Integral calculus & Fourier series

unit - I

Definite Integrals - Integration By parts and reduction formulae.

unit - II

geometric application of Integration - Area under plane curves: cartesian co-ordinates - Area of a closed curves - Examples - Area in polar co-ordinates.

unit - III

Double Integrals - changing the order of Integrals - triple Integrals.

unit - IV

Beta and Gamma functions and the relation between them - Integration using Beta and Gamma functions.

unit - V

Fourier series - Definition - Fourier series expansion of periodic functions with period 2π - use of odd and even functions in Fourier series - half range Fourier series - Developement in cosine series - Developement in sine series.

TEXT BOOK

1) S. Narayanan and T.K Manica vachagam pillai, calculus volume II

2) S. Narayanan and T.K Manica vachagam pillai, caculus volume III

=) unit - I 66-950

chapter 1 section 11,12,13 of [I]

=) unit - II 112-126

chapter 2 section 1.1, 1.2, 1.3, 1.4 of [I]

=) unit - III 203-222

chapter 5 section 2.1, 2.12 & 2.14 of [I]

=) unit - IV

chapter 7 section 2.1-2.5 of [I] 276-240

=) unit - V

chapter 6 section 1,2,3,4,5.1,5.2 of [II]

In Every street in adirai, there is a house in a street, In which there is a person in the house who study at KMC.

Ans:-

S(x): x is street in adirai.

H(y,x): y is a house in street x

P(z,y): z is a person in house y.

K(z): person z studies at KMC

$\forall x[S(x)] \rightarrow \exists y[H(y,x)] \wedge \exists z[P(z,y) \wedge K(z)]$

Integration

Given $\frac{dy}{dx} = f(x)$. The process of finding y in terms of x is called Integration.

we write symbolically $y(x) = \int f(x)dx$, where $f(x)$ is called integrent and x is called variable of integration.

Here, $\int f(x)dx$ is called indefinite Integral.

Definite Integration

Let $\int f(x)dx = F(x) + c$, where c is an arbitrary constant of integration. The value of integral when $x=a$ is $F(a)+c$ & when $x=b$ is $F(b)+c$

Subracting we get

$F(b) - F(a) =$ value of Integral when $x=b$ − value of Integral when $x=a$

The symbol $\int_a^b f(x)dx$ denotes The value of Integral when $x=b$ minus value of Integral when $x=a$. and Thus is $F(b) - F(a)$. Integral $\int_a^b f(x)dx$ is called The Definite Integral, a and b are called The limits of The integration, a being The lower limit and b being The upper limit

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properties of Definite integral.

1) $\int_a^b f(x)dx = -\int_b^a f(x)dx$. This is obvious

from the definition of a definite integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

Let $\int f(x)dx = F(x)+c$.

Now $\int_a^b f(x)\,dx = F(b) - F(a)$ &

$$\int_b^a f(x)\,dx = F(a) - F(b)$$

L.H.S $\int_a^b f(x)dx = F(b) - F(a)$

$$= -[F(a) - F(b)]$$

$$= -\int_b^a f(x)\,dx = \text{R.H.S}$$

$\therefore \int_a^b f(x)dx = -\int_b^a f(x)dx$ Hence proved

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2). $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where c is some value of x between a and b.

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Let $\int f(x)dx = F(x)+c$

Now $\int_a^b f(x)dx = F(b) - F(a)$

$$\int_a^c f(x)dx = F(c) - F(a) \;\&$$

$$\int_c^b f(x)dx = F(b) - F(c)$$

R.H.S

$$\int_a^c f(x)dx + \int_c^b f(x)dx = \cancel{F(c)} - F(a) + F(b) - \cancel{F(c)}$$

$$= F(b) - F(a)$$

$$= \int_a^b f(x)dx = L.H.S$$

$$\therefore \int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Hence proved

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3) $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$, if $f(x)$ is an <u>even</u> function of x. $f(x) = f(-x)$

$$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_0^a f(x)dx.$$

$$= -\int_0^{-a} f(x)dx + \int_0^a f(x)dx.$$

$\int_0^{-a} f(x)dx$ | $y = -x$, $\frac{dy}{dx} = -1$, $dy = -dx$

$-\int_0^{-a} f(-x)dx$ | $x = 0 \Rightarrow y = 0$

$-\int_0^{a} f(y)dy$ | $x = -a \Rightarrow y = -(-a)$, $y = a$

$$= \int_0^a f(x)dx + \int_0^a f(x)dx.$$

$y = x$

$\frac{dy}{dx} = 1 \Rightarrow dy = dx$

$y = 0 \Rightarrow x = 0$

$y = a \Rightarrow x = a$

$$= 2\int_0^a f(x)dx.$$

$\therefore$ Hence proved.

4) $\int_{-a}^{a} f(x)\,dx = 0$, If $f(x)$ is odd function of x.

$$\int_{-a}^{a} f(x)\,dx = 0$$

L.H.S

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

[$\because$ odd Function
$f(x) = -f(-x)$]

$$= -\int_{0}^{-a} -f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$y = -x$

$\frac{dy}{dx} = -1$

$dy = -dx$

$x = 0 \Rightarrow y = 0$

$x = -a \Rightarrow y = a$

$$= \int_{0}^{-a} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$y = x$

$dy = dx$

$x = 0 \Rightarrow y = 0$

$x = a \Rightarrow y = a$

$$= -\int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 0 = \text{R.H.S.}$$

$\therefore \int_{-a}^{a} f(x)\,dx = 0$, Hence proved.

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5) $\int_{0}^{a} f(x)\,dx = \int_{0}^{a} f(a-x)\,dx$.

In R.H.S $\int_{0}^{a} f(a-x)\,dx$

put $y = a - x$

Diff with respect to 'x'

$$\frac{dy}{dx} = 0 - 1 = -1$$

$$dy = -dx$$

$$x = 0 \Rightarrow y = a - 0$$
$$y = a$$

$$x = a \Rightarrow y = a - a$$
$$y = 0$$

R.H.S

$$\int_0^a f(a-x)\,dx = -\int_a^0 f(y)\,dy$$

$$= \int_0^a f(y)\,dy$$

put $y = x$

$$\frac{dy}{dx} = 1$$

$$dy = dx$$

$y = 0 \Rightarrow x = 0$, $y = a \Rightarrow x = a$

$$= \int_0^a f(x)\,dx = \text{L.H.S}$$

$$\therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx,$$

Hence proved

Example. 1.

prove that $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx.$$

R.H.S

$$\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n(\pi/2 - x)\,dx \quad \left[\text{by } \sin(90^\circ - \theta) = \cos\theta\right]$$

$$\left[\text{by } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi/2} \sin^n x\,dx = \text{L.H.S}$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$$

Hence proved.

Example - 2

$$\int_0^{\pi/2} \frac{(\sin x)^{3/2}}{(\sin x)^{3/2} + (\cos x)^{3/2}} dx = \pi/4$$

L.H.S

$$I = \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} dx.$$

$$\left[\text{by } \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$$

$$\left[\text{by } \begin{array}{l} \sin(90^\circ - \theta) = \cos\theta \\ \cos(90^\circ - \theta) = \sin\theta \end{array}\right]$$

$$= \int_0^{\pi/2} \frac{\sin^{3/2}\left(\frac{\pi}{2} - x\right)}{\sin^{3/2}(\pi/2 - x) + \cos^{3/2}\left(\frac{\pi}{2} - x\right)} dx$$

$$= \int_0^{\pi/2} \frac{\cos^{3/2} x}{\cos^{3/2} x + \sin^{3/2} x} dx = I$$

$$I + I = \int_0^{\pi/2} \left(\frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} + \frac{\cos^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \right) dx$$

$$2I = \int_0^{\pi/2} \frac{\cancel{\sin^{3/2} x + \cos^{3/2} x}}{\cancel{\sin^{3/2} x + \cos^{3/2} x}} dx$$

$$2I = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \left(\frac{\pi}{2} - 0\right)$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4} = R.H.S$$

$$\therefore \int_0^{\pi/2} \frac{\sin^{3/2} x}{\sin^{3/2} x + \cos^{3/2} x} \, dx = \frac{\pi}{4}$$

Hence proved.

Example . 3

$$\int_0^{\pi/4} \log(1+\tan\theta) d\theta = \frac{\pi}{8} \log 2$$

L.H.S

$$I = \int_0^{\pi/4} \log(1+\tan\theta) d\theta$$

$$\left[\text{by } \int_0^a f(x) dx = \int_0^a f(a-x) dx\right]$$

$$= \int_0^{\pi/4} \log\left[1+\tan\left(\pi/4 - \theta\right)\right] d\theta.$$

$$\left[\text{by } \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}\right]$$

$$= \int_0^{\pi/4} \log\left[1+ \frac{\tan \pi/4 - \tan\theta}{1+\tan \pi/4 \tan\theta}\right] d\theta.$$

$$\left[\text{by } \tan \pi/4 = 1\right]$$

$$= \int_0^{\pi/4} \log\left[1+ \frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\cancel{\tan\theta}+1-\cancel{\tan\theta}}{1+\tan\theta}\right] d\theta$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$I + I = \int_0^{\pi/4} \left[\log(1+\tan\theta) + \log\left(\frac{2}{1+\tan\theta}\right)\right] d\theta$$

[by $\log a + \log b = \log(ab)$]

$$2I = \int_0^{\pi/4} \log\left[\cancel{1+\tan\theta} \times \frac{2}{\cancel{1+\tan\theta}}\right] d\theta$$

$$2I = \int_0^{\pi/4} \log 2 \, d\theta = \log 2 \int_0^{\pi/4} d\theta$$

$$2I = \log 2 \, [\theta]_0^{\pi/4}$$

$$= \log 2 \left[\frac{\pi}{4} - 0\right]$$

$$2I = \frac{\pi}{4} \log 2$$

$$I = \frac{\pi}{8} \log 2 = R.H.S$$

$$\therefore \int_0^{\pi/4} \log(1+\tan\theta) d\theta = \frac{\pi}{8} \log 2$$

Hence proved

Example: 4.

$$\int_0^{\pi} \theta \sin^3\theta \, d\theta = \frac{2\pi}{3}$$

Let.

$$I = \int_0^{\pi} \theta \sin^3\theta \, d\theta$$

$$\left[\text{by} \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right]$$

$$= \int_0^{\pi} (\pi-\theta) \sin^3(\pi-\theta)\,d\theta.$$

$$I = \int_0^{\pi} (\pi-\theta) \sin^3\theta \, d\theta$$

$$I = \int_0^{\pi} \pi \sin^3\theta - \theta \sin^3\theta \, d\theta$$

$$I + I = \int_0^{\pi} (\cancel{\theta \sin^3\theta} + \pi \sin^3\theta - \cancel{\theta \sin^3\theta})\,d\theta$$

$$2I = \int_0^{\pi} \pi \sin^3\theta \, d\theta.$$

$$= \pi \int_0^{\pi} \sin^3\theta \, d\theta = \pi \int_0^{\pi} \sin^2\theta \sin\theta \, d\theta$$

$$\left[\text{by } y = \cos\theta. \right.$$

$$\frac{dy}{d\theta} = -\sin\theta \Rightarrow dy = -\sin\theta \, d\theta$$

$$d\theta = -\frac{dy}{\sin\theta}.$$

$$\theta = 0 \Rightarrow y = \cos 0 = 1$$

$$\left. \theta = \pi \Rightarrow y = \cos\pi = -1 \right]$$

$$\left[\text{by } \sin^2\theta + \cos^2\theta = 1 \right.$$

$$\left. \sin^2\theta = 1 - \cos^2\theta = 1 - y^2. \right]$$

$$2I = -\pi \int_1^{-1} (1-y^2)\,\cancel{\sin\theta}\,\frac{dy}{\cancel{\sin\theta}}$$

$$2I = -\pi \int_{1}^{-1} (1-y^2)\,dy$$

$$= -\pi \left[\int_{1}^{-1} dy - \int_{1}^{-1} y^2\,dy \right]$$

$$\left[\text{by } \int y^n\,dy = \frac{y^{n+1}}{n+1} \right]$$

$$= -\pi \left[y - \frac{y^3}{3} \right]_{1}^{-1}$$

$$= -\pi \left[\left(-1 + \frac{1}{3}\right) - \left(1 - \frac{1}{3}\right) \right]$$

$$= -\pi \left[\left(\frac{-3+1}{3}\right) - \left(\frac{3-1}{3}\right) \right]$$

$$= -\pi \left(\frac{-2}{3} - \frac{2}{3} \right) = -\pi\left(\frac{-4}{3}\right)$$

$$2I = \frac{4}{3}\pi$$

$$I = \frac{4}{3\times 2}\pi$$

$$I = \frac{2}{3}\pi$$

$\therefore \int_0^{\pi} \theta \sin^3\theta\,d\theta = \frac{2}{3}\pi$, Hence proved

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Example: 5

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \text{ if } f(2a-x) = f(x)$$

and $= 0$ if $f(2a-x) = -f(x)$

$$\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx, \quad \int_0^{2a} f(x)\,dx = 0$$

$$\begin{cases} 2\int_0^a f(x)dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^{2a} f(x)dx$$

Substitute

$$y = 2a - x \Rightarrow x = 2a - y$$

$$\frac{dy}{dx} = 0 - 1$$

$$dy = -dx$$

$$x = a \Rightarrow y = 2a - a = a$$

$$x = 2a \Rightarrow y = 2a - 2a = 0$$

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_a^0 f(2a-y)(-dy)$$

$$= \int_0^a f(x)dx - \int_a^0 f(2a-y)\,dy$$

$$= \int_0^a f(x)dx + \int_0^a f(2a-y)dy$$

$$\therefore \int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(2a-x)\,dx. \quad \text{—}①$$

$\Rightarrow$ If $f(2a-x) = f(x)$, by ①

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(x)dx$$

$$= 2\int_0^a f(x)dx$$

$\therefore \; 2\int_0^a f(x)dx$, if $f(2a-x) = f(x)$ proved

$\Rightarrow$ If $f(2a-x) = -f(x)$ by ①

$$\int_0^{2a} f(x)dx = \int_0^a f(x)dx - \int_0^a f(x)dx = 0$$

$\therefore$ 0, If $f(2a-x) = -f(x)$ proved.

Example. 6.

Evalute $I = \int_0^{\pi/2} \log \sin x \, dx$.

$$I = \int_0^{\pi/2} \log \sin x \, dx.$$

$$\left[\text{by} \int_0^a f(x)dx = \int_0^a f(a-x)dx \right]$$

$$= \int_0^{\pi/2} \log \sin\left(\frac{\pi}{2} - x\right) dx$$

$$I = \int_0^{\pi/2} \log \cos x \, dx$$

$$I + I = \int_0^{\pi/2} [\log(\sin x) + \log(\cos x)] dx$$

[by $\log a + \log b = \log(ab)$]

$$2I = \int_0^{\pi/2} \log(\sin x \cos x) \, dx$$

[by $\sin 2x = 2 \sin x \cos x \Rightarrow$

$\frac{1}{2} \sin 2x = \sin x \cos x$]

$$2I = \int_0^{\pi/2} \log\left(\frac{1}{2} \times \sin 2x\right) dx$$

$$\left[\log(a\times b) = \log a + \log b\right]$$

$$2I = \int_0^{\pi/2} \left[\log \sin 2x + \log\left(\tfrac{1}{2}\right)\right] dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx + \int_0^{\pi/2} \log\left(\tfrac{1}{2}\right) dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx + \log\left(\tfrac{1}{2}\right) \int_0^{\pi/2} dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx + \log\left(\tfrac{1}{2}\right) [x]_0^{\pi/2}$$

$$2I = \int_0^{\pi/2} \log \sin 2x \, dx + \frac{\pi}{2} \log\left(\tfrac{1}{2}\right) \quad ①$$

substitute.

$z = 2x$	$z = x$
$\frac{dz}{dx} = 2$	$\frac{dz}{dx} = 1$
$dz = 2dx$	$dz = dx$
$dx = \frac{1}{2} dz$	
$x = 0 \Rightarrow z = 2\times 0 = 0$	$z = 0 \Rightarrow x = 0$
$x = \frac{\pi}{2} \Rightarrow z = \cancel{2} \times \frac{\pi}{\cancel{2}} = \pi$.	$z = \pi \Rightarrow x = \pi$

$$\int_0^{\pi/2} \log \sin 2x \, dx = \frac{1}{2} \int_0^{\pi} \log \sin z \, dz$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin x \, dx$$

[by Example 5 cor. $\int_0^{\pi} f(\sin x) dx = 2 \int_0^{\pi/2} f(\sin x) dx$]

$$\int_0^{\pi/2} \log \sin 2x\,dx = \frac{1}{2} \times 2 \int_0^{\pi/2} \log \sin x\,dx$$

$$= \int_0^{\pi/2} \log \sin x \cdot dx. \quad \text{——} (2)$$

Substitude (2) in (1)

$$2I = \int_0^{\pi/2} \log \sin x\,dx + \frac{\pi}{2} \log\left(\frac{1}{2}\right)$$

$$\left[\text{by} \int_0^{\frac{\pi}{2}} \log \sin x\,dx = I\right]$$

$$2I = I + \frac{\pi}{2} \log\left(\frac{1}{2}\right)$$

$$2I - I = \frac{\pi}{2} \log\left(\frac{1}{2}\right)$$

$$I = \frac{\pi}{2} \log\left(\frac{1}{2}\right)$$

$\therefore$ Evaluated $I = \int_0^{\pi/2} \log \sin x\,dx = \frac{\pi}{2} \log\left(\frac{1}{2}\right)$

Integration by parts.

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\int \frac{d}{dx}(uv)\,dx = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx$$

$$uv = \int u\,dv + \int v\,du$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Example . 1.

$$\int xe^x dx$$

$$\Rightarrow \int xe^x dx$$

Let $u = x$ | $\int dv = \int e^x dx$

$\therefore du = dx$ | $v = e^x$

we know That

$$\left[\because \int u\,dv = uv - \int v\,du.\right]$$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$\left[\text{by } \int e^x dx = e^x + c\right]$$

$$= xe^x - e^x$$

$$= e^x(x-1)$$

$$\boxed{\int xe^x dx = e^x(x-1)}$$

Example . 2

$$\int x \sin 2x\, dx$$

$$\Rightarrow \int x \sin 2x\, dx.$$

Let $u = x$ | $\int dv = \int \sin 2x\, dx.$

$du = dx$ | $v = -\dfrac{\cos 2x}{2}$ $\left[\text{by } \int \sin x = -\cos x +\right.$

we know That

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x \sin 2x\, dx = -\frac{x\cos 2x}{2} - \left[-\int \frac{\cos 2x}{2}\, dx\right]$$

$$= -\frac{x\cos 2x}{2} + \int \frac{\cos 2x}{2}\, dx$$

$$\left[\text{by } \int \cos x\, dx = \sin x + c\right]$$

$$\int x\sin 2x\,dx = -\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}$$

Example. 3

$$\int x^n \log x\,dx.$$

$$\Rightarrow \int x^n \log x\,dx.$$

Let.

$u = \log x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x}dx.$

$\int dv = \int x^n dx.$

$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

$v = \frac{x^{n+1}}{n+1}$

we know That.

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x^n \log x\,dx = \log x \frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}\frac{1}{x}dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^{n+1}\frac{1}{x}dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^{n+1}.x^{-1}dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^{n+\cancel{1}-\cancel{1}}dx$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\int x^n dx$$

$$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{x^{n+1}}{n+1}\log x - \frac{1}{n+1}\left(\frac{x^{n+1}}{n+1}\right)$$

$$\boxed{\int x^n \log x \, dx = \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2}}$$

Example - 4.

$$\int \sin^{-1} x \, dx.$$

$$\Rightarrow \int \sin^{-1} x \, dx$$

Let

$$u = \sin^{-1} x$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$du = \frac{1}{\sqrt{1-x^2}} dx$$

$$\int dv = \int dx$$

$$v = x.$$

we know that

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

Let $x = \sin\theta$

$$\frac{dx}{d\theta} = \cos\theta$$

$$dx = \cos\theta \, d\theta$$

$$1 - x^2 = 1 - \sin^2\theta$$

$$1 - x^2 = \cos^2\theta \Rightarrow$$

$$\cos\theta = \sqrt{1-x^2}$$

$$\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{\sin\theta}{\sqrt{\cos^2\theta}} \cos\theta \, d\theta$$

$$= x \sin^{-1} x - \int \frac{\sin\theta}{\cos\theta} \cos\theta \, d\theta$$

$$= x \sin^{-1} x - \int \sin\theta \, d\theta.$$

$$\left[\text{by} \int \sin\theta \, d\theta = -\cos\theta + c\right]$$

$$= x \sin^{-1} x - (-\cos\theta)$$

$$= x \sin^{-1} x + \cos\theta.$$

$$\left[\text{by } \cos\theta = \sqrt{1-x^2}\right]$$

$$\therefore \boxed{\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}}$$

Example. 5

$$\int \tan^{-1}x\,dx.$$

$$\Rightarrow \int \tan^{-1}x\,dx.$$

Let.

$u = \tan^{-1}x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}dx$

$\int dv = \int dx$

$v = x.$

we know that

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int \tan^{-1}x\,dx = x\tan^{-1}x - \int \frac{x}{1+x^2}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$$

$$= x\tan^{-1}x - \frac{1}{2}\log(1+x^2)$$

$$\boxed{\int \tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\log(1+x^2)}$$

Example. 6

$$\int x^2\tan^{-1}x\,dx.$$

$$\Rightarrow \int x^2\tan^{-1}x\,dx.$$

Let $u = \tan^{-1}x$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}dx$

$\int dv = \int x^2 dx$

$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

$v = \frac{x^3}{3}$

We know that

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x^2 \tan^{-1}x\,dx = \frac{x^3}{3}\tan^{-1}x - \int \frac{x^3}{3(1+x^2)}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int \frac{x^3}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\int x - \frac{x}{1+x^2}dx$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left[\int x\,dx - \frac{1}{2}\int \frac{2x}{1+x^2}dx\right]$$

$$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$\left[\text{by } \int \frac{nx}{1+x^n}dx = \log(1+x^n) + c\right]$$

$$= \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\right]$$

$$\boxed{\int x^2 \tan^{-1}x\,dx = \frac{x^3}{3}\tan^{-1}x - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\log(1+x^2)\right]}$$

Example . 7

$$\int (\log x)^2 dx$$

$$\Rightarrow \int (\log x)^2 dx$$

Let. $u = (\log x)^2$

$\cancel{(\log x)^n = n(\log x)^{n-1}\frac{1}{x}}$

$$\left[\text{by } (\log x)^n \frac{d}{dx} = n(\log x)^{n-1}\frac{1}{x}\right]$$

$$\frac{du}{dx} = 2\log x \cdot \frac{1}{x}$$

$$du = \left(2\log x \cdot \frac{1}{x}\right)dx$$

$$\int dv = \int dx$$

$$v = x$$

we know that

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int (\log x)^2 dx = x(\log x)^2 - \int x \cdot 2\log x \frac{1}{x} dx$$

$$= x(\log x)^2 - 2\int \log x \, dx$$

Let

$u = \log x$

$\frac{du}{dx} = \frac{1}{x}$

$du = \frac{1}{x} dx$

$\int dv = \int dx$

$v = x$

$$\int (\log x)^2 dx = x(\log x)^2 - 2\left(x\log x - \int x \cdot \frac{1}{x} dx\right)$$

$$= x(\log x)^2 - 2\left(x\log x - \int dx\right)$$

$$= x(\log x)^2 - 2(x\log x - x)$$

$$= x(\log x)^2 - 2x\log x + 2x$$

$$\boxed{\int (\log x)^2 dx = x(\log x)^2 - 2x\log x + 2x}$$

Example. 8.

Example : 8 :

$$\int \sqrt{a^2+x^2}\,dx$$

$$\Rightarrow \int \sqrt{a^2+x^2}\,dx$$

Let

$$u=\sqrt{a^2+x^2}$$

$$\left[\text{by } \sqrt{x}\ \frac{d}{dx}=\frac{1}{2\sqrt{x}}\right]$$

$$\frac{du}{dx}=\frac{1}{\cancel{2}\sqrt{a^2+x^2}}\cdot\cancel{2}x$$

$$du=\frac{x}{\sqrt{a^2+x^2}}\,dx$$

$$\int dv=\int dx$$

$$v=x.$$

we know That

$$\left[\because \int u\,dv=uv-\int v\,du\right]$$

$$\int\sqrt{a^2+x^2}\,dx=x\sqrt{a^2+x^2}-\int\frac{x^2}{\sqrt{a^2+x^2}}\,dx.$$

$$=x\sqrt{a^2+x^2}-\int\frac{x^2+a^2-a^2}{\sqrt{a^2+x^2}}\,dx$$

$$=x\sqrt{a^2+x^2}-\left[\int\frac{a^2+x^2}{\sqrt{a^2+x^2}}\,dx-\int\frac{a^2}{\sqrt{a^2+x^2}}\,dx\right]$$

$$=x\sqrt{a^2+x^2}-\int\frac{a^2+x^2\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}\sqrt{a^2+x^2}}\,dx+a^2\int\frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$=x\sqrt{a^2+x^2}-\int\frac{a^2+x^2\sqrt{a^2+x^2}}{(\sqrt{a^2+x^2})^{\cancel{2}}}\,dx+a^2\int\frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$\int \sqrt{a^2+x^2}\,dx = x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + a^2 \int \frac{1}{\sqrt{a^2+x^2}}\,dx$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\,dx + a^2 \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\left[\text{by } \int \sqrt{a^2+x^2}\,dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \log\left|x+\sqrt{a^2+x^2}\right|\right]$$

$$= x\sqrt{a^2+x^2} - \left[\frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2} \log\left|x+\sqrt{a^2+x^2}\right|\right] + a^2 \sinh^{-1}\left(\frac{x}{a}\right)$$

$$\left[\because \text{ by } \log\left|x+\sqrt{a^2+x^2}\right| = \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

$$= x\sqrt{a^2+x^2} - \frac{x}{2}\sqrt{a^2+x^2} - \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + a^2 \sinh^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{2} x\sqrt{a^2+x^2} + \frac{1}{2} a^2 \sinh^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{1}{2}\left[x\sqrt{a^2+x^2} + a^2 \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}\left[x\sqrt{a^2+x^2} + a^2 \sinh^{-1}\left(\frac{x}{a}\right)\right]$$

Example. 9.

$$\int \frac{x+\sin x}{1+\cos x}\,dx$$

$$\Rightarrow \int \frac{x+\sin x}{1+\cos x}\,dx.$$

$$= \int \frac{x}{1+\cos x}\,dx + \int \frac{\sin x}{1+\cos x}\,dx.$$

$\left[\text{by } 1+\cos x = 2\cos^2 \frac{x}{2}\right]$

$\left[\text{by } \sin x = 2\sin \frac{x}{2} \cos \frac{x}{2}\right]$

$$= \int \frac{x}{2\cos^2 \frac{x}{2}}\,dx + \int \frac{\cancel{2}\sin x/_2 \, \cancel{\cos} x/_2}{\cancel{2}\cos^{\cancel{2}} x/_2}\,dx$$

$$= \int \frac{x}{2\cos^2 x/_2}\,dx + \int \frac{\sin x/_2}{\cos x/_2}\,dx$$

$$= \frac{1}{2}\int x \cdot \frac{1}{\cos^2 x/_2}\,dx + \int \tan x/_2\,dx.$$

$$= \frac{1}{2}\int x \sec^2 x/_2\,dx + \int \tan x/_2\,dx. \quad \text{—— (1)}$$

Let

$u = x$

$du = dx$

$\int dv = \int \sec^2 x/_2\,dx$

$v = \frac{\tan x/_2}{1/_2} = \tan x/_2 \times \frac{2}{1}$

$v = 2\tan x/_2$

we know that

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x \sec^2 x/2 \, dx = 2x\tan x/2 - 2\int \tan x/2 \, dx \quad \text{--- (2)}$$

substitute (2) in (1).

$$\int \frac{x+\sin x}{1+\cos x}dx = \frac{1}{2}\left[2x\tan x/2 - 2\int \tan x/2 \, dx\right] + \int \tan x/2 \, dx$$

$$= \frac{1}{2}\times 2x\tan x/2 - \frac{1}{\cancel{2}}\times\cancel{2}\int \tan x/2\,dx + \int \tan x/2 \, dx$$

$$= x\tan x/2 - \cancel{\int \tan x/2\,dx} + \cancel{\int \tan x/2\,dx}$$

$$= x\tan x/2$$

$$\boxed{\int \frac{x+\sin x}{1+\cos x}dx = x\tan x/2}$$

Example. 10. $\int e^x \frac{x+1}{(x+2)^2}dx.$

$$\int e^x \frac{x+1}{(x+2)^2}dx = \int e^x\left[\frac{x+1+1-1}{(x+2)^2}\right]dx$$

$$= \int e^x\left[\frac{x+2-1}{(x+2)^2}\right]dx$$

$$= \int e^x \left[\frac{x+2}{(x+2)^2} - \frac{1}{(x+2)^2} \right] dx$$

$$\int e^x \left[\frac{x+1}{(x+2)^2}\right] dx = \int e^x \left[\frac{1}{x+2} - \frac{1}{(x+2)^2} \right] dx.$$

- We know that the formula for Integration by parts is

$$\int u\,dv = uv - \int v\,du$$

- A useful Identity for integrals of the form

$$\left[\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c\right]$$

$$\int e^x \left[\frac{x+1}{(x+2)^2}\right] dx = \int e^x \left[\frac{1}{x+2} - \frac{1}{(x+2)^2} \right] dx.$$

$$= e^x \frac{1}{x+2} = \frac{e^x}{x+2}$$

$$\boxed{\int e^x \left[\frac{x+1}{(x+2)^2}\right] dx = \frac{e^x}{x+2}}$$

Example 11 $\int e^x (\sin x + \cos x)\, dx$.

$$\int e^x (\sin x + \cos x)\, dx.$$

we know that Integrals of the form

$$\left[\because \int e^x [f(x) + f'(x)]\, dx = e^x f(x) + c\right]$$

$$f(x) = \sin x \quad \Big| \quad e^x f(x) = e^x \sin x.$$
$$f'(x) = \cos x$$

$$\boxed{\int e^x (\sin x + \cos x)\, dx = e^x \sin x = \sin x . e^x}$$

Reduction formula.

1) problem. 1. (13.1)

$I_n = \int x^n e^{ax} dx$, where n is a positive Integer.

$$I_n = \int x^n e^{ax} dx.$$

Let $u = x^n$ | $\int dv = \int e^{ax} dx$

$\frac{du}{dx} = nx^{n-1}$ | $v = \frac{e^{ax}}{a}$

$du = nx^{n-1} dx$

we know That Integration By parts.

$$\left[\because \int u dv = uv - \int v du.\right]$$

$$I_n = \int x^n e^{ax} dx = x^n \frac{e^{ax}}{a} - \int n x^{n-1} \frac{e^{ax}}{a} dx$$

$$= \frac{1}{a}\left[x^n e^{ax} - n \int x^{n-1} e^{ax} dx\right]$$

$$\left[\text{by } I_{n-1} = \int x^{n-1} e^{ax} dx.\right]$$

$$I_n = \frac{1}{a}\left(x^n e^{ax} - n I_{n-1}\right)$$

$$\boxed{I_n = \int x^n e^{ax} dx = \frac{1}{a}\left(x^n e^{ax} - n I_{n-1}\right)}$$

Problem : 2 (13.2).

② $I_n = \int x^n \cos ax\, dx$, where n is a positive Integer.

$$I_n = \int x^n \cos ax\, dx.$$

Let $u = x^n$ | $\int dv = \int \cos ax\, dx$

$\frac{du}{dx} = nx^{n-1}$ | $v = \frac{\sin ax}{a}$

$du = nx^{n-1} dx$

We know that Integration by parts

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$I_n = \int x^n \cos ax\,dx$$

$$= \frac{x^n \sin ax}{a} - \int n x^{n-1} \frac{\sin ax}{a}\,dx$$

$$I_n = \frac{x^n \sin ax}{a} - \frac{n}{a}\int x^{n-1} \sin ax\,dx \quad \text{--- (1)}$$

Let $u = x^{n-1}$ | $\int dv = \int \sin ax\,dx$

$\frac{du}{dx} = n-1\, x^{n-1-1}$ | $v = -\frac{\cos ax}{a}$

$du = n-1\, x^{-2}\,dx$

$\left[\text{by } \int u\,dv = uv - \int v\,du\right]$

$$\int x^{n-1} \sin ax\,dx = -\frac{x^{n-1}\cos ax}{a} + \int (n-1) x^{n-2} \frac{\cos ax}{a}\,dx.$$

$$= -\frac{x^{n-1}\cos ax}{a} + \frac{(n-1)}{a}\int x^{n-2} \cos ax\,dx$$

$$\left[\text{by } I_n = \int x^n \cos ax\,dx\right]$$

$$\int x^{n-1} \sin ax\,dx = -\frac{x^{n-1}\cos ax}{a} + \frac{(n-1)}{a} I_{n-2} \quad \text{--- (2)}$$

substitude (2) in (1)

$$I_n = \frac{x^n \sin ax}{a} - \frac{n}{a}\left[-\frac{x^{n-1}\cos ax}{a} + \frac{(n-1)}{a} I_{n-2}\right.$$

$$= \frac{x^n \sin ax}{a} + \frac{n x^{n-1}}{a^2}\cos ax - \frac{n(n-1)}{a^2} I_{n-2}$$

$$I_n = \int x^n \cos ax\, dx = \frac{x^n \sin ax}{a} + \frac{n x^{n-1}}{a^2} \cos ax - \frac{n(n-1)}{a^2} I_{n-2}$$

Reduction formula.

problam. 1.

(1) The ultimate Integral is either $\int x e^{ax} dx$ or $\int e^{ax} dx$ according as n is odd or even.

(i) $\int x e^{ax} dx$. (odd)

$$\int x e^{ax} dx.$$

Let

$u = x$ | $\int dv = \int e^{ax} dx$

$\frac{du}{dx} = 1$ | $v = \frac{e^{ax}}{a}$

$du = dx$

Integration By parts

$$\left[\because \int u\,dv = uv - \int v\,du \right]$$

$$\int x e^{ax} dx = x \frac{e^{ax}}{a} - \int \frac{e^{ax}}{a} dx.$$

$$= x \frac{e^{ax}}{a} - \frac{1}{a} \int e^{ax} dx$$

$$= x \frac{e^{ax}}{a} - \frac{1}{a} \cdot \frac{e^{ax}}{a}.$$

$$= x \frac{e^{ax}}{a} - \frac{e^{ax}}{a^2}.$$

$$\boxed{\int x e^{ax} dx = x \frac{e^{ax}}{a} - \frac{e^{ax}}{a^2}}$$

(ii) $\int e^{ax}\,dx$: (Even)

$$\boxed{\int e^{ax}\,dx = \frac{e^{ax}}{a}.}$$

problam. 2.

The ultimate Integral is either.

$\int x\cos ax\,dx$ or $\int \cos ax\,dx$ according.

(2) as n is a odd or even.

(i) $\int x\cos ax\,dx$ (odd.)

$$\int x\cos ax\,dx.$$

Let $u = x$ | $\int dv = \int \cos ax\,dx.$

$\frac{du}{dx} = 1$ | $v = \frac{\sin ax}{a}.$

$du = dx$

Integration By parts.

$$\left[\because \int u\,dv = uv - \int v\,du.\right]$$

$$\int x\cos ax\,dx = \frac{x\sin ax}{a} - \int \frac{\sin ax}{a}\,dx$$

$$= \frac{x\sin ax}{a} - \frac{1}{a}\int \sin ax\,dx.$$

$$= \frac{x\sin ax}{a} - \frac{1}{a}\left(-\frac{\cos ax}{a}\right)$$

$$= \frac{x\sin ax}{a} + \frac{1}{a^2}\cos ax.$$

$$\boxed{\int x\cos ax\,dx = \frac{x\sin ax}{a} + \frac{1}{a^2}\cos ax}$$

(ii) $\int \cos ax \, dx$. (Even.)

$$\boxed{\int \cos ax \, dx = \frac{\sin ax}{a}}$$

HOME WORK.

$I_n = \int x^n \sin ax \, dx$, where n is a positive Integer.

$$I_n = \int x^n \sin ax \, dx.$$

Let

$u = x^n$

$\frac{du}{dx} = n x^{n-1}$

$du = n x^{n-1} \, dx$

$\int dv = \int \sin ax \, dx$

$v = -\frac{\cos ax}{a}$

Integration By parts.

$$\int u \, dv = uv - \int v \, du.$$

$$I_n = \int x^n \sin ax \, dx = -\frac{x^n \cos ax}{a} - \int -n x^{n-1} \frac{\cos ax}{a} \, dx$$

$$I_n = -\frac{x^n \cos ax}{a} + \frac{n}{a} \int x^{n-1} \cos ax \, dx \quad \text{--- ①}$$

Let

$u = x^{n-1}$

$\frac{du}{dx} = n-1 \, x^{n-1-1}$

$du = n-1 \, x^{n-2} \, dx$

$\int dv = \int \cos ax \, dx$.

$v = \frac{\sin ax}{a}$.

[by $\int u \, dv = uv - \int v \, du$]

$$\int x^{n-1} \cos ax \, dx = \frac{x^{n-1} \sin ax}{a} - \int n-1 \, x^{n-2} \frac{\sin ax}{a} \, dx$$

$$\left[\text{by } I_n = \int x^n \sin ax \, dx.\right]$$

$$\int x^{n-1} \cos ax \, dx = \frac{x^{n-1} \sin ax}{a} - \frac{(n-1)}{a} I_{n-2} \quad \text{---} (2)$$

Substitute (2) in (1).

$$I_n = -\frac{x^n \cos ax}{a} + \frac{n}{a}\left[\frac{x^{n-1}\sin ax}{a} - \frac{(n-1)}{a} I_{n-2}\right]$$

$$= -\frac{x^n \cos ax}{a} + \frac{n x^{n-1}}{a^2}\sin ax - \frac{n(n-1)}{a^2} I_{n-2}$$

$$\boxed{I_n = \int x^n \sin ax \, dx = -\frac{x^n \cos ax}{a} + \frac{n x^{n-1}}{a^2}\sin ax - \frac{n(n-1)}{a^2} I_{n-2}}$$

The ultimate Integral is either $\int x \sin ax \, dx$ or $\int \sin ax \, dx$ according as n is a odd or eve

(i) $\int x \sin ax \, dx$ (Odd)

$$\int x \sin ax \, dx.$$

Let $u = x$ $\qquad \int dv = \int \sin ax \, dx$

$\frac{du}{dx} = 1$

$du = dx$ $\qquad v = -\frac{\cos ax}{a}$

Integration By parts.

$$\int u\,dv = uv - \int v\,du$$

$$\int x \sin ax \, dx = -\frac{x \cos ax}{a} - \int -\frac{\cos ax}{a} dx$$

$$= -\frac{x \cos ax}{a} + \frac{1}{a}\int \cos ax \, dx$$

$$= -\frac{x \cos ax}{a} + \frac{1}{a} \cdot \frac{\sin ax}{a}$$

$$\boxed{\int x \sin ax\,dx = -\frac{x\cos ax}{a} + \frac{1}{a^2}\sin ax.}$$

(ii) $\int \sin ax\,dx$. (Even)

$$\boxed{\int \sin ax\,dx = -\frac{\cos ax}{a}}$$

Problam. 3 (13.3)

$I_n = \int \sin^n x\,dx$, where n being a positive Integer. (Theoram)

$I_n = \int \sin^n x\,dx = \int \sin^{n-1} x \cdot \sin x\,dx.$

Let $u = \sin^{n-1} x$ $\qquad \int dv = \int \sin x\,dx$

$\frac{du}{dx} = n-1 \sin^{n-2} x$ $\qquad v = -\cos x$

$du = n-1 \sin^{n-2} x\,dx$

Integration By parts.

$$\int u\,dv = uv - \int v\,du.$$

$I_n = -\sin^{n-1} x \cos x + \int (n-1) \cos^2 x \sin^{n-2} x\,d$

$= -\sin^{n-1} x \cos x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x\,d$

$= -\sin^{n-1} x \cos x + (n-1)\left[\int \sin^{n-2} x\,dx - \int \sin^{n-2}\right.$

$= -\sin^{n-1} x \cos x + (n-1)\left[\int \sin^{n-2} x\,dx - \int \sin^{n} x\right.$

$\left[\text{by } I_n = \int \sin^n x\,dx\right]$

$I_n = -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n$

$$I_n + (n-1) I_n = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$$

$$\cancel{I_n} + nI_n - \cancel{I_n} = -\sin^{n-1}x \cos x + (n-1) I_{n-2}$$

$$\boxed{I_n = \frac{1}{n}\left[-\sin^{n-1}x \cos x + (n-1) I_{n-2}\right]}$$

The ultimate Integral is $\int \sin x\,dx$ or $\int dx$ according as n is odd or even

(i) $\int \sin x\,dx$ (odd)

$$\boxed{\int \sin x\,dx = -\cos x}$$

(ii) $\int \sin^0 x\,dx = \int dx$ (even)

$$\boxed{\int dx = x.}$$

corollary:

$$I_n = \int_0^{\pi/2} \sin^n x\,dx.$$

$$I_n = \int_0^{\pi/2} \sin^n x\,dx$$

$$= \frac{1}{n}\left(\left[-\sin^{n-1}x \cos x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2} dx\right)$$

$$\left[\text{by } I_n = \int_0^{\pi/2} \sin^n x\,dx\right]$$

$$I_n = \frac{1}{n}\left([0-0] + (n-1) I_{n-2}\right)$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

Now

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}$$

$$I_{n-4} = \frac{n-5}{n-4} I_{n-6}$$

$$\vdots$$

$\therefore I_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} I_{n-4}$

$I_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots\cdots$

The ultimate Integral is $\int_0^{\pi/2} \sin x\, dx$ or $\int_0^{\pi/2} dx$. according as n is odd or even

(i) $\int_0^{\pi/2} \sin x\, dx$ (odd)

$I_1 = \int_0^{\pi/2} \sin x\, dx = \left[-\cos x\right]_0^{\pi/2}$

$= -\left[\cos\frac{\pi}{2} - \cos 0\right] = -(0-1) = 1$

$I_1 = 1$

$\therefore I_n = \int_0^{\pi/2} \sin^n x\, dx = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \cdots\cdots \frac{2}{3} \cdot 1$

(ii) $\int_0^{\pi/2} dx$ (even)

$I_0 = \int_0^{\pi/2} dx = \left[x\right]_0^{\pi/2} = \frac{\pi}{2}$

$I_0 = \frac{\pi}{2}$

$\therefore I_n = \int_0^{\pi/2} dx = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \cdots\cdots \frac{1}{2} \frac{\pi}{2}$

---

problam. 4. (13.4)

$I_n = \int \cos^n x\, dx$, where n being a positive Integer.

$I_n = \int \cos^n x\, dx = \int \cos^{n-1} x \cos x\, dx$

Let $u = \cos^{n-1} x$ $\qquad \int dv = \int \cos x \, dx$

$\frac{du}{dx} = -n-1 \cos^{n-2} x \sin x$ $\qquad v = \sin x.$

$du = -n-1 \cos^{n-2} x \sin x \, dx$

Integration By parts.

$\int u \, dv = uv - \int v \, du.$

$I_n = \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \, dx.$

$= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x \, dx.$

$= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x \, dx$

$= \cos^{n-1} x \sin x + (n-1) \left[\int \cos^{n-2} x \, dx - \int \cos^{n-2+2} x \, dx \right]$

$= \cos^{n-1} x \sin x + (n-1) \left[\int \cos^{n-2} x \, dx - \int \cos^{2} x \, dx \right]$

$= \cos^{n-1} x \sin x + (n-1) \left[I_{n-2} - I_n \right]$

$I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2} + (n-1) I_n$

$I_n + (n-1) I_n = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$\cancel{I_n} + n I_n - \cancel{I_n} = \cos^{n-1} x \sin x + (n-1) I_{n-2}$

$$\boxed{I_n = \frac{1}{n} \left[\cos^{n-1} x \sin x + (n-1) I_{n-2} \right]}$$

The ultimate Integrals is $\int \cos x \, dx$ or $\int dx$ according as n is odd or even.

