OSCULATING GEOMETRY AND HIGHER-ORDER DISTANCE LOCI

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ABSTRACT. We discuss the problem of optimizing the distance function from a given point, subject to polynomial constraints. A key algebraic invariant that governs its complexity is the *Euclidean distance degree*, which pertains to first-order tangency.

We focus on the *data locus* of points possessing at least one critical point of the distance function that is normal to a higher-order osculating space. We propose a novel definition of *higher-order distance degree* as an intersection-theoretic invariant involving jet bundles and higher-order polar classes. Our research yields closed formulas for generic maps, Veronese embeddings, and toric embeddings. We place particular emphasis on the Bombieri-Weyl metric, revealing that the chosen metric profoundly influences both the degree and birationality of the higher-order projection maps. Additionally, we introduce a tropical framework that represents these degrees as stable intersections with Bergman fans, facilitating effective combinatorial computation in toric settings.

CONTENTS

1. INTRODUCTION

Motivation and Background. Metric algebraic geometry investigates the geometry of algebraic varieties through notions of distance and approximation [BKS24]. At its core lies the problem of identifying points on a variety that are closest to a given point in ambient space, typically with respect to the Euclidean norm. A key measure of the intrinsic complexity of these problems is the *Euclidean distance degree (EDD)*. This algebraic invariant counts the number of complex critical points of the squared Euclidean distance function when restricted to a given variety. Introduced in the foundational work [DHO$^+$16], the EDD is central in the theory of optimization over algebraic varieties and has catalyzed recent advances at the interface of geometry and computation.

The classical EDD reflects first-order geometry: critical points are governed by tangent spaces and orthogonality conditions between the variety and the direction vector from a data point. However, many applications across data science, signal processing, and numerical optimization require a more refined notion of proximity, one that accounts for curvature, torsion, or higher-order contact.

For instance, in manifold learning [MM10, FMN16, FIK+18, AK24, KWW25], trajectory planning [BH75, Zha02], nonlinear regression, and geometric data fitting [CP05, SW89], it is not sufficient to minimize the Euclidean distance merely; how a model surface curves toward the data is crucial for both interpretability and accuracy. Such problems require critical configurations that exhibit higher-order contact, modeled via osculating spaces rather than tangents alone. This motivated us to study new algebraic invariants that generalize Euclidean distance degrees and encode higher-order tangency. Related recent contributions include [BW24, Hor24, BRW25].

Geometric Setup and State of the Art. We start by fixing an $(n+1)$-dimensional real vector space $V_{\mathbb{R}}$ and a positive definite quadratic form $q\colon V_{\mathbb{R}} \to \mathbb{R}$. We denote by Q the quadric hypersurface defined by q. For any data point $u \in V_{\mathbb{R}}$, we denote by d_u the distance function from u, defined by $d_u(v) := \sqrt{q(u-v)}$ for all $v \in V_{\mathbb{R}}$. For any fixed subset $S \subseteq V_{\mathbb{R}}$ such that $u \notin S$, it is natural to study the set of points $x \in S$ attaining a local minimum of d_u restricted to S. In this paper, we restrict ourselves to subsets S that are affine cones over real algebraic varieties. As a prototypical example, we focus on affine cones over projective toric varieties [DDRP14], and in particular varieties of rank at most one symmetric tensors, seen as homogeneous polynomials [Ban38]. More precisely, we let $Y_{\mathbb{R}}$ be the image of a real morphism $f\colon X_{\mathbb{R}} \to \mathbb{P}(V_{\mathbb{R}}) = \mathbb{P}^n_{\mathbb{R}}$ and $S = C(Y_{\mathbb{R}})$ be the affine cone over $Y_{\mathbb{R}}$. Firstly, despite being naturally a problem defined over the real numbers, the algebraic study of the locus of critical points of d_u requires working over the field of complex numbers. Secondly, we observe that a similar study may be carried out for more general affine varieties that are not cones. However, we restrict ourselves to affine cones over projective varieties to apply classical results of intersection theory in projective space. Motivated by these facts, we consider the complex vector space $V := V_{\mathbb{R}} \otimes \mathbb{C}$, the Zariski closure Y of $Y_{\mathbb{R}}$ in $\mathbb{P}(V) = \mathbb{P}^n$, and its affine cone $C(Y) \subseteq V$. In particular, we will always work with morphisms $f\colon X \to \mathbb{P}^n$, where $Y = f(X)$. Additionally, since the squared distance function d_u^2 is polynomial, we consider it as a polynomial function $d_u^2\colon V \to \mathbb{C}$. We emphasize that the real quadratic form q induces only a bilinear symmetric form over V and not a Hermitian inner product. We denote by Q the nonsingular quadric hypersurface in $\mathbb{P}^n$ defined by the quadratic form q.

A generic complex data point typically has a nonsingular critical point for the distance function restricted to a reduced variety. Introducing additional conditions on the nature of critical points leads to richer algebraic structures known as *data loci*. For example, *conditional ED data loci* encode data points with at least one critical point on a prescribed subvariety [Hor17, HR22, DRGS25]. Other well-studied data loci are *ε-offsets*, representing points with a critical point at distance ε, *bisector hypersurfaces*, defined by data having two equidistant critical points [HW19, OS20], and *Voronoi cells*, consisting of all data points closest to a fixed point lying on a given variety [CRSW22].

In this work, we introduce and study the *higher-order distance degree* $\mathrm{DD}_k(f,Q)$ and *higher-order distance loci* $\mathrm{DL}_k(f,Q)$ of a morphism $f\colon X \to \mathbb{P}^n$ with respect to a fixed nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$. This invariant describes the algebraic complexity for f to satisfy a prescribed osculating condition. In the following, we summarize the main steps that motivate our definition.

For a fixed integer $k \geq 1$ and a point $p \in X$, the *kth osculating space* $\mathbb{T}^k_p(f) \subseteq \mathbb{P}^n$ is defined as the projective span of all partial derivatives of the components of f at p up to order k, equivalently, the image of the kth jet map at p. This space captures all infinitesimal directions of the morphism f at p up to order k and is the smallest projective linear subspace with this property. For $k = 1$, the osculating space $\mathbb{T}^1_p(f)$ coincides with the projective tangent space $\mathbb{T}_p(f)$, which has dimension $m = \dim X$ at nonsingular points. If f is an embedding, the tangent space has constant dimension by definition. However, for $k \geq 2$, the dimension of the osculating spaces $\mathbb{T}^k_p(f)$ may vary. We say that a morphism $f\colon X \to \mathbb{P}^n$ is *globally k-osculating* if the dimension of $\mathbb{T}^k_p(f)$ is constant across all points of X (see Definition 2.2).

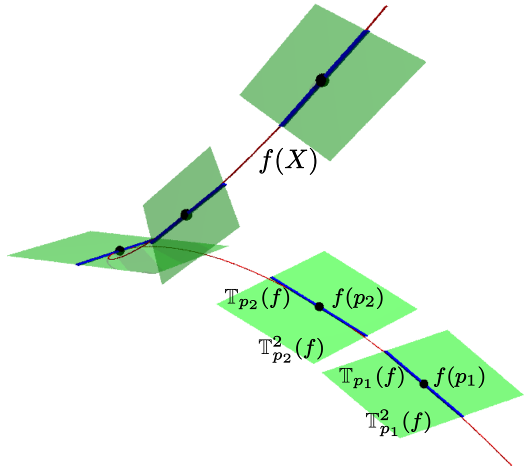

FIGURE 1. Tangent lines and osculating planes of the twisted cubic image of the Veronese embedding $f = \nu_1^3 \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ in the affine chart $\{u_0 = 1\} \cong \mathbb{R}^3$.

In particular, a globally 1-osculating parametrization is a closed embedding. Globally k-osculating morphisms are particularly well-suited to geometric analysis, as they admit a natural system of invariants known as *higher-order polar classes* $p_{k,i}(f)$ for $i \in \{0, \dots, m\}$. Introduced by Piene in [Pie78], these classes are subvarieties of codimension i in X that encode the geometric behavior of the kth order osculating spaces. In Sections 2 and 3, we recall and further develop the geometric framework of osculating functions and higher-order polar geometry, which forms the foundation for our main results.

Section 4 contains the novel definitions of this paper. Given a morphism $f \colon X \to \mathbb{P}^n$ and a point $u \in V$, we say that $[v] = f(p) \in f(X)$ is *critical of order* k for the squared distance function d_u^2 if

$$[\nabla d_u^2(v)] \in \mathbb{N}_p^k(f, Q) \coloneqq \langle \mathbb{T}_p^k(f)^\perp, f(p) \rangle \,,$$

where $\langle S \rangle$ denotes the projective span of a subset $S \subseteq \mathbb{P}^n$, and $\mathbb{T}_p^k(f)^\perp$ is the *polar subspace* of the kth order osculating space $\mathbb{T}_p^k(f)$ with respect to the quadric Q, see (4.2). We refer to $\mathbb{N}_p^k(f, Q)$ as the *kth order normal space* of f at p.

The central object of study is the *kth order distance correspondence*:

$$\mathrm{DC}_k(f, Q) \coloneqq \overline{\{([v], [u]) \in \mathbb{P}^n \times \mathbb{P}^n \mid [v] \in f(X) \text{ is critical of order } k \text{ for } d_u^2\}} \subseteq \mathbb{P}^n \times \mathbb{P}^n \,,$$

and its projection to the second factor $\mathbb{P}^n$, the *kth order distance locus* of (f, Q):

$$\mathrm{DL}_k(f, Q) \coloneqq \mathrm{pr}_2(\mathrm{DC}_k(f, Q)) \subseteq \mathbb{P}^n \,.$$

For $k = 1$, these constructions recover the classical *projective ED correspondence* $\mathrm{DC}(f, Q) \coloneqq \mathrm{DC}_1(f, Q)$ and the *Euclidean distance degree* $\mathrm{DD}(f, Q) \coloneqq \mathrm{DD}_1(f, Q)$, defined as the degree of the surjective projection $\mathrm{pr}_2|_{\mathrm{DC}(f,Q)} \colon \mathrm{DC}(f, Q) \to \mathbb{P}^n = \mathrm{DL}_1(f, Q)$; see [DHO+16, Theorem 4.4]. The description of higher-order distance loci $\mathrm{DL}_k(f, Q)$ becomes increasingly intricate for $k \geq 2$: indeed, for generic $[u] \in \mathbb{P}^n$, a critical point of order k may not exist on $f(X)$. Hence, unlike the case $k = 1$, the data locus $\mathrm{DL}_k(f, Q)$ forms a proper algebraic subvariety of $\mathbb{P}^n$, reflecting the rarity and structural complexity of higher-order critical interactions.

To quantify its complexity, we consider the surjective morphism $\varphi_{2,k} \colon \mathrm{DC}_k(f, Q) \to \mathrm{DL}_k(f, Q)$ induced by pr_2 and we define the *k-th order distance degree* of (f, Q) as

$$\mathrm{DD}_k(f, Q) \coloneqq \deg \mathrm{DL}_k(f, Q) \cdot \deg \varphi_{2,k} \,,$$

as formalized in Definition 4.7. In particular, when the pair (f, Q) is in *general k-osculating position* (see Definition 4.9), we prove that the morphism $\varphi_{2,k}$ is generically finite over its image and that $\mathrm{DD}_k(f, Q)$ attains its maximum value, called the *generic kth order distance degree of f* and denoted by $\mathrm{gDD}_k(f)$. In particular, this invariant is independent of Q. It can be computed via intersection-theoretic techniques, as seen in Theorem 4.11.

Main Results. The main result of Section 5 provides a closed formula for the kth order distance degree of a generic polynomial map. Its proof is given in Section 5.2.

Theorem 1.1. *Let $f_0, \ldots, f_n$ be $n+1$ generic polynomials in $\mathbb{C}[x_0, \ldots, x_m]_d$ and consider a non-negative integer $k \leq d$. Assume that $n > \binom{m+k}{k} - 1$ and let $f \colon \mathbb{P}^m \to \mathbb{P}^n$ be the associated globally k-osculating morphism. Then for any nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$ such that (f, Q) is in general k-osculating position, the morphism $\varphi_{2,k}$ is birational and*

$$\mathrm{DD}_k(f, Q) = \deg \mathrm{DL}_k(f, Q) = \sum_{i=0}^{m} \binom{\binom{m+k}{k}}{i} (d-k)^i d^{m-i}\,.$$

This result relies on an interesting interplay between generic and special metric properties. We begin by exploring the connection between higher-order osculating optimization and a concrete, widely studied problem in data approximation: the so-called best (symmetric) rank-one tensor approximation [Ban38, EY36, dSL08, OSS14]. In Definition 5.4, we introduce *higher-order eigenvectors* of symmetric tensors, a generalization of the notions introduced in [Lim05, Qi05]. In Proposition 5.7, we relate them to higher-order critical points of the nearest-point problem to the Veronese variety with respect to the Bombieri-Weyl quadratic form Q_{BW}, see Definition 5.1. Furthermore, we compute the higher-order distance degrees of Veronese embeddings with respect to the Bombieri-Weyl quadratic form, and examine how the geometry of the higher-order distance loci $\mathrm{DL}_k(f, Q)$ varies with the choice of metric. In Section 5.1, we prove the following result.

Theorem 1.2. *Let $\nu_m^d \colon \mathbb{P}^m \hookrightarrow \mathbb{P}(\mathbb{C}[x_0, \ldots, x_m]_d)$ be the degree-d Veronese embedding of $\mathbb{P}^m$, and equip $\mathbb{R}[x_0, \ldots, x_m]_d$ with the Bombieri-Weyl inner product. For all $k \leq d$,*

1. *If $(m, k) = (1, d-1)$, then $\deg \varphi_{2,d-1} = 2$ and $\deg \mathrm{DL}_{d-1}(\nu_1^d, Q_{\mathrm{BW}}) = d-1$.*
2. *If $(m, k) \neq (1, d-1)$, then $\varphi_{2,k}$ is birational and $\mathrm{DD}_k(\nu_m^d, Q_{\mathrm{BW}}) = \deg \mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}})$ equals the coefficient of the monomial $h_1^m h_2^{\binom{m+k}{k}-m-1}$ in the expansion of*

$$(-1)^{\binom{m+k}{k}-1} \sum_{j=0}^{\infty} (-1)^j (dh_1 + h_2 + k(d-k)h_1^2 + kh_1h_2)^j \in \frac{\mathbb{Z}[h_1, h_2]}{\langle h_1^{m+1}, h_2^{\binom{m+d}{d}} \rangle}\,. \tag{1.1}$$

Furthermore, for all $k \leq d$, if (ν_m^d, Q) is in general k-osculating position, then $\varphi_{2,k}$ is birational and

$$\deg \mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}}) = \mathrm{gDD}_k(\nu_m^d) = \sum_{i=0}^{m} \binom{\binom{m+k}{k}}{i} (d-k)^i d^{m-i}\,. \tag{1.2}$$

When $k = 1$, Equation (1.1) specializes to the distance degree $\mathrm{DD}(\nu_m^d, Q_{\mathrm{BW}})$ computed in [DHO+16, Corollary 8.7], a consequence of [CS13, Theorem 5.6], see also [OO13, Theorem 3.4] and [FO14, Theorem 12] for related formulas in a more general setting.

In Section 6, we focus on the case of *k-regular embeddings*, that is, embeddings $f \colon X \hookrightarrow \mathbb{P}^n$ whose kth order osculating spaces attain the maximal possible dimension globally. For such morphisms, we derive explicit formulas that are particularly well-suited to computational applications. We demonstrate their use in the context of embeddings of varieties of dimension at most three, with special focus on toric embeddings.

Beyond the projective-geometric and intersection-theoretic frameworks, we consider a tropical approach, which is suited to both combinatorial and computational perspectives. In Section 7, we study the tropicalization of the higher-order conormal variety $\operatorname{Trop} W_k(f,Q)$ associated with a morphism f. This viewpoint offers a powerful framework for analyzing the behavior of higher-order critical loci under degenerations and for computing higher-order polar degrees combinatorially. We show that higher-order polar degrees can be expressed in terms of stable intersections between $\operatorname{Trop} W_k(f,Q)$ and Bergman fans $\operatorname{Berg}(M_{n+1}^{n-j+1} \times M_{n+1}^{j+m_k-m+2})$ corresponding to uniform matroids, obtaining the following tropical formula, whose proof is given in Section 7.1.

Theorem 1.3. *Let f be a monomial k-osculating embedding of generic k-osculating dimension m_k. The generic kth order distance degree of f is the sum of the degrees*

$$\mathrm{gDD}_k(f) = \sum_{j=0}^{m+n-m_k-1} \int \operatorname{Trop} W_k(f,Q) \cdot \operatorname{Berg}(M_{n+1}^{n-j+1} \times M_{n+1}^{j+m_k-m+2})\,.$$

When f is a toric embedding, we further make these results effective and give a combinatorial description of $\operatorname{Trop} W_k(f,Q)$ in Proposition 7.7. We accompany our theoretical findings with an easy-to-use `Julia`-based software implementation, which enables the computation of higher-order distance degrees and polar degrees in this toric setting, similar to results from [HS18].

In the following, we present an example that provides a clear illustration of how we can explicitly compute and visualize all our constructions, including osculating spaces, higher-order normal bundles, polar loci, and distance degrees. Additionally, it connects our intersection-theoretic framework with issues related to low-rank tensor approximation, which we revisit in Section 5.

Illustrative Example. Consider the space $V = (\mathbb{C}^2)^* = \mathbb{C}[x_0,x_1]_1$ of linear forms in $x=(x_0,x_1)$, $X = \mathbb{P}(V) = \mathbb{P}^1$, and the morphism $f\colon \mathbb{P}^1 \to \mathbb{P}(S^3V) \cong \mathbb{P}^3$ defined by $f([\ell]) := [\ell^3]$ for every $\ell \in V$. Its image $f(\mathbb{P}^1)$ is a nonsingular twisted cubic in $\mathbb{P}^3$. In coordinates we write $\ell(x) = t_0x_0 + t_1x_1$, hence $\ell^3(x) = \sum_{i=0}^3 \binom{3}{i}(t_0x_0)^{3-i}(t_1x_1)^i$ and we define $f([t_0:t_1]) := [t_0^3 : t_0^2t_1 : t_0t_1^2 : t_1^3]$. In particular, f corresponds to the Veronese embedding $\nu_1^3\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, and its image consists of all cubic binary forms of rank at most one. Given $p=[t_0:t_1]$, we consider the matrix

$$A_p^{(2)}(f) = \begin{array}{c} \begin{pmatrix} t_0^3 & t_0^2t_1 & t_0t_1^2 & t_1^3 \\ 3\,t_0^2 & 2\,t_0t_1 & t_1^2 & 0 \\ 0 & t_0^2 & 2\,t_0t_1 & 3\,t_1^2 \\ 6\,t_0 & 2\,t_1 & 0 & 0 \\ 0 & 2\,t_0 & 2\,t_1 & 0 \\ 0 & 0 & 2\,t_0 & 6\,t_1 \end{pmatrix} \end{array} \begin{array}{l} f \\ \frac{\partial f}{\partial t_0} \\ \frac{\partial f}{\partial t_1} \\ \frac{\partial^2 f}{\partial t_0^2} \\ \frac{\partial^2 f}{\partial t_0 \partial t_1} \\ \frac{\partial^2 f}{\partial t_1^2} \end{array},$$

whose rows correspond to all partial derivatives of order at most 2 of the components of f. On the one hand $\mathbb{T}_p^2(f) = \mathbb{P}(\operatorname{rowspan} A_p^{(2)}(f))$. One verifies that $\dim \operatorname{rowspan} A_p^{(2)}(f) = 3$, hence $\dim \mathbb{T}_p^2(f) = 2$. In Figure 1, we display the tangent and second-order osculating spaces of f at some points. On the other hand, the right kernel of $A_p^{(2)}(f)$ is one-dimensional and is generated by the vector $\eta = (-t_1^3, 3\,t_0t_1^2, -3\,t_0^2t_1, t_0^3)^\intercal$.

Consider a quadric hypersurface $Q \subseteq \mathbb{P}^3$ and let M_Q be the positive-definite symmetric matrix defining the corresponding quadratic form in $S^3\mathbb{R}^2$. Then the 2nd order distance locus $\mathrm{DL}_2(f,Q)$ is equal to the union of the normal spaces

$$\mathbb{N}_p^2(f,Q) = \langle \mathbb{T}_p^2(f)^\perp, f(p)\rangle = \langle [M_Q\eta], f(p)\rangle$$

for every $p \in \mathbb{P}^1$, in particular it is a projective surface in $\mathbb{P}^3$. Applying Theorem 4.11 and Proposition 4.12, we have $\mathrm{gDD}_2(f) = \mu_{2,0}(f) + \mu_{2,1}(f) = 3 + 3 = 6$.

Firstly, let Q_{ED} be the standard Euclidean quadric of equation $\sum_{i=0}^3 u_i^2 = 0$, hence $M_{Q_{\mathrm{ED}}}$ is the identity matrix. We verified in `Macaulay2` [GS97] that $\mathrm{DL}_2(f, Q_{\mathrm{ED}})$ is a surface of degree $6 = \mathrm{gDD}_2(f)$ (displayed in light blue on the left-hand side of Figure 2) and that $\varphi_{2,2}$ is a birational morphism. If instead we consider $Q_{\mathrm{BW}}\colon \sum_{i=0}^3 \binom{3}{i} u_i^2 = 0$, corresponding to the *Bombieri-Weyl* inner product in $S^3\mathbb{R}^2$, then two interesting properties hold: firstly $\mathrm{DD}_2(f, Q_{\mathrm{BW}}) = 4 < 6 = \mathrm{gDD}_2(f)$, equivalently there is a "2nd-order ED defect" equal to 2, a behaviour similar to the classical ED defects studied in [MRW20]. Secondly, the morphism $\varphi_{2,2}$ is not birational, indeed $\mathrm{DL}_2(f, Q_{\mathrm{BW}})$ is the nonsingular quadric surface of equation $u_1^2 - u_0u_2 + u_2^2 - u_1u_3 = 0$ (displayed in yellow on the right-hand side of Figure 2), where $[u_0 : u_1 : u_2 : u_3]$ are homogeneous coordinates in $\mathbb{P}^3$, while $\deg \varphi_{2,2} = 2$. The smaller value of $\mathrm{DD}_2(f, Q_{\mathrm{BW}})$ follows because each 2nd order normal line $\mathbb{N}_p^2(f, Q_{\mathrm{BW}})$ meets the twisted cubic $f(X)$ at another point $f(p')$ such that $p = [v]$, $p' = [v']$, and the vectors v, v' are orthogonal, implying that $\mathbb{N}_p^2(f, Q_{\mathrm{BW}}) = \mathbb{N}_{p'}^2(f, Q_{\mathrm{BW}})$. We explain this graphically on the right-hand side of Figure 2, in particular with the points $p = [1 : 1]$ and $p' = [1 : -1]$. The `Macaulay2` code to compute these data loci is available in [DRRS25].

