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Page

Question-1

$\mathbb{P}(X=x_0) = .2$

$\mathbb{P}(X=x_1) = .8$

$E(x_0)$ = Event that x_0 is transmitted

$E(x_1)$ = Event that x_1 is transmitted.

$\mathbb{P}(y_0 | x_0) = .6$

$\mathbb{P}(y_1 | x_0) = .4$

$\mathbb{P}(y_0 | x_1) = .4$, $\mathbb{P}(y_1 | x_1) = .6$

Since , ~~occuring of~~ Events $E(x_0)$ and $E(x_1)$ is mutually exclusive and exhaustive. we can say,

a) $P(y_0) = P(y_0|x_0) \cdot P(x_0) + P(y_0|x_1) \cdot P(x_1)$

$= .6 \times .2 + .4 \times .8$

$= .12 + .32 = .44$

$P(y_1) = P(y_1|x_0) \cdot P(x_0) + P(y_1|x_1) \cdot P(x_1)$

$= .4 \times .2 + .6 \times .8$

$= .08 + .48 = .56$

b) $P(x_0|y_0) = \frac{P(y_0|x_0) \cdot P(x_0)}{P(y_0)}$

$= \frac{.2 \times .6}{.44} = \frac{.12}{.44}$

$= \frac{3}{11}$

$P(x_0|y_1) = \frac{P(y_1|x_0) \cdot P(x_0)}{P(y_1)}$

$= \frac{.4 \times .2}{.56} = \frac{8}{56} = \frac{1}{7}$

$P(x_1|y_0) = \frac{P(y_0|x_1) \cdot P(x_1)}{P(y_0)}$

$= \frac{.4 \times .8}{.44} = \frac{8}{11}$

$P(x_1|y_1) = \frac{P(y_1|x_1) \cdot P(x_1)}{P(y_1)}$

$= \frac{.6 \times .8}{.56} = \frac{48}{56} = \frac{6}{7}$

Question-2

Let's suppose the three coins are:

a: Two heads
b: normal coins
c: Two tails.

A be the event of choosing coin a

B be the event of choosing coin b

C be the event of choosing coin c

H_i be the ~~prob~~ event of the lower face of coin to be head in i^{th} toss.

$P(A) = 2/5$ $P(B) = 2/5$

$P(C) = 1/5$

a) $P(H_1) = P(H_1/A)\cdot P(A) + P(H_1/B)\cdot P(B)$

$+ P(H_1/C)\cdot P(C)$

$= 1 \times \frac{2}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{3}{5}$

b) Let H_i' be the event of the top face of coin being head.

So, $P(H_1') = P(H_1'|A)\cdot P(A) + P(H_1'|B)\cdot P(B)$

$+ P(C)\cdot P(H_1'|C)$

$= 1 \times \frac{2}{5} + \frac{1}{2} \times \frac{2}{5} = \frac{3}{5}$

$P(H_1|H_1') = \frac{P(H_1 \cap H_1')}{P(H_1')}$

$= \frac{P(H_1|A)\cdot P(A)}{P(H_1')}$

$= \frac{1 \times 2/5}{3/5} = \frac{2}{3}$

c) $P(H_2 | H_1')$

$$= \frac{P(H_2 \cap H_1')}{P(H_1')}$$

$$= \frac{P(H_1'/A) \cdot P(H_2/A) \cdot P(A) + P(H_1'|B) \cdot P(H_2|B) \cdot P(B)}{P(H_1')}$$

$$= \frac{1 \times 1 \times \frac{2}{5} + \frac{1}{4} \times \frac{2}{5}}{\frac{3}{5}}$$

$$= (2 + 1/2)/3 = \frac{5}{6}$$

d) $P(H_2 | (H_2' \cap H_1'))$

$$= \frac{P(H_2 \cap H_2' \cap H_1')}{P(H_2' \cap H_1')}$$

$$= \frac{P(H_2'|A) P(H_1'|A) \cdot P(A)}{\left(P(H_2'|A) \cdot P(H_1'|A) \cdot P(A) + P(H_2'|B) P(H_1'|B) \cdot P(B)\right)}$$