(i) $\int \cos x \, dx$ (odd)

$\therefore \int \cos x \, dx = \sin x.$

(ii) $\int \cos^0 x \, dx = \int dx$ (even)

$\therefore \int dx = x.$

corollary.

$$I_n = \int_0^{\pi/2} \cos^n x \, dx.$$

$$I_n = \int_0^{\pi/2} \cos^n x \, dx$$

$$= \frac{1}{n}\left(\left[\cos^{n-1}x \sin x\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2}x \, dx\right)$$

$$= \frac{1}{n}\left(\left[\cos^{n-1}x \sin x\right]_0^{\pi/2} + (n-1) I_{n-2}\right)$$

$$= \frac{1}{n}\left([0-0] + (n-1) I_{n-2}\right)$$

$$I_n = \frac{(n-1)}{n} I_{n-2}$$

Now

$$I_{n-2} = \frac{(n-3)}{(n-2)} I_{n-4}$$

$$I_{n-4} = \frac{(n-5)}{(n-4)} I_{n-6}$$

$$\vdots$$

$$\therefore I_n = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots$$

The ultimate Integral is $\int_0^{\pi/2} \cos x \, dx$ $\int_0^{\pi/2} dx$ according as n is odd or even

(i) $\int_0^{\pi/2} \cos x \, dx$. (odd)

$$I_1 = \int_0^{\pi/2} \cos x \, dx = \left[\sin x\right]_0^{\pi/2}$$

$$= \left[\sin \tfrac{\pi}{2} - \sin 0\right] = 1 - 0 = 1$$

$$I_1 = 1$$

$$\therefore I_n = \int_0^{\pi/2} \cos x \, dx = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \cdots \frac{2}{3} \cdot 1$$

(ii) $\int_0^{\pi/2} dx$ (even)

$$I_0 = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$I_0 = \frac{\pi}{2}$$

$$\therefore I_n = \int_0^{\pi/2} dx = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}$$

---

problam. 5 (13.5)

$I_{m,n} = \int \sin^m x \cos^n x \, dx$, where m, n being positive Integer.

$$I_{m,n} = \int \sin^m x \cos^n x \, dx$$

$$= \int \sin^m x \cos^{n-1} x \cos x \, dx$$

$$\left[\text{by } D(\sin^{m+1} x) = (m+1) \sin^m x \cos x \, dx \right.$$

$$\left. D\left(\frac{\sin^{m+1} x}{m+1}\right) = \sin^m x \cos x \, dx. \right]$$

$$I_{m,n} = \int \cos^{n-1} x \sin^m x \cos x \, dx.$$

$$= \int \cos^{n-1} x \cdot D\left(\frac{\sin^{m+1} x}{m+1}\right)$$

Let

$u = \cos^{n-1} x$

$\frac{du}{dx} = (n-1) \cos^{n-2} x (-\sin x)$

$du = -(n-1) \cos^{n-2} x \sin x \, dx$

$dv = D\left(\frac{\sin^{m+1} x}{m+1}\right)$

$v = \frac{\sin^{m+1} x}{m+1}$

Integration By parts.

$$\int u\,dv = uv - \int v\,du$$

$$I_{m,n} = \int \cos^{n-1}x \cdot D\left(\frac{\sin^{m+1}x}{m+1}\right)$$

$$= \frac{\cos^{n-1}x\,\sin^{m+1}x}{m+1} + \int \frac{\sin^{m+1}x}{m+1}(n-1)\cos^{n-2}x\,\sin x\,dx$$

$$= \frac{\cos^{n-1}x\,\sin^{m+1}x}{m+1} + \left(\frac{n-1}{m+1}\right)\int \sin^{m+1+1}x\,\cos^{n-2}x\,dx$$

$$= \frac{\cos^{n-1}x\,\sin^{m+1}x}{m+1} + \left(\frac{n-1}{m+1}\right)\int \sin^{m+2}x\,\cos^{n-2}x\,dx$$

$$= \frac{\cos^{n-1}x\,\sin^{m+1}x}{m+1} + \left(\frac{n-1}{m+1}\right)\int \sin^{m}x\cdot\sin^{2}x\cdot\cos^{n-2}x\,dx$$

$$\left[\text{by } \sin^2 x = 1 - \cos^2 x\right]$$

$$= \frac{\cos^{n-1}x\,\sin^{m+1}x}{m+1} + \frac{(n-1)}{(m+1)}\int \sin^{m}x\,(1-\cos^2 x)\cos^{n-2}x\,dx$$

$$= \frac{1}{m+1}\left[\cos^{n-1}x\,\sin^{m+1}x + (n-1)\left[\int \sin^m x\cos^{n-2}x\,dx - \int \sin^m x\cos^{n-2+2}x\,dx\right]\right]$$

$$(m+1)I_{m,n} = \cos^{n-1}x\,\sin^{m+1}x + (n-1)\left[\int \sin^m x\cos^{n-2}x\,dx - \int \sin^m x\cos^n x\,dx\right]$$

$$= \cos^{n-1}x\,\sin^{m+1}x + (n-1)\left[I_{m,n-2} - I_{m,n}\right.$$

$$(m+1) I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2} - (n-1) I_{m,n}$$

$$(m+1) I_{m,n} + (n-1) I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}$$

$$I_{m,n} (m\cancel{+1}+n\cancel{-1}) = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}$$

$$(m+n) I_{m,n} = \cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}$$

$$I_{m,n} = \frac{\cos^{n-1} x \sin^{m+1} x}{m+n} + \frac{(n-1)}{(m+n)} I_{m,n-2}$$

$$\boxed{I_{m,n} = \frac{1}{m+n}\left[\cos^{n-1} x \sin^{m+1} x + (n-1) I_{m,n-2}\right]}$$

The ultimate Integral m or n be an odd Integer. say n applying The formula

$$\cancel{I_{m,1} = \frac{1}{m+1}\left[\cos^{1-1} x \sin\right.}$$

$$I_{m,1} = \int \sin^m x \cos^1 x \, dx = \int \sin^m x \cos x \, dx.$$

$$\left[\text{by } \int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}\right]$$

$$\therefore I_{m,1} = \int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1}$$

The ultimate Integral m or n be an even Integer. say n applying The formula..

$$I_{m,0} = \int \sin^m x \cos^0 x \, dx = \int \sin^m x \, dx.$$

$$\left[\text{by} \int \sin^m x \, dx = \frac{1}{m}\left(-\sin^{m-1} x \cos x + (m-1) I_{m-2}\right)\right]$$

$\therefore I_{m,0} = \int \sin^m x \, dx = \frac{1}{m}\left(-\sin^{m-1} x \cos x + (m-1) I_{m-2}\right)$

Corollary:

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx.$$

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx.$$

$$= \frac{1}{m+n}\left(\left[\cos^{n-1} x \sin^{m+1} x\right]_0^{\pi/2} + (n-1) \int_0^{\pi/2} \sin^m x \cos^{n-2} x \, dx\right)$$

$$= \frac{1}{m+n}\left(\left[\cos^{n-1}\tfrac{\pi}{2} \sin^{m+1}\tfrac{\pi}{2} - \cos^{n-1} 0 \sin^{m+1} 0\right] + (n-1) I_{m,n-2}\right)$$

$\left[\text{by } \cos\frac{\pi}{2} = 0, \ \cos 0^\circ = 1 \right.$

$\left. \sin\pi/2 = 1, \ \sin 0^\circ = 0 \right]$

$$I_{m,n} = \frac{1}{m+n}\left(\left[(0 \times 1) - (1 \times 0)\right] + (n-1) I_{m,n-2}\right)$$

$$I_{m,n} = \frac{1}{m+n}\left[(n-1) I_{m,n-2}\right]$$

$$I_{m,n} = \frac{n-1}{m+n} I_{m,n-2}$$

Now,

$$I_{m,n-2} = \frac{n-2-1}{m+n-2} I_{m,n-2-2} = \frac{n-3}{m+n-2} I_{m,n-4}$$

$$I_{m,n-4} = \frac{n-4-1}{m+n-4} I_{m,n-4-2}$$

$$= \frac{n-5}{m+n-4} I_{m,n-6}$$

$$\therefore I_{m,n} = \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\,\frac{n-5}{m+n-4}\cdots\cdots$$

The ultimate Integral m or n be an odd Integer. say n applying The formula.

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx.$$

$$I_{m,1} = \int_0^{\pi/2} \sin^m x \cos x\,dx$$

$$= \left[\frac{\sin^{m+1} x}{m+1}\right]_0^{\pi/2}$$

$$= \frac{1}{m+1}\left[\sin^{m+1} x\right]_0^{\pi/2}$$

$$= \frac{1}{m+1}\left[\sin^{m+1}\pi/2 - \sin^{m+1} 0\right]$$

$$= \frac{1}{m+1}(1-0) = \frac{1}{m+1}$$

$$I_{m,1} = \frac{1}{m+1}$$

$$\therefore I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{n-1}{m+n}\,\frac{n-3}{m+n-2}\cdots\frac{2}{m+3}\,\frac{1}{m-}$$

The ultimate Integral m or n be an even Integer, Say n applying The formula.

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\,dx$$

$$I_{m,0} = \int_0^{\pi/2} \sin^m x\,dx.$$

$$= \frac{1}{m}\left(\left[-\sin^{m-1} x \cos x\right]_0^{\pi/2} + (m-1) I_{m-2}\right)$$

$$= \frac{1}{m}\left(\left[-\sin^{m-1}\pi/2 \cos\frac{\pi}{2} - (-\sin^{m-1} 0 \cdot \cos 0)\right]\right.$$

$$\left. + (m-1) I_{m-2}\right)$$

$$= \frac{1}{m}\left([0-0] + (m-1) I_{m-2}\right)$$

$$I_{m,0} = \frac{m-1}{m} I_{m-2}$$

$$\therefore I_{m,n} = \int_0^{\pi/2} \sin^m x \, dx = \frac{m-1}{m} \cdot \frac{m-3}{m-2} \cdot \frac{m-5}{m-4} \cdots \frac{1}{2} \frac{\pi}{2}$$

according to corollary.

$$I_{m,n} = \frac{n-1}{m+n} \frac{n-3}{m+n-2} \frac{n-5}{m+n-4} \cdots I_{m,1} \text{ or } I_{m,0}$$

according as n is odd or even.

problam. 6 (13.6)

$I_n = \int \tan^n x \, dx$, where n being a positive Integer.

$$I_n = \int \tan^n x \, dx$$

$$= \int \tan^{n-2} x \tan^2 x \, dx.$$

$$\left[\text{by } \tan^2 x = \sec^2 x - 1 \right]$$

$$I_n = \int \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \int (\tan^{n-2} x \sec^2 x - \tan^{n-2} x) \, dx$$

$$= \int \tan^{n-2} x \sec^2 x \, dx - \int \tan^{n-2} x \, dx$$

$t = \tan x.$

$$\frac{dt}{dx} = \sec^2 x$$

$$dt = \sec^2 x \, dx.$$

$$I_n = \int t^{n-2} \, dt - \int \tan^{n-2} x \, dx.$$

$$\left[\text{by } \int t^n \, dt = \frac{t^{n+1}}{n+1} \right]$$

$$\left[\text{by } I_{n-2} = \int \tan^{n-2} x\, dx.\right]$$

$$I_n = \frac{t^{n-2+1}}{n-2+1} - I_{n-2}$$

$$= \frac{t^{n-1}}{n-1} - I_{n-2}$$

$$\left[\text{by } t = \tan x\right]$$

$$\therefore \boxed{I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}}$$

* The ultimate Integral is $\int \tan x\, dx$ or $\int dx$ according as n is odd or even

(i) $\int \tan x\, dx$ (odd.)

$$I_n = \int \tan^n x\, dx$$

$$I_1 = \int \tan^1 x\, dx. = \int \tan x\, dx.$$

$$I_1 = \log|\sec x|$$

$$\therefore I_n = \int \tan x\, dx = \log|\sec x|$$

(ii) $\int dx$ (even)

$$I_n = \int \tan^n x\, dx$$

$$I_0 = \int \tan^0 x\, dx = \int dx.$$

$$I_0 = x$$

$$\therefore I_n = \int dx = x.$$

---

problam. 7 (13.7)

$I_n = \int \cot^n x\, dx$, where n being a positive Integer.

$$I_n = \int \cot^n x\, dx.$$

$I_n = \int \cot^{n-2} x \cot^2 x \, dx.$

$\left[\text{by } \cot^2 x = \operatorname{cosec}^2 x - 1\right]$

$= \int \cot^{n-2} x \,(\operatorname{cosec}^2 x - 1)\, dx$

$= \int (\cot^{n-2} x \operatorname{cosec}^2 x - \cot^{n-2} x)\, dx.$

$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - \int \cot^{n-2} x \, dx.$

$t = \cot x$

$\frac{dt}{dx} = -\operatorname{cosec}^2 x$

$dt = -\operatorname{cosec}^2 x \, dx$

$-dt = \operatorname{cosec}^2 x \, dx$

$= \int t^{n-2} x \,(-dt) - \int \cot^{n-2} x \, dx$

$= -\int t^{n-2} x \, dt - \int \cot^{n-2} x \, dx.$

$\left[\text{by } \int t^n dt = \frac{t^{n+1}}{n+1}\right]$

$\left[\text{by } I_{n-2} = \int \cot^{n-2} x \, dx.\right]$

$I_n = -\frac{t^{n-2+1}}{n-2+1} - I_{n-2} = -\frac{t^{n-1}}{n-1} - I_{n-2}$

$\left[\text{by } t = \cot x\right]$

$$\boxed{I_n = -\frac{\cot^{n-1} x}{n-1} - I_{n-2}}$$

The ultimate Integral is $\int \cot x \, dx$ or $\int dx$ according as n is odd or even

(i) $\int \cot x \, dx$ - (odd.)

$I_n = \int \cot^n x \, dx$

$I_1 = \int \cot x \, dx$.

$I_1 = \log|\sin x|$

$\therefore I_n = \int \cot x \, dx = \log|\sin x|$

(ii) $\int dx$ (even)

$I_n = \int \cot^n x \, dx$

$I_0 = \int \cot^0 x \, dx = \int dx$

$I_0 = x$.

$\therefore I_n = \int dx = x$

---

problam .8 (13.8)

$I_n = \int \sec^n x \, dx$, where n being a positive Integer.

$$I_n = \int \sec^n x \, dx$$

$$I_n = \int \sec^{n-2} x \sec^2 x \, dx.$$

$[\text{by } D(\tan x) = \sec$

$$= \int \sec^{n-2} x . D(\tan x) \, dx$$

Let

$u = \sec^{n-2} x$

$\frac{du}{dx} = n-2 \sec^{n-3} x \sec x \tan x$

$du = n-2 \sec^{n-3+1} x \tan x \, dx$

$du = n-2 \sec^{n-2} x \tan x \, dx.$

$dv = D(\tan x$

$v = \tan x.$

Integration By parts

$$\int u \, dv = uv - \int v \, du.$$

$$I_n = \int \underbrace{\sec^{n-2}x}_{u} \; \underbrace{D(\tan x)}_{dv}$$

$$= \sec^{n-2}x \tan x - \int (n-2) \sec^{n-2}x \tan^2 x$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x \tan^2 x$$

$$[\text{by } \tan^2 x = \sec^2 x - 1]$$

$$= \sec^{n-2}x \tan x - (n-2) \int \sec^{n-2}x (\sec^2 x$$

$$= \sec^{n-2}x \tan x - (n-2) \left[\int \sec^{n-2+2}x \, dx - \int \sec\right.$$

$$= \sec^{n-2}x \tan x - (n-2)\left[\int \sec^n x \, dx - \int \sec^{n-2}x \, dx\right.$$

$$\left[\text{by } I_n = \int \sec^n x \, dx\right]$$

$$I_n = \sec^{n-2}x \tan x - (n-2)\left[I_n - I_{n-2}\right]$$

$$I_n = \sec^{n-2}x \tan x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = \sec^{n-2}x \tan x + (n-2) I_{n-2}$$

$$I_n + n I_n - 2 I_n = \sec^{n-2}x \tan x + (n-2) I_{n-2}$$

$$n I_n - I_n = \sec^{n-2}x \tan x + (n-2) I_{n-2}$$

$$(n-1) I_n = \sec^{n-2}x \tan x + (n-2) I_{n-2}$$

$$\boxed{I_n = \frac{1}{n-1}\left[\sec^{n-2}x \tan x + (n-2) I_{n-2}\right]}$$

The ultimate Integral is $\int \sec x \, dx$ or $\int dx$ according as n is odd or even.

(i) $\int \sec x \, dx$ (odd)

$I_n = \int \sec^n x \, dx$.

$I_1 = \int \sec x \, dx$

$I_1 = \log (\tan x + \sec x)$

$\therefore I_n = \int \sec x dx = \log (\tan x + \sec x)$

(ii) $\int dx$ (even)

$I_n = \int \sec^n x \, dx$

$I_0 = \int \sec^0 x \, dx = \int dx$

$I_0 = x$

$\therefore I_n = \int dx = x$

---

problam. 9. (13.9)

$I_n = \int \operatorname{cosec}^n x \, dx$, where n being a positive Integer

$$I_n = \int \operatorname{cosec}^n x \, dx$$

$$I_n = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x \, dx.$$

$$[D(-\cot x) = \operatorname{cosec}^2 x \, dx]$$

$I_n = + \int \operatorname{cosec}^{n-2} x \cdot D(-\cot x)$

Let

$u = \operatorname{cosec}^{n-2} x$

$\frac{du}{dx} = n-2 \operatorname{cosec}^{n-3} x (- \operatorname{cosec} x \cot x)$

$\frac{du}{dx} = -(n-2) \operatorname{cosec}^{n-3+1} x \cot x$

$du = -(n-2) \operatorname{cosec}^{n-2} x \cot x \cdot dx$

$dv = D(\cot x)$

$v = -\cot x$.

Integration By parts.

$$\int u\,dv = uv - \int v\,du$$

$$I_n = +\int \underset{u}{\operatorname{cosec}^{n-2} x} \cdot \underset{dv}{D(-\cot x)}$$

$$= \operatorname{cosec}^{n-2} x\,(-\cot x) - \left[\int -(n-2) \operatorname{cosec}^{n-2} x \cot x\,(-\cot x)\,dx\right]$$

$$= -\operatorname{cosec}^{n-2} x \cdot \cot x - (n-2)\int \operatorname{cosec}^{n-2} x \cot^2 x\,dx$$

$$\left[\text{by } \cot^2 x = \operatorname{cosec}^2 x - 1\right]$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\int \operatorname{cosec}^{n-2} x\,(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\operatorname{cosec}^{n-2} x \cot x - (n-2)\left[\int \operatorname{cosec}^{n-2+2} x\,dx - \int \operatorname{cosec}^{n-2} x\,dx\right]$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2)\left[\int \operatorname{cosec}^{n} x\,dx - \int \operatorname{cosec}^{n-2} x\,dx\right]$$

$$\left[\text{by } I_n = \int \operatorname{cosec}^{n} x\,dx\right]$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2)\left[I_n - I_{n-2}\right]$$

$$I_n = -\operatorname{cosec}^{n-2} x \cot x - (n-2) I_n + (n-2) I_{n-2}$$

$$I_n + (n-2) I_n = -\operatorname{cosec}^{n-2} x \cot x + (n-2) I_{n-2}$$

$I_n + nI_n - 2I_n = -\operatorname{cosec}^{n-2}x \cot x + (n-2) I_{n-2}$

$nI_n - I_n = -\operatorname{cosec}^{n-2}x \cot x + (n-2) I_{n-2}$

$(n-1) I_n = -\operatorname{cosec}^{n-2}x \cot x + (n-2) I_{n-2}$

$$\boxed{I_n = \frac{1}{n-1}\left[-\operatorname{cosec}^{n-2}x \cot x + (n-2) I_{n-2}\right]}$$

The ultimate Integral is $\int \operatorname{cosec} x\, dx$ or $\int dx$ according as n is odd or even.

(i) $\int \operatorname{cosec} x\, dx$ (odd.)

$I_n = \int \operatorname{cosec}^n x\, dx$

$I_1 = \int \operatorname{cosec} x\, dx$

$I_1 = \log(\operatorname{cosec} x - \cot x)$

$I_1 = -\log(\operatorname{cosec} x + \cot x)$

$\therefore I_n = \int \operatorname{cosec} x\, dx = -\log(\operatorname{cosec} x + \cot x)$

(ii) $\int dx$ (even)

$I_n = \int \operatorname{cosec}^n x\, dx$

$I_0 = \int \operatorname{cosec}^0 x\, dx = \int dx$

$I_0 = x$

$\therefore I_n = \int dx = x$

properties of Definite Integral.

3) $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$, If $f(x)$ is an even Function of x.

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

L.H.S

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{-a} f(x)dx + \int_{0}^{a} f(x)dx.$$

[by even Function.

$f(x) = f(-x)$]

$$= -\int_{0}^{-a} f(-x)dx + \int_{0}^{a} f(x)dx.$$

put $y = -x$.

$\frac{dy}{dx} = -1$

$dy = -dx$.

$-dy = dx$

$x = 0 \Rightarrow y = 0$

$x = -a \Rightarrow y = -(-a)$

$y = a$

$$= -\int_{0}^{a} f(y)(-dy) + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(y)dy + \int_{0}^{a} f(x)dx$$

put $y = x$

$$\frac{dy}{dx} = 1$$

$$dy = dx$$

$$y = 0 \Rightarrow x = 0$$

$$y = a \Rightarrow x = a$$

$$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= 2\int_{0}^{a} f(x)\,dx = \text{R.H.S}$$

$$\therefore \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx ,$$

Hence proved.

---

4) $\int_{-a}^{a} f(x)\,dx = 0$, If $f(x)$ is an odd Function of x.

$$\int_{-a}^{a} f(x)\,dx = 0$$

L.H.S

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{-a} f(x)\,dx + \int_{0}^{a} f(x)\,dx.$$

[by odd Function

$f(x) = -f(-x)$]

$$= -\int^{-a} -f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$\int_{-a}^{a} f(x)dx = \int_{0}^{-a} f(-x)\,dx + \int_{0}^{a} f(x)dx.$$

put $y = -x$

$$\frac{dy}{dx} = -1$$

$$dy = -dx$$

$$-dy = dx.$$

$x = 0 \Rightarrow y = 0$

$x = -a \Rightarrow y = -(-a) = a$

$$= -\int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(x)dx.$$

put $y = x$

$$\frac{dy}{dx} = 1$$

$$dy = dx$$

$x = 0 \Rightarrow y = 0$

$x = a \Rightarrow y = a$

$$= -\int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)dx$$

$= 0 = R.H.S$

$\therefore \int_{-a}^{a} f(x)dx = 0$, Hence proved.

---

problam. 10 (13.10)

$I_{m,n} = \int x^m (\log x)^n dx$; where m and n

$$I_{m,n} = \int x^m (\log x)^n dx$$

$$\left[\text{by } D\left(\frac{x^{m+1}}{m+1}\right) = x^m dx \right]$$

$$= \int (\log x)^n \, D\left(\frac{x^{m+1}}{m+1}\right)$$

$u \qquad v$

[by Integration By parts. $\int u dv = uv - \int v du$] $\left[u = (\log x)^n \quad du = n(\log x)^{n-1} \frac{1}{x} \cdot dx \right.$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \int \frac{x^{m+1}}{m+1} n(\log x)^{n-1} \frac{1}{x} dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \int \frac{x^{m+1}}{m+1} n(\log x)^{n-1} x^{-1} dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int x^{m+1-1} (\log x)^{n-1} dx$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} \int x^m (\log x)^{n-1} dx$$

$$\left[\text{by } I_{m,n} = \int x^m (\log x)^n dx \right]$$

$$= (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

$$\boxed{I_{m,n} = (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}}$$

The ultimate Integral is.

$I_{m,0} = \int x^m dx.$

$I_{m,n} = \int x^m (\log x)^n dx$

$I_{m,0} = \int x^m (\log x)^0 dx = \int x^m dx$

$\therefore I_{m,0} = \int x^m dx = \dfrac{x^{m+1}}{m+1}$

$$I_{m,n} = \int x^m (\log x)^n dx = (\log x)^n \frac{x^{m+1}}{m+1} - \frac{n}{m+1} I_{m,n-1}$$

$$I_{4,3} = \int x^4 (\log x)^3 dx = (\log x)^3 \frac{x^5}{5} - \frac{3}{5} I_{4,2}$$

$$I_{4,2} = \int x^4 (\log x)^2 dx = (\log x)^2 \frac{x^5}{5} - \frac{2}{5} I_{4,1}$$

$$I_{4,1} = \int x^4 (\log x)^1 dx = (\log x)^1 \frac{x^5}{5} - \frac{1}{5} I_{4,0}$$

$$I_{m,0} = \int x^m dx = \frac{x^{m+1}}{m+1}$$

$$I_{4,0} = \int x^4 dx = \frac{x^{4+1}}{4+1} = \frac{x^5}{5}.$$

---

(14)

$\int e^{ax} \cos bx \, dx$, a and b are constants.

Let $c = \int e^{ax} \cos bx \, dx$

$s = \int e^{ax} \sin bx \, dx$.

$$C + is = \int e^{ax} \cos bx \, dx + \int e^{ax} i \sin bx \, dx$$

$$= \int e^{ax} (\cos bx + i \sin bx) \, dx$$

[by Euler's Formula,
$e^{i\theta} = \cos\theta + i\sin\theta$]

$$= \int e^{ax} . e^{ibx} \, dx$$

$$= \int e^{(a+ib)x} dx$$

[by $\int e^{ax} dx = \frac{e^{ax}}{a}$]

$$c+is = \frac{e^{(a+ib)x}}{a+ib} \times \frac{a-ib}{a-ib}$$

$$= \frac{e^{(a+ib)x} \times (a-ib)}{(a+ib)(a-ib)}$$

$$\left[\text{by } (a+b)(a-b) = a^2-b^2 \right]$$

$$= \frac{e^{(a+ib)x} \times (a-ib)}{a^2-(ib)^2}$$

$$= \frac{e^{ax} \cdot e^{ibx} (a-ib)}{a^2 - i^2 b^2}$$

$$= \frac{e^{ax} (a-ib) e^{ibx}}{a^2 - (-1) b^2}$$

$$\left[\text{by } e^{ibx} = \cos bx + i \sin bx \right]$$

$$\boxed{c+is = \frac{e^{ax} (a-ib)(\cos bx + i \sin bx)}{a^2+b^2}}$$

c = Real part of

$$\frac{e^{ax} (a-ib)(\cos bx + i \sin bx)}{a^2+b^2}$$

Sol.

$$= e^{ax} \frac{(a \cos bx - i^2 b \sin bx)}{a^2+b^2}$$

$$= e^{ax} \frac{(a \cos bx - (-1) b \sin bx)}{a^2+b^2}$$

$$= e^{ax} \frac{(a \cos bx + b \sin bx)}{a^2+b^2}$$

$\therefore$ c = Real part of

$$e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{a^2+b^2} = e^{ax}\frac{(a\cos bx + b\sin bx)}{a^2+b^2}$$

s = Imaginary part of

$$e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$$

Sol.

$$= e^{ax}\frac{(a-b)(\sin bx + \cos bx)}{a^2+b^2}$$

$$= e^{ax}\frac{(a\sin bx - b\cos bx)}{a^2+b^2}$$

$\therefore$ s = Imaginary part of

$$e^{ax}\frac{(a-ib)(\cos bx + i\sin bx)}{a^2+b^2} = e^{ax}\frac{(a\sin bx - b\cos bx)}{a^2+b^2}$$

§14. $\int e^{ax} \cos bx \, dx$, a and b are constants.

Let.

$$C = \int e^{ax} \cos bx \, dx.$$

$$S = \int e^{ax} \sin bx \, dx.$$

$$C + iS = \int e^{ax} \cos bx \, dx + \int e^{ax} i \sin bx \, dx.$$

$$= \int e^{ax} (\cos bx + i \sin bx) \, dx.$$

by Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$$C + iS = \int e^{ax} . e^{ibx} \, dx.$$

$$= \int e^{(a+ib)x} \, dx.$$

$$\text{by } \int e^{ax} \, dx = \frac{e^{ax}}{a} + c$$

$$C + iS = \frac{e^{(a+ib)x}}{a+ib} \times \frac{a-ib}{a-ib}$$

$$= \frac{e^{(a+ib)x} \times (a-ib)}{(a+ib)(a-ib)}$$

$$\text{by } (a+b)(a-b) = a^2 - b^2$$

$$= \frac{e^{ax+ibx} \times (a-ib)}{a^2 - (ib)^2}$$

$$c+is = \frac{e^{ax} \cdot e^{ibx} (a-ib)}{a^2 - i^2 b^2} = \frac{e^{ax} \cdot e^{ibx} (a-ib)}{a^2 - (-1) b^2}$$

$$= \frac{e^{ax} \cdot e^{ibx} (a-ib)}{a^2+b^2}$$

$$\boxed{\text{by } e^{ibx} = \cos bx + i \sin bx}$$

$$c+is = \frac{e^{ax} (a-ib)(\cos bx + i \sin bx)}{a^2+b^2}$$

$$c = \text{Real part of. } \frac{e^{ax} (a-ib)(\cos bx + i \sin bx)}{a^2+b^2}$$

Sol.

$$c = \int e^{ax} \cos bx\,dx = \text{Real part of } \frac{e^{ax} (a-ib)(\cos bx + i \sin bx)}{a^2+b^2}.$$

$$= \text{Real part of } \frac{e^{ax} (a-ib)(\cos bx + i \sin bx)}{a^2+b^2}$$

$$= \text{Real part of } \frac{e^{ax} (a \cos bx - i^2 b \sin bx)}{a^2+b^2}$$

$$= \frac{e^{ax} [a \cos bx - (-1) b \sin bx]}{a^2+b^2}$$

$$= \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2+b^2}$$

$$\therefore \quad c = \int e^{ax} \cos bx\,dx = \frac{e^{ax} (a \cos bx + b \sin bx)}{a^2+b^2}$$

$$S = \text{Imaginary part of } \frac{e^{ax}(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$$

$$S = \int e^{ax} \sin bx \, dx$$

$$S = \text{Imaginary part of } \frac{e^{ax}(a-ib)(\cos bx + i\sin bx)}{a^2+b^2}$$

$$S = \frac{e^{ax}(a-b)(\cos bx + \sin bx)}{a^2+b^2}$$

$$= \frac{e^{ax}(a-b)(\sin bx + \cos bx)}{a^2+b^2}$$

$$= \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

$$\therefore S = \int e^{ax} \sin bx \, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}.$$

Examples.

Example. 1

$$\int e^{2x} \cdot \cos 3x \, dx.$$

$$C = \int e^{2x} \cos 3x \, dx.$$

$$\boxed{C = \int e^{ax} \cos bx \, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}}$$

$$a = 2 \; , \; b = 3$$

$$C = \int e^{2x} \cos 3x \, dx = \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{(2)^2 + (3)^2}$$

$$= \frac{e^{2x}(2\cos 3x + 3\sin 3x)}{4+9}$$

$$= \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x)$$

$$\therefore c = \int e^{2x}\cos 3x\,dx = \frac{e^{2x}}{13}(2\cos 3x + 3\sin 3x)$$

Example. 2

$$\int e^{-x}\sin^2 x\,dx.$$

$$\int e^{-x}\sin^2 x\,dx = \int e^{-x}\left(\frac{1-\cos 2x}{2}\right)dx.$$

$$= \int \frac{e^{-x} - e^{-x}\cos 2x}{2}\,dx.$$

$$= \int \frac{e^{-x}}{2}dx - \int \frac{e^{-x}\cos 2x}{2}\,dx.$$

$$= \frac{1}{2}\int e^{-x}dx - \frac{1}{2}\int e^{-x}\cos 2x\,dx.$$

$$\boxed{\int e^{-x}dx = \frac{e^{-x}}{-1} = -e^{-x} + c}$$

$$= -\frac{e^{-x}}{2} - \frac{1}{2}\int e^{-x}\cos 2x\,dx. \quad (1)$$

put $c = \int e^{-x}\cos 2x\,dx.$

$$\boxed{c = \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}}$$

$\therefore \quad a = -1, \quad b = 2$

$$c = \int e^{-x} \cos 2x \, dx = \frac{e^{-x}(-\cos 2x + 2\sin 2x)}{(-1)^2 + (2)^2}$$

$$= \frac{e^{-x}(-\cos 2x + 2\sin 2x)}{1+4}$$

$$c = \frac{e^{-x}(-\cos 2x + 2\sin 2x)}{5} \quad \text{---} \quad (2)$$

substitude eqn (2) in (1).

$$\int e^{-x} \sin^2 x \, dx = -\frac{e^{-x}}{2} - \frac{1}{2} \, \frac{e^{-x}(-\cos 2x + 2\sin 2x)}{5}$$

$$\therefore \int e^{-x} \sin^2 x \, dx = -\frac{e^{-x}}{2} - \frac{1}{2} e^{-x} \, \frac{-\cos 2x + 2\sin 2x}{5}.$$

Example. 3

$$\int e^{ax} \cos mx \cos nx \, dx.$$

$$\int e^{ax} \cos mx \cos nx \, dx.$$

$$\boxed{\cos Ax \cos Bx = \frac{1}{2}\left[\cos(A+B)x + \cos(A-B)x\right]}$$

$$\int e^{ax} \cos mx \cos nx \, dx = \int e^{ax} \frac{1}{2}\left[\cos(m+n)x + \cos(m-n)x\right]dx$$

$$= \frac{1}{2}\int e^{ax}\left[\cos(m+n)x + \cos(m-n)x\right]dx$$

$$= \frac{1}{2}\int \left[e^{ax}\cos(m+n)x + e^{ax}\cos(m-n)x\right]dx$$

$$= \frac{1}{2}\left[\int e^{ax}\cos(m+n)x \, dx + \int e^{ax}\cos(m-n)x \, dx\right]$$

$$\int e^{ax}\cos mx \cos nx\,dx = \frac{1}{2}\left[\int e^{ax}\cos(m+n)x\,dx + \int e^{ax}\cos(m-n)x\,dx\right] \quad \text{—— (1)}$$

Let $c_1 = \int e^{ax}\cos(m+n)x\,dx.$

$c_2 = \int e^{ax}\cos(m-n)x\,dx.$

$$c = c_1 + c_2 = \int e^{ax}\cos(m+n)x\,dx + \int e^{ax}\cos(m-n)x\,dx$$

$$\boxed{c = \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}}$$

$c_1 \Rightarrow a = a, \quad b = (m+n)$

$c_2 \Rightarrow a = a, \quad b = (m-n)$

$$c = \frac{e^{ax}\left[a\cos(m+n)x + (m+n)\sin(m+n)x\right]}{a^2+(m+n)^2} + \frac{e^{ax}\left[a\cos(m-n)x + (m-n)\sin(m-n)x\right]}{a^2+(m-n)^2}$$

$$c = e^{ax}\left[\frac{a\cos(m+n)x + (m+n)\sin(m+n)x}{a^2+(m+n)^2} + \frac{a\cos(m-n)x + (m-n)\sin(m-n)x}{a^2+(m-n)^2}\right] \quad \text{—— (2)}$$

Substitude eqn (2) in (1).

$$\therefore \int e^{ax} \cos mx \cos nx \, dx = \frac{1}{2} e^{ax}$$

$$\left[\frac{a\cos(m+n)x + (m+n)\sin(m+n)x}{a^2+(m+n)^2} + \frac{a\cos(m-n)x + (m-n)\sin(m-n)x}{a^2+(m-n)^2}\right]$$

Exercise 20.

Integrate.

1). $e^x \sin 2x$

$$S = \int e^x \sin 2x \cdot dx$$

$$\boxed{S = \int e^{ax} \sin bx \, dx = e^{ax} \frac{a\sin bx - b\cos bx}{a^2+b^2}.}$$

$$a = 1 \quad , b = 2.$$

$$S = \int e^x \sin 2x \, dx = \frac{e^x(\sin 2x - 2\cos 2x)}{(1)^2 + (2)^2}$$

$$= \frac{e^x(\sin 2x - 2\cos 2x)}{1+4.}$$

$$= \frac{e^x}{5}(\sin 2x - 2\cos 2x)$$

$$\therefore S = \int e^x \sin 2x \, dx = \frac{e^x}{5}(\sin 2x - 2\cos 2x)$$

2) $e^{-3x} \sin \frac{x}{2}$

$$S = \int e^{-3x} \sin \frac{x}{2} \, dx$$

$$s=\int e^{-3x}\sin\frac{1}{2}x\,dx.$$

$$\boxed{s=\int e^{ax}\sin bx\,dx = e^{ax}\,\frac{a\sin bx - b\cos bx}{a^2+b^2}}$$

$$a=-3 \quad , \ b=\frac{1}{2}$$

$$s=\int e^{-3x}\sin\frac{1}{2}x\,dx.$$

$$= e^{-3x}\left[\frac{-3\sin\frac{1}{2}x-\frac{1}{2}\cos\frac{1}{2}x}{(-3)^2+\left(\frac{1}{2}\right)^2}\right]$$

$$= e^{-3x}\left[\frac{-3\sin\frac{x}{2}-\frac{1}{2}\cos\frac{x}{2}}{9+\frac{1}{4}}\right]$$

$$= e^{-3x}\left[\frac{-3\sin\frac{x}{2}-\frac{1}{2}\cos\frac{x}{2}}{\frac{36+1}{4}}\right]$$

$$= -e^{-3x}\left[\frac{3\sin\frac{x}{2}+\frac{1}{2}\cos\frac{x}{2}}{\frac{37}{4}}\right]$$

$$s=-4e^{-3x}\left(\frac{3\sin\frac{x}{2}+\frac{1}{2}\cos\frac{x}{2}}{37}\right)$$

$$\therefore s=\int e^{-3x}\sin\frac{x}{2}\,dx = -4e^{-3x}\,\frac{\left(3\sin\frac{x}{2}+\frac{1}{2}\cos\frac{x}{2}\right)}{37}$$

3) $e^{4x}\cos 3x$.

$$C=\int e^{4x}\cos 3x\,dx.$$

$$\boxed{C=\int e^{ax}\cos bx\,dx=\frac{e^{ax}(a\cos bx+b\sin bx)}{a^2+b^2}}$$

$$a=4\quad,\quad b=3.$$

$$C=\int e^{4x}\cos 3x\,dx=\frac{e^{4x}(4\cos 3x+3\sin 3x)}{(4)^2+(3)^2}$$

$$=\frac{e^{4x}(4\cos 3x+3\sin 3x)}{16+9}.$$

$$=\frac{e^{4x}}{25}(4\cos 3x+3\sin 3x)$$

$$\therefore C=\int e^{4x}\cos 3x\,dx=\frac{e^{4x}}{25}(4\cos 3x+3\sin 3x)$$

---

5) $e^{ax}\sin(bx+c)$

$$S=\int e^{ax}\sin(bx+c)\,dx.$$

$$\boxed{S=\int e^{ax}\sin bx\,dx=e^{ax}\,\frac{a\sin bx-b\cos bx}{a^2+b^2}}$$

$$a=a\quad,\ b=b.$$

$$S=\int e^{ax}\sin(bx+c)\,dx.$$

$$=\frac{e^{ax}[a\sin(bx+c)-b\cos(bx+c)]}{a^2+b^2}$$

$\therefore s=\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin(bx+c)-b\cos(bx+c)\right]$

6) $e^x\cos^2 x$

$$\int e^x\cos^2 x\,dx = \int e^x\frac{(1+\cos 2x)}{2}\,dx.$$

$$=\int \frac{e^x+e^x\cos 2x}{2}\,dx$$

$$=\int \frac{e^x}{2}\,dx+\int\frac{e^x\cos 2x}{2}\,dx.$$

$$=\frac{1}{2}\int e^x dx+\frac{1}{2}\int e^x\cos 2x\,dx.$$

$$\boxed{\int e^x dx = e^x + c}$$

$$=\frac{e^x}{2}+\frac{1}{2}\int e^x\cos 2x\,dx \quad \text{—— (1)}$$

put. $c=\int e^x\cos 2x\,dx.$

$$\boxed{c=\int e^{ax}\cos bx\,dx = e^{ax}\frac{(a\cos bx+b\sin bx)}{a^2+b^2}}$$

$a=1\ ,\ b=2$

$$c=\int e^x\cos 2x\,dx = e^x\frac{(1\cos 2x+2\sin 2x)}{(1)^2+(2)^2}$$

$$=\frac{e^x(\cos 2x+2\sin 2x)}{1+4}$$

$$c = \int e^{x} \cos 2x \, dx = e^{x} \frac{(\cos 2x + 2\sin 2x)}{5} \quad \text{--- (2)}$$

Substitude eqn (2) in (1).

$$\int e^{x} \cos^2 x \, dx = \frac{e^{x}}{2} + \frac{1}{2} e^{x} \frac{(\cos 2x + 2\sin 2x)}{5}$$

$$\therefore \int e^{x} \cos^2 x \, dx = \frac{e^{x}}{2}\left[1 + \frac{\cos 2x + 2\sin 2x}{5}\right]$$

7) $e^{2x} \cos(3x+4)$

$$c = \int e^{2x} \cos(3x+4) \, dx.$$

$$\boxed{c = \int e^{ax} \cos bx \, dx = e^{ax} \frac{(a\cos bx + b\sin bx)}{a^2+b^2}.}$$

$$a = 2, \quad b = 3.$$

$$c = \int e^{2x} \cos(3x+4) \, dx = e^{2x} \frac{[2\cos(3x+4) + 3\sin(3x+}{(2)^2 + (3)^2}$$

$$= e^{2x} \frac{[2\cos(3x+4) + 3\sin(3x+4)}{4+9}.$$

$$= \frac{e^{2x}}{13}[2\cos(3x+4) + 3\sin(3x+4)$$

$$\therefore c = \int e^{2x} \cos(3x+4) \, dx = \frac{e^{2x}}{13}\left[2\cos(3x+4) + 3\sin(3x+4)\right]$$

8) $e^{x} \sin 3x \cos 2x$

$$\int e^{x} \sin 3x \cos 2x$$

$$\boxed{\sin Ax \cos Bx = \frac{1}{2}\left[\sin(A+B)x + \sin(A-B)x\right]}$$

$$\int e^x \sin 3x \cos 2x = \int e^x \frac{1}{2}\left[\sin(3+2)x + \sin(3-2)x\right]dx$$

$$= \frac{1}{2}\int e^x \left[\sin 5x + \sin x\right]dx$$

$$= \frac{1}{2}\left[\int e^x \sin 5x\,dx + \int e^x \sin x\,dx\right] \quad \text{—— (1)}$$

Let

$$S_1 = \int e^x \sin 5x\,dx$$

$$S_2 = \int e^x \sin x\,dx.$$

$$S = S_1 + S_2. = \int e^x \sin 5x\,dx + \int e^x \sin x\,dx$$

$$\boxed{S = \int e^{ax} \sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}}$$

$S_1 \Rightarrow a = 1, \; b = 5$

$S_2 \Rightarrow a = 1, \; b = 1$

$$S = \frac{e^x(\sin 5x - 5\cos 5x)}{(1)^2 + (5)^2} + \frac{e^x(\sin x - \cos x)}{(1)^2 + (1)^2}$$

$$= \frac{e^x(\sin 5x - 5\cos 5x)}{1+25} + \frac{e^x(\sin x - \cos x)}{1+1}$$

$$= \frac{e^x(\sin 5x - 5\cos 5x)}{26} + \frac{e^x(\sin x - \cos x)}{2}$$

$$s = \frac{e^x}{12}\left[\frac{\sin 5x - 5\cos 5x}{13} + \sin x - \cos x\right] \quad \text{——— ②}$$

substitude eqn ② in ①

$$\int e^x \sin 3x \cos 2x = \frac{1}{2}\left(\frac{e^x}{13}\left[\frac{\sin 5x - 5\cos 5x}{2} + \sin x - \cos x\right]\right)$$

$$= \frac{e^x}{2}\left[\frac{\sin 5x - 5\cos 5x}{26} + \frac{\sin x - \cos x}{2}\right]$$

$$= \frac{e^x}{4}\left[\frac{\sin 5x - 5\cos 5x}{13} + \sin x - \cos x\right]$$

$$\therefore \int e^x \sin 3x \cos 2x = \frac{e^x}{4}\left[\frac{\sin 5x - 5\cos 5x}{13} + \sin x - \cos x\right]$$

---

9) $e^{2x} \cos 5x \cos 4x.$

$$\int e^{2x} \cos 5x \cos 4x \, dx$$

$$\boxed{\cos Ax \cos Bx = \frac{1}{2}\left[\cos(A+B)x + \cos(A-B)x\right]}$$

$$\int e^{2x} \cos 5x \cos 4x = \int e^{2x} \frac{1}{2}\left[\cos(5+4)x + \cos(5-4)x\right] dx$$

$$= \frac{1}{2}\int e^{2x}\left[\cos 9x + \cos x\right] dx$$

$$= \frac{1}{2}\left[\int e^{2x} \cos 9x \, dx + \int e^{2x} \cos x \, dx\right] \quad \text{——— ①}$$

Let

$$c_1 = \int e^{2x} \cos 9x \, dx$$

$$c_2 = \int e^{2x} \cos x \, dx.$$

$$c = c_1 + c_2 = \int e^{2x} \cos 9x \, dx + \int e^{2x} \cos x \, dx$$

$$\boxed{c = \int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}}$$

$c_1 \Rightarrow a = 2, \quad b = 9$

$c_2 \Rightarrow a = 2, \quad b = 1$

$$c = \frac{e^{2x}(2\cos 9x + 9\sin 9x)}{(2)^2 + (9)^2} + \frac{e^{2x}(2\cos x + \sin x)}{(2)^2 + (1)^2}$$

$$= \frac{e^{2x}(2\cos 9x + 9\sin 9x)}{4 + 81} + \frac{e^{2x}(2\cos x + \sin x)}{4 + 1}$$

$$= \frac{e^{2x}(2\cos 9x + 9\sin 9x)}{85} + \frac{e^{2x}(2\cos x + \sin x)}{5}$$

$$c = \frac{e^{2x}}{5}\left[\frac{2\cos 9x + 9\sin 9x}{17} + 2\cos x + \sin x\right] \quad \text{---} (2)$$

substitude (2) in (1)

$$\int e^{2x} \cos 5x \cos 4x = \frac{1}{2} \frac{e^{2x}}{5}\left[\frac{2\cos 9x + 9\sin 9x}{17} + 2\cos x + \sin x\right]$$

$$= \frac{e^{2x}}{10}\left[\frac{2\cos 9x + 9\sin 9x}{17} + 2\cos x + \sin x\right]$$

$$\therefore \int e^{2x} \cos 5x \cos 4x \, dx = \frac{e^{2x}}{10}\left[\frac{2\cos 9x + 9\sin 9x}{17} + 2\cos x + \sin x\right]$$

10) $e^{-3x} \sin 3x \sin 2x$.

$$\int e^{-3x} \sin 3x \sin 2x \, dx.$$

$$\boxed{\sin Ax \sin Bx = \frac{1}{2}\left[\cos(A-B)x - \cos(A+B)x\right]}$$

$$\int e^{-3x} \sin 3x \sin 2x \, dx = \int e^{-3x} \frac{1}{2}\left[\cos(3-2)x - \cos(3+2)x\right] dx$$

$$= \frac{1}{2}\int e^{-3x}\left[\cos x - \cos 5x\right] dx$$

$$= \frac{1}{2}\left[\int e^{-3x} \cos x \, dx - \int e^{-3x} \cos 5x \, dx\right] \quad \text{--- (1)}$$

let

$$c_1 = \int e^{-3x} \cos x \, dx$$

$$c_2 = \int e^{-3x} \cos 5x \, dx.$$

$$c = c_1 + c_2 = \int e^{-3x} \cos x \, dx - \int e^{-3x} \cos 5x \, dx.$$

$$\boxed{c = \int e^{ax} \cos bx \, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2+b^2}}$$

$c_1 \Rightarrow a = -3 \;, b = 1$

$c_2 \Rightarrow a = -3 \;, b = 5$

$$c = \frac{e^{-3x}(-3\cos x + \sin x)}{(-3)^2 + (1)^2} - \frac{e^{-3x}(-3\cos 5x + 5\sin 5x)}{(-3)^2 + (5)^2}$$

$$= \frac{e^{-3x}(-3\cos x + \sin x)}{9+1} - \frac{e^{-3x}(-3\cos 5x + 5\sin 5x)}{9+25}$$

$$= \frac{e^{-3x}(\sin x - 3\cos x)}{10} + \frac{e^{-3x}(3\cos 5x - 5\sin 5x)}{34}$$

$$c = \frac{e^{-3x}}{2}\left[\frac{\sin x - 3\cos x}{5} + \frac{3\cos 5x - 5\sin 5x}{17}\right] \quad \text{--- (2)}$$

substitude equ (2) in (1)

$$\int e^{-3x}\sin 3x \sin 2x\,dx = \frac{1}{2}\times\frac{e^{-3x}}{2}\left[\frac{\sin x - 3\cos x}{5} + \frac{3\cos 5x - 5\sin 5x}{17}\right]$$

$$= \frac{e^{-3x}}{4}\left[\frac{\sin x - 3\cos x}{5} + \frac{3\cos 5x - 5\sin 5x}{17}\right]$$

$$\therefore \int e^{-3x}\sin 3x \sin 2x\,dx = \frac{e^{-3x}}{4}\left[\frac{\sin x - 3\cos x}{5} + \frac{3\cos 5x - 5\sin 5x}{17}\right]$$

---

11) $a^x \sin x$

$$\int a^x \sin x \cdot dx$$

$$\int \log a^x \sin x\,dx$$

$$\int x\log a \sin x\,dx$$

$$\boxed{\int a x \sin bx\,dx = \frac{a^x(a\sin bx - b\cos bx)}{a^2+b^2}}$$

$a = \log a$, $b = 1$

$$\int x\log a \sin x\,dx = \frac{\log a^x(\log a \sin x - \cos x)}{(\log a)^2 + (1)^2}$$

$$= \frac{x\log a^x(\sin x \log a - \cos x)}{(\log a)^2 + 1}$$

$$\therefore \int a^x \sin x\,dx = \frac{a^x(\sin x \log a - \cos x)}{(\log a)^2 + 1}$$

12) $3^x \sin 2x$

$$\int 3^x \sin 2x \, dx$$

$$\int \log 3^x \sin 2x \, dx$$

$$\int x \log 3 \sin 2x \, dx.$$

$$\boxed{\int a^x \sin bx \, dx = \frac{a^x (a \sin bx - b \cos bx)}{a^2 + b^2}}$$

$a = \log 3$, $b = 2$

$$\int x \log 3 \sin 2x \, dx = \frac{\log 3^x (\log 3 \sin 2x - 2\cos 2x)}{(\log 3)^2 + (2)^2}$$

$$= \frac{x \log 3 (\sin 2x \log 3 - 2\cos 2x)}{(\log 3)^2 + 4}$$

$$= \frac{3^x (\sin 2x \log 3 - 2\cos 2x)}{(\log 3)^2 + 4}$$

$$= 3^x \frac{\log 3 . \sin 2x - 2\cos 2x}{(\log 3)^2 + 4}$$

$$\therefore \int 3^x \sin 2x \, dx = 3^x \frac{\log 3 . \sin 2x - 2\cos 2x}{(\log 3)^2 + 4}$$

15.1 Bernoulli's Formula

This formula is merely an extension of the formula of integration by parts.