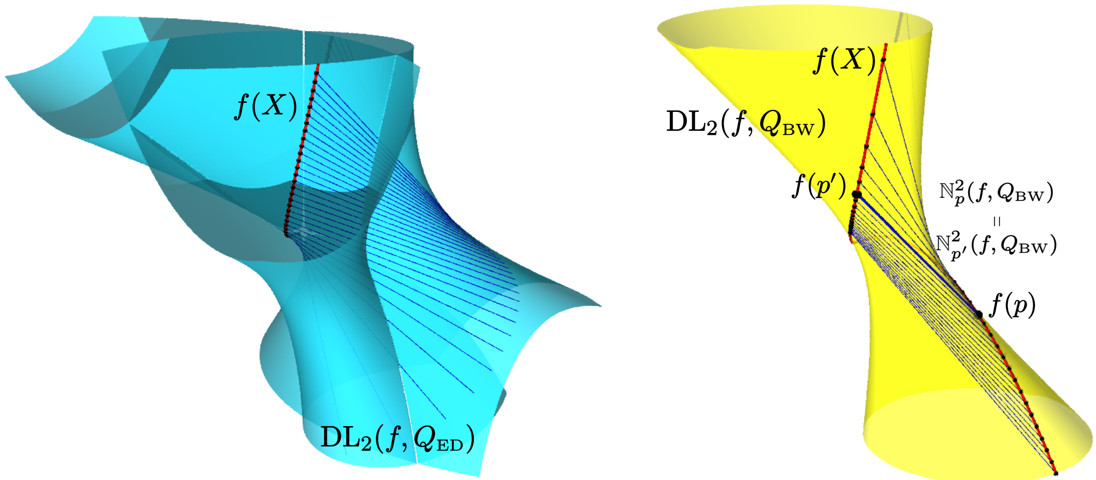

FIGURE 2. Second-order distance loci of the Veronese embedding $f = \nu_1^3\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, with respect to the standard (left) and Bombieri-Weyl (right) quadratic form in $S^3\mathbb{R}^2$, displayed in the real affine chart $\{u_0 = 1\}$.

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2. Osculating functions

Notation. Let V be an $(n+1)$-dimensional vector space over $\mathbb{C}$, or more generally an algebraically closed field. We denote by $\mathbb{P}^n$ the projective space $\mathbb{P}(V)$, while $(\mathbb{P}^n)^\vee$ denotes the dual projective space of hyperplanes in $\mathbb{P}^n$. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, we define $|\alpha| \coloneqq \alpha_1 + \cdots + \alpha_m$, and for any vector $x = (x_1, \ldots, x_m)$, we call $x^\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

Let $f\colon X \to \mathbb{P}^n$ be an algebraic morphism. We may assume that it is defined by the global sections of the line bundle $\mathcal{O}_X(1) = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$, in particular $f(p) = (\sigma_0(p), \dots, \sigma_n(p))$ for a given basis $(\sigma_0, \dots, \sigma_n)$ of $H^0(X, \mathcal{O}_X(1))$. Because f is a morphism, at each $p \in X$ there is at least one nonvanishing global section. This allows us to identify V with $H^0(X, \mathcal{O}_X(1))$. Let $p \in X$ and $\sigma \in H^0(X, \mathcal{O}_X(1))$. Recall that any $\sigma \in H^0(X, \mathcal{O}_X(1))$ can be represented in a neighborhood of p as a polynomial in local coordinates $(x_1, \dots, x_m)$. The *kth jet* of σ at p is the $\binom{m+k}{k}$-tuple

$$j_{k,p}(\sigma) \coloneqq \left(\frac{\partial^{|\alpha|}\sigma}{\partial x^\alpha}(p)\right)_{\substack{\alpha\in\mathbb{N}^m\\ 0\le|\alpha|\le k}} \in \mathcal{J}_{k,p}(f) \coloneqq H^0\left(X, \mathcal{O}_X(1) \otimes \frac{\mathcal{O}_{X,p}}{\mathfrak{m}_p^{k+1}}\right) \cong \mathbb{C}^{\binom{m+k}{k}},$$

where $\mathfrak{m}_p$ denotes the maximal ideal at p. The vector space $\mathcal{J}_{k,p}(f)$ is the fiber at p of a vector bundle $\mathcal{J}_k(f)$ of rank $\binom{m+k}{k}$, called the *kth jet bundle* of the morphism f or the principal part bundle. Gluing together the maps $j_{k,p}$ yields a morphism of vector bundles on X:

$$j_k\colon H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \to \mathcal{J}_k(f)\,. \tag{2.1}$$

Jet bundles, also called bundles of principal parts, were first introduced in [Poh62, Pie77], where an algebraic definition is given.

Definition 2.1. For every point $p \in X$, the *kth osculating space of f at $f(p) \in f(X)$* is

$$\mathbb{T}_p^k(f) \coloneqq \mathbb{P}(\operatorname{im} j_{k,p})\,.$$

If f is a closed embedding, by convention we identify X with its image $f(X)$ and write p instead of $f(p)$. Moreover, when f is a closed embedding, the first osculating space of X at the point p coincides with the projective tangent space of f at p:

$$\mathbb{T}_p^1(f) = \mathbb{T}_p(f)\,.$$

Furthermore, osculating spaces form an ascending sequence of inclusions:

$$\{p\} \subseteq \mathbb{T}_p(f) = \mathbb{T}_p^1(f) \subseteq \cdots \subseteq \mathbb{T}_p^k(f) \subseteq \cdots \subseteq \mathbb{P}^n\,. \tag{2.2}$$

For a generic $p \in X$, the rank of $j_{k,p}$ is constant. This motivates the following definition.

Definition 2.2. Given $f\colon X \to \mathbb{P}^n$, let $X_k^\circ \subseteq X$ be the dense open subset of X of points p such that $\operatorname{rank} j_{k,p}$ is constant. We define the *generic k-osculating dimension* of f as $m_k \coloneqq \operatorname{rank} j_{k,p} - 1$ for any $p \in X_k^\circ$. We say that the morphism f is *globally k-osculating* if $X_k^\circ = X$. In this case, m_k is referred to as the *k-osculating dimension* of f.

Remark 2.3. The generic 1-osculating dimension of a morphism $f\colon X \to \mathbb{P}^n$ is $m_1 = m$. Indeed, the open dense subset $X_1^\circ \subseteq X$ coincides with the nonsingular locus of X, and $\mathbb{T}_p^1(f) = \mathbb{T}_p(f) = \mathbb{P}^m$ for every $p \in X_1^\circ$. Furthermore, the morphism f is globally 1-osculating if and only if it is a closed embedding.

Let $(x_1, \dots, x_m)$ be a system of local coordinates around the point $p = (0, \dots, 0)$. After choosing a basis $(\sigma_0, \dots, \sigma_n)$ of $V = H^0(X, \mathcal{O}_X(1))$ one can define the $\binom{m+k}{k} \times (n+1)$ matrix

$$A_p^{(k)}(f) \coloneqq \left(\frac{\partial^{|\alpha|}\sigma_\beta}{\partial x^\alpha}(p)\right)_{\substack{\alpha\in\mathbb{N}^m\\ 0\le|\alpha|\le k\\ 0\le\beta\le n}} = \Big(\ j_{k,p}(\sigma_0)^\mathsf{T} \ \Big|\ j_{k,p}(\sigma_1)^\mathsf{T} \ \Big|\ \cdots \ \Big|\ j_{k,p}(\sigma_n)^\mathsf{T}\ \Big) \tag{2.3}$$

obtained by concatenating horizontally the $n+1$ column vectors $j_{k,p}(\sigma_\beta)^\mathsf{T}$ of length $\binom{m+k}{k}$ for every $\beta \in \{0, \dots, k\}$. A basis of $\operatorname{im} j_{k,p}$ is given by $m_k + 1$ linearly independent vectors in the column span of $A_p^{(k)}(f)$. Notice that $X_k^\circ = \{p \in X \mid \operatorname{rank} A_p^{(k)}(f) = m_k + 1\}$ and that this identity is independent of the choice of a basis of V.

Definition 2.4. The morphism $f\colon X \to \mathbb{P}^n$ is *k-regular* if $X_k^\circ = X$ and if the vector bundle morphism j_k in (2.1) is surjective, equivalently $m_k + 1 = \binom{m+k}{k}$.

Proposition 2.5. *Let $f\colon X \to \mathbb{P}^n$ be a globally k-osculating morphism for some $k \geq 1$. Then it is ℓ-osculating for all $\ell \leq k$. Similarly, if f is k-regular, then it is ℓ-regular for all $\ell \leq k$.*

Proof. Assume that $f\colon X \to \mathbb{P}^n$ is a globally k-osculating morphism for some $k \geq 1$, and let $\ell \leq k$. Notice that

$$A_p^{(k)}(f) = \begin{pmatrix} A_p^{(\ell)}(f) \\ B_p(f) \end{pmatrix}$$

where $B_p(f)$ is an $\left(\binom{m+k}{k} - \binom{m+t}{t}\right) \times n$ matrix for every $p \in X$. Assuming that there exists a special point p where $A_p^{(t)}$ does not have the maximal generic rank while $A_p^{(k)}$ has the maximal generic rank is impossible. Moreover, if $A_p^{(k)}(f)$ has maximal rank, then $A_p^{(\ell)}(f)$ will also have maximal rank. $\square$

Remark 2.6. A globally 1-osculating morphism is always 1-regular. For $k \geq 2$, this is no longer the case. See Example 2.9 below.

2.1. **A gallery of toric examples.** For varieties endowed with a maximal dimensional torus action, i.e., toric varieties, the torus action leads to remarkably simple criteria dictated by the size of the associated polytope.

Let $f\colon X \to \mathbb{P}^n$ be a nonsingular, globally k-osculating, torus-equivariant embedding of a toric variety X of dimension m. Then f is defined by an integer matrix

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_{n+1} \end{pmatrix} \in \mathbb{Z}^{(m+1)\times(n+1)}$$

of full rank, containing in its row span the vector $\mathbf{1} = (1, \dots, 1)$. We also denote by A the subset $\{a_1, \dots, a_{n+1}\} \subseteq \mathbb{Z}^m$. Let $P(f) = \operatorname{conv}(A)$ be the associated polytope and $\operatorname{vert} P(f)$ the set of vertices of $P(f)$. When f is defined by an ample divisor, the polytope $P(f)$ is of maximal dimension and every $v \in \operatorname{vert} P(f)$ corresponds to a fixed point $p_v \in X$. The variety has an affine cover $X = \bigcup_{v \in \operatorname{vert} P(f)} U_v$ where $U_v \subseteq \mathbb{C}^m$ is a Zariski open subset containing p_v for every $v \in \operatorname{vert} P(f)$. Moreover, assuming X nonsingular, we can choose a lattice basis where $v = 0$ and consequently local coordinates $(x_1, \dots, x_m)$ around $p_v = 0$. It follows that the first lattice vectors on the m edges through v correspond to a generating set for the dual lattice and $f|_{U_v}(x) = (x^\alpha)_{\alpha \in A}$. Let $p = \mathbf{1}$ be the generic point in the torus. For every $k \geq 1$, we denote by $A^{(k)}$ the matrix $A_{\mathbf{1}}^{(k)}(f)$ defined in (2.3). The construction of the matrices is explained in detail in [DDRP14, Section 5.2]. In particular $A^{(1)}$ is obtained adding the row $\mathbf{1}$ on top of A. The augmented matrix $A^{(k)}$ determines the generic k-osculating rank. More generally, for every fixed point $p_v \in X$, we denote by $A_v^{(k)}$ the matrix $A_{p_v}^{(k)}(f)$. The following result characterizes globally k-osculating toric embeddings.

Proposition 2.7. *A nonsingular toric embedding f is globally k-osculating of osculating dimension m_k if and only if $\dim \mathbb{T}_{p_v}^k(f) = \operatorname{rank} A_v^{(k)} - 1 = \operatorname{rank} A^{(k)} - 1$ for every fixed point p_v.*

Proof. From the description above, it is clear that $m_k + 1 = \operatorname{rank} A^{(k)}$. Note that if the locus of points where the kth osculating space has a lower dimension than m_k is nonempty, the torus action ensures that it contains fixed points. It follows that to ensure global kth osculation is enough that the kth osculation dimension at each fixed point is m_k. $\square$

Example 2.8. Consider $\pi\colon X \to \mathbb{P}^2$, the blow-up of $\mathbb{P}^2$ at the three points and the embedding $f_1\colon X \to \mathbb{P}^6$, a Del Pezzo surface of degree six defined by the global sections of the line bundle

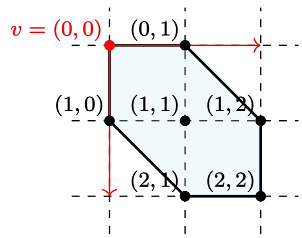

FIGURE 3. The polygon and the vectors of exponents of the monomial map representing the embedding f_1 of Example 2.8.

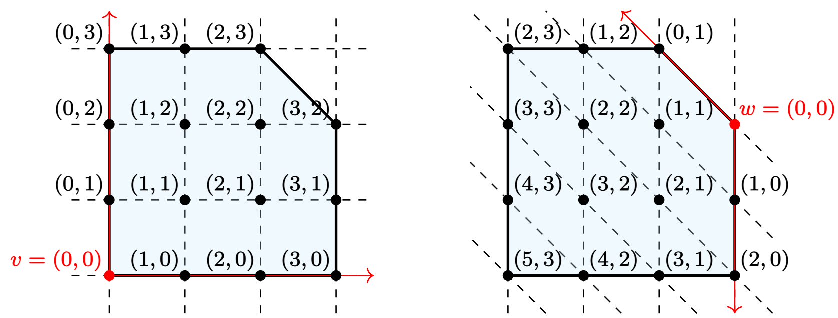

FIGURE 4. The polygon and the vectors of exponents of the monomial maps representing the embedding f_2 of Example 2.9.

$\mathcal{O}_X(1) = \pi^*\mathcal{O}_{\mathbb{P}^2}(3) - E_1 - E_2 - E_3$, where E_i are the exceptional divisors. On an affine patch around the fixed point p_v, where v is indicated in Figure 3, the monomial map is:

$$(x_1, x_2) \mapsto (1, x_1, x_2, x_1x_2, x_1x_2^2, x_1^2x_2, x_1^2x_2^2)\,.$$

Consider $k = 2$. In this case $\operatorname{rank} A^{(2)} = 6$, and thus the generic 2-osculating dimension of f_1 is $m_2 = 5$, but f_1 is not globally 2-osculating since $\dim \mathbb{T}^2_{p_v}(f_1) = 3$ for every fixed point $p_v \in X$. $\diamondsuit$

Example 2.9. Consider $\pi\colon X \to \mathbb{P}^1 \times \mathbb{P}^1$, the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point and the embedding $f_2\colon X \to \mathbb{P}^{14}$, a surface of degree 17 defined by the global sections of the line bundle $\mathcal{O}_X(1) = \pi^*\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(3,3) - E$, where E is the exceptional divisor. On the affine patches around the fixed points p_v and p_w, where v and w are indicated in Figure 4, the monomial maps are respectively

$$\begin{aligned}
&(x_1, x_2) \mapsto (1, x_1, x_1^2, x_1^3, x_2, x_1x_2, x_1^2x_2, x_1^3x_2, x_2^2, x_1x_2^2, x_1^2x_2^2, x_1^3x_2^2, x_2^3, x_1x_2^3, x_1^2x_2^3)\\
&(y_1, y_2) \mapsto (1, y_1, y_1^2, y_2, y_1y_2, y_1^2y_2, y_1^3y_2, y_1y_2^2, y_1^2y_2^2, y_1^3y_2^2, y_1^4y_2^2, y_1^2y_2^3, y_1^3y_2^3, y_1^4y_2^3, y_1^5y_2^3)\,.
\end{aligned}$$

Consider $k = 2$. One verifies that $\operatorname{rank} A^{(2)}_v = 6$ and $\operatorname{rank} A^{(2)}_w = 5$, therefore the embedding f_2 is not globally 2-osculating. $\diamondsuit$

A special class of toric varieties that are always globally k-osculating consists of the Segre-Veronese embeddings of a product of projective spaces. Consider two r-tuples of positive integers $\boldsymbol{m} = (m_1, \dots, m_r)$ and $\boldsymbol{d} = (d_1, \dots, d_r)$, with $m_1 \le \cdots \le m_r$. Define $d := d_1 + \cdots + d_r$. Let $V_1, \dots, V_r$ be vector spaces of dimensions $m_1 + 1 \le \cdots \le m_r + 1$, and consider the product $\mathbb{P}^{\boldsymbol{m}} = \prod_{i=1}^r \mathbb{P}(V_i)$. The line bundle $\mathcal{O}_{\mathbb{P}^{\boldsymbol{m}}}(\boldsymbol{d})$ induces an embedding

$$\sigma\nu^{\boldsymbol{d}}_{\boldsymbol{m}}\colon \mathbb{P}^{\boldsymbol{n}} \hookrightarrow \mathbb{P}\left(\bigotimes_{i=1}^{r} \operatorname{Sym}^{d_i} V_i\right), \tag{2.4}$$

whose image $SV^{\boldsymbol{d}}_{\boldsymbol{n}} = \sigma\nu^{\boldsymbol{d}}_{\boldsymbol{m}}(\mathbb{P}^{\boldsymbol{n}})$ is the *Segre-Veronese variety*. When $r = 1$, we use the notation $\boldsymbol{m} = m$, $\boldsymbol{d} = d$, $\nu^d_m := \sigma\nu^d_m$, and $V^d_m := SV^d_m$ is the *Veronese variety*.

A complete description of the osculating spaces of Segre-Veronese varieties is given in [AMR19, Proposition 2.5]. In particular, it is shown that, for all $\boldsymbol{n}$ and $\boldsymbol{d}$, and for all integer $k \geq 1$, the Segre-Veronese embedding $\sigma\nu_{\boldsymbol{m}}^{\boldsymbol{d}}$ is globally k-osculating of k-osculating dimension

$$m_k = \begin{cases} \sum_{i=1}^{k} \sum_{|\boldsymbol{s}|=i\,,\,\boldsymbol{s}\leq\boldsymbol{d}} \binom{n_1+s_1-1}{s_1} \cdots \binom{n_r+s_r-1}{s_r} & \text{if } 1 \leq k \leq |\boldsymbol{d}| \\ \prod_{\ell=1}^{r} \binom{n_\ell+d_\ell}{d_\ell} - 1 & \text{if } k > |\boldsymbol{d}|. \end{cases} \tag{2.5}$$

The following is an immediate corollary of Proposition 2.7 and also of [DR99, Theorem 4.2, Proposition 4.5]. It characterizes the k-regularity of Segre-Veronese embeddings. For additional references, see [MR19, CGG02, BCGI07, BF03].

Corollary 2.10. *Let $\boldsymbol{m}$ and $\boldsymbol{d}$ be two r-tuples of positive integers. The Segre-Veronese embedding $\sigma\nu_{\boldsymbol{m}}^{\boldsymbol{d}}$ in (2.4) is k-regular if and only if $k \leq \min \boldsymbol{d}$.*

3. Higher-order polar classes

The definition of polar classes is classical in Algebraic Geometry, going back to Severi [Sev02] and Todd [Tod37]. A generalization to higher-order polar classes has been recently given by Piene in [Pie22].

With notation as above, assume that the morphism $f\colon X \to \mathbb{P}^n$ is globally k-osculating, and thus a closed embedding. Then the image $\operatorname{im} j_k$ and the kernel $\ker j_k$ of the morphism j_k in (2.1) are also vector bundles over X of ranks m_k+1 and $n-m_k$ respectively. We then obtain the following short exact sequence of vector bundles over X:

$$0 \to \ker j_k \hookrightarrow H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \twoheadrightarrow \operatorname{im} j_k \to 0\,. \tag{3.1}$$

Definition 3.1. Let $f\colon X \to \mathbb{P}^n$ be a globally k-osculating morphism and let $0 \leq i \leq \dim X = m$ be an integer. The *ith polar class of f of order k* is the Chern class:

$$p_{k,i}(f) = c_i(\operatorname{im} j_k)\,. \tag{3.2}$$

The *ith polar degree of f of order k* is $\mu_{k,i}(f) \coloneqq \deg p_{k,i}(f)$.

In [Pie22, Section 3], Piene defines the higher-order polar loci

$$P_{k,i}(f,L) \coloneqq \overline{\{x \in X_k^\circ \mid \dim(\mathbb{T}_p^k(f) \cap L) > m_k - \operatorname{codim}(L)\}} \text{ for all } i \in \{0,\dots,\dim X = m\}, \tag{3.3}$$

where $L \subseteq \mathbb{P}^n$ is a linear subspace of dimension $n-m_k+i-2$, and proves the identity (3.2) in [Pie22, Theorem 3.1], without any smoothness assumption on X nor the assumption of global osculation for f. In our approach, we consider globally k-osculating morphisms $f\colon X \to \mathbb{P}^n$ and adopt the definition given by (3.2) directly. For completeness, in Proposition 3.7 we include a self-contained proof of the equivalence between $p_{k,i}(f)$ and the classes of the loci $P_{k,i}(f,L)$ for a generic L, using slightly different methods under the assumption of global k-osculation.

Definition 3.2. Let $p \in X_k^\circ$ and $H \in (\mathbb{P}^N)^\vee$. Then H is *k-osculating at* $f(p)$ if $H \supset \mathbb{T}_p^k(f)$. The kth *conormal variety* of $f\colon X \to \mathbb{P}^n$ is the Zariski closure $W_k(f) \coloneqq \overline{W_k^\circ(f)} \subseteq \mathbb{P}^n \times (\mathbb{P}^n)^\vee$, where:

$$W_k^\circ(f) \coloneqq \{(x,H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid x = f(p) \text{ for some } p \in X_k^\circ \text{ and } H \text{ is } k\text{-osculating at } x\}\,. \tag{3.4}$$

Remark 3.3. Since $\dim \mathbb{T}_p^k(f) = m_k$ for every $p \in X_k^\circ$, we have

$$\dim W_k(f) = n - 1 + m - m_k\,. \tag{3.5}$$

Let the morphism f be globally k-osculating. Then f is a closed embedding and, using (2.1),

$$W_k^\circ(f) = W_k(f) = \mathbb{P}((\ker j_k)^\vee) \subseteq X \times (\mathbb{P}^n)^\vee\,.$$

In particular, if $k=1$, then $W(f) = W_1(f)$ is the classical conormal variety of dimension $n-1$ in $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$.

Let pr_1 and pr_2 be the projections of $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$ to the factors $\mathbb{P}^n$ and $(\mathbb{P}^n)^\vee$, respectively.

Definition 3.4. [Pie83] The *dual variety of order k* of $f\colon X \to \mathbb{P}^N$ is $X_k^\vee \coloneqq \mathrm{pr}_2(W_k(f))$.

Consider the surjective morphisms $\pi_{1,k}\colon W_k(f) \to X$ and $\pi_{2,k}\colon W_k(f) \to X_k^\vee$ induced by the projections pr_1 and pr_2. More precisely $\pi_{i,k} = \mathrm{pr}_i \circ \iota_k$ for $i = 1, 2$, where $\iota_k\colon W_k(f) \hookrightarrow \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ is the inclusion. Given $H \in X_k^\vee$, the *contact locus of H of order k* is

$$\mathrm{Cont}_k(H, X) \coloneqq \pi_{1,k}(\pi_{2,k}^{-1}(H)) = \overline{\{p \in X_k^\circ \mid H \supset \mathbb{T}_p^k(f)\}}\,. \tag{3.6}$$

For a generic $H \in X_k^\vee$, the dimension of $\mathrm{Cont}_k(H, X)$ is equal to the dimension of a generic fiber of $\pi_{2,k}$. From (3.5) it follows that

$$\dim X_k^\vee = n - (m_k - m + 1) - \dim \mathrm{Cont}_k(H, X) \text{ for a generic hyperplane } H \in X_k^\vee.$$

Generally, one expects that $\dim X_k^\vee = n - (m_k - m + 1)$, or equivalently that the generic contact locus is zero-dimensional.