$$P\left(H_2 \mid (H_2' \cap H_1)\right)$$

$$= \frac{1 \times 1 \times \frac{2}{5}}{1 \times 1 \times \frac{2}{5} + \frac{1}{2} \times \frac{1}{2} \times \frac{2}{5}}$$

$$= \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{10}} = \frac{\frac{2}{5}}{\frac{5}{10}}$$

$$= \frac{4}{5}$$

Q-3

Ans - $R =$ Number of Brown balls

$B =$ Number of Black balls

A_K is the event of choosing 1st brown ball in kth time.

Here we are assumming same colour balls to be identical

$P_K = P(A_K)$

Probability of finding Brown ball in 1st time $= \frac{B}{B+R}$

Probability of finding Brown ball in 2nd time $= \frac{B}{B+R} \cdot \frac{R}{B+R-1}$

Probability of finding ball in kth time

$=$ if $k > B+1$

$P(A_K) = 0$

For $k \leq B+1$

$$P(A_K) = \frac{(B)}{(B+R)} \cdot \frac{(B-1)}{(B+R-1)} \cdot \frac{(B-2)}{(B+R-2)} \cdots \frac{(B-(k-2))}{B+R-(k-2)} \cdot \frac{R}{B+R-(k-1)}.$$

$$P(A_K) = \frac{RB!\,(B+R-k)!}{(B+R)!\,(B-k+1)!}$$

$$P(A_K) = \begin{cases} \frac{RB!\,(B+R-k)!}{(B+R)!\,(B-k+1)!}, & -k \geq -(B+1) \\ 0, & k > 0 \end{cases}$$

$\alpha = \frac{R}{B+R}$, let $B+R = T$, $B = (1-\alpha)T$, $R = T\alpha$.

$$\lim_{\substack{R \to \infty \\ B \to \infty}} (R+B) = \lim_{T \to \infty} T$$

$$\lim_{T \to \infty} P(A_k) = \left[\frac{(1-\alpha)T}{T}\right] \left[\frac{(1-\alpha)T-1}{T-1}\right] \cdots \frac{\alpha T}{T-(k-1)}, \quad \text{for } k \leq B+1$$

$$= (1-\alpha) \cdot \left[\frac{(1-\alpha)-1/T}{1-1/T}\right] \cdots \left[\frac{\alpha}{1-\left[\frac{k-1}{T}\right]}\right]$$

$$\lim_{T \to \infty} P(A_k) = \begin{cases} (1-\alpha)^{k-1}\alpha & k \leq B+1 \\ 0 & k > B+1. \end{cases}$$

Question - 4

Given :

n urns.

r-th urn contains : $r-1$ brown balls
$n-r$ black balls

Sol^n:
probability of picking the r-th urn at random $= \frac{1}{n}$ (since, all n-urns are equiprobable)

Now, we pick two balls at random without replacement from the r-th urn.

Let,

A_1 be the event that → the first ball is brown & the second ball is black.

A_2 be the event that → the first ball is black & the second ball is black.

A be the event that → the second ball is black.

$$\therefore \quad A = A_1 \cup A_2 .$$

$$\therefore P(A) = P(A_1) + P(A_2) \quad [\because P(A_1 \cap A_2) = 0].$$

probability of event A after selecting the r-th urn with probability $\frac{1}{n}$,

$$P(A) = \sum_{r=1}^{n} \underbrace{\left[\frac{r-1}{n-1} \cdot \frac{n-r}{n-2}\right]\frac{1}{n}}_{P(A_1) \text{ for } r\text{-th urn}} + \underbrace{\left[\frac{n-r}{n-1} \cdot \frac{n-r-1}{n-2}\right]\frac{1}{n}}_{P(A_2) \text{ for } r\text{-th urn}}$$

$$\therefore P(A) = \sum_{r=1}^{n} \frac{(n-r)}{n(n-1)(n-2)} \left[r-1+n-r-1\right]$$

$$= \sum_{r=1}^{n} \frac{(n-r)}{n(n-1)(n-2)} \left[n-2\right]$$

$$= \sum_{r=1}^{n} \frac{1}{n(n-1)} \cdot (n-r)$$

$$= \frac{1}{n(n-1)} \left[n^2 - \frac{n(n+1)}{2}\right]$$

$$= \frac{1}{n(n-1)} \left[\frac{n^2-n}{2}\right]$$

$$\therefore \boxed{P(A) = \frac{1}{2}}$$

Let A_3 be the event that → the first ball is black.