Let dashes denote successive differentiation and suffixes denote successive integration

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

The advantage of the above formula can be seen in examples where the tediouness of successive integration by parts is avoided.

Examples

Ex. 1

Evaluate $\int x^4 e^x\,dx$.

$$\int x^4 e^x\,dx.$$

Here

$u = x^4$

$u' = 4x^3$

$u'' = 4(3)x^2$

$= 12x^2$

$u''' = 12(2)x$

$= 24x$

$u'''' = 24$

$\int dv = \int e^x\,dx$

$v = e^x$

$v_1 = e^x$

$v_2 = e^x$

$v_3 = e^x$

$v_4 = e^x$.

Bernoulli's formula

$$\boxed{\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + u''''v_4 - \cdots}$$

$$\therefore \int x^4 e^x\,dx = x^4e^x - 4x^3e^x + 12x^2e^x - 24xe^x + 24e^x$$

Example. 2

Evaluate $\int x^3 \cos 2x\,dx$.

$$\int x^3 \cos 2x\,dx$$

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Here

$u = x^3$

$u' = 3x^2$

$u'' = 3(2)x$

$= 6x$

$u''' = 6$

$\int dv = \int \cos 2x \, dx$

$v = \frac{\sin 2x}{2}$

$v_1 = -\frac{\cos 2x}{2\times 2} = -\frac{\cos 2x}{4}$

$v_2 = -\frac{\sin 2x}{4\times 2} = -\frac{\sin 2x}{8}$

$v_3 = -\left(-\frac{\cos 2x}{8\times 2}\right)$

$= \frac{\cos 2x}{16}$

$$\int x^3 \cos 2x \, dx = x^3 \frac{\sin 2x}{2} - 3x^2\left(-\frac{\cos 2x}{4}\right) + \cancel{6}^{3}x\left(-\frac{\sin 2x}{\cancel{8}_4}\right) - \cancel{6}^{3}\left(\frac{\cos 2x}{\cancel{16}_8}\right)$$

$$= \frac{x^3 \sin 2x}{2} + \frac{3x^2 \cos 2x}{4} - \frac{3x \sin 2x}{4} - \frac{3\cos 2x}{8}$$

$$= \frac{1}{2}\left[x^3 \sin 2x + \frac{3x^2 \cos 2x}{2} - \frac{3x \sin 2x}{2} - \frac{3\cos 2x}{4}\right]$$

$$\therefore \int x^3 \cos 2x \, dx = \frac{1}{2}\left[x^3 \sin 2x + \frac{3x^2 \cos 2x}{2} - \frac{3x \sin 2x}{2} - \frac{3\cos 2x}{4}\right]$$

Exercises 21

Integrate:

1) $x^3 e^{-2x}$

2) $x^4 \sin x$

3) $x^3 \sin 3x$

4) $x^2 (e^x + e^{-x})$

5) $x^5 \cos \frac{x}{2}$

6) $x^3 \sin nx$

Exercise. 20

4) Integrate $e^{m\cos^{-1}x}$

$$\int e^{m\cos^{-1}x}\,dx.$$

- $u = \cos^{-1}x$

$x = \cos u$

$\frac{dx}{du} = -\sin u$

$dx = -\sin u\,du$

$$\int e^{m\cos^{-1}x}\,dx = \int e^{mu}(-\sin u)\,du$$

$$= -\int e^{mu}\sin u\,du$$

$$\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}}$$

$a = m$, $b = 1$

$$-\int e^{mu}\sin u\,du = -\frac{e^{mu}(m\sin u - \cos u)}{m^2+1^2}$$

$$= -\frac{e^{m\cos^{-1}x}}{1+m^2}\left[m\sin(\cos^{-1}x) - \cos(\cos^{-1}x)\right.$$

- $\cos(\cos^{-1}x) = x$
- $\sin(\cos^{-1}x) = ?$

$\theta = \cos^{-1}x \Rightarrow \cos\theta = x$

$\sin^2\theta + \cos^2\theta = 1$

$\sin^2\theta = 1 - \cos^2\theta$

$\sin^2\theta = 1 - x^2$

$\sin\theta = \sqrt{1-x^2}$

$\sin(\cos^{-1}x) = \sin\theta = \sqrt{1-x^2}$

$$\int e^{m\cos^{-1}x}\,dx = -\frac{e^{m\cos^{-1}x}}{1+m^2}\left[m\sqrt{1-x^2} - x\right]$$

$$= e^{m\cos^{-1}x}\left(\frac{x \cancel{+} m\sqrt{1-x^2}}{1+m^2}\right)$$

$$\therefore \int e^{m\cos^{-1}x}\,dx = e^{m\cos^{-1}x}\left(\frac{x-m\sqrt{1-x^2}}{1+m^2}\right)$$

Exercise . 21

Integrate

1) x^3e^{-2x}

$$\int x^3e^{-2x}\,dx.$$

$$\boxed{\begin{array}{c}\text{Bernoulli's formula}\\ \int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots\end{array}}$$

Here.

$u = x^3$

$u' = 3x^2$

$u'' = 3(2)x$

$= 6x$

$u''' = 6$

$\int dv = \int e^{-2x}\,dx$

$v = \frac{e^{-2x}}{-2} = -\frac{e^{-2x}}{2}$

$v_1 = -\frac{e^{-2x}}{2\times -2} = \frac{e^{-2x}}{4}$

$v_2 = \frac{e^{-2x}}{4\times -2} = -\frac{e^{-2x}}{8}$

$v_3 = -\frac{e^{-2x}}{8\times -2} = \frac{e^{-2x}}{16}$

$$\int x^3e^{-2x}\,dx = x^3\left(-\frac{e^{-2x}}{2}\right) - 3x^2\left(\frac{e^{-2x}}{4}\right) + 6x\left(-\frac{e^{-2x}}{8}\right) - 6\left(\frac{e^{-2x}}{16}\right)$$

$$= e^{-2x}\left[-\frac{x^3}{2} - \frac{3x^2}{4} - \frac{6x}{8} - \frac{\cancel{6}\,3}{\cancel{16}\,8}\right]$$

$$= -e^{-2x}\left[\left(\frac{x^3}{2}\times\frac{4}{4}\right) + \left(\frac{3x^2}{4}\times\frac{2}{2}\right) + \frac{6x}{8} + \frac{3}{8}\right]$$

$$= -e^{-2x}\left[\frac{4x^3}{8}+\frac{6x^2}{8}+\frac{6x}{8}+\frac{3}{8}\right]$$

$$= -\frac{e^{-2x}}{8}\left[4x^3+6x^2+6x+3\right]$$

$$\therefore \int x^3 e^{-2x}\,dx = -\frac{e^{-2x}}{8}\left[4x^3+6x^2+6x+3\right]$$

---

2) $x^4 \sin x$

$$\int x^4 \sin x \, dx.$$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + u''''v_4 - \cdots$$

Here.

$u = x^4$

$u' = 4x^3$

$u'' = 4(3)x^2$
$= 12x^2$

$u''' = 12(2)x$
$= 24x$

$u'''' = 24$

$\int dv = \int \sin x \, dx$

$v = -\cos x$

$v_1 = -\sin x$

$v_2 = -(-\cos x)$
$= \cos x$

$v_3 = \sin x$

$v_4 = -\cos x.$

$$\int x^4 \sin x\,dx = x^4(-\cos x) - 4x^3(-\sin x) + 12x^2\cos x - 24x\sin x + 24(-\cos x)$$

$$= -x^4\cos x + 4x^3\sin x + 12x^2\cos x - 24x\sin x - 24\cos x$$

$$= -x^4\cos x + 12x^2\cos x - 24\cos x + 4x^3\sin x - 24x\sin x$$

$$= \cos x(-x^4+12x^2-24) + 4x\sin x(x^2-6$$

$$= \cos x(-x^4+12x^2-24)+4x(x^2-6)\sin x$$

$$\therefore \int x^4 \sin x\,dx = \cos x(-x^4+12x^2-24)+4x(x^2-6)\sin x$$

---

3) $x^3 \sin 3x$

$$\int x^3 \sin 3x\,dx$$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Here

$u = x^3$

$u' = 3x^2$

$u'' = 3(2)x$

$= 6x$

$u''' = 6$

$\int dv = \int \sin 3x\,dx$

$v = -\frac{\cos 3x}{3}$

$v_1 = -\frac{\sin 3x}{3\times 3} = -\frac{\sin 3x}{9}$

$v_2 = -\left(-\frac{\cos 3x}{9\times 3}\right)$

$= \frac{\cos 3x}{27}$

$v_3 = \frac{\sin 3x}{27\times 3} = \frac{\sin 3x}{81}$

$$\int x^3 \sin 3x\,dx = x^3\left(-\frac{\cos 3x}{3}\right) - 3x^2\left(-\frac{\sin 3x}{9}\right) + 6x\left(\frac{\cos 3x}{27}\right) - 6\left(\frac{\sin 3x}{81}\right)$$

$$= -\frac{x^3 \cos 3x}{3} + \frac{\overset{2}{\cancel{6}}x \cos 3x}{\underset{9}{\cancel{27}}} + \frac{3x^2 \sin 3x}{9} - \frac{\overset{2}{\cancel{6}} \sin 3x}{\underset{27}{\cancel{81}}}$$

$$= -\frac{x^3 \cos 3x}{3} + \frac{2x \cos 3x}{9} + \frac{3x^2 \sin 3x}{9} - \frac{2 \sin 3x}{27}$$

$$= -\left(\frac{x^3\cos 3x}{3} \times \frac{9}{9}\right) + \left(\frac{2x\cos 3x}{9} \times \frac{3}{3}\right)$$

$$+\left(\frac{3x^2 \sin 3x}{9} \times \frac{3}{3}\right) - \frac{2\sin 3x}{27}$$

$$= -\frac{9x^3\cos 3x}{27} + \frac{6x\cos 3x}{27} + \frac{9x^2\sin 3x}{27} - \frac{2\sin 3x}{27}$$

$$= \frac{1}{27}\left[-9x^3\cos 3x + 6x\cos 3x + 9x^2 \sin 3x - 2\sin 3x\right]$$

$$= \frac{1}{27}\left[6x\cos 3x - 9x^3\cos 3x + 9x^2\sin 3x - 2\sin 3x\right]$$

$$= \frac{1}{27}\left[3x(2-3x^2)\cos 3x + \sin 3x(9x^2-2)\right]$$

$$\therefore \int x^3 \sin 3x\, dx = \frac{1}{27}\left[3x(2-3x^2)\cos 3x + \sin 3x(9x^2-2)\right]$$

4) $x^2(e^x + e^{-x})$

$\int x^2(e^x + e^{-x})\,dx$

Bernoulli's formula.

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

Here:

$u = x^2$

$u' = 2x$

$u'' = 2$

$\int dv = \int (e^x + e^{-x})\,dx$

$v = e^x + \frac{e^{-x}}{-1} = e^x - e^{-x}$

$v_1 = e^x - \frac{e^{-x}}{-1}$

$= e^x + e^{-x}$

$v_2 = e^x + \frac{e^{-x}}{-1} = e^x - e^{-x}$

$$\int x^2(e^x+e^{-x})dx = x^2(e^x-e^{-x}) - 2x(e^x+e^{-x}) + 2(e^x-e^{-x})$$

$$= x^2e^x - x^2e^{-x} - 2xe^x - 2xe^{-x} + 2e^x - 2e^{-x}$$

$$= (x^2e^x - 2xe^x + 2e^x) - x^2e^{-x} - 2xe^{-x} - 2e^{-x}$$

$$= e^x(x^2-2x+2) - e^{-x}(x^2+2x+2)$$

$$\therefore \int x^2(e^x+e^{-x})dx = e^x(x^2-2x+2) - e^{-x}(x^2+2x+2)$$

---

5) $x^5 \cos\frac{x}{2}$

$$\int x^5 \cos\frac{x}{2}\, dx$$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + u''''v_4 - u'''''v_5 + u''''''v_6 - \cdots$$

Here,

$u = x^5$

$u' = 5x^4$

$u'' = 5(4)x^3 = 20x^3$

$u''' = 20(3)x^2 = 60x^2$

$u'''' = 60(2)x = 120x$

$u''''' = 120$

$\int dv = \int \cos\frac{x}{2}\, dx$

$v = \frac{\sin x/2}{1/2} = 2\sin x/2$

$v_1 = \frac{2(-\cos x/2)}{1/2} = -4\cos x/2$

$v_2 = -\frac{4\sin x/2}{1/2} = -8\sin x/2$

$v_3 = -\frac{8(-\cos x/2)}{1/2} = 16\cos x/2$

$v_4 = \frac{16(\sin x/2)}{1/2}$

$= 32\sin x/2$

$v_5 = \frac{32(-\cos x/2)}{1/2}$

$= -64\cos x/2$

$\int x^5 \cos x/2 \, dx = x^5(2\sin x/2) - 5x^4(-4\cos x/2) + 20x^3(-8\sin x/2) - 60x^2(16\cos x/2) + 120x(32\sin x/2) - 120(-64\cos x/2)$

120×32
$= 3840$

$= 2x^5 \sin x/2 - 160x^3 \sin x/2 + 3840 \sin x/2 + 20x^4 \cos x/2 - 960x^2 \cos x/2 + 7680 \cos x/2$

$= 2x(x^4 - 80x^2 + 1920) \sin x/2 + 20\cos x/2 \, (x^4 - 48x^2 + 384)$

$\therefore \int x^5 \cos x/2 \, dx = 2x[x^4 - 8x^2 + 1920] \sin x/2 + 20\cos x/2 \, [x^4 - 48x^2 + 384]$

---

6) $x^3 \sin nx$

$\int x^3 \sin nx \, dx$

Bernoulli's formula.

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

Here.

$u = x^3$

$u' = 3x^2$

$u'' = 3(2)x = 6x$

$u''' = 6$

$\int dv = \int \sin nx \, dx$

$v = -\dfrac{\cos nx}{n}$

$v_1 = -\dfrac{\sin nx}{n \times n} = -\dfrac{\sin nx}{n^2}$

$v_2 = -\left(-\dfrac{\cos nx}{n^2 \times n}\right) = \dfrac{\cos nx}{n^3}$

$v_3 = \dfrac{\sin nx}{n^3 \times n} = \dfrac{\sin nx}{n^4}$

$$\therefore \int x^3 \sin nx\, dx = x^3\left(-\frac{\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)$$

$$= -\frac{x^3\cos nx}{n} + \frac{3x^2\sin nx}{n^2} + \frac{6x\cos nx}{n^3} - \frac{6\sin nx}{n^4}$$

$$\therefore \int x^3 \sin nx\, dx = -\frac{x^3\cos nx}{n} + \frac{3x^2\sin nx}{n^2} + \frac{6x\cos nx}{n^3} - \frac{6\sin nx}{n^4}$$

double Integral.

CHAPTER-2

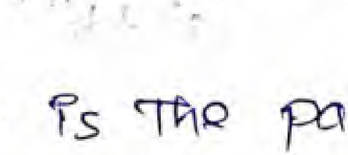
GEOMETRICAL APPLICATIONS OF INTEGRATION.

Example. 1 (1.1)

Find the area bounded by the curve $y^2 = 4ax$, the x-axis and the ordinate $x=h$

The curve is the parabola which, we know, passes. through the ~~Hea~~ origin. The limits for the area in question are 0 and h.

first ▷



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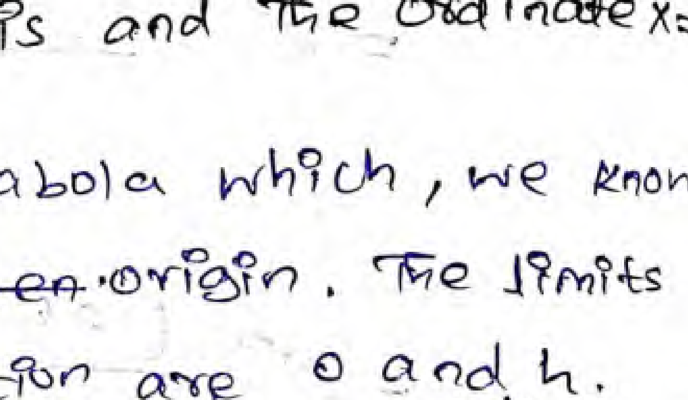


Hence The required area is

$$f(x) = y$$

$$y^2 = 4ax \Rightarrow y = \sqrt{4ax}$$

$$\int_0^h f(x)dx = \int_0^h y\, dx$$

$$= \int_0^h \sqrt{4ax}.dx$$

$$= 2\sqrt{a}\int_0^h \sqrt{x}\, dx$$

$$= 2\sqrt{a}\int_0^h (x)^{1/2}\, dx$$

$$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + c \right]$$

$$= 2\sqrt{a} \times \left[\frac{x^{1/2+1}}{1/2+1}\right]_0^h$$

$$= 2\sqrt{a} \times \left[\frac{x^{3/2}}{3/2}\right]_0^h$$

$$= \frac{4\sqrt{a}}{3}\left[x^{3/2}\right]_0^h$$

$$= \frac{4\sqrt{a}}{3}\left[h^{3/2} - 0^{3/2}\right]$$

$$= \frac{4\sqrt{a}}{3}\left(h^1 \times h^{1/2}\right)$$

$$= \frac{4h\sqrt{a}\sqrt{h}}{3}$$

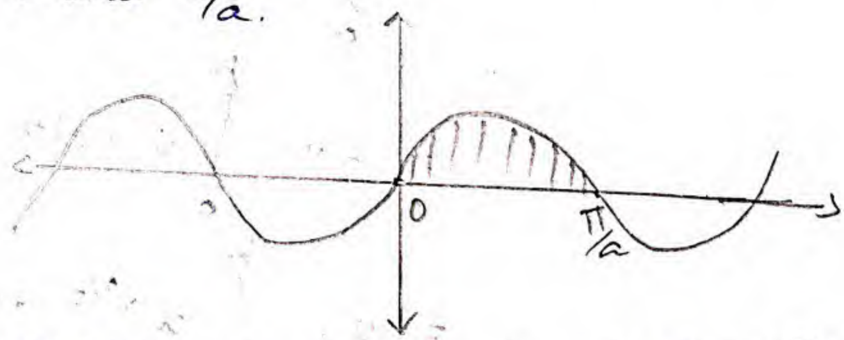
$$= \frac{4h\sqrt{ah}}{3}$$

$$\therefore \int_0^h \sqrt{4ax}\, dx = \frac{4h\sqrt{ah}}{3}$$

Example. 2 (1.1).

Find the area bounded by one arch of the curve. $y = \sin ax$ and the x-axis.

The curve crosses the x-axis when $x = 0$ and $\frac{n\pi}{a}$, where n is a positive or negative Integer. The limits for one arch are 0 and π/a.



So the limits for one arch are $0 \to \pi/a$.

Hence the area is

$$f(x) = y = \sin ax$$

$$\int_0^{\pi/a} f(x)\,dx = \int_0^{\pi/a} y\,dx = \int_0^{\pi/a} \sin ax\,dx$$

$$\left[\text{by } \int \sin x\,dx = -\cos x + c\right]$$

$$= -\left[\frac{\cos ax}{a}\right]_0^{\pi/a}$$

$$= -\frac{1}{a}\left[\cos ax\right]_0^{\pi/a}$$

$$= -\frac{1}{a}\left[\cos a \times \pi/a - \cos a \times 0\right]$$

$$= -\frac{1}{a}\left[\cos \pi - \cos 0\right]$$

$$\left[\text{by } \cos\pi = -1 \quad \cos 0 = 1\right]$$

$$= -\frac{1}{a}\left[-1 - 1\right]$$

$$\int_0^{\pi/a} \sin ax \, dx = -\frac{1}{a}(-2) = \frac{2}{a}$$

$$\therefore \int_0^{\pi/a} \sin ax \, dx = \frac{2}{a}.$$

Example. 3 (1.1)

Find the area bounded by one arch of the cycloid. $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$ and its base.

As the point p describes one arch, the parameter θ varies from 0 to 2π

$$x = a(\theta - \sin\theta); \quad y = a(1-\cos\theta)$$

$$\frac{dx}{d\theta} = a(1-\cos\theta).$$

Hence the required area is

$$\int_0^{2\pi} y \frac{dx}{d\theta} d\theta = \int_0^{2\pi} a(1-\cos\theta) \times a(1-\cos\theta) d\theta.$$

$$= \int_0^{2\pi} a^2 (1-\cos\theta)^2 d\theta.$$

$$\left[\text{by } (a-b)^2 = a^2 - 2ab + b^2 \right]$$

$$= a^2 \int_0^{2\pi} (1 - 2\cos\theta + \cos^2\theta) d\theta.$$

$$= a^2 \left[\int_0^{2\pi} d\theta - 2\int_0^{2\pi} \cos\theta \, d\theta + \int_0^{2\pi} \cos^2\theta \, d\theta \right]$$

$$\left[\text{by } \int \cos\theta \, d\theta = \sin\theta + c \right]$$

$$\left[\text{by } \cos^2\theta = \frac{1+\cos 2\theta}{2} \right]$$

$$= a^2 \left[[\theta]_0^{2\pi} - 2[\sin\theta]_0 + \int_0 \frac{1+\cos 2\theta}{2} d\theta \right.$$

$$= a^2 \left([2\pi - 0] - 2[\sin 2\pi - \sin 0^\circ] + \int_0^{2\pi} \left(\frac{1}{2} + \frac{\cos 2\theta}{2} \right) \right.$$

$$[\text{by } \sin\pi = \sin 0^\circ = 0]$$

$$= a^2 \left[2\pi - 2(0-0) + \int_0^{2\pi} \frac{1}{2} d\theta + \int_0^{2\pi} \frac{\cos 2\theta}{2} d\theta \right.$$

$$= a^2 \left[2\pi + \frac{1}{2} \int_0^{2\pi} d\theta + \frac{1}{2} \int_0^{2\pi} \cos 2\theta \, d\theta \right]$$

$$\left[\text{by } \int \cos a\theta \, d\theta = \frac{\sin a\theta}{a} \right]$$

$$= a^2 \left[2\pi + \frac{1}{2} [\theta]_0^{2\pi} + \frac{1}{2} \left[\frac{\sin 2\theta}{2} \right]_0^{2\pi} \right]$$

$$= a^2 \left[2\pi + \frac{1}{2} [2\pi - 0] + \frac{1}{4} [\sin(2 \times 2\pi) - \sin(\right.$$

$$= a^2 \left[2\pi + \frac{\cancel{2}\pi}{\cancel{2}} + \frac{1}{4} [\sin 4\pi - \sin 0^\circ] \right]$$

$$[\text{by } \sin\pi = \sin 0 = 0]$$

$$= a^2 \left[2\pi + \pi + \frac{1}{4}(0-0) \right]$$

$$= a^2 (3\pi) = 3\pi a^2$$

$$\therefore \int_0^{2\pi} y \frac{dx}{d\theta} d\theta = 3\pi a^2$$

Example ...

Find the area of loop of the curve $y^2 = x^2\frac{a+x}{a-x}$

The limits for the loop are $-a$ and 0.

As the curve is symmetrical about the x-axis,

The area of the loop = twice the area of the loop above the x-axis.

$y^2 = x^2\frac{(a+x)}{a-x}$

$y=0$

$0 = x^2(a+x)$

$a+x=0$

$x=-a$

$x=0$

$$y^2 = x^2\,\frac{a+x}{a-x}$$

$$y = \sqrt{x^2}\,\sqrt{\frac{a+x}{a-x}}$$

$$y = x\left(\frac{a+x}{a-x}\right)^{1/2}$$

twice the area of the loop above the x-axis

$$I = 2\int_0^{-a} y\,dx = 2\int_0^{-a} x\left(\frac{a+x}{a-x}\right)^{1/2} dx.$$

put $x = a\cos 2\theta$

$$\frac{dx}{d\theta} = a(-\sin 2\theta)\cdot 2$$

$$dx = -2a\sin 2\theta\, d\theta.$$

$\cos 2\theta = \frac{x}{a}$

$x=0 \Rightarrow \cos 2\theta = \frac{0}{a}$

$\cos 2\theta = 0$

$\cos 2\theta = \cos(\pi/2)$

$2\theta = \pi/2$

$\theta = \pi/4$

$x=-a \Rightarrow \cos 2\theta = -1$

$\cos 2\theta = \cos\pi$

$\theta = \pi/2$

$$I = 2\int_0^{-a} y\,dx = 2\int_{\pi/4}^{\pi/2} a\cos 2\theta\left(\frac{a+a\cos 2\theta}{a-a\cos 2\theta}\right)^{1/2}(-2a\sin 2\theta)\,d\theta$$

$$I = -4a^2\int_{\pi/4}^{\pi/2} \cos 2\theta\left[\frac{\cancel{a}(1+\cos 2\theta)}{\cancel{a}(1-\cos 2\theta)}\right]^{1/2}\sin 2\theta\, d\theta$$

[by $(1+\cos 2\theta) = 2\cos^2\theta$]

[by $(1-\cos 2\theta) = 2\sin^2\theta$]

[by $\sin 2\theta = 2\sin\theta\cos\theta$]

$$I = -4a^2 \int_{\pi/4} \cos 2\theta \left[\frac{\cancel{2}\cos^2\theta}{\cancel{2}\sin^2\theta}\right](2\sin\theta\cos\theta)\,d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta \sqrt{\frac{\cos^2\theta}{\sin^2\theta}}\,(\sin\theta\cos\theta)\,d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta\,\frac{\cos\theta}{\cancel{\sin\theta}}\,(\cancel{\sin\theta}\cos\theta)\,d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta\,\cos^2\theta\,d\theta.$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= -8a^2 \times \frac{1}{2} \int_{\pi/4}^{\pi/2} \cos 2\theta\,(1+\cos 2\theta)\,d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \cos 2\theta\,\cos^2\theta\,d\theta$$

$$\left[\text{by } \cos 2\theta = 2\cos^2\theta - 1\right]$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} (2\cos^2\theta - 1)\cos^2\theta\,d\theta$$

$$I = -8a^2 \int_{\pi/4}^{\pi/2} (2\cos^4\theta - \cos^2\theta)\,d\theta$$

using $\cos^2\theta = \frac{1+\cos 2\theta}{2}$ and

$$\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{(1+\cos 2\theta)^2}{2^2}$$

$$= \frac{(1+\cos 2\theta)^2}{4}$$

$[\text{by } (a+b)^2 = a^2+2ab+b^2]$

$$= \frac{1+2\cos 2\theta + \cos^2 2\theta}{4}$$

$\left[\text{by } \cos^2 2\theta = \frac{1+\cos 4\theta}{2}\right]$

$$= \frac{1+2\cos 2\theta + \frac{1+\cos 4\theta}{2}}{4}$$

$$= \frac{\frac{2}{2} + \frac{4\cos 2\theta}{2} + \frac{1+\cos 4\theta}{2}}{4}$$

$$= \frac{\frac{2+4\cos 2\theta + 1 + \cos 4\theta}{2}}{4}$$

$$\cos^4\theta = \frac{3+4\cos 2\theta + \cos 4\theta}{8}$$

$\cos^4\theta, \cos^2\theta$ in (I).

$$I = -8a^2 \int_{\pi/4}^{\pi/2} \left[\cancel{2}\left(\frac{3+4\cos 2\theta + \cos 4\theta}{\cancel{8}\,4}\right) - \frac{1+\cos 2\theta}{2}\right] d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \left(\frac{3+4\cos 2\theta + \cos 4\theta}{4} - \frac{2+2\cos 2\theta}{4}\right) d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \left(\frac{3+4\cos 2\theta + \cos 4\theta - 2 - 2\cos 2\theta}{4} \right) d\theta$$

$$= -8a^2 \int_{\pi/4}^{\pi/2} \left(\frac{1+2\cos 2\theta + \cos 4\theta}{4} \right) d\theta$$

$$= -\frac{8a^2}{4} \int_{\pi/4}^{\pi/2} (1 + 2\cos 2\theta + \cos 4\theta) d\theta$$

$$= -2a^2 \left[\theta + \frac{\cancel{2}\sin 2\theta}{\cancel{2}} + \frac{\sin 4\theta}{4} \right]_{\pi/4}^{\pi/2}$$

$$= -2a^2 \left[\theta + \sin 2\theta + \frac{1}{4}\sin 4\theta \right]_{\pi/4}^{\pi/2}$$

$$I = -2a^2 \left[\left(\pi/2 + \sin(\cancel{2} \times \pi/\cancel{2}) + \frac{1}{4}\sin(4^2 \times \pi/\cancel{2}) \right) - \left(\pi/4 + \sin(\cancel{2} \times \pi/4^2) + \frac{1}{4}\sin(\cancel{4} \times \pi/\cancel{4}) \right) \right]$$

$$= -2a^2 \left[\left(\pi/2 + \sin\pi + \frac{1}{4}\sin 2\pi \right) - \left(\pi/4 + \sin\pi/2 + \frac{1}{4}\sin\pi \right) \right]$$

$$= -2a^2 \left[(\pi/2 + 0 + 0) - (\pi/4 + 1 + 0) \right]$$

$$= -2a^2 \left[\frac{\pi}{2} - (\pi/4 + 1) \right]$$

$$= -2a^2 \left[\frac{\pi}{2} - \pi/4 - 1 \right]$$

$$= -2a^2 \left[\frac{2\pi}{4} - \pi/4 - 1 \right] = -2a^2 \left[\frac{2\pi - \pi}{4} - 1 \right]$$

$I = -2a^2\left[\pi/4 - 1\right]$

$I = 2a^2\left[-\pi/4 + 1\right]$

$I = 2a^2\left[1 - \pi/4\right]$

$$\boxed{I = 2a^2 - \frac{2a^2\pi}{4}}$$

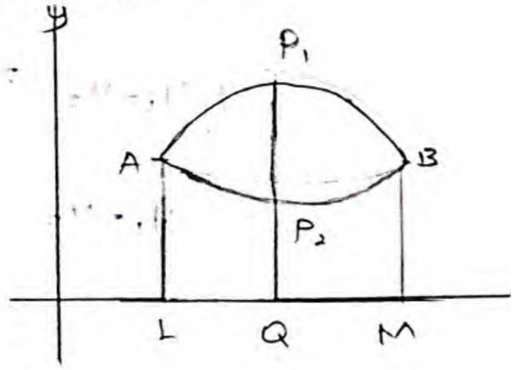
$\therefore$ Hence the area required is

$2a^2 \cdot (1 - \pi/4)$

1.2 Area of a closed curve.

Let AL and BM be the tangents to the closed curve parallel to the y-axis. Let an intermediate ordinate meet the curve in two points P_1 and P_2, where P_1 is (x, y_1) and P_2 is (x, y_2). Let $y_1 > y_2$ Denoting OL and OM by a and b respectively, area $LAP_1BM = \int_a^b y_1\,dx$ and area $LAP_2BM = \int_a^b y_2\,dx$

By subtraction, we get the area of the closed curve to be $\int_a^b (y_1 - y_2)\,dx$.



closed curve to be $\int_a^b (y_1 - y_2)\,dx$.

This integral gives the area whether the x-axis cuts the curve or not. The values y_1 and y_2 corresponding to any value of x are found by solving the equation of the curve as a quadratic in y.

Example.

Find the area of the ellipse.

$$x^2+4y^2-6x+8y+9=0$$

$$x^2+4y^2-6x+8y+9=0$$

$$4y^2+8y+(x^2-6x+9)=0$$

$$y^2+2y+\left(\frac{x^2-6x+9}{4}\right)=0$$

$$a=1,\ b=2,\ c=\frac{x^2-6x+9}{4}$$

$$\left[\because \alpha_1+\alpha_2=-b\right]$$

$$y_1+y_2=-2$$

$$\left[\because \alpha_1\alpha_2=c\right]$$

$$y_1 * y_2 = y_1y_2 = \frac{x^2-6x+9}{4}$$

Hence.

$$\Rightarrow (y_1-y_2)^2=(y_1)^2+(y_2)^2-2y_1y_2$$

$$=[(y_1)^2+(y_2)^2+2y_1y_2]-2y_1y_2-2y_1y_2$$

$$\left[\because (a+b)^2=a^2+b^2+2ab\right]$$

$$=(y_1+y_2)^2-2y_1y_2-2y_1y_2$$

$$=(y_1+y_2)^2-4y_1y_2$$

$$(y_1-y_2)^2=(y_1+y_2)^2-4y_1y_2$$

$$y_1-y_2=\sqrt{(y_1+y_2)^2-4y_1y_2}$$

$$=\sqrt{(-2)^2-\cancel{4}\left(\frac{x^2-6x+9}{\cancel{4}}\right)}$$

$$=\sqrt{4-(x^2-6x+9}$$

$$=\sqrt{4-x^2+6x-9}$$

$$=\sqrt{-x^2+6x-5}$$

$$y_1 - y_2 = \sqrt{-x^2+6x-5}$$

put $-x^2+6x-5$

$$\begin{array}{c|c} -5 & \cdot 6 \\ \hline -5\times -1 & -5-1 \end{array}$$

$$-(x-1)(x-5)$$

$$(1-x)(x-5)$$

$$y_1 - y_2 = \sqrt{(1-x)(x-5)}$$

The two values y_1 and y_2 are equal when $x=1$, and $x=5$

These are the abscissae of the points at which the tangents are parallel to the y-axis.

Hence the area of the ellipse $= \int_1^5 (y_1-y_2)\,dx$.

$$\int_1^5 (y_1-y_2)\,dx = \int_1^5 \sqrt{(1-x)(x-5)}\,dx.$$

putting.

$$x = \sin^2\theta + 5\cos^2\theta$$

$$\frac{dx}{d\theta} = 2\sin\theta\cos\theta + (5\times 2)\cos\theta(-\sin\theta)$$

$$\frac{dx}{d\theta} = 2\sin\theta\cos\theta - 10\sin\theta\cos\theta$$

$$dx = -8\sin\theta\cos\theta\,d\theta.$$

$$x=1 \Rightarrow 1 = \sin^2\pi/2 + 5\cos^2 0$$

$$\theta = \pi/2$$

$$x=5 \Rightarrow 5 = \sin^2 5 + 5\cos^2 5$$

$$\theta = 0$$

$$\int_1^5 (y_1-y_2)\,dx = \int_{\pi}^{0} \sqrt{(1-\sin^2\theta - 5\cos^2\theta)(\sin^2\theta + 5\cos^2\theta - 5)}\,(-8\cos\theta\sin\theta)\,d\theta$$

$$= \int_{\pi/2}^{0} \sqrt{(1-\sin^2\theta - 5\cos^2\theta)\left[\sin^2\theta + 5(\cos^2\theta - 1)\right.}$$

$$(-8\sin\theta\cos\theta)\,d\theta.$$

$\left[\text{by } 1-\sin^2\theta = \cos^2\theta\right]$

$\left[\text{by } \cos^2\theta - 1 = -\sin^2\theta\right]$

$$= \int_{\pi/2}^{0} \sqrt{(\cos^2\theta - 5\cos^2\theta)\left[\sin^2\theta + 5(-\sin^2\theta)\right.}$$

$$(-8\sin\theta\cos\theta)\,d\theta.$$

$$= \int_{\pi/2}^{0} \sqrt{(\cos^2\theta - 5\cos^2\theta)\left[\sin^2\theta - 5\sin^2\theta\right]}$$

$$(-8\sin\theta\cos\theta)\,d\theta$$

$$= \int_{\pi/2}^{0} \sqrt{\cos^2\theta\,(1-5)\;\sin^2\theta\,(1-5)}$$

$$(-8\sin\theta\cos\theta)\,d\theta$$

$$= \int_{\pi/2}^{0} \sqrt{(-4)\cos^2\theta\,(-4)\,\sin^2\theta}$$

$$(-8\sin\theta\cos\theta)\,d\theta$$

$$= \int_{\pi/2}^{0} \sqrt{16\cos^2\theta\sin^2\theta}\,(-8\sin\theta\cos\theta)\,d\theta$$

$$= \int_{\pi/2}^{0} 4\cos\theta\sin\theta\,(-8\sin\theta\cos\theta)\,d\theta$$

$$= -\int_{\pi/2}^{0} 32\sin^2\theta\,\cos^2\theta\,d\theta.$$

$$= -32\int_{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= 32\int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

[by when m and n is even

$$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdots\cdots \frac{1}{2} \cdot \frac{\pi}{2}]$$

$$= 32 \cdot \frac{2-1}{2+2} \cdot \frac{1}{2} \frac{\pi}{2} = 32 \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 32 \frac{1}{4} \cdot \frac{\pi}{4} = 32 \frac{\pi}{16} = 2\pi$$

$$\therefore \int_1^5 (y_1 - y_2)\, dx = 2\pi$$

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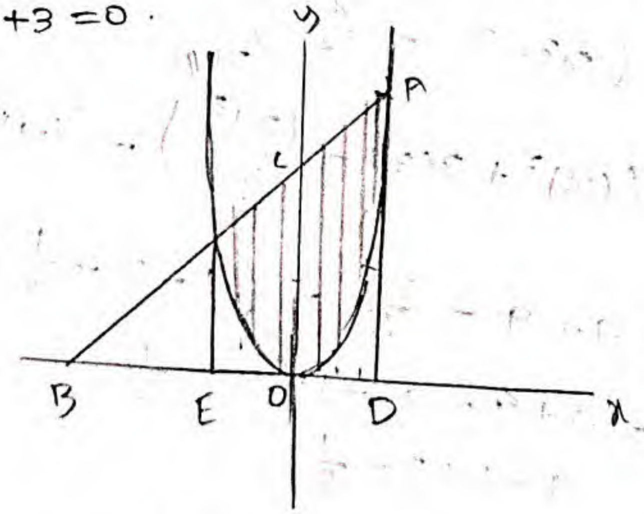
Example. 2. (1.2)

Find the area enclosed between the parabola $y = x^2$ and the straight line. $2x - y + 3 = 0$.



$y_2 \rightarrow y = x^2$

$y_1 \rightarrow y = 2x + 3$

$2x - y + 3 = 0$

$2x - x^2 + 3 = 0$

$2x + 3 = x^2$

$x^2 - 2x - 3 = 0.$

$x^2 - 2x - 3 = 0$ $\quad \begin{array}{l|l} & -3 \\ \hline 1x-3 & 1-3 \end{array}$

$(x-3)(x+1)=0$

$x-3=0 \quad | \quad x+1=0$

$x=3 \quad | \quad x=-1$

$x = 3$ or -1. Hence $OD = 3$ and $OE = -1$

Area required $= \int_{-1}^{3} (y_1 - y_2)\,dx$

$y_1 = 2x+3$

$y_2 = x^2$

$$A = \int_{-1}^{3} (2x+3-x^2)\,dx$$

$$= \int_{-1}^{3} 2x\,dx + 3\int_{-1}^{3} dx - \int_{-1}^{3} x^2\,dx$$

$$\left[\because \int x^n\,dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \left(\frac{\cancel{2}x^2}{\cancel{2}} + 3x - \frac{x^3}{3}\right)_{-1}^{3}$$

$$= \left(x^2 + 3x - \frac{x^3}{3}\right)_{-1}^{3}$$

$$= \left((3)^2 + 3(3) - \frac{(3)^3}{3}\right) - \left((-1)^2 + 3(-1) - \frac{(-1)^3}{3}\right)$$

$$= 9 + 9 - \frac{\cancel{27}^{9}}{\cancel{3}_{1}} - \left(1 - 3 + \frac{1}{3}\right)$$

$$= 9 + \cancel{9} - \cancel{9} - 1 + 3 - \frac{1}{3}$$

$$= 9 - 1 + 3 - \frac{1}{3}$$

$$= 11 - \frac{1}{3}$$

$$= \frac{33-1}{3} = \frac{32}{3}$$

$$= 10\frac{2}{3}$$

$$\begin{array}{r|l} & 10 \\ 3 & 32 \\ & 30 \\ \hline & 2 \end{array} \qquad = 10\frac{2}{3}$$

Hence The area $= 10\frac{2}{3}$

Exercise. 14

Integration By parts.

