Homework 1: Solutions

ISyE 6420
Fall 2022

1. Circuit. A circuit S consisting of seven independent elements $E_1, \ldots, E_7$ is connected

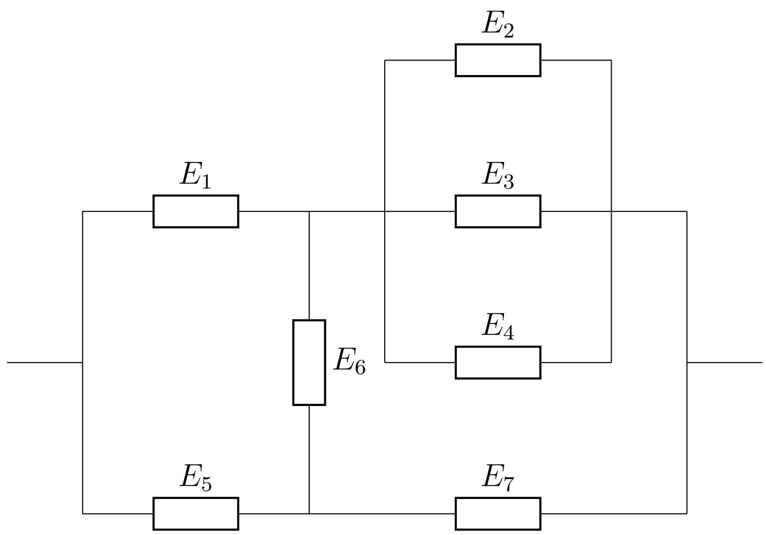


Figure 1: Circuit S with seven independent elements

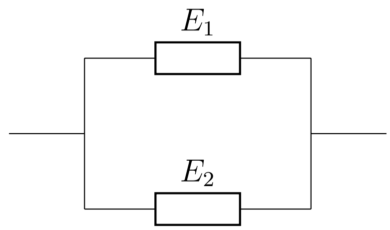
as in Figure 1. The elements are operational during time interval T with probabilities

	E_1	E_2	E_3	E_4	E_5	E_6	E_7
Probability of working (p)	0.5	0.4	0.1	0.6	0.9	0.8	0.7

Let the probability of an element E_i of a system S being operational during time interval T as p_{E_i}. Suppose all elements work independently. We consider the following two ways of connection of elements.

- For sequential connection of elements: $p_S = p_{E_1} p_{E_2}$

- For parallel connection of elements: $p_S = p_{E_1 \cup E_2} = p_{E_1} + p_{E_2} - p_{E_1} p_{E_2}$



(a) Find the probability that the circuit is operational during time interval T.

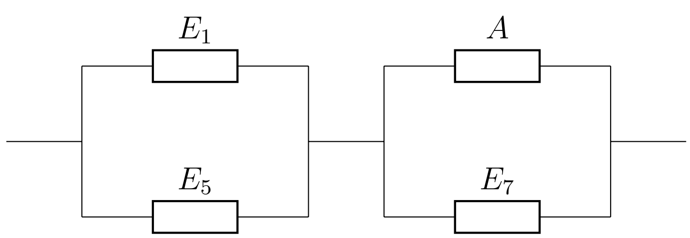
We consider the following two hypotheses:

- H_1: E_6 works during time interval T;
- H_2: E_6 fails during time interval T.

We know that $P(H_1) = 0.8$ and $P(H_2) = 0.2$. By the law of total probability, we have

$$P(S) = P(S|H_1)P(H_1) + P(S|H_2)P(H_2)$$

Under hypothesis H_1, S is equivalent to the following circuit where A is a sub-circuit that E_2, E_3 and E_4 are connected in parallel.