A_4 be the event that → the second ball is black.

Observe that, $A_2 = A_3 \cap A_4$.

We have to find, $P\left(\frac{A_4}{A_3}\right)$

i.e. $\frac{\mathbb{P}(A_3 \cap A_4)}{\mathbb{P}(A_3)} = \frac{\mathbb{P}(A_2)}{\mathbb{P}(A_3)}$

$$\mathbb{P}(A_2) = \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{n-r}{n-1} \cdot \frac{n-r-1}{n-2}$$

$$= \sum_{r=1}^{n} \frac{1}{n(n-1)(n-2)} \left[n^2 + r^2 - 2nr - n + r \right]$$

$$= \sum_{r=1}^{n} \frac{1}{n(n-1)(n-2)} \left[(n^2-n) + r(1-2n) + r^2 \right]$$

$$= \frac{1}{n(n-1)(n-2)} \left\{ (n^2-n)n + \frac{n(n+1)}{2}(1-2n) + \frac{n(n+1)(2n+1)}{6} \right\}$$

$$= \frac{1}{(n-1)(n-2)} \left\{ n(n-1) + \frac{(n+1)(1-2n)}{2} + \frac{(n+1)(2n+1)}{6} \right\}$$

$$= \frac{1}{(n-1)(n-2)} \left\{ n(n-1) + \frac{n+1}{2} \left(\frac{2n+1}{3} + 1 - 2n \right) \right\}$$

$$= \frac{1}{(n-1)(n-2)} \left\{ n(n-1) + \frac{n+1}{2} \left(\frac{2n+1+3-6n}{3} \right) \right\}$$

$$= \frac{1}{(n-1)(n-2)} \left\{ n(n-1) + \frac{n+1}{2} \left(\frac{4-4n}{3} \right) \right\}$$

$$= \frac{1}{(n-1)(n-2)} \left\{ n(n-1) + \frac{2}{3}(n+1)(1-n) \right\}$$

$$= \frac{1}{n-2} \left\{ n - \frac{2}{3}(n+1) \right\}$$

$$= \frac{1}{n-2} \left\{ \frac{3n-2n-2}{3} \right\}$$

$$\therefore \boxed{P(A_2) = \frac{1}{3}}$$

Now,

$$P(A_3) = \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{n-r}{n-1}$$

$$= \frac{1}{n(n-1)} \left\{ n^2 - \frac{(n)(n+1)}{2} \right\}$$

$$= \frac{1}{n(n-1)} \left\{ \frac{2n^2 - n^2 - n}{2} \right\}$$

$$\therefore \boxed{P(A_3) = \frac{1}{2}}$$

$$\therefore P\left(\frac{A_4}{A_3}\right) = \frac{P(A_2)}{P(A_3)} = \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{3}$$

$$\therefore \boxed{P\left(\frac{A_4}{A_3}\right) = \frac{2}{3}}$$

Question-5

X_i be a random variable for i^{th} vertex.

$X_i \sim$ Bernoulli distripbution (P)

$X_i = 1$: vertex is in black region

$X_i = 0$: vertex is in white region.

$P(X_i = 1) = .9$

$P(X_i = 0) = .1$

Lets consider another Random variable.

$$Y = \sum_{i=1}^{8} X_i$$

Y takes value in $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

and $y=i$ represent that i vertices are in black region.