1) $\int \log x \, dx.$

$$\int \log x \, dx.$$

Let

$u = \log x$ | $\int dv = \int dx$

$\frac{du}{dx} = \frac{1}{x}$ | $v = x.$

$du = \frac{1}{x} dx.$

we know that Integration By parts is

$$\left[\because \int u\,dv = uv - \int v\,du.\right]$$

$$\int \log x \, dx = x \log x - \int x \cdot \frac{1}{x} dx.$$

$$= x \log x - \int dx.$$

$$= x \log x - x.$$

$$\boxed{\int \log x \, dx = x \log x - x.}$$

---

2) $\int x^3 \log x \, dx.$

$$\int x^3 \log x \, dx.$$

Let

$u = \log x$ | $\int dv = \int x^3 dx$

$\frac{du}{dx} = \frac{1}{x}$ | $\left[\text{by} \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

$du = \frac{1}{x} \cdot dx$ | $v = \frac{x^4}{4}$

we know that Integration By parts is

$$\left[\because \int u\,dv = uv - \int v\,du.\right]$$

$$\int x^3 \log x \, dx = \frac{x^4 \log x}{4} - \int \frac{x^4}{4x} dx$$

$$\int x^3 \log x \, dx = \frac{x^4}{4}\log x - \int \frac{x^3}{4} dx$$

$$= \frac{x^4}{4}\log x - \frac{1}{4}\int x^3 dx$$

$$\left[\text{by} \int x^n dx = \frac{x^{n+1}}{n+1} + c \right]$$

$$= \frac{x^4}{4}\log x - \left(\frac{1}{4} \times \frac{x^4}{4}\right)$$

$$= \frac{x^4}{4}\log x - \frac{x^4}{16}$$

$$\boxed{\int x^3 \log x \, dx = \frac{x^4}{4}\log x - \frac{x^4}{16}}$$

3) $\int x \log(x+1) dx$.

$$\int x \log(x+1) dx$$

Let

$u = \log(x+1)$ | $\int dv = \int x \, dx$

$\frac{du}{dx} = \frac{1}{x+1}$ | $\left[\text{by} \int x^n dx = \frac{x^{n+1}}{n+1} + c \right]$

$du = \frac{1}{x+1} dx$ | $v = \frac{x^2}{2}$

We know That Integration By parts is

$$\left[\because \int u\,dv = uv - \int v\,du \right]$$

$$\int x \log(x+1) dx = \frac{x^2}{2}\log(x+1) - \int \frac{x^2}{2} \cdot \frac{1}{x+1} dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \frac{x^2}{x+1} dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\int \left[x - \frac{x}{x+1} \right] dx$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\int x\,dx - \int \frac{x}{x+1}dx\right]$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\int x\,dx - \left[\int 1 - \frac{1}{x+1}\,dx\right]\right]$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\int x\,dx - \left[\int dx - \int \frac{1}{x+1}dx\right]\right]$$

$$\left[\text{by} \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$\left[\text{by} \int dx = x + c\right]$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\frac{x^2}{2} - \left[x - \log(x+1)\right]\right]$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\left[\frac{x^2}{2} - x + \log(x+1)\right]$$

$$= \frac{x^2}{2}\log(x+1) - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\log(x+1)$$

$$= \frac{x^2}{2}\log(x+1) - \frac{1}{2}\log(x+1) - \frac{x^2}{4} + \frac{x}{2}$$

$$= \frac{1}{2}\left[x^2\log(x+1) - \log(x+1)\right] - \frac{x^2}{4} + \frac{x}{2}$$

$$= \frac{1}{2}\log(x+1)\left[x^2 - 1\right] - \frac{x^2}{4} + \frac{x}{2}$$

$$\boxed{\int x\log(x+1)dx = \frac{1}{2}\log(x+1)\left[x^2-1\right] - \frac{x^2}{4} + \frac{x}{2}}$$

4) $\int \cos^{-1}\left(\frac{x}{a}\right) dx.$

$$\int \cos^{-1}\left(\frac{x}{a}\right) dx.$$

Let

$u = \cos^{-1}\frac{x}{a}$ | $\int dv = \int dx$

$v = x.$

$$\frac{du}{dx} = -\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a}$$

$$du = -\frac{1}{\sqrt{1-\frac{x^2}{a^2}}} \cdot \frac{1}{a}\, dx.$$

$$du = -\frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a} \cdot dx.$$

$$du = -1 \times \sqrt{\frac{a^2}{a^2-x^2}} \cdot \frac{1}{a}\, dx.$$

$$du = -\frac{a}{\sqrt{a^2-x^2}} \cdot \frac{1}{a}\, dx$$

$$du = -\frac{1}{\sqrt{a^2-x^2}}\, dx. \quad , \quad v = x.$$

we know that Integration By parts is.

$$\left[\because \int u\,dv = uv - \int v\,du. \right]$$

$$\int \cos^{-1}\left(\frac{x}{a}\right) dx = x \cos^{-1}\left(\frac{x}{a}\right) - \int \frac{x}{\sqrt{a^2-x^2}}\, dx.$$

$$= x \cos^{-1}\left(\frac{x}{a}\right) - \int \frac{1}{\sqrt{a^2-x^2}}\, x\, dx$$

Substitudes.

$$u = a^2 - x^2$$

$$\frac{du}{dx} = 0 - 2x$$

$$du = -2x\, dx$$

$$-\frac{1}{2} du = x\, dx.$$

$$\int \cos^{-1}\left(\frac{x}{a}\right) dx = x\cos^{-1}\left(\frac{x}{a}\right) - \int -\frac{1}{2}\cdot\frac{1}{\sqrt{u}}\, du$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\int \frac{1}{\sqrt{u}}\, du$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\int \frac{1}{(u)^{1/2}}\, du$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\int (u)^{-1/2}\, du$$

$$\left[\text{by } \int u^n du = \frac{u^{n+1}}{n+1} + c\right]$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\left[\frac{u^{-1/2+1}}{-1/2+1}\right]$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right]$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \frac{1}{\cancel{2}}\left(\cancel{2}\sqrt{u}\right)$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{u}$$

$$\left[\text{by } u = a^2 - x^2\right]$$

$$= x\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}$$

$$\boxed{\int \cos^{-1}\left(\frac{x}{a}\right) dx = x\cos^{-1}\left(\frac{x}{a}\right) + \sqrt{a^2 - x^2}}$$

5) $\int x \sin^{-1} x \, dx.$

$$\int x \sin^{-1} x \, dx.$$

Let.

$u = \sin^{-1} x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$du = \frac{1}{\sqrt{1-x^2}} dx$

$\int dv = \int x \, dx$

$v = \frac{x^2}{2}.$

we know that integration by parts is

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \int \frac{x^2}{2}\frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx \quad \text{——— (1)}$$

$\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ we can't directly integrate

so we can split the $\frac{x^2}{\sqrt{1-x^2}}$

$$\frac{x^2}{\sqrt{1-x^2}} = \left[\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right] \quad \text{——— (2)}$$

(2) in (1)

$$\int x\sin^{-1}x\,dx = \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\int\left[\frac{1}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right]dx$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\left[\int\frac{1}{\sqrt{1-x^2}}\,dx - \int\sqrt{1-x^2}\,dx\right]$$

$$\left[\text{by } \int\frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + c\right]$$

$$\left[\text{by } \int\sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right]$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\left[\sin^{-1}x - \left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}x\right)\right]$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\left[\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x\right.$$

$$= \frac{x^2}{2}\sin^{-1}x - \frac{1}{2}\sin^{-1}x + \frac{x}{4}\sqrt{1-x^2} + \frac{1}{4}\sin^{-1}x.$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\sin^{-1}x + \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}$$

$$\boxed{\int x\sin^{-1}x\,dx = \frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}\sin^{-1}x + \frac{1}{4}x\sqrt{1-x^2}}$$

6) $\int x\tan^{-1}x\,dx.$

$$\int x\tan^{-1}x\,dx.$$

Let

$u = \tan^{-1}x.$

$\frac{du}{dx} = \frac{1}{1+x^2}$

$du = \frac{1}{1+x^2}dx$

$\int dv = \int x\,dx$

$v = \frac{x^2}{2}$

we know that Integration By parts.

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \int \frac{x^2}{2}\cdot\frac{1}{1+x^2}dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}dx.$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[\int dx - \int\frac{1}{1+x^2}dx\right]$$

$$\left[\text{by } \int \frac{1}{1+x^2}dx = \tan^{-1}x + C\right]$$

$$\int x\tan^{-1}x\,dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\left[x - \tan^{-1}x\right]$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x$$

$$= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{1}{2}\tan^{-1}x$$

$$= \frac{1}{2}\left[x^2\tan^{-1}x + \tan^{-1}x - x\right]$$

$$= \frac{1}{2}\left[(x^2+1)\tan^{-1}x - x\right]$$

$$\boxed{\int x\tan^{-1}x\,dx = \frac{1}{2}\left[(x^2+1)\tan^{-1}x - x\right]}$$

+7) $\int x^2\sin^{-1}x\,dx.$

$$\int x^2\sin^{-1}x\,dx.$$

Let

$u = \sin^{-1}x$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$

$du = \frac{1}{\sqrt{1-x^2}}dx$

$\int dv = \int x^2\,dx$

$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + C\right]$

$v = \frac{x^3}{3}$

we know that Integration By parts is

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x^2\sin^{-1}x\,dx = \frac{x^3}{3}\sin^{-1}x - \int \frac{x^3}{3}\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}}dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x^2 \cdot x}{\sqrt{1-x^2}}\,dx \quad \text{——— (1)}$$

substitution.

Let $u = 1-x^2 \Rightarrow x^2 = 1-u$

$$\frac{du}{dx} = 0 - 2x$$

$$du = -2x\,dx.$$

$$-\frac{1}{2}du = x\,dx.$$

$$\int \frac{x^2 \cdot x}{\sqrt{1-x^2}}\,dx = \int \frac{(1-u)}{\sqrt{u}}\left(-\frac{1}{2}\right)du$$

$$= -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du$$

$$= -\frac{1}{2}\left[\int \frac{1}{\sqrt{u}}\,du - \int \frac{u}{\sqrt{u}}\,du\right]$$

$$= -\frac{1}{2}\left[\int (u)^{-1/2}\,du - \int u^1(u)^{-1/2}\,du\right]$$

$$= -\frac{1}{2}\left[\int (u)^{-1/2}\,du - \int u^{1-1/2}\,du\right]$$

$$= -\frac{1}{2}\left[\int (u)^{-1/2}\,du - \int u^{1/2}\,du\right]$$

$$= -\frac{1}{2}\int (u^{-1/2} - u^{1/2})\,du$$

~~by~~

$$= -\frac{1}{2}\left[\int (u)^{-1/2}\,du - \int u^{1/2}\,du\right]$$

$$\left[\text{by} \int u^n\,du = \frac{u^{n+1}}{n+1} + c\right]$$

$$= -\frac{1}{2}\left[\frac{u^{-1/2+1}}{-1/2+1} - \frac{u^{1/2+1}}{1/2+1}\right]$$

$$= -\frac{1}{2}\left[\frac{u^{1/2}}{1/2} - \frac{u^{3/2}}{3/2}\right]$$

$$= -\frac{1}{2}\left[2u^{1/2} - \frac{2}{3}u^{3/2}\right]$$

$$= -\frac{2}{2}u^{1/2} + \left(\frac{1}{\cancel{2}} \times \frac{\cancel{2}}{3}u^{3/2}\right)$$

$$= -u^{1/2} + \frac{1}{3}u^{3/2}$$

$$\left[\text{by } u = 1 - x^2\right]$$

$$= -(1-x^2)^{1/2} + \frac{1}{3}(1-x^2)^{3/2}$$

$$\int \frac{x^2 \cdot x}{\sqrt{1-x^2}}\,dx = -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} \quad \text{--- (2)}$$

(2) in (1) :-

$$\int x^2 \sin^{-1}x\,dx = \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[\frac{1}{3}(1-x^2)^{3/2} - \sqrt{1-x^2}\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{9}(1-x^2)^{3/2} + \frac{1}{3}\sqrt{1-x^2}$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{9}(1-x^2)^1(1-x^2)^{1/2} + \frac{1}{3}\sqrt{1-x^2}$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{\sqrt{1-x^2}\,(1-x^2)}{9} + \frac{\sqrt{1-x^2}}{3}$$

$$\boxed{\int x^2 \sin^{-1}x\,dx = \frac{x^3}{3}\sin^{-1}x - \frac{\sqrt{1-x^2}\,(1-x^2)}{9} + \frac{\sqrt{1-x^2}}{3}}$$

1) $\int x^2 \sin^{-1}x \, dx$:

$$\int x^2 \sin^{-1}x \, dx$$

let $u = \sin^{-1}x$ $\qquad \int dv = \int x^2 dx$

$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}}$ $\qquad \left[\because \text{by} \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$

$v = \frac{x^3}{3}$

$du = \frac{1}{\sqrt{1-x^2}} dx$

we know that Integration By parts.

$$\left[\because \int u\,dv = uv - \int v\,du\right]$$

$$\int x^2 \sin^{-1}x \, dx = \frac{x^3}{3}\sin^{-1}x - \int \frac{x^3}{3} \cdot \frac{1}{\sqrt{1-x^2}} dx$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x^3}{\sqrt{1-x^2}} dx.$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{x^2 \cdot x}{\sqrt{1-x^2}} dx$$

substitudes.

$1-x^2 = t^2 \Rightarrow x^2 = 1-t^2$

$-2x\frac{dx}{dt} = 2t$

$-\cancel{2}x\,dx = \cancel{2}t\,dt$

$x\,dx = -t\,dt$

$$\int x^2 \sin^{-1}x \, dx = \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{-(1-t^2)\,t\,dt}{\sqrt{t^2}}$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int \frac{(t^2-1)\cancel{t}\,dt}{\cancel{t}}$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\int (t^2-1)\,dt$$

$$\int x^2 \sin^{-1}x\, dx = \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[\int t^2 dt - \int dt\right]$$

$$\left[\because \text{ by } \int t^n dt = \frac{t^{n+1}}{n+1} + c\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[\frac{t^3}{3} - t\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{t^3}{9} + \frac{t}{3}$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[t\left(\frac{t^2}{3} - 1\right)\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[\sqrt{1-x^2}\left(\frac{1-x^2}{3} - 1\right)\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[\sqrt{1-x^2}\left(\frac{1-x^2-3}{3}\right)\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[\sqrt{1-x^2}\left(\frac{-2-x^2}{3}\right)\right]$$

$$= \frac{x^3}{3}\sin^{-1}x - \frac{1}{3}\left[-\frac{1}{3}\sqrt{1-x^2}(2+x^2)\right]$$

$$= \frac{x^3}{3}\sin^{-1}x + \frac{1}{9}\sqrt{1-x^2}(2+x^2)$$

$$\boxed{\int x^2 \sin^{-1}x\, dx = \frac{x^3}{3}\sin^{-1}x + \frac{1}{9}\sqrt{1-x^2}(2+x^2)}$$

GEOMETRICAL APPLICATIONS OF INTEGRATION

1.1 Areas under plane curves: cartesian co-ordinates.

∴ we shall find a formula for the area bounded by the arc of the curve $y = f(x)$

The ordinates: $x=a$, $x=b$ and the portion of the x-axis between them.

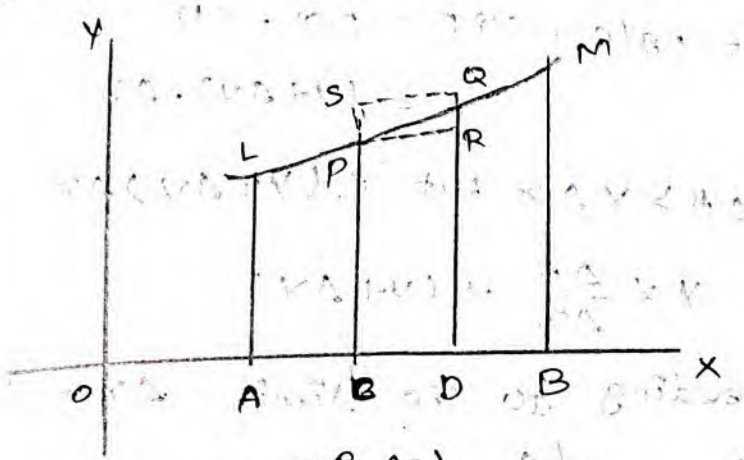


Fig (2)

For definiteness, let us suppose that $a<b$. The graph of $y=f(x)$ is in the figure. The graph is LPQM. Let $OA=a$ and $OB=b$. Then the ordinates AL and BM are $y=f(a)$ and $y=f(b)$ respectively.

We shall prove that the area bounded by the arc LM, the ordinates AL and BM and the portions AB of the x-axis is $\int_a^b f(x)dx$.

Let P be any point (x, y) on the curve and Q a neighbouring point $(x+\Delta x, y+\Delta y)$ on it. Draw the ordinates PC and QD and draw PR and QS perpendicular to QD and CP respectively.

Let A represent the area bounded by the arc LP, the ordinates AL, CP and the portion AC of the x-axis.

Then the area ALQD can be represented by $A+\Delta A$ so that the area CPQD is ΔA.

From the figure, it can be seen that the area CPQD is greater than the inner rectangle CPRD and is less than the outer rectangle CSQD.

$= y. \Delta x$

rectangle $CSQD = DQ \cdot CD$

$= (y+\Delta y) \cdot \Delta x$

$\Delta A > y \Delta x$ but $< (y+\Delta y) \Delta x$.

$$y < \frac{\Delta A}{\Delta x} < (y+\Delta y)$$

proceeding to the limit $\Delta x \to 0$,

$\frac{\Delta A}{\Delta x} \to \frac{dA}{dx}$ and $y+\Delta y \to y$.

$\therefore \frac{dA}{dx}$ lies between y and a quantity which tends to y in the limit.

$$\text{Hence } \frac{dA}{dx} = y$$

$\therefore A = \int y\,dx + c$, where c is the constant of Integration,

$$A = \int y\,dx + c$$

$$[\because f(x) = y]$$

$$A = \int f(x)\,dx + c.$$

$$\text{Let } \int f(x)\,dx = F(x) + c$$

$$A = F(x) + c$$

$\Rightarrow$ when $x = a$, $A = 0$ as A is, by definition, area $ALPC$

$$A = F(x) + c$$

$$[x = a,\ A = 0]$$

$$0 = F(a) + c \quad \text{——— (1)}$$

$\Rightarrow$ ~~when $x = b$, A = area ALMP which is sought.~~

~~$[\because$ The required area $ALMP = F(b) + c]$~~

~~$A = F(x) + c$.~~

~~$\left[x = b, \; A = \text{area } ALMP \right.$~~

~~$\left. ALMP = F(b) + c \right]$~~

~~area $ALMP = F(b) + c$~~

~~$F(b) + c =$~~

when $x = b$, $A =$ area $ALMP$ which is sought

$$A = F(x) + c$$

The required area $ALMP = F(b) + c$ — (2)

(2) = (1) The required area ALMP = The required area ALPC.

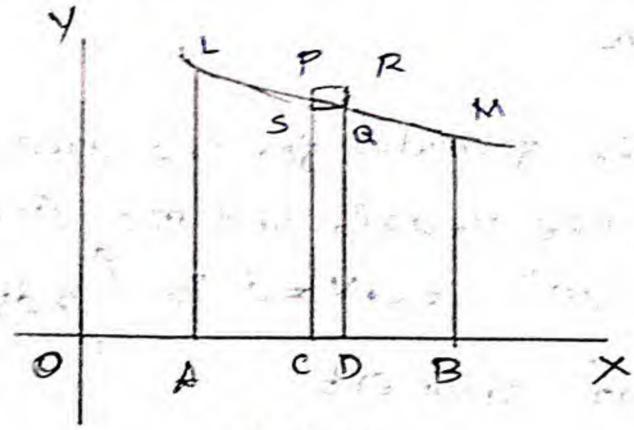
$$F(b) + c = F(a) + c$$

$$F(b) + c - F(a) - c = 0$$

$$= F(b) - F(a)$$

$$= \left[F(x) \right]_a^b$$

$$= \int_a^b f(x)\,dx$$



Fig(3)

Note :-

(i) There is a point in the above proof which deserves our notice. In the curve we have drawn. y increases with x. If y decreases as x increases, as in the figure ... the area holds good with the ... that

$\Delta A < Y\Delta x$ and $> (Y+\Delta Y)\Delta x$

$Y\Delta x > \Delta A > (Y+\Delta Y)\Delta x$.

$\div$ by Δx

$$\frac{Y\cancel{\Delta x}}{\cancel{\Delta x}} > \frac{\Delta A}{\Delta x} > \frac{(Y+\Delta Y)\cancel{\Delta x}}{\cancel{\Delta x}}$$

$$\therefore Y > \frac{\Delta A}{\Delta x} > (Y+\Delta Y)$$

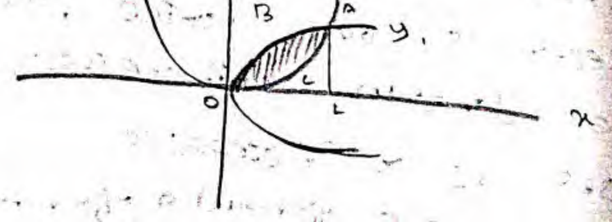
These inequalities are revereseed in the case of an increasing function. But, in the limit, when $\Delta x \to 0$, $\frac{dA}{dx} = y$ and the rest of the proof is the same as before, whether y increases or decreases with x, the above formula holds good

(ii) one other point also deserves special mention. when part of the area is below the x-axis, the corresponding ordinates are negative and Δx being taken to be positive, the area will be negative.

(iii) The formula for the area under a curve, the y-axis and the lines. (abscissae) $y=c, y=d$ is $\int_c^d x\,dy$ by a similar argument.

Example: 2 (1.2)

Find the area bounded by the parabolas; $y^2=4ax$ and $x^2=4by$



$y^2 = 4ax$ —— (1)

$x^2 = 4by$ —— (2)

(1) $\Rightarrow$ $y^2 = 4ax$

$y = \sqrt{4ax}$

$y_1 = 2\sqrt{a}\sqrt{x}$ —— (3)

(2) $\Rightarrow$ $x^2 = 4by$

$y_2 = \frac{x^2}{4b}$ —— (4)

eqn (4) in (1)

$\left(\frac{x^2}{4b}\right)^2 = 4ax$

$\frac{x^4}{16b^2} = 4ax$

$x^4 = 4ax(16b^2)$

$x^4 = 64ab^2x$

$x^4 - 64ab^2x = 0$

$x(x^3 - 64ab^2) = 0$

$x = 0$

$x^3 - 64ab^2 = 0$

$x^3 = 64ab^2$

$x = \sqrt[3]{64ab^2}$

$x = \sqrt[3]{4\times4\times4\, ab^2}$

$x = 4a^{1/3}b^{2/3}$

$A = \int_0^{4a^{1/3}b^{2/3}} y_1 - y_2 \, dx$

$A = \int_0^{4a^{1/3}b^{2/3}} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right) dx$

$A = \int_0^{4a^{1/3}b^{2/3}} 2\sqrt{a}\sqrt{x}\, dx - \frac{1}{4b}\int_0^{4a^{1/3}b^{2/3}} x^2\, dx.$

$= 2\sqrt{a}\int^{4a^{1/3}b^{2/3}} (x)^{1/2} dx - \frac{1}{4b}\int_0^{4a^{1/3}a^{2/3}} x^2 dx$

$$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + c \right]$$

$$A = 2\sqrt{a}\left[\frac{x^{1/2+1}}{1/2+1}\right]_0^{4a^{1/3}b^{2/3}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4a^{1/3}b^{2/3}}$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a^{1/3}b^{2/3}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4a^{1/3}b^{2/3}}$$

$$= 2\sqrt{a} \times \frac{2}{3}\left[x^{3/2}\right]_0^{4a^{1/3}b^{2/3}} - \frac{1}{12b}\left[x^3\right]_0^{4a^{1/3}b^{2/3}}$$

$$= \frac{4\sqrt{a}}{3}\left(4a^{1/3}b^{2/3}\right)^{3/2} - \frac{1}{12b}\left(4a^{1/3}b^{2/3}\right)^3$$

$$= \frac{4\sqrt{a}}{3}(4)^{3/2}(a^{1/3})^{3/2}(b^{2/3})^{3/2} - \frac{1}{12b}(4)^3(a^{1/3})^3(b^{2/3})^3$$

$$= \frac{4\sqrt{a}}{3}(4)^1(4)^{1/2}(a)^{1/2}b - \frac{1}{12b}64ab^2$$

$$= \frac{4\sqrt{a}\times\sqrt{a}}{3}\times 4\sqrt{4}\,b - \frac{\cancel{64}^{16}ab^2}{\cancel{12}_{3}b}$$

$$= \frac{16\times 2\times ab}{3} - \frac{16ab^{\cancel{2}}}{3\cancel{b}}$$

$$= \frac{32}{3}ab - \frac{16}{3}ab$$

$$= \frac{32ab - 16ab}{3} = \frac{16ab}{3}$$

$\therefore$ The required area $\frac{32}{3}ab - \frac{16}{3}$

1.4 Areas in polar co-ordinates

Example. 1.

Find the area of the cardioid $r = a(1+\cos\theta)$

$$\text{The Area}(A) = \frac{1}{2}\int_a^b r^2 d\theta.$$

$0, \pi$

$$r^2 = a^2(1+\cos\theta)^2$$

$$2A = 2 \times \frac{1}{2}\int_0^\pi a^2(1+\cos\theta)^2 d\theta$$

$$= a^2\int_0^\pi (1+\cos\theta)^2 d\theta$$

$$\left[\because \text{by } 1+\cos\theta = 2\cos^2\theta/_2\right]$$

$$= a^2\int_0^\pi (2\cos^2\theta/_2)^2 d\theta$$

$$= a^2\int_0^\pi 4\cos^4\theta/_2\, d\theta$$

$$\left[\begin{array}{l|l} \text{by putting } \theta/_2 = \phi & \phi = \frac{\theta}{2} \\ \theta = 2\phi & \theta = 0 \Rightarrow \phi = \frac{0}{2} = 0 \\ \frac{d\theta}{d\phi} = 2\phi & \theta = \pi \Rightarrow \phi = \pi/_2 \\ d\theta = 2\phi\, d\phi & \end{array}\right]$$

$$= 8a^2\int_0^{\pi/2} \cos^4\phi\, d\phi$$

$$\left[\text{by } \int_0^{\pi/2} \cos^n\theta\, d \quad (\text{even})\right.$$

$$\left. = \frac{(n-1)}{} \frac{(n-3)}{} \frac{(n-5)}{} \cdots \frac{1}{2}\cdot\frac{\pi}{2}\right]$$

$$2A = 8a^2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 8a^2 \cdot \frac{3}{4} \cdot \frac{\pi}{4}$$

$$= \frac{\cancel{24}^{3} a^2 \pi}{\cancel{16}\,\cancel{8}_{2}}$$

$$2A = \frac{3a^2 \pi}{2}$$

$\therefore$ The area $2A = \frac{3\pi a^2}{2}$

Example. 2

Find the entire area of the lemniscase of Bernouli $r^2 = a^2 \cos 2\theta$

The area consists of two loops and is symmetrical about the initial line.

The area required $= 4 \times$ the area of one-half loop of the curve above the initial line.

The area required $= 4A$

$$4A = 4 \times \frac{1}{2} \int_a^b r^2 \, d\theta$$

$$= \cancel{4}^{2} \times \frac{1}{\cancel{2}_{1}} \int_0^{\pi/4} a^2 \cos 2\theta$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{2a^2}{2} \left[\sin 2\theta \right]_0^{\pi/4}$$

$$= a^2 \left[\sin \cancel{2} (\pi/4) - \sin 2(0) \right]$$

$$= a^2 \left[\sin(\pi/2) - \sin 0 \right]$$

$4A = a^2[1 - 0]$

$4A = a^2.$

$\therefore$ The area required $= 4A = a^2.$

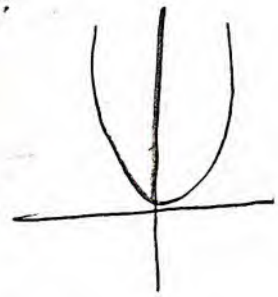
Home work.

1) Find the area bounded by curve.

$x^2 = 4y$, x-axis, $x = 2$

given

$x^2 = 4y$



$$y = \frac{x^2}{4}.$$

$$\text{Area} = \int_0^2 y\,dx$$

$$= \int_0^2 \frac{x^2}{4}\,dx.$$

$$= \frac{1}{4}\int_0^2 x^2\,dx$$

$$\left[\text{by } \int x^n dx = \frac{x^{n+1}}{n+1} + c\right]$$

$$= \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \frac{1}{12}\left[x^3\right]_0^2$$

$$= \frac{1}{12}\left[(2)^3 - 0^3\right]$$

$$= \frac{1}{12}(8) = \frac{\cancel{8}\,2}{\cancel{12}\,3}$$

$$\text{Area} = \frac{2}{3}$$

2). Find the area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}$$

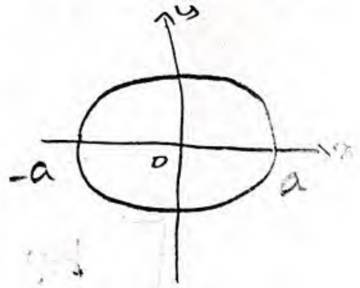
$$\text{Area} = 4\int_0^a y\,dx$$

$$= 4\int_0^a \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}\,dx$$

$$= 4\int_0^a \sqrt{b^2 - \frac{b^2x^2}{a^2}}\,dx$$

$$= 4\int_0^a \sqrt{b^2(1 - x^2/a^2)}\,dx$$

2) Find The area enclosed by ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Given

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$y = \sqrt{b^2(1 - x^2/a^2)}$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}} = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}$$

$$y = b\left(\frac{a^2 - x^2}{a^2}\right)^{1/2}$$

$$y = b\,\frac{(a^2 - x^2)^{1/2}}{(a^2)^{1/2}}$$

$$y = b\,\frac{\sqrt{a^2 - x^2}}{\sqrt{a^2}}$$

$$y = \frac{b}{a}\sqrt{a^2-x^2}$$

$$\text{Area} = \int_0^a y\, dx.$$

$$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\, dx$$

$$= \frac{b}{a}\int_0^a \sqrt{a^2-x^2}\, dx.$$

$$\left[\text{by } \int\sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+c\right]$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$$

$$= \frac{b}{a}\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right]$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\sin^{-1}(1)\right]$$

$$= \frac{b}{a}\left[\frac{a^2}{2}\cancel{\sin^{-1}}\left(\cancel{\sin}\,\pi/2\right)\right]$$

$$= \frac{b}{\cancel{a}}\left[\frac{a^{\cancel{2}}}{2}\cdot\frac{\pi}{2}\right].$$

$$\text{Area} = \frac{a \cdot b \cdot \pi}{4}$$

1.11 Areas in polar co-ordinates.

We propose to find a formula for the area bounded by the curve whose polar equation is $r=f(\theta)$ and two radii vectors OA and OB.

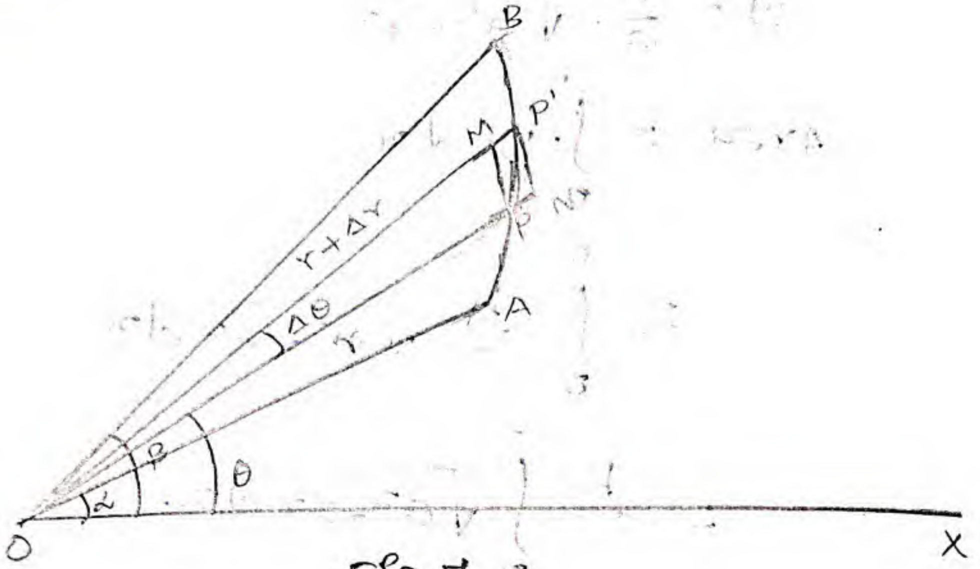


Fig. 7

Let $X\hat{O}A$ and $X\hat{O}B$ be respectively α and β

Let P be a point (r, θ) on the curve and P' a neighbouring point $(r+\Delta r, \theta+\Delta\theta)$ on it

$$P = (r, \theta)$$
$$P' = (r+\Delta r, \theta+\Delta\theta)$$

If we denote the area AOP by A, then the area denoted by AOP' is $A+\Delta A$ so that area POP' is ΔA.

$$AOP = A$$
$$AOP' = A+\Delta A$$
$$POP' = \cancel{A}+\Delta A-\cancel{A} = \Delta A$$

Let the circle centre O and radius OP cut OP' at M and the circle centre O and radius OP' cut OP produced at N

Area POP' lies between the areas of the circular sectors OPM and OP'N.

$$POP' = \Delta A$$
$$OPM = \frac{1}{2} r^2 \Delta\theta$$
$$OP'N = \frac{1}{2}(r+\Delta r)^2 \Delta\theta$$

$\therefore \frac{1}{2}(r+\Delta r)^2 \Delta\theta > \Delta A > \frac{1}{2} r^2 \Delta\theta$

Dividing by $\Delta\theta$.

$$\frac{\frac{1}{2}(r+\Delta r)^2 \cancel{\Delta\theta}}{\cancel{\Delta\theta}} > \frac{\Delta A}{\Delta\theta} > \frac{1}{2}\frac{r^2 \cancel{\Delta\theta}}{\cancel{\Delta\theta}}$$

$$\frac{1}{2}(r+\Delta r)^2 > \frac{\Delta A}{\Delta\theta} > \frac{1}{2} r^2$$

proceeding to the limit as $\Delta\theta \to 0$, we have.

$$\frac{dA}{d\theta} = \frac{1}{2} r^2$$

$$\int dA = \int \frac{1}{2} r^2 \, d\theta$$

$$A = \frac{1}{2}\int r^2 d\theta + c$$

$$A = F(\theta) + c \quad \text{———} \; (1)$$

$$A = A.$$

$$F(\theta) + c = \tfrac{1}{2}\int r^2 d\theta + c$$

$$F(\theta) = \tfrac{1}{2}\int r^2 d\theta \cancel{+c} \cancel{-c}$$

$$F(\theta) = \tfrac{1}{2}\int r^2 d\theta. \quad \text{———} \; (2).$$

putting $\theta = \alpha$, $A = 0$ in eqn (1).

$(1) \Rightarrow \quad A = F(\theta) + c$

$$0 = F(\alpha) + c \quad \text{———} \; (i)$$

putting $\theta = \beta$, $A = \text{area } OAB$ in eqn (1)

$(1) \Rightarrow \quad A = F(\theta) + c$

$$\text{area } OAB = F(\beta) + c \quad \text{———} \; (ii)$$

By subtraction, eqn (ii) of area OAB − eqn (i)

$$\text{area } OAB = F(\beta) + c - [F(\alpha) + c]$$

$$= F(\beta) \cancel{+c} - F(\alpha) \cancel{-c}$$

$$= F(\beta) - F(\alpha)$$

$$= \left[F(\theta) \right]_{\alpha}^{\beta}$$

[by eqn (2)

$F(\theta) = \frac{1}{2} \int r^2 \, d\theta$]

$$\left[F(\theta) \right]_{\alpha}^{\beta} = \frac{1}{2} \int_{\alpha}^{\beta} r^2 \, d\theta$$

Unit - III

$$\left[F(\theta)\right]_{\alpha} = \frac{1}{2}\int_{\alpha} r^2 d\theta$$

Unit - III

Multiple Integrals.

Double Integral.

The Double Integral is defined by $\iint_A f(x,y)\,dA = \int_{x=a}^{b}\int_{y=\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\,dx.$

1) Evaluate. $\int_0^a\int_0^b (x^2+y^2)\,dx\,dy.$

$$\int_0^a\int_0^b (x^2+y^2)\,dx\,dy = \int_0^a\left(\int_0^b (x^2+y^2)\,dy\right)dx$$

$$= \int_0^a\left[\int_0^b x^2\,dy + \int_0^b y^2\,dy\right]dx.$$

$$\left[\because \int y^n\,dy = \frac{y^{n+1}}{n+1}\right]$$

$$= \int_0^a\left[x^2y + \frac{y^3}{3}\right]_0^b dx.$$

$$= \int_0^a \left[x^2 b + \frac{b^3}{3} \right] dx$$

$$\left[\because \int x^n \, dx = \frac{x^{n+1}}{n+1} \right]$$

$$= \left[\frac{x^3 b}{3} + \frac{b^3 x}{3} \right]_0^a$$

$$= \frac{a^3 b}{3} + \frac{a b^3}{3}$$

---

2) Evaluate $\int_0^3 \int_1^2 xy(x+y) \, dy \, dx$

$$\int_0^3 \int_1^2 xy(x+y) \, dy \, dx = \int_0^3 \int_1^2 (x^2 y + x y^2) \, dy \, dx.$$

$$= \int_{x=0}^3 \left[\int_{y=1}^2 (x^2 y + x y^2) \, dy \right] dx.$$

$$\left[\because \int y^n \, dy = \frac{y^{n+1}}{n+1} \right]$$

$$= \int_{x=0}^3 \left[\frac{x^2 y^2}{2} + \frac{x y^3}{3} \right]_{y=1}^2 dx.$$

$$= \int_{x=0}^3 \left(\frac{4x^2}{2} + \frac{8x}{3} - \frac{x^2}{2} - \frac{x}{3} \right) dx.$$

$$= \int_{x=0}^3 \left(\frac{4x^2}{2} - \frac{x^2}{2} + \frac{8x}{3} - \frac{x}{3} \right) dx$$

$$= \int_{x=0} \left(\frac{3x^2}{2} + \frac{7x}{3} \right) dx .$$

$$\left[\because \int x^n \, dx = \frac{x^{n+1}}{n+1} \right]$$

$$= \left[\frac{3x^3}{2\times 3} + \frac{7x^2}{3\times 2} \right]_0^3$$

$$= \left[\frac{3x^3}{6} + \frac{7x^2}{6} \right]_0^3$$

$$= \frac{3(3)^3}{6} + \frac{7(3)^2}{6}$$

$$= \frac{\cancel{3}\times 27}{\cancel{6}\,2} + \frac{7\times \cancel{9}\,3}{\cancel{6}\,2} = \frac{27}{2} + \frac{21}{2}$$

$$= \frac{27+21}{2}$$

$$= \frac{48}{2}$$

$$= 24$$

$$\therefore \int_0^3 \int_1^2 xy(x+y)\,dy\,dx = 24.$$

HOME WORK.

3). Evaluate $\int_0^2 \int_0^3 xy^2 \, dy \, dx$.

$$\int_{x=0}^{x=2}\int_{y=0}^{y=3} xy^2\,dy\,dx = \int_{x=0}^{2}\left(\int_{y=0}^{3} xy^2\,dy\right)dx$$

$$= \int_{x=0}^{2}\left[\frac{xy^3}{3}\right]_0^3 dx$$

$$= \int_{x=0}^{2}\left[\frac{x(3)^3}{3}\right]dx$$

$$= \int_{x=0}^{2}\left(\frac{27x}{3}\right)dx.$$

$$= \left[\frac{27x^2}{3\times 2}\right]_0^2$$

$$= \left[\frac{27x^2}{6}\right]_0^2$$

$$= \frac{27\times(2)^2}{6} = \frac{\overset{9}{\cancel{27}}\times \cancel{4}^{2}}{\cancel{6}}$$

$$= 9\times 2. = 18.$$

$$\therefore \int_{x=0}^{2}\int_{y=0}^{3} xy^2\,dy\,dx = 18.$$

---

4) Evaluate. $\int_0^1\int_0^1 xy\,dx\,dy.$

$$\int_0^1\int_0^1 xy\,dy\,dx = \int_{x=0}^{1}\left(\int_{y=0}^{1} xy\,dy\right)dx$$

$$= \int_{x=0}^{1}\left[\frac{xy^2}{2}\right]_{y=0}^{1} dx$$

$$= \int_{x=0}^{1}\left(\frac{x(1)^2}{2}\right)dx$$

$$= \int_{x=0}^{1}\left(\frac{x}{2}\right)dx$$

$$= \left[\frac{x^2}{2\times 2}\right]_0^1$$

$$= \left[\frac{x^2}{4}\right]_0^1$$

$$= \frac{1^2}{4}$$

$$= \frac{1}{4}.$$

5) Evaluate $\int_1^2 \int_1^3 xy^2\,dy\,dx$

$$\int_{x=1}^{2} \int_{y=1}^{3} xy^2\,dy\,dx = \int_{x=1}^{2} \left[\int_{y=1}^{3} xy^2\,dy \right] dx.$$

$$\left[\because \int y^n\,dx = \frac{y^{n+1}}{n+1} \right]$$

$$= \int_{x=1}^{2} \left[\frac{xy^3}{3} \right]_{y=1}^{3} dx$$

$$= \int_{x=1}^{2} \left(\left[\frac{x(3)^3}{3} \right] - \left[\frac{x(1)^3}{3} \right] \right) dx$$

$$= \int_{x=1}^{2} \left[\frac{27x}{3} - \frac{x}{3} \right] dx.$$

$$\left[\because \int x^n\,dx = \frac{x^{n+1}}{n+1} \right]$$

$$= \left[\frac{27x^2}{3 \times 2} - \frac{x^2}{3 \times 2} \right]_{x=1}^{2}$$

$$= \left[\frac{27x^2}{6} - \frac{x^2}{6} \right]_{x=1}^{2}$$

$$= \frac{1}{6} \left[27x^2 - x^2 \right]_{x=1}^{2}$$

$$= \frac{1}{6} \left[27(2)^2 - (2)^2 - (27(1)^2 - 1^2) \right]$$

$$= \frac{1}{6} \left[27(4) - 4 - 27 + 1 \right]$$

$$= \frac{1}{6}(108 - 4 - 27 + 1)$$

$$= \frac{1}{6}(109 - 31)$$

$$= \frac{\cancel{78}^{13}}{\cancel{6}_{1}}$$

$$= 13$$

$$\therefore \int_{x=1}^{2} \int_{y=1}^{3} xy^2 \, dy \, dx = 13.$$

b) Evaluate. $\int_0^{\pi/2} \int_0^a dr \, d\theta$.

$$\int_{\theta=0}^{\pi/2} \int_{r=0}^{a} dr \, d\theta = \int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{a} dr \right) d\theta.$$

$$= \int_{\theta=0}^{\pi/2} [r]_0^a \, d\theta.$$

$$= \int_{\theta=0}^{\pi/2} (a - 0) \, d\theta.$$

$$= \int_{\theta=0}^{\pi/2} a \, d\theta.$$

$$= a \int_{\theta=0}^{\pi/2} d\theta.$$

$$= a \, [\theta]_0^{\pi/2}$$

$$= a \left[\pi/2 - 0 \right] =$$

$$\therefore \int_{\theta=0} \int_{r=0} dr\,d\theta = \frac{a\pi}{2}$$

7) Evaluate. $\displaystyle\int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^2\,dr\,d\theta.$

$$\int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2\,dr\,d\theta = \int_{\theta=-\pi/2}^{\pi/2} \left[\int_{r=0}^{2\cos\theta} r^2\,dr \right] d\theta.$$

$$\left[\because \int r^n\,dr = \frac{r^{n+1}}{n+1} \right]$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{r^3}{3} \right]_0^{2\cos\theta} d\theta.$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \left[\frac{(2\cos\theta)^3}{3} \right] d\theta.$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \left(\frac{2^3\cos^3\theta}{3} \right) d\theta$$

$$= \int_{\theta=-\pi/2}^{\pi/2} \left(\frac{8\cos^3\theta}{3} \right) d\theta$$

$$= \frac{8}{3} \int_{\theta=-\pi/2}^{\pi/2} (\cos^3\theta)\,d\theta.$$

$$\left[\because \int_{-a}^{a} f(x)\,dx = 2\int^{a} f(x)\,dx. \right]$$

$$= \frac{8}{3} \times 2 \int_0^{\pi/2} \cos^3\theta \, d\theta .$$

$$= \frac{16}{3} \int_0^{\pi/2} \cos^2\theta \cdot \cos\theta \, d\theta .$$

put $y = \sin\theta$

$$\frac{dy}{d\theta} = \cos\theta$$

$$dy = \cos\theta \, d\theta .$$

$\theta = 0 \quad \Rightarrow \quad y = \sin 0^\circ = 0$

$\theta = \pi/2 \quad \Rightarrow \quad y = \sin \pi/2 = 1$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta$$

$$= 1 - y^2 .$$

$$= \frac{16}{3} \int_0^1 (1 - y^2) \, dy$$

$$= \frac{16}{3} \int_0^1 dy - \int_0^1 y^2 \, dy$$

$$\left[\because \int y^n \, dy = \frac{y^{n+1}}{n+1} \right]$$

$$= \frac{16}{3} \left[y - \frac{y^3}{3} \right]_0^1$$

$$= \frac{16}{3} \left(1 - \frac{1^3}{3} \right) = \frac{16}{3} \left(\frac{1}{1} - \frac{1}{3} \right)$$

$$= \frac{16}{3} \left(\frac{3-1}{\ } \right) = \frac{16}{3} \left(\frac{2}{\ } \right) = \frac{32}{\ }$$

$$\therefore \int_{-\pi/2}^{\pi/2} \int_{0}^{2\cos\theta} r^2 \, dr \, d\theta = \frac{32}{9}.$$

8) Evaluate. $\displaystyle\int_{-\pi}^{\pi} \int_{a}^{a(1+\cos\theta)} r \, d\theta \, d\theta.$

$$\int_{\theta=-\pi}^{\pi} \int_{r=a}^{a(1+\cos\theta)} r \, dr \, d\theta = \int_{\theta=-\pi}^{\pi} \left[\int_{r=a}^{a(1+\cos\theta)} r \, dr \right] d\theta.$$

$$= \int_{-\pi}^{\pi} \left[\frac{r^2}{2} \right]_{a}^{a(1+\cos\theta)} d\theta.$$

$$= \int_{-\pi}^{\pi} \left(\frac{[a(1+\cos\theta)]^2}{2} - \frac{a^2}{2} \right) d\theta.$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \left(a^2(1+\cos\theta)^2 - a^2 \right) d\theta.$$

$$= \frac{a^2}{2} \int_{-\pi}^{\pi} (1+\cos\theta)^2 - 1 \, d\theta.$$

$$= \frac{a^2}{2} \int_{-\pi}^{\pi} (\cancel{1} + 2\cos\theta + \cos^2\theta \cancel{-1}) \, d\theta$$

$$= \frac{a^2}{2} \int_{-\pi}^{\pi} (\cos^2\theta + 2\cos\theta) \, d\theta.$$

$$= \frac{a^2}{2} \left(\int_{-\pi}^{\pi} \cos^2\theta \, d\theta + 2 \int_{-\pi}^{\pi} \cos\theta \, d\theta \right)$$

$$= \frac{a^2}{2}\left(\int_{-\pi} \frac{1+\cos 2\theta}{2}\, d\theta + 2\int_{-\pi} \cos\theta\, d\theta\right)$$

$$= \frac{a^2}{2}\left[\int_{-\pi}^{\pi} \frac{1}{2}\, d\theta + \int_{-\pi}^{\pi} \cos 2\theta \frac{d\theta}{2} + 2\int_{-\pi}^{\pi} \cos\theta\, d\theta\right.$$

put.

$$2\theta = x$$

$$\theta = \frac{x}{2}$$

$$\frac{d\theta}{dx} = \frac{1}{2}$$

$$d\theta = \frac{dx}{2}.$$

$$\theta = -\pi \Rightarrow x = -2\pi$$

$$\theta = \pi \Rightarrow x = 2\pi$$

$$= \frac{a^2}{2}\left[\frac{1}{2}\int_{-\pi}^{\pi} d\theta + \int_{-2\pi}^{2\pi} \cos x \frac{dx}{2\times 2} + 2\left[\sin\theta\right]_{-\pi}^{\pi}\right)$$

$$= \frac{a^2}{2}\left(\frac{1}{2}\left[\theta\right]_{-\pi}^{\pi} + \frac{1}{4}\int_{-2\pi}^{2\pi} \cos x\, dx + 2\left[\sin\pi - (\sin(-\pi)\right.\right.$$

$$= \frac{a^2}{2}\left(\frac{1}{2}\left[\pi - (-\pi)\right] + \frac{1}{4}\left[\sin x\right]_{-2\pi}^{2\pi} + 2\left[\sin\pi + \sin\pi\right.\right.$$

$\sin\pi = 0$

$$= \frac{a^2}{2}\left(\frac{1}{2}\left[\pi + \pi\right] + \frac{1}{4}\left[\sin 2\pi - (\sin(-2\pi))\right] + 0\right)$$

$\sin 2\pi = 0$

$$= \frac{a^2}{\cancel{2}}\left(\frac{\cancel{2}\pi}{2}\right) = \frac{a^2\pi}{2}$$

$$\therefore \int_{-\pi}^{\pi} \int_{a}^{a(1+\cos\theta)} r\,dr\,d\theta = \frac{a^2\pi}{2}$$

HOME WORK.

a) Evaluate $\displaystyle\int_{-\pi}^{\pi} \int_{0}^{a(1+\cos\theta)} r\,dr\,d\theta$.

