Maximum Likelihood Reconstruction for Emission Tomography

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Results:

Number of detectors = 40

- Reconstruction of Single Pixel Impulse

  1. Resolution = 4 (9*9 image)

<table>
<tr><td colspan="4">Input Images</td></tr>
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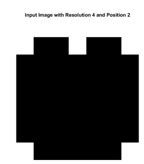
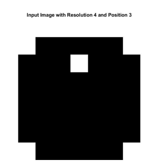
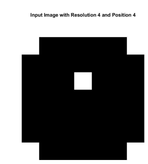
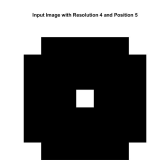
<td></td>
<td></td>
<td></td>
<td></td>
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<tr><td colspan="4">EM Output</td></tr>
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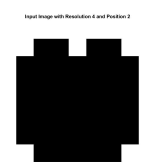
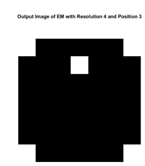
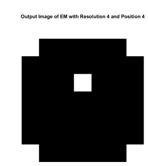
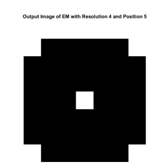
<td></td>
<td></td>
<td></td>
<td></td>
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<tr><td colspan="4">Inversion Output</td></tr>
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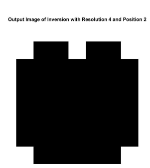
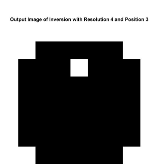
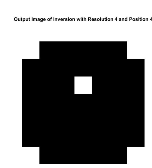
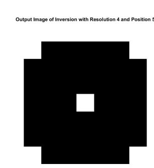
<td></td>
<td></td>
<td></td>
<td></td>
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2. Resolution = 8 (17*17 image)