If we find Expectation of x_i,
then $E(x_i) = 1 \cdot P(x_i = 1) + 0 \cdot P(x_i = 0)$

$$= \frac{9}{10}$$

$$E(Y) = E\left(\sum_{i=1}^{8} x_i\right)$$

$$= \sum_{i=1}^{8} E(x_i) = \frac{72}{10} = 7.2$$

$E(x_i)$ does not depend on i, and even though x_i's are not independent

$E\left(\sum_{i=1}^{8} x_i\right) = \sum_{i=1}^{8} E(x_i)$, because

Expectation can be distributed over summation

~~Sin~~ Now, $E(Y) = 7.2$

this ~~means~~ means the $P(Y=8)$ is non-zero. Because, if in any possible distribution of colour on sphere $P(Y=8) = 0$, then

$E(Y)$ could take maximum value as 7 but since it is 7.2, so $P(Y=8)$ must be non-zero.

So, there always exist a cube whose all 8 vertices lies in black region.

Question - 6

PART A

here, n is the total fish population.

We catch 100 fish, mark them, then release them back into the lake.

Now, we catch 100 fish again, out of which 10 were marked before,
and thus 90 we unmarked.
Call this event A.

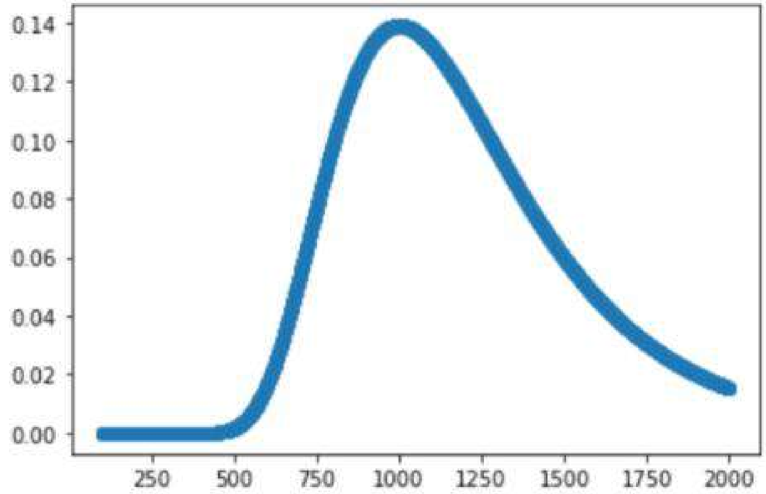
Since we have n fishes, out of which 100 are marked & $n-100$ are unmarked.

Number of ways of picking 10 marked fishes out of 100 marked fish $= {}^{100}C_{10}$

Number of ways of picking 90 unmarked fish out of $n-100$ unmarked fish $= {}^{n-100}C_{90}$.

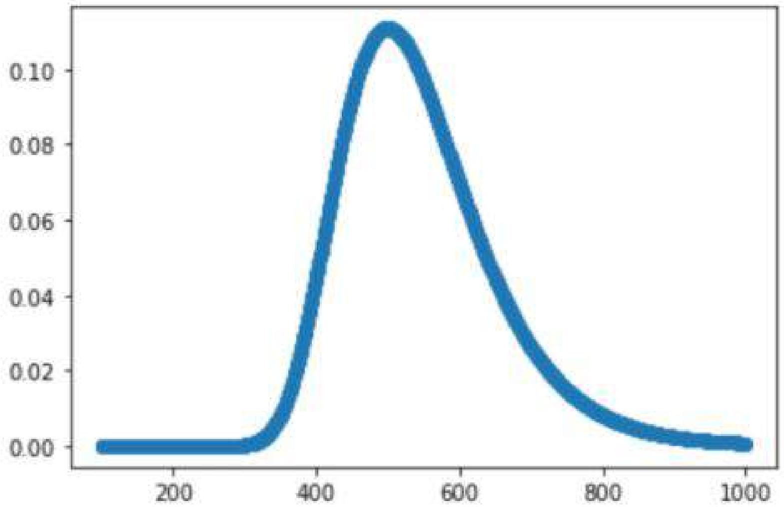
$$\therefore \boxed{\mathbb{P}(A) = \frac{{}^{100}C_{10} \cdot {}^{n-100}C_{90}}{{}^{n}C_{100}}}$$

m = 100, p = 10 :