10) Evaluate $\displaystyle\int_{0}^{a} \int_{0}^{2\sqrt{ax}} x^2\,dx\,dy$.

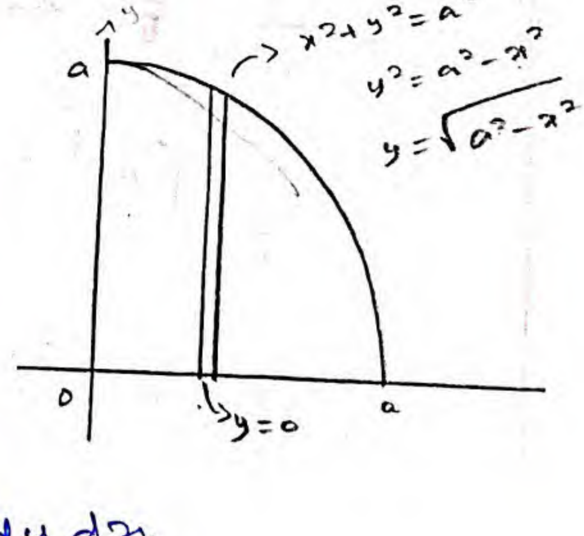
11) Evaluate $\displaystyle\int_{0}^{1} \int_{\sqrt{y}}^{2-y} x^2\,dx\,dy$.

1) Evaluate $\iint xy\,dx\,dy$. taken over the positive quadrant of the circle $x^2+y^2=a^2$

Limits.

$x : 0 \to a$

$y : 0 \to \sqrt{a^2-x^2}$



$$\text{Area} = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy\,dx.$$

$$= \int_{x=0}^{a} \left(\int_{y=0}^{\sqrt{a^2-x^2}} xy\,dy \right) dx.$$

$$= \int^{a} \left[\frac{xy^2}{2} \right]^{\sqrt{a^2-x^2}} dx$$

$$\text{Area} = \int_{x=0} \frac{x}{2}\left[y^2\right]_{y=0}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=0}^{a} \frac{x}{2}\left(\sqrt{a^2-x^2}\right)^2 dx$$

$$= \int_{x=0}^{a} \frac{x(a^2-x^2)}{2} dx$$

$$= \int_{x=0}^{a} \frac{a^2x - x^3}{2} dx$$

$$= \int_{x=0}^{a} \frac{a^2x}{2}dx - \int_{x=0}^{a}\frac{x^3}{2}dx$$

$$\cancel{= \frac{1}{2}\left[\frac{a^2}{2}\int_{x=0}^{a} x\,dx\right.}$$

$$= \frac{1}{2}\left[a^2\int_{x=0}^{a} x\,dx - \int_{x=0}^{a} x^3\,dx\right]$$

$$= \frac{1}{2}\left[a^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left(\frac{a^2a^2}{2} - \frac{a^4}{4}\right)$$

$$= \frac{1}{2}\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = \frac{1}{2}\left(\frac{2a^4}{4} - \frac{a^4}{4}\right)$$

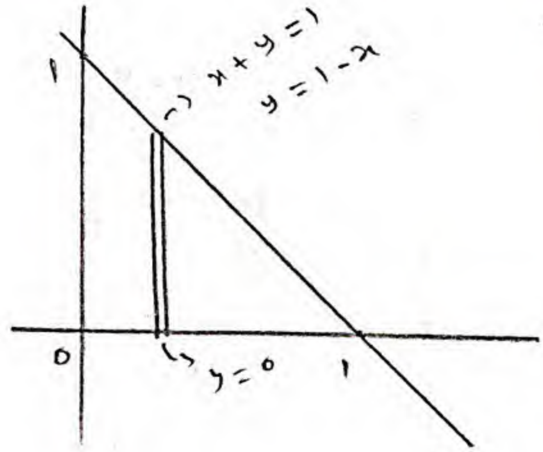
$$= \frac{1}{2}\left(\frac{2a^4 - a^4}{4}\right) = \frac{1}{2}\left(\frac{a^4}{4}\right)$$

$$= \frac{a^4}{8}$$

$$\therefore \text{Area} = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} xy \, dy \, dx = \frac{a^4}{8}$$

2) Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region for which x & y both greader than or equal to $x+y \le 1$

limits.



when $y=0$, $x+y=1$

$\Rightarrow x=1$

$\therefore$ x : $0 \to 1$

y : $0 \to 1-x$.

$$\text{Area} = \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2+y^2)\,dy\,dx.$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{1-x} (x^2+y^2)\,dy \right) dx.$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{1-x} x^2\,dy + \int_{y=0}^{1-x} y^2\,dy \right) dx.$$

$$= \int_{x=0}^{1} \left(x^2 \int_{y=0}^{1-x} dy + \int_{y=0}^{1-x} y^2\,dy \right) dx$$

$$= \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^{1-x} dx$$

$$= \int_{x=0} \left[x^2(1-x) + \frac{(1-x)^3}{3} \right] dx$$

$$= \int_{x=0}^{1} \left(x^2 - x^3 + \frac{(1-x)^3}{3} \right) dx$$

$$= \int_{x=0}^{1} x^2\,dx - \int_{x=0}^{1} x^3\,dx + \frac{1}{3}\int_{x=0}^{1} (1-x)^3\,dx$$

$$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

$$= \int_{x=0}^{1} x^2\,dx - \int_{x=0}^{1} x^3\,dx + \frac{1}{3}\int_{x=0}^{1} (1 - 3x + 3x^2 - x^3)\,dx$$

$$= \int_{x=0}^{1} x^2\,dx - \int_{x=0}^{1} x^3\,dx + \frac{1}{3}\left[\int_{x=0}^{1} dx - 3\int_{x=0}^{1} x\,dx + 3\int_{x=0}^{1} x^2\,dx - \int_{x=0}^{1} x^3\,dx \right]$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\left[x - \frac{3x^2}{2} + \frac{3x^3}{3} - \frac{x^4}{4} \right] \right]_0^1$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{x}{3} - \frac{\cancel{3}x^2}{\cancel{6}\,2} + \frac{\cancel{3}x^3}{\cancel{9}\,3} - \frac{x^4}{12} \right]_0^1$$

$$= \left[\frac{x}{3} - \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^4}{12} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{3} - \frac{1}{4} - \frac{1}{12}$$

$$= -\frac{1}{2} + \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \left(\frac{1}{4} \times \frac{3}{3}\right) - \frac{1}{12}$$

$$= -\frac{1}{2} + \frac{3}{3} - \frac{3}{12} - \frac{1}{12}$$

$$= -\frac{1}{2} + \frac{1}{1} - \frac{(3+1)}{12}$$

$$= \frac{-1+2}{2} - \frac{4}{12} = \frac{1}{2} - \frac{4}{12}$$

$$= \left(\frac{1}{2} \times \frac{6}{6}\right) - \frac{4}{12}$$

$$= \frac{6}{12} - \frac{4}{12} = \frac{6-4}{12}$$

$$= \frac{\cancel{2}\,1}{\cancel{12}\,6}$$

$$= \frac{1}{6}$$

$$\therefore \text{Area} = \int_{x=0}^{1} \int_{y=0}^{1-x} (x^2+y^2)\, dy\, dx = \frac{1}{6}$$

HOME WORK.

9) Evaluate $\int_{-\pi}^{\pi} \int_{0}^{a(1+\cos\theta)} r\, dr\, d\theta$.

$$\int_{\theta=-\pi}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\, dr\, d\theta = \int_{\theta=-\pi}^{\pi} \left(\int_{r=0}^{a(1+\cos\theta)} r\, dr \right) d\theta$$

$$= \int_{\theta=-\pi}^{\pi} \left[\frac{r^2}{2} \right]_{r=0}^{a(1+\cos\theta)} d\theta .$$

$$= \int_{\theta=-\pi}^{\pi} \frac{1}{2} \left[r^2 \right]_{0}^{a(1+\cos\theta)} d\theta .$$

$$= \frac{1}{2} \int_{\theta=-\pi}^{\pi} \left(a^2(1+\cos\theta)^2 - 0^2 \right) d\theta .$$

$$= \frac{1}{2} \int_{\theta=-\pi}^{\pi} a^2(1+\cos\theta)^2 \, d\theta$$

$$= \frac{a^2}{2} \int_{-\pi}^{\pi} (1+\cos\theta)^2 \, d\theta$$

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

$$= \frac{a^2}{2} \int_{-\pi}^{\pi} (1 + 2\cos\theta + \cos^2\theta) \, d\theta$$

$$= \frac{a^2}{2} \left[\int_{-\pi}^{\pi} d\theta + 2\int_{-\pi}^{\pi} \cos\theta \, d\theta + \int_{-\pi}^{\pi} \cos^2\theta \, d\theta \right]$$

$$= \frac{a^2}{2} \left[\int_{-\pi}^{\pi} d\theta + 2\int_{-\pi}^{\pi} \cos\theta \, d\theta + \int_{-\pi}^{\pi} \frac{1+\cos 2\theta}{2} \, d\theta \right]$$

$$= \frac{a^2}{2} \left[\int_{-\pi}^{\pi} d\theta + 2\int_{-\pi}^{\pi} \cos\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} d\theta + \frac{1}{2}\int_{-\pi}^{\pi} \cos 2\theta \right.$$

put $2\theta = x$

$$\theta = \frac{x}{2}$$

$$\frac{d\theta}{dx} = \frac{1}{2}$$

$$d\theta = \frac{dx}{2}$$

$\theta = -\pi \Rightarrow x = -2\pi$

$\theta = \pi \Rightarrow x = 2\pi$

$$= \frac{a^2}{2}\left[\int_{-\pi}^{\pi} d\theta + 2\int_{-\pi}^{\pi} \cos\theta \, d\theta + \frac{1}{2}\int_{-\pi}^{\pi} d\theta + \frac{1}{2}\int_{-2\pi}^{2\pi} \cos x \frac{dx}{2}\right]$$

$$= \frac{a^2}{2}\left([\theta]_{-\pi}^{\pi} + 2[\sin\theta]_{-\pi}^{\pi} + \frac{1}{2}[\theta]_{-\pi}^{\pi} + \frac{1}{4}[\sin x]_{-2\pi}^{2\pi}\right)$$

$$= \frac{a^2}{2}\left(\left[\pi - (-\pi)\right] + 2\left[\sin\pi - (\sin(-\pi))\right] + \frac{1}{2}\left[\pi - (-\pi)\right] + \frac{1}{4}\left[\sin 2\pi - \sin(-2\pi)\right]\right)$$

$$= \frac{a^2}{2}\left([\pi + \pi] + 2[\sin\pi + \sin\pi] + \frac{1}{2}[\pi + \pi] + \frac{1}{4}[\sin 2\pi + \sin 2\pi]\right)$$

$$\boxed{\begin{array}{l} \sin\pi = \sin 180^\circ = 0 \\ \sin 2\pi = \sin 360^\circ = 0 \end{array}}$$

$$= \frac{a^2}{2}\left(2\pi + 0 + \frac{2\pi}{2} + 0 \right)$$

$$= \frac{a^2}{2}\left[2\pi + \frac{\cancel{2}\pi}{\cancel{2}} \right] = \frac{a^2}{2}(2\pi + \pi)$$

$$= \frac{3a^2\pi}{2}$$

$$\therefore \int_{-\pi}^{\pi}\int_{0}^{a(1+\cos\theta)} r\,dr\,d\theta = \frac{3a^2\pi}{2}$$ 229

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10) Evaluate. $\int_0^a \int_0^{2\sqrt{ax}} x^2\,dx\,dy$.

$$\int_{x=0}^{a}\int_{y=0}^{2\sqrt{ax}} x^2\,dx\,dy = \int_{x=0}^{a} x^2\left(\int_{y=0}^{2\sqrt{ax}} dy\right)dx.$$

$$= \int_{x=0}^{a} x^2\,\left[y\right]_{y=0}^{2\sqrt{ax}}\,dx.$$

$$= \int_{x=0}^{a} x^2\,\left[2\sqrt{ax}-0\right]dx$$

$$= \int_{x=0}^{a} x^2\,2\sqrt{ax}\,dx.$$

$$= 2\sqrt{a}\int_{x=0}^{a} x^2\sqrt{x}\,dx$$

$$= 2\sqrt{a}\int_{x=0}^{a} x^2\cdot x^{1/2}\,dx$$

$$= 2\sqrt{a}\int_{x=0}^{a} x^{2+\frac{1}{2}}\,dx$$

$$= 2\sqrt{a}\int_{x=0}^{a} x^{5/2}\,dx$$

$$= 2\sqrt{a}\left[\frac{x^{5/2+1}}{5/2+1}\right]_0^a$$

$$= 2\sqrt{a}\left[\frac{x^{7/2}}{7/2}\right]_0^a$$

$$= 2\sqrt{a} \times \frac{2}{7}\left[x^{7/2}\right]_0^a$$

$$= \frac{4\sqrt{a}}{7}\left[a^{7/2}\right]$$

$$= \frac{4\sqrt{a}}{7} \times a^3 \cdot a^{1/2}$$

$$= \frac{4\sqrt{a} \times \sqrt{a}}{7} \times a^3$$

$$= \frac{4a \times a^3}{7}$$

$$= \frac{4a^4}{7}$$

$$\therefore \int_0^a \int_0^{2\sqrt{ax}} x^2\,dx\,dy = \frac{4a^4}{7}$$

11) Evaluate. $\displaystyle\int_0^1 \int_{\sqrt{y}}^{2-y} x^2\,dx\,dy.$

$$\int_{x=0}^1 \int_{y=\sqrt{y}}^{2-y} x^2\,dx\,dy = \int_{x=0}^1 \left(\int_{\sqrt{y}}^{2-y} x^2\,dx\right) dy.$$

$$= \int_{x=0}^1 \left[\frac{x^3}{3}\right]_{\sqrt{y}}^{2-y} dy.$$

$$= \frac{1}{3}\int_{x=0}^1 \left[x^3\right]_{\sqrt{y}}^{2-y} dy$$

$$= \frac{1}{3}\int_{x=0} \left[(2-y)^3 - (\sqrt{y})^3 \right] dy$$

$$\boxed{(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3}$$

$$= \frac{1}{3}\int_{x=0}^{1} \left[(2)^3 - 3(2)^2(y) + 3(2)(y)^2 - y^3 - (y^{1/2})^3 \right] dy$$

$$= \frac{1}{3}\int_{x=0}^{1} (8 - 12y + 6y^2 - y^3 - y^{3/2})\, dy.$$

$$= \frac{1}{3}\left[8\int_{x=0}^{1} dy - 12\int_{x=0}^{1} y\, dy + 6\int_{x=0}^{1} y^2 dy - \int_{x=0}^{1} y^3 dy - \int_{x=0}^{1} y^{3/2} dy \right]$$

$$= \frac{1}{3}\left[8y - \frac{\overset{6}{\cancel{12}}y^2}{\cancel{2}_1} + \frac{\overset{2}{\cancel{6}}y^3}{\cancel{3}_1} - \frac{y^4}{4} - \frac{y^{3/2+1}}{3/2+1} \right]_0^1$$

$$= \frac{1}{3}\left[8y - 6y^2 + 2y^3 - \frac{y^4}{4} - \frac{y^{5/2}}{5/2} \right]_0^1$$

$$= \frac{1}{3}\left[8(1) - 6(1)^2 + 2(1)^3 - \frac{(1)^4}{4} - \frac{(1)^{5/2}}{5/2} \right]$$

$$= \frac{1}{3}\left[8 - 6 + 2 - \frac{1}{4} - \frac{1}{5/2} \right]$$

$$= \frac{1}{3}\left[8 - 6 + 2 - \frac{1}{4} - \frac{2}{5} \right]$$

$$= \frac{1}{3}\left[4 - \frac{1}{4} - \frac{2}{5} \right] = \frac{1}{3}\left[\frac{16-1}{4} - \frac{2}{5} \right]$$

$$= \frac{1}{3}\left[\frac{15}{4} - \frac{2}{5} \right] = \frac{1}{3}\left(\frac{75-8}{20} \right)$$

$$= \frac{1}{3}\left(\frac{67}{20} \right) = \frac{67}{60}$$

$$\therefore \int_{x=0} \int_{y=\sqrt{y}} x^2 \, dx \, dy = \frac{67}{60}$$

HOME WORK.

3) Find the value of $\iint (a^2-x^2)\, dx\,dy$ taken over the half circle $x^2+y^2=a^2$ in the positive quardinant. $\left(\frac{3\pi a^2}{16}\right)$

4) Evaluate $\iint x^2y^2 \, dy\, dx$ over the circuler area $x^2+y^2 \le 1$ and $x, y \ge 0$ $\left(\frac{\pi}{24}\right)$

polar.

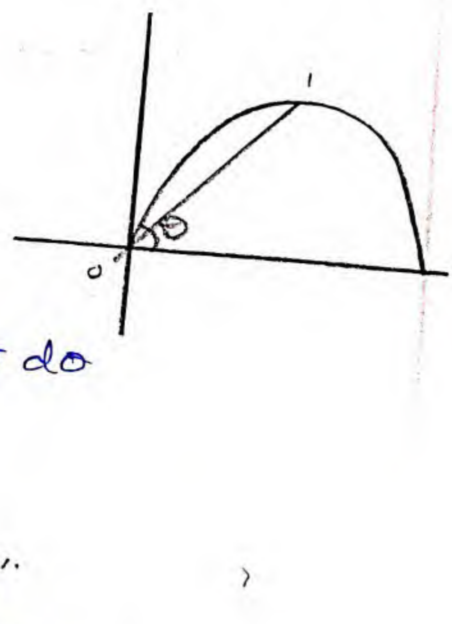
5) Evaluate $\iint r\sqrt{a^2-r^2}\, dr\, d\theta$ over the upper half of the circle $r = a\cos\theta$.

limits.

VIVY

$\theta : 0 \to \pi/2$

$r : 0 \to a\cos\theta$.



$$Area = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$$

put.

$a^2 - r^2 = t^2$

Diff w.r. to "t"

$0 - 2r\frac{dr}{dt} = 2t$

$-2r\,dr = 2t\,dt$

$-r\,dr = \frac{\cancel{2}t}{\cancel{2}}\,dt$

$r\,dr = -t\,dt.$

$r=0 \Rightarrow t^2 = a^2 - 0$

$t^2 = a^2$

$t = a$

$r = a\cos\theta \Rightarrow t^2 = a^2 - (a\cos\theta)^2$

$= a^2 - a^2\cos^2\theta$

$= a^2(1-\cos^2\theta)$

$t^2 = a^2\sin^2\theta$

$t = a\sin\theta.$

$$\text{Area} = \int_{\theta=0}^{\pi/2}\int_{t=a}^{a\sin\theta} t(-t)\,dt\,d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\left[\int_{t=a}^{a\sin\theta} t^2\,dt\right]d\theta.$$

$$= -\int_{\theta=0}^{\pi/2}\left[\frac{t^3}{3}\right]_a^{a\sin\theta} d\theta$$

$$= -\int_{\theta=0}^{\pi/2}\frac{1}{3}\left[t^3\right]_a^{a\sin\theta} d\theta.$$

$$= -\frac{1}{3}\int_{\theta=0}^{\pi/2}(a^3\sin^3\theta - a^3)\,d\theta$$

$$= -\frac{1}{3}\left[a^3\int_{\theta=0}^{\pi/2}\sin^3\theta\,d\theta - a^3\int_{\theta=0}^{\pi/2}d\theta\right]$$

$$\boxed{\int_0^{\pi/2}\sin^n x\,dx\ (\text{odd}) = \frac{(n-1)}{n}\frac{(n-3)}{(n-2)}\cdots\frac{2}{3}}$$

$$= -\frac{1}{3}\left[a^3\left(\frac{2}{3}\right) - a^3\left[\theta\right]_0^{\pi/2} \right]$$

$$= -\frac{1}{3}\left[\frac{2a^3}{3} - a^3\left[\frac{\pi}{2} - 0\right] \right]$$

$$= -\frac{1}{3}\left[\frac{2a^3}{3} - \frac{\pi a^3}{2} \right]$$

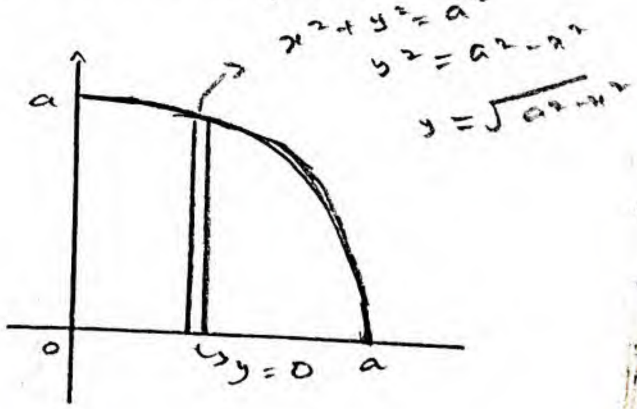
$$= -\frac{2a^3}{a} + \frac{\pi a^3}{6}$$

$$= \frac{a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3} \right]$$

$$\therefore \text{Area} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta = \frac{a^3}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right]$$

HOME WORK.

3) Find the value of $\iint (a^2-x^2)\,dx\,dy$ taken over the half circle $x^2+y^2=a^2$ in the positive quardinant.



Jimits

x : $0 \to a$

y : $0 \to \sqrt{a^2-x^2}$

$$\text{Area} = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} (a^2-x^2)\,dx\,dy.$$

$$= \int_{x=0}^{a} \left[\int_{y=0}^{\sqrt{a^2-x^2}} (a^2-x^2)\,dy \right] dx.$$

The Integral is given by $\iint (a^2-x^2)\,dx\,dy$ over the half circle $x^2+y^2=a^2$ in the positive quadrant.

In polar coordinates, we have

$x = r\cos\theta$ and $y = r\sin\theta$ and $dx\,dy = r\,dr\,d\theta$

The region is the part of the circle $r=a$ in the first quadrant, so the limits for

$r : 0 \to a$

$\theta : 0 \to \pi/2$

Substituting these into the Integral.

$$\text{Area} = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \left[a^2 - (r\cos\theta)^2\right] r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} (a^2 - r^2\cos^2\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left(\int_{r=0}^{a} (a^2 r - r^3\cos^2\theta)\,dr\right)d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[a^2\int_{r=0}^{a} r\,dr - \cos^2\theta\int_{r=0}^{a} r^3\,dr\right]d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left[\frac{a^2r^2}{2} - \frac{r^4}{4}\cos^2\theta\right]_0^a d\theta$$

$$= \int_{0}^{\pi/2}\left(\frac{a^2(a^2)}{2} - \frac{a^4}{4}\cos^2\theta\right)d\theta$$

$$= \int_0^{\pi/2} \left[\left(\frac{a^4}{2} \times \frac{2}{2} \right) - \frac{a^4}{4} \cos^2\theta \right] d\theta$$

$$= \int_0^{\pi/2} \left(\frac{2a^4}{\cancel{2}\,4} - \frac{a^4}{4} \cos^2\theta \right) d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} (2 - \cos^2\theta) d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} \left(2 - \frac{1+\cos 2\theta}{2} \right) d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} \left(\frac{4 - 1 - \cos 2\theta}{2} \right) d\theta.$$

$$= \frac{a^4}{4} \times \frac{1}{2} \int_0^{\pi/2} (3 - \cos 2\theta) d\theta$$

$$= \frac{a^4}{8} \left[3\int_0^{\pi/2} d\theta - \int_0^{\pi/2} \cos 2\theta \, d\theta \right]$$

$$= \frac{a^4}{8} \left[3\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{a^4}{8} \left[\left(3\pi/2 - \frac{\sin \cancel{2} (\pi/\cancel{2})}{2} \right) - \left(0 - \frac{\sin(0)}{2} \right) \right]$$

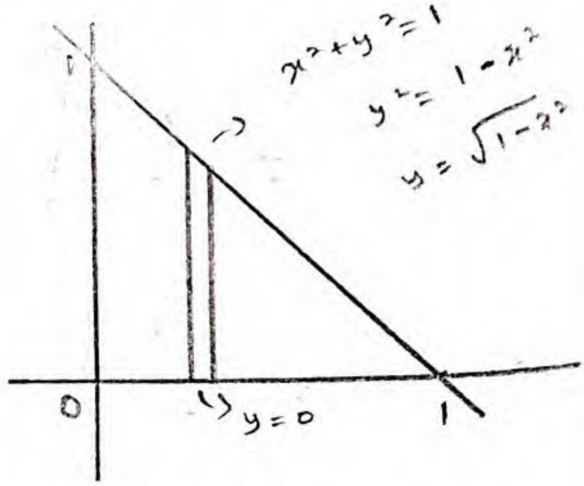
$$= \frac{a^4}{8} \left[\left(\frac{3\pi}{2} - \frac{\sin \pi}{2} \right) - 0 + 0 \right]$$

$$= \frac{a^4}{8} \left[\frac{3\pi}{2} - 0 - 0 + 0 \right]$$

$$= \frac{a^4}{8} \left(\frac{3\pi}{2} \right) = \frac{3\pi a^4}{16}$$

$\therefore$ The value of the double Integral is $\frac{3\pi a^4}{16}$

4) Evaluate $\iint x^2y^2 \, dy \, dx$ over the circular area $x^2+y^2 \leq 1$ and $x, y \geq 0$



limits

when $y=0$, $x^2+y^2=1$

$x^2+0=1$

$x^2=1$

$x=\sqrt{1}$

$x=1$

$\therefore x: 0 \to 1$

$y: 0 \to \sqrt{1-x^2}$

$$\text{Area} = \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} x^2y^2 \, dy \, dx$$

$$= \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} x^2y^2 \, dy \right) dx$$

$$= \int_{x=0}^{1} \left(x^2 \int_{y=0}^{\sqrt{1-x^2}} y^2 \, dy \right) dx$$

$$= \int_{x=0}^{1} \left[\frac{x^2 y^3}{3} \right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{1}{3} \int_{x=0}^{1} x^2 \left(\sqrt{1-x^2}\right)^3 dx$$

$x = \sin\theta$

$\frac{dx}{d\theta} = \cos\theta$

$dx = \cos\theta \, d\theta.$

when $x = 0$, $0 = \sin\theta$ $\theta = 0$

when, $x = 1$, $1 = \sin\theta$ $\theta = \pi/2$

$$x^2 = \sin^2\theta$$

$$1 - x^2 = 1 - \sin^2\theta = \cos^2\theta.$$

$$\text{Area} = \frac{1}{3}\int_0^{\pi/2} \sin^2\theta \left(\sqrt{\cos^2\theta}\right)^3 \cos\theta\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin^2\theta \cos^3\theta \cos\theta\, d\theta.$$

$$= \frac{1}{3}\int_0^{\pi/2} \sin^2\theta\, \overset{+\text{ b even}}{\cos^4\theta}\, d\theta.$$

(even)

$$\boxed{I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x\, dx = \frac{n-1}{m+n} \cdot \frac{n-3}{m+n-2} \cdot \frac{n-5}{m+n-4} \cdots \frac{1}{2} \cdot \frac{\pi}{2}}$$

$$= \frac{1}{3}\left[\frac{(4-1)}{(2+4)} \cdot \frac{(4-3)}{2+4-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{1}{3}\left[\frac{3}{6} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}\right]$$

$$= \frac{1}{\cancel{3}}\left[\frac{\cancel{3} \times \pi}{6 \times 4 \times 2 \times 2}\right]$$

$$= \frac{\pi}{96}$$

∴ The value of the double Integral is $\frac{\pi}{96}$.

double Integral.

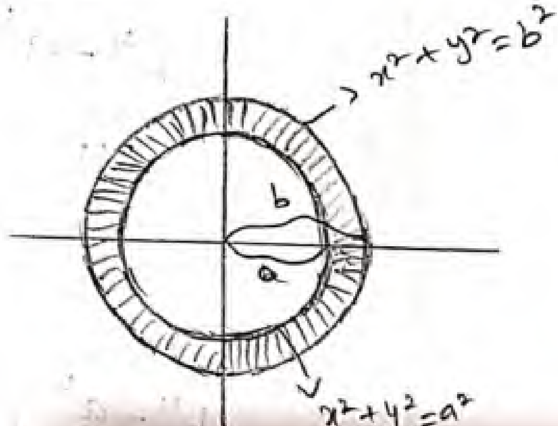
b) By transforming into the polar co-ordinates $\Rightarrow$ Evaluate $\iint \frac{x^2y^2}{x^2+y^2}\,dy\,dx$ over the annulat region between the circle $x^2+y^2=a^2$ and $x^2-y^2=b^2$

$$\iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy.$$

$x^2+y^2=a^2$ and $x^2-y^2=b^2$

polar coordinates are

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$dx\,dy = r\,dr\,d\theta.$$



Limits.

$$r : a \to b$$
$$\theta : 0 \to 2\pi$$

Now,

$$\text{Area} = \iint \frac{x^2y^2}{x^2+y^2}\,dx\,dy$$

$$\text{Area} = \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\cos^2\theta\, r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=a}^{b} \frac{r^5 \cdot \cos^2\theta \sin^2\theta}{r^2(\cos^2\theta + \sin^2\theta)}\, dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\left(\int_{r=a}^{b} r^3 \cos^2\theta\,(1-\cos^2\theta)\,dr\right)d\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_a^b (\cos^2\theta - \cos^4\theta)\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{b^4}{4} - \frac{a^4}{4} \cdot (\cos^2\theta - \cos^4\theta)\,d\theta$$

$$= \frac{b^4-a^4}{4}\left[\int_0^{2\pi}\cos^2\theta\,d\theta - \int_0^{2\pi}\cos^4\theta\,d\theta\right]$$

$$\boxed{I_n = \int \cos^n\theta\,d\theta = \frac{1}{n}\left[\cos^{n-1}x \sin x + (n-1)I_{n-2}\right]}$$

$$= \frac{b^4-a^4}{4}\left[\frac{1}{2}\left(\cos x \sin x + (2-1)I_{2-2}\right) - \frac{1}{4}\left(\cos^3 x \sin x + (4-1)I_{4-2}\right)\right]_0^{2\pi}$$

$$= \frac{b^4-a^4}{4}\left[\frac{1}{2}(\cos x \sin x + x) - \frac{1}{4}\left(\cos^3 x \sin x + 3 \times \frac{1}{2}(\cos x \sin x + x)\right)\right]$$

$$= \frac{b^4-a^4}{4}\left[\frac{1}{2}\cos x\sin x+\frac{1}{2}x-\frac{1}{4}\left(\cos^3 x\sin x+\frac{3}{2}\cos x\sin x+\frac{3}{2}x\right)\right]_0^{2\pi}$$

$$= \frac{b^4-a^4}{4}\times\frac{1}{2}\left[\cos x\sin x+x-\frac{1}{2}\cos^3 x\sin x-\frac{3}{4}\cos x\sin x-\frac{3}{4}x\right]_0^{2\pi}$$

$$= \frac{b^4-a^4}{8}\left[\frac{1}{4}\cos x\sin x+\frac{1}{4}x-\frac{1}{2}\cos^3 x\sin x\right]_0^{2\pi}$$

$$= \frac{b^4-a^4}{16}\left[\frac{1}{2}\cos x\sin x+\frac{1}{2}x-\cos^3 x\sin x\right]_0^{2\pi}$$

$\frac{1}{2}(1)(0)+\pi-(1)(0)-\frac{1}{2}(1)(0)-0+(1)(0)$

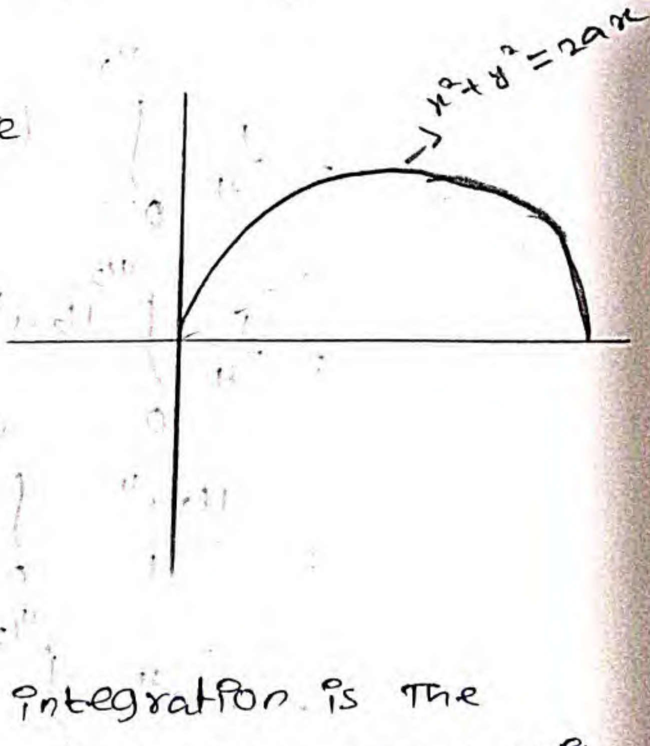
$$= \frac{b^4-a^4}{16}\left[\frac{1}{2}\cos(2\pi)\sin(2\pi)+\frac{1}{2}(2\pi)-\cos^3(2\pi)\sin(2\pi)-\frac{1}{2}\cos 0\sin 0+\frac{1}{2}(0)-\cos^3(0)\sin(0)\right]$$

$\sin 2\pi = \sin 0 = 0$
$\cos 2\pi = \cos 0 = 1$

$\boxed{\text{Area} = \pi\,\frac{b^4-a^4}{16}}$

7) By changing into polar co-ordinates. evaluate. The Integral $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy$

$$\int_0^{2a}\int_0^{\sqrt{2ax-x^2}}(x^2+y^2)\,dx\,dy.$$



polar co-ordinates are

$x = r\cos\theta$

$y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$

Limits.

$y=\sqrt{2ax-x^2}$

$\Rightarrow x^2+y^2=2ax.$

The region of integration is the semi-circle $x^2+y^2=2ax$ above the x-axis changing into polars, the region becomes $r=2a\cos\theta$ from $\theta=0$ to $\theta=\pi/2$

$$\text{Area} = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\, dx\, dy$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} (r^2\cos^2\theta + r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2a\cos\theta} r^3 (\cos^2\theta + \sin^2\theta)\, dr\, d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{2a\cos\theta} r^3\, dr \right) d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4} \right]_0^{2a\cos\theta} d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} (2a\cos\theta)^4\, d\theta$$

$$= \frac{1}{4} \int_0^{\pi/2} 16a^4 \cos^4\theta\, d\theta$$

$$= \frac{16a^4}{4} \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$= 4a^4 \int_0^{\pi/2} \cos^4\theta\, d\theta$$

$$\boxed{I_n = \int \cos^n x\, dx = \frac{(n-1)}{n} \frac{(n-3)}{(n-2)} \frac{(n-5)}{(n-4)} \cdots \frac{1}{2} \cdot \frac{\pi}{2}}$$

(even)

$$= \frac{1}{4}\left(\frac{a^4 a^2}{4} - \frac{a^2 a^4}{4} + \frac{a^6}{12}\right)$$

$$= \frac{1}{4}\left(\cancel{\frac{a^6}{4}} - \cancel{\frac{a^6}{4}} + \frac{a^6}{12}\right)$$

$$= \frac{1}{4}\left(\frac{a^6}{12}\right)$$

$$= \frac{a^6}{48}$$

$\therefore$ The value of triple integral is $\frac{a^6}{48}$.

Triple Integral.

$$\int f(x,y,z)\, dv = \int_{x=x_1}^{x_2} \int_{y=\phi_1(x)}^{\phi_2(x)} \int_{z=f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz\, dy\, dx$$

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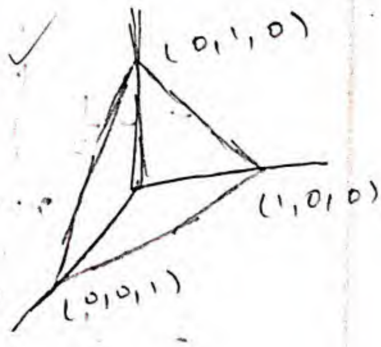
9) [Evaluate $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ taken over the volume bounded by the planes $x=0, y=0, z=0, x+y+z=1$.]



Limits.

Given $x+y+z=1$ ——— ①

$z = 1-x-y$

$z : 0 \to 1-x-y$

put $z=0$ in ①

$x+y=1$ $\quad$ $y : 0 \to 1-x$

$y = 1-x$

put $z=y=0$ in ①

$x=1$

$x : 0 \to 1$

$$\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}\frac{dx\,dy\,dz}{(x+y+z+1)^3}=\int_{x=0}^{1}\int_{y=0}^{1-x}\left[\frac{(x+y+z+1)^{-3+1}}{-3+1}\right.$$

$$=-\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left[(\cancel{x}+\cancel{y}+1-\cancel{x}-\cancel{y}+1)^{-2}-(x+y+1)^{-2}\right]dx$$

$$=-\frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x}\left(2^{-2}-(x+y+1)^{-2}\right)dy\,dx$$

$$=-\frac{1}{2}\int_{x=0}^{1}\left[\int_{y=0}^{1-x}\frac{1}{2^2}\,dy-\int(x+y+1)^{-2}dy\right]dx$$

$$=-\frac{1}{2}\int_{x=0}^{1}\left[\frac{1}{4}y-\frac{(x+y+1)^{-2+1}}{-2+1}\right]_{y=0}^{1-x}dx$$

$$=-\frac{1}{2}\int_{x=0}^{1}\left(\frac{1}{4}(1-x)-\frac{(\cancel{x}+1-\cancel{x}+1)^{-1}}{-1}+\frac{(x+1)^{-1}}{-1}\right)$$

$$=-\frac{1}{2}\int_{x=0}^{1}\left(\frac{1-x}{4}+(2)^{-1}-\frac{1}{x+1}\right)dx$$

$$=-\frac{1}{2}\int_{x=0}^{1}\left(\frac{1}{4}+\frac{1}{2}-\frac{x}{4}-\frac{1}{x+1}\right)dx$$

$$=-\frac{1}{2}\int_{x=0}^{1}\left(\frac{1}{4}+\frac{2}{4}-\frac{x}{4}-\frac{1}{x+1}\right)dx$$

$$=-\frac{1}{2}\int_{x=0}^{1}\left(\frac{3}{4}-\frac{x}{4}-\frac{1}{x+1}\right)dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}\int_{x=0}^{1} dx - \frac{1}{4}\int_{x=0}^{1} x\,dx - \int_{x=0}^{1} \frac{1}{x+1}\,dx\right]$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{4\times 2} - \log(1+x)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{3}{4}x - \frac{x^2}{8} - \log(1+x)\right]_0^1$$

$$= -\frac{1}{2}\left[\left(\frac{3}{4} - \frac{1}{8} - \log(1+1)\right) - \log 1\right]$$

$$= -\frac{1}{2}\left(\frac{6}{8} - \frac{1}{8} - \log(2)\right)$$

$$= -\frac{1}{2}\left[\frac{5}{8} - \log(2)\right]$$

$$= -\frac{5}{16} + \frac{1}{2}\log(2)$$

$$= \frac{1}{2}\log(2) - \frac{5}{16}.$$

$$\boxed{b\log a = \log a^b}$$

$$= \log 2^{1/2} - \frac{5}{16}$$

$$= \log\sqrt{2} - \frac{5}{16}.$$

$\therefore$ The triple Integral value is

$$\log\sqrt{2} - \frac{5}{16}.$$

---

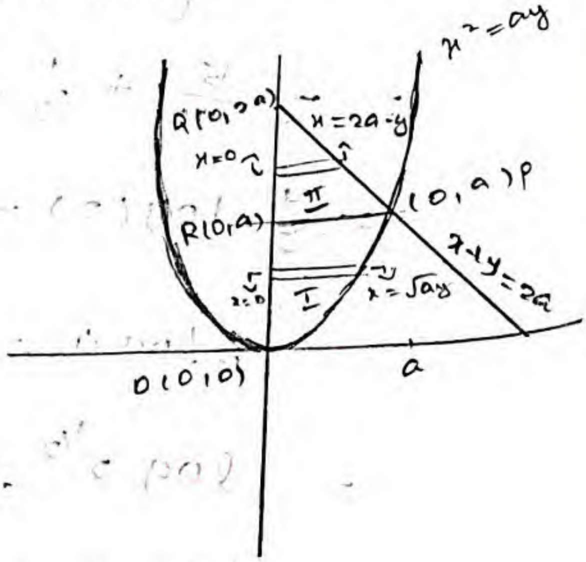
10) Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ for all positive values of x, y, z for which the integral is real.

11) Evaluate $\iiint (x+y+z)\, dx\,dy\,dz$ taken over the region bounded by the surface $x^2+y^2=a^2$, $z: 0 \to h$

12) Evaluate $\iiint x^2yz\, dx\,dy\,dz$ over the tetrahedron bounded by $x=0, y=0, z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Change in the order of Integration.

13) By changing the order of integration $\int_0^a \int_{x^2/a}^{2a-x} xy\, dx\,dy$. Evaluate it.