Definition 3.5. The *dual defect of order k* of the morphism $f\colon X \to \mathbb{P}^n$ is defined as

$$\mathrm{def}_k(f) \coloneqq \dim \mathrm{Cont}_k(H, X) \text{ for a generic } H \in (\mathbb{P}^n)^\vee. \tag{3.7}$$

We say that X is *dual defective of order k* if $\mathrm{def}_k(f) > 0$. Otherwise, X is *dual nondefective of order k*.

Example 3.6. Consider the embedding $f_2\colon X \to \mathbb{P}^6$ in Example 2.8. The conormal variety $W(f)$ has dimension $n - 1 = 5$. We verified that the dual variety $X^\vee \subseteq (\mathbb{P}^6)^\vee$ is a hypersurface of degree 12, in particular $\mathrm{def}(f) = \mathrm{def}_1(f) = 0$. For $k = 2$, we already observed in Example 2.8 that $m_2 = \mathrm{rank}\, A^{(2)} - 1 = 5$ generically, hence the identity (3.5) gives $\dim W_2(f) = 6 - 1 + 2 - 5 = 2$. Furthermore, we verified that $\dim X_2^\vee = 2$ and $\deg X_2^\vee = 6$, in particular $\mathrm{def}_2(f) = 0$. ♢

When the morphism f is globally k-osculating, the defect $\mathrm{def}_k(f)$ is governed by nonvanishing higher-order polar classes.

Proposition 3.7. *Let $f\colon X \to \mathbb{P}^n$ be a globally k-osculating morphism. The following holds.*

(1) *$p_{k,i}(f) = [P_{k,i}(f, L)]$ for a generic subspace $L \subseteq \mathbb{P}^n$ of dimension $n - m_k + i - 2$.*
(2) *$p_{k,i}(f) = 0$ for $i > m - \mathrm{def}_k(f)$.*
(3) *$p_{k,m-\mathrm{def}_k(f)}(f) \neq 0$ and $\deg X_k^\vee = \mu_{k,m-\mathrm{def}_k(f)}(f)$.*

Proof. Recall that $\mathrm{im}\, j_k$ is globally generated by the sections of $\mathcal{O}_X^{\oplus(n+1)} = X \times H^0(X, \mathcal{O}_X(1))$ via the vector bundle map j_k, and $\mathrm{rank}\,\mathrm{im}\, j_k = m_k + 1$. Using [GH78, §III.3], for every integer $i \geq 0$, the ith Chern class $c_i(\mathrm{im}\, j_k)$ is the rational class $[D(j_k(\sigma_1), \ldots, j_k(\sigma_{m_k-i+2}))]$ of the degeneracy locus

$$D(j_k(\sigma_1), \ldots, j_k(\sigma_{m_k-i+2})) \coloneqq \{p \in X \mid j_k(\sigma_1)(p) \wedge \cdots \wedge j_k(\sigma_{m_k-i+2})(p) = 0\}$$

for the generic global sections $j_k(\sigma_1), \ldots, j_k(\sigma_{m_k-i+2}) \in H^0(X, \mathrm{im}\, j_k)$. This is equivalent to requiring that $\dim(\mathbb{T}_p^k(f) \cap L) > i - 2 = m_k - \mathrm{codim}(L)$ for a generic linear space L of codimension $m_k - i + 2$.

Consider the kth dual variety $X_k^\vee$ introduced in Definition 3.4, in particular

$$X_k^\vee = \overline{\{H \in (\mathbb{P}^n)^\vee \mid \mathbb{T}_p^k(f) \subseteq H \text{ for some } p \in X_k^\circ\}}\,.$$

Note that a generic linear subspace L of dimension $\mathrm{codim}(X_k^\vee) - 1 = m_k - m + \mathrm{def}_k(f)$ does not meet $X_k^\vee$ and hence $j_k(\sigma) \neq 0$ for a generic linear combination $\sigma = \sum_{i=1}^{m_k-m+\mathrm{def}_k(f)+1} \lambda_i \sigma_i$. From what proven above it follows that $p_{k,m-\mathrm{def}_k(f)+1}(f) = c_{m-\mathrm{def}_k(f)+1}(\mathrm{im}\, j_k) = 0$. Similarly one sees that $p_{k,m-\mathrm{def}_k(f)}(f) = c_{m-\mathrm{def}_k(f)}(\mathrm{im}\, j_k) \neq 0$ and that $\mu_{k,m-\mathrm{def}_k(f)}(f) = \deg X_k^\vee$. □

Let H and H' be hyperplanes in $\mathbb{P}^n$ and $(\mathbb{P}^n)^\vee$, respectively, and let $[H]$ and $[H']$ be the corresponding generating classes in the Chow rings $A^*(\mathbb{P}^n)$ and $A^*((\mathbb{P}^n)^\vee)$. We use the shorthand

$$h = \mathrm{pr}_1^*([H]) = [H \times (\mathbb{P}^n)^\vee]\,, \quad h' = \mathrm{pr}_2^*([H']) = [\mathbb{P}^n \times H']\,. \tag{3.8}$$

Recalling (3.5), the class $[W_k(f)] \in A^*(\mathbb{P}^n \times (\mathbb{P}^n)^\vee)$ can be written as

$$[W_k(f)] = \sum_{i=0}^{n-m_k+m-1} \delta_{k,i}(f) h^{n-i}(h')^{m_k-m+1+i}\,. \tag{3.9}$$

The numbers $\delta_{k,i}(f)$ are the multidegrees of $W_k(f)$. Given $0 \le i \le n-m_k+m-1$, let $L_1 \subseteq \mathbb{P}^n$ and $L_2 \subset (\mathbb{P}^n)^\vee$ be generic subspaces of dimensions $n-i$ and $m_k-m+1+i$, respectively. In particular $[L_1] = [H]^i$ and $[L_2] = [H']^{n-m_k+m-1-i}$. Then $\delta_{k,i}(f)$ is the cardinality of the finite intersection $W_k(f) \cap (L_1 \times L_2)$, or equivalently

$$\delta_{k,i}(f) = \int \mathrm{pr}_1^*([L_1]) \cdot \mathrm{pr}_2^*([L_2]) \cdot [W_k(f)] = \int h^i (h')^{n-m_k+m-1-i}[W_k(f)]\,. \tag{3.10}$$

We can now prove a higher-order version of the celebrated Kleiman duality formula (see [Kle86, Prop. (3), p. 187]). In particular, we show a correspondence between the kth order polar degrees $\mu_{k,j}(f)$ and the multidegrees $\delta_{k,j}(f)$ of the rational equivalence class $[W_k(f)]$.

Proposition 3.8. *Consider a morphism $f\colon X \to \mathbb{P}^n$, that is globally k-osculating of k-osculating dimension $m_k < n$. For all $0 \le j \le \dim X = m$, we have $\mu_{k,j}(f) = \delta_{k,m-j}(f)$.*

Proof. Since the morphism f is k-osculating of k-osculating dimension m_k, in particular X is nonsingular and the variety $W_k(f)$ has the structure of a projective bundle over X, hence it is also a nonsingular variety. Consider the surjective morphisms $\pi_{k,1}\colon W_k(f) \to X$, $\pi_{k,2}\colon W_k(f) \to X_k^\vee$. Let $L \subseteq \mathbb{P}^n$ be a generic projective subspace of dimension $n - m_k + j - 2$. Then $L_k^\vee = L^\vee$ and $\dim L_k^\vee = n - 1 - \dim L = m_k - j + 1$. In particular $[L_k^\vee] = [H']^{n-1-m_k+j}$, using the notation in (3.8). Following [Pie22, Proposition 3.4] a similar approach in [DRGS25, Lemma 3.24], one can show that

$$P_{k,j}(f) = \pi_{k,1}(\pi_{k,2}^{-1}(L_k^\vee))\,. \tag{3.11}$$

Let $\iota_k\colon W_k(f) \hookrightarrow \mathbb{P}^n \times (\mathbb{P}^n)^\vee$ be the inclusion. Then

$$\begin{aligned}
\mu_{k,j}(f) &= \deg P_{k,j}(f) = \int [H]^{m-j} \cdot [P_{k,j}(f)] \\
&\overset{(*)}{=} \int [H]^{m-j} \cdot [\pi_{1,k}(\pi_{2,k}^{-1}(L_k^\vee))] = \int [H]^{m-j} \cdot \pi_{1,k*}(\pi_{2,k}^*([L_k^\vee])) \\
&= \int [H]^{m-j} \cdot \pi_{1,k*}(\pi_{2,k}^*([H']^{n-1-m_k+j})) \overset{(**)}{=} \int \pi_{1,k*}(\pi_{1,k}^*([H]^{m-j}) \cdot \pi_{2,k}^*([H']^{n-1-m_k+j})) \\
&= \int \pi_{1,k*}(\pi_{1,k}^*([H])^{m-j} \cdot \pi_{2,k}^*([H'])^{n-1-m_k+j}) = \int \pi_{1,k*}(\iota_k^*(h^{m-j}) \cdot \iota_k^*((h')^{n-1-m_k+j})) \\
&= \int \pi_{1,k*}(\iota_k^*(h^{m-j} \cdot (h')^{n-1-m_k+j})) = \int \pi_{1,k*}(h^{m-j} \cdot (h')^{n-1-m_k+j} \cdot [W_k(f)]) \\
&\overset{(***)}{=} \int h^{m-j} \cdot (h')^{n-1-m_k+j} \cdot [W_k(f)] = \delta_{k,m-j}(f)\,.
\end{aligned}$$

In $(*)$, we applied (3.11). In $(**)$, we applied the general projection formula in [Ful98, Theorem 3.2]. In $(***)$, we used the fact that $h^{m-j}(h')^{n-1-m_k+j}[W_k(f)]$ is a zero-dimensional cycle, so the degree is preserved under projection. □

4. Higher-order distance loci

In this section, we introduce a *higher-order distance degree*, [DHO+16]. Furthermore, we define the *higher-order distance loci.* These concepts and their geometry strongly depend on the choice of an underlying metric. Our ambient space is an $(n+1)$-dimensional vector space V, and $\{x_0,\dots,x_n\}$ are coordinates in V. To introduce a notion of "normality", we fix a nonsingular quadric hypersurface $Q=\mathbb{V}(q)\subseteq\mathbb{P}(V)\cong\mathbb{P}^n$, where $q\in\operatorname{Sym}^2 V^*$. The hypersurface Q is the *isotropic quadric* associated with q. A standard choice is to consider $q(x)=q_{\mathrm{ED}}(x)=\sum_{i=0}^n x_i^2$, but there is no need to restrict to the standard choice. The quadric Q induces the following *polarity* or *reciprocity* map,

$$\begin{array}{cccc} \partial_q\colon & \mathbb{P}^n & \longrightarrow & (\mathbb{P}^n)^\vee \\ & p=[p_0:\cdots:p_n] & \longmapsto & \left[\frac{\partial q}{\partial x_0}(p):\cdots:\frac{\partial q}{\partial x_n}(p)\right]. \end{array} \tag{4.1}$$

Let $L\subseteq\mathbb{P}^n$ be a subspace. We define

$$L^\perp \coloneqq \partial_q(L)^\vee=\left\{y\in\mathbb{P}^n \;\middle|\; \sum_{i=0}^n \frac{\partial q}{\partial x_i}(p)y_i=0 \text{ for every } p\in L\right\}. \tag{4.2}$$

Observe that $\mathbb{P}(L)^\perp=\mathbb{P}(L^\perp)$ considering the product $\langle u,v\rangle_Q=\sum_{i=0}^n\frac{\partial q}{\partial x_i}(u)v_i=0$. Observe also that if $\dim L=r$, then $\dim L^\perp=n-r-1$. In particular, if $p=[p_0:\cdots:p_n]\in\mathbb{P}^n$, then $p^\perp$ is the polar hyperplane $H_p\subseteq\mathbb{P}^n$ of equation $\sum_{i=0}^n\frac{\partial q}{\partial x_i}(p)x_i=0$. Furthermore $(\mathbb{P}^n)^\perp=\emptyset$.

Definition 4.1. [Pie22] Consider a morphism $f\colon X\to\mathbb{P}^n$, a nonsingular quadric hypersurface $Q\subseteq\mathbb{P}^n$, and let $p\in X_k^\circ$. The *kth order normal space of* (f,Q) *at* p is

$$\mathbb{N}_p^k(f,Q)\coloneqq\langle(\mathbb{T}_p^k(f))^\perp,f(p)\rangle\subseteq\mathbb{P}^n.$$

When $k=1$, we call it the *normal space of* (f,Q) *at* p and we denote it by $\mathbb{N}_p(f,Q)$.

In particular, if $f\colon X\to\mathbb{P}^n$ is globally 1-osculating, then the normal space of (f,Q) at p is $\mathbb{N}_p(f,Q)=\langle\mathbb{T}_p(f)^\perp,p\rangle$.

Remark 4.2. Given a point $p\in\mathbb{P}^n\setminus Q$ and the inclusion $\iota\colon\{p\}\hookrightarrow\mathbb{P}^n$, the normal space of (ι,Q) is $\mathbb{N}_p(\iota,Q)=\langle H_p,p\rangle=\mathbb{P}^n$. Furthermore, when $f=\mathrm{id}\colon\mathbb{P}^n\to\mathbb{P}^n$, the normal space of (id,Q) at $p\in\mathbb{P}^n$ is $\mathbb{N}_p(\mathrm{id},Q)=\langle(\mathbb{P}^n)^\perp,p\rangle=\{p\}$. The analog of the sequence (2.2) for higher-order normal spaces is

$$\{p\}\subseteq\mathbb{N}_p^k(f,Q)\subseteq\cdots\subseteq\mathbb{N}_p^2(f,Q)\subseteq\mathbb{N}_p(f,Q)\subseteq\mathbb{P}^n.$$

Remark 4.3. Observe that, as p varies in X, the dimension of $\mathbb{N}_p^k(f,Q)$ might change. Even if we assume that the map f is k-osculating, the dimension of the span $\langle(\mathbb{T}_p^k(f))^\perp,p\rangle$ drops if $p\in\mathbb{T}_p^k(f)^\perp$, which happens if X intersects Q nontransversally.

Definition 4.4. Consider a morphism $f\colon X\to\mathbb{P}^n$ and a nonsingular quadric hypersurface $Q\subseteq\mathbb{P}^n$. The *kth order distance correspondence of* (f,Q) is the incidence correspondence

$$\mathrm{DC}_k(f,Q)\coloneqq\overline{\left\{([v],[u])\in\mathbb{P}^n\times\mathbb{P}^n\mid[v]=f(p)\text{ for some }p\in X_k^\circ\text{ and }[\nabla d_u(v)]\in\mathbb{N}_p^k(f,Q)\right\}}.$$

The following lemma is proven similarly to the first part of [DHO+16, Theorem 4.1].

Lemma 4.5. *Consider a morphism $f\colon X\to\mathbb{P}^n$ and a nonsingular quadric hypersurface $Q\subseteq\mathbb{P}^n$. Assume that X is irreducible of dimension m and that $f(X)\not\subseteq Q$. Then $\mathrm{DC}_k(f,Q)$ is irreducible of dimension $m+n-m_k$ in $\mathbb{P}^n\times\mathbb{P}^n$. The projection of $\mathrm{DC}_k(f,Q)$ onto the first factor $\mathbb{P}^n$ is locally trivial over $f(X_k^\circ)\setminus Q$ with fibers of dimension $n-m_k+1$.*

Definition 4.6. Consider a morphism $f\colon X \to \mathbb{P}^n$ and a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$. The *kth order distance locus of* (f, Q) is

$$\mathrm{DL}_k(f,Q) \coloneqq \mathrm{pr}_2(\mathrm{DC}_k(f,Q)) = \overline{\bigcup_{p\in X_k^\circ} \mathbb{N}_p^k(f,Q)}\,, \tag{4.3}$$

where pr_2 is the projection of $\mathbb{P}^n \times \mathbb{P}^n$ onto the second factor. The *distance loci sequence* is:

$$\mathrm{DL}_k(f,Q) \subseteq \mathrm{DL}_{k-1}(f,Q) \subseteq \cdots \subseteq \mathrm{DL}_1(f,Q) = \mathbb{P}^n\,.$$

In the following, we denote by $\varphi_{1,k}\colon \mathrm{DC}_k(f,Q) \to f(X)$ and $\varphi_{2,k}\colon \mathrm{DC}_k(f,Q) \to \mathrm{DL}_k(f,Q)$ the surjective morphisms induced by the projections pr_1 and pr_2 of $\mathbb{P}^n \times \mathbb{P}^n$ onto its factors.

Definition 4.7. Consider a morphism $f\colon X \to \mathbb{P}^n$ and a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$. Assume that the morphism $\varphi_{2,k}$ is generically finite. The *kth order distance degree of* (f, Q) is

$$\mathrm{DD}_k(f,Q) \coloneqq \deg \mathrm{DL}_k(f,Q) \cdot \deg \varphi_{2,k}\,,$$

where $\deg \varphi_{2,k} = \deg \varphi_{2,k}^{-1}(u)$ for a generic $u \in \mathrm{DL}_k(f,Q)$.

For $k = 1$ we use the shorthand $\mathrm{DC}_1(f,Q) = \mathrm{DC}(f,Q)$, while $\varphi_{1,1} = \varphi_1$ and $\varphi_{2,1} = \varphi_2$. In this case $\dim \mathrm{DC}(f,Q) = n$ and $\mathrm{DL}_1(f,Q) = \mathbb{P}^n$. Hence, the classical ED degree of (f, Q) introduced in [DHO+16] is $\mathrm{DD}_1(f,Q) = \deg \varphi_2$. This is not generally the case for $k \geq 2$, see the running example in the introduction.

We always have the chain of inclusions $f(X) \subseteq \mathrm{DL}_k(f,Q) \subseteq \mathbb{P}^n$. In fact, there always exists an integer $k \geq 1$ such that $f(X) = \mathrm{DL}_k(f,Q)$. The following lemma examines the two extreme cases: when $f(X) = \mathrm{DL}_k(f,Q)$ and when $\mathrm{DL}_k(f,Q) = \mathbb{P}^n$.

Proposition 4.8. *Let f be a globally k-osculating morphism and Q a generic quadric. Then*

1. *There always exists a k for which $f(X) = \mathrm{DL}_k(f,Q)$. Moreover $(X, f) = (\mathbb{P}^m, \nu_m^k)$ is the kth Veronese embedding if and only if $f(X) = \mathrm{DL}_k(f,Q)$ and $m_k = \binom{m+k}{k} - 1$.*
2. *If $\mathrm{DL}_k(f,Q) = \mathbb{P}^n$ for some $k \geq 1$ then $\mathrm{DL}_\ell(f,Q) = \mathbb{P}^n$ for every $\ell \in [k]$.*

Proof. (1) Given a closed embedding $f\colon X \to \mathbb{P}^n$ we have $m_k = n$ for a large enough k. It follows that $\operatorname{rank} \operatorname{im} j_k = n+1$ and thus j_k is an isomorphism of vector bundles, imposing $\operatorname{im} j_k \cong \mathcal{O}_X^{\oplus(n+1)}$. This implies that the morphism $\varphi_{2,k}$ is birational, hence $\mathrm{DC}_k(f,Q) \cong f(X)$ and $\mathrm{DL}_k(f,Q) = f(X)$. If $(X, f) = (\mathbb{P}^m, \nu_m^k)$ is the Veronese embedding, then f is k-regular, namely $\operatorname{im} j_k = \mathcal{J}_k(f) \cong \mathcal{O}_X^{\oplus(m_k+1)}$ with $m_k + 1 = \binom{m+k}{k}$, and the map j_k is an isomorphism. This implies, as above, that $\varphi_{2,k}$ is birational and that its image is $f(X)$. Viceversa if $\mathrm{DL}_k(f,Q) = f(X)$ and $m_k + 1 = \binom{m+k}{k}$, then $\operatorname{im} j_k = \mathcal{J}_k(f) = \mathcal{O}_X^{\oplus(m_k+1)}$ and thus the projectivized jet bundle is a product. Applying [DRS01, Theorem 3.1], then f is either the kth Veronese embedding ν_m^k or a morphism of an abelian variety. The last possibility can be excluded since f is assumed to be a closed embedding.

(2) Assume now that $\mathrm{DL}_k(f,Q) = \mathbb{P}^n$. Then $m_k = m_{k-1} = \cdots = m$, the morphism $\varphi_{2,k} = \cdots = \varphi_{2,1}$ is finite, and $\mathrm{DL}_k(f,Q) = \cdots = \mathrm{DL}_1(f,Q) = \mathbb{P}^n$. □

Definition 4.9. Consider a morphism $f\colon X \to \mathbb{P}^n$ and a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$. The pair (f, Q) is *in general k-osculating position* if f is globally k-osculating and $f(X)$ intersects Q transversally.

If (f, Q) is in general k-osculating position, then $\dim \mathbb{N}_p^k(f,Q)$ is constant at all points, in particular for every $p \in X$

$$\dim \mathbb{N}_p^k(f,Q) = \dim \mathbb{T}_p^k(f)^\perp + 1 = n - \dim \mathbb{T}_p^k(f) = n - m_k\,. \tag{4.4}$$

In the following, we recall the construction of higher-order Euclidean normal bundles, see [Pie22, Section 4]. Recall that $V = H^0(X, \mathcal{O}_X(1))$. The quadratic form q induces an isomorphism

$$\varphi_q \colon H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \longrightarrow H^0(X, \mathcal{O}_X(1))^\vee \otimes \mathcal{O}_X .$$

In the following, we assume that f is globally k-osculating of k-osculating dimension m_k, in the sense of Definition 2.2. Consider the k-jet exact sequence

$$0 \to \ker j_k \hookrightarrow H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \twoheadrightarrow \operatorname{im} j_k \to 0 . \tag{4.5}$$

From the inclusion $\ker j_k \hookrightarrow H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X$ and the isomorphism φ_q we get a quotient $H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \twoheadrightarrow (\ker j_k)^\vee$. Considering also the point map $H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{O}_X(1)$, we get a surjection

$$H^0(X, \mathcal{O}_X(1)) \otimes \mathcal{O}_X \twoheadrightarrow (\ker j_k)^\vee \oplus \mathcal{O}_X(1) . \tag{4.6}$$

Definition 4.10. Consider a morphism $f \colon X \to \mathbb{P}^n$ and a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$. Assume that (f, Q) is in general k-osculating position. The *kth order normal bundle* of (f, Q) is the image $\mathcal{E}_k(f, Q) \coloneqq (\ker j_k)^\vee \oplus \mathcal{O}_X(1)$ of the surjection in (4.6).

Following notation as in [Ful98], we consider the projectivized bundle $\mathbb{P}(\mathcal{E})$ of a vector bundle $\mathcal{E}$ on X as the projective bundle defined by the quotient spaces of the fibers of $\mathcal{E}$. This means that $\mathbb{P}(\mathcal{E})_x = \mathbb{P}(\mathcal{E}_x^\vee)$ for all $x \in X$. Additionally, let $\mathcal{O}_{\mathcal{E}}(1)$ be the tautological line bundle of $\mathcal{E}$ on $\mathbb{P}(\mathcal{E})$.

