ISyE 6669 HW 3 Solutions

Fall 2021

1. Consider the following linear optimization problem

$$\begin{aligned} \min \quad & x + y \\ \text{s.t.} \quad & x + y = 1, \\ & x \leq 0,\ y \leq 0. \end{aligned}$$

   Does this problem have an optimal solution? Is this problem feasible? Explain your answer.

   Solution: This problem does not have an optimal solution and is infeasible. We have that

$$x \leq 0,\ y \leq 0$$

   which implies that

$$x + y \leq 0.$$

   But we also have the constraint:

$$x + y = 1.$$

   It is not possible for

$$x + y \leq 0$$

   and

$$x + y = 1.$$

   Thus the problem is infeasible.

2. Consider the following optimization problem

$$\begin{aligned} \min \quad & (x \cdot \sin(x))^2 \\ \text{s.t.} \quad & x \in \mathbb{R}. \end{aligned}$$

   (a) Find all the global minimum solutions. Explain how you find them. Hint: there may be multiple ones.

   (b) Is there any local minimum solution that is not a global minimum solution?

   (c) Is the objective function $f(x) = (x \cdot \sin(x))^2$ a convex function on $\mathbb{R}$?

Solution:

(a) The global minimum solutions are $x = 0$ and $x = \pi n$, where $n \in \mathcal{Z}$. The optimal objective value is 0. We find the global minima by noting that

$$(x \cdot \sin(x))^2 \geq 0.$$

Thus the global minima are the x values such that

$$(x \cdot \sin(x))^2 = 0.$$

Thus, the problem becoming a root-finding one. Clearly, $x = 0$ makes $(x \cdot \sin(x))^2 = 0$. Also, $x = \pi n$, where $n \in \mathcal{Z}$ makes $\sin(x) = 0$ and hence $(x \cdot \sin(x))^2 = 0$

(b) There is no local minimum solution that is not a global minimum solution.

(c) The objective function $f(x) = (x \cdot \sin(x))^2$ is not a convex function on $\mathbb{R}$. We show that

$$f(\alpha x + (1-\alpha)y) > \alpha f(x) + (1-\alpha)f(y)$$

for a specific $x, y \in \mathbb{R}$ and $\alpha \in [0, 1]$. Let $x = 0$, $y = \pi$ and $\alpha = \frac{1}{2}$. We have that

$$\begin{aligned} f(\alpha x + (1-\alpha)y) &= f(0 + \frac{\pi}{2}) \\ &= (\frac{\pi}{2} \cdot \sin(\frac{\pi}{2}))^2 \\ &= (\frac{\pi}{2})^2 \\ &= \frac{\pi^2}{4} \end{aligned}$$

Meanwhile, we have:

$$\begin{aligned} \alpha f(x) + (1-\alpha)y &= \frac{1}{2}f(0) + \frac{1}{2}f(\pi) \\ &= 0 + \frac{1}{2}(\pi \cdot \sin(\pi))^2 \\ &= 0 \end{aligned}$$

Note that

$$\frac{\pi^2}{4} > 0,$$

so $f(x)$ is not a convex function on $\mathbb{R}$

3. Consider the following optimization problem

$$\begin{aligned} \min \quad & \frac{1}{x} \\ \text{s.t.} \quad & x \geq 0. \end{aligned}$$

Does this problem have an optimal solution? Why?

Solution: This problem does not have an optimal solution. Note that for $x \geq 0$, $\frac{1}{x}$ is monotonically decreasing and we have that

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Thus, note that for any $x_1 \geq 0$, we are always able to find an $x_2 > x_1 \geq 0$ so that

$$\frac{1}{x_2} < \frac{1}{x_1}.$$

Hence, no optimal solution exists.

4. Consider the following problem

$$\begin{aligned} \min \quad & x + f(x) \\ \text{s.t.} \quad & x \in \mathbb{R}, \end{aligned}$$

where the function $f(x)$ is defined as

$$f(x) = \begin{cases} 0, & -1 < x < 1 \\ 1, & x = 1 \\ 2, & x = -1 \\ +\infty, & x > 1 \text{ or } x < -1 \end{cases}.$$

(a) Is the objective function a convex function defined on $\mathbb{R}$? Explain your answer by checking the criterion of convexity.