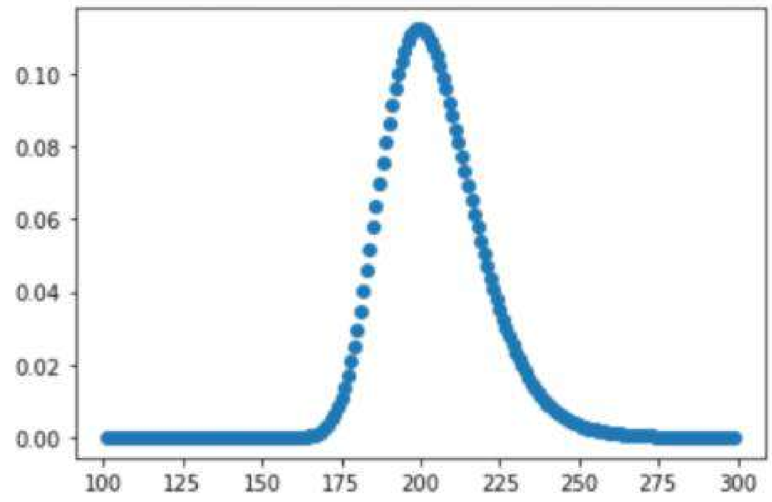
n1_cap ~ 1000

m = 100, p = 20 :



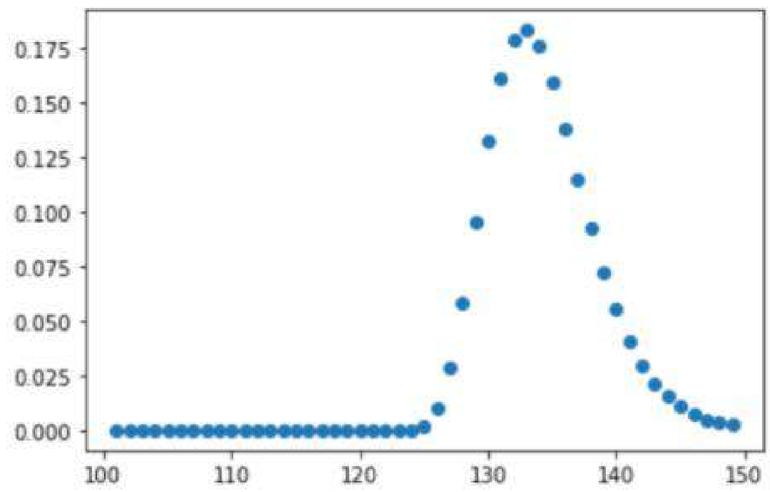
n2_cap ~ 500

m = 100, p = 50 :



n3_cap ~ 200

m = 100, p = 75 :



n4_cap ~ 135

PART B:

After generating plots for $P_{m,p}(n)$ as function of n for:

$m = 100$

& $p = 10, 20, 50, 75,$

We observe that:

each of the graph has a maxima for a specific value of n (estimate).

the estimates are as follows:

$m = 100, p = 10, \boxed{\hat{n}_1 = 1000}$

$m = 100, p = 20, \boxed{\hat{n}_2 = 500}$

$m = 100, p = 50, \boxed{\hat{n}_3 = 200}$

$m = 100, p = 75, \boxed{\hat{n}_4 = 135}$

PART C:

Simulating the experiment 500 times to obtain the value of 'p' and taking it's average:

$\hat{n}_1 = 1000$	$m = 100$	$P_1 = 10.108$
$\hat{n}_2 = 500$	$m = 100$	$P_2 = 20.092$
$\hat{n}_3 = 200$	$m = 100$	$P_3 = 49.968$
$\hat{n}_4 = 135$	$m = 100$	$P_4 = 74.116$

the values calculated are close compared to the values in part B.