The key result of this section gives a closed formula in terms of higher-order polar classes, generalizing [DHO+16, Theorem 5.4]. The formula is inspired by the geometrical construction of higher-order polar loci in [Pie22, Section 5].

Theorem 4.11. *Let $f \colon X \to \mathbb{P}^n$ be a morphism and $Q \subseteq \mathbb{P}^n$ a nonsingular quadric hypersurface. If (f, Q) is in general k-osculating position, then the morphism $\varphi_{2,k}$ is generically finite over its image and*

$$\mathrm{DD}_k(f, Q) = \sum_{i=0}^{m} \mu_{k,i}(f) . \tag{4.7}$$

where $\mu_{k,i}(f)$ are the degrees of the kth polar classes of f. In particular, $\mathrm{DD}_k(f, Q)$ is independent of Q, and we refer to it as the generic kth order distance degree *of f, denoted by* $\mathrm{gDD}_k(f)$.

Proof. Consider a morphism $f \colon X \to \mathbb{P}^n$ and a nonsingular quadric hypersurface $Q \subseteq \mathbb{P}^n$ such that (f, Q) is in general k-osculating position. Let $\mathcal{E}_k(f, Q)$ be the kth order normal bundle of (f, Q) introduced in Definition 4.10. Then $\mathcal{E}_k(f, Q)$ is a vector bundle on X of rank $n - m_k + 1$. The tautological line bundle $\mathcal{O}_{\mathcal{E}_k(f,Q)^\vee}(1)$, whose sections are given by $H^0(X, (\ker j_k)^\vee \oplus \mathcal{O}_X(1))$, is globally generated and hence defines a morphism on $\mathbb{P}(\mathcal{E}_k(f, Q)^\vee)$. The fact that (f, Q) is in general k-osculating position implies that $\mathbb{P}(\mathcal{E}_k(f, Q))_p = \mathbb{N}_p^k(f, Q)$ for every $p \in X$, hence $\mathrm{DC}_k(f, Q) = \mathbb{P}(\mathcal{E}_k(f, Q)^\vee)$, and the morphism defined by its global sections coincides with $\varphi_{2,k}$.

Since $\mathcal{O}_{\mathcal{E}_k(f,Q)^\vee}(1)$ is globally generated, it is said to be "big" if its top self-intersection is positive. Recall that, for a vector bundle $\mathcal{E}$ on X of rank r, if $\pi \colon \mathbb{P}(\mathcal{E}) \to X$ denotes the projection map, then $H^0(\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathcal{E}}(1)) = H^0(X, \mathcal{E}^\vee)$ and $c_1(\mathcal{O}_{\mathcal{E}}(1))^{r-1+i} = s_i(\mathcal{E})$, where $s_i(\mathcal{E})$ is the ith Segre class of $\mathcal{E}$, see [Ful98]. In the following, we denote by $s(\mathcal{E})$ the total Segre class of $\mathcal{E}$. These facts and equation (4.4) yield the relation $c_1(\mathcal{O}_{\mathcal{E}_k(f,Q)^\vee}(1))^{m+n-m_k} = s_m(\mathcal{E}_k(f, Q)^\vee)$. It follows that

$$s(\mathcal{E}^k(f, Q)^\vee) = s((\ker j_k) \oplus \mathcal{O}_X(-1)) = s(\ker j_k) s(\mathcal{O}_X(-1)) = c(\ker j_k)^{-1} c(\mathcal{O}_X(-1))^{-1} .$$

From the k-jet exact sequence (4.5) we derive the identity $c(\ker j_k)^{-1} = c(\operatorname{im} j_k)$, hence

$$c(\ker j_k)^{-1} c(\mathcal{O}_X(-1))^{-1} = c(\operatorname{im} j_k) \sum_{i=0}^{m} c_1(\mathcal{O}_X(1))^i \tag{4.8}$$

which implies that

$$c_1(\mathcal{O}_{\mathcal{E}_k(f,Q)^\vee}(1))^{m+n-m_k} = s_m(\mathcal{E}_k(f,Q)^\vee) = \sum_{i=0}^{m} c_i(\operatorname{im} j_k)\cdot c_1(\mathcal{O}_X(1))^{m-i}\,. \tag{4.9}$$

The hypothesis of global k-osculation implies that $\mathcal{O}_X(1)$ is very ample. Hence, the top self intersection $c_1(\mathcal{O}_X(1))^m$ is always positive. For the same reason, the other terms $c_i(\operatorname{im} j_k)\cdot c_1(\mathcal{O}_X(1))^{m-i} = 0$ are equal to zero if $c_i(\operatorname{im} j_k) = 0$, otherwise they are positive. Therefore, the top self intersection $c_1(\mathcal{O}_{\mathcal{E}_k(f,Q)^\vee}(1))^{m+n-m_k}$ of the tautological line bundle $\mathcal{O}_{\mathcal{E}_k(f,Q)^\vee}(1)$ is always positive. For this reason, we conclude that the morphism $\varphi_{2,k}$ is generically finite over its image and

$$c_1(\mathcal{O}_{\mathcal{E}_k(f,Q)^\vee}(1))^{m+n-m_k} = \deg \operatorname{im}\varphi_{2,k}\cdot \deg \varphi_{2,k}^{-1}(u) \tag{4.10}$$

for a generic $u \in \operatorname{im}\varphi_{2,k} = \mathrm{DL}_k(f,Q)$. The right-hand side of (4.10) is equal to $\mathrm{DD}_k(f,Q)$ by Definition 4.7. The formula in (4.7) follows from (4.8) and (4.9). □

We conclude this section with the first application of Theorem 4.11 to Veronese embeddings. The coming result generalizes the formula in [DHO+16, Proposition 7.10] for $k=1$.

Proposition 4.12. *Consider the Veronese embedding $\nu_m^d\colon \mathbb{P}^m \hookrightarrow \mathbb{P}^{\binom{m+d}{d}-1}$. Then for every $k\in[d]$*

$$\mathrm{gDD}_k(\nu_m^d) = \sum_{i=0}^{m}\binom{\binom{m+k}{k}}{i}(d-k)^i d^{m-i}\,. \tag{4.11}$$

Proof. The generic kth order distance degree of ν_m^d is, by Theorem 4.11, equal to

$$\mathrm{gDD}_k(\nu_m^d) = \sum_{i=0}^{m}\mu_{k,i}(\nu_m^d) = \sum_{i=0}^{m}\int_X c_i(\operatorname{im} j_k)\,. \tag{4.12}$$

Since ν_m^d is k-regular for all $k\in[d]$, we have $\operatorname{im} j_k = \mathcal{J}_k(\nu_m^d) = \mathcal{O}_{\mathbb{P}^m}(d-k)^{\oplus\binom{m+k}{k}}$. Hence, letting $h = c_1(\mathcal{O}_{\mathbb{P}^m}(1))$, we have $c(\mathcal{J}_k(\nu_m^d)) = (1+(d-k)h)^{\binom{m+k}{k}} = \sum_{i=0}^m \binom{\binom{m+k}{k}}{i}(d-k)^i h^i$, in particular

$$\int_X c_i(\mathcal{J}_k(\nu_m^d)) = \int_X \binom{\binom{m+k}{k}}{i}(d-k)^i h^i\cdot (dh)^{m-i} = \binom{\binom{m+k}{k}}{i}(d-k)^i d^{m-i}\,,$$

giving the desired identity. □

5. Osculating eigenvectors of symmetric tensors

We start by setting some preliminary notation. Given a vector $v=(v_0,\dots,v_m)\in\mathbb{C}^{m+1}$, we use the shorthand v^α for the product $v_0^{\alpha_0}\cdots v_m^{\alpha_m}$ for all $\alpha=(\alpha_0,\dots,\alpha_m)\in\mathbb{N}^{m+1}$. Furthermore we define $|\alpha| = \alpha_0+\cdots+\alpha_m$. For all $k\in\mathbb{N}$ we define the vector $v^k := (v^\alpha)_{\alpha\in\mathbb{N}^{m+1},|\alpha|=k}\in\mathbb{C}^{\binom{m+k}{k}}$.

A symmetric tensor of order d over $\mathbb{C}^{m+1}$ is an element of the tensor space $\mathrm{Sym}^d(\mathbb{C}^{m+1})^*$, which we identify with the space $\mathbb{C}[x_0,\dots,x_m]_d$ of homogeneous polynomials of degree d in $m+1$ variables. In particular, we write an element $f\in\mathbb{C}[x_0,\dots,x_m]_d$ as

$$f(x_0,\dots,x_m) = \sum_{|\alpha|=d}\binom{d}{\alpha} f_\alpha x^\alpha\,, \tag{5.1}$$

where $\binom{d}{\alpha} = \frac{d!}{\alpha_0!\cdots\alpha_m!}$. In particular, we identify f with the vector $(f_\alpha)_{|\alpha|=d}\in\mathbb{C}^{\binom{m+d}{d}}$. The monomials $(x^\alpha)_{|\alpha|=d}$ form a basis for the Veronese embedding ν_m^d.

The geometry induced by a specific quadric hypersurface may differ from the generic behavior. In the following, we analyze the geometry of the Bombieri-Weyl metric.

Definition 5.1. Given $f = (f_\alpha)_{|\alpha|=d}$ and $g = (g_\alpha)_{|\alpha|=d}$ with real coordinates, the Bombieri-Weyl inner product between f and g is

$$\langle f, g\rangle_{\mathrm{BW}} \coloneqq \sum_{|\alpha|=d} \binom{d}{\alpha} f_\alpha g_\alpha\,.$$

In particular, the Bombieri-Weyl norm of $f = (f_\alpha)_{|\alpha|=d}$ is the square root of the quadratic form $q_{\mathrm{BW}}(f) \coloneqq \sum_{|\alpha|=d}\binom{d}{\alpha} f_\alpha^2$, while $\sqrt{q_{\mathrm{BW}}(f-g)}$ is the Bombieri-Weyl distance between the polynomials f and g. In this section, we aim to study the higher-order critical points of the Bombieri-Weyl distance minimization problem

$$\min_{\ell\in(\mathbb{R}^{n+1})^*} q_{\mathrm{BW}}(f-\ell^d)\,,\quad f\in\mathbb{R}[x_0,\dots,x_m]_d\,. \tag{5.2}$$

As for the general case, we relax the assumptions that f and ℓ have real coefficients; therefore, we study the higher-order critical points of the complex-valued polynomial function $q_{\mathrm{BW}}(f-\ell^d)$. Let $q(x) = \sum_{i=0}^m x_i^2$ be the standard Euclidean quadratic form in $(\mathbb{R}^{m+1})^*$ and $S_q^m = \mathbb{V}(q(x)-1) \subseteq \mathbb{R}^{m+1}$ the m-dimensional unit sphere associated with q. Consider the map

$$S_q^m\times\mathbb{R} \longrightarrow \mathbb{R}[x_0,\dots,x_m]_d\,,\quad (v,\lambda)\mapsto \lambda\langle v,x\rangle^d\,. \tag{5.3}$$

Then the subset $\{\ell^d \mid \ell\in(\mathbb{C}^{m+1})^*\}\subseteq\mathbb{C}[x_0,\dots,x_m]_d$ is the Zariski closure of the image of the map in (5.3). More precisely, it equals the affine cone over the Veronese variety $V_m^d\subseteq\mathbb{P}(\mathbb{C}[x_0,\dots,x_m]_d)$ defined in (2.4). Furthermore, two important identities coming from the definition of the Bombieri-Weyl quadratic form are

$$\begin{gathered} q_{\mathrm{BW}}(\langle v,x\rangle^d) = q(v)^d \quad \forall\, v\in\mathbb{C}^{m+1}\\ \left\langle \prod_{j=1}^d\langle x,v_j\rangle, \prod_{j=1}^d\langle x,w_j\rangle\right\rangle_{\mathrm{BW}} = \prod_{j=1}^d\langle v_j,w_j\rangle \quad \forall\, v_j,w_j\in\mathbb{C}^{m+1}\,. \end{gathered} \tag{5.4}$$

Remark 5.2. The minimization problem (5.2) can be rephrased as

$$\min_{v\in S_q^m}\min_{\lambda\in\mathbb{R}} q_{\mathrm{BW}}(f-\lambda\,\ell^d)\,,\quad f\in\mathbb{R}[x_0,\dots,x_m]_d\,, \tag{5.5}$$

where $\ell=\ell(v)\coloneqq\langle x,v\rangle\in(\mathbb{C}^{m+1})^*$. Let $\langle\ell^d\rangle$ be the line spanned by ℓ^d. Then the projection of f onto $\langle\ell^d\rangle$ is denoted by $P_{\langle\ell^d\rangle}(f)$. We also consider the orthogonal complement $\langle\ell^d\rangle^\perp$ of $\langle\ell^d\rangle$ with respect to the Bombieri-Weyl inner product. By Pythagorean identity, we have that

$$q_{\mathrm{BW}}(f) = q_{\mathrm{BW}}(P_{\langle\ell^d\rangle}(f)) + q_{\mathrm{BW}}(P_{\langle\ell^d\rangle^\perp}(f))\,. \tag{5.6}$$

Furthermore, we have that $\min_{\lambda\in\mathbb{R}} q_{\mathrm{BW}}(f-\lambda\,\ell^d) = q_{\mathrm{BW}}(P_{\langle\ell^d\rangle^\perp}(f))$. Hence, we can say that the original problem (5.5) is now rephrased as

$$\min_{v\in S_q^m} q_{\mathrm{BW}}(P_{\langle\ell^d\rangle^\perp}(f)) \iff \max_{v\in S_q^m} q_{\mathrm{BW}}(P_{\langle\ell^d\rangle}(f))\,, \tag{5.7}$$

and the equivalence is an immediate consequence of (5.6). The identity (5.4) allows us to write $P_{\langle\ell^d\rangle}(f) = \langle f,\ell^d\rangle_{\mathrm{BW}}\,\ell^d \quad \forall\, v\in S_q^m$, where $q_{\mathrm{BW}}(\ell^d)=1$ thanks to (5.4). But then the second part of (5.7) is in turn equivalent to

$$\max_{v\in S_q^m}\langle f,\ell^d\rangle_{\mathrm{BW}}\,. \tag{5.8}$$

In particular, $\langle f,\ell^d\rangle_{\mathrm{BW}}$ is a polynomial objective function of degree d in the coordinates of v. In short, minimizing the Bombieri-Weyl distance between f and the affine cone CV_m^d is equivalent to maximizing the function $\langle f,\ell^d\rangle_{\mathrm{BW}}$ for $v\in S_q^m$.

Lemma 5.3. *The Veronese embedding $\nu_m^d\colon \mathbb{P}^m \hookrightarrow \mathbb{P}(\mathbb{C}[x_0,\dots,x_m]_d)$ is k-regular for all $k \le d$. Furthermore, for all $\ell \in (\mathbb{C}^{m+1})^* \setminus \{0\}$ and $k \le d$, we have*

$$\mathbb{T}^k_{[\ell^d]}(\nu_m^d) = \{[\ell^{d-k}g] \mid g \in \mathbb{C}[x_0,\dots,x_m]_k\}\,. \tag{5.9}$$

Definition 5.4. Let $f \in \mathbb{C}[x_0,\dots,x_m]_d$ and $k \le d$. A nonzero vector $v \in \mathbb{C}^{m+1}$ is a *normalized eigenvector of f of order k* if $q(v) = 1$ and there exists $\lambda \in \mathbb{C}$ such that

$$\frac{(d-k)!}{d!}\nabla_k f(v) = \lambda\, v^k\,, \tag{5.10}$$

where $\nabla_k f \coloneqq \left(\frac{\partial^k f}{\partial x^\alpha}\right)_{|\alpha|=k} \in \mathbb{C}^{\binom{m+k}{k}}$. The value λ is the *eigenvalue of f of order k* associated with v. The pair (v,λ) is called a *normalized eigenpair* of f of order k.

Remark 5.5. If d and k are odd or d is odd and k is even, then if (v,λ) is a solution of (5.10), then also $(-v,-\lambda)$ is. Instead, if d is even and k is odd or d and k are even, then if (v,λ) is a solution of (5.10), then also $(v,-\lambda)$ is.

Lemma 5.6. *Let $f \in \mathbb{C}[x_0,\dots,x_m]_d$ and $k \le d$. For every $v \in \mathbb{C}^{m+1}$, we define $\ell = \ell(v) \coloneqq \langle x, v\rangle \in (\mathbb{C}^{m+1})^*$. For any $\alpha \in \mathbb{N}^{m+1}$ with $|\alpha| = k$*

$$\langle f, \ell^{d-k}x^\alpha\rangle_{\mathrm{BW}} = \frac{(d-k)!}{d!}\frac{\partial^k f}{\partial x^\alpha}(v)\,.$$

Proof. Writing f as in (5.1) and applying Definition 5.1, we get

$$\begin{aligned}\langle f, \ell^{d-k}x^\alpha\rangle_{\mathrm{BW}} &= \sum_{|\gamma|=d}\sum_{|\beta|=d-k}\binom{d-k}{\beta}\left\langle \binom{d}{\gamma}x^\gamma, x^{\alpha+\beta}\right\rangle_{\mathrm{BW}} f_\gamma v^\beta = \sum_{\substack{|\gamma|=d\\ \gamma-\alpha\ge 0}}\binom{d-k}{\gamma-\alpha}f_\gamma v^{\gamma-\alpha}\\ &= \frac{(d-k)!}{d!}\sum_{\substack{|\gamma|=d\\ \gamma-\alpha\ge 0}}\binom{d}{\gamma-\alpha}f_\gamma v^{\gamma-\alpha} = \frac{(d-k)!}{d!}\sum_{\substack{|\gamma|=d\\ \gamma-\alpha\ge 0}}\binom{d}{\gamma}f_\gamma\left[\prod_{j=0}^m\frac{\gamma_j!}{(\gamma_j-\alpha_j)!}v^{\gamma-\alpha}\right]\\ &= \frac{(d-k)!}{d!}\sum_{\substack{|\gamma|=d\\ \gamma-\alpha\ge 0}}\binom{d}{\gamma}f_\gamma\left[\frac{\partial^k x^\gamma}{\partial x^\alpha}(v)\right] = \frac{(d-k)!}{d!}\frac{\partial^k f}{\partial x^\alpha}(v)\,,\end{aligned}$$

namely the desired identity. □

Proposition 5.7. *Let $f \in \mathbb{C}[x_0,\dots,x_m]_d$ and $k \le d$. For every $v \in \mathbb{C}^{m+1}$, we define $\ell = \ell(v) \coloneqq \langle x, v\rangle \in (\mathbb{C}^{m+1})^*$. A vector $\lambda\,\ell^d$ with $v \in S_q^m$ and $\lambda \in \mathbb{C}\setminus\{0\}$ is a critical point of order k of the function $q_{\mathrm{BW}}(f - \lambda\,\ell^d)$ if and only if (v,λ) is a normalized eigenpair of f of order k.*

Proof. Consider the linear form $\ell(v) = \langle x, v\rangle \in (\mathbb{C}^{m+1})^*$. The tensor $\lambda\,\ell^d$ is critical of order k of $q_{\mathrm{BW}}(f - \lambda\,\ell^d)$ if and only if $\langle f - \lambda\,\ell^d, w\rangle_{\mathrm{BW}} = 0$ for every $w \in \mathbb{T}^k_{[\ell^d]}(\nu_m^d)$. Using Lemma 5.3 and linearity of the inner product, we can simplify this problem by imposing that

$$\langle f - \lambda\,\ell^d, \ell^{d-k}x^\alpha\rangle_{\mathrm{BW}} = 0 \quad \forall\,\alpha \in \mathbb{N}^{m+1}\ \ |\alpha| = k\,.$$

For any $\alpha \in \mathbb{N}^{m+1}$ with $|\alpha| = k$, we compute $\langle f, \ell^{d-k}x^\alpha\rangle_{\mathrm{BW}}$ and $\langle \ell^d, \ell^{d-k}x^\alpha\rangle_{\mathrm{BW}}$. The former is computed in Lemma 5.6. The latter is computed applying Lemma 5.6 replacing f with ℓ^d:

$$\langle \ell^d, \ell^{d-k}x^\alpha\rangle_{\mathrm{BW}} = q(\ell)^{d-k}\langle \ell^k, x^\alpha\rangle_{\mathrm{BW}} = \sum_{|\beta|=k}\binom{d}{\alpha}^{-1} v^\beta \left\langle \binom{d}{\beta}x^\beta, \binom{d}{\alpha}x^\alpha\right\rangle_{\mathrm{BW}}$$

$$= \sum_{|\beta|=k} \binom{d}{\alpha}^{-1} v^\beta \binom{d}{\alpha} \delta_{\beta,\alpha} = v^\alpha\,,$$

where the first and second equalities follow from (5.4) and $\delta_{\beta,\alpha} \neq 0$ if and only if $\beta = \alpha$, in which case $\delta_{\beta,\beta} = 1$. Summing up, the point $\lambda\,\ell^d$ with $v \in S^m_q$ is critical of order k of $q_{\mathrm{BW}}(f - \lambda\,\ell^d)$ if and only if $\frac{(d-k)!}{d!}\frac{\partial^k f}{\partial x^\alpha}(v) = \lambda\, v^\alpha$ for all $\alpha \in \mathbb{N}^{m+1}$ with $|\alpha| = k$, or equivalently, if and only if (v, λ) is a normalized eigenpair of f. $\square$

In Proposition 5.7, it is fundamental to use the Bombieri-Weyl quadratic form q_{BW}. If we replace Q_{BW} with a generic quadric Q, then $\mathrm{DD}_k(\nu^d_m, Q) = \mathrm{gDD}_k(\nu^d_m)$, and the latter is computed in Proposition 4.12. Our next goal is to compute $\mathrm{DD}_k(\nu^d_m, Q_{\mathrm{BW}})$ instead.