(b) Find an optimal solution, or explain why there is no optimal solution.

Solution:

(a) We have an objective function $g : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ given by:

$$g(x) = \begin{cases} x, & -1 < x < 1 \\ 2, & x = 1 \\ 1, & x = -1 \\ +\infty, & x > 1 \text{ or } x < -1 \end{cases}.$$

Method 1: We will show that function $g(x)$ is convex using the definition of a convex function, i.e. we will show that

$$g(\lambda a + (1-\lambda)b) \leq \lambda g(x) + (1-\lambda)g(x)$$

for any $a, b \in \mathbb{R}$ and any $\lambda \in (0,1)$.

– If $a, b \in (-1,1)$ then $\lambda a + (1-\lambda)b \in (-1,1)$ and:

$$g(\lambda a + (1-\lambda)b) = \lambda a + (1-\lambda)b = \lambda g(a) + (1-\lambda)g(b)$$

We can also notice that in this case objective function is equal to $x+0$ which is linear function, and thus convex.

– If $a = -1$ and $b \in (-1,1)$ then $\lambda(-1) + (1-\lambda)b \in (-1,1)$ and:

$$g(-\lambda+(1-\lambda)b) = -\lambda+(1-\lambda)b \leq \lambda+(1-\lambda)b = \lambda g(-1)+(1-\lambda)g(b)$$

– If $a \in (-1,1)$ and $b = 1$ then $\lambda a + (1-\lambda)(1) \in (-1,1)$ and

$$g(\lambda a+(1-\lambda)(1)) = \lambda a+1-\lambda \leq \lambda a+2(1-\lambda) = \lambda g(a)+(1-\lambda)g(1)$$

– If either of the points a or b belongs to $(-\infty,-1)\cup(1,+\infty)$ then value of the function $g(x)$ at that point will be $+\infty$. In that case we have $\lambda g(a) + (1-\lambda)g(b) = +\infty$ and:

$$g(\lambda a + (1-\lambda)b) \leq +\infty = \lambda g(a) + (1-\lambda)g(b)$$

Method 2: Lets look at function $h : [-1,1] \to \mathbb{R}$ given by:

$$h(x) = \begin{cases} x, & -1 < x < 1 \\ 2, & x = 1 \\ 1, & x = -1 \end{cases}.$$

Its plot is given by:

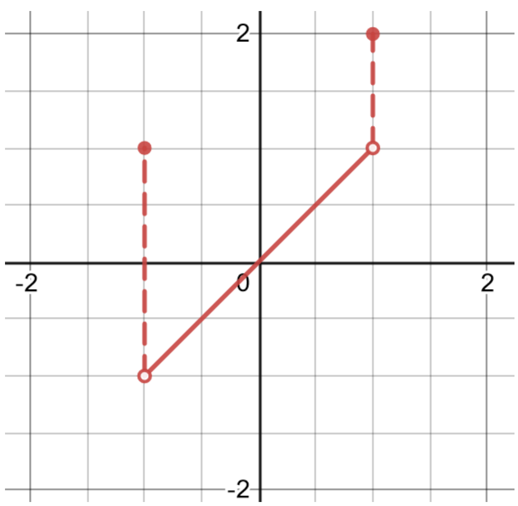

Figure 2: Plot of function h

and we can see it is a convex function on its domain. If function $h(x)$ is convex on its domain $dom(h) = [-1, 1]$ then its extension $g : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ given by:

$$g(x) = \begin{cases} h(x), & x \in [-1,1] \\ +\infty, & x \notin [-1,1] \end{cases}.$$

is also convex (same argument as in the last bullet point from Method 1).

