Supplementary Material for Onset of thermocapillary convection in hotter and cooler liquid layers under non-uniform heating

Introduction

This supplementary material provides detailed governing equations, derivations using the Galerkin method, trial functions, full expressions of the Marangoni number for all six temperature profiles, and extended tables and figure captions supporting the main article. This ensures reproducibility and transparency of the theoretical analysis.

1. Governing Equations

The linearized governing equations for the vertical velocity component w and temperature perturbation θ in a horizontal liquid layer of thickness d are:

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w = 0, \tag{1}$$

$$(1-\alpha_2 T_0)\left(\frac{\partial\theta}{\partial t} - f(z)w\right) = \kappa\nabla^2\theta, \tag{2}$$

where ν is the kinematic viscosity, κ the thermal diffusivity, $f(z)$ the normalized non-uniform temperature profile, and $\alpha_2 T_0$ the thermal modulation parameter. The Laplacian is $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

2. Boundary Conditions

The lower boundary ($z = 0$) is stress-free and perfectly conducting:

$$w = 0, \quad \frac{\partial^2 w}{\partial z^2} = 0, \quad \theta = 0, \tag{3}$$

and the upper free surface ($z = d$) is insulating:

$$w = 0, \quad \rho\nu\frac{\partial^2 w}{\partial z^2} = \sigma\nabla_1^2\theta, \quad \frac{\partial\theta}{\partial z} = 0. \tag{4}$$

3. Non-dimensional Eigenvalue Problem

Applying normal mode solutions:

$$w = W(z)e^{i(a_x x + a_y y)+pt}, \quad \theta = \Theta(z)e^{i(a_x x + a_y y)+pt},$$

and non-dimensionalization, the neutral stability equations (assuming $p = 0$) reduce to:

$$(D^2 - a^2)^2 W = 0, \tag{5}$$
$$(D^2 - a^2)\Theta = -(1 - \alpha_2 T_0) f(z) W, \tag{6}$$

with $D = d/dz$.

4. Galerkin Formulation and Trial Functions

Using single-term Galerkin approximation:

$$W(z) = AW_1(z), \quad \Theta(z) = B\Theta_1(z),$$

where $W_1(z)$ and $\Theta_1(z)$ satisfy boundary conditions:

$$W_1 = z(1 - z^2), \quad \Theta_1 = z(1 - z/2).$$

Integrals:

$$I_1 = \int_0^1 [(D^2W_1)^2 + 2a^2(DW_1)^2 + a^4W_1^2]\, dz,$$
$$I_2 = a^2(DW_1)(1)\Theta_1(1), \quad I_3 = \int_0^1 W_1\Theta_1 f(z)\, dz, \quad I_4 = \int_0^1 [(D\Theta_1)^2 + a^2\Theta_1^2]\, dz.$$

The Marangoni number M is then given by:

$$M = -\frac{I_1 I_4}{(1 - \alpha_2 T_0) I_2 I_3}.$$

5. Closed-form Expressions for Six Temperature Profiles

- **Case 1: Uniform $f(z) = 1$**

$$M_1 = \frac{32(315 + 42a^2 + 2a^4)(5 + 2a^2)}{1155\, a^2(1 - \alpha_2 T_0)}, \quad M_{c1} = \frac{46.21}{1 - \alpha_2 T_0},\ a_c = 1.89$$

- **Case 2: Piecewise heating from below $f(z) = \frac{1}{\epsilon}$, $0 \le z < \epsilon$**

$$M_{c2} = \frac{43.67}{1 - \alpha_2 T_0}, \quad \epsilon_2 = 0.88$$

- **Case 3: Piecewise cooling from above $f(z) = \frac{1}{\epsilon}$, $1 - \epsilon \le z \le 1$**

$$M_{c3} = \frac{33.56}{1 - \alpha_2 T_0}, \quad \epsilon_3 \approx 0.55$$

- **Case 4: Parabolic $f(z) = 2z$**

$$M_{c4} = \frac{38.67}{1 - \alpha_2 T_0},\ a_c = 1.89$$

- **Case 5: Inverted parabolic** $f(z) = 2(1 - z)$

$$M_{c5} = \frac{57.38}{1 - \alpha_2 T_0}, \; a_c = 1.89$$

- **Case 6: Localized heating** $f(z) = \delta(z - \epsilon)$

$$M_{c6} = \frac{25.71}{1 - \alpha_2 T_0}, \; \epsilon_6 \approx 0.66$$

6. Extended Table of Critical Marangoni Numbers

$\alpha_2 T_0$	Mc1	Mc2	ϵ_2	Mc3	ϵ_3	Mc4	Mc5	Mc6	ϵ_6
0.0	46.21	43.67	0.88	33.56	0.55	38.67	57.38	25.71	0.66
0.1	51.35	48.53	0.88	37.29	0.55	42.97	63.76	28.57	0.66
0.2	57.76	54.58	0.88	41.95	0.55	48.34	71.73	32.14	0.66
0.3	66.02	62.38	0.88	47.94	0.55	55.24	81.97	36.73	0.66
0.4	77.02	72.78	0.88	55.94	0.55	64.45	95.64	42.85	0.66
0.5	92.42	87.34	0.88	67.12	0.55	77.34	114.76	51.42	0.66

7. Figure Captions

- **Figure S1:** Neutral stability curves showing variation of M_1, M_4, and M_5 with horizontal wavenumber a for different $\alpha_2 T_0$.
- **Figure S2:** Critical Marangoni numbers for localized and piecewise profiles as a function of thermal depth ϵ.