Economic Growth and Sustainable Development

The growth model with an exogenous saving rate (Solow-Swan)

Federico Di Pace

ENSAI

January – March 2025

Long-term growth

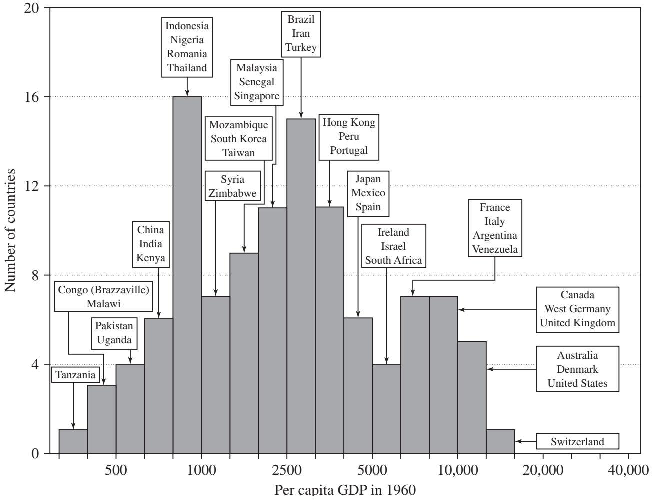
- "**Growth**": growth of the per-capita Gross Domestic Product (GDP)
- Growth is a relatively recent phenomenon:

Year	1500	1820	1992
World population (millions)	425	1068	5441
Per-capita world GDP ($ of 1990)	565	651	5145

Source: Maddison (1995)

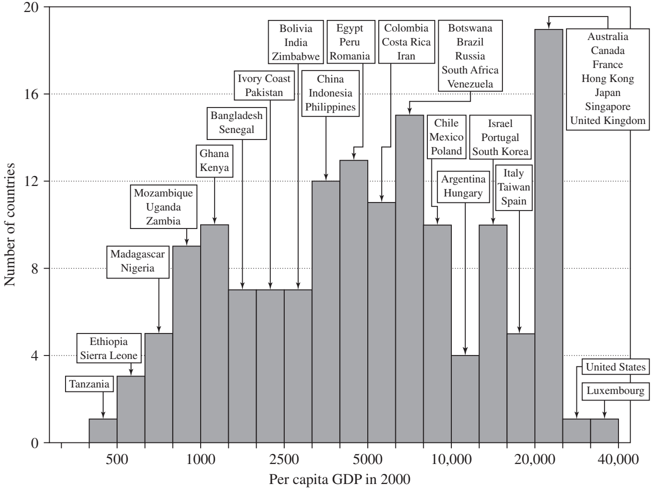
- The average annual world-GDP growth rate is
  - 0,04% from 1500 to 1820,
  - 1,21% from 1820 to 1992

Dispersion of per-capita GDPs across countries in 1960



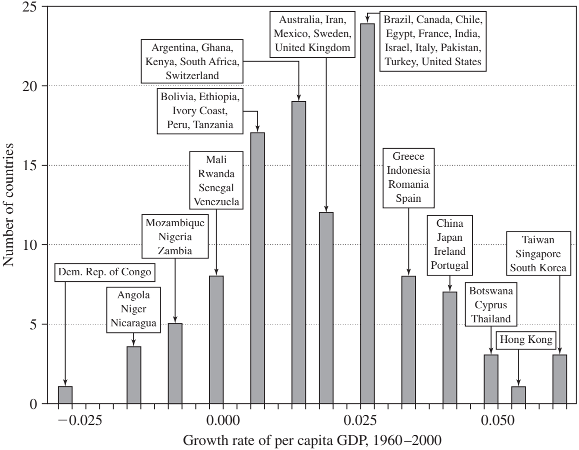
Source: Barro and Sala-i-Martin (2004). Per-capita GDP expressed in $ of 1996

Dispersion of per-capita GDPs across countries in 2000



Source: Barro and Sala-i-Martin (2004). Per-capita GDP expressed in $ of 1996

Dispersion of growth rates across countries, 1960-2000



Source: Barro and Sala-i-Martin (2004). "Growth rate of per capita GDP, 1960-2000": average annual growth rate of per-capita GDP from 1960 to 2000 (e.g., 0.02 = 2% per year)