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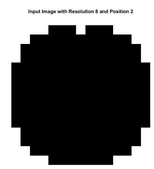
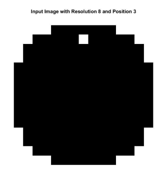
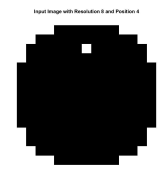
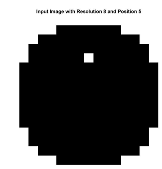
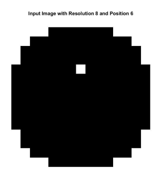
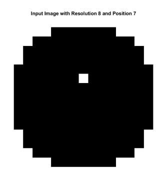
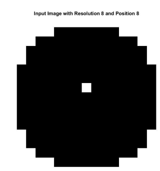
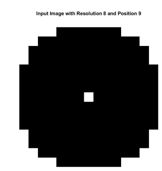
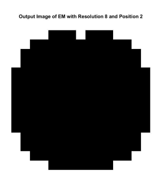
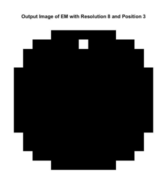
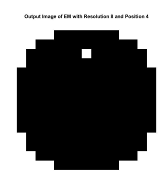
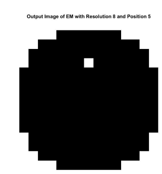