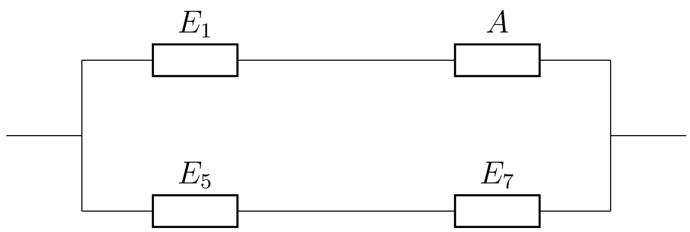
We know that the probability that A is operational during time interval T is

$$p_A = 1 - q_{E_2} q_{E_3} q_{E_4} = 1 - (0.6)(0.9)(0.4) = 0.784$$

Thus, we have

$$p_{S|H_1} = (1 - q_{E_1} q_{E_5})(1 - q_A q_{E_7}) = (1 - (0.5)(0.1))(1 - (0.216)(0.3)) = 0.88844$$

Under hypothesis H_2, S is equivalent to the following circuit where A is a sub-circuit that E_2, E_3 and E_4 are connected in parallel.



We have:

$$\begin{aligned} p_{S|H_2} &= 1-(1-p_{E_1}p_A)(1-p_{E_5}p_{E_7}) \\ &= 1-(1-(0.5)(0.784))(1-(0.9)(0.7)) \\ &= 0.77504 \end{aligned}$$

Hence:

$$P(S) = (0.88844)(0.8) + (0.77504)(0.2) = 0.86576$$

(b) If the circuit was found operational at the time T, what is the probability that the element E_6 was operational?

By Bayes' rule:

$$P(H_1|S) = \frac{P(S \mid H_1)\, P(H_1)}{P(S)} = \frac{(0.88844)(0.8)}{0.86576} = 0.82096$$

.

2. Two Batches. There are two batches of the same product. In batch 1 5% of products are non-conforming. Batch 2 contains 10% non-conforming products. A batch is selected at random and one randomly selected product from that batch is inspected. The inspected product was found conforming and was returned back to its batch.

(a) What is the probability that the second product, randomly selected from the same batch, is found non-conforming?

We let the first batch be the batch with 5% of products non-conforming, and let the second batch be the batch that contains 10% non-conforming products. We define the following events:

- H_1: The first batch is selected;
- H_2: The second batch is selected;
- A: The first selected product from the selected batch is conforming;

- B: The second selected product is non-conforming.

We know that $P(H_1) = P(H_2) = .5$ as the batch is selected randomly. We also know that $P(A|H1) = .95$ and $P(A|H2) = 0.9$ based on the problem setup. We hope to find $P(B|A)$. By the law of total probability, we have:

$$P(A) = P(A|H_1)P(H_1) + P(A|H_2)P(H_2) = (.95)(0.5) + (0.9)(0.5) = 0.925$$

and

$$\begin{aligned} P(B|A) &= P(B, H_1|A) + P(B, H_2|A) \\ &= P(B|H_1, A)P(H_1|A) + P(B|H_2, A)P(H_2|A) \end{aligned}$$

By Bayes rule, we have

$$P(H_1|A) = \frac{P\left(A \mid H_1\right) P\left(H_1\right)}{P(A)} = \frac{(0.95)(0.5)}{0.925} = 0.5135$$

.

$$P(H_2|A) = \frac{P\left(A \mid H_2\right) P\left(H_2\right)}{P(A)} = \frac{(0.9)(0.5)}{0.925} = 0.4865$$

.

Hence:

$$\mathrm{P}(B \mid A) = (0.05)(.5135) + (0.1)(.4865) = .0743$$

(b) Assuming the second product was indeed non-conforming, what is the probability it came from batch 2?