Questions

- Main questions addressed in the course:
  - how do we explain long-term growth?
  - how to explain this dispersion of per-capita GDPs and of growth rates across countries?
  - what economic policy to conduct in order to “optimize” long-term growth?
- Questions that can be judged more important, for human welfare, than questions about short-term macroeconomics fluctuations (Lucas, 2003)

Growth theories

- "**Exogenous-growth theory (resp. endogenous-growth theory)**" $\equiv$ theory in which the long-term growth rate is equal (resp. is not equal) to an exogenous technical progress

- Exogenous-growth theories:
  - the model with an exogenous saving rate (studied in Chapter 1)
  - the model with an endogenous saving rate (studied in Chapter 2)
  - endogenous saving rate and climate (studied in Chapter 4)

- Endogenous-growth theories:
  - the model with learning by doing (Chapter 3)
  - if time allows, the model with product variety (Chapter 5)
  - the Schumpeterian model (not covered in the course)

- **Joseph A. Schumpeter**: Austrian economist, born in 1883 in Triesch, deceased in 1950 in Salisbury, professor at Harvard University from 1927 to 1950.

Solow-Swan model

- The model with an exogenous saving rate, built independently by Solow (1956) and Swan (1956), is called the "**Solow-Swan model**"

- **Robert M. Solow**: American economist, born in 1924 in New York, professor at MIT since 1950, laureate of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 1987 "*for his contributions to the theory of economic growth*"

- **Trevor W. Swan**: Australian economist, born in 1918 in Sydney, deceased in 1989, professor at the Australian National University from 1950 to 1983

- This model is not micro-founded, unlike the other models studied in the course, but it is nonetheless studied in Chapter 1 because
  - it remains a very useful benchmark to understand economic growth
  - it serves to introduce some concepts used in the other models

Stocks and flows

- In continuous time,
  - a **stock** is a variable that has a meaning only at a given time
  - a **flow** is a variable that has a meaning only over an arbitrarily short period
- For instance, capital K_t is a stock, investment I_t is a flow:
  - at time t, capital is K_t
  - from time t to time $t + dt$, where $dt \to 0^+$, investment is $I_t dt$
- The derivative of a stock with respect to time is a flow
- For instance, absent capital depreciation,

$$\dot{K}_t \equiv \lim_{dt \to 0^+} \frac{K_{t+dt} - K_t}{dt} = I_t.$$

- Unlike flows, stocks are necessarily continuous functions of time (except in the presence of particular shocks like "earthquake shocks")

Instantaneous growth rate of a stock or a flow

- Let X_t denote a stock or a flow, and dt a duration arbitrarily close to 0.

- From time t to time $t + dt$, the growth rate of X_t is

$$\frac{X_{t+dt} - X_t}{X_t}$$

- Per unit of time, this growth rate is

$$\frac{X_{t+dt} - X_t}{X_t dt}$$

- At time t, the **instantaneous growth rate** of X_t is

$$\lim_{dt \to 0^+} \frac{X_{t+dt} - X_t}{X_t dt} = \frac{\dot{X}_t}{X_t}$$

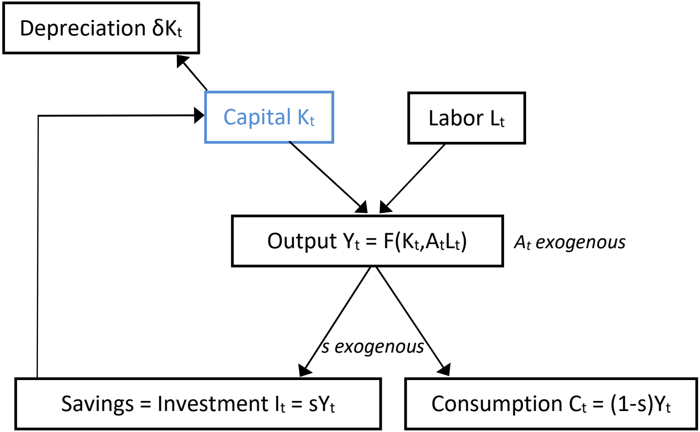
Chapter outline

Presentation of the model

General overview of the model I

- **Capital** (stock) and **labor** (flow) are used to produce **goods** (flow)
- **Goods** (flow) are used for **consumption** (flow) and **investment** in new capital (flow)
- The **saving rate** (quantity of non-consumed, or saved, or invested goods / total quantity of produced goods) is **exogenous**
- **Capital** (stock) evolves over time due to **investment** (flow) and capital **depreciation** (flow)