To Find.

$$I=\int_0^a \int_{x^2/a}^{2a-x} xy\, dx\,dy$$

$$= \int_0^a \left(\int_{x^2/a}^{2a-x} xy\, dy \right) dx.$$

Given the limits of y are.

$$y=\frac{x^2}{a} \text{ and } y=2a-x$$

$$x^2=ay \text{ and } x+y=2a$$

Given the limits of x are.

$$x=0 \text{ and } x=a.$$

Hence the region of integration is OPQ from the figure. In changing the order of integration, we first integrate with respect to x keeping y as constant.

With elementary strips paralal to x-axis covering the region OPQ, the end of the strips extant to the line $x+y=2a$ and to the curve $y=\frac{x^2}{a}$ or $x^2=ay$. Hence we divided the region into two parts By $y=a$ line passes Through P and R.

Region-I: OPR.

Limits.

$x: 0 \to \sqrt{ay}$

$y: 0 \to a$

now,

$$I_1 = \int_{y=0}^{a} \left(\int_{x=0}^{\sqrt{ay}} xy\, dx \right) dy$$

$$= \int_{y=0}^{a} y\left[\frac{x^2}{2}\right]_0^{\sqrt{ay}} dy$$

$$= \frac{1}{2}\int_{y=0}^{a} y(\sqrt{ay})^2\, dy$$

$$= \frac{1}{2}\int_{y=0}^{a} ay^2\, dy$$

$$= \frac{a}{2}\int_{y=0}^{a} y^2\, dy$$

$$= \frac{a}{2}\left[\frac{y^3}{3}\right]_0^{a}$$

$$= \frac{a}{2}\left(\frac{a^3}{3}\right)$$

$$I_1 = \frac{a^4}{6}$$

Region - II : PQR.

Limits.

$x : 0 \to 2a-y$

$y : a \to 2a$.

$$I_2 = \int_{y=a}^{2a} \left(\int_{x=0}^{2a-y} xy \, dx \right) dy$$

$$= \int_{y=a}^{2a} y \left[\frac{x^2}{2} \right]_0^{2a-y} dy$$

$$= \frac{1}{2} \int_{y=a}^{2a} y \, (2a-y)^2 \, dy$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$= \frac{1}{2} \int_{y=a}^{2a} y \, (4a^2 + y^2 - 4ay) \, dy$$

$$= \frac{1}{2} \int_{y=a}^{2a} (4a^2y + y^3 - 4ay^2) \, dy$$

$$= \frac{1}{2} \left[\frac{4a^2y^2}{2} + \frac{y^4}{4} - \frac{4ay^3}{3} \right]_a^{2a}$$

$$= \frac{1}{2} \left[\left(\frac{4a^2(2a)^2}{2} + \frac{(2a)^4}{4} - \frac{4a(2a)^3}{3} \right) - \left(\frac{4a^2(a)^2}{2} + \frac{a^4}{4} - \frac{4a(a)^3}{3} \right) \right]$$

$$= \frac{1}{2} \left[\frac{16a^4}{2} + \frac{16a^4}{4} - \frac{32a^4}{3} - \left(\frac{4a^4}{2} + \frac{a^4}{4} - \frac{4a^4}{3} \right) \right]$$

$$= \frac{1}{2} \left[\frac{16a^4}{2} + \frac{16a^4}{4} - \frac{32a^4}{3} - \frac{4a^4}{2} - \frac{a^4}{4} + \frac{4a^4}{3} \right]$$

$$= \frac{1}{2}\left[\frac{12a^4}{2} + \frac{15a^4}{4} - \frac{28a^4}{3}\right]$$

$$= \frac{a^4}{2}\left[6 + \frac{15}{4} - \frac{28}{3}\right]$$

$$= \frac{a^4}{2}\left[6 + \frac{45-112}{12}\right]$$

$$= \frac{a^4}{2}\left[6 - \frac{67}{12}\right]$$

$$= \frac{a^4}{2}\left[\frac{72-67}{12}\right]$$

$$= \frac{a^4}{12}\left(\frac{5}{12}\right)$$

$$I_2 = \frac{5a^4}{24}$$

$$I = I_1 + I_2$$

$$I = \frac{a^4}{6} + \frac{5a^4}{24}$$

$$= \left(\frac{a^4}{6} \times \frac{4}{4}\right) + \frac{5a^4}{24}$$

$$= \frac{4a^4}{24} + \frac{5a^4}{24}$$

$$= \frac{9a^4}{24}$$

$$\boxed{I = \frac{3a^4}{8}}$$

$$\therefore \ I = \int_0^a \int_{x^2/a}^{2a-x} xy \, dx\, dy = \frac{3a^4}{8}$$

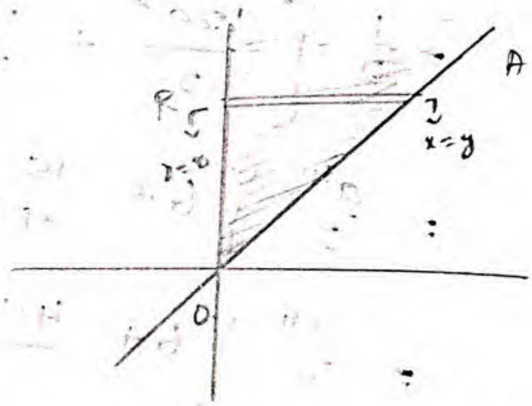
---

(4) By changing the order of integration evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dx\, dy$

Given That.

$y : x \to \infty$

$x : 0 \to \infty$



Integrate with respect to y from x to ∞ and Then integrate with respect to x from 0 to ∞.

Let OA be The straight line $y=x$
Region of integration is R above OA.
Reverse The order of integration, so we integrate with respect to x keep y as constant.

y constant

$x : 0 \to y$

$y : 0 \to \infty$

$$I_1 = \int_{y=0}^{\infty} \left(\int_{x=0}^{y} \frac{e^{-y}}{y} dx \right) dy$$

$$= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y \, dy$$

$$= \int_{y=0}^{\infty} \left(\frac{e^{-y}}{y} \times y \right) dy$$

$$= \int_{y=0}^{\infty} e^{-y} dy$$

$$= \left[\frac{e^{-y}}{-1} \right]_0^{\infty}$$

$$= \left[-e^{-y} \right]_0^{\infty}$$

$$= \left[-e^{-\infty} - (-e^{0})\right]$$

$$= (-0+1)$$

$$= 1$$

$$\therefore \int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dx\, dy = 1$$

Triple Integral.

10) Evaluate $\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ for all positive values of x, y, z for which. The Integral is real.

Limits.

$$\sqrt{1-x^2-y^2-z^2} \geq 0$$

$$1-x^2-y^2-z^2=0$$

$$z^2 = 1-x^2-y^2$$

$$z = \sqrt{1-x^2-y^2}$$

$$z: 0 \to \sqrt{1-x^2-y^2}$$

put $z=0$,

$$1-x^2-y^2=0$$

$$y^2 = 1-x^2$$

$$y = \sqrt{1-x^2}$$

$$y: 0 \to \sqrt{1-x^2}$$

put $z=y=0$.

$$1-x^2=0$$

$$x^2=1$$

$$x=\sqrt{1}=1$$

$$x=1$$

$$x: 0 \to 1$$

$$\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left(\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{1-x^2-y^2-z^2}}\right) dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left(\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{dz}{\sqrt{(1-x^2-y^2)-z^2}}\right) dy\,dx$$

Let $A = 1-x^2-y^2$.

Then limits are $z : 0 \rightarrow \sqrt{A}$.

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}}\left(\int_{z=0}^{\sqrt{A}} \frac{dz}{\sqrt{(\sqrt{A})^2-z^2}}\right) dy\,dx$$

$$\boxed{\int \frac{dz}{\sqrt{a^2-z^2}} = \sin^{-1}\left(\frac{z}{a}\right)+c}$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \sin^{-1}\left[\frac{z}{\sqrt{A}}\right]_{z=0}^{\sqrt{A}} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \sin^{-1}\left[\frac{z}{\sqrt{1-x^2-y^2}}\right]_{z=0}^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \sin^{-1}\left(\frac{\cancel{\sqrt{1-x^2-y^2}}}{\cancel{\sqrt{1-x^2-y^2}}}\right) dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \sin^{-1}(1)\, dy\,dx.$$

$$= \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} \cancel{\sin^{-1}}(\cancel{\sin}\,\pi/2)\, dy\,dx.$$

$$= \int_{x=0}^{1} \int_{y=0}^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx$$

$$= \pi/_2 \int_{x=0}^{1} \left(\int_{y=0}^{\sqrt{1-x^2}} dy \right) dx.$$

$$= \pi/_2 \int_{x=0}^{1} \left[y \right]_{y=0}^{\sqrt{1-x^2}}$$

$$= \frac{\pi}{2} \int_{x=0}^{1} \sqrt{1-x^2}\, dx.$$

$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + c}$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \sin^{-1}\left(\frac{x}{1}\right) + \frac{x}{2}\sqrt{1-x^2} \right]_0^1$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \sin^{-1}(1) + \frac{1}{2}\sqrt{1-1} \right]$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \cancel{\sin^{-1}}(\cancel{\sin}\, \pi/_2) + 0 \right]$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \times \frac{\pi}{2} \right]$$

$$= \frac{\pi}{2} \left(\frac{\pi}{4} \right)$$

$$= \frac{\pi^2}{8}$$

$\therefore$ The triple Integral value is

$$\iiint \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \frac{\pi^2}{8}$$

11) Evaluate $\iiint (x+y+z)\, dx\, dy\, dz$ taken the region bounded by the surface $x^2+y^2=a^2$, $z: 0 \to h$.

Step 1: to find limits.

$$z: 0 \to h.$$

$$x^2+y^2=a^2$$

$$y^2=a^2-x^2$$

$$y=\pm\sqrt{a^2-x^2}$$

$$y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}$$

put $y=0$.

$$x^2+0=a^2$$

$$x^2=a^2$$

$$x=\pm\sqrt{a^2}$$

$$x=\pm a$$

$$x: -a \to a$$

$\therefore$ limits.

$$x: -a \to a$$

$$y: -\sqrt{a^2-x^2} \to \sqrt{a^2-x^2}.$$

$$z: 0 \to h.$$

Step 2: Evaluation.

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{z=0}^{h} (x+y+z)\, dz\, dy\, dx.$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(x\int_{z=0}^{h} dz + y\int_{z=0}^{h} dz + \int_{z=0}^{h} z\, dz \right) dy\, dx$$

$$= \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left[xz + yz + \frac{z^2}{2} \right]_0^h dy\, dx$$

$$= \int_{x=-a} \left(\int_{y=-\sqrt{a^2-x^2}} \left[xh + yh + \frac{h^2}{2} \right] dy \right) dx.$$

$$= \int_{x=-a}^{a} \left[xhy + \frac{y^2 h}{2} + \frac{h^2 y}{2} \right]_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{x=-a}^{a} \left[xh\left(\sqrt{a^2-x^2} + \sqrt{a^2-x^2}\right) + \frac{h}{2}\left[\left(\sqrt{a^2-x^2}\right)^2 - \left(-\sqrt{a^2-x^2}\right)^2\right] + \frac{h^2}{2}\left(\sqrt{a^2-x^2} + \sqrt{a^2-x^2}\right) \right] dx$$

$$= \int_{x=-a}^{a} \left[xh\left(2\sqrt{a^2-x^2}\right) + \frac{h}{2}\left(\cancel{a^2} - \cancel{x^2} - \cancel{a^2} + \cancel{x^2}\right) + \frac{h^2}{2}\left(2\sqrt{a^2-x^2}\right) \right] dx$$

$$= \int_{x=-a}^{a} \left(2xh\sqrt{a^2-x^2} + \frac{\cancel{2}h^2}{\cancel{2}} \sqrt{a^2-x^2} \right) dx$$

$$= 2\int_{x=-a}^{a} xh\sqrt{a^2-x^2}\, dx + \int_{x=-a}^{a} h^2\sqrt{a^2-x^2}\, dx.$$

$$= 2h \int_{x=-a}^{a} x\sqrt{a^2-x^2}\, dx + h^2 \int_{x=-a}^{a} \sqrt{a^2-x^2}\, dx.$$

Let

$f(x) = x\sqrt{a^2-x^2}$

$f(-x) = -x\sqrt{a^2-(-x)^2}$

$= -x\sqrt{a^2-x^2}$

$= -f(x)$

$f(-x) = -f(x)$

$f(x) = -f(-x) \Rightarrow f(x)$ is odd.

$\int_{-a}^{a} f(x)\,dx = 0$ if $f(x)$ is odd.

$$= 2h \int^{a} f(x)\,dx + h^2 \int^{a} \sqrt{a^2-x^2}\, dx$$

$$= 2h(0) + h^2 \int_{x=-a}^{a} \sqrt{a^2-x^2}\, dx$$

$$= h^2 \int_{x=-a}^{a} \sqrt{a^2-x^2}\, dx.$$

$$\boxed{\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c}$$

$$= h^2 \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) \right]_{-a}^{a}$$

$$= h^2 \left[\left(\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right) \right) - \left(\frac{-a}{2}\sqrt{a^2-(-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{-a}{a}\right) \right) \right]$$

$$= h^2 \left[\left(0 + \frac{a^2}{2}\sin^{-1}(1) \right) - \left(0 + \frac{a^2}{2}\sin^{-1}(-1) \right) \right]$$

$$= h^2 \left[\frac{a^2}{2}\sin^{-1}(1) - \frac{a^2}{2}\sin^{-1}(-1) \right]$$

$$= h^2 \left[\frac{a^2}{2}\cancel{\sin^{-1}}(\cancel{\sin}\, \pi/2) - \frac{a^2}{2}\cancel{\sin^{-1}}(\cancel{\sin}(-\pi/2)) \right]$$

$$= h^2 \left[\frac{a^2}{2} \times \frac{\pi}{2} - \frac{a^2}{2}\left(-\pi/2\right) \right]$$

$$= h^2 \left[\frac{a^2\pi}{4} + \frac{a^2\pi}{4} \right]$$

$$= h^2 \left(\frac{\cancel{2}^{1}\pi a^2}{\cancel{4}_2} \right)$$

$$= \frac{\pi a^2 h^2}{2}$$

$\therefore$ The triple Integral value is

$$\iiint (x+y+z)\, dx\, dy\, dz = \frac{\pi a^2 h^2}{2}$$

12) Evaluate $\iiint x^2yz\,dx\,dy\,dz$ taken over the tetrahedron bounded by the planes. $x=0,\ y=0,\ z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Step 1: To find limits.

Given

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$\frac{z}{c}=1-\frac{x}{a}-\frac{y}{b}$$

$$z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$z: 0 \to c\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

put $z=0$.

$$\frac{x}{a}+\frac{y}{b}=1$$

$$\frac{y}{b}=1-\frac{x}{a}$$

$$y=b\left(1-\frac{x}{a}\right)$$

$$y: 0 \to b\left(1-\frac{x}{a}\right)$$

put $z=y=0$

$$\frac{x}{a}=1$$

$$x=a$$

$$x: 0 \to a.$$

The limits of Integration are

$$0 \le z \le b\left(1-\frac{x}{a}-\frac{y}{b}\right)$$

$$0 \le y \le b\left(1-\frac{x}{a}\right)$$

$$0 \le x \le a$$

Step 2: Evaluation.

$$\iiint x^2yz\,dx\,dy\,dz = \int_{x=0}^{a}\int_{y=0}^{b(1-x/a)}\int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} x^2yz\,dz\,dy\,dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} \left(\int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} x^2 y\, z\, dz \right) dy\, dx$$

$$= \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} x^2 y \left[\frac{z^2}{2} \right]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \int_{y=0}^{b(1-\frac{x}{a})} x^2 y \left[z^2 \right]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\, dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left(\int_{y=0}^{b(1-\frac{x}{a})} x^2 y\, c^2 \left(1-\frac{x}{a}-\frac{y}{b}\right)^2 dy \right) dx$$

Let $u = 1-\frac{x}{a}$, Then the upper limit for y is

$$= \frac{1}{2} \int_{x=0}^{a} \left(\int_{y=0}^{bu} x^2 y\, c^2 \left(u-\frac{y}{b}\right)^2 dy \right) dx.$$

Let

$$v = u - \frac{y}{b} \Rightarrow \frac{y}{b} = u - v$$

$$y = b(u-v)$$

$$\frac{dv}{dy} = 0 - \frac{1}{b}$$

$$dv = -\frac{1}{b} dy$$

$$dy = -b\, dv$$

$$y=0 \Rightarrow v = u - \frac{0}{b}$$

$$v = u$$

$$y = bu \Rightarrow v = u - \frac{bu}{b} = \frac{bu-bu}{b}$$

$$v = 0$$

$$= \frac{1}{2} \int_{x=0}^{a} \left(x^2 c^2 \int_{v=u}^{0} b(u-v)\, v^2 (-b)\, dv \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left(x^2 c^2 \int_{v=u} -b^2 (uv^2 - v^3)\, dv \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{a} \left(x^2 c^2 \int_{v=0}^{u} b^2 (uv^2 - v^3)\, du \right) dx$$

$$= \frac{1}{2} \int_{x=0}^{a} x^2 c^2 b^2 \left[\frac{uv^3}{3} - \frac{v^4}{4} \right]_0^u dx$$

$$= \frac{1}{2} \int_{x=0}^{a} x^2 c^2 b^2 \left[\frac{4uv^3 - 3v^4}{12} \right]_0^u dx$$

$$= \frac{1}{2} \times \frac{1}{12} \int_{x=0}^{a} x^2 c^2 b^2 \left[4uu^3 - 3u^4 \right] dx$$

$$= \frac{1}{24} \int_{x=0}^{a} x^2 c^2 b^2 \left[4u^4 - 3u^4 \right] dx$$

$$= \frac{1}{24} \int_{x=0}^{a} x^2 c^2 b^2 u^4 \, dx$$

$$\boxed{u = 1 - \frac{x}{a}}$$

$$= \frac{1}{24} \int_{x=0}^{a} x^2 c^2 b^2 \left(1 - \frac{x}{a}\right)^4 dx$$

Let.

$w = 1 - \frac{x}{a} \Rightarrow \frac{x}{a} = 1 - w$

$x = a(1-w)$

$\frac{dw}{dx} = 0 - \frac{1}{a}$

$dw = -\frac{1}{a} dx$

$dx = -a\, dw$

$x = 0 \Rightarrow w = 1 - \frac{0}{a} = 1$

$w = 1$

$x = a \Rightarrow w = 1 - \frac{a}{a} = 1 - 1 = 0$

$w = 0$

$$= \frac{1}{24}c^2b^2 \int_{w=1}^{0} a^2(1-w)^2 w^4 (-a)\, dw$$

$$= \frac{1}{24}c^2b^2 \int_{w=1}^{0} -a^3(1^2-2w+w^2) w^4\, dw$$

$$= \frac{1}{24}c^2b^2 \int_{w=0}^{1} a^3(w^4-2w^5+w^6)\, dw$$

$$= \frac{1}{24}c^2b^2a^3 \int_{w=0}^{1} (w^4-2w^5+w^6)\, dw$$

$$= \frac{1}{24}a^3b^2c^2 \left[\frac{w^5}{5} - \frac{2w^6}{6} + \frac{w^7}{7}\right]_0^1$$

$$= \frac{1}{24}a^3b^2c^2 \left(\frac{1}{5} - \frac{1}{3} + \frac{1}{7}\right)$$

$$= \frac{1}{24}a^3b^2c^2 \left(\frac{3-5}{15} + \frac{1}{7}\right)$$

$$= \frac{1}{24}a^3b^2c^2 \left(-\frac{2}{15} + \frac{1}{7}\right)$$

$$= \frac{1}{24}a^3b^2c^2 \left(\frac{-14+15}{105}\right)$$

$$= \frac{1}{24}a^3b^2c^2 \left(\frac{1}{105}\right)$$

$$= \frac{1}{2520}a^3b^2c^2$$

$$= \frac{a^3b^2c^2}{2520}$$

$\therefore$ The value of the triple Integral is

$$\iiint x^2yz\, dx\,dy\,dz = \frac{a^3b^2c^2}{2520}$$

Beta and Gamma functions

Unit - IV

Beta function

Beta function is defined as

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \text{ for } m>0 \text{ and } n>0$$

Gamma function

Gamma function is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx, \text{ for } n>0$$

properties of Beta function

1) prove That $\beta(m,n) = \beta(n,m)$

soln:-

we know That

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$$

put

$$1-x = y$$

$$x = 1-y$$

$$\frac{dx}{dy} = 0-1$$

$$dx = -dy$$

when $x=0 \Rightarrow y=1$

when $x=1 \Rightarrow y=0$

$$\beta(m,n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= \int_1^0 -(1-y)^{m-1} y^{n-1}\,dy$$

$$= \int_0^1 y^{n-1}(1-y)^{m-1}\,dy$$

$$= \beta(n,m)$$

$\therefore \beta(m,n) = \beta(n,m)$, Hence proved.

2) $\beta(m,n)$ can be expressed as a definite Integral with $0, \infty$ as limits

(or)

prove that $\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy$.

<u>Solution:-</u>

we know that

$$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx.$$

$$x = \frac{y}{1+y}$$

$$-x = -\frac{y}{1+y}$$

$$1-x = 1-\frac{y}{1+y}$$

$$= \frac{1+y-y}{1+y}$$

$$1-x = \frac{1}{1+y}$$

$$x = \frac{y}{1+y} = \frac{u}{v}$$

$$\boxed{\frac{u}{v} = \frac{vu' - uv'}{v^2}}$$

$$\frac{dx}{dy} = \frac{(1+y)(1) - y(0+1)}{(1+y)^2}$$

$$dx = \frac{(1+y)dy - ydy}{(1+y)^2}$$

$$= \frac{dy + \cancel{ydy} - \cancel{ydy}}{(1+y)^2}$$

$$dx = \frac{dy}{(1+y)^2}$$

when $x=0$

$$\frac{y}{1+y}=0$$

$$y=0$$

when $x=1$

$$\frac{y}{1+y}=1$$

$$y=1+y$$

$\div y$

$$1=\frac{1}{y}+1$$

$$\frac{1}{y}=1-1=0$$

$$y=\frac{1}{0}$$

$$y=\infty$$

$$\beta(m,n)=\int_0^\infty\left(\frac{y}{1+y}\right)^{m-1}\left(\frac{1}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$=\int_0^\infty\frac{y^{m-1}}{(1+y)^{m-1}}\frac{1}{(1+y)^{n-1}}\frac{1}{(1+y)^2}dy$$

$$=\int_0^\infty\frac{y^{m-1}}{(1+y)^{m+n-\cancel{2}+\cancel{2}}}dy$$

$$=\int_0^\infty\frac{y^{m-1}}{(1+y)^{m+n}}dy.$$

$$\therefore\ \beta(m,n)=\int_0^\infty\frac{y^{m-1}}{(1+y)^{m+n}}dy,$$ Hence proved.

3) prove That $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}x\cos^{2n-1}x\,d$

Soln:

we know That

$$\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

put $x=\sin^2\theta$

$$1-x=1-\sin^2\theta$$

$$=\cos^2\theta$$

$$x = \sin^2\theta$$

$$\therefore \quad dx = 2\sin\theta\cos\theta\, d\theta.$$

When $x=0$, $\Rightarrow \sin^2\theta = 0$

$$\sin\theta = 0$$

$$\sin\theta = \sin 0^\circ$$

$$\theta = 0$$

When $x = 1$, $\Rightarrow \sin^2\theta = 1$

$$\sin\theta = 1$$

$$\sin\theta = \sin \pi/2$$

$$\theta = \pi/2$$

$$B(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1} (\cos^2\theta)^{n-1}\, 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta \, \sin\theta \, \cos^{2n-2}\theta \, \cos\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2+1}\theta \, \cos^{2n-2+1}\theta \, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta$$

$$\therefore B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \, \cos^{2n-1}\theta \, d\theta,$$

Hence proved.

Let

$$I_{m,n} = \int_0^{\pi/2} \sin^m\theta \, \cos^n\theta \, d\theta$$

$2m-1 = m$
$m = \frac{m+1}{2}$

$$\therefore I_{m,n} = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$$

$2n-1 = n$
$n = \frac{n+1}{2}$

properties of Gamma function

1) prove That $\Gamma(n+1) = n\Gamma(n)$, If $n>0$

or

Recurrence formula for Gamma function

soln:

we know That

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx.$$

$$\Gamma(n+1) = \int_0^\infty x^{n-1+1} e^{-x} dx$$

$$= \int_0^\infty x^{n} e^{-x} dx.$$

let
$u = x^n$; $\int dv = \int e^{-x} dx$

$du = nx^{n-1} dx$; $v = -e^{-x}$

$$\boxed{\int u dv = uv - \int v du}$$

$$\Gamma(n+1) = \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty (-e^{-x}) n x^{n-1} dx$$

$$= 0 - \int_0^\infty (-e^{-x}) n x^{n-1} dx$$

$$= n \int_0^\infty x^{n-1} e^{-x} dx$$

$$= n\Gamma(n)$$

$\therefore \Gamma(n+1) = n\Gamma(n)$, Hence proved.

2) prove That $\Gamma(n+1) = n!$

solution

$$\Gamma(n+1) = n!$$

we know That.

$$\Gamma(n+1) = n\Gamma(n)$$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$= n(n-1)(n-2)(n-3)\Gamma(n-3)$$

$$= n(n-1) \ldots 3.2.1\,\Gamma(1) \quad \text{—①}$$

Now

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\,dx$$

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\,dx$$

$$= \int_0^\infty e^{-x}\,dx$$

$$= \left[-e^{-x}\right]_0^\infty$$

$$= -\left[e^{-x}\right]_0^\infty$$

$$= -\left[e^{-\infty} - e^{0}\right]$$

$$= -\left[\frac{1}{e^\infty} - 1\right]$$

$$= -(0-1)$$

$$\Gamma(1) = 1$$

From equation ①, we get

$$\Gamma(n+1) = n(n-1) \ldots 3.2.1$$

$$= n!$$

$\therefore \Gamma(n+1) = n!$, Hence proved.

3) prove That $\Gamma(n+a) = (a+n-1)(a+n-2)\cdots a\Gamma(a)$

Solution

$$\Gamma(n+a) = (n+a-1)\Gamma(n+a-1)$$
$$=(n+a-1)(n+a-2)\Gamma(n+a-2)$$
$$=(a+n-1)(a+n-2)(a+n-3)\Gamma(a+n-3)$$
$$=(a+n-1)(a+n-2)\cdots\cdots(a+\cancel{n}-\cancel{n})\Gamma(a+\cancel{n}-\cancel{n})$$
$$=(a+n-1)(a+n-2)\cdots a\Gamma(a)$$

$$\therefore \Gamma(n+a) = (a+n-1)(a+n-2)\cdots\cdots a\Gamma(a).$$

Hence proved.

Relation Between Beta and Gamma.

~~Derive between~~

Derive The Relation Between Beta and Gamma functions.

prove That $\beta(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

$$\Gamma(m) = \int_0^\infty x^{m-1} e^{-x} dx.$$

put

$x = t^2$

$x^{m-1} = t^{2m-2}$

$e^{-x} = e^{-t^2}$

$x = t^2$

$\frac{dx}{dt} = 2t$

$dx = 2t\,dt$

when $x=0$, $t^2=0$

$\Rightarrow t = 0$

when $x=\infty$, $t^2=\infty$

$\Rightarrow t = \infty$

$$\Gamma(m) = \int_0^\infty t^{2m-2} e^{-t^2} 2t\,dt$$

$$\Gamma(m) = 2\int_0^\infty t^{2m-2}\, t\, e^{-t^2}\, dt$$

$$= 2\int_0^\infty t^{2m-1} e^{-t^2}\, dt$$

$$\Gamma(m) = 2\int_0^\infty x^{2m-1} e^{-x^2}\, dx$$

$$\Gamma(n) = 2\int_0^\infty y^{2n-1} e^{-y^2}\, dy.$$

$$\Gamma(m)\Gamma(n) = \left(2\int_0^\infty x^{2m-1} e^{-x^2} dx\right)\left(2\int_0^\infty y^{2n-1} e^{-y^2} dy\right)$$

$$= 4\int_0^\infty\int_0^\infty x^{2m-1} y^{2n-1} e^{-x^2} e^{-y^2}\, dx\, dy$$

$$\Gamma(m)\Gamma(n) = 4\int_0^\infty\int_0^\infty x^{2m-1} y^{2n-1} e^{-(x^2+y^2)}\, dx\, dy$$

Changing into polar co-ordinates

$x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = r\,dr\,d\theta$.

Limits.

$r : 0 \to \infty$

$\theta : 0 \to \pi/2$

$$\Gamma(m)\Gamma(n) = 4\int_0^{\pi/2}\int_0^\infty r^{2m-1}\cos^{2m-1}\theta\; r^{2n-1}\sin^{2n-1}\theta\; e^{-(r^2\cos^2\theta + r^2\sin^2\theta)}\, r\,dr\,d\theta$$

$$= 4\int_0^{\pi/2}\int_0^\infty r^{2m+2n-2+1}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, e^{-r^2}\, dr\,d\theta$$

$$= 4\left(\int_0^\infty r^{2(m+n)-1} e^{-r^2} dr\right)\left(\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\, d\theta\right)$$

$$\Gamma(m)\Gamma(n) = 4\left(\int_0^{\infty} r^{2(m+n)} r^{-1} e^{-r^2} dr\right)\left(\int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta\right)$$

——— ①

By III property of Beta function, we have.

$$\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \cos^{2m-1}\theta \sin^{2n-1}\theta \, d\theta \quad \text{——— ②}$$

$$\Rightarrow \int_0^{\infty} r^{2(m+n)} r^{-1} e^{-r^2} dr$$

Let.

$$r^2 = t$$

$$2r\frac{dr}{dt} = 1$$

$$2r\,dr = dt$$

$$dr = \frac{dt}{2r}.$$

when $r=0$ $t=0$

when $r=\infty$ $t=\infty$

$$\int_0^{\infty} r^{2(m+n)} r^{-1} e^{-r^2} dr = \int_0^{\infty} t^{(m+n)} e^{-t} r^{-1} \frac{dt}{2r}$$

$$= \frac{1}{2}\int_0^{\infty} t^{(m+n)} e^{-t} \frac{1}{r}\frac{dt}{r}$$

$$= \frac{1}{2}\int_0^{\infty} t^{(m+n)} e^{-t} \frac{1}{r^2} dt$$

$$= \frac{1}{2}\int_0^{\infty} t^{(m+n)} e^{-t} \frac{1}{t} dt$$

$$= \frac{1}{2}\int_0^{\infty} t^{(m+n)} e^{-t} t^{-1} dt$$

$$= \frac{1}{2}\int_0^{\infty} t^{(m+n)-1} e^{-t} dt$$

$$\int_0^{\infty} r^{2(m+n)} r^{-1} e^{-r^2} dr = \frac{1}{2}\Gamma(m+n) \quad \text{——— ③}$$

From eqn ② and ③ in ①

$$\Gamma(m)\Gamma(n) = 4 \cdot \frac{1}{2}\Gamma(m+n)\frac{1}{2}\beta(m,n)$$

$$= 4 \cdot \frac{1}{4}\Gamma(m+n)\beta(m,n)$$

$$\Gamma(m)\Gamma(n) = \Gamma(m+n)\beta(m,n)$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$, Hence proved.

corollary (i) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

put $m = n = \frac{1}{2}$

$$\beta(\tfrac{1}{2},\tfrac{1}{2}) = \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(1/2+1/2)}$$

$$= \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(\cancel{2}/\cancel{2})}$$

$$\beta(1/2,1/2) = \frac{[\Gamma(1/2)]^2}{\Gamma(1)} \qquad [\because \Gamma(1) = 1]$$

$$\beta(1/2,1/2) = [\Gamma(1/2)]^2 \text{ ——— ①}$$

$$\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

put $m=n=\frac{1}{2}$

$$\frac{1}{2}\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\int_0^{\pi/2}\sin^{\cancel{2}(\frac{1}{\cancel{2}})-1}\theta\,\cos^{\cancel{2}(\frac{1}{\cancel{2}})-1}\theta\,d\theta$$

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right)=2\int_0^{\pi/2}\sin^{1-1}\theta\,\cos^{1-0}\theta\,d\theta$$

$$=2\int_0^{\pi/2}\sin^{0}\theta\,\cos^{0}\theta\,d\theta.$$

$$=2\int_0^{\pi/2}d\theta$$

$$=2\left[\theta\right]_0^{\pi/2}$$

$$=\cancel{2}\left(\frac{\pi}{\cancel{2}}\right)$$

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right)=\pi \qquad \text{---} \quad (2)$$

subs equation (2) in (1)

$$\pi=\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2$$

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2=\pi$$

$$\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

$\therefore\ \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$, Hence proved.

---

corollary (ii)

In $\beta(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

put $m=n$ and $n=1-n$.

$$\beta(m,1-n)=\frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(\cancel{n}+1-\cancel{n})}$$

$$\beta(n, 1-n) = \frac{\Gamma(n)\Gamma(1-n)}{\Gamma(1)}$$

$$\Gamma(1) = 1$$

$$\beta(n, 1-n) = \Gamma(n)\Gamma(1-n)$$

put $n = 1/2$

$$\beta(1/2, 1-1/2) = \Gamma(1/2)\,\Gamma(1-1/2)$$

$$\beta\left(1/2, \frac{2-1}{2}\right) = \Gamma(1/2) \cdot \Gamma\frac{2-1}{2}$$

$$\beta(1/2, 1/2) = \Gamma(1/2)\Gamma(1/2)$$

$$\beta(1/2, 1/2) = [\Gamma(1/2)]^2 \quad \text{———} \quad (1)$$

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

put $m = n$ and $n = 1-n$

$$\beta(n, 1-n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{\cancel{n}+1-\cancel{n}}}\,dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^1}\,dx$$

$$= \int_0^\infty \frac{x^{n-1}}{1+x}\,dx$$

$$\beta(n, 1-n) = \frac{\pi}{\sin n\pi}$$ [we shall assume this result]

put $n = 1/2$

$$\beta(1/2, 1-1/2) = \frac{\pi}{\sin(1/2 \times \pi)}$$

$$\beta(1/2, 1/2) = \frac{\pi}{\sin \pi/2}$$

$\sin \frac{\pi}{2} = 1$

$\beta(1/2, 1/2) = \frac{\pi}{1} = \pi$ ——— ②

subs equation ② in ①

$\pi = [\Gamma(1/2)]^2$

$[\Gamma(1/2)]^2 = \pi$

$\Gamma(1/2) = \sqrt{\pi}$

$\therefore \Gamma(1/2) = \sqrt{\pi}$, Hence proved.

Example. 1

Evaluate $\int_0^1 x^m \cdot (\log \frac{1}{x})^n dx$.

$\int_0^1 x^m (\log \frac{1}{x})^n dx$. $\quad [\because \log_a 1 = 0]$

put $\log_e(\frac{1}{x}) = t$

$\log_e e^{(1/x)} = e^t$

$\frac{1}{x} = e^t$

$\frac{1}{e^t} = x$

$x = e^{-t}$

$\frac{dx}{dt} = -e^{-t}$

$dx = -e^{-t} dt$

$x = 0 \Rightarrow t = \log e(\frac{1}{0})$

$t = \infty$

$x = 1 \Rightarrow t = \log e(\frac{1}{1})$

$= \log e^1$

$t = 0$

$\cancel{x = 0 \Rightarrow t = \log e(\frac{1}{0})}$

$\cancel{t = 0}$

$\cancel{x = 0 \Rightarrow t = \infty}$

$\int_0^1 x^m (\log \frac{1}{x})^n dx = \int_\infty^0 (e^{-t})^m t^n (-e^{-t} dt)$

$= \int_\infty^0 - e^{-tm} \cdot e^{-t} t^n dt$

$= \int_0^\infty e^{-tm-t} t^n dt$

$$\int_0^1 x^m\left(\log\frac{1}{x}\right)^n dx = \int_0^\infty e^{-(m+1)t}\, t^n\, dt$$

put $(m+1)t = y$

$$t = \frac{y}{m+1} \Rightarrow t^n = \frac{y^n}{(m+1)^n}$$

$$\frac{dt}{dy} = \frac{1}{m+1}$$

$$dt = \frac{1}{m+1}\, dy$$

$$t = 0 \Rightarrow y = 0(m+1)$$

$$y = 0$$

$$t = \infty \Rightarrow y = \infty(m+1)$$

$$y = \infty$$

Then the Integral on this substitution becomes.

$$\int_0^1 x^m\left(\log\frac{1}{x}\right)^n dx = \int_0^\infty e^{-y}\, \frac{y^n}{(m+1)^n} \cdot \frac{1}{m+1}\, dy$$

$$= \int_0^\infty \frac{e^{-y}\cdot y^n}{(m+1)^n} \cdot \left(\frac{1}{m+1}\right) dy$$

$$= \int_0^\infty \frac{y^n e^{-y}}{(m+1)^{n+1}}\, dy$$

$$= \frac{1}{(m+1)^{n+1}} \int_0^\infty y^n e^{-y}\, dy$$

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

$$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$$

$\therefore$ Evaluation $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$

Example. 2

Evaluate $\int_0^\infty e^{-x^2} dx$.

$$\int_0^\infty e^{-x^2} dx$$

put $x^2 = t \Rightarrow x = \sqrt{t}$

$$2x \frac{dx}{dt} = 1$$

$$2x\,dx = dt$$

$$dx = \frac{dt}{2x}$$

$$dx = \frac{1}{2x} dt$$

$$dx = \frac{1}{2\sqrt{t}} dt.$$

$$x = 0 \Rightarrow t = 0$$

$$x = \infty \Rightarrow t = \infty$$

$$\int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-t} \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} \frac{1}{(t)^{1/2}} dt$$

$$= \frac{1}{2} \int_0^\infty e^{-t} (t)^{-1/2} dt$$

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\int_0^\infty (t)^{-1/2} e^{-t} dt$$

$$= \frac{1}{2}\int_0^\infty (t)^{1/2 - 1} e^{-t} dt$$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

$$\int_0^\infty e^{-x^2} dx = \frac{1}{2}\Gamma(1/2) \qquad \Gamma(1/2) = \sqrt{\pi}$$

$$= \frac{1}{2}\sqrt{\pi}$$

$$= \frac{\sqrt{\pi}}{2}$$

$\therefore$ Evaluation $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

Example. 3.

Evaluate (i) $\int_0^1 x^7 (1-x)^8 dx$.

$$\int_0^1 x^7 (1-x)^8 dx.$$

$$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx.$$

$$\int_0^1 x^7 (1-x)^8 dx = \int_0^1 x^{8-1} (1-x)^{9-1} dx$$

$$= \beta(8, 9)$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^1 x^7(1-x)^8\,dx = \frac{\Gamma(8)\Gamma(9)}{\Gamma(8+9)}$$

$$= \frac{\Gamma(8)\Gamma(9)}{\Gamma(17)}$$

$$\Gamma(n+1) = n!$$

$$= \frac{\Gamma(7+1)\,\Gamma(8+1)}{\Gamma(16+1)}$$

$$= \frac{7!\,8!}{16!}$$

$$= \frac{7!\,\cancel{8!}}{16\times15\times14\times13\times12\times11\times10\times9\times\cancel{8!}}$$

$$= \frac{\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times\cancel{2}\times1}{\cancel{16}\times\cancel{15}\times\cancel{14}\times13\times\cancel{12}\times11\times10\times9}$$

$$= \frac{1}{4\times13\times2\times11\times10\times9}$$

$$= \frac{1}{13\times8\times11\times90}$$

$$= \frac{1}{13\times8\times990}$$

$$= \frac{1}{102960}$$

$\therefore$ Evaluation

$$\int_0^1 x^7(1-x)^8\,dx = \frac{1}{102960}$$

(ii) $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta$$

$$\frac{1}{2} \beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta$$

$$\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{2} \beta\left(\frac{7+1}{2}, \frac{5+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{8}{2}, \frac{6}{2}\right)$$

$$= \frac{1}{2} \beta(4, 3)$$

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{2} \frac{\Gamma(4)\,\Gamma(3)}{\Gamma(4+3)}$$

$$= \frac{1}{2} \frac{\Gamma(4)\,\Gamma(3)}{\Gamma(7)}$$

$$= \frac{1}{2} \frac{\Gamma(3+1)\,\Gamma(2+1)}{\Gamma(6+1)}$$

$$\Gamma(n+1) = n!$$

$$= \frac{1}{2} \frac{3!\,2!}{6!}$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{3!}\,\cancel{2}\times 1}{6\times 5\times 4\times \cancel{3!}}$$

$$= \frac{1}{120}$$

$\therefore$ Evaluation $\int_0^{\pi/2} \sin^7\theta \cos^5\theta \, d\theta = \frac{1}{120}$

(iii) $\int_0^{\pi/2} \sin^{10}\theta \, d\theta$.

$$\int_0^{\pi/2} \sin^{10}\theta \cos^0\theta \, d\theta = \int_0^{\pi/2} \sin^{10}\theta \, d\theta.$$

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \int_0^{\pi/2} \sin^{10}\theta \cos^0\theta \, d\theta.$$

$$\frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta.$$

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \frac{1}{2}\beta\left(\frac{10+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{11}{2}, \frac{1}{2}\right)$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{2} \frac{\Gamma(11/2)\Gamma(1/2)}{\Gamma\left(\frac{11}{2}+\frac{1}{2}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(11/2)\Gamma(1/2)}{\Gamma\left(\frac{\cancel{12}\,6}{\cancel{2}\,1}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(11/2)\Gamma(1/2)}{\Gamma(6)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{9}{2}+1\right)\Gamma(1/2)}{\Gamma(6)}$$

$$\Gamma(n+1) = n\Gamma n = n(n-1)\cdots\cdots 3.2.1.\Gamma(1)$$

$$\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \frac{1}{2} \, \frac{\frac{9}{2}\Gamma\left(\frac{9}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(6)}$$

$$= \frac{1}{2} \, \frac{\frac{9}{2}\left(\frac{9}{2}-1\right)\left(\frac{7}{2}-1\right)\cdots\cdots\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(6)}$$

$$= \frac{1}{2} \, \frac{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(6)}$$

$$= \frac{1}{2} \, \frac{9.7.5.3.\left(\sqrt{\pi}\right)^2}{\Gamma(5+1)\, 2^5}$$

$$= \frac{9.7.\cancel{5}.\cancel{3}\;\pi}{\cancel{5}.4.\cancel{3}.2.1.\;2^6}$$

$$= \frac{63\pi}{8.2^6}$$

$$= \frac{63\pi}{8\times 8\times 8}$$

$$= \frac{63\pi}{512}$$

$\therefore$ Evaluation $\displaystyle\int_0^{\pi/2} \sin^{10}\theta \, d\theta = \frac{63\pi}{512}$

(iv) $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta.$

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} (\tan\theta)^{1/2}\, d\theta$$

$$= \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{1/2} d\theta$$

$$= \int_0^{\pi/2} \frac{(\sin\theta)^{1/2}}{(\cos\theta)^{1/2}}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \cdot d\theta$$

$$\frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right) = \int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta.$$

$$= \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{\frac{1+2}{2}}{2}, \frac{\frac{-1+2}{2}}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1/4)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$= \frac{1}{2} \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma\left(\frac{4}{4}\right)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma(1/4)}{\Gamma(1)}$$

$$\Gamma(1) = 1$$

$$= \frac{1}{2}\Gamma(1/4)\Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)$$

$$\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma(\cancel{2}\times\frac{1}{\cancel{4}_2})}{2^{(\cancel{2}\times\frac{1}{\cancel{4}_2})-1}}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2})}{2^{\frac{1}{2}-1}}$$

$$= \frac{1}{2} \frac{\sqrt{\pi} \times \sqrt{\pi}}{2^{-1/2}}$$

$$= \frac{2^{1/2}(\sqrt{\pi})^2}{2}$$

$$= \frac{\sqrt{2}\,\pi}{2}$$

$$= \frac{\sqrt{2}\,\pi}{\sqrt{2}\times\sqrt{2}}$$

$$= \frac{\pi}{\sqrt{2}}$$

$\therefore$ Evaluation of $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$

Relation between Beta and Gamma functions.

corollary (iii)

$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n)$ is often expressed is the following form.

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n)$$

putting $2m = p$, $2n = q$

$m = \frac{p}{2}$, $n = \frac{q}{2}$

$$\int_0^{\pi/2} \sin^{(\cancel{2}\times\frac{p}{\cancel{2}})-1}\theta \cos^{(\cancel{2}\times\frac{q}{\cancel{2}})-1}\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p}{2}, \frac{q}{2}\right)$$

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta \, d\theta = \frac{1}{2} \beta\left(\frac{p}{2}, \frac{q}{2}\right)$$

$$\beta(m;n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{p}{2}+\frac{q}{2}\right)}$$

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{q-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}{\Gamma\left(\frac{p+q}{2}\right)} \quad \text{——— (1)}$$

we put $q=1$ in equation (1), we get.