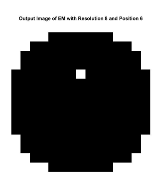
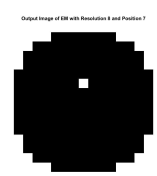
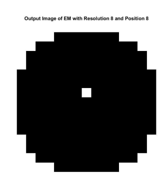
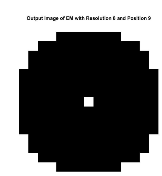
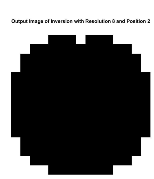
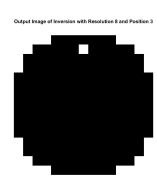
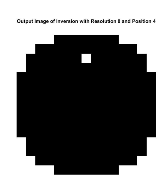
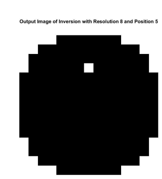
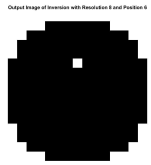
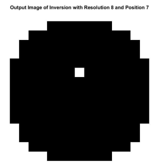
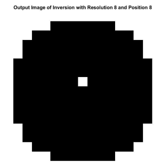
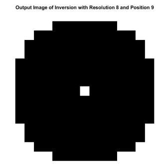
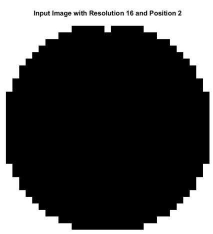