General overview of the model II



(In blue: stock; in black: flow)

Exogenous variables

- **Neither flows nor stocks**:
  - continuous time, indexed by t,
  - saving rate s, such that $0 < s < 1$
- **Flow**:
  - labor $= 1$ per person
- **Stocks**:
  - initial capital $K_0 > 0$,
  - population $L_t = L_0 e^{nt}$, where $L_0 > 0$ and $n \geq 0$,
  - productivity parameter $A_t = A_0 e^{gt}$, where $A_0 > 0$ and $g \geq 0$

Endogenous variables

- **Flows**:
  - production Y_t,
  - consumption C_t
- **Stock**:
  - capital K_t (except at $t = 0$)
- Solving the model $\equiv$ getting each endogenous variable as a function of only exogenous variables

Production function I

- **Production function** F: $Y_t = F(K_t, A_t L_t)$ (technological progress increasing labor's efficiency, called "Harrod-neutral" technological progress)
- **Roy F. Harrod**: English economist, born in 1900 in London, deceased in 1978 in Holt, professor at Oxford University from 1923 to 1967
- Denoting by F_j the first derivative of F and $F_{j,j}$ its second derivative with respect to its j^{th} argument for $j \in \{1, 2\}$, we make the following assumptions on F:

1. F: $\mathbb{R}^{+2} \to \mathbb{R}^{+}$, $(x, y) \mapsto F(x, y)$; $\forall (x, y) \in \mathbb{R}^{+2}$, $F(x, 0) = F(0, y) = 0$
2. F is **strictly increasing** in each of its arguments: $\forall (x, y) \in \mathbb{R}^{+2}$, $F_1(x, y) > 0$ and $F_2(x, y) > 0$ (the marginal productivities of capital and effective labor are strictly positive)

Production function II

3. F is **strictly concave** in each of its arguments: $\forall (x, y) \in \mathbb{R}^{+2}$, $F_{1,1}(x, y) < 0$ and $F_{2,2}(x, y) < 0$ (the marginal productivities of capital and effective labor are strictly decreasing)

4. F is **homogeneous of degree 1** (or "constant returns to scale"): $\forall (x, y, \lambda) \in \mathbb{R}^{+3}$, $F(\lambda x, \lambda y) = \lambda F(x, y)$.

5. F satisfies the **Inada conditions** (Inada, 1963):

$$\begin{aligned} \forall y &\in \mathbb{R}^{+}, \lim_{x \to 0^{+}} F_1(x, y) = +\infty \text{ and } \lim_{x \to +\infty} F_1(x, y) = 0, \\ \forall x &\in \mathbb{R}^{+}, \lim_{y \to 0^{+}} F_2(x, y) = +\infty \text{ and } \lim_{y \to +\infty} F_2(x, y) = 0. \end{aligned}$$

- **Example** of function satisfying these assumptions: Cobb-Douglas function $F(x, y) = x^{\alpha} y^{1-\alpha}$ with $0 < \alpha < 1$

Re-writing the production function

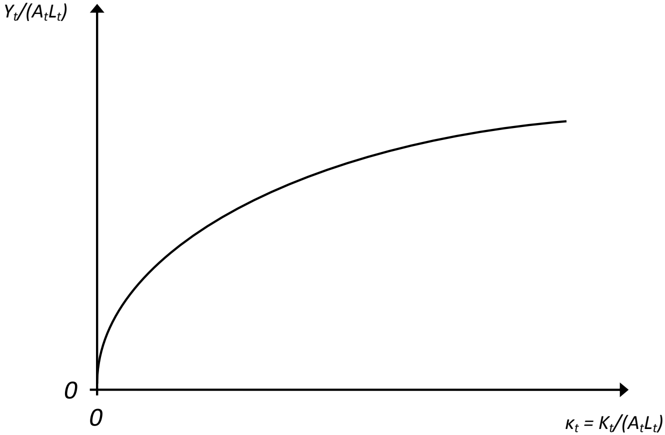
- Denoting by $\kappa_t \equiv \frac{K_t}{A_t L_t}$ the stock of capital per effective-labor unit, we get

$$\frac{Y_t}{A_t L_t} = \frac{1}{A_t L_t} F(K_t, A_t L_t) = F(\kappa_t, 1) \equiv f(\kappa_t)$$

where f has the following properties:

1. f: $\mathbb{R}^+ \to \mathbb{R}^+$, $z \mapsto f(z)$, with $f(0) = 0$,
2. f is **strictly increasing**: $\forall z \in \mathbb{R}^+$, $f'(z) > 0$,
3. f is **strictly concave**: $\forall z \in \mathbb{R}^+$, $f''(z) < 0$,
4. f satisfies the **Inada conditions**: $\lim_{z \to 0^+} f'(z) = +\infty$ and $\lim_{z \to +\infty} f'(z) = 0$

Shape of the production function f



Other production functions

- Other production functions, which do not necessarily satisfy the same conditions:
  - $Y_t = K_t^{\alpha} H_t^{\beta} (A_t L_t)^{1-\alpha-\beta}$, where H_t represents human capital,
  - $Y_t = K_t^{\alpha} R^{\beta} (A_t L_t)^{1-\alpha-\beta}$, where R represents a stock of natural resources in fixed quantity (like land),

  with $\alpha > 0$, $\beta > 0$ and $\alpha + \beta < 1$

Assumptions on capital dynamics

1. From t to $t + dt$, an exogenous and constant fraction s of output $Y_t dt$ is saved and invested in new capital, with $0 < s < 1$
2. From t to $t + dt$, an exogenous and constant fraction δdt of the capital stock K_t disappears because of capital depreciation, with $\delta > 0$

$\hookrightarrow$ The capital-stock dynamics is thus governed by the equation

$$\dot{K}_t = \underbrace{sY_t}_{\text{savings}} - \underbrace{\delta K_t}_{\text{depreciation}}$$

Resolution

Differential equation

- Dividing $\dot{K}_t = sY_t - \delta K_t$ by $A_t L_t$ and using $\kappa_t \equiv K_t/(A_t L_t)$ and $Y_t/(A_t L_t) = f(\kappa_t)$, we get

$$\frac{\dot{K}_t}{K_t}\kappa_t = sf(\kappa_t) - \delta\kappa_t$$

- Then, using

$$\frac{\dot{K}_t}{K_t} = \ln \dot{K}_t = \ln \dot{\kappa}_t + \ln \dot{A}_t + \ln \dot{L}_t = \frac{\dot{\kappa}_t}{\kappa_t} + \frac{\dot{A}_t}{A_t} + \frac{\dot{L}_t}{L_t} = \frac{\dot{\kappa}_t}{\kappa_t} + g + n,$$

we get the **differential equation**

$$\dot{\kappa}_t = \underbrace{sf(\kappa_t)}_{\text{savings}} - \underbrace{(n+g+\delta)\kappa_t}_{\substack{\text{dilution plus}\\ \text{depreciation}}}$$

to be solved for a given κ_0

Steady state I

- **Steady state** (or stationary growth path, or balanced-growth path) $\equiv$ situation in which κ_0 is such that all quantities are non-zero and grow at constant rates

- Dividing $\dot{\kappa}_t = sf(\kappa_t) - (n + g + \delta)\kappa_t$ by κ_t, we get that

$$\frac{\dot{\kappa}_t}{\kappa_t} \text{ is constant over time } \Rightarrow \frac{f(\kappa_t)}{\kappa_t} \text{ is constant over time}$$

- We show in the appendix that the function $z \mapsto f(z)/z$ is strictly decreasing

Steady state II

- The function $z \mapsto f(z)/z$ is therefore bijective, which implies that

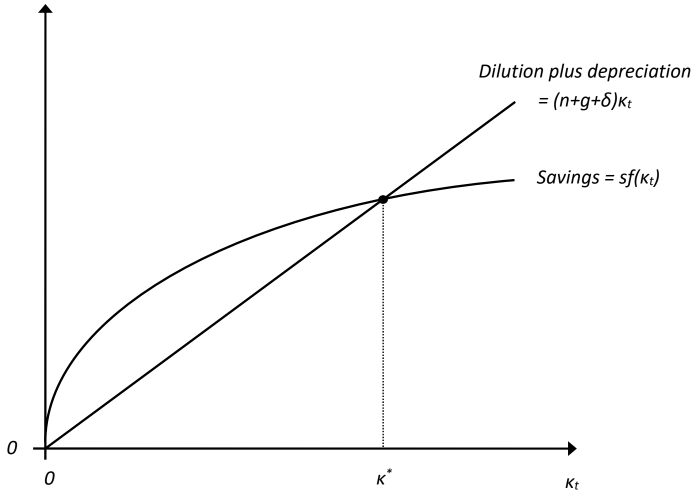
$$\frac{f(\kappa_t)}{\kappa_t} \text{ is constant over time } \Rightarrow \kappa_t \text{ is constant over time}$$