(b) Problem does not have a solution.
Value $x^* \notin (-1, 1)$ can't be the solution, because for example

$$g(0) = 0 < g(x^*) = \begin{cases} 1, & x = -1 \\ 2, & x = 1 \\ +\infty, & x \notin [-1,1] \end{cases}.$$

Value $x^* \in (-1, 1)$ can't be solution because we can always find $x^{**} \in (-1, 1)$ such that $x^{**} < x^*$ and thus:

$$g(x^{**}) = x^{**} < x^* = g(x^*).$$

- if $-1 < x < 1$ then the optimal value is $-1 + \epsilon$ where ϵ is a very small positive number
- if $x = 1$ then the optimal value is 2
- if $x = -1$ then the optimal value is 1
- if $x > 1$ *or* $x < -1$ then optimal value is $+\infty$ which doesn't exist

5. For each of the statements below, state whether it is true or false. Justify your answer.

(a) Consider the optimization problem

$$\min f(\boldsymbol{x}) \text{ s.t. } g(\boldsymbol{x}) \geq 0.$$

Suppose the current optimal objective value is v. Now, if I change the right-hand-side of the constraint from 0 to 1 and resolve the problem, the new optimal objective value will be less than or equal to v.

(b) Consider the following optimization problem:

$$\min f(\boldsymbol{x})^4 \text{ s.t. } \boldsymbol{x} \in X$$

where $f(\boldsymbol{x})$ is a nonconvex function and X is a non-empty set. Suppose at a feasible solution $\boldsymbol{x}^* \in X$, $f(\boldsymbol{x}^*) = 0$, then $\boldsymbol{x}^*$ must be a global optimal solution.

(c) Consider the following optimization problem

$$\begin{aligned} (P) \quad & \max \quad f(\boldsymbol{x}) \\ & \text{s.t.} \quad g_i(\boldsymbol{x}) \geq b_i, \quad \forall i \in I. \end{aligned}$$

Suppose the optimal objective value of (P) is v_P. Then, the Lagrangian dual of (P) is given by

$$(D) \quad \min\{\mathcal{L}(\boldsymbol{\lambda}) : \boldsymbol{\lambda} \geq \mathbf{0}\}, \tag{1}$$

where $\mathcal{L}(\boldsymbol{\lambda}) = \max_{\boldsymbol{x}}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i(g_i(x) - b_i)\}$. Furthermore, suppose the optimal objective value of (D) is v_D, then $v_P \leq v_D$.

Solution:

(a) False.
Restricting the solution space cannot result in a better solution. Note that any solution of the new problem is also a feasible solution of the original problem because

$$g(x) \geq 1 \implies g(x) \geq 0.$$

Let $\hat{x}$ be the optimal solution of the new problem, and $\hat{v} = f(\hat{x})$ be the optimal value. Since $\hat{x}$ is also feasible to the original problem,

$$v \leq f(\hat{x}) \qquad (\because v \text{ is optimal value of original}) \tag{2}$$

$$\therefore \quad v \leq \hat{v} \tag{3}$$

(b) True.
Note that $f(x)^4 \geq 0, \forall x$. Since $f(x^*) = 0$, we have $f(x)^4 \geq f(x^*), \forall x$. Thus, by the definition of global optimum, x^* is a global optimal solution.

(c) True.
Let $S = \{x : g_i(x) \geq b_i, \forall i \in I\}$. Let $x^* \in S$ be the optimal solution of (P). Thus, $v_P = f(x^*)$. Also, let $\lambda^* \geq 0$ be the optimal solution of (D) and so $v_D = \mathcal{L}(\lambda^*)$. Now, for any $\lambda \geq 0$,

$$\begin{aligned} \mathcal{L}(\lambda) &= \max_{\boldsymbol{x}}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i(g_i(x) - b_i)\} && (4) \\ &\geq \max_{\boldsymbol{x} \in S}\{f(\boldsymbol{x}) + \sum_{i \in I} \lambda_i(g_i(x) - b_i)\} && (5) \\ &\geq \max_{\boldsymbol{x} \in S}\{f(\boldsymbol{x})\} && (6) \\ &= v_P && (7) \end{aligned}$$

where Eq.(5) follows because the feasible space is being restricted (refer to Q5 (a) if unclear) and Eq.(6) follows because $\lambda \geq 0$ and $x \in S$ imply that $\lambda_i(g_i(x) - b_i) \geq 0$ for all $i \in I$. Since this holds for all $\lambda \geq 0$, it also holds for λ^*. Therefore, $v_D = \mathcal{L}(\lambda^*) \geq v_P$.