There are at least two ways to do this.
The first method is to start from the beginning.

$$P(H_2|A, B) = \frac{P(A, B|H_2)P(H_2)}{P(A, B)}$$
$$P(A, B) = Pr(A, B|H_1)Pr(H_1) + Pr(A, B|H_2)Pr(H_2)$$

$$\begin{aligned} Pr(A, B|H_1) &= P(A|H_1)P(B|H_1) \\ &= (.95)(.05) \\ &= .0475 \end{aligned}$$

$$
\begin{aligned}
Pr(A,B|H_2) &= P(A|H_2)P(B|H_2) \\
&= (.9)(.1) \\
&= .09
\end{aligned}
$$

$$
\begin{aligned}
P(A,B) &= Pr(A,B|H_1)Pr(H_1) + Pr(A,B|H_2)Pr(H_2) \\
&= (.0475)(.5) + (.09)(.5) \\
&= .06875
\end{aligned}
$$

$$
\begin{aligned}
P(H_2|A,B) &= \frac{P(A,B|H_2)P(H_2)}{P(A,B)} \\
&= \frac{(.09)(.5)}{(.06875)} \\
&= .6545
\end{aligned}
$$

The second method is to recognize that we already have $P(H_2|A) = .4865$ and $P(B) = .0743$, now we need $P(H_2|B)$. By using $P(H_2|A)$ as our prior in the below equation, we are incorporating the knowledge related to the first selected product into our calculation.

$$
\begin{aligned}
P(H_2|B) &= \frac{P(B|H_2)P(H_2)}{P(B)} \\
&= \frac{(.1)(.4865)}{(.0743)} \\
&= .6548
\end{aligned}
$$

The second result is slightly off because of some rounding we did in part (a), otherwise the two methods would come out the same.

3. Machine. A machine has four independent components, three of which fail with probability $q = 1 - p$, and one with probability 1/2. The machine is operational as long as at least three components are working.

We let M be the component whose failure probability is 1/2, and consider the following events:

- A: The machine is working;
- B: Component M is working.

We use $\bar{(\cdot)}$ to denote the complement of an event.

(a) What is the probability that the machine will fail? Evaluate this probability for $p = 0.8$.

Based on the law of total probability, we know that

$$P(A) = P(A, B) + P(A, \bar{B}) = P(A|B)P(B) + P(A|\bar{B})P(\bar{B})$$

We know that the machine fails when all components fail or three components fail. We condition on whether component M is operational (event B).

If component M is working, then we need at least two of the three other components working:

$$P(A|B) = p^3 + \binom{3}{2}(1-p)p^2$$

If component M has failed, then the machine fails when any of other three components fail, so we need all of them working.

$$P(A|\bar{B}) = p^3$$

Thus, we have

$$\begin{aligned}
P(A) &= P(A|B)P(B) + P(A|\bar{B})P(\bar{B}) \\
&= \left(p^3 + \binom{3}{2}p^2(1-p)\right)\left(\frac{1}{2}\right) + p^3\left(\frac{1}{2}\right) \\
&= ((0.8)^3 + 3(0.8)^2(0.2)))(0.5) + (0.8)^3(0.5) \\
&= 0.704
\end{aligned}$$

The probability that the machine will fail is $P(\bar{A}) = 1 - P(A)$, so 0.296.

(b) If the machine failed, what is the probability in terms of p that the component which fails with probability 1/2 actually failed?

We want to find $P(\bar{B}|\bar{A})$. By Bayes' rule, we have

$$P(\bar{B}|\bar{A}) = \frac{P(\bar{A}|\bar{B})P(\bar{B})}{P(\bar{A})}$$

We already know that

$$P(\bar{A}|\bar{B}) = 1 - p^3$$

and

$$
\begin{aligned}
P(\bar{A}) &= 1 - \left(\left(p^3 + \binom{3}{2} p^2 (1-p) \right) \left(\frac{1}{2} \right) + p^3 \left(\frac{1}{2} \right) \right) \\
&= 1 - 1.5p^2 + .5p^3
\end{aligned}
$$

so we have

$$
\begin{aligned}
P(\bar{B}|\bar{A}) &= \frac{(1-p^3)(.5)}{1 - 1.5p^2 + .5p^3} \\
&= \frac{1-p^3}{2 - 3p^2 + p^3}
\end{aligned}
$$

If that were evaluated at $p = .8$, then the probability of component M failing in this situation would be .8243.