

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are colored alternately black and white (as on a chessboard).

For any pair of positive integers m and n , consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths m and n , lie along edges of the squares.

Let S_1 be the total area of the black part of triangle and S_2 be the total area of white part. Let

$$f(m, n) = |S_1 - S_2|. \quad (1)$$

- (a) calculate $f(m, n)$ for all positive integers m and n which are either both even or both odd.
 - (b) Prove that $f(m, n) \leq \frac{1}{2} \max\{m, n\}$ for all m and n .
 - (c) Show that there is no constant C such that $f(m, n) < C$ for all m and n .
2. The angle at A is the smallest angle of triangle ABC . The point B and C divide the circumcircle of the triangle into two arcs. Let U be an interior point of the arc between B and C which does not contain A . The perpendicular bisectors of AB and AC meet the line AU at V and W , respectively. The lines BV and CW meet at T . Show that

$$AU = TB + TC. \quad (2)$$

3. Let x_1, x_2, \dots, x_n be the real numbers satisfying the conditions

$$|x_1 + x_2 + \dots + x_n| = 1 \quad (3)$$

and

$$|x_i| \leq \frac{n+1}{2} i = 1, 2, \dots, n. \quad (4)$$

Show that there exists a permutation y_1, y_2, \dots, y_n of x_1, x_2, \dots, x_n such that

$$|y_1 + 2y_2 + \dots + ny_n| \leq \frac{n+1}{2}. \quad (5)$$

4. A $n \times n$ matrix whose entries come from the set $S = \{1, 2, \dots, 2n-1\}$ is called a silver matrix if, for each $i = 1, 2, \dots, n$, the i th row and i th column together contain all elements of S . Show that

- (a) there is no silver matrix for $n = 1997$;
- (b) silver matrices exist for infinitely many values of n .

5. Find all pairs (a, b) of integers $a, b \geq 1$ that satisfy the equation

$$a^{b^2} = b^a. \quad (6)$$

6. For each positive integer n , let $f(n)$ denote the number of ways of representing n as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways;

$$4; 2 + 2; 2 + 1 + 1; 1 + 1 + 1 + 1. \quad (7)$$

Prove that, for any integer $n \geq 3$,

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}. \quad (8)$$