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{1-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}$$

$$\int_0^{\pi/2} \sin^{p-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)} \quad \text{——— (2)}$$

we put $p=q$ in (1), we get.

$$\int_0^{\pi/2} \sin^{p-1}\theta \cos^{p-1}\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p}{2}\right)}{\Gamma\left(\frac{p+p}{2}\right)}$$

$$= \frac{1}{2} \frac{\left[\Gamma\left(\frac{p}{2}\right)\right]^2}{\Gamma\left(\frac{\cancel{2}p}{\cancel{2}}\right)}$$

$$\int_0^{\pi/2} (\sin\theta . \cos\theta)^{p-1} d\theta = \frac{1}{2} \frac{\left[\Gamma\left(\frac{p}{2}\right)\right]^2}{\Gamma(p)}$$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

$$\int_0^{\pi/2} \left(\frac{1}{2}\left[\sin(\theta+\theta) - \sin(\theta-\theta)\right]\right)^{p-1} d\theta = \frac{1}{2} \frac{\left[\Gamma\left(\frac{p}{2}\right)\right]^2}{\Gamma(p)}$$

$$\int_0^{\pi/2} \frac{1}{2^{p-1}}\left[\sin(2\theta) - \sin 0\right]^{p-1} d\theta = \frac{1}{2} \frac{\left[\Gamma\left(\frac{p}{2}\right)\right]^2}{\Gamma(p)}$$

$$\frac{1}{2^{P-1}}\int_0^{\pi/2}[\sin(2\theta)-0]^{P-1}\,d\theta=\frac{1}{2}\frac{[\Gamma(\frac{P}{2})]^2}{\Gamma(P)}$$

$$\frac{1}{2^{P-1}}\int_0^{\pi/2}\sin^{P-1}2\theta\,d\theta=\frac{[\Gamma(\frac{P}{2})]^2}{2\Gamma(P)}$$

putting $2\theta=\varphi$, we get.

$$\theta=\frac{\varphi}{2}$$

$$\frac{d\theta}{d\varphi}=\frac{1}{2}$$

$$d\theta=\frac{1}{2}d\varphi$$

$$2d\theta=d\varphi$$

$$\theta=0 \Rightarrow \varphi=2\times 0=0$$

$$\theta=\pi/2 \Rightarrow \varphi=\cancel{2}\times\frac{\pi}{\cancel{2}}=\pi$$

$$\frac{1}{2^{P-1}}\int_0^{\pi}\sin^{P-1}\varphi\,\frac{1}{2}\,d\varphi=\frac{[\Gamma(\frac{P}{2})]^2}{2\Gamma(P)}$$

$$\frac{1}{2^{P-1}}\cdot\cancel{\frac{1}{2}}\int_0^{\pi}\sin^{P-1}\varphi\,d\varphi=\cancel{\frac{1}{2}}\frac{[\Gamma(\frac{P}{2})]^2}{\Gamma(P)}$$

$$\frac{1}{2^{P-1}}\int_0^{2\times\frac{\pi}{2}}\sin^{P-1}\varphi\,d\varphi=\frac{[\Gamma(\frac{P}{2})]^2}{\Gamma(P)}$$

$$\int_0^{2a}f(x)dx=2\int_0^{a}f(x)dx,\ \text{If}\ f(2a-x)=f(x)$$

$$\frac{1}{2^{P-1}}\,2\int_0^{\pi/2}\sin^{P-1}\varphi\,d\varphi=\frac{[\Gamma(\frac{P}{2})]^2}{\Gamma(P)}$$

using equation ②, we get

$$② \Rightarrow 2\int_0^{\pi/2}\sin^{P-1}\varphi\,d\varphi=\frac{\Gamma(\frac{P}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{P+1}{2})}$$

$$\frac{\cancel{\Gamma(\frac{P}{2})}\Gamma(\frac{1}{2})}{2^{P-1}\Gamma(\frac{P+1}{2})}=\frac{[\Gamma(\frac{P}{2})]^{\cancel{2}\,1}}{\Gamma(P)}$$

$$\frac{\sqrt{\pi}}{2^{p-1}}\Gamma(p) = \Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p+1}{2}\right)$$

$$\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{p+1}{2}\right) = \frac{\sqrt{\pi}}{2^{p-1}}\Gamma(p) \quad \text{———} \ (3)$$

putting $p=2n$, we have.

$$\Gamma\left(\frac{\cancel{2}n}{\cancel{2}}\right)\Gamma\left(\frac{2n+1}{2}\right) = \frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n)$$

$$\Gamma(n)\Gamma\left(\frac{\cancel{2}n}{\cancel{2}}+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

$$\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

put $n=\frac{1}{4}$,

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\cancel{2}\times\frac{1}{\cancel{4}2}\right)}{2^{\left(\cancel{2}\times\frac{1}{\cancel{4}2}\right)-1}}$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}+\frac{2}{4}\right) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}{2^{\frac{1}{2}-1}}$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1+2}{4}\right) = \frac{\sqrt{\pi}\times\sqrt{\pi}}{2^{\frac{1-2}{2}}}$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi}{2^{-1/2}}$$

$$= 2^{1/2}\pi$$

$$\boxed{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi}$$

Example. 4.

Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of Gamma function and evaluate. The Integral

$$\int_0^1 x^5(1-x^3)^{10}\,dx.$$

$$\int_0^1 x^m(1-x^n)^p\,dx.$$

put $x^n = y$

$x = y^{1/n}$

$\frac{dx}{dy} = \frac{1}{n} y^{\frac{1}{n}-1}$

$x=0 \Rightarrow y=0$

$x=1 \Rightarrow y=1$

$dx = \frac{1}{n} y^{1/n-1}\,dy$

$$\int_0^1 x^m(1-x^n)^p\,dx = \int_0^1 (y^{1/n})^m (1-y)^p \left(\frac{1}{n} y^{1/n-1}\right) dy$$

$$= \frac{1}{n}\int_0^1 y^{m/n}(1-y)^p\, y^{1/n-1}\,dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m}{n}}\, y^{1/n-1}(1-y)^p\,dy$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m}{n}+\frac{1}{n}-1}(1-y)^p\,dy$$

$$= \frac{1}{n}\int_0^1 y^{\left(\frac{m+1}{n}\right)-1}(1-y)^{(p+1)-1}\,dy.$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= \frac{1}{n}\,\beta\left(\frac{m+1}{n},\ p+1\right)$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\,\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

put $m=5$, $n=3$, $p=10$.

$$\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{3}\,\frac{\Gamma\left(\frac{5+1}{3}\right)\Gamma(10+1)}{\Gamma\left(\frac{5+1}{3}+10+1\right)}$$

$$= \frac{1}{3}\,\frac{\Gamma\left(\frac{\cancel{6}^{\,2}}{\cancel{3}}\right)\Gamma(11)}{\Gamma\left(\frac{\cancel{6}^{\,2}}{\cancel{3}}+11\right)}$$

$$= \frac{1}{3}\,\frac{\Gamma(2)\,\Gamma(11)}{\Gamma(2+11)}$$

$$= \frac{1}{3}\,\frac{\Gamma(2)\,\Gamma(11)}{\Gamma(13)}$$

$$= \frac{1}{3}\,\frac{\Gamma(1+1)\,\Gamma(10+1)}{\Gamma(12+1)}$$

$\Gamma(n+1) = n! = n(n-1)(n-2)\ldots\ldots 3.2.1.\Gamma(1)$

$$= \frac{1}{3}\,\frac{1!\;10!}{12!}$$

$$= \frac{1}{3}\,\frac{1\;\cancel{10!}}{12\times11\times\cancel{10!}}$$

$$= \frac{1}{3}\,\frac{1}{132}$$

$$= \frac{1}{396}.$$

$\therefore$ Evaluation of $\int_0^1 x^5(1-x^3)^{10}\,dx = \frac{1}{396}$

Example. 5

prove That $\int_0^{\pi/2} \frac{\cos^{2m-1}\theta \cdot \sin^{2n-1}\theta \, d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} = \frac{\beta(m,n)}{2a^m b^n}.$

Let

$$I = \int_0^{\pi/2} \frac{\cos^{2m-1}\theta \ \sin^{2n-1}\theta \, d\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}}$$

putting $\tan\theta = t$

$$\sec^2\theta \frac{d\theta}{dt} = 1$$

$$\sec^2\theta \, d\theta = dt$$

$$d\theta = \frac{dt}{\sec^2\theta}$$

$$= \frac{dt}{1+\tan^2\theta}$$

$$d\theta = \frac{1}{1+t^2} dt$$

when $\theta = 0$, $t = \tan(0) = 0$

when $\theta = \pi/2$, $t = \tan(\pi/2) = \infty$

Also, we can express $\sin\theta$ and $\cos\theta$ in terms of t.

$$\sin\theta = \frac{\tan\theta}{\sqrt{1+\tan^2\theta}} = \frac{t}{\sqrt{1+t^2}}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+t^2}}$$

$$I = \int_0^{\pi/2} \frac{(\cos\theta)^{2m-1}(\sin\theta)^{2n-1}}{[a(\cos\theta)^2 + b(\sin\theta)^2]^{m+n}} \, d\theta$$

$$I = \int_0^{\infty} \frac{\left(\frac{1}{\sqrt{1+t^2}}\right)^{2m-1}\left(\frac{t}{\sqrt{1+t^2}}\right)^{2n-1}}{\left[a\left(\frac{1}{\sqrt{1+t^2}}\right)^2 + b\left(\frac{t}{\sqrt{1+t^2}}\right)^2\right]^{m+n}} \frac{dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{\frac{1}{[(1+t^2)^{1/2}]^{2m-1}} \frac{t^{2n-1}}{[(1+t^2)^{1/2}]^{2n-1}}}{\left[\frac{a}{[(1+t^2)^{1/\cancel{2}}]^{\cancel{2}}} + \frac{bt^2}{[(1+t^2)^{1/\cancel{2}}]^{\cancel{2}}}\right]^{m+n}} \frac{dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{\frac{1}{(1+t^2)^{\frac{\cancel{2}m}{\cancel{2}}-\frac{1}{2}}} \frac{t^{2n-1}}{(1+t^2)^{\frac{\cancel{2}n}{\cancel{2}}-\frac{1}{2}}}}{\left(\frac{a}{1+t^2} + \frac{bt^2}{1+t^2}\right)^{m+n}} \frac{dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{\frac{1}{(1+t^2)^{m-1/2}} \cdot \frac{t^{2n-1}}{(1+t^2)^{n-1/2}}}{\left(\frac{a+bt^2}{1+t^2}\right)^{m+n}} \frac{dt}{1+t^2}$$

$$= \int_0^{\infty} \frac{\frac{t^{2n-1}}{(1+t^2)^{m+n-1/2-\frac{1}{2}}}}{\frac{(a+bt^2)^{m+n}}{(1+t^2)^{m+n}}} \frac{dt}{1+t^2}$$

$$I = \int_0^\infty \frac{t^{2n-1}}{(1+t^2)^{m+n-1}} \times \frac{(1+t^2)^{m+n}}{(a+bt^2)^{m+n}} \frac{1}{1+t^2}\, dt$$

$$= \int_0^\infty \frac{t^{2n-1}}{(1+t^2)^{m+n-1}} \times \frac{(1+t^2)^{m+n}(1+t^2)^{-1}}{(a+bt^2)^{m+n}}\, dt$$

$$= \int_0^\infty \frac{t^{2n-1}}{\cancel{(1+t^2)^{m+n-1}}} \times \frac{\cancel{(1+t^2)^{m+n-1}}}{(a+bt^2)^{m+n}}\, dt$$

$$I = \int_0^\infty \frac{t^{2n-1}}{(a+bt^2)^{m+n}}\, dt$$

Let.

$y = bt^2 \Rightarrow t^2 = \frac{y}{b}$

$t = \sqrt{\frac{y}{b}}$

$\frac{dy}{dt} = b2t.$

$dy = 2bt\, dt.$

$dt = \frac{dy}{2bt} = \frac{dy}{2b\sqrt{\frac{y}{b}}}$

when $t = 0 \Rightarrow y = b(0)^2 = 0$

when $t = \infty \Rightarrow y = b(\infty)^2 = \infty$

$$I = \int_0^\infty \frac{\left(\sqrt{\frac{y}{b}}\right)^{2n-1}}{(a+y)^{m+n}} \frac{dy}{2b\sqrt{\frac{y}{b}}}$$

$$= \int_0^\infty \frac{\left[\left(\frac{y}{b}\right)^{1/2}\right]^{2n-1}}{(a+y)^{m+n}} \cdot \frac{dy}{2b\frac{\sqrt{y}}{\sqrt{b}}}$$

$$\int_0^\infty \left(\frac{y}{b}\right)^{\frac{2n-1}{2}} \quad dy$$

$$I = \int_0^\infty \frac{\left(\frac{y}{b}\right)^{\frac{2n}{2}-\frac{1}{2}}}{(a+y)^{m+n}} \cdot \frac{dy}{2\sqrt{b}\sqrt{y}}$$

$$= \int_0^\infty \frac{\left(\frac{y}{b}\right)^{n-1/2}}{(a+y)^{m+n}} \cdot \frac{dy}{2(b)^{1/2}(y)^{1/2}}$$

$$= \int_0^\infty \frac{\left(\frac{y^{n-1/2}}{b^{n-1/2}}\right)}{(a+y)^{m+n}} \cdot \frac{dy}{2(b)^{1/2}(y)^{1/2}}$$

$$= \frac{1}{2}\int_0^\infty \frac{y^{n-1/2}\, b^{-n+1/2}}{(a+y)^{m+n}} (b)^{-1/2}(y)^{-1/2}\, dy$$

$$= \frac{1}{2}\int_0^\infty \frac{y^{n-1/2} \cdot y^{-1/2} \times b^{-n+1/2} \cdot b^{-1/2}}{(a+y)^{m+n}}\, dy$$

$$= \frac{1}{2}\int_0^\infty \frac{y^{n-1/2-1/2} \times b^{-n+\cancel{1/2}-\cancel{1/2}}}{(a+y)^{m+n}}\, dy$$

$$= \frac{1}{2}\int_0^\infty \frac{y^{n-1} \times b^{-n}}{(a+y)^{m+n}}\, dy.$$

$$I = \frac{b^{-n}}{2}\int_0^\infty \frac{y^{n-1}}{(a+y)^{m+n}}\, dy$$

$$\beta(m,n) = \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\, dt.$$

Let.

$$y = ax \Rightarrow x = \frac{y}{a} \qquad\Big|\qquad y=0,\ x=\frac{0}{a}=0$$

$$\frac{dy}{dx} = a \Rightarrow dy = a\,dx \qquad\Big|\qquad y=\infty,\ x=\frac{\infty}{a}=\infty$$

$$I = \frac{b^{-n}}{2}\int_0^\infty \frac{(ax)^{n-1}}{(a+ax)^{m+n}}\, a\,dx$$

$$= \frac{b^{-n}}{2}\int_0^\infty \frac{a^{n-1}\, x^{n-1}}{[a(1+x)]^{m+n}}\, a\,dx$$

$$= \frac{b^{-n}}{2}\int_0^\infty \frac{a^{n-1}\cdot a^{1}\, x^{n-1}}{a^{m+n}\,(1+x)^{m+n}}\,dx$$

$$= \frac{b^{-n}}{2}\int_0^\infty \frac{a^{n\cancel{-1}\cancel{+1}}\, x^{n-1}}{a^{m+n}\,(1+x)^{m+n}}\,dx$$

$$= \frac{b^{-n}}{2}\,\frac{a^{n}}{a^{m+n}}\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{b^{-n}}{2}\, a^{n}\, a^{-(m+n)}\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{b^{-n}}{2}\, a^{\cancel{n}-m-\cancel{n}}\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \frac{a^{-m}\, b^{-n}}{2}\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx.$$

$$\beta(n,m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\beta(m,n) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dx$$

$$\beta(n,m) = \beta(m,n)$$

$$I = \frac{a^{-m} b^{-n}}{2} \beta(m,n)$$

$$I = \frac{\beta(m,n)}{2a^m b^n}.$$

$$\therefore \quad I = \int_0^{\pi/2} \frac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} d\theta = \frac{\beta(m,n)}{2a^m b^n}$$

Hence proved.

Unit - IV

Hence proved.

Unit - IV

Fourier series.

periodic function.

A function $f: R \to R$ is said to be periodic if there exists a real number ω (omega) such that $f(x) = f(\omega + x)$ for all real number x.

Note :-

1). ω is called period of function f

2) If a periodic function has a smallest positive period ω (omega). Then omega is called primitive period of the function f.

Example. 1

$\sin\theta$, $\cos\theta$ as period 2π, since It is the smallest positive period. Than 2π is the primitive period.

Example. 2

$f(x) = c$, it is a periodic function,

since $f(a+x) = f(x)$. which is equals to c.

Example. 3

$$f: R \to R \text{ defined by } f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1, & \text{if } x \text{ is rational} \end{cases}$$

it is a periodic function

Trigonometric series :

The series of the form

$$\frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \ldots\ldots + a_n \cos nx + b_n \sin nx + \ldots\ldots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where a's and b's are constant,

This is called Trigonometry.

Fourier series

The series $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$,

where a_0, a_n, b_n are given By Euler's formula:

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx, \quad n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx, \quad n = 1, 2, 3, \ldots$$

is called the Fourier series of $f(x)$.

a_0, a_n, b_n are called Fourier coefficient's.

odd and even function.

A Real function $f(x)$, is said to be

(i) odd function if for all x $f(-x) = -f(x)$
or $f(x) = -f(-x)$

(ii) Even function if for all x $f(-x) = f(x)$
or $f(x) = f(-x)$

Example $\cos nx$ is an even fn and $\sin nx$ is an odd fn.

Remark. 1:

(i) If $f(x)$ is even function, Then

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

(ii) If $f(x)$ is odd function. Then

$$\int_{-a}^{a} f(x)\,dx = 0.$$

Remark 2:

(i) The product of two even function is even function.

(ii) The product of two odd function is even function.

(iii) The product of an odd function and an even function is an odd function.

Remark 3!

If $f(x)$ is an odd function. Then

$f(x)\cos nx$ is an odd function. and

$f(x)\sin nx$ is an even function

$$\therefore \quad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0, \quad n = 1, 2, 3, \dots$$

~~And Hence The fourier series~~

~~become~~

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx, \quad n = 1, 2, 3, \dots$$

Remark 4!

If $f(x)$ is an even function. Then

$f(x)\cos nx$ is an ~~odd~~ even function and

$f(x)\sin nx$ is an ~~even~~ odd function

$$\therefore \quad a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx, \quad n = 1, 2, 3, \dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0, \quad n = 1, 2, 3, \dots$$

Bernoulli's Formula. (Result 1)

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \dots$$

where $u' = \dfrac{du}{dx}$, $u'' = \dfrac{d^2u}{dx^2}, \dots$

$$v_1 = \int v\,dx, \quad v_2 = \int v_1\,dx, \dots$$

Result 2

(i) $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

(ii) $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

Working Rules:

Step 1:

Check whether $f(x)$ is odd or even

Step 2:

If $f(x)$ is even $b_n = 0$ Find a_0, a_n

If $f(x)$ is odd $a_0, a_n = 0$ Find b_n

Step 3:

If $f(x)$ is neither odd nor even

Then calculate fourier coefficients using Euler's formula.

Problam. 1

Show That $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}$ in The

Integral $-\pi \leq x \leq \pi$.

Deduce That (i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}\left[x^3\right]_{-\pi}^{\pi}$$

$$= \frac{1}{3\pi}\left[\pi^3 - (-\pi)^3\right]$$

$$= \frac{1}{3\pi}\left[\pi^3 - (-\pi^3)\right]$$

$$= \frac{1}{3\pi}\left[\pi^3 + \pi^3\right]$$

$$= \frac{2\pi^{\cancel{3}\,2}}{3\cancel{\pi}}$$

$$\boxed{a_0 = \frac{2\pi^2}{3}}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx.$$

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx, \text{ If } f(x) \text{ is even. fn.}$$

$$a_n = \frac{1}{\pi}\times 2\int_{0}^{\pi} x^2\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx.$$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots\cdots$$

Let

$u = x^2$ | $\int dv = \int \cos nx \, dx$

$v = \frac{\sin nx}{n} = \frac{1}{n}(\sin nx)$

$u' = 2x$

$v_1 = \frac{1}{n}\left(-\frac{\cos nx}{n}\right) = -\frac{1}{n^2}(\cos nx$

$u'' = 2$

$v_2 = -\frac{1}{n^2}\left(\frac{\sin nx}{n}\right) = -\frac{1}{n^3}(\sin n$

Then

$$a_n = \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} - \left(-\frac{2x \cos nx}{n^2}\right) + \left(-\frac{2\sin nx}{n^3}\right.\right.$$

$$= \frac{2}{n\pi}\left[x^2 \sin nx\right]_0^\pi + \frac{4}{\pi n^2}\left[x \cos nx\right]_0^\pi - \frac{4}{\pi n^3}\left[\sin nx\right]_0^\pi$$

$$= \frac{2}{n\pi}\left[\pi^2 \sin n\pi - 0^2 \sin n(0)\right] + \frac{4}{\pi n^2}\left[\pi \cos n\pi - 0 \cos n(0)\right] - \frac{4}{\pi n^3}\left[\sin n\pi - \sin\right.$$

$$= \frac{2}{n\pi}\left[\pi^2(0) - 0\right] + \frac{4}{n^2\pi}\left[\pi(-1)^n - 0\right] - \frac{4}{n^3\pi}\left[0 - 0\right]$$

$$= 0 + \frac{4\cancel{\pi}(-1)^n}{n^2\cancel{\pi}} - 0$$

$$\boxed{a_n = \frac{4(-1)^n}{n^2}}$$

When n is odd, $a_n = \frac{-4}{n^2}$, $n = 1, 3, 5, 7, \ldots$

When n is even, $a_n = \frac{4}{n^2}$, $n = 2, 4, 6, 8, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx.$$

$f(x)=x^2$ is an even function.

$f(x)=\sin nx$ is an odd function.

Remark 2:

(iii) The product of an even function and an odd function is an "odd function"

which means. $\int_{-a}^{a} f(x)\,dx = 0$ [odd function]

$$\therefore \; b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx$$

$$b_n = \frac{1}{\pi} \times 0$$

$$\boxed{b_n = 0}$$

Let the series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$x^2 = \frac{\cancel{2}\pi^2}{3\times\cancel{2}} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n \cos nx}{n^2} + 0\right]$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

$$\therefore \quad x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2},$$

This proves the first part of the problem

Deduce that

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$ [Fourier Series]

substitute $x=0$ into the fourier series

$$0^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n.0)}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \cos 0^\circ}{n^2}$$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n 1}{n^2}$$

$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

$$-\frac{\pi^2}{3} = 4\left[\frac{(-1)^1}{1^2} + \frac{(-1)^2}{2^2} + \frac{(-1)^3}{3^2} + \cdots\right]$$

$$-\frac{\pi^2}{3} = 4\left(-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right)$$

$$\cancel{-}\frac{\pi^2}{3\times 4} = \cancel{-}\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots$$

$$\therefore \quad \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$$

This proves the first deduction.

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

Fourier series.

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$$

substitute $x = \pi$ in to the fourier series

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n^2}$$

$$\cos(n\pi) = (-1)^n$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n (-1)^n}{n^2}$$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$(-1)^{2n} = 1, \quad n = 1, 2, 3, 4, \cdots$$

$$\therefore \pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{3\pi^2 - \pi^2}{3 \times 4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\cancel{2}\pi^2}{\cancel{12}\,6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\therefore \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

This proves the second deduction.

(iii) $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

add the results from (i) and (ii)

Let $S_1=\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$

Let $S_2=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$

$$S_1+S_2=\left(\frac{1}{1^2}-\cancel{\frac{1}{2^2}}+\frac{1}{3^2}-\cancel{\frac{1}{4^2}}+\frac{1}{5^2}-\cdots\right)+\left(\frac{1}{1^2}+\cancel{\frac{1}{2^2}}+\frac{1}{3^2}+\cancel{\frac{1}{4^2}}+\frac{1}{5^2}+\cdots\right)$$

$$\frac{\pi^2}{12}+\frac{\pi^2}{6}=\frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+\cdots$$

$$\frac{\pi^2}{12}+\left(\frac{\pi^2}{6}\times\frac{2}{2}\right)=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\frac{\pi^2}{12}+\frac{2\pi^2}{12}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\frac{\pi^2+2\pi^2}{12}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\frac{{}^1\cancel{3}\pi^2}{{}_4\cancel{12}}=2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\frac{\pi^2}{4\times 2}=\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

$$\therefore\ \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

This proves the third deduction.

Example. 2.

Express $f(x)=\frac{1}{2}(\pi-x)$ as a fourier series with period 2π, to be valid in the interval 0 to 2π

$$f(x)=\frac{1}{2}(\pi-x) \quad , \text{ limit } 0 \to 2\pi$$

Let the fourier series.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx.$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)dx.$$

$$=\frac{1}{2\pi}\left[\pi\int_0^{2\pi}dx-\int_0^{2\pi}x\,dx\right]$$

$$=\frac{1}{2\pi}\left[\pi x-\frac{x^2}{2}\right]_0^{2\pi}$$

$$=\frac{1}{2\pi}\left[\left(\pi(2\pi)-\frac{(2\pi)^2}{2}\right)-\left(\pi(0)-\frac{0^2}{2}\right)\right]$$

$$=\frac{1}{2\pi}\left[2\pi^2-\frac{4\pi^2}{2}\right]$$

$$=\frac{1}{2\pi}\left[2\pi^2-2\pi^2\right]$$

$$=\frac{1}{2\pi}(0)$$

$$\boxed{a_0=0}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)dx.$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi-x)\cos(nx)dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos(nx)\,dx.$$

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \ldots\ldots$$

Let

$u = \pi - x$

$u' = 0 - 1$

$u' = -1$

$\int dv = \int \cos(nx)\,dx$

$v = \frac{\sin nx}{n} = \frac{1}{n}\sin nx$

$v_1 = \frac{1}{n}\left(-\frac{\cos nx}{n}\right) = -\frac{1}{n^2}\cos$

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{1}{n^2}\cos nx\right)\right.$$

$$= \frac{1}{2\pi}\left[\frac{(\pi - x)}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(\frac{(\pi - 2\pi)}{n}\sin(2n\pi) - \frac{1}{n^2}\cos(2n\pi)\right) - \right.$$

$$\left.\left(\frac{(\pi - 0)}{n}\sin(n\times 0) - \frac{1}{n^2}\cos(n\times 0)\right)\right]$$

$\sin(2n\pi) = 0$, $\cos(2n\pi) = 1^n = 1$ $[\because \cos 2\pi =$

$\sin(0) = 0$, $\cos(0) = 1$

$$= \frac{1}{2\pi}\left[\left(-\frac{\pi}{n}(0) - \frac{1}{n^2}(1)\right) - \left(\frac{\pi}{n}(0) - \frac{1}{n^2}(1)\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(0 - \frac{1}{n^2}\right) - \left(0 - \frac{1}{n^2}\right)\right]$$

$$= \frac{1}{2\pi}\left(-\cancel{\frac{1}{n^2}} + \cancel{\frac{1}{n^2}}\right)$$

$$= \frac{1}{2\pi}(0)$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_0 f(x)\sin nx\, dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{1}{2}(\pi - x)\sin nx\, dx$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\, dx.$$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

Let $u = \pi - x$ | $\int dv = \int \sin nx\, dx$

$u' = 0 - 1$ | $v = -\frac{\cos(nx)}{n} = -\frac{1}{n}\cos(nx)$

$u' = -1$ | $v_1 = -\frac{1}{n}\left(\frac{\sin nx}{n}\right) = -\frac{1}{n^2}\sin(nx)$

$$b_n = \frac{1}{2\pi}\left[(\pi - x)\left(-\frac{\cos(nx)}{n}\right) - (-1)\left(-\frac{\sin(nx)}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{(\pi - x)}{n}\cos(nx) - \frac{1}{n^2}\sin(nx)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left(-\frac{(\pi - 2\pi)}{n}\cos(2n\pi) - \frac{1}{n^2}\sin(2n\pi)\right) - \left(-\frac{(\pi - 0)}{n}\cos(n\times 0) - \frac{1}{n^2}\sin(n\times 0)\right)\right]$$

$$\cos 2\pi = 1$$

$$\cos(2n\pi) = 1^n = 1, \quad \sin(2n\pi) = 0$$

$$\cos(0^\circ) = 1, \quad \sin(0^\circ) = 0$$

$$= \frac{1}{2\pi}\left[\left(-\frac{(-\pi)}{n}(1) - \frac{1}{n^2}(0)\right) - \left(-\frac{\pi}{n}(1) - \frac{1}{n^2}(0)\right)\right]$$

$$= \frac{1}{2\pi}\left[\left(\frac{\pi}{n} - 0\right) - \left(-\frac{\pi}{n} - 0\right)\right]$$

$$b_n = \frac{1}{2\pi}\left[\frac{\pi}{n} - \left(-\frac{\pi}{n}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n} + \frac{\pi}{n}\right]$$

$$= \frac{1}{\cancel{2\pi}}\left(\frac{\cancel{2\pi}}{n}\right)$$

$$\boxed{b_n = \frac{1}{n}}$$

Let the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$\frac{1}{2}(\pi - x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left[0 \cdot \cos(nx) + \frac{1}{n}\sin(nx)\right]$$

$$\frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)$$

$\therefore$ The fourier series for $f(x) = \frac{1}{2}(\pi - x)$ is

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin(nx) \ldots$$

$\Rightarrow$ The Fourier series.

$$\frac{1}{2}(\pi - x) = \sum_{n=1}^{\infty}\frac{1}{n}\sin(nx)$$

~~In this series we put x = π/2~~.

$$\frac{1}{2}(\pi - x) = \frac{1}{1}\sin(1 \times x) + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \cdots$$

$$\frac{1}{2}(\pi - x) = \sin x + \frac{1}{2}\sin(2x) + \frac{1}{3}\sin(3x) + \frac{1}{4}\sin(4x) + \cdots$$

In this series we put $x = \pi/2$, we get,

$$\frac{1}{2}\left(\pi - \frac{\pi}{2}\right) = \sin\frac{\pi}{2} + \frac{1}{2}\sin\left(\cancel{2} \times \frac{\pi}{\cancel{2}}\right) + \frac{1}{3}\sin\left(3 \times \frac{\pi}{2}\right) + \frac{1}{4}\sin\left(\overset{2}{\cancel{4}} \times \frac{\pi}{\cancel{2}}\right) + \cdots$$

$$\frac{1}{2}\left(\frac{2\pi-\pi}{2}\right) = \sin(\pi/2) + \frac{1}{2}\sin(\pi) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin(2\pi) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \cdots\cdots$$

$$\frac{1}{2}\left(\frac{\pi}{2}\right) = 1 + \frac{1}{2}(0) + \frac{1}{3}(-1) + \frac{1}{4}(0) + \frac{1}{5}(1) + \cdots$$

$$\frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} + 0 - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\cdots$$

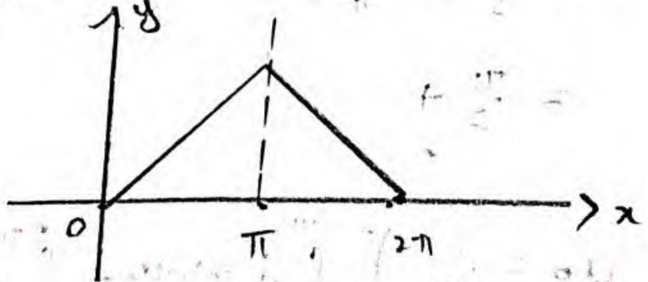
Example 3.

A function $f(x)$ is defined within the range $(0, 2\pi)$ by the relations.

$$f(x) = \begin{cases} x, & \text{for } 0 < x < \pi \\ 2\pi - x, & \text{for } \pi < x < 2\pi \end{cases}$$

Express $f(x)$ as a fourier series in the range$(0,$

If we draw the curve $f(x)$ in the range$(0, 2\pi$



Let the fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx.$$

$$f(x) = \begin{cases} x, & \text{for } 0 < x < \pi \\ 2\pi - x, & \text{for } \pi < x < 2\pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_0^{2\pi}(2\pi - x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi}\left[2\pi\int_0^{2\pi} dx - \int_0^{2\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{2}\left[x^2\right]_0^{\pi} + \frac{1}{\pi}\left[2\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\pi^2 - 0^2\right] + \frac{1}{\pi}\left[\left(2\pi(2\pi) - \frac{(2\pi)^2}{2}\right) - \left(2\pi(0) - \frac{0^2}{2}\right)\right]$$

$$= \frac{\pi^2}{2\pi} + \frac{1}{\pi}\left[\left(4\pi^2 - \frac{4\pi^2}{2}\right) - 0\right]$$

$$= \frac{\pi}{2} + \frac{1}{\pi}\left(4\pi^2 - 2\pi^2\right)$$

$$= \frac{\pi}{2} + \frac{1}{\pi}\left(2\pi^2\right)$$

$$= \frac{\pi}{2} +$$

$$a_0 = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\left[\int_{\pi}^{2\pi} 2\pi\,dx - \int_{\pi}^{2\pi} x\,dx\right]$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\left[2\pi\int_{\pi}^{2\pi} dx - \int_{\pi}^{2\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{\pi}\left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2\pi}\left[\pi^2 - 0\right] + \frac{1}{\pi}\left[\left(2\pi(2\pi) - \frac{(2\pi)^2}{2}\right) - \left(2\pi(\pi) - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{\pi^{\cancel{2}}}{2\cancel{\pi}} + \frac{1}{\pi}\left[\left(4\pi^2 - \frac{\overset{2}{\cancel{4}}\pi^2}{\cancel{2}}\right) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right]$$

$$= \frac{\pi}{2} + \frac{1}{\pi}\left[\left(4\pi^2 - 2\pi^2\right) - \left(\frac{4\pi^2 - \pi^2}{2}\right)\right]$$

$$= \frac{\pi}{2} + \frac{1}{\pi}\left[(2\pi^2) - \left(\frac{3\pi^2}{2}\right)\right]$$

$$= \frac{\pi}{2} + \frac{1}{\pi}\left[2\pi^2 - \frac{3\pi^2}{2}\right]$$

$$= \frac{\pi}{2} + \frac{2\pi^{\cancel{2}}}{\cancel{\pi}} - \frac{3\pi^{\cancel{2}}}{2\cancel{\pi}}$$

$$= \frac{\pi}{2} + \left(2\pi \times \frac{2}{2}\right) - \frac{3\pi}{2}$$

$$= \frac{\pi}{2} + \frac{4\pi}{2} - \frac{3\pi}{2}$$

$$= \frac{\pi + 4\pi - 3\pi}{2}$$

$$= \frac{2\pi}{2}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} x\cos(nx)\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos(nx)\,dx\right] \quad \text{— (1)}$$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 + \cdots\cdots$$

For The first Integral.

$$\int_0^{\pi} x\cos(nx)\,dx$$

Let

$$u = x \qquad \int dv = \int \cos(nx)\,dx$$

$$u' = 1 \qquad v = \frac{\sin nx}{n}$$

$$v_1 = -\frac{\cos nx}{n\times n} = -\frac{\cos nx}{n^2}$$

$$\int_0^{\pi} x\cos(nx)\,dx = \left[x\frac{\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \left[\frac{1}{n}(x\sin nx) + \frac{1}{n^2}(\cos nx)\right]_0^{\pi}$$

$$= \left[\frac{1}{n}\left[\left(\pi\sin(n\pi) + \frac{1}{n^2}\cos(n\pi)\right) - \left(\frac{1}{n}(0\sin(n\times 0)) + \frac{1}{n^2}\cos(n\times 0)\right)\right]\right]$$

$$\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n$$

$$= \left[\frac{1}{n}(\pi(0)) + \frac{1}{n^2}(-1)^n\right] - \left[\frac{1}{n}(0) + \frac{1}{n^2}(1)\right]$$

$$= \left(0 + \frac{1}{n^2}(-1)^n - 0 - \frac{1}{n^2}\right)$$

$$\int_0^{\pi} x\cos(nx)\,dx = \frac{1}{n^2}\left[(-1)^n - 1\right] \quad \text{——} \ (2)$$

For The second Integral.

$$\int_{\pi}^{2\pi} (2\pi - x)\cos(nx)\,dx.$$

Bernoulli's formula use.

Let

$u = 2\pi - x$	$\int dv = \int \cos(nx)\,dx$
$u' = 0 - 1$	$v = \frac{\sin(nx)}{n}$
$u' = -1$	$v_1 = -\frac{\cos(nx)}{n \times n} = -\frac{\cos(nx)}{n^2}$

$$\int_{\pi}^{2\pi} (2\pi - x)\cos(nx)\,dx = \left[\frac{(2\pi - x)}{n}(\sin nx) - (-1)\left(-\frac{\cos(nx)}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \left[\frac{(2\pi - x)}{n}\sin(nx) - \frac{1}{n^2}\cos(nx)\right]_{\pi}^{2\pi}$$

$$= \left[\left(\frac{(2\pi - 2\pi)}{n}\sin(2n\pi) - \frac{1}{n^2}\cos(2n\pi)\right) - \left(\frac{(2\pi - \pi)}{n}\sin(n\pi) - \frac{1}{n^2}\cos(n\pi)\right)\right]$$

$\sin(2n\pi) = 0, \quad \cos(2n\pi) = 1^n = 1$

$\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n$

$$= \frac{0}{n}(0) - \frac{1}{n^2}(1) - \left(\frac{\pi}{n}(0) - \frac{1}{n^2}(-1)^n\right)$$

$$= -\frac{1}{n^2} + \frac{1}{n^2}(-1)^n$$

$$\int_{\pi}^{2\pi} (2\pi - x)\cos(nx)\,dx = \frac{1}{n^2}\left[(-1)^n - 1\right] \quad \text{③}$$

Substitude equation ② and ③ in ①

$$a_n = \frac{1}{\pi}\left[\frac{1}{n^2}\left[(-1)^n - 1\right] + \frac{1}{n^2}\left[(-1)^n - 1\right]\right]$$

$$= \frac{1}{\pi}\left(\frac{2.\left[(-1)^n - 1\right]}{n^2}\right)$$

$$\boxed{a_n = \frac{2\left[(-1)^n - 1\right]}{n^2\pi}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\xrightarrow{\sin(nx)} dx + \int_{\pi}^{2\pi} f(x)\sin(nx)\,dx\right]$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} x\sin(nx)\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin(nx)\,dx\right] \quad \text{④}$$

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

For the first Integral

$$\int_0^{\pi} x\sin(nx)\,dx$$

Let

$u = x$

$u' = 1$

$$\int dv = \int \sin(nx)\,dx$$

$$v = -\frac{\cos(nx)}{n}$$

$$v_1 = -\frac{1}{n}\left(\frac{\sin nx}{n}\right) = -\frac{\sin(nx)}{n^2}$$

$$\int_0^{\pi} x\sin(nx)\,dx = \left[-\frac{x\cos(nx)}{n} - (1)\left(-\frac{\sin(nx)}{n^2}\right)\right]_0^{\pi}$$

$$= \left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi}$$

$$= \left[\left(-\frac{\pi\cos(n\pi)}{n} + \frac{\sin(n\pi)}{n^2}\right) - \left(-\frac{0\cos(n.0)}{n} + \frac{\sin(n.0)}{n^2}\right)\right]$$

$$\cos(n\pi) = (-1)^n, \quad \sin(n\pi) = 0$$

$$\cos(0^\circ) = 1, \quad \sin(0^\circ) = 0$$

$$= -\frac{\pi(-1)^n}{n} + \frac{0}{n^2} + 0 - \frac{0}{n^2}$$

$$= -\frac{\pi(-1)^n}{n} + 0 + 0 - 0$$

$$\int_0^{\pi} x\sin(nx)\,dx = -\frac{\pi(-1)^n}{n} \quad \text{———} \quad (5)$$

For, The second Integral.

$$\int_{\pi}^{2\pi} (2\pi - x)\sin(nx)\,dx.$$

Bernoulli's formula use.

let.

$u = 2\pi - x$	$\int dv = \int \sin(nx)\,dx$
$u' = 0 - 1$	$v = -\frac{\cos(nx)}{n}$
$u' = -1$	$v_1 = -\frac{\sin(nx)}{n \times n}$
	$= -\frac{\sin(nx)}{n^2}$

$$\int_{\pi}^{2\pi}(2\pi-x)\sin(nx)\,dx = \left[-\frac{(2\pi-x)}{n}\cos(nx)-(-1)\left(-\frac{\sin(nx)}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \left[\left(-\frac{(2\pi-2\pi)}{n}\cos(2n\pi)-\frac{1}{n^2}\sin(2n\pi)\right)-\right.$$

$$\left.\left(-\frac{(2\pi-\pi)}{n}\cos(n\pi)-\frac{1}{n^2}\sin(n\pi)\right)\right]$$

$$\cos(2n\pi)=(1)^n=1, \quad \sin(2n\pi)=0$$

$$\cos(n\pi)=(-1)^n, \quad \sin(n\pi)=0$$

$$= \left[\left(-\frac{0}{n}(1)-\frac{1}{n^2}(0)\right)-\left(-\frac{\pi}{n}(-1)^n-\frac{1}{n^2}(0)\right)\right]$$

$$= \left[0+\frac{\pi(-1)^n}{n}\right]$$

$$\int_{\pi}^{2\pi}(2\pi-x)\sin(nx)\,dx = \frac{\pi(-1)^n}{n} \quad \text{——— (6)}$$

substitude eqnation (5) and (6) in (4)

$$b_n = \frac{1}{\pi}\left[-\frac{\cancel{\pi(-1)^n}}{n}+\frac{\cancel{\pi(-1)^n}}{n}\right]$$

$$b_n = \frac{1}{\pi}(0)$$

$$\boxed{b_n = 0}$$

Let the fourier series

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$f(x) = \frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2[(-1)^n-1]}{n^2\pi}\cos nx + 0$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} \cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos nx$$

when n is even, $1-(-1)^n = 0$; $n = 2, 4, 6, 8, \ldots$

when n is odd, $1-(-1)^n = 2$; $n = 1, 3, 5, 7, \ldots$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left(\frac{2}{1^2}\cos x + \frac{0}{2^2}\cos 2x + \frac{2}{3^2}\cos 3x + \frac{0}{4^2}\cos 4x + \ldots \right)$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\left(2\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \ldots \right)$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots \right)$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \frac{1}{7^2}\cos 7x + \ldots \right).$$

whe $x = 0$, $f(x) = 0$.

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos 0^\circ + \frac{1}{3^2}\cos(3\times 0^\circ) + \frac{1}{5^2}\cos(5\times 0^\circ) + \ldots \right)$$

$\cos 0^\circ = 1$.

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \right)$$

$$\frac{\cancel{-}\frac{\pi}{2}}{\cancel{-}\frac{4}{\pi}} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

when $x = \pi$, $f(x) = \pi$

$$\pi - \frac{\pi}{2} = -\frac{4}{\pi}\left(\cos\pi + \frac{1}{3^2}\cos 3\pi + \frac{1}{5^2}\cos 5\pi + \ldots \right)$$

$$+\frac{\pi^2}{8} = -\left[(-1) + \frac{1}{3^2}(-1)^3 + \frac{1}{5^2}(-1)^5 + \ldots \right]$$

$$\frac{\pi^2}{8} = -\left[-1 - \frac{1}{3^2} - \frac{1}{5^2} - \ldots \right]$$

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Half Range Fourier Series

It is often convenient to obtain a fourier expansion of a function to hold for a range which is half the period of the fourier series, That is to expand $f(x)$ as the fourier series in the range $(0, \pi)$.

In the half range fourier series of $f(x)$ a series contains cosines alone or sines alone.

Results

1) $\int_0^{\pi} \cos(mx)\,dx = 0$, if m is an Integer.

2) $\int_0^{\pi} \cos(mx)\cos(nx)\,dx = 0$, If $m \neq n$ and m, n are Integers.

3) $\int_0^{\pi} \sin(mx)\sin(nx)\,dx = 0$, If $m \neq n$ and m, n are Integers

4) $\int_0^{\pi} \cos(mx)\cos(nx) = \int_0^{\pi} \cos^2(mx)\,dx$, If $m=n$

$$= \frac{\pi}{2}$$

5) $\int_0^{\pi} \sin(mx)\sin(nx)\,dx = \int_0^{\pi} \sin^2(mx)\,dx$, If $m=n$

$$= \pi/2$$

Full Range Fourier Series

Example.4.

Find the range $-\pi$ to π, a Fourier series for $y = \begin{cases} 1+x, & \text{For } 0 < x < \pi \\ -1+x; & \text{For } -\pi < x < 0 \end{cases}$

$$y = \begin{cases} 1+x, & \text{For } 0 < x < \pi \\ -1+x, & \text{For } -\pi < x < 0 \end{cases}$$

Let the Fourier series

$$y = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} y\,dx.$$

$$\therefore a_0 = \frac{1}{\pi}\left[\int_{-\pi}^{0} (-1+x)\,dx + \int_{0}^{\pi} (1+x)\,dx\right]$$

$$= \frac{1}{\pi}\left(\left[-x+\frac{x^2}{2}\right]_{-\pi}^{0} + \left[x+\frac{x^2}{2}\right]_{0}^{\pi}\right)$$

$$= \frac{1}{\pi}\left(\left[(0-0)-\left(\pi+\frac{\pi^2}{2}\right)\right] + \left[\left(\pi+\frac{\pi^2}{2}\right)-(0+0)\right]\right)$$

$$= \frac{1}{\pi}\left(\cancel{-\pi}-\cancel{\frac{\pi^2}{2}}+\cancel{\pi}+\cancel{\frac{\pi^2}{2}}\right)$$

$$= \frac{1}{\pi}(0)$$

$$\boxed{a_0 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y\cos nx\,dx.$$

$$a_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1+x)\cos nx\,dx + \int_{0}^{\pi}(1+x)\cos nx\,dx\right] \quad \text{--- (1)}$$

For The first Integral $\int_{-\pi}^{0}(-1+x)\cos nx\,dx$.

