

1. We are given a positive integer  $r$  and a rectangular board  $ABCD$  with dimensions  $|AB| = 20$ ,  $|BC| = 12$ . The rectangle is divided into a grid of  $20 \times 12$  unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is  $\sqrt{r}$ . The task is to find a sequence of moves leading from the square with  $A$  as a vertex to the square with  $B$  as a vertex.
  - (a) Show that the task cannot be done if  $r$  is divisible by 2 or 3.
  - (b) Prove that the task is possible when  $r = 73$ .
  - (c) Can the task be done when  $r = 97$ ?
2. Let  $P$  be a point inside triangle  $ABC$  such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let  $D, E$  be the incenters of triangles  $APB, APC$ , respectively. Show that  $AP, BD, CE$  meet at a point.

3. Let  $S$  denote the set of nonnegative integers. Find all functions  $f$  from  $S$  to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \forall m, n \in S.$$

4. The positive integers  $a$  and  $b$  are such that the numbers  $15a + 16b$  and  $16a - 15b$  are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?
5. Let  $ABCDEF$  be a convex hexagon such that  $AB$  is parallel to  $DE$ ,  $BC$  is parallel to  $EF$ , and  $CD$  is parallel to  $FA$ . Let  $R_A, R_C, R_E$  denote the circumradii of triangles  $FAB, BCD, DEF$ , respectively, and let  $P$  denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

6. Let  $p, q, n$  be three positive integers with  $p + q < n$ . Let  $(x_0, x_1, \dots, x_n)$  be an  $(n + 1)$ -tuple of integers satisfying the following conditions:
  - (a)  $x_0 = x_n = 0$ .

- (b) For each  $i$  with  $1 \leq i \leq n$ , either  $x_i - x_{i-1} = p$  or  $x_i - x_{i-1} = -q$ .  
Show that there exist indices  $i < j$  with  $(i, j) \neq (0, n)$ , such that  $x_i = x_j$ .