Proposition 5.8. *Let ν^d_m be the degree-d Veronese embedding of $\mathbb{P}^m$, and equip $\mathbb{R}[x_0,\dots,x_m]_d$ with the Bombieri-Weyl inner product. For all $k \le d$, the kth order distance degree $\mathrm{DD}_k(\nu^d_m, Q_{\mathrm{BW}})$ equals the coefficient of the monomial $h_1^m h_2^{\binom{m+k}{k}-m-1}$ in the expansion of (1.1). Furthermore, the kth order distance locus of $(\nu^d_m, Q_{\mathrm{BW}})$ is irreducible of dimension $m + \binom{m+d}{d} - \binom{m+k}{k}$.*

Proof. By Proposition 5.7, the kth order critical points of the Bombieri-Weyl distance function from f restricted to $\nu^d_m(\mathbb{P}^m)$ correspond to the kth order eigenvectors of f. Consider the $\binom{m+k}{k} \times 2$ matrix

$$M_k(x, f) \coloneqq \begin{pmatrix} \nabla_k f(x) & x^k \end{pmatrix}\,. \tag{5.11}$$

We consider the pair $([x],[f]) \in Z \coloneqq \mathbb{P}^m \times \mathbb{P}(\mathbb{C}[x_0,\dots,x_m]_d)$. Note that the columns of $M_k(x,f)$ have bidegrees $(k,0)$ and $(d-k,1)$ in the coordinates of x and f, respectively. This means that the transpose of $M_k(x,f)$ defines the fiber $\varphi_{[x],[f]}$ over $([x],[f])$ of the vector bundle morphism

$$\varphi\colon \mathcal{E} \to \mathcal{F}\,,\quad \mathcal{E} \coloneqq \mathcal{O}_Z^{\binom{m+k}{k}}\,,\quad \mathcal{F} \coloneqq \mathcal{O}_Z(k,0) \oplus \mathcal{O}_Z(d-k,1)\,. \tag{5.12}$$

The degeneracy locus $D(\varphi) \coloneqq \{z \in Z \mid \operatorname{rank}\varphi_z \le 1\}$ coincides with the kth order distance correspondence $\mathrm{DC}_k(\nu^d_m, Q_{\mathrm{BW}})$ introduced in Definition 4.4. Observe that $\mathrm{DC}_k(\nu^d_m, Q_{\mathrm{BW}}) \neq \emptyset$ and

$$\dim \mathrm{DC}_k(\nu^d_m, Q_{\mathrm{BW}}) = m + \binom{m+d}{d} - \binom{m+k}{k} = \dim Z - (\operatorname{rank}\mathcal{E} - 1)(\operatorname{rank}\mathcal{F} - 1)$$

coincides with the expected dimension of $D(\varphi)$. Therefore the degree of $D(\varphi)$, or the degree of $\mathrm{DC}_k(\nu^d_m, Q_{\mathrm{BW}})$, can be computed applying Porteous' formula [ACGH85, (4.2), p. 86]:

$$\deg D(\varphi) = (-1)^{\binom{m+k}{k}-1} c_{\binom{m+k}{k}-1}(\mathcal{E} - \mathcal{F})\,, \tag{5.13}$$

where

$$\begin{aligned} c_t(\mathcal{E} - \mathcal{F}) = \frac{c_t(\mathcal{E})}{c_t(\mathcal{F})} &= \frac{1}{(1 + kh_1)(1 + (d-k)h_1 + h_2)} \\ &= \frac{1}{1 + dh_1 + h_2 + k(d-k)h_1^2 + kh_1h_2} \\ &= \sum_{j=0}^{\infty} (-1)^j (dh_1 + h_2 + k(d-k)h_1^2 + kh_1h_2)^j\,. \end{aligned}$$

The kth order distance locus $\mathrm{DL}_k(\nu^d_m, Q_{\mathrm{BW}}) = \varphi_{2,k}(\mathrm{DC}_k(\nu^d_m, Q_{\mathrm{BW}}))$ is irreducible of dimension $m + \binom{m+d}{d} - \binom{m+k}{k}$ by Lemma 4.5, or of codimension $\binom{m+k}{k} - m - 1$. Hence $\mathrm{DD}_k(\nu^d_m, Q_{\mathrm{BW}})$ equals the coefficient of the monomial $h_1^m h_2^{\binom{m+k}{k}-m-1}$ in the previous expansion. $\square$

We determine close formulas for $\mathrm{DD}_k(\nu^d_m, Q_{\mathrm{BW}})$ for small values of m.

Corollary 5.9. *For all $k \leq d$, we have*

$$\begin{aligned}
\mathrm{DD}_k(\nu_1^d, Q_{\mathrm{BW}}) &= k(d-k+1)\\
\mathrm{DD}_k(\nu_2^d, Q_{\mathrm{BW}}) &= \frac{(d-k)^2}{2}\binom{k+2}{k}^2 - \frac{(d-k)(3d-5k)}{2}\binom{k+2}{k} + (d-2k)^2\,.
\end{aligned}$$

Proof. Let $m = 1$ and $k \leq d$. By Proposition 5.8, $\mathrm{DD}_k(\nu_1^d, Q_{\mathrm{BW}})$ is the coefficient of $h_1 h_2^{k-1}$ in the expansion of

$$(-1)^k \sum_{j=0}^{\infty} (-1)^j (dh_1 + h_2 + kh_1h_2)^j \in \frac{\mathbb{Z}[h_1,h_2]}{\langle h_1^2, h_2^{d+1}\rangle}\,. \tag{5.14}$$

Observe that

$$\sum_{j=0}^{\infty} (-1)^j (dh_1 + h_2 + kh_1h_2)^j = \sum_{j=0}^{\infty} (-1)^j \sum_{i_1+i_2+i_3=j} \binom{j}{i_1,i_2,i_3} d^{i_1} k^{i_3} h_1^{i_1+i_3} h_2^{i_2+i_3}\,.$$

To extract the coefficient of $h_1h_2^{k-1}$, we need to impose that $i_1+i_3 = 1$, hence $(i_1, i_3) \in \{(1,0),(0,1)\}$ and $i_2 = j-1$, therefore, from the inner sum we extract the two summands $jdh_1h_2^{j-1} + jkh_1h_2^j$. The coefficient of $h_1h_2^{k-1}$ is obtained selecting $j \in \{k-1, k\}$, hence

$$\mathrm{DD}_k(\nu_1^d, Q_{\mathrm{BW}}) = (-1)^k[(-1)^{k-1}(k-1)k + (-1)^k kd] = k(d-k+1)\,.$$

Now consider $m = 2$ and $k \leq d$. In this case $\mathrm{DD}_k(\nu_2^d, Q_{\mathrm{BW}})$ is the coefficient of $h_1^2 h_2^{\binom{k+2}{2}-3}$ in the expansion of

$$(-1)^{\binom{k+2}{k}-1} \sum_{j=0}^{\infty} (-1)^j (dh_1 + h_2 + k(d-k)h_1^2 + kh_1h_2)^j \in \frac{\mathbb{Z}[h_1,h_2]}{\langle h_1^3, h_2^{\binom{d+2}{d}}\rangle}\,. \tag{5.15}$$

Observe that

$$(dh_1 + h_2 + k(d-k)h_1^2 + kh_1h_2)^j = \sum_{|i|=j} \binom{j}{i} d^{i_1} k^{i_3+i_4} (d-k)^{i_3} h_1^{i_1+2i_3+i_4} h_2^{i_2+i_4}\,.$$

To extract the coefficient of $h_1^2 h_2^{\binom{k+2}{2}-3}$, we need to assume that $i_1 + 2i_3 + i_4 = 2$. This implies that $(i_1, i_3, i_4) \in \{(2,0,0),(1,0,1),(0,1,0),(0,0,2)\}$. Using $|i| = j$, we obtain $i_2 = j - i_1 - i_3 - i_4$ and the four possible summands

$$\frac{j(j-1)}{2} d^2 h_1^2 h_2^{j-2} + j(j-1)dkh_1^2h_2^{j-1} + jk(d-k)h_1^2h_2^{j-1} + \frac{j(j-1)}{2}k^2h_1^2h_2^j\,.$$

Therefore, the coefficient of $h_1^2 h_2^{\binom{k+2}{2}-3}$ is obtained selecting the indices $j \in \{\binom{k+2}{2}-3, \binom{k+2}{2}-2, \binom{k+2}{2}-1\}$. Calling $S := \binom{k+2}{k}$, we obtain the value

$$\begin{aligned}
\mathrm{DD}_k(\nu_2^d, Q_{\mathrm{BW}}) &= \frac{(S-3)(S-4)}{2}k^2 - (S-2)(S-3)dk - (S-2)k(d-k)\\
&\quad + \frac{(S-1)(S-2)}{2}d^2\\
&= \frac{(d-k)^2}{2}S^2 - \frac{(d-k)(3d-5k)}{2}S + (d-2k)^2\,,
\end{aligned} \tag{5.16}$$

where we have already multiplied every summand by the common coefficient $(-1)^{\binom{k+2}{k}-1}$. □

It is interesting to compare the formulas in Corollary 5.9 with

$$\mathrm{gDD}_k(\nu_1^d) = d + (k+1)(d-k)$$
$$\mathrm{gDD}_k(\nu_2^d) = \frac{(d-k)^2}{2}S^2 + \frac{(d+k)(d-k)}{2}S + d^2\,,$$

computed using Proposition 4.12. For example $\mathrm{DD}_2(\nu_2^3, Q_{\mathrm{BW}}) = 22 < 42 = \mathrm{gDD}_2(\nu_2^3)$.

5.1. **Proof of Theorem 1.2.** The next proof computes the degrees $\deg\varphi_{2,k}$ and $\deg \mathrm{DL}_k(\nu_m^d, Q)$, namely the two factors of $\mathrm{DD}_k(f,Q)$, when either $Q = Q_{\mathrm{BW}}$ or (f,Q) is in general k-osculating position. It is a necessary step towards the proof of the main Theorem 1.1.

Proof of Theorem 1.2. By definition, the Veronese embedding $\nu_m^d\colon \mathbb{P}^m \hookrightarrow \mathbb{P}(\mathbb{C}[x_0,\dots,x_m]_d)$ sends the class $[\ell] \in \mathbb{P}^m$ of the linear form $\ell = \langle v, x\rangle$ for some $v = (v_0,\dots,v_m) \in \mathbb{C}^{m+1}$ to $[\ell^d] \in \mathbb{P}(\mathbb{C}[x_0,\dots,x_m]_d)$. For all $[\ell] \in \mathbb{P}^m$, we have the identity

$$\mathbb{T}^k_{[\ell^d]}(\nu_m^d)^\perp = \left\{[f] \in \mathbb{P}(\mathbb{C}[x_0,\dots,x_m]_d) \;\middle|\; \frac{\partial^k f}{\partial x^\alpha}(v) = 0 \text{ for all } \alpha \in \mathbb{N}^{m+1} \text{ with } |\alpha| = k\right\}, \tag{5.17}$$

which is a generalization of a result attributed to Lasker, see [Ott13, §1.2, Proposition 1], that follows by Lemma 5.6.

We consider first the case $(m,k) = (1, d-1)$. Using (5.17), one verifies that $\mathbb{T}^k_{[\ell^d]}(\nu_1^d)^\perp = \{[(\ell^\perp)^{k+1}g] \in \mathbb{P}^d \mid g \in \mathbb{C}[x_0,x_1]_{d-k-1}\}$ for all $k \le d$, where $\ell^\perp = \langle v^\perp, x\rangle$ and $v^\perp = (v_1, -v_0)$. In particular $\mathbb{T}^{d-1}_{[\ell^d]}(\nu_1^d)^\perp = \{[(\ell^\perp)^d]\}$, hence $\mathbb{N}^{d-1}_{[\ell^d]}(\nu_1^d, Q_{\mathrm{BW}})$ is the projective subspace generated by $[\ell^d]$ and $[(\ell^\perp)^d]$. If v is generic, in particular if $v_0^2 + v_1^2 \neq 0$, then $\mathbb{N}^{d-1}_{[\ell^d]}(\nu_1^d, Q_{\mathrm{BW}})$ is a projective line; otherwise, it coincides with the point $[\ell^d]$.

Now consider a generic point $[f] \in \mathrm{DL}_{d-1}(\nu_1^d, Q_{\mathrm{BW}})$, in particular $f = \lambda\ell^d + \mu(\ell^\perp)^d$ for some $(\lambda,\nu) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and $\ell = \langle v, x\rangle$ for some $v = (v_0, v_1) \in \mathbb{C}^2$ such that $v_0^2 + v_1^2 \neq 0$. From the previous considerations, we conclude that $f \in \mathbb{N}^{d-1}_{[\ell^d]}(\nu_1^d, Q_{\mathrm{BW}}) = \mathbb{N}^{d-1}_{[(\ell^\perp)^d]}(\nu_1^d, Q_{\mathrm{BW}})$, implying that $\varphi_{1,d-1}(\varphi_{2,d-1}^{-1}([f])) = \{[\ell^d], [(\ell^\perp)^d]\}$. We conclude that $\deg\varphi_{2,d-1} = 2$. Furthermore, by Corollary 5.9 we have $\mathrm{DD}_{d-1}(\nu_1^d, Q_{\mathrm{BW}}) = 2(d-1)$, hence necessarily $\deg \mathrm{DL}_{d-1}(\nu_1^d, Q_{\mathrm{BW}}) = d-1$.

Now assume that $(m,k) \neq (1, d-1)$. To prove our statement, it is enough to find a point $[f] \in \mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}})$ such that $\varphi_{2,k}^{-1}([f])$ is zero-dimensional, reduced, and of degree one. Indeed, this would imply that $\varphi_{2,k}^{-1}([f])$ consists of a simple point for all $[f]$ in an open dense subset of $\mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}})$, thus yielding the birationality of $\varphi_{2,k}$ over its image $\mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}})$. Thanks to Proposition 5.7, our claim is equivalent to finding a homogeneous polynomial $f \in \mathbb{C}[x_0,\dots,x_m]_d$ with a unique normalized eigenvector of order k. Fix the standard basis vector $e_0 = (1,0,\dots,0) \in \mathbb{C}^{m+1}$, and consider the point $[x_0^d] = [\langle e_0, x\rangle^d] \in \nu_m^d(\mathbb{P}^m)$. Using (5.17), one verifies that

$$\mathbb{T}^k_{[x_0^d]}(\nu_m^d)^\perp = \{[f] \in \mathbb{P}(\mathbb{C}[x_0,\dots,x_m]_d) \mid \deg_{x_0}(f) \le d-k-1\}$$

for all $k \le d-1$, while $\mathbb{T}^d_{[x_0^d]}(\nu_m^d)^\perp = \emptyset$. This means that $\mathbb{N}^d_{[x_0^d]}(\nu_m^d, Q_{\mathrm{BW}}) = \{[x_0^d]\}$, hence the morphism $\varphi_{2,k}$ is birational and $\mathrm{DL}_d(\nu_m^d, Q_{\mathrm{BW}}) = \nu_m^d(\mathbb{P}^m)$. Now, assume $k \le d-1$. A generic element of $\mathbb{N}^k_{[x_0^d]}(\nu_m^d, Q_{\mathrm{BW}})$ is the class of a polynomial of the form $f = x_0^d + \sum_{i=0}^{d-k-1} x_0^{d-k-1-i} g_i$, where $g_i \in \mathbb{C}[x_1,\dots,x_m]_{k+1+i}$ for all $i \in \{0,\dots,d-k-1\}$. In particular, we consider the point $[f] = [x_0^d + g] \in \mathbb{N}^k_{[x_0^d]}(\nu_m^d, Q_{\mathrm{BW}})$ for some generic $g \in \mathbb{C}[x_1,\dots,x_m]_d$. The kth order eigenvectors of f are the vectors $x \in \mathbb{C}^{m+1}$ such that $\operatorname{rank} M_k(x,f) \le 1$, where $M_k(x,f)$ is defined in (5.11). Let

$\widetilde{x} = (x_1, \dots, x_m)$. For our choice of f,

$$M_k(x,f) = \left(\begin{array}{c|ccc|c|ccc|c} \frac{d!}{(d-k)!}x_0^{d-k} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \nabla_k g(\widetilde{x}) \\ x_0^d & x_0^{d-1}x_1 & \cdots & x_0^{d-1}x_m & \cdots & x_0x_1^d & \cdots & x_0x_m^d & \widetilde{x}^k \end{array}\right)^{\mathsf{T}}$$

Observe that the block matrix with rows $\nabla_k g(\widetilde{x})$ and $\widetilde{x}^k$ coincides with $A_k(\widetilde{x}, g)$. Since $g \in \mathbb{C}[x_1, \dots, x_m]_d$ is generic and $(m,k) \neq (1, d-1)$, then $\operatorname{rank} M_k(\widetilde{x}, g) \leq 1$ if and only if $\widetilde{x} = 0$. This implies that the unique solution of $\operatorname{rank} M_k(x,f) \leq 1$, or the unique kth order eigenvector of f, is $x = e_0$. Therefore we have found a point $[f] \in \mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}})$ such that $\varphi_{2,k}^{-1}([f]) = \{([x_0^d], [f])\}$, hence $\varphi_{2,k}$ is birational over its image $\mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}})$. The last part of the statement descends by Proposition 5.8.

Now we prove the last part of the statement. First, let $(m,k) = (1, d-1)$ with $d \geq 1$ and consider a pair (ν_1^d, Q) in general $(d-1)$-osculating position. In this case $\operatorname{im} j_{d-1} = \mathcal{J}_{d-1}(\nu_1^d) = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d}$, hence $\dim\ker j_{d-1} = 1$, in particular $\ker j_{d-1} = \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \in \mathbb{Z}$. To compute a, we use the sequence (3.1) which in this case simplifies to

$$0 \to \mathcal{O}_{\mathbb{P}^1}(a) \to \mathcal{O}_{\mathbb{P}^1}^{\oplus(d+1)} \to \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d} \to 0\,,$$

from which we obtain the identity $0 = c_1(\mathcal{O}_{\mathbb{P}^1}^{\oplus(d+1)}) = c_1(\mathcal{O}_{\mathbb{P}^1}(a)) + c_1(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus d})$, giving $a = -d$. As a consequence $\mathcal{E}_{d-1}(\nu_1^d, Q) = (\ker j_{d-1})^\vee \oplus \mathcal{O}_{\mathbb{P}^1}(d) = \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus 2}$, in particular the tautological line bundle $\mathcal{O}_{\mathcal{E}_{d-1}(\nu_1^d,Q)^\vee}(1)$, whose sections are given by $H^0(X, (\ker j_{d-1})^\vee \oplus \mathcal{O}_X(1)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)^{\oplus 2})$, is very ample, therefore the morphism $\varphi_{2,d-1}\colon \mathbb{P}(\mathcal{E}_{d-1}(\nu_1^d, Q)^\vee) = \mathrm{DC}_{d-1}(\nu_1^d, Q) \to \mathrm{DL}_{d-1}(\nu_1^d, Q)$, induced by $\mathcal{O}_{\mathcal{E}_{d-1}(\nu_1^d,Q)^\vee}(1)$, is birational.

Finally we consider the case $(m,k) \neq (1, d-1)$ with $k \leq d$. Consider the open subset $\mathcal{U} \subseteq \mathbb{P}(\mathbb{C}[x_0, \dots, x_m]_2)$ of nondegenerate quadratic forms on $\mathbb{C}^{m+1}$ and the incidence variety

$$\Sigma := \overline{\left\{([v],[u],[Q]) \mid [Q] \in \mathcal{U} \text{ and } [v] \in \nu_m^d(\mathbb{P}^m) \text{ is critical of order } k \text{ for } d_u^2\right\}}$$

together with the morphism $\psi_{23}\colon \Sigma \to \mathbb{P}^n \times \mathbb{P}(\mathbb{C}[x_0, \dots, x_m]_2)$ induced by the projection of Σ onto the last two components. For any $[Q] \in \mathcal{U}$ and $[u] \in \mathrm{DL}_k(\nu_m^d, Q)$, we have $\psi_{23}^{-1}([u],[Q]) = \varphi_{2,k}^{-1}([u])$. From the first part of the proof we derive that, if $(m,k) \neq (1, d-1)$, then $\psi_{23}^{-1}([u],[Q_{\mathrm{BW}}])$ is a point for a generic $[u] \in \mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}})$, in particular ψ_{23} is birational. As a consequence, if we define

$$\mathcal{V} := \bigcup_{\substack{Q \in \mathcal{U} \\ \varphi_{2,k} \text{ birational}}} \mathrm{DL}_k(\nu_m^d, Q) \times \{[Q]\} \subseteq \mathbb{P}^n \times \mathbb{P}(\mathbb{C}[x_0, \dots, x_m]_2)\,,$$

then $\mathcal{V} \neq \emptyset$ because $\mathrm{DL}_k(\nu_m^d, Q_{\mathrm{BW}}) \times \{[Q_{\mathrm{BW}}]\} \subseteq \mathcal{V}$. Furthermore $\overline{\mathcal{V}} = \psi_{23}(\Sigma)$, or equivalently $\Sigma = \psi_{23}^{-1}(\overline{\mathcal{V}})$. Now consider the projection $\psi_3\colon \Sigma \to \mathbb{P}(\mathbb{C}[x_0, \dots, x_m]_2)$, which is dominant because $\psi_3^{-1}([Q]) \cong \mathrm{DC}_k(\nu_m^d, Q)$ for all $Q \in \mathcal{U}$. Then $\psi_3(\overline{\mathcal{V}}) = \mathbb{P}(\mathbb{C}[x_0, \dots, x_m]_2)$, which is equivalent to say that, for a generic $[Q] \in \mathbb{P}(\mathbb{C}[x_0, \dots, x_m]_2)$, the morphism $\varphi_{2,k}$ is birational. The last formula (1.2) for the degree of the data locus $\mathrm{DL}_k(\nu_m^d, Q)$ is obtained from Proposition 4.12. □

5.2. **Proof of Theorem 1.1.** We are now ready to prove the main result of this section.

Proof of Theorem 1.1. Let $f_0, \dots, f_n$ be $n+1$ generic polynomials in $\mathbb{C}[x_0, \dots, x_m]_d$ and consider a nonnegative integer $k \leq d$. Assume that $n > \binom{m+k}{k} - 1$ and let $f\colon \mathbb{P}^m \to \mathbb{P}^n$ the associated globally k-osculating morphism. Since the polynomials $f_0, \dots, f_n$ are generic, the morphism f can be expressed as the composition $f = \pi \circ \nu_m^d$, where $\nu_m^d\colon \mathbb{P}^m \hookrightarrow \mathbb{P}^{\binom{m+d}{d}-1}$ denotes the Veronese embedding, and $\pi\colon \mathbb{P}^{\binom{m+d}{d}-1} \dashrightarrow \mathbb{P}^n$ is a generic linear projection.

We now establish the following two claims. Let $f\colon X \to \mathbb{P}^n$ be a globally k-osculating morphism of k-osculating dimension m_k, and let $Q \subseteq \mathbb{P}^n$ be a nonsingular quadric hypersurface such that (f, Q) is in general k-osculating position. Suppose $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^r$ is a generic linear projection with center $\Lambda = \mathbb{P}^{n-r-1}$ and $r > m_k$, and define $f' \coloneqq \pi \circ f$. Then:

Claim 1: $\mathrm{gDD}_k(f) = \mathrm{gDD}_k(f')$.

Claim 2: There exists a nonsingular quadric hypersurface $Q' \subseteq \mathbb{P}^r$ such that

$$\deg \mathrm{DL}_k(f, Q) = \deg \mathrm{DL}_k(f', Q')\,.$$

These two assertions imply that $\mathrm{gDD}_k(f) = \mathrm{gDD}_k(\nu_m^d)$ and that there exists a nonsingular quadric $Q' \subseteq \mathbb{P}^{\binom{m+k}{k}-1}$ satisfying $\deg \mathrm{DL}_k(f, Q) = \deg \mathrm{DL}_k(\nu_m^d, Q')$. Consequently, the degrees of the morphisms $\varphi_{2,k}(f, Q)\colon \mathrm{DC}_k(f, Q) \to \mathrm{DL}_k(f, Q)$ and $\varphi_{2,k}(\nu_m^d, Q')\colon \mathrm{DC}_k(\nu_m^d, Q') \to \mathrm{DL}_k(\nu_m^d, Q')$ are equal. The degree of $\varphi_{2,k}(\nu_m^d, Q')$ is one by Theorem 1.2. The final identity follows by (4.11).

The proof of Claim 1 follows from the identity $\mu_{k,j}(f) = \mu_{k,j}(f')$, as established in [Pie22] and [Pie78]. For completeness, we provide a self-contained argument using the notation and methods developed herein.

