1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard). For any pair of positive integers m and n, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n, lie along edges of the squares.
   Let S_1 be the total area of the black part of triangle ans S_2 be the total area of white part. Let

$$f(m,n) = \left|S_1 - S_2\right|.$$

   (a) calculate $f(m,n)$ for all positive integers m and n which are either both even or both odd.
   (b) Prove that $f(m,n) \leq \frac{1}{2} max\{m,n\}$ for all m and n.
   (c) Show that there is no constant C such that $f(m,n) < c$ for all m and n.

2. The angle at A is the smallest angle of triangle ABC. The point B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A. The perpendicular bisectors of AB and AC meet the line AU at V and W, respsctively. The lines BV and CW meet at T. Show that

$$AU = TB + TC.$$

3. Let $x_1, x_2, ..., x_n$ be the real numbers satisfying the conditions

$$\left|x_1 + x_2 + ... + x_n\right| = 1$$

   and

$$\left|x_i\right| \leq \frac{n+1}{2} i = 1, 2, ..., n.$$

   Show that there exists a permutation $y_1, y_2, ..., y_n$ of $x_1, x_2, ..., x_n$ such that

$$\left|y_1 + 2y_2 + ... + ny_n\right| \leq \frac{n+1}{2}.$$

4. A $n \times n$ matrix whose entires come from the set $S = \{1, 2, .., 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, ...n$, the ith row and ith column together contain all elements of S. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of n.

5. Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a.$$

6. For each positive integer n, let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways;

$$4; 2+2; 2+1+1; 1+1+1+1.$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$