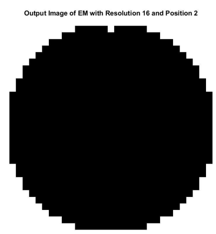
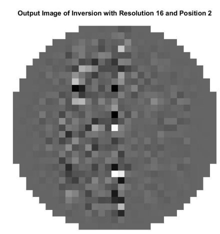
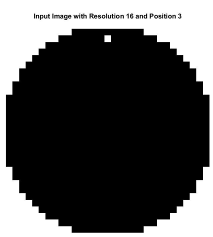
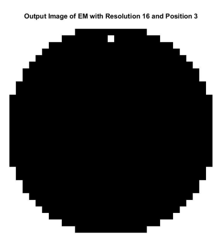
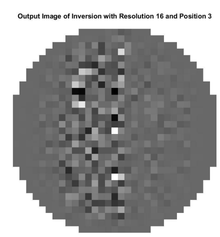
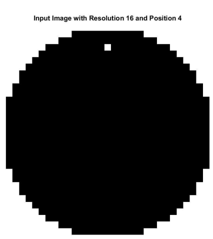
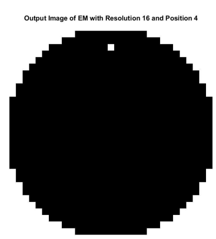
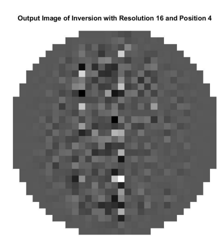
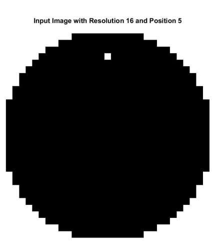
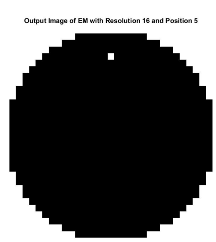
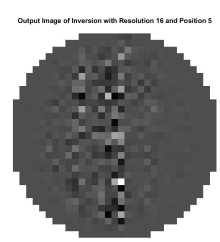
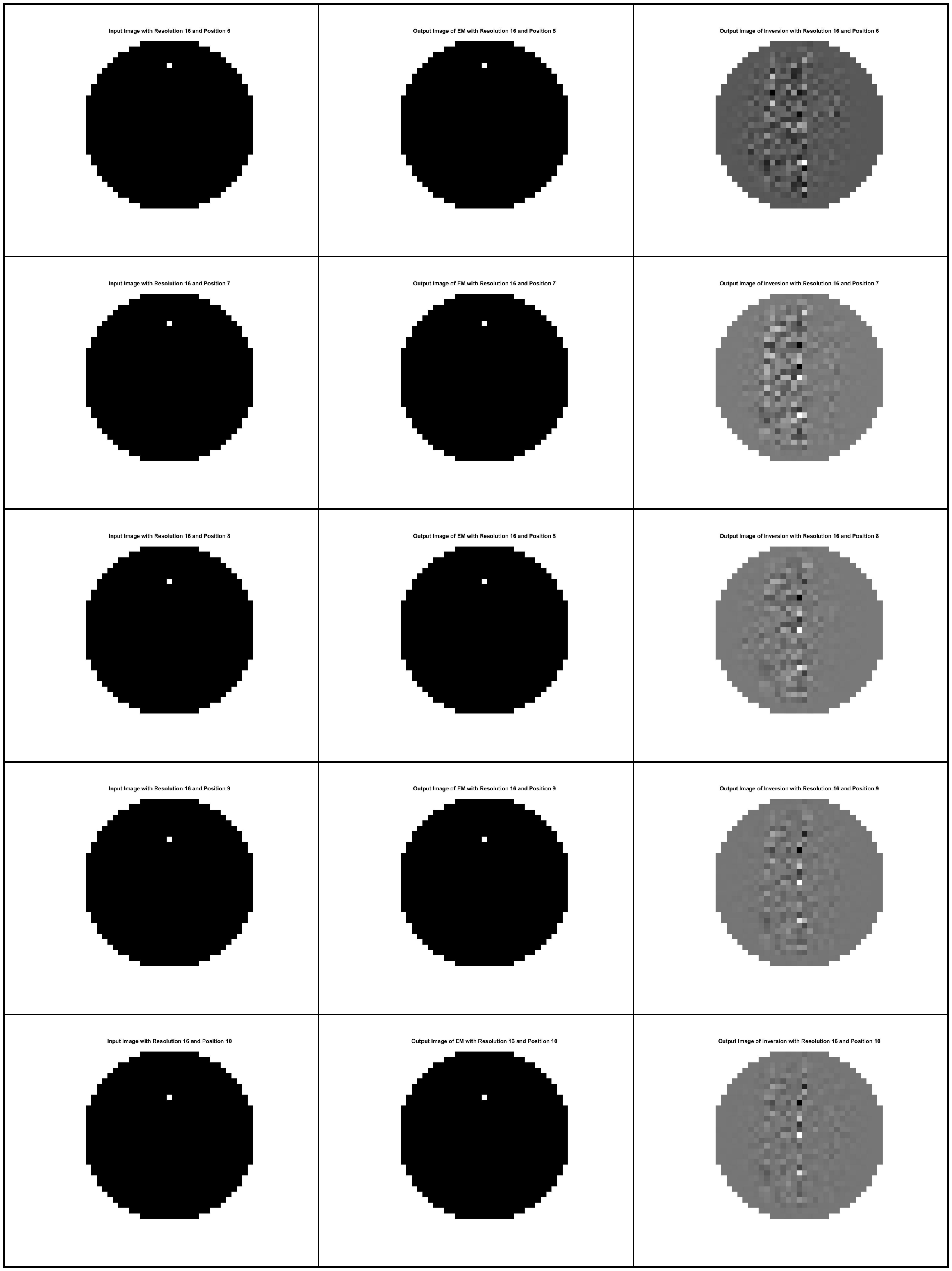
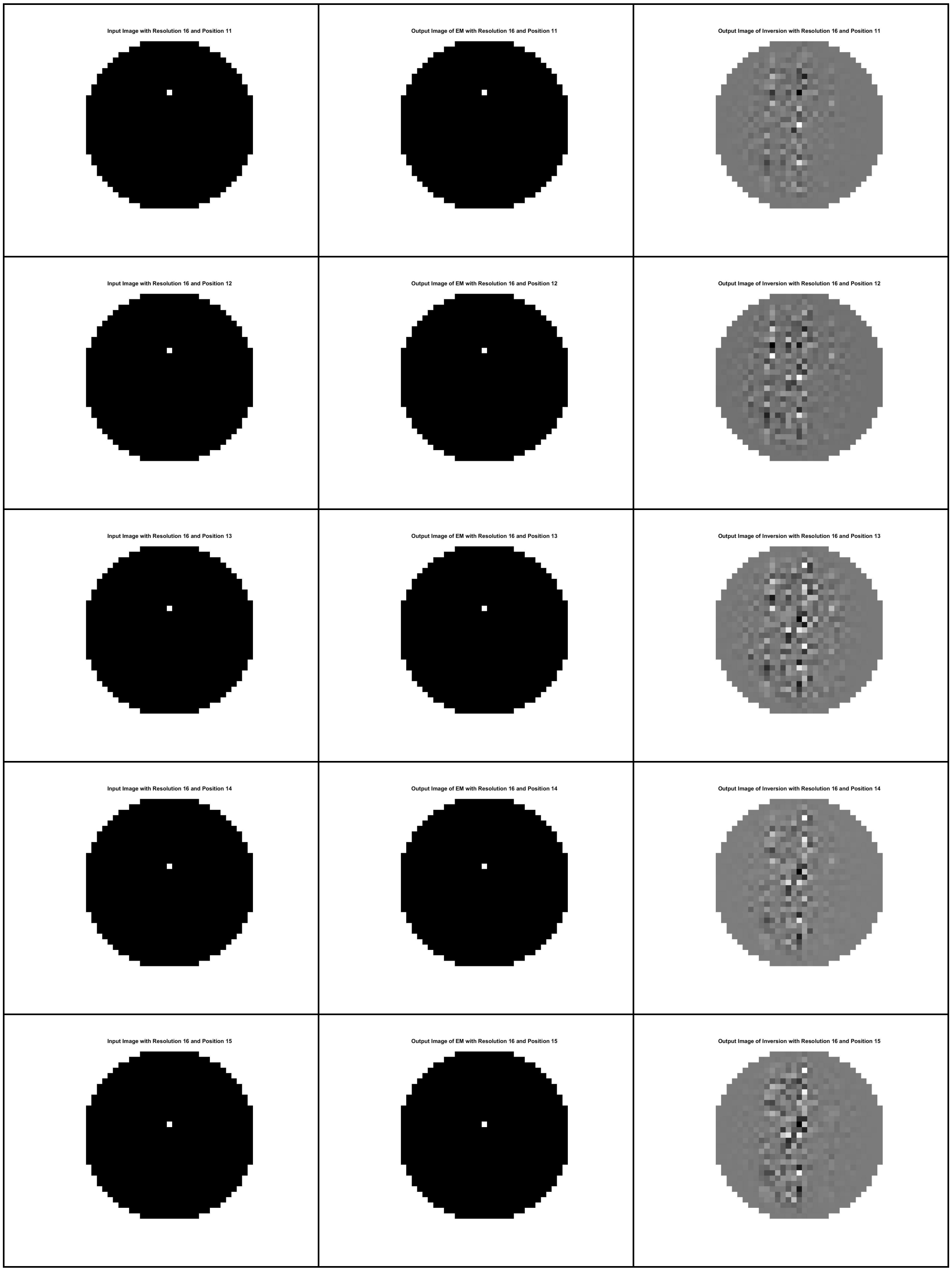
			