Let $\mathbb{P}^n \cong \mathbb{P}(V)$ and $\mathbb{P}^r \cong \mathbb{P}(W)$. Let $M\colon V \to W$ be the linear map defining π with center $\Lambda = \mathbb{P}(\ker M) \subseteq \mathbb{P}^n$, which is disjoint from X by genericity. Consider the kth jet morphisms $j_k\colon X \times V \to \mathcal{J}_k(f)$ and $j_k'\colon X \times W \to \mathcal{J}_k(f')$, associated to f and f', respectively. Choose bases $(\sigma_0, \dots, \sigma_n)$ of V and $(\tau_0, \dots, \tau_r)$ of W, and write:

$$f(p) = (\sigma_0(p), \dots, \sigma_n(p)), \quad f'(p) = (\tau_0(p), \dots, \tau_r(p)).$$

The generic kth osculating dimension of f (and similarly f') corresponds to the rank of the matrices:

$$A_p^{(k)}(f) \coloneqq \left(\ j_{k,p}(\sigma_0)^\intercal \ \middle|\ j_{k,p}(\sigma_1)^\intercal \ \middle|\ \cdots \ \middle|\ j_{k,p}(\sigma_n)^\intercal\ \right),$$
$$A_p^{(k)}(f') \coloneqq \left(\ j_{k,p}(\tau_0)^\intercal \ \middle|\ j_{k,p}(\tau_1)^\intercal \ \middle|\ \cdots \ \middle|\ j_{k,p}(\tau_r)^\intercal\ \right).$$

Since π is generic, the columns of $A_p^{(k)}(f')$ are generic linear combinations of those of $A_p^{(k)}(f)$. Hence,

$$\mathrm{rank}(A_p^{(k)}(f')) = \min\{r, m_k\} = m_k,$$

by the assumption $r > m_k$. It follows that $\mathbb{T}_p^k(f') \cong \mathbb{T}_p^k(f)$ for all $p \in X$. The isomorphism is induced by π, in particular

$$\pi(\mathbb{T}_p^k(f)) = \mathbb{T}_p^k(f')\,. \tag{5.18}$$

At the level of vector bundles, the map M induces a surjection $\mathrm{im}\, j_k(f) \twoheadrightarrow \mathrm{im}\, j_k(f')$, which is an isomorphism due to equal rank. Since $\pi|_X\colon X \to \pi(X)$ is an isomorphism, we have $\pi|_X^*(\mathrm{im}\, j_k(f')) \cong \mathrm{im}\, j_k(f)$. Therefore,

$$\begin{aligned}\mu_{k,j}(f) &= \int_X c_j(\mathrm{im}\, j_k(f)) = \int_X c_j(\pi|_X^* \,\mathrm{im}\, j_k(f'))\\ &= \int_X \pi|_X^*(c_j(\mathrm{im}\, j_k(f'))) = \int_{\pi(X)} c_j(\mathrm{im}\, j_k(f')) = \mu_{k,j}(f'),\end{aligned}$$

using the projection formula [Ful98, Theorem 3.2].

It remains to prove Claim 2. Consider the orthogonal decomposition $V = W \oplus (\ker M)$ with respect to the bilinear form induced by Q, identifying the quotient $\mathbb{P}(\mathbb{C}^{n+1}/\ker M)$ with the subspace $\mathbb{P}(W) \subseteq \mathbb{P}^n$. Define Q' as the restriction of Q to $\mathbb{P}(W)$, in other words

$$\langle \bar{v}, \bar{w}\rangle_{Q'} = q'(\bar{v}, \bar{w})) = q(v, w) = \langle v, w\rangle_Q \text{ for all } \bar{v}, \bar{w} \in V/\ker M\,.$$

The quadric Q' remains nonsingular by genericity. Let $(\,)^{\perp}$ and $(\,)^{\perp'}$ denote the orthogonal complements in $\mathbb{P}^n$ and $\mathbb{P}(W)$ associated with Q and Q', respectively. We show that

$$\mathrm{DL}_k(f',Q') \cong \mathrm{DL}_k(f,Q) \cap \mathbb{P}(W)\,. \tag{5.19}$$

Observe that $\mathbb{T}_p^k(f')^{\perp'} = \mathbb{T}_p^k(f)^{\perp} \cap \mathbb{P}(W)$ for every $p \in X$. In fact:

$$\begin{aligned}
[\bar{w}] \in \mathbb{T}_p^k(f')^{\perp'} &\iff \langle \bar{w}, \bar{v}\rangle_{Q'} = 0 \quad \forall\, [\bar{v}] \in \mathbb{T}_p^k(f')\\
&\iff \langle w, v\rangle_{Q} = 0 \quad \forall\, [v] \in \mathbb{T}_p^k(f)\\
&\iff [w] \in \mathbb{T}_p^k(f)^{\perp} \cap \mathbb{P}(W).
\end{aligned}$$

where we used (5.18). Finally, for all $p \in X$, we have:

$$\begin{aligned}
\mathbb{N}_p^k(f',Q') &= \langle \mathbb{T}_p^k(f')^{\perp'}, f'(p)\rangle = \langle \mathbb{T}_p^k(f)^{\perp} \cap \mathbb{P}(W), f'(p)\rangle\\
&\cong \langle \mathbb{T}_p^k(f)^{\perp}, f(p)\rangle \cap \mathbb{P}(W) = \mathbb{N}_p^k(f,Q) \cap \mathbb{P}(W)\,,
\end{aligned}$$

where the isomorphism is induced by π. Using (4.3), we establish the isomorphism (5.19), that yields the identity $\deg \mathrm{DL}_k(f',Q') = \deg \mathrm{DL}_k(f,Q)$. $\square$

We conclude this section with an example highlighting the isomorphism (5.19).

Example 5.10. Consider the morphism $f\colon \mathbb{P}^1 \to \mathbb{P}^4$ defined by sending the point $p = [t_0 : t_1]$ to the class of the column vector

$$\begin{pmatrix}
t_1(t_0^3 + t_0t_1^2 + t_1^3)\\
t_0(t_0 - t_1)^2(t_0 + t_1)\\
-(t_0^2 + t_1^2)(t_0^2 - t_0t_1 - t_1^2)\\
-t_1(t_0 - t_1)(t_0 + t_1)^2\\
-(t_0 + t_1)(t_0^3 - t_0^2t_1 + t_1^3)
\end{pmatrix}.$$

The image $f(\mathbb{P}^1)$ is a rational normal curve of degree 4. Choosing $k = 2$, one verifies that the right kernel of $A_p^{(2)}(f)$ is two-dimensional and is generated by the columns of the matrix

$$\begin{pmatrix}
-(t_0^3 - 3\,t_0^2t_1 + 9\,t_0t_1^2 + t_1^3) & -(t_0^3 - 9\,t_0^2t_1 - 3\,t_0t_1^2 - t_1^3)\\
t_0^2(t_0 + 3t_1) & -t_0^3\\
t_0^3 + 6\,t_0t_1^2 + t_1^3 & -t_1(6\,t_0^2 + 3\,t_0t_1 + t_1^2)\\
-(t_0^3 + 3\,t_0t_1^2 + t_1^3) & 3\,t_0t_1(t_0 + t_1)\\
t_1(3\,t_0^2 - 6\,t_0t_1 - t_1^2) & -t_0(t_0^2 - 6\,t_0t_1 - 3\,t_1^2)
\end{pmatrix}.$$

If $Q = Q_{\mathrm{ED}}$ is the nonsingular quadric threefold in $\mathbb{P}^4$ of equation $\sum_{i=0}^4 x_i^2 = 0$, then the 2nd order normal space $\mathbb{N}_p^2(f, Q_{\mathrm{ED}})$ at p is the projective plane in $\mathbb{P}^4$ spanned by the two column vectors above and by $f(p)$. The polynomial components defining f have been chosen generic enough so that (f, Q_{ED}) is in general 2-osculating position. Applying Theorem 1.1 with $d = 4$ and $k = 2$, we have that $\mathrm{DL}_2(f, Q_{\mathrm{ED}})$ is a threefold of degree $\sum_{i=0}^1 \binom{3}{i} 2^i 4^{1-i} = 4 + 6 = 10$ in $\mathbb{P}^4$.

Now let $\mathbb{P}(W)$ be the hyperplane of equation $x_4 = 0$, and consider the projection $\pi\colon \mathbb{P}^4 \dashrightarrow \mathbb{P}(W)$. In particular the vertex of π is the point $\Lambda = [0:0:0:0:1] \notin f(\mathbb{P}^1)$. Hence, considering the new morphism $f' = \pi \circ f$, the image $f'(\mathbb{P}^1)$ is a nonsingular quartic curve in $\mathbb{P}^3$, isomorphic to $f(\mathbb{P}^1)$ and parametrized by the first four polynomial components of f. One verifies that the right kernel of $A_p^{(2)}(f')$ is one-dimensional and is generated by the column vector

$$\begin{pmatrix}
(t_0^2 + t_1^2)(t_0^4 - 6\,t_0^3t_1 - 10\,t_0^2t_1^2 + 6\,t_0t_1^3 + t_1^4)\\
-t_0^3(t_0 - 8\,t_1)(t_0 + t_1)^2\\
-(t_0^6 - 6\,t_0^5t_1 - 15\,t_0^4t_1^2 - 8\,t_0^3t_1^3 - 3\,t_0^2t_1^4 + 6\,t_0t_1^5 + t_1^6)\\
t_0^2(t_0^4 - 6\,t_0^3t_1 - 9\,t_0^2t_1^2 - 8\,t_0t_1^3 + 6\,t_1^4)
\end{pmatrix}$$

The restriction of Q_{ED} to $\mathbb{P}(W)$ is the nonsingular quadric surface $Q'_{\mathrm{ED}} = \mathbb{V}(\sum_{i=0}^{3} x_i^2)$. Again (f', Q'_{ED}) is in general 2-osculating position, and $\mathrm{DL}_2(f', Q'_{\mathrm{ED}})$ is a surface of degree 10 in $\mathbb{P}(W)$, isomorphic to $\mathrm{DL}_k(f, Q_{\mathrm{ED}}) \cap \mathbb{P}(W)$. The two surfaces are displayed in Figure 5. $\diamondsuit$

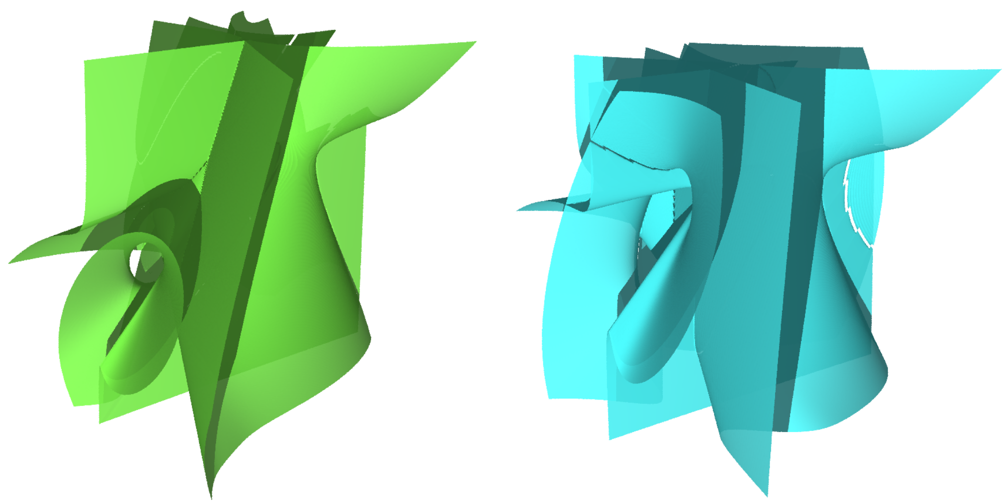

FIGURE 5. The projective surfaces $\mathrm{DL}_2(f, Q_{\mathrm{ED}}) \cap \mathbb{P}(W)$ and $\mathrm{DL}_2(f', Q'_{\mathrm{ED}})$ in the affine chart $\{x_3 = 1\} \subseteq \mathbb{P}^3$.

6. Higher-order distance degrees of regular embeddings

In this subsection, we apply the formula (4.7) to compute the generic kth order distance degree of a k-regular embedding $f\colon X \hookrightarrow \mathbb{P}^n$. Recall that f is defined by the global sections of the line bundle $\mathcal{O}_X(1) = f^*(\mathcal{O}_{\mathbb{P}^n}(1))$. The k-regularity of f implies that $\mu_{k,i}(f)$ equals the degree of $c_i(\mathcal{J}_k(f))$ for all i and k.

In Proposition 4.12, we already computed the generic higher-order distance degree of the Veronese embedding $\nu_m^d\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^{\binom{m+d}{d}-1}$, using that its jet bundles split as a sum of line bundles. More in general, given a k-regular embedding $f\colon X \hookrightarrow \mathbb{P}^n$, the short exact sequences

$$\begin{gathered} 0 \to \Omega_X \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{J}_1(f) \longrightarrow \mathcal{O}_X(1) \to 0 \\ 0 \to \mathrm{Sym}^k \Omega_X \otimes \mathcal{O}_X(1) \longrightarrow \mathcal{J}_k(f) \longrightarrow \mathcal{J}_{k-1}(f) \to 0 \quad \forall\, k \geq 2 \end{gathered} \tag{6.1}$$

allow us to compute the Chern classes of the vector bundles $\mathcal{J}_k(f)$ via a recursive formula, provided that the Chern classes of $\mathrm{Sym}^k\Omega_X \otimes \mathcal{O}_X(1)$ and of $\mathrm{Sym}^k\Omega_X$ are known. Regarding the first ones, the Chern classes of the tensor product between a vector bundle $\mathcal{F}$ of rank r and a line bundle $\mathcal{L}$ on a nonsingular variety X are given by the following formula.

$$c_i(\mathcal{F} \otimes \mathcal{L}) = \sum_{j=0}^{i} \binom{r-j}{i-j} c_j(\mathcal{F}) c_1(\mathcal{L})^{i-j}\,. \tag{6.2}$$

In the following, we write the total Chern classes of $\mathcal{T}_X$ and $\mathcal{O}_X(1)$ as $c(\mathcal{O}_X(1)) = 1 + L$ and $c(\mathcal{T}_X) = c(X) = 1 + \sum_{i=1}^{m} c_i$. Note that $c_i(\Omega_X) = (-1)^i c_i$ for all i. From the first exact sequence in (6.1) we get $c(\mathcal{J}_1(f)) = c(\Omega_X \otimes \mathcal{O}_X(1)) \cdot c(\mathcal{O}_X(1))$. Using Lemma 6.2 we get for all $i \geq 1$

$$c_i(\mathcal{J}_1(f)) = c_i(\Omega_X \otimes \mathcal{O}_X(1)) + c_{i-1}(\Omega_X \otimes \mathcal{O}_X(1)) \cdot L$$

$$\begin{aligned}
&= \sum_{j=0}^{i} \binom{m-j}{i-j} c_j(\Omega_X)\cdot L^{i-j} + \sum_{j=0}^{i-1} \binom{m-j}{i-1-j} c_j(\Omega_X)\cdot L^{i-j}\\
&= c_i(\Omega_X) + \sum_{j=0}^{i-1} \left[\binom{m-j}{i-j} + \binom{m-j}{i-1-j}\right] c_j(\Omega_X)\cdot L^{i-j}\\
&= (-1)^i c_i + \sum_{j=0}^{i-1} \binom{m+1-j}{i-j} (-1)^j c_j \cdot L^{i-j}\\
&= \sum_{j=0}^{i} \binom{m+1-j}{i-j} (-1)^j c_j \cdot L^{i-j}\,,
\end{aligned}$$

which corresponds to [Hol88, §3]. Then, using the second exact sequence in (6.1), we have $c(\mathcal{J}_k(f)) = c(\mathrm{Sym}^k\Omega_X \otimes \mathcal{O}_X(1))\cdot c(\mathcal{J}_{k-1}(f))$, hence

$$\begin{aligned}
c_i(\mathcal{J}_k(f)) &= \sum_{\ell=0}^{i} c_\ell(\mathrm{Sym}^k\Omega_X\otimes\mathcal{O}_X(1))c_{i-\ell}(\mathcal{J}_{k-1}(f))\\
&= \sum_{\ell=0}^{i}\left[\sum_{j=0}^{\ell}\binom{\binom{m+k-1}{k}-j}{\ell-j} c_j(\mathrm{Sym}^k\Omega_X)\cdot L^{\ell-j}\right] c_{i-\ell}(\mathcal{J}_{k-1}(f))\,,
\end{aligned}$$

where we used the fact that $\operatorname{rank}\mathrm{Sym}^k\Omega_X = \binom{m+k-1}{k}$ for all $k\geq 1$. Summing up, we have the following recursive formula:

$$\begin{cases} c_i(\mathcal{J}_1(f)) = \sum_{j=0}^{i}\binom{m+1-j}{i-j}(-1)^j c_j\cdot L^{i-j}\\ c_i(\mathcal{J}_k(f)) = \sum_{\ell=0}^{i}\left[\sum_{j=0}^{\ell}\binom{\binom{m+k-1}{k}-j}{\ell-j} c_j(\mathrm{Sym}^k\Omega_X)\cdot L^{\ell-j}\right] c_{i-\ell}(\mathcal{J}_{k-1}(f))\,. \end{cases} \tag{6.3}$$

We implemented a `Macaulay2` code [GS97] that computes the Chern classes $c_i(\mathcal{J}_k(f))$ for any input k and m using (6.3); see [DRRS25]. Our code computes the generic kth order distance degree of any k-regular embedding $f\colon X\hookrightarrow\mathbb{P}^n$ of a nonsingular projective variety X of dimension m.

In the following, we derive closed formulas for the classes $c_i(\mathcal{J}_k(f))$ for $m\in[3]$, using the following lemma, whose proof is a direct application of the splitting principle [Ful98, Remark 3.2.3]. More involved identities can be derived for larger m.

Lemma 6.1. *Let $\mathcal{F}$ be a vector bundle on a nonsingular surface X. Then for any $k\geq 1$,*

$$c(\mathrm{Sym}^k\mathcal{F}) = \begin{cases} 1 + k\,c_1(\mathcal{F}) & \text{if } \operatorname{rank}\mathcal{F}=1\\ 1+\binom{k+1}{2}c_1(\mathcal{F}) + \binom{k+1}{3}\frac{3k+2}{4}c_1^2(\mathcal{F}) + \binom{k+2}{3}c_2(\mathcal{F}) & \text{if } \operatorname{rank}\mathcal{F}=2\\ 1+\binom{k+2}{3}c_1(\mathcal{F}) + \frac{1}{2}\binom{k+3}{3}\binom{k+1}{3}c_1^2(\mathcal{F}) + \binom{k+3}{4}c_2(\mathcal{F}) & \\ +\binom{k+3}{5}\frac{5k^4+20k^3-5k^2-50k-12}{54}c_1^3(\mathcal{F}) & \\ +\binom{k+3}{5}\frac{5k^2+20k+6}{6}c_1(\mathcal{F})c_2(\mathcal{F}) + \binom{k+3}{4}\frac{2k+3}{5}c_3(\mathcal{F}) & \text{if } \operatorname{rank}\mathcal{F}=3. \end{cases}$$

Proposition 6.2. *Consider a k-regular embedding $f\colon C\hookrightarrow\mathbb{P}^n$ of a nonsingular projective curve C. Write $c(X)=1+c_1$ and $c(\mathcal{O}_X(1))=1+L$. Then*

$$\mathrm{gDD}_k(f) = \int_C (k+2)L - \binom{k+1}{2}c_1\,. \tag{6.4}$$

Proof. We claim that, for all $k \geq 1$,

$$c(\mathcal{J}_k(f)) = 1 + (k+1)L - \binom{k+1}{2} c_1 \,. \tag{6.5}$$

Assuming the claim is true, then

$$\mathrm{gDD}_k(f) = \mu_{k,0}(f) + \mu_{k,1}(f) = \int_C L + c_1(\mathcal{J}_k(f)) = \int_C (k+2)L - \binom{k+1}{2} c_1 \,,$$

which is the desired formula. To prove the claim, consider the recursive formula (6.3). The first equation yields $c(\mathcal{J}_1(f)) = 1 + 2L - c_1$, which is identity (6.5) for $k = 1$. Now assume (6.5) true for $k-1$. Using the second exact sequence in (6.1), we obtain that

$$\begin{aligned} c(\mathcal{J}_k(f)) &= c(\mathrm{Sym}^k\Omega_C \otimes \mathcal{O}_C(1)) c(\mathcal{J}_{k-1}(f)) = (1 + L + c_1(\mathrm{Sym}^k\Omega_C)) \left(1 + kL - \binom{k}{2} c_1\right) \\ &= (1 + L - kc_1)\left(1 + kL - \binom{k}{2} c_1\right) = 1 + (k+1)L - \binom{k+1}{2} c_1 \,, \end{aligned}$$

thus proving the identity (6.5) at the step k. This completes the proof. □

Example 6.3. We apply Proposition 6.2 for the Veronese embedding $\nu_1^d \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$, which is k-regular for any $k \leq d$. Let h denote the class of a point in $\mathbb{P}^1$. In particular $c(\mathbb{P}^1) = (1+h)^2 = 1+2h$ and $L = dh$, hence $\mathrm{gDD}_k(\nu_1^d) = \int_{\mathbb{P}^1} (k+2)dh - \binom{k+1}{2} 2h = (k+2)d - k(k+1)$. ♢

Proposition 6.4. *Consider a k-regular embedding $f \colon S \hookrightarrow \mathbb{P}^n$ of a nonsingular projective surface S. Write $c(S) = 1 + c_1 + c_2$ and $c(\mathcal{O}_S(1)) = 1 + L$. Then*

$$\mathrm{gDD}_k(f) = \int_X \alpha_1 \, L^2 + \alpha_2 \, c_1 L + \alpha_3 \, c_1^2 + \alpha_4 \, c_2 \,, \tag{6.6}$$

where

$$\begin{aligned} \alpha_1 &= 1 + \binom{k+2}{2} + 3\binom{k+3}{4} \,, \quad \alpha_2 = -\binom{k+2}{2}\binom{k+2}{3} \,, \\ \alpha_3 &= \frac{1}{2}\binom{k+1}{3}\binom{k+3}{3} \,, \quad \alpha_4 = \binom{k+3}{4} \,. \end{aligned} \tag{6.7}$$

Proof. We claim that, for all $k \geq 1$,

$$\begin{aligned} c_1(\mathcal{J}_k(f)) &= \binom{k+2}{2} L - \binom{k+2}{3} c_1 \\ c_2(\mathcal{J}_k(f)) &= \frac{1}{2}\binom{k+1}{2}\binom{k+3}{2} L^2 - 2k\binom{k+3}{4} c_1 L + \frac{1}{2}\binom{k+1}{3}\binom{k+3}{3} c_1^2 + \binom{k+3}{4} c_2 \,. \end{aligned} \tag{6.8}$$