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$$

Let.

$u = -1+x$ | $\int dv = \int \cos nx\,dx$

$u' = 0+1$ | $v = \frac{\sin nx}{n}$

$u' = 1$ | $v_1 = -\frac{\cos nx}{n\times n} = -\frac{\cos nx}{n^2}$

$$\int_{-\pi}^{0}(-1+x)\cos nx\,dx = \left[\frac{(-1+x)\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$= \left[\frac{(-1+x)\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{-\pi}^{0}$$

$$= \left[\left(\frac{(-1+0)\sin(n\times 0)}{n} + \frac{\cos(n\times 0)}{n^2}\right) - \left(\frac{(-1-\pi)\sin(-n\pi)}{n} + \frac{\cos(-n\pi)}{n^2}\right)\right]$$

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$.

$$= \left[\left(-\frac{\sin 0}{n} + \frac{\cos 0}{n^2}\right) - \left(-\frac{(-1-\pi)\sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2}\right)\right]$$

$$= \left[\left(-\frac{0}{n} + \frac{1}{n^2}\right) - \left(\frac{(1+\pi)\sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2}\right)\right]$$

$\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n$

$$= \left[\frac{1}{n^2} - \left(\frac{(1+\pi)\times 0}{n} + \frac{(-1)^n}{n^2} \right) \right]$$

$$= \frac{1}{n^2} - \frac{(-1)^n}{n^2}$$

$$\int_{-\pi}^{0} (-1+x) \cos nx \, dx = \frac{1-(-1)^n}{n^2} \quad \text{——— (2)}$$

For the second Integral $\cdot \int_0^{\pi} (1+x) \cos nx \, dx$.

Bernoulli's formula

Let

$u = 1+x$ | $\int dv = \int \cos nx \, dx$

$u' = 0+1$ | $v = \frac{\sin nx}{n}$

$u' = 1$ | $v_1 = -\frac{\cos nx}{n^2}$

$$\int_0^{\pi} (1+x) \cos nx \, dx = \left[\frac{(1+x)\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right) \right]_0^{\pi}$$

$$= \left[\frac{\sin nx}{n} + \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right]_0^{\pi}$$

$$= \left[\left(\frac{\sin(n\pi)}{n} + \frac{\pi \sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2} \right) - \left(\frac{\sin(n\times 0)}{n} + \frac{0 \sin(n\times 0)}{n} + \frac{\cos(n\times 0)}{n^2} \right) \right]$$

$\sin n\pi = 0, \cos(n\pi) = (-1)^n$

$\sin 0^\circ = 0, \cos 0^\circ = 1$

$$= \left[\left(0 + 0 + \frac{(-1)^n}{n^2} \right) - \left(0 + 0 + \frac{1}{n^2} \right) \right]$$

$$= \frac{(-1)^n}{n^2} - \frac{1}{n^2}$$

$$= \frac{(-1)^n - 1}{n^2}$$

$$\int_0^{\pi} (1+x) \cos nx \, dx = -\frac{1-(-1)^n}{n^2} \quad \text{——— (3)}$$

Substitude equation ② and ③ in ①.

$$a_n = \frac{1}{\pi}\left[\frac{1-\cancel{(-1)^n}}{n^2} - \frac{1-\cancel{(-1)^n}}{n^2}\right] = \frac{1}{\pi}(0)$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y\sin nx\,dx.$$

$$b_n = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-1+x)\sin nx\,dx + \int_{0}^{\pi}(1+x)\sin nx\,dx.\right] \quad ④$$

For The first Integral. $\int_{-\pi}^{0}(-1+x)\sin nx\,dx$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

Let.

$u = -1+x$	$\int dv = \int \sin nx\,dx$
$u' = 0+1$	$v = -\frac{\cos nx}{n}$
$u' = 1$	$v_1 = -\frac{\sin nx}{n\times n} = -\frac{\sin nx}{n^2}$

$$\int_{-\pi}^{0}(-1+x)\sin nx\,dx = \left[(-1+x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_{-\pi}^{0}$$

$$= \left[-\frac{(-1+x)\cos nx}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0}$$

$$= \left[\frac{(1-x)\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0}$$

$$= \left[\frac{\cos(nx)}{n} - \frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{-\pi}^{0}$$

$$\int_{-\pi}^{0}(-1+x)\sin nx\,dx = \left[\left(\frac{\cos(n.0)}{n} - \frac{0\cos(n.0)}{n} + \frac{\sin(n.0)}{n^2}\right) - \left(\frac{\cos(-n\pi)}{n} - \frac{(-\pi)\cos(-n\pi)}{n} + \frac{\sin(-n\pi)}{n^2}\right)\right]$$

$$\cos(-\theta) = \cos\theta, \quad \sin(-\theta) = -\sin\theta$$

$$\cos 0^\circ = 1, \quad \sin 0^\circ = 0$$

$$= \left[\left(\frac{1}{n} - 0 + 0\right) - \left(\frac{\cos(n\pi)}{n} + \frac{\pi\cos(n\pi)}{n} + \frac{\sin(n\pi)}{n^2}\right)\right]$$

$$\cos(n\pi) = (-1)^n, \quad \sin(n\pi) = 0$$

$$= \left[\frac{1}{n} - \left(\frac{(-1)^n}{n} + \frac{\pi(-1)^n}{n} - 0\right)\right]$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} - \frac{\pi(-1)^n}{n}$$

$$\int_{-\pi}^{0}(-1+x)\sin nx\,dx = \frac{1-(-1)^n - \pi(-1)^n}{n} \quad \text{——— (5)}$$

For the second Integral: $\int_0^\pi (1+x)\sin nx\,dx$.

Bernoulli's formula

Let $u = 1+x$, $u' = 0+1$, $u' = 1$ | $\int dv = \int \sin nx\,dx$, $v = -\frac{\cos nx}{n}$, $v_1 = -\frac{\sin nx}{n\times n} = -\frac{\sin nx}{n^2}$

$$\int_0^\pi (1+x)\sin nx\,dx = \left[(1+x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^\pi$$

$$= \left[(-1-x)\left(\frac{\cos nx}{n}\right) + \frac{\sin(nx)}{n^2}\right]_0^\pi$$

$$= \left[-\frac{\cos(nx)}{n} - \frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^\pi$$

$$= \left[\left(-\frac{\cos(n\pi)}{n} - \frac{\pi\cos(n\pi)}{n} + \frac{\sin(n\pi)}{n^2}\right) - \left(-\frac{\cos(n.0)}{n} - \frac{0\cos(n.0)}{n} + \frac{\sin(n.0)}{n^2}\right)\right]$$

$\cos(n\pi) = (-1)^n ; \quad \sin(n\pi) = 0$

$\cos 0^\circ = 1 \quad , \sin 0^\circ = 0.$

$$\int_0^{\pi} (1+x)\sin nx\,dx = \left[\left(-\frac{(-1)^n}{n} - \frac{\pi(-1)^n}{n} + 0\right) - \left(-\frac{1}{n} - 0 + 0\right)\right]$$

$$= \left[-\frac{(-1)^n}{n} - \frac{\pi(-1)^n}{n} - \left(-\frac{1}{n}\right)\right]$$

$$= \left[-\frac{(-1)^n}{n} - \frac{\pi(-1)^n}{n} + \frac{1}{n}\right]$$

$$= \frac{1}{n} - \frac{(-1)^n}{n} - \frac{\pi(-1)^n}{n}$$

$$\int_0^{\pi} (1+x)\sin nx\,dx = \frac{1-(-1)^n - \pi(-1)^n}{n} \quad \text{——— (6)}$$

Substitade equation (5) and (6) in (4)

$$b_n = \frac{1}{\pi}\left[\frac{1-(-1)^n - \pi(-1)^n}{n} + \frac{1-(-1)^n - \pi(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{1-(-1)^n - \pi(-1)^n + 1 - (-1)^n - \pi(-1)^n}{n}\right]$$

$$= \frac{1}{n\pi}\left[2 - 2(-1)^n - 2\pi(-1)^n\right]$$

$$= \frac{2}{n\pi} - \frac{2(-1)^n}{n\pi} - \frac{2\cancel{\pi}(-1)^n}{n\cancel{\pi}}$$

$$= \frac{2-2(-1)^n}{n\pi} - \frac{2(-1)^n}{n}$$

$$\boxed{b_n = \frac{2[1-(-1)^n]}{n\pi} - \frac{2(-1)^n}{n}}$$

Let the fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$y = \frac{0}{2} + \sum_{n=1}^{\infty} \left[0 . \cos nx + \frac{2[1-(-1)^n]}{n\pi} - \frac{2(-1)^n}{n} \sin(nx) \right]$$

$$y = \sum_{n=1}^{\infty} \frac{2[1-(-1)^n]}{n\pi} - \frac{2(-1)^n}{n} \sin(nx)$$

when n is even, $n = 2, 4, 6, 8, 10, \ldots$

$$b_n = \frac{2[1-(-1)^n]}{n\pi} - \frac{2(-1)^n}{n} = -\frac{2}{n}$$

when n is odd, $n = 1, 3, 5, 7, 9, \ldots$

$$b_n = \frac{2[1-(-1)^n]}{n\pi} - \frac{2(-1)^n}{n} = \frac{2(\pi+2)}{n\pi}$$

$$y = \frac{2(\pi+2)}{1\times\pi} \sin(1\times x) - \frac{\cancel{2}^{1}}{\cancel{2}} \sin 2x + \frac{2(\pi+2)}{3\pi} \sin 3x - \frac{\cancel{2}^{1}}{\cancel{4}_{2}} \sin 4x + \frac{2(\pi+2)}{5\pi} \sin 5x - \ldots$$

$$y = \frac{2(\pi+2)}{\pi} \sin x - \sin 2x + \frac{2(\pi+2)}{3\pi} \sin 3x - \frac{1}{2} \sin 4x + \frac{2(\pi+2)}{5\pi} \sin 5x - \frac{1}{3} \sin 6x + \ldots$$

when $x = \pi/2$, $y = 1 + x$, for $0 < x < \pi$

$$y = 1 + \frac{\pi}{2} = \frac{2+\pi}{2}$$

when $x = \pi/2$, the right side of the series becomes.

$$y = \frac{2(\pi+2)}{\pi} \sin(\pi/2) - \sin \cancel{2}(\pi/\cancel{2}) + \frac{2(\pi+2)}{3\pi} \sin\left(\frac{3\pi}{2}\right) - \frac{1}{2} \sin \cancel{4}^{2}(\pi/\cancel{2}) + \frac{2(\pi+2)}{5\pi} \sin\left(\frac{5\pi}{2}\right) - \frac{1}{3} \sin(\cancel{6}^{3} \pi/\cancel{2}) + \ldots$$

$$\frac{2+\pi}{2} = \frac{2(\pi+2)}{\pi}\sin\left(\pi/2\right) - \sin\pi + \frac{2(\pi+2)}{3\pi}\sin\left(\frac{3\pi}{2}\right)$$

$$-\frac{1}{2}\sin(2\pi) + \frac{2(\pi+2)}{5\pi}\sin\left(\frac{5\pi}{2}\right) -$$

$$\frac{1}{3}\sin(3\pi) + \cdots\cdots$$

$\sin 90^\circ\ (\pi/2),\ \sin 450^\circ\left(\frac{5\pi}{2}\right),\ \sin 810^\circ\left(\frac{9\pi}{2}\right),$

$\cdots\cdots\cdots = 1$

$\sin 180^\circ(\pi),\ \sin 360^\circ(2\pi),\ \sin 560^\circ(3\pi),$

$\sin 720^\circ(4\pi,\ \sin 900^\circ(5\pi),\ \cdots = 0$

$\sin 270^\circ(3\pi/2),\ \sin 630^\circ(7\pi/2),\ \sin 9900^\circ\left(\frac{11\pi}{2}\right),$

$\cdots\cdots = -1$

$$\frac{2+\pi}{2} = \frac{2(\pi+2)}{\pi}(1) - 0 + \frac{2(\pi+2)}{3\pi}(-1) - \frac{1}{2}(0) +$$

$$\frac{2(\pi+2)}{5\pi}(1) - \frac{1}{3}(0) + \frac{2(\pi+2)}{7\pi}(-1) - \cdots$$

$$= \frac{2(\pi+2)}{\pi} - \frac{1}{3}\frac{2(\pi+2)}{\pi} + \frac{1}{5}\frac{2(\pi+2)}{\pi}$$

$$-\frac{1}{7}\frac{2(\pi+2)}{\pi} + \cdots$$

$$\therefore \frac{2+\pi}{2} = \frac{2(\pi+2)}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$$

$$\frac{2+\pi}{2} = \frac{2(\pi+2)}{\pi}\left(\frac{\pi}{4}\right)$$

$$\frac{2+\pi}{2} = \frac{2\pi(\pi+2)}{4\pi} = \frac{2\pi^2 + 4\pi}{4\pi}$$

$$\frac{2+\pi}{2} = \frac{\cancel{2}\pi^{\cancel{2}}}{2\cancel{4}\cancel{\pi}} + \frac{\cancel{4\pi}}{\cancel{4\pi}}$$

$$\frac{2+\pi}{2} = \frac{\pi}{2} + \frac{1}{1}$$

$$2+\pi = \pi+2$$

§.3. Even and odd function.

If $f(x) = f(-x)$, Then $f(x)$ is called to be an even function

If $f(x) = -f(-x)$, Then $f(x)$ is called to be an odd function.

§ 3.1. properties of odd and even function.

Remark 1:

(i). If $f(x)$ is odd function, Then

$$\therefore \int_{-a}^{a} f(x)dx = 0$$

If $f(x)$ in even function. Then

$$\text{(ii)} \int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

$$= 2\int_{0}^{a} f(x)dx$$

$$\therefore \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

§ 3.2.

These properties of odd and even functions can be used to shorten the computation when we have to find the fourier series of either an even or odd functions for the interval $-\pi < x < \pi$

If $f(x)$ be expanded as a fourier series of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

We have.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad n = 0,1,2,\ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n = 1,2,3,\ldots$$

case (i)

$\Rightarrow$ If $f(x)$ is an odd function, then a_0 is also an odd function.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$f(x)$ is an odd function

$$\int_{-a}^{a} f(x)\,dx = 0$$

$$a_0 = \frac{1}{\pi}\times 0$$

$$\therefore a_0 = 0$$

$\Rightarrow$ If $f(x)$ is an even function, then a_0 is also an ~~odd~~ even function

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

$f(x)$ is an even function.

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\times 2\int_{0}^{\pi} f(x)\,dx$$

$$\therefore a_0 = \frac{2}{\pi}\int^{\pi} f(x)\,dx$$

case (ii)

$\Rightarrow f(x)$ is an odd function, then $f(x)\cos nx$ is also an odd function. [Remark 21 (iii)]

even

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

$f(x)$ is an odd function

$$\int_{-a}^{a} f(x)\,dx = 0$$

$$a_n = \frac{1}{\pi} \times 0$$

$\therefore$ Hence $a_n = 0$

$\Rightarrow f(x)$ is an odd function, then $f(x)\sin nx$ is also an even function. [Remark 21 (ii)]

odd

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$f(x)$ is an od e

case (ii).

1) $f(x)$ is an odd function, the $f(x)\cos nx$ is also an odd function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

$f(x)$ is ~~and~~ odd function

$\cos nx$ is an even function

Remark 21

(iii) The product of an odd function and an even function is an odd function.

odd function

$$\int^{a} f(x)\cos nx\, dx = 0$$

$$a_n = \frac{1}{\pi} \times 0$$

$\therefore$ Hence $a_n = 0$

2) If $f(x)$ is an odd function, then $f(x)\sin nx$ is an even function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

$f(x)$ is an odd function

$\sin nx$ is an odd function.

Remark 2:

(ii) The product of two odd function is an even function.

$$\int_{-a}^{a} f(x)\sin nx\, dx = 2\int_{0}^{a} f(x)\sin nx\, dx$$ even function

$$b_n = \frac{1}{\pi} \times 2\int_{0}^{\pi} f(x)\sin nx\, dx$$

$$\therefore \text{Hence } b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\, dx.$$

---

case (iii).

1) If $f(x)$ is an even function, then $f(x)\cos nx$ is an even function.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

$f(x)$ is an even function

$\cos nx$ is an even function

Remark 2:

(i) The product of two even function is an even function.

even function

$$\int_{-a}^{a} f(x)\cos nx\, dx = 2\int_{0}^{a} f(x)\cos nx\, dx$$

$$a_n = \frac{1}{\pi} \times 2\int_{0}^{\pi} f(x)\cos nx\, dx$$

$\therefore$ Hence $a_n = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} f(x)\cos nx\, dx.$

2) If $f(x)$ is an even function. Then $f(x)\sin nx$ is an odd function.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx.$$

$f(x)$ is an even function.

$\sin nx$ is an odd function.

Remark 2!

(iii) The product of an even function and an odd is an odd function.

odd function.

$$\int_{-a}^{a} f(x)\sin nx\, dx = 0$$

$$b_n = \frac{1}{\pi} \times 0$$

$\therefore$ Hence $b_n = 0$.

Hence we get the result that.

i) If $f(x)$ is an even function, $f(x)$ can be expanded as a series of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

in the Interval $(-\pi < x < \pi)$ where.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad (n = 0,1,2,\ldots)$$

(iii) If $f(x)$ is an odd function, $f(x)$ can be expanded as a series of the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

in the Interval $(-\pi < x < \pi)$ where.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Examples.

Example 1.

Express $f(x) = x$, for $-\pi < x < \pi$ as a fourier series with period 2π.

$f(x) = x$ is an odd Function.

Hence in the expansion, the cosine terms are absent.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$x = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{------} \quad (1)$$

In the Interval $(-\pi < x < \pi)$ where.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x \sin nx\,dx.$$

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

let

$u = x$

$u' = 1$

$\int dv = \int \sin nx \, dx$

$v = -\frac{\cos nx}{n}$

$v_1 = -\frac{\sin nx}{n \times n} = -\frac{\sin nx}{n^2}$

$$b_n = \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(-\frac{\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2}\right) - \left(-\frac{0\cos(n.0)}{n} + \frac{\sin(n.0)}{n^2}\right)\right]$$

$\sin n\pi = 0$, $\cos(n\pi) = (-1)^n$

$\sin(0^\circ) = 0$, $\cos(0^\circ) = 1$

$$= \frac{2}{\pi}\left[\left(-\frac{\pi(-1)^n}{n} + \frac{0}{n^2}\right) - (0 + 0)\right]$$

$$= \frac{2}{\pi}\left(-\frac{\pi(-1)^n}{n}\right)$$

$$= -\frac{2\cancel{\pi}(-1)^n}{\cancel{\pi} n}$$

$$b_n = -\frac{2}{n}(-1)^n$$

substitude b_n in the equation (1)

$$x = \sum_{n=1}^{\infty} -\frac{2}{n}(-1)^n \sin nx$$

when n is odd, $b_n = -\frac{2}{n}(-1)^n = \frac{2}{n}$, $n = 1, 3, 5, 7, \ldots$

when n is even, $b_n = -\frac{2}{n}(-1)^n = -\frac{2}{n}$, $n = 2, 4, 6, 8, \ldots$

$$x = -\frac{2}{1}(-1)^1 \sin(1 \times x) - \frac{2}{2}(-1)^2 \sin(2 \times x) - \frac{2}{3}(-1)^3$$
$$\sin(3 \times x) - \frac{2}{4}(-1)^4 \sin(4 \times x) - \ldots\ldots$$

$$x = \frac{2}{1}\sin x - \frac{2}{2}\sin 2x + \frac{2}{3}\sin 3x - \frac{2}{4}\sin 4x + \frac{2}{5}\sin 5x - \frac{2}{6}\sin 6x + \cdots$$

$$\therefore x = 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \frac{1}{5}\sin 5x - \frac{1}{6}\sin 6x + \cdots\right)$$

Exampe.2

If $f(x) = \begin{cases} -x & , \text{in } -\pi < x < 0 \\ x & , \text{in } 0 \le x < \pi \end{cases}$

expand $f(x)$ as fourier series in the interval $-\pi$ ~~to~~ π.

Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

The given function is $f(x) = |x|$ in the Interval $[-\pi, \pi]$. because $f(x) = -x$ for $-\pi < x < 0$ and $f(x) = x$ for $0 \le x < \pi$.

since $f(-x) = |-x| = |x| = f(x)$ the Function is an even function. For an even function.

The fourier series only contains cosine terms, so the coefficients $b_n = 0$ for all n.

~~The fourier series~~

$f(-x) = |-x| = |x| = f(x)$ the function is an even function. The fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

In the Interval $(-\pi < x < \pi)$. where.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{—— ①}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{2\pi}\left[\pi^2 - 0\right]$$

$$= \frac{\cancel{2}\pi^{\cancel{2}}}{\cancel{2}\cancel{\pi}}$$

$$\boxed{a_0 = \pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx.$$

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

let

$u = x$	$\int dv = \int \cos nx\,dx$
$u' = 1$	$v = \frac{\sin nx}{n}$
	$v_1 = -\frac{\cos nx}{n\times n} = -\frac{\cos nx}{n^2}$

$$a_n = \frac{2}{\pi}\left[\frac{x\sin nx}{n} - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi\sin(n\pi)}{n} + \frac{\cos(n\pi)}{n^2}\right) - \left(\frac{0\sin(n.0)}{n} + \frac{\cos(n.0)}{n^2}\right)\right]$$

$\sin(n\pi) = 0, \quad \cos(n\pi) = (-1)^n$

$\sin(0^\circ) = 0, \quad \cos(0^\circ) = 1$

$$a_n = \frac{2}{\pi}\left[\left(\frac{\pi(0)}{n} + \frac{(-1)^n}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

$$\boxed{a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]}$$

substitude a_0, a_n in the equation ①

$$x = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left[(-1)^n - 1\right]}{\pi n^2} \cos(nx)$$

$$x = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos(nx)$$

when n is odd, $\frac{(-1)^n - 1}{n^2} = -\frac{2}{n^2}$, $n = 1, 3, 5, 7, \ldots$

when n is even, $\frac{(-1)^n - 1}{n^2} = 0$, $n = 2, 4, 6, 8, \ldots$

$$x = \frac{\pi}{2} + \frac{2}{\pi}\left[-\frac{2}{1^2}\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \frac{2}{5^2}\cos 5x + 0 - \frac{2}{7^2}\cos 7x + \cdots\right]$$

$$\therefore x = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$$

when $x = 0$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos 0^\circ}{1^2} + \frac{\cos(3 \cdot 0^\circ)}{3^2} + \frac{\cos(5 \cdot 0^\circ)}{5^2} + \cdots\right]$$

$\cos 0° = 1$

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\right)$$

$$\frac{\frac{\pi}{2}}{\frac{4}{\pi}} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

$$\frac{\pi}{2} \cdot \frac{\pi}{4} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\therefore \quad \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

This proves the deduction.

---

§4. Half Range fourier series.

§.5.1 Development in cosine series.

Let $f(x)$ be expanded as a series containing cosines only and let.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{———} \quad (1)$$

If we Integrate both sides of the equation (1) between limits 0 and π, then.

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{a_0}{2}\,dx + \sum_{n=1}^{\infty} \int_0^{\pi} a_n \cos nx\,dx$$

$$\int_0^{\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_0^{\pi} \cos nx\,dx.$$

$$\int_0^{\pi} \cos(mx)\,dx = 0, \text{ if } m \text{ is an Integer.}$$

$$\int_0^\pi f(x)dx = \frac{a_0}{2}[x]_0^\pi + \sum_{n=1} a_n (0)$$

$$\int_0^\pi f(x)dx = \frac{a_0}{2}[\pi - 0] + 0$$

$$\int_0^\pi f(x)dx = \frac{a_0 \pi}{2}$$

$$\therefore a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx.$$

If we Multiply $\cos nx$ in both sides of the equation ① and Integrate between 0 and π, then.

$$\int_0^\pi f(x)\cos nx\,dx = \int_0^\pi \frac{a_0}{2}\cos nx + \sum_{n=1}^{\infty}\int_0^\pi a_n . \cos nx . \cos nx$$

$$\int_0^\pi f(x)\cos nx\,dx = \frac{a_0}{2}\int_0^\pi \cos nx + \sum_{n=1}^{\infty} a_n \int_0^\pi \cos^2(nx)dx$$

$$\int_0^\pi \cos(mx) = 0,\ \text{If } m \text{ is an integer.}$$

$$\int_0^\pi \cos(mx)\cos(nx)dx = \int_0^\pi \cos^2(mx)\,dx, \quad \text{If } m=n$$

$$= \pi/2$$

$$\int_0^\pi f(x)\cos(nx)dx = \frac{a_0}{2}(0) + \sum_{n=1}^{\infty} a_n \pi/2$$

Since all the terms except the term containing a_n vanish.

$$\int_0^\pi f(x)\cos(nx)dx = a_n \pi/2$$

$$\therefore a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$$

§5.2 Development in sine series.

Let $f(x)$ be expanded as a series containing sines only and let.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{—— (1)}$$

Multiply $\sin(nx)$ in both sides of the equation (1) and integrate from 0 to π

$$\int_0^{\pi} f(x)\sin(nx)\,dx = \sum_{n=1}^{\infty} b_n \int_0^{\pi} \sin(nx).\sin(nx).\,dx$$

$$\int_0^{\pi} f(x)\sin(nx)\,dx = \sum_{n=1}^{\infty} b_n \int_0^{\pi} \sin^2(nx)\,dx$$

$$\int_0^{\pi} \sin(mx)\sin(nx)\,dx = \int_0^{\pi} \sin^2(mx)\,dx, \text{ If } m=n$$

$$= \pi/2$$

$$\int_0^{\pi} f(x)\sin(nx)\,dx = \sum_{n=1}^{\infty} b_n \,\pi/2$$

since all the terms except the term containing b_n vanish.

$$\int_0^{\pi} f(x)\sin(nx)\,dx = b_n \,\pi/2$$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

Examples.

Example. 1.

Find a sine series for $f(x) = c$ in the range 0 to π

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

$$f(x) = c$$

$$c = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{--- (1)}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} c \sin(nx)\,dx$$

$$= \frac{2c}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= \frac{2c}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi}$$

$$= \frac{2c}{n\pi}\left[-\cos(nx)\right]_0^{\pi}$$

$$= \frac{2c}{n\pi}\left[-\cos(n\pi) - (-\cos(n.0))\right]$$

$$\cos(n\pi) = (-1)^n, \quad \cos 0^\circ = 1$$

$$= \frac{2c}{n\pi}\left[-(-1)^n - (-1)\right]$$

$$= \frac{2c}{n\pi}\left[-(-1)^n + 1\right]$$

$$\boxed{b_n = \frac{2c}{n\pi}\left[1 - (-1)^n\right]}$$

substitude b_n in The equation (1)

$$c = \sum_{n=1}^{\infty} \frac{2c\left[1-(-1)^n\right]}{n\pi}\sin(nx)$$

when n is even, $b_n = \frac{2c\left[1-(-1)^n\right]}{n\pi} = 0, \quad n = 2,4,6\ldots$

when n is odd, $b_n = \frac{2c\left[1-(-1)^n\right]}{n\pi} = \frac{4c}{n\pi}, \quad n = 1,3,5\ldots$

$$c = \frac{4c}{1\pi}\sin x + 0 + \frac{4c}{3\pi}\sin 3x + 0 + \frac{4c}{5\pi}\sin 5x$$

$$+ 0 + \frac{4c}{7\pi}\sin 7x + \cdots$$

$$\therefore c = \frac{4c}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right]$$

putting $x = \pi/2$

$$c = \frac{4c}{\pi}\left[\sin(\pi/2) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{5}\sin(5\pi/2) + \frac{1}{7}\sin(7\pi/2) + \cdots\right]$$

$$\sin 90^\circ\,(\pi/2);\ \sin 450^\circ\,(5\pi/2), \text{---} = 1$$

$$\sin 270^\circ\,(3\pi/2),\ \sin 630^\circ\,(7\pi/2), \text{---} = -1$$

$$\frac{c}{\frac{4c}{\pi}} = \left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \cdots\right]$$

$$\cancel{c} \times \frac{\pi}{4\cancel{c}} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Example . 2

$$\text{If } f(x) = \begin{cases} x, & \text{when } 0 < x < \pi/2 \\ \pi - x, & \text{when } x > \pi/2 \end{cases}$$

Expand $f(x)$ as a sine series in the interval $(0, \pi)$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$

$$f(x) = x, \text{ when } 0 < x < \pi/2$$
$$= \pi - x, \text{ when } x > \pi/2$$

$$\cancel{x}+\pi-\cancel{x} = \sum_{n=1} b_n \sin(nx)$$

$$\pi = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \text{——— (1)}$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\sin(nx)\,dx + \int_{\pi/2}^{\pi} f(x)\sin(nx)\,dx\right]$$

$$b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x\sin(nx)\,dx + \int_{\pi/2}^{\pi}(\pi-x)\sin(nx)\,dx\right] \quad \text{——— (2)}$$

For The first Integral.

$$\int_0^{\pi/2} x\sin(nx)\,dx.$$

Bernoulli's formula.

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

Let $u = x$, $u' = 1$ | $\int dv = \int \sin(nx)\,dx$, $v = -\frac{\cos(nx)}{n}$, $v_1 = -\frac{\sin(nx)}{n\times n} = -\frac{\sin(nx)}{n^2}$

$$\int_0^{\pi/2} x\sin(nx)\,dx = \left[x\left(-\frac{\cos(nx)}{n}\right) - (1)\left(-\frac{\sin(nx)}{n^2}\right)\right]_0^{\pi/2}$$

$$= \left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{\pi/2}$$

$$= \left[\left(-\frac{\pi/2\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}\right) - \left(-\frac{0\cos(n.0)}{n} + \frac{\sin(n.0)}{n^2}\right)\right]$$

$\cos(n\pi/2) = 0$, $\cancel{\sin(n\pi/2) = (-1)^n = 1}$

$\cos 0^\circ = 1$, $\sin(0^\circ) = 0$

$$\int_0^{\pi/2} x\sin(nx)\,dx = \left[\left(-0 + \frac{1}{n^2}\overset{\sin(n\pi/2)}{}\right) - (0+0)\right]$$

$$= \frac{1}{n^2}\sin\left(\frac{n\pi}{2}\right) \text{ --- } (3)$$

For the second Integral.

$$\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,dx$$

Bernoulli's formula

Let $u = \pi - x$, $u' = 0 - 1 = -1$

$\int dv = \int \sin(nx)\,dx$

$v = -\frac{\cos(nx)}{n}$

$v_1 = -\frac{\sin(nx)}{n \times n} = -\frac{\sin(nx)}{n^2}$

$$\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,dx = \left[(\pi - x)\left(-\frac{\cos(nx)}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \left[-\frac{(\pi - x)\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \left[-\frac{\pi\cos(nx)}{n} + \frac{x\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \left[\left(-\frac{\pi\cos(n\pi)}{n} + \frac{\pi\cos(n\pi)}{n} - \frac{\sin(n\pi)}{n^2}\right) - \left(-\frac{\pi\cos(n\pi/2)}{n} + \frac{\pi/2\cos(n\pi/2)}{n} - \frac{\sin(n\pi/2)}{n^2}\right)\right]$$

$\cos(n\pi) = (-1)^n$, $\sin(n\pi) = 0$

$\cos(n\pi/2) = 0$, $\cancel{\sin(n\pi/2) = (-1)^n = 1}$

$$= \left[\left(-\cancel{\frac{\pi(-1)^n}{n}} + \cancel{\frac{\pi(-1)^n}{n}} - 0\right) - \left(0 + 0 - \frac{1}{n^2}\overset{\sin(n\pi/2)}{}\right)\right]$$

$$\int_{\pi/2}^{\pi} (\pi - x)\sin(nx)\,dx = \frac{1}{n^2}\sin(n\pi/2) \quad (4)$$

substitude equation (3) and (4) in equation (2)

$$b_n = \frac{2}{\pi}\left[\frac{1}{n^2} + \frac{1}{n^2}\right]\sin(n\pi/2)$$

$$= \frac{2}{\pi}\left(\frac{1+1}{n^2}\right)\sin(n\pi/2)$$

$$= \frac{2}{\pi}\left(\frac{2}{n^2}\right)\sin(n\pi/2)$$

$$\boxed{b_n = \frac{4}{\pi n^2}\sin(n\pi/2)}$$

substitude b_n in the equation (1)

$$\pi = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \xrightarrow{\sin(n\pi/2)} \sin(nx)$$

$$\pi = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

$$\pi = \frac{4}{\pi}\sin(n\pi/2) \sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

The value of $\frac{4}{\pi}\sin(n\pi/2)$ depends on n:

when n is odd, $\frac{4}{\pi}\sin(n\pi/2) = \frac{4}{\pi}$ $[n = 1, 5, 9, \ldots]$

, $\frac{4}{\pi}\sin(n\pi/2) = -\frac{4}{\pi}$, $[n = 3, 7, 11, \ldots]$

when n is even, $\frac{4}{\pi}\sin\left(\frac{n\pi}{2}\right) = 0$, $[n = 2, 4, 6, 8, \ldots]$

even $\Rightarrow$ $[\sin\pi, \sin 2\pi, \sin 3\pi, \sin 4\pi, \ldots = 0]$

odd $\Rightarrow$ $[\sin \pi/2, \sin 5\pi/2, \sin 9\pi/2, \ldots = 1]$

$\Rightarrow$ $[\sin 3\pi/2, \sin 7\pi/2, \sin \frac{11\pi}{2}, \ldots = -1]$

$$\pi = \frac{4}{\pi}\sin\left(\frac{\pi}{2}\right)\frac{\sin x}{1^2} + \frac{4}{\pi}\sin\left(\frac{2\pi}{2}\right)\frac{\sin 2x}{2^2} +$$

$$\frac{4}{\pi}\sin\left(\frac{3\pi}{2}\right)\frac{\sin 3x}{3^2} + \frac{4}{\pi}\sin\left(\frac{4\pi}{2}\right)\frac{\sin 4x}{4^2} +$$

$$\frac{4}{\pi}\sin\left(\frac{5\pi}{2}\right)\frac{\sin 5x}{5^2} + \frac{4}{\pi}\sin\left(\frac{6\pi}{2}\right)\frac{\sin 6x}{6^2} +$$

$$\frac{4}{\pi}\sin\left(\frac{7\pi}{2}\right)\frac{\sin 7x}{7^2} + \cdots$$

$$\pi = \frac{4}{\pi}(1)\frac{\sin x}{1^2} + 0 + \frac{4}{\pi}(-1)\frac{\sin 3x}{3^2} + 0$$

$$+ \frac{4}{\pi}(1)\frac{\sin 5x}{5^2} + 0 + \frac{4}{\pi}(-1)\frac{\sin 7x}{7^2} + \cdots$$

$$\pi = \frac{4}{\pi}\frac{\sin x}{1^2} - \frac{4}{\pi}\frac{\sin 3x}{3^2} + \frac{4}{\pi}\frac{\sin 5x}{5^2} -$$

$$\frac{4}{\pi}\frac{\sin 7x}{7^2} + \cdots$$

$$\pi = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \frac{\sin 7x}{7^2} + \cdots\right]$$

$$f(x) = x + \pi - x = \pi$$

$$\therefore f(x) = \frac{4}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \cdots\right]$$

Example. 3

Find a cosine series in the range 0 to π for

$$f(x) = x, \quad \left(0 < x < \frac{\pi}{2}\right)$$

$$= \pi - x, \quad \left(\frac{\pi}{2} < x < \pi\right)$$

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \quad \text{—— (1)}$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x\,dx + \frac{2}{\pi}\left[\pi\int_{\frac{\pi}{2}}^{\pi} dx - \int_{\pi/2}^{\pi} x\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2} + \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}$$

$$= \frac{2}{2\pi}\left[\left(\frac{\pi}{2}\right)^2 - 0^2\right] + \frac{2}{\pi}\left[\left(\pi(\pi) - \frac{\pi^2}{2}\right) - \left(\pi\left(\frac{\pi}{2}\right) - \frac{\left(\frac{\pi}{2}\right)^2}{2}\right)\right]$$

$$= \frac{2}{2\pi}\left(\frac{\pi^2}{4}\right) + \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - \left(\frac{\pi^2}{2} - \frac{\pi^2}{4\times 2}\right)\right]$$

$$= \frac{2\pi^2}{8\pi} + \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{2\pi^2}{8\pi} + \frac{2}{\pi}\left[\pi^2 - \frac{\cancel{2}\pi^2}{\cancel{2}} + \frac{\pi^2}{8}\right]$$

$$= \frac{2\pi^2}{8\pi} + \frac{2}{\pi}\left[\cancel{\pi^2} - \cancel{\pi^2} + \frac{\pi^2}{8}\right]$$

$$= \frac{2\pi^2}{8\pi} + \frac{2\pi^2}{8\pi}$$

$$= \frac{\cancel{4}^{1}\pi^{\cancel{2}1}}{\cancel{8}_{2}\cancel{\pi}}$$

$$\boxed{a_0 = \frac{\pi}{2}}$$

$$a_n = \frac{2}{\pi}\int_0 f(x)\cos nx\,dx.$$

$$a_n = \frac{2}{\pi}\left[\int_0^{\frac{\pi}{2}} x\cos(nx)\,dx + \int_{\pi/2}^{\pi}(\pi - x)\cos(nx)\,dx\right] \quad \text{——— (2)}$$

For The First Integral.

$$\int_0^{\pi/2} x\cos(nx)\,dx.$$

Bernoulli's formula

$$\int u\,dv = uv - u'v_1 + u''v_2 - \cdots$$

Let

$u = x$

$u' = 1$

$\int dv = \int \cos(nx)\,dx.$

$v = \frac{\sin(nx)}{n}$

$v_1 = -\frac{\cos(nx)}{n\times n} = -\frac{\cos(nx)}{n^2}$

$$\int_0^{\pi/2} x\cos(nx)\,dx = \left[\frac{x\sin(nx)}{n} - (1)\left(-\frac{\cos(nx)}{n^2}\right)\right]_0^{\pi/2}$$

$$= \left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi/2}$$

$$= \left[\left(\frac{\pi\sin(n\pi/2)}{2n} + \frac{\cos(n\pi/2)}{n^2}\right) - \left(\frac{0\sin(n.0)}{n} + \frac{\cos(n.0)}{n^2}\right)\right]$$

$\sin(n\pi/2) = (1)^n = 1$

$\sin(0^\circ) = 0 \quad , \quad \cos 0^\circ = 1$

$$\int_0^{\pi/2} x\cos(nx)\,dx = \frac{\pi}{2n} + \frac{1}{n^2}\cos(n\pi/2) - \frac{1}{n^2} \quad \text{——— (B)}$$

For the second integral:

$$\int_{\pi/2}^{\pi} (\pi - x)\cos(nx)\,dx.$$

Bernoulli's formula.

Let

$u = \pi - x$

$u' = 0 - 1$

$= -1$

$\int dv = \int \cos(nx)\,dx$

$v = \frac{\sin(nx)}{n}$

$v_1 = -\frac{\cos(nx)}{n \times n} = -\frac{\cos(nx)}{n^2}$

$$\int_{\pi/2}^{\pi} (\pi - x)\cos(nx)\,dx = \left[\frac{(\pi - x)\sin(nx)}{n} - (-1)\left(-\frac{\cos(nx)}{n^2}\right)\right]_{\pi/2}^{\pi}$$

$$= \left[\frac{\pi \sin(nx)}{n} - \frac{x\sin(nx)}{n} - \frac{\cos(nx)}{n^2}\right]_{\pi/2}^{\pi}$$

$$= \left[\left(\frac{\cancel{\pi \sin(n\pi)}}{n} - \frac{\cancel{\pi \sin(n\pi)}}{n} - \frac{\cos(n\pi)}{n^2}\right) - \left(\frac{\pi \sin(n\pi/2)}{n} - \frac{\frac{\pi}{2}\sin(n\pi/2)}{n} - \frac{1}{n^2}\cos(n\pi/2)\right)\right]$$

$\sin(n\pi/2) = (1)^n = 1$.

$\cos(n\pi) = (-1)^n$.

$$= \left[-\frac{1}{n^2}\cos(n\pi) - \left(\frac{\pi}{n} - \frac{\pi}{2n} - \frac{1}{n^2}\cos(n\pi/2)\right)\right]$$

$$= -\frac{1}{n^2}(-1)^n - \frac{\pi}{n} + \frac{\pi}{2n} + \frac{1}{n^2}\cos(n\pi/2)$$

$$= -\frac{(-1)^n}{n^2} - \frac{2\pi - \pi}{2n} + \frac{1}{n^2}\cos(n\pi/2)$$

$$\int_{\pi/2}^{\pi} (\pi - x)\cos(nx)\,dx = -\frac{\pi}{2n} + \frac{1}{n^2}\cos(n\pi/2) - \frac{(-1)^n}{n^2} \quad \text{(ii)}$$

substitude equation ③ and ④ in equation ②.

$$a_n = \frac{2}{\pi}\left[\cancel{\frac{\pi}{2n}} + \frac{1}{n^2}\cos(n\pi/_2) - \frac{1}{n^2} - \cancel{\frac{\pi}{2n}} + \right.$$

$$\left. \frac{1}{n^2}\cos(n\pi/_2) - \frac{(-1)^n}{n^2}\right]$$

$$= \frac{2}{\pi}\left[-\frac{1}{n^2} - \frac{(-1)^n}{n^2} + \frac{2\cos(n\pi/_2)}{n^2}\right]$$

$$\boxed{a_n = \frac{2}{\pi}\left[\frac{-1-(-1)^n+2\cos(n\pi/_2)}{n^2}\right]}$$

substitude a_0, a_n in the equation ①

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-1-(-1)^n+2\cos(n\pi/_2)}{n^2}\cos(nx)$$

- when. n is odd, $a_n = \dfrac{-1-(-1)^n+2\cos(\frac{n\pi}{2})}{n^2} = 0, \; n = 1,3,5,7,\ldots$

$$\cos\left(\frac{\pi}{2}\right), \cos\left(\frac{3\pi}{2}\right), \cos\left(\frac{5\pi}{2}\right), \cos\left(\frac{7\pi}{2}\right), \ldots\ldots = 0$$

- when n is even, $a_n = \dfrac{-1-(-1)^n+2\cos(n\pi/_2)}{n^2} = 0, \; n = 4,8,12,\ldots$

$$\cos(2\pi), \cos(4\pi), \cos(6\pi), \ldots\ldots = 1$$

$$a_n = \frac{-1-(-1)^n+2\cos(n\pi/_2)}{n^2} = -\frac{4}{n^2}$$

$n = 2, 6, 10, \ldots\ldots$

$$\cos(\pi), \cos(3\pi), \cos(5\pi), \ldots\ldots = -1$$

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[0 - \frac{4}{2^2}\cos 2x + 0 + 0 + 0 - \frac{4}{6^2}\right.$$

$$\cos 6x + 0 + 0 + 0 - \frac{4}{10^2}\cos 10x$$

$$\left. + \ldots\ldots \right]$$

$$f(x) = \frac{\pi}{4} + \frac{2}{\pi}\left[-\frac{4}{2^2}\cos 2x - \frac{4}{2^2 \times 3^2}\cos 6x - \frac{4}{2^2 \times 5^2}\cos 10x - \cdots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{4}{2^2}\right)\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$$

$$= \frac{\pi}{4} - \frac{8}{4\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$$

$$\therefore f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$$

Exercise 19.

(8) If $\int_0^{\pi/2} \cos^m x \cos nx\,dx = f(m,n)$, prove that

$f(m,n) = \frac{m}{m+n} f(m-1,n-1)$, Hence prove

That $f(m,m) = \frac{\pi}{2^{n+1}}$

Solution.

$$\int_0^{\pi/2} \cos^m x \cos nx\,dx.$$

$$D\left(\frac{\sin nx}{n}\right) = \cos nx\,dx$$

$$\int_0^{\pi/2} \cos^m x \cos nx\,dx = \int_0^{\pi/2} \overset{u}{\cos^m x}\, D\left(\overset{v}{\frac{\sin nx}{n}}\right)$$

Integration By parts.

$$\int u\,dv = uv - \int v\,du.$$

let.

$u = \cos^m x$

$\frac{du}{dx} = m\cos^{m-1}x\,(-\sin x)$

$du = -m\cos^{m-1}x\,\sin x.$

$dv = D\left(\frac{\sin nx}{n}\right)$

$v = \frac{\sin nx}{n}$

$$\int_0^{\pi/2} \cos^m x \cos nx\,dx = \left[\frac{\cos^m x \sin nx}{n}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin nx}{n}\left(-m\cos^{m-1}x\,\sin x\right)dx$$

$$\int_0^{\pi/2} \cos^m x \cos nx \, dx = \frac{1}{n}\Big[\cos^m x \sin nx\Big]_0^{\pi/2} +$$

$$\frac{m}{n}\int_0^{\pi/2} \sin nx \cos^{m-1} x \sin x \, dx$$

$$= \frac{1}{n}\Big[\cos^m \tfrac{\pi}{2} \cdot \sin(n\tfrac{\pi}{2}) - \cos^m 0 \cdot \sin(n$$

$$+ \frac{m}{n}\int_0^{\pi/2} \cos^{m-1} x \sin nx \sin x \, dx.$$