			
EM Output			
			
			
Inversion Output			
			

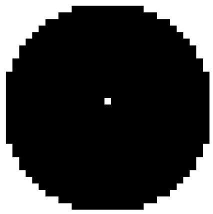
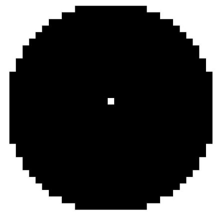
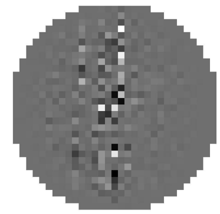
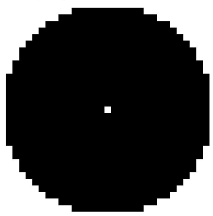
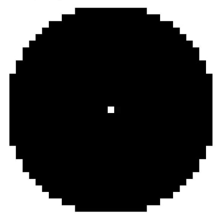
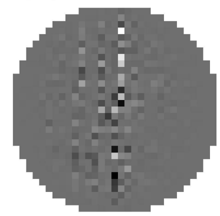
			

3. Resolution = 16 (33*33 image)

Input Image	EM Output	Inversion
		
		
		
		

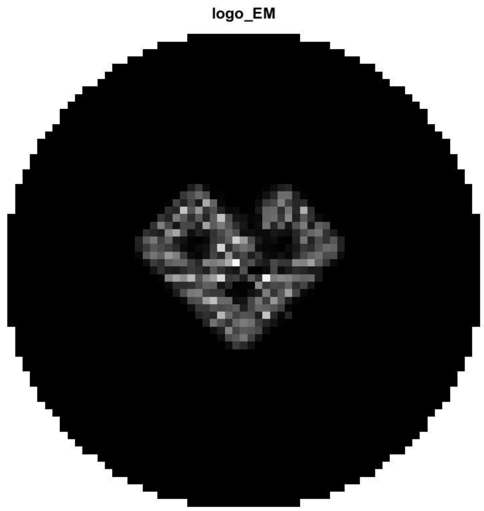
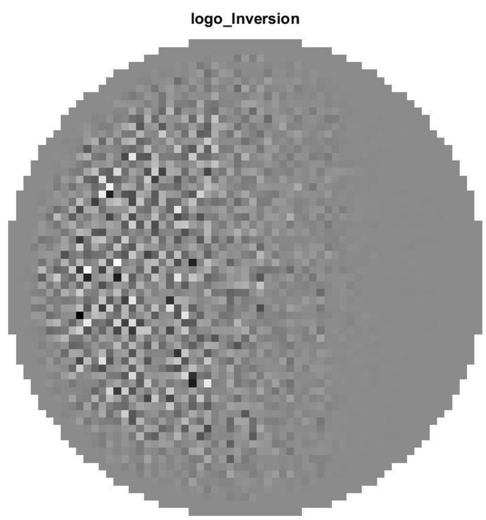

Input Image with Resolution 16 and Position 6
Output Image of EM with Resolution 16 and Position 6
Output Image of Inversion with Resolution 16 and Position 6
Input Image with Resolution 16 and Position 7
Output Image of EM with Resolution 16 and Position 7
Output Image of Inversion with Resolution 16 and Position 7
Input Image with Resolution 16 and Position 8
Output Image of EM with Resolution 16 and Position 8
Output Image of Inversion with Resolution 16 and Position 8
Input Image with Resolution 16 and Position 9
Output Image of EM with Resolution 16 and Position 9
Output Image of Inversion with Resolution 16 and Position 9
Input Image with Resolution 16 and Position 10
Output Image of EM with Resolution 16 and Position 10
Output Image of Inversion with Resolution 16 and Position 10


Input Image with Resolution 16 and Position 11
Output Image of EM with Resolution 16 and Position 11
Output Image of Inversion with Resolution 16 and Position 11
Input Image with Resolution 16 and Position 12
Output Image of EM with Resolution 16 and Position 12
Output Image of Inversion with Resolution 16 and Position 12
Input Image with Resolution 16 and Position 13
Output Image of EM with Resolution 16 and Position 13
Output Image of Inversion with Resolution 16 and Position 13
Input Image with Resolution 16 and Position 14
Output Image of EM with Resolution 16 and Position 14
Output Image of Inversion with Resolution 16 and Position 14
Input Image with Resolution 16 and Position 15
Output Image of EM with Resolution 16 and Position 15
Output Image of Inversion with Resolution 16 and Position 15

Input Image with Resolution 16 and Position 16 	Output Image of EM with Resolution 16 and Position 16 	Output Image of Inversion with Resolution 16 and Position 16 
Input Image with Resolution 16 and Position 17 	Output Image of EM with Resolution 16 and Position 17 	Output Image of Inversion with Resolution 16 and Position 17 

- Reconstruction of Binary Image
  Number of detectors = 80
  Resolution = 32 (65*65 image)

Images	Description
logo_input 	Input Binary Image (logo)

	EM Output
	Inversion Output

Theory Concerning Project

Here 'b' denote sites, There are 'B' sites in total

'd' denote detector tube and there are 'D' tubes in total.