Assuming the claim is true, then

$$\begin{aligned} \mathrm{gDD}_k(f) &= \mu_{k,0}(f) + \mu_{k,1}(f) + \mu_{k,2}(f) = \int_X L^2 + c_1(\mathcal{J}_k(f))L + c_2(\mathcal{J}_k(f)) \\ &= \int_X \left[1 + \binom{k+2}{2} + 3\binom{k+3}{4}\right] L^2 - \left[\binom{k+2}{3} + 2k\binom{k+3}{4}\right] c_1 L + \frac{1}{2}\binom{k+1}{3}\binom{k+3}{3} c_1^2 + \binom{k+3}{4} c_2 \,, \end{aligned}$$

which, after simplifying, is equal to the desired formula. To prove the claim, consider the recursive formula (6.3). The first equation yields the identities $c_1(\mathcal{J}_1(f)) = 3L - c_1$ and $c_2(\mathcal{J}_1(f)) = 3L^2 -$

$2c_1L + c_2$, thus proving (6.8) for $k = 1$. Now assume (6.8) true for $k-1$. Observe that $\mathrm{Sym}^k\Omega_S$ has rank $k+1$ for all $k \geq 1$. Using Lemma 6.1, we get the Chern classes

$$c_1(\mathrm{Sym}^k\Omega_S) = -\binom{k+1}{2}c_1\,,\quad c_2(\mathrm{Sym}^k\Omega_S) = \frac{3k+2}{4}\binom{k+1}{3}c_1^2 + \binom{k+2}{3}c_2\,. \tag{6.9}$$

Applying Lemma 6.2, we obtain that

$$\begin{aligned}
c(\mathrm{Sym}^k\Omega_S\otimes\mathcal{O}_S(1)) &= \sum_{i=0}^{k+1} c_i(\mathrm{Sym}^k\Omega_S\otimes\mathcal{O}_S(1)) = \sum_{i=0}^{k+1}\sum_{j=0}^{i}\binom{k+1-j}{i-j}c_j(\mathrm{Sym}^k\Omega_S)\cdot L^{i-j}\\
&= 1 + (k+1)L + c_1(\mathrm{Sym}^k\Omega_S) + \binom{k+1}{2}L^2 + kc_1(\mathrm{Sym}^k\Omega_S)L + c_2(\mathrm{Sym}^k\Omega_S)\\
&= 1 + (k+1)L - \binom{k+1}{2}c_1 + \binom{k+1}{2}L^2 - k\binom{k+1}{2}c_1L + \frac{3k+2}{4}\binom{k+1}{3}c_1^2 + \binom{k+2}{3}c_2\,.
\end{aligned}$$

Using the second exact sequence in (6.1) and the induction step, we obtain that

$$\begin{aligned}
c(\mathcal{J}_k(f)) &= c(\mathrm{Sym}^k\Omega_S\otimes\mathcal{O}_S(1))\cdot c(\mathcal{J}_{k-1}(f))\\
&= \left(1 + (k+1)L - \binom{k+1}{2}c_1 + \binom{k+1}{2}L^2 - k\binom{k+1}{2}c_1L + \frac{3k+2}{4}\binom{k+1}{3}c_1^2 + \binom{k+2}{3}c_2\right)\\
&\quad\cdot\left(1 + \binom{k+1}{2}L - \binom{k+1}{3}c_1 + \frac{1}{2}\binom{k}{2}\binom{k+2}{2}L^2 - 2(k-1)\binom{k+2}{4}c_1L + \frac{1}{2}\binom{k}{3}\binom{k+2}{3}c_1^2 + \binom{k+2}{4}c_2\right)\,.
\end{aligned}$$

Expanding the product, one verifies that

$$\begin{aligned}
c_1(\mathcal{J}_k(f)) &= \left[(k+1) + \binom{k+1}{2}\right]L - \left[\binom{k+1}{2} + \binom{k+1}{3}\right]c_1 = \binom{k+2}{2}L - \binom{k+2}{3}c_1\,,\\
c_2(\mathcal{J}_k(f)) &= \left[(k+2)\binom{k+1}{2} + \frac{1}{2}\binom{k}{2}\binom{k+2}{2}\right]L^2\\
&\quad - \left[k\binom{k+1}{2} + \binom{k+1}{2}^2 + (k+1)\binom{k+1}{3} + 2(k-1)\binom{k+2}{4}\right]c_1L\\
&\quad + \left[\frac{3k+2}{4}\binom{k+1}{3} + \binom{k+1}{2}\binom{k+1}{3} + \frac{1}{2}\binom{k}{3}\binom{k+2}{3}\right]c_1^2 + \left[\binom{k+2}{3} + \binom{k+2}{4}\right]c_2\,,
\end{aligned}$$

which, after simplifying, correspond to the identities in (6.5) at the step k. □

We also compute the generic kth order distance degree of any k-regular embedding $f\colon X\hookrightarrow\mathbb{P}^n$ of a nonsingular projective threefold X. Its proof uses again the recursive formula (6.3), Lemma 6.1, and is similar to Proposition 6.4; therefore, we omit it.

Proposition 6.5. *Consider a k-regular embedding $f\colon X\hookrightarrow\mathbb{P}^n$ of a nonsingular projective threefold X. Write $c(X) = 1 + c_1 + c_2 + c_3$ and $c(\mathcal{O}_X(1)) = 1 + L$. Then*

$$\mathrm{gDD}_k(f) = \int_X \beta_1\,L^3 + \beta_2\,c_1L^2 + \beta_3\,c_1^2L + \beta_4\,c_2L + \beta_5\,c_1^3 + \beta_6\,c_1c_2 + \beta_7\,c_3\,, \tag{6.10}$$

where

$$
\begin{aligned}
\beta_1 &= \frac{(k+4)(k^2+2\,k+3)(k^6+12\,k^5+58\,k^4+138\,k^3+157\,k^2+66\,k+216)}{1296}\\
\beta_2 &= -\binom{k+3}{4}\frac{k^6+12\,k^5+58\,k^4+138\,k^3+157\,k^2+66\,k+72}{72}\\
\beta_3 &= \binom{k+3}{5}\frac{k(k^2+6\,k+11)(5\,k^3+35\,k^2+90\,k+72)}{288}\\
\beta_4 &= \binom{k+4}{5}\frac{k(k^2+6\,k+11)}{6}\\
\beta_5 &= -\binom{k+3}{5}\frac{5\,k^7+65\,k^6+355\,k^5+931\,k^4+816\,k^3-1404\,k^2-3312\,k-1152}{3456}\\
\beta_6 &= -\binom{k+4}{6}\frac{k^3+7\,k^2+18\,k+8}{4}\\
\beta_7 &= -\binom{k+4}{5}\frac{k+2}{3}\,.
\end{aligned}
\tag{6.11}
$$

Example 6.6. We apply Propositions 6.4 and 6.5 for the Segre-Veronese embedding $\nu_{\boldsymbol{m}}^{\boldsymbol{d}}$ defined in (2.4) and with $\boldsymbol{m}=(1,\dots,1)\in\mathbb{N}^r$ and $r\in\{2,3\}$. Recall that $\nu_{\boldsymbol{m}}^{\boldsymbol{d}}$ is k-regular if and only if $k\le\min\boldsymbol{d}$ by Corollary 2.10. Let h_i be the class of a point in the ith factor of $\mathbb{P}^{\boldsymbol{m}}=(\mathbb{P}^1)^{\times r}$. In particular $L=\sum_{i=1}^r d_ih_i$ and $c(\mathbb{P}^{\boldsymbol{m}})=\prod_{i=1}^r(1+h_i)^2=\sum_{j=0}^r 2^je_j(h_1,\dots,h_r)$, where e_j is the jth elementary symmetric polynomial in its arguments, with $e_0:=1$.

For $r=2$, the relations needed are $L^2=2d_1d_2h_1h_2$, $c_1L=2(d_1+d_2)h_1h_2$, $c_1^2=8h_1h_2$, and $c_2=4h_1h_2$. Plugging these relations in (6.6) and simplifying, one verifies that

$$
\mathrm{gDD}_k(\nu_{\boldsymbol{m}}^{\boldsymbol{d}})=\frac{(k^2+k+2)(k^2+5k+8)}{4}d_1d_2-2\binom{k+2}{3}\binom{k+2}{2}(d_1+d_2)+\frac{4}{3}(2k^2+1)\binom{k+3}{4}
$$

for any $k\le\min\{d_1,d_2\}$. Instead for $r=3$, using (6.10), one verifies that

$$
\mathrm{gDD}_k(\nu_{\boldsymbol{m}}^{\boldsymbol{d}})=2\,[3\,\beta_1\,e_3(\boldsymbol{d})+2\,\beta_2\,e_2(\boldsymbol{d})+2(2\,\beta_3+\beta_4)e_1(\boldsymbol{d})+24\,\beta_5+12\,\beta_6+4\,\beta_7]
\tag{6.12}
$$

for any $k\le\min\{d_1,d_2,d_3\}$. For example, if $k=2\le\min\{d_1,d_2,d_3\}$, then $\mathrm{gDD}_2(\nu_{\boldsymbol{m}}^{\boldsymbol{d}})=8(132\,e_3(\boldsymbol{d})-115\,e_2(\boldsymbol{d})+108\,e_1(\boldsymbol{d})-110)$. ◊

Example 6.7. Let $f\colon X\hookrightarrow\mathbb{P}^n$ be an embedded nonsingular complete intersection variety of dimension m. Let $d=(d_1,\dots,d_{n-m})$ be the vector of degrees of the polynomials defining X. Then

$$
c(X)=\frac{(1+L)^{n+1}}{(1+d_1L)\cdots(1+d_{n-m}L)}=\sum_{i=0}^m\left(\sum_{j=0}^i(-1)^j\binom{n+1}{i-j}e_j(d)\right)L^i\,,
$$

where $e_j(d)$ is the jth elementary symmetric function of d. One can apply Propositions 6.2 and 6.4, or more in general the recursive formula 6.3 to derive a formula for the generic kth order distance degree of f as a polynomial in d. For example, if $m=1$, then

$$
\mathrm{gDD}_k(f)=d_1\cdots d_{n-1}\left[k+2-\binom{k+1}{2}(n+1-d_1-\cdots-d_{n-1})\right]
$$

whenever f is k-regular. ◊

The formulas (6.4), (6.6), and (6.10) have an interesting geometric interpretation in toric geometry. In the following, we adopt the notation used in [DDRP14]. For an m-dimensional toric embedding $f\colon X \hookrightarrow \mathbb{P}^n$ with corresponding lattice polytope $P = P(f) \subseteq \mathbb{R}^m$, setting $L = c_1(\mathcal{O}_X(1))$ one has (see [Dan78, Corollary 11.5])

$$c_i = \sum_{F\subseteq P,\, \mathrm{codim}(F)=i} [F]\,, \quad \int_X c_i L^{m-i} = \sum_{F\subseteq P,\, \mathrm{codim}(F)=i} \mathrm{Vol}(F) \tag{6.13}$$

for every $i \in \{0,\dots,m\}$, where $[F]$ denotes the class of the invariant subvariety of X associated to the face F of P, and $\mathrm{Vol}(F)$ indicates the lattice volume of F measured with respect to the lattice induced by $\mathbb{Z}^m$ in the linear span of F, in particular $\mathrm{Vol}(F)$ is equal to $(\dim F)!$ times the Euclidean volume of F. Since the following formulas deal with toric varieties of dimension $m \leq 3$, we also define

$$\mathcal{V} := |\{\text{vertices of } P\}|\,, \quad \mathcal{E} := \sum_{\xi \text{ edge of } P} \mathrm{Vol}(\xi)\,, \quad \mathcal{F} := \sum_{F \text{ facet of } P} \mathrm{Vol}(F)\,, \quad \mathcal{P} := \mathrm{Vol}(P)\,.$$

Corollary 6.8. *Let $f\colon X \hookrightarrow \mathbb{P}^n$ be a k-regular toric embedding of a nonsingular toric surface X and let $P = P(f)$ be the associated polygon. Then:*

$$\mathrm{gDD}_k(f) = \alpha_1\,\mathcal{P} + \alpha_2\,\mathcal{E} + (\alpha_4 - \alpha_3)\mathcal{V} + 12\,\alpha_3\,, \tag{6.14}$$

where the coefficients $\alpha_i(k)$ are displayed in (6.7).

Proof. Recall that $L = c_1(\mathcal{O}_X(1))$. Firstly, using the relations in (6.13), one obtains that $\int_X L^2 = \mathcal{P}$, $\int_X c_1 L = \mathcal{E}$, and $\int_X c_2 = \mathcal{V}$. Secondly, Noether's Formula $\chi(\mathcal{O}_X) = (K_X^2 + \chi(X))/12$ [GH78, IV,§1] gives $\int_X c_1^2 = \int_X K_X^2 = 12\chi(\mathcal{O}_X) - \chi(X) = 12\chi(\mathcal{O}_X) - \mathcal{V}$. Applying [Dan78, Corollary 7.4], a consequence of Demazure vanishing for toric varieties [Dem70, Proposition 6, p. 564], we have $\chi(\mathcal{O}_X) = 1$. Hence $\int_X c_1^2 = 12 - \mathcal{V}$. The statement descends by (6.6) after plugging in the relations obtained before. $\square$

Example 6.9. For $k = 1$, the formula (6.14) simplifies to $\mathrm{gDD}(f) = 7\,\mathcal{P} - 3\,\mathcal{E} + \mathcal{V}$, which agrees with [DHO+16, Corollary 5.11] when $m = 2$. We also point out that, for $k = 2$, the formula (6.14) simplifies to $\mathrm{gDD}_2(f) = 22\,\mathcal{P} - 24\,\mathcal{E} + 60$, in particular it does not depend on $\mathcal{V}$. $\diamondsuit$

Let $f\colon X \hookrightarrow \mathbb{P}^n$ be the toric embedding of a nonsingular toric threefold with associated lattice polytope P. Denote by $K_X = c_1(\bigwedge^3 \Omega_X)$ the canonical divisor of X. Recall the notation $L = c_1(\mathcal{O}_X(1))$. If the divisor $K_X + L$ is nef [Laz04, Definition 1.4.1], then the corresponding polytope is $P^\circ := \mathrm{conv}(\mathrm{int}(P) \cap \mathbb{Z}^3)$. We also define

$$\mathcal{F}_1 := \sum_{F \text{ facet of } P^\circ} \mathrm{Vol}(F)\,, \quad \mathcal{P}_1 := \mathrm{Vol}(P^\circ)\,.$$

We conclude this section with a classification of k-regular toric embeddings of nonsingular toric threefolds, with their respective generic 2nd order distance degrees.

Proposition 6.10. *Let $f\colon X \hookrightarrow \mathbb{P}^n$ be a k-regular toric embedding of a nonsingular toric threefold. Then one of the following possibilities occurs:*

(1) *$k = 1$ and $\mathrm{gDD}(f) = \mathrm{gDD}_1(f) = 15\mathcal{P} - 7\mathcal{F} + 1/40\mathcal{E} - \mathcal{V}$*
(2) *$k \in \{2,3\}$ and $(X,f) = (\mathbb{P}^3, \nu_3^d)$ with $d \in \{2,3\}$. In this case*

$$\mathrm{gDD}_k(f) = \begin{cases} 8 & \text{if } k=2, d=2 \\ 370 & \text{if } k=2, d=3. \\ 27 & \text{if } k=3, d=3. \end{cases} \tag{6.15}$$

(3) *$k = 2$ and $X = \mathbb{P}(\mathcal{E})$ for $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c)$ with $a \geq b \geq c \geq 2$ and f is the embedding defined by $2\,\xi_{\mathcal{E}}$. In this case*

$$\mathrm{gDD}_2(f) = 162(a+b+c) - 154\,. \tag{6.16}$$

(4) *Otherwise*

$$\mathrm{gDD}_2(f) = \beta_1\,\mathcal{P} + \beta_2\,\mathcal{P}_1 - \beta_3\,\mathcal{F} + \beta_4\,\mathcal{F}_1 + \beta_5\,\mathcal{E} - \beta_6\,\mathcal{V} - \beta_7\,. \tag{6.17}$$

Proof. Following the list of exceptions in [DRHNP13, A1], we see that the exceptions of $K_X + L$ being nef have $k = 1$ or they are as in cases (1) and (2). In particular, simple blow-ups at a fixed point and Cayley sums have $k = 1$ as they contain linear spaces of degree one.

In Case (1), the value $\mathrm{gDD}_2(f)$ is computed applying Proposition 6.5 with $k = 1$, giving $\mathrm{gDD}(f) = \mathrm{gDD}_1(f) = \int_X 15L^3 - 7c_1L^2 + 3c_2L - c_3$. The desired formula is obtained considering the polytope interpretation of Chern classes as in (6.13).

In Case (2), the value of $\mathrm{gDD}_2(f)$ is computed using (4.12).

Case (3) deals with threefolds $\mathbb{P}(\mathcal{E}) = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \oplus \mathcal{O}_{\mathbb{P}^1}(c))$ with $a \geq b \geq c \geq 2$ and the 2-regular embedding $f\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^{4a+4b+4c+5}$ defined by $\mathcal{O}_{\mathcal{E}}(2)$. Recall that $c_1 = \mathcal{O}_{\mathcal{E}}(3) - (a+b+c-2)F$ and $c_2 = 3c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))^2 - 2(a+b+c-3)c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))F$, where $F \cong \mathbb{P}^2$ is the divisor class of the fiber of the projection map $\pi\colon \mathbb{P}(\mathcal{E}) \to \mathbb{P}^1$. It follows that $\int_X L^3 = 8(a+b+c)$, $\int_X c_1L^2 = 8(a+b+c+1)$, $\int_X c_1^2L = 6(a+b+c+4)$, $\int_X c_2L = 2(a+b+c+6)$, $\int_X c_1^3 = 54$, $\int_X c_1c_2 = 24$, and $\int_X c_3 = 6$. Plugging these values in the identity (6.10) for $k = 2$, that is

$$\mathrm{gDD}_2(f) = \int_X 176\,L^3 - 230\,c_1L^2 + 81\,c_1^2L - 7\,c_1^3 + 54\,c_2L - 20\,c_1c_2 - 8\,c_3\,, \tag{6.18}$$

we derive the desired identity (6.16).

Finally, we assume that we are not in cases (1) and (2); therefore, the divisor $K_X + L$ is nef. Recall that $L = c_1(\mathcal{O}_X(1))$. Firstly, using the relations in (6.13), one obtains that $\int_X L^3 = \mathcal{P}$, $\int_X c_1L^2 = \mathcal{F}$, $\int_X c_2L = \mathcal{E}$, and $\int_X c_3 = \mathcal{V}$. It remains to compute the degrees of c_1c_2, c_1^2L, and c_1^3. For a nonsingular threefold, the Riemann-Roch theorem gives $\frac{1}{24}\chi(\mathcal{O}_X) = \int_X c_1c_2$. Since $\chi(\mathcal{O}_X) = 1$ (see the proof of Corollary 6.8), we conclude that $\int_X c_1c_2 = 24$. Since $c_1 = -K_X$, it follows that $\mathcal{P}_1 = \int_X (L - c_1)^3$ and $\mathcal{F}_1 = \int_X c_1(L - c_1)^2$, from which we derive the identities $\int_X c_1^3 = 2(\mathcal{P}_1 - \mathcal{P}) + 3(\mathcal{F}_1 + \mathcal{F})$ and $\int_X c_1^2L = \mathcal{P}_1 - \mathcal{P} + \mathcal{F}_1 + 2\mathcal{F}$. Plugging these values in (6.18), we obtain (6.17). □

Example 6.11. We apply (6.17) in a slightly more general version of [DDRP14, Example 3.8]. Given an integer $a \geq 2$, we consider the 2-regular toric threefold given by the Segre-Veronese embedding of $(\mathbb{P}^1)^{\times 3}$ with the line bundle $\mathcal{O}_{(\mathbb{P}^1)^{\times 3}}(a, a, a)$. The lattice polytope associated with the embedding is a cube with edges of length a, hence $\mathcal{P} = 3! \cdot a^3 = 6\,a^3$, $\mathcal{F} = 6 \cdot 2! \cdot a^2 = 12\,a^2$, $\mathcal{E} = 12\,a$, and $\mathcal{V} = 8$. Furthermore P° is also a cube with edges of length $a-2$, hence $\mathcal{P}_1 = 6(a-2)^3$ and $\mathcal{F}_1 = 12(a-2)^2$. Applying (6.17), after simplification we obtain that $\mathrm{gDD}_2(\nu_{\boldsymbol{m}}^{\boldsymbol{d}}) = 1056\,a^3 - 2760\,a^2 + 2592\,a - 880$. The result is confirmed by the formula (6.12) for $k = 2$ and $\boldsymbol{d} = (a, a, a)$. ♢

7. Tropical geometry of distance optimization

In this chapter, we present a new combinatorial framework for studying distance optimization, based on tropical geometry. To enable this analysis, we introduce an additional parameter t to the coefficients of our variety X, and replace our base field of complex numbers $\mathbb{C}$ with the valued field of complex Puiseux series $\mathbb{C}\{\{t\}\}$. The limiting behavior of X as t goes to 0 is captured by the tropicalization $\operatorname{Trop} X$, a polyhedral complex that encodes many of the geometric properties of X. Studying this limiting behavior allows us to characterize both higher distance degrees and the sensitivity of critical points of the distance function with respect to the parameter t.

We begin by fixing some notation. Throughout, $\mathcal{C}$ denotes an algebraically closed field with a nontrivial, non-Archimedean valuation $\nu\colon \mathcal{C}^* \to \mathbb{R}$. We denote by $\mathcal{C}^\circ := \{x \in \mathcal{C} \mid \nu(x) \geq 0\}$ and by

$\mathcal{C}^{\circ\circ} := \{x \in \mathcal{C} \mid \nu(x) > 0\}$ the valuation ring of $\mathcal{C}$ and its unique maximal ideal, respectively. The residue field $\mathcal{C}^\circ/\mathcal{C}^{\circ\circ}$ is denoted by $\widetilde{\mathcal{C}}$.

We recall that a nonzero *formal Puiseux series* is of the form

$$x(t) = \sum_{k=k_0}^{\infty} c_k t^{\frac{k}{N}} \text{ for some } k_0 \in \mathbb{Z},\, N \in \mathbb{N},\, c_k \in \mathbb{C},\, c_{k_0} \neq 0\,.$$

The valuation $\nu(x(t)) = \frac{k_0}{N}$ is defined as the smallest exponent of t in $x(t)$ with nonzero coefficient, it is a measure of the degree to which $x(t)$ depends on t.

From now on, let $f\colon X \to \mathbb{P}^n = \mathbb{P}(\mathcal{C})$ denote the closed embedding of an m-dimensional variety. We denote by $\operatorname{Trop} X$ the tropicalization of the cone over $f(X)$.

$$\operatorname{Trop} X := \overline{\{\nu(x) \;:\; x \in (\mathcal{C}^*)^{n+1} \mid \;[x] \in f(X)\}} \subseteq \mathbb{R}^{n+1},$$

This is the closure of the image under the element-wise valuation map, taken in Euclidean topology.

