution of θ given $\mathbf{y}$ and τ^2, we have:

$$\begin{aligned}
\pi(\theta|\mathbf{y}, \tau^2) &\propto \exp\{-0.5(-2\theta \sum_{i=1}^{n} y_i + n\theta^2 + \theta^2/\tau^2)\} \\
&\propto \exp\{-0.5(-2\theta \sum_{i=1}^{n} y_i + (n + 1/\tau^2)\theta^2)\} \\
&\propto \exp\{-0.5(n + 1/\tau^2)(\theta^2 - 2\frac{\sum_{i=1}^{n} y_i}{(n + 1/\tau^2)}\theta)\} \\
&\propto \exp\{-0.5(n + 1/\tau^2)(\theta^2 - 2\frac{\sum_{i=1}^{n} y_i}{n + 1/\tau^2}\theta + \left(\frac{\sum_{i=1}^{n} y_i}{n + 1/\tau^2}\right)^2)\} \\
&\propto \exp\{-0.5(n + 1/\tau^2)\left(\theta - \frac{\sum_{i=1}^{n} y_i}{n + 1/\tau^2}\right)^2\}
\end{aligned}$$

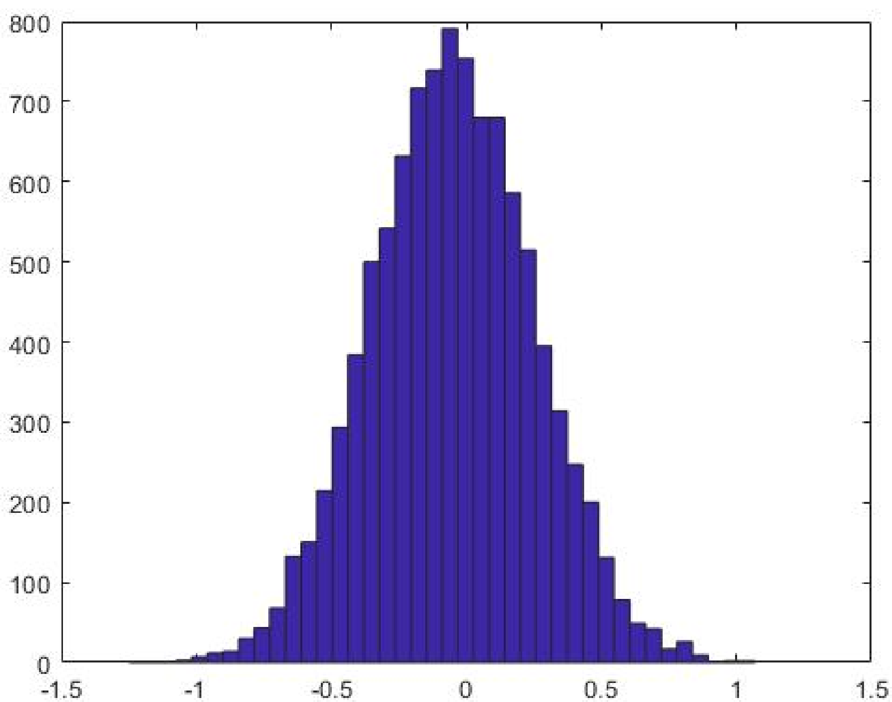
which is the kernel of a $N(\frac{\sum_{i=1}^n y_i}{n+1/\tau^2}, \frac{1}{n+1/\tau^2})$ distribution.

To find the conditional distribution of τ^2 given $\mathbf{y}$ and θ, we have:

$$\begin{aligned}\pi(\tau^2|\mathbf{y},\theta) &\propto \exp\{-0.5(\theta^2/\tau^2)\} \cdot (\tau^2)^{-2.5} e^{-1/\tau^2} \\ &\propto \exp\{-(0.5\theta^2+1)/\tau^2\} \cdot (\tau^2)^{-2.5} \\ &\sim IG(1.5, 0.5\theta^2+1)\end{aligned}$$

Now that we have the full conditional distributions, we can set up the Gibbs sampler and answer the questions as specified. Specific details in code file / appendix.

(1) posterior density of θ



(2) posterior mean of θ: Using matlab, we obtain `mean_theta =  -0.0571`

(3) 95% equi-tailed credible interval of θ: Using matlab, we obtain the interval `[-0.6468, 0.5367]`