$P(b, d)$ is the probability that emission in 'b' is detected in tube 'd'. The emission at site 'b' is assumed to be poisson with mean $\lambda(b)$

$n^*(d)$ is the total number of emission detected in tube 'd' during acquisition (real data)

$n(b, d)$ denote the number of emission in 'b' detected in 'd'

$n(b, d)$ are mutually independent poisson random variable, across b, across d. (Split of poisson process independent)

$n(b, d) \sim \text{Poisson}\ (\lambda(b, d))$ $\qquad \sum_{d=1}^{D} P(b, d) = 1$

$\lambda(b, d) = \lambda(b) \times P(b, d)$

$$n^*(d) = \sum_{b=1}^{B} n(b, d) \Rightarrow \lambda^*(d) = E[n^*(d)] = \sum_{b=1}^{B} \lambda(b)\, P(b, d)$$

$\Rightarrow$ Best estimate of $\lambda^*(d) = n^*(d)$

$$\Rightarrow \quad n^*(d) = \sum_{b=1}^{B} \lambda(b)\, P(b, d) \quad \text{for } d = 1, \ldots D.$$

Above set of linear equations form the basis for inversion reconstruction.

Now let's move on to expectation maximisation. If we have $n(b, d)$ as hidden random variable then estimation of $\lambda(b)$ becomes very easy.

So X = Hidden Random Variable $= n(b,d)$

Y = observed data $= n^*(d)$

$$\text{Likelihood} = P(X,Y) = \prod_{\substack{b=1\ldots B \\ d=1\ldots D}} e^{-\lambda(b,d)} \frac{(\lambda(b,d))^{n(b,d)}}{n(b,d)!}$$

($\because$ each $n(b,d)$ poisson process is independent)

$$\left(\sum_{b=1}^{b} n(b,d) = n^*(d) \quad \forall d \right)$$

$$\text{log likelihood} = \sum_{d=1}^{D} \sum_{b=1}^{B} -\lambda(b,d) + n(b,d) \log(\lambda(b,d)) - \log(n(b,d)!)$$

↑ of no use as does not contain parameter $\lambda(b,d)$

$$E[n(b,d) / n^*(d)] = \frac{n^*(d)\, \lambda^i(b,d)}{\sum_{b'=1}^{B} \lambda^i(b',d)}$$

$\lambda^i(b,d)$ = estimate at i^{th} iteration

$$\Rightarrow E[\text{log likelihood}] = \sum_{d=1}^{D} \sum_{b=1}^{B} -\lambda(b,d) + \frac{n^*(d)\, \lambda^i(b,d)}{\sum_{b'=1}^{B} \lambda^i(b',d)} \log(\lambda(b,d)) = Q(\theta' \theta^i)$$

Differentiating above with respect to $\lambda(b,d)$ to get maxima

$$\Rightarrow \lambda^{i+1}(b,d) = \frac{n^*(d)\, \lambda^i(b,d)}{\sum_{b'=1}^{B} \lambda^i(b',d)}$$

$$\lambda^i(b,d) = \lambda^i(b)\, P(b,d)$$

$$\lambda(b) = \sum_{d=1}^{D} \lambda(b,d)$$

$$\Rightarrow \lambda^{new}(b) = \lambda^{old}(b) \sum_{d=1}^{D} \frac{n^*(d)\, P(b,d)}{\sum_{b'=1}^{B} \lambda_{old}(b')\, P(b',d)}$$

Final formula for update is given below

$$\boxed{\lambda^{new}(b) = \lambda^{old}(b) \sum_{d=1}^{D} \frac{n^*(d)\, P(b,d)}{\sum_{b'=1}^{B} \hat{\lambda}_{old}(b')\, P(b',d)}}$$