Recall the notation given in the introduction and at the beginning of Section 4. Consider $q \in \operatorname{Sym}^2 V^*$, the nonsingular quadric hypersurface $Q = \mathbb{V}(q) \subseteq \mathbb{P}(V) = \mathbb{P}^n$, and the squared distance function $d_u^2(x) = q(u-x)$, seen as a function $d_u^2\colon V \to \mathbb{C}$. Throughout most of this section, we assume $Q = Q_{\mathrm{ED}} = \mathbb{V}(x_0^2 + \cdots + x_n^2)$, which is most relevant for Euclidean distance optimization. Note that, identifying the projective space $\mathbb{P}^n$ and its dual $(\mathbb{P}^n)^\vee$ through the reciprocity map (4.1), induces a closed embedding $\mathrm{id} \times \partial_q^{-1}\colon W_k(f) \subseteq \mathbb{P}^n \times \mathbb{P}^n$ of the conormal variety, and denote the embedded variety by $W_k(f,Q) := \mathrm{id} \times \partial_q^{-1}(W_k(f))$. The varieties $W_k(f,Q)$ and $W_k(f,Q_{\mathrm{ED}})$ are related through a linear change of coordinates induced by M_Q. We now show that the tropicalization $\operatorname{Trop} W_k(f,Q)$ contains crucial information on the kth-order polar degrees of f for arbitrary k.

7.1. Proof of Theorem 1.3.

Proof of Theorem 1.3. We can express higher polar degrees of f in terms of the tropicalized conormal variety $\operatorname{Trop} W_k(f,Q)$ purely combinatorially. To this end, we denote for $0 \le h \le n+1$ by M_{n+1}^h the uniform matroid of rank h on $n+1$ elements, and by $\operatorname{Berg}(M_{n+1}^h)$ the corresponding Bergman fan (see [MS15, §4.2], and for several characterizations see [FS05, Proposition 3.6]). We claim that for every $0 \le j \le m+n-m_k-1$, the *jth multidegree $\delta_{k,j}(f)$ of order k* is equal to the degree of the stable intersection $\operatorname{Trop} W_k(f,Q) \cdot \operatorname{Berg}(M_{n+1}^{n-j+1} \times M_{n+1}^{j+m_k-m+2})$.

The tropicalization of $L_1 \times L_2$ is equal to the product of the Bergman fans of the uniform matroids M_{n+1}^{n-j+1} and $M_{n+1}^{j+m_k-m+2}$, respectively [MS15, Example 5.2.7]. As in equation (3.10), the multidegree $\delta_{k,j}(f)$ is equal to the cardinality of the finite intersection $W_k(f,Q) \cap (L_1 \times L_2)$, where $L_1, L_2 \subseteq \mathbb{P}^n$ are generic linear spaces of dimension $n-j$ and $j+m_k-m+1$ respectively. It follows from [OP13, Theorem 5.3.3] that the cardinality of $W_k(f,Q) \cap (L_1 \times L_2)$ is equal to the degree of the stable intersection of $\operatorname{Trop} W_k(f,Q) \cdot \operatorname{Trop}(L_1 \times L_2)$. Alternatively, note that the multidegree $\delta_{k,j}(f)$ is the degree of the cycle $[W_k(f,Q)] \cdot [L_1 \times L_2]$ in the Chow ring of $\mathbb{P}^n \times \mathbb{P}^n$. To show equality with the degree of the tropical cycle $\operatorname{Trop} W_k(f,Q) \cdot \operatorname{Trop}(L_1 \times L_2)$ we employ [FS97, Theorem 3.1], together with [Kat12, Theorem 4.4]. We obtain

$$\delta_{k,j}(f) = \operatorname{Trop} W_k(f,Q) \cdot \operatorname{Berg}(M_{n+1}^{n-j+1} \times M_{n+1}^{j+m_k-m+2}).$$

Finally, the equality

$$\mathrm{gDD}_k(f) = \sum_{j=0}^{m+n-m_k-1} \int \operatorname{Trop} W_k(f,Q) \cdot \operatorname{Berg}(M_{n+1}^{n-j+1} \times M_{n+1}^{j+m_k-m+2})$$

follows from Proposition 3.8 and Theorem 4.11. ◻

Going beyond Theorem 1.3, for $k = 1$ we can not only compute the generic Euclidean distance degree of f, but we can also determine the tropical critical points on X.

Definition 7.1. We call $w \in \mathbb{R}^n$ a *tropical critical point* of X with respect to u, if it holds $w = \nu(x(t))$ for some critical point $x(t)$ of d_u^2 and $[x(t)] \in X(\mathcal{C})$.

Proposition 7.2. *Assume that the pair (f, Q_{ED}) is in general 1-osculating position. Define $Z_u := \{(x, y) \in \mathcal{C}^{n+1} \times \mathcal{C}^{n+1} \mid x + y = u\}$. The set of critical points of X with respect to u is the image of the set*

$$W_1(f, Q_{\mathrm{ED}}) \cap Z_u$$

under the natural projection $\mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ to the first component $\mathbb{C}^{n+1}$.

Proof. It follows from [DHO+16, Lemma 2.8] that for every critical point x we can write the data vector u as a sum $u = x + y$ for some pair (x, y) in the cone over the conormal variety $W_1(f, Q_{\mathrm{ED}})$. This immediately implies the desired statement. $\square$

The above statement has the following tropical analogue. However, we need the slightly stronger genericity assumption that f is in general position under the torus action.

Corollary 7.3. *Let $t \in \mathcal{C}^{n+1}$ be a generic element of the algebraic torus with element-wise vanishing valuation: $\nu(t) = 0$. Then the set of tropical critical points of $t \cdot X$ with respect to u is the image of the stable intersection*

$$\operatorname{Trop} W_1(t \cdot f, Q_{\mathrm{ED}}) \cdot \operatorname{Trop} Z_u$$

under the natural projection $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ to the first component $\mathbb{R}^{n+1}$.

Proof. By Proposition 7.2, it suffices to show the equality

$$\operatorname{Trop} W_1(t \cdot f, Q_{\mathrm{ED}}) \cdot \operatorname{Trop} Z_u = \operatorname{Trop}(W_1(t \cdot f, Q_{\mathrm{ED}}) \cap Z_u)\,.$$

The desired equality follows from [OP13, Theorem 5.3.3]. $\square$

Example 7.4. As a simple example, we consider the rational projective curve

$$f\colon \mathbb{P} \to \mathbb{P}^3\,, \quad [t_0 : t_1] \mapsto [t_0^4 : t_0^2 t_1^2 : t_0^1 t_1^3 : t_1^4]\,,\ \text{defined by } A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{pmatrix}.$$

The projective dual variety is the hypersurface

$$X^\vee = \mathbb{V}(4\,x_1^3 x_2^2 + 27\,x_0 x_2^4 - 16\,x_1^4 x_3 - 144\,x_0 x_1 x_2^2 x_3 + 128\,x_0 x_1^2 x_3^2 - 256\,x_0^2 x_3^3) \subseteq (\mathbb{P}^3)^\vee\,.$$

In Figure 6 we show a green curve $u(t) = (1, t^2, t^3, t^4)$ of data points, where $(t_0, t_1) = (1, t)$, and a red curve $x(t) \in X$ of points with a minimal Euclidean distance to $u(t)$. We display the image of the cone over X under the projection $(x_0, x_1, x_2, x_3) \mapsto (x_1, x_2, x_3)$ in 3 dimensional space. Drawn on logarithmic paper, Figure 6 becomes approximately linear. We show the tropicalization $\operatorname{Trop} X = \operatorname{rowspan} A$ in Figure 7. In this example, the tropical conormal variety $\operatorname{Trop} W_1(f, Q_{\mathrm{ED}})$ is of dimension 4, and the union of the four maximal cones

$$\sigma_1 = \mathbb{R}_+ \cdot e_5 + L,\ \sigma_2 = \mathbb{R}_+ \cdot e_6 + L,\ \sigma_3 = \mathbb{R}_+ \cdot e_7 + L,\ \sigma_4 = \mathbb{R}_+ \cdot e_8 + L,$$

where L denotes the three-dimensional lineality space

$$L = \operatorname{rowspan} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 & 0 & -2 & -3 & -4 \end{pmatrix}.$$

The tropicalization Z_u is a four-dimensional polyhedral fan comprising 80 maximal cones. We check that the stable intersection of $\operatorname{Trop} Z_u$ and $\operatorname{Trop} W_k(f, Q_{\mathrm{ED}})$ contains the point $(0, 2, 3, 4, 0, 4, 3, 2)$, indicating that there exists a critical point $x(t)$ with valuation $\nu(x(t)) = (0, 2, 3, 4)$.

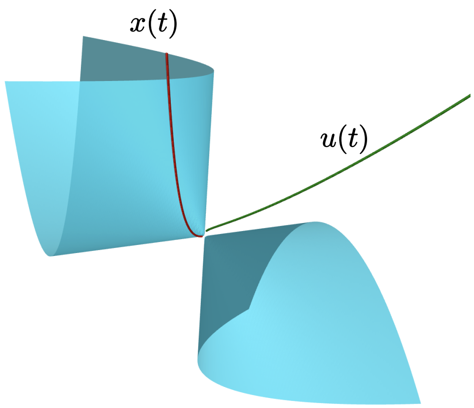

FIGURE 6. A one parameter family of data vectors $u(t)$ and critical points $x(t)$.

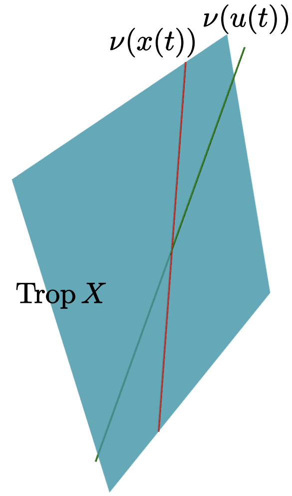

FIGURE 7. The tropical picture.

7.2. **Toric varieties.** In this chapter, we characterize the generic higher-order distance degrees and polar degrees of toric varieties. Similar to [DFS07], we use implicitization techniques to tropicalize higher-order conormal varieties, based on an analogue of the Horn uniformization map.

Recall the notation given in Section 2.1. Let $f\colon X \to \mathbb{P}^n$ be a nonsingular, globally k-osculating, torus-equivariant embedding of a toric variety X of dimension m. Then f is defined by an integer matrix $A \in \mathbb{Z}^{(m+1)\times(n+1)}$ of full rank, containing the all ones vector $\mathbf{1} = (1, \dots, 1)$ in its row span. We denote by $A^{(k)}$ the matrix $A_{\mathbf{1}}^{(k)}(f)$ defined in (2.3). In particular, the kth osculating space of X at $\mathbf{1}$, $\mathbb{T}_{\mathbf{1}}^k(f)$, is equal to the row span of $A^{(k)}$. We again denote by M_Q the symmetric matrix defining Q. The following is analogous to the Horn parametrization of the dual variety $X^\vee$, see, for example, [DFS07, Proposition 4.1]. In fact, the kth order dual variety is the projection of $W_k(f)$ onto the second factor. Our exposition differs in that we allow k to be larger than one.

Proposition 7.5. *The kth order conormal variety $W_k(f, Q_{\mathrm{ED}})$ is the closure of the image of the map*

$$\begin{aligned}\gamma\colon (\mathbb{C}^*)^m \times \mathbb{P}(\ker(A^{(k)})) &\longrightarrow \mathbb{P}^n \times \mathbb{P}^n\\ (t, u) &\longmapsto (t^A \cdot \mathbf{1}, t^{-A} \cdot u).\end{aligned}$$

Proof. Both $\overline{\operatorname{im}\gamma}$ and $W_k(f, Q_{\mathrm{ED}})$ are irreducible varieties of dimension $n - m_k + m - 1$, and we are left with showing the inclusion $\overline{\operatorname{im}\gamma} \subseteq W_k(f, Q_{\mathrm{ED}})$. We first note that, by the construction of $W_k(f)$ we have the inclusion $\{\mathbf{1}\} \times \mathbb{T}_{\mathbf{1}}^k(f)^\perp \subseteq W_k(f, Q_{\mathrm{ED}})$. By [DDRP14, Lemma 5.2] it holds $\mathbb{T}_{\mathbf{1}}^k(f) = \mathbb{P}(\operatorname{rowspan} A^{(k)})$, and one checks directly that, following (4.2), the orthogonal complement $\mathbb{P}(\operatorname{rowspan} A^{(k)})^\perp$ is equal to the kernel $\mathbb{P}(\ker(A^{(k)}))$. Together, we obtain the inclusion

$$\{\mathbf{1}\} \times \mathbb{P}(\ker(A^{(k)})) \subseteq W_k(f, Q_{\mathrm{ED}})$$

Now $W_k(f, Q_{\mathrm{ED}})$ is stable under the $(\mathbb{C}^*)^m$-action $t \cdot (x, T) = (t^A \cdot x, t^{-A} \cdot T)$ for all $(x, T) \in W_k(f, Q_{\mathrm{ED}})$. This finishes the proof. □

Before providing an algorithm for computing higher-order polar degrees, we recall basic results on the tropicalization of certain unirational varieties. Let C and D be integer matrices of size $r \times d$ and $s \times r$ respectively. The rows of D are $w_1, \dots, w_s \in \mathbb{Z}^r$. We denote by λ_C the map defined by C, and by μ_D the monomial map specified by D:

$$\begin{aligned} \lambda_C \colon (\mathcal{C}^*)^d &\dashrightarrow (\mathcal{C}^*)^r \\ v &\longmapsto C\,v \end{aligned} \qquad\qquad \begin{aligned} \mu_D \colon (\mathcal{C}^*)^r &\longrightarrow (\mathcal{C}^*)^s \\ x &\longmapsto (x^{w_1}, \ldots, x^{w_s}). \end{aligned} \tag{7.1}$$

The composition of these maps gives the unirational variety $Y_{C,D} = \overline{\mathrm{im}(\mu_D \circ \lambda_C)}$ in $(\mathcal{C}^*)^s$. Its tropicalization $\mathrm{Trop}\, Y_{C,D}$ is obtained by tropicalizing the map $\mu_D \circ \lambda_C$. The tropical linear space $\mathrm{Trop}(\mathrm{im}\,\lambda_C)$ is computed purely combinatorially, as the *Bergman fan* of the matroid of U. An effective algorithm for computing this object was introduced in [Rin13]. An implementation can be found in `polymake` [GJ00]. The monomial map μ_D tropicalizes to the linear map $V \colon \mathbb{R}^r \to \mathbb{R}^s$. The following result is from [DFS07, Theorem 3.1] and [MS15, Theorem 5.5.1]. Linear projections of balanced polyhedral complexes, and the underlying weights, are described in [MS15, Lemma 3.6.3].

Lemma 7.6. *The tropical variety* $\mathrm{Trop}\, Y_{C,D}$ *is the image, as a balanced fan, of the Bergman fan* $\mathrm{Trop}(\mathrm{im}\,\lambda_C)$ *under the linear map* $\mathbb{R}^r \to \mathbb{R}^s$ *given by* V.

In particular, Lemma 7.6 allows us to compute not only the support of $\mathrm{Trop}\, Y_{C,D}$, but also the multiplicities of its maximal cones [Stu93, Definition 3.4.3].

The following result is a direct consequence of Proposition 7.5 together with Lemma 7.6.

Proposition 7.7. *Fix an* $(n+1) \times (n+1-m_k)$ *matrix* B *whose columns span the kernel of* $A^{(k)}$*, called the* Gale dual *of* $A^{(k)}$*. Then the affine cone over* $W_k(f, Q_{\mathrm{ED}})$ *is* $Y_{C,D}$*, where*

$$U = \begin{pmatrix} 0 & B \\ I_{m+1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} -A^t & 0 \\ A^t & I_{n+1} \end{pmatrix}. \tag{7.2}$$

Here $Y_{C,D} = \overline{\mathrm{im}(\mu_D \circ \lambda_C)}$ *and* λ_C *and* μ_D *are defined as in equation* (7.1) *with* $d = n+m+2-m_k$*,* $r = n+m+2$*, and* $s = 2n+2$*.*

Furthermore, the tropical variety $\mathrm{Trop}\, W_k(f, Q_{\mathrm{ED}})$ *is supported on the image of* $\mathrm{Trop}\, Y_{C,D}$ *in* $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$*. This is the Minkowski sum*

$$\mathrm{Trop}\, W_k(f, Q_{\mathrm{ED}}) = \{0\} \times \mathrm{Trop}(\ker A^{(k)}) + \mathrm{rowspan}\begin{pmatrix} -A & A \end{pmatrix}.$$

Remark 7.8. Compare the description of $\mathrm{Trop}\, W_k(f, Q_{\mathrm{ED}})$ in Proposition 7.7 to the description of the tropicalized dual variety $\mathrm{Trop}\, X^{(k)}$, given in [DFS07, Corollary 4] for $k = 1$, and for general k in [DDRP14, Theorem 5.3]. We recover both results by projecting $\mathrm{Trop}\, W_k(f, Q_{\mathrm{ED}}) \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ onto the second factor.

Proposition 7.7 leads to Algorithm 1 below for computing $\mathrm{Trop}\, W_k(f, Q_{\mathrm{ED}})$. Algorithm 1 returns a list of pairs (m_τ, τ), where $\tau \subseteq \mathbb{R}^{2n}$ is a polyhedral cone, and m_τ is a positive integer. The tropical variety $\mathrm{Trop}\, W_k(f, Q_{\mathrm{ED}})$ is the union of all cones τ, and the multiplicity of $\mathrm{Trop}\, W_k(f, Q_{\mathrm{ED}})$ at a generic point x is the sum $\sum_{x \in \tau} m_\tau$. We note that, although the union of all cones τ forms the support of a fan, the collection of cones itself is generally not a fan. Together with Theorem 1.3, Algorithm 1 allows us to compute higher-order polar degrees of toric varieties. We provide an experimental implementation in the `Julia` package `TropicalImplicitization` [RST25]. It can be found at the supplementary website `https://github.com/kemalrose/TropicalImplicitization.jl`.

Example 7.9. To recover the second order polar degrees $(\mu_{2,0}(\nu_1^3), \mu_{2,1}(\nu_1^3)) = (3, 3)$ of the cubic Veronese embedding $\nu_1^3 \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, defined in coordinates as $[t_1 : t_2] \mapsto [t_1^3 : t_1^2 t_2 : t_1 t_2^2 : t_2^3]$, we download the `Julia` software package `TropicalImplicitization` from the source and we run the following commands:

```
A = [1 1 1 1; 0 1 2 3]
cone_list, weight_list = get_tropical_conormal_variety(A, 2)
extract_polar_degrees(cone_list, weight_list)
```

This allows us to recover the 4-dimensional tropical conormal variety $\operatorname{Trop} W_2(f, Q_{\mathrm{ED}})$ as the union of four maximal cones

$$\sigma_1 = \mathbb{R}_+ \cdot e_5 + L\,, \quad \sigma_2 = \mathbb{R}_+ \cdot e_6 + L\,, \quad \sigma_3 = \mathbb{R}_+ \cdot e_7 + L\,, \quad \sigma_4 = \mathbb{R}_+ \cdot e_8 + L\,.$$

Here L denotes the lineality space. We can also just run the command

```
compute_polar_degrees(A,2)
```

The output then reads:

```
The toric variety is of degree 3.
The generic distance degree of order 2 is 6.
The dual variety of order 2 is of degree 3 and of codimension 2.
The polar degrees of order 2 are [3, 3].
```

Example 7.10. Consider the triples $\boldsymbol{m} = (1,1,1)$ and $\boldsymbol{d} = (1,1,2)$. As a more challenging example, we compute the polar degrees of order 2 of the Segre-Veronese embedding $\nu_{\boldsymbol{m}}^{\boldsymbol{d}} \colon \mathbb{P}^{\boldsymbol{m}} \hookrightarrow \mathbb{P}^{11}$, see (2.4). This toric embedding is defined by the 4×12 matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \end{pmatrix}.$$

We note that $\nu_{\boldsymbol{m}}^{\boldsymbol{d}}$ is 2-osculating, but not 2-regular, thanks to Corollary 2.10. Our software reveals that $(\mu_{2,0}(\nu_{\boldsymbol{m}}^{\boldsymbol{d}}), \ldots, \mu_{2,3}(\nu_{\boldsymbol{m}}^{\boldsymbol{d}})) = (12, 28, 36, 28)$, hence $\mathrm{gDD}_2(f) = \sum_{i \geq 0} \mu_{2,i}(\nu_{\boldsymbol{m}}^{\boldsymbol{d}}) = 88$. Note that this number cannot be computed using (6.12). Based on Proposition 3.7, we see that the second-order dual variety is of dimension 6 and of degree 28.

$$\operatorname{codim}(X_2^\vee) = m_2 - m + \operatorname{def}_2(X) + 1 = 7 - 3 + 0 + 1 = 5.$$

Algorithm 1 is based on Proposition 7.7 together with Lemma 7.6. In particular, $\operatorname{Trop} W_k(f, Q_{\mathrm{ED}})$ is a linear projection of the tropicalized column span of $U = \begin{pmatrix} B & 0 \\ 0 & I_{m+1} \end{pmatrix}$ under $V = \begin{pmatrix} 0 & -A^t \\ I_{n+1} & A^t \end{pmatrix}$.

To describe the underlying weights of $\operatorname{Trop} W_k(f, Q_{\mathrm{ED}})$, let $y \in \operatorname{Trop} W_k(f, Q_{\mathrm{ED}})$ be a generic point inside a top-dimensional cone $\tau \subseteq \operatorname{Trop} W_k(f, Q_{\mathrm{ED}})$. Following [ST08, Theorem 3.12], we can express the multiplicity m_τ of τ as a sum of lattice indices. Here the sum runs over all (finitely many) points x in the tropicalized column span of U, which V maps to y. We denote by σ_x a top-dimensional cone in $\operatorname{Trop}(\operatorname{im} \lambda_C)$ containing x, and by $\mathbb{L}_\tau$ and $\mathbb{L}_\sigma$ the linear span of $\tau - y$ and $\sigma - x$ respectively. The multiplicity m_τ can be expressed as:

$$m_\tau \;=\; \sum_{Vx = y} \operatorname{index}\big(\, \mathbb{L}_\tau \cap \mathbb{Z}^{2n+2} : V(\mathbb{L}_{\sigma_x} \cap \mathbb{Z}^{n+m+2})\,\big).$$

Together, the following algorithm is correct. Compare to the chapter "Tropical Implicitization Revisited" in [DEF+25].

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Algorithm 1: Tropicalizing higher conormal varieties.

Input : An integer matrix $A \in \mathbb{Z}^{(m+1)\times(n+1)}$, of full rank containing the all ones vector $(1, 1, \ldots, 1)$ in its row span.

Output: The tropical variety $\operatorname{Trop} W_k(f, Q_{\mathrm{ED}}) \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$

$B \to$ Gale dual of $A^{(k)}$

$U \to \begin{pmatrix} 0 & B \\ I_{m+1} & 0 \end{pmatrix}$

$V \to \begin{pmatrix} -A^t & 0 \\ A^t & I_{n+1} \end{pmatrix}$

$M \to$ matroid of U

$\operatorname{Trop}(\operatorname{im} \lambda_C) \to$ Bergman fan of M

$\operatorname{Trop} W_k(f, Q_{\mathrm{ED}}) \to \emptyset$

for $(m_\sigma, \sigma) \in \operatorname{Trop}(\operatorname{im} \lambda_C)$ **do**

  $\tau \to V\sigma$

  $m_{\mathrm{lattice}} \to \operatorname{index}(\mathbb{L}_\tau \cap \mathbb{Z}^{2n+2} : V(\mathbb{L}_\sigma \cap \mathbb{Z}^{n+m+2}))$

  $\operatorname{Trop} W_k(f, Q_{\mathrm{ED}}) \to \operatorname{Trop} W_k(f, Q_{\mathrm{ED}}) \cup \{(m_\sigma \cdot m_{\mathrm{lattice}}, \tau)\}$

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