- As a consequence, at the steady state, κ_t is constant over time

- Replacing $\dot{\kappa}_t$ with 0 in the differential equation and using the bijectivity of $z \mapsto f(z)/z$, we get that κ_t at the steady state is equal to the unique value $\kappa^* > 0$ such that

$$sf(\kappa^*) = (n + g + \delta)\kappa^*$$

Steady state III



Steady state IV

- Differentiating $sf(\kappa^*) = (n+g+\delta)\,\kappa^*$ with respect to s, n, g or δ, and using $sf'(\kappa^*) < n+g+\delta$, we get that κ^* **is**
  - **increasing in s,**
  - **decreasing in n, g, δ,**

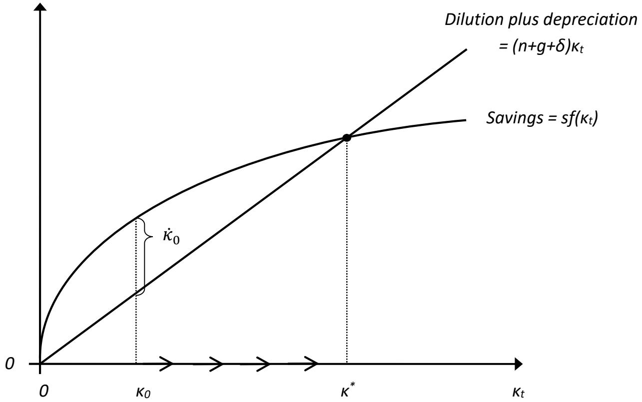
  as the previous figure illustrates

- If $F(x,y) = x^\alpha y^{1-\alpha}$ with $0 < \alpha < 1$ ($\equiv$ "Cobb-Douglas case"), then we have $f(z) = z^\alpha$ and hence

$$\kappa^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

Convergence to the steady state

- Graphical representation of $\dot{\kappa}_t = sf(\kappa_t) - (n+g+\delta)\kappa_t$:



- κ_t therefore converges to κ^*

Interpretation of the convergence to the steady state I

(In italics: per effective-labor unit)

marginal productivity of capital $F_1(K_t, A_t L_t)$ decreases
from $+\infty$ (when $K_t \to 0$) to 0 (when $K_t \to +\infty$)

$\Downarrow$

marginal productivity of capital $f'(\kappa_t)$ decreases
from $+\infty$ (when $\kappa_t \to 0$) to 0 (when $\kappa_t \to +\infty$)

$\Downarrow$

average productivity of capital $\frac{f(\kappa_t)}{\kappa_t}$ decreases
from $+\infty$ (when $\kappa_t \to 0$) to 0 (when $\kappa_t \to +\infty$)

$\Downarrow$

$\vdots$

Interpretation of the convergence to the steady state II

$$\Downarrow$$

ratio $\frac{\textit{savings } sf(\kappa_t)}{\textit{dilution plus depreciation } (n+g+\delta)\kappa_t}$ decreases from $+\infty$ (when $\kappa_t \to 0$) to 0 (when $\kappa_t \to +\infty$)

$$\Downarrow$$

$$\textit{savings } sf(\kappa_t) \gtreqless \textit{dilution plus depreciation } (n+g+\delta)\,\kappa_t$$
$$\text{when } \kappa_t \lesseqgtr \kappa^*$$

$$\Downarrow$$

$$\dot{\kappa}_t \gtreqless 0 \text{ when } \kappa_t \lesseqgtr \kappa^*$$

The convergence of κ_t to κ^* is thus due the decreasing nature of capital productivity

Resolution in the Cobb-Douglas case I

- If $F(x,y) = x^{\alpha} y^{1-\alpha}$ with $0 < \alpha < 1$, then the differential equation becomes

$$\dot{\kappa}_t = s\kappa_t^{\alpha} - (n+g+\delta)\,\kappa_t.$$

- Using $u_t \equiv \kappa_t^{1-\alpha}$, we get $\dot{u}_t = (1-\alpha)\,\kappa_t^{-\alpha}\dot{\kappa}_t$ and the differential equation can thus be rewritten as

$$\frac{\dot{u}_t}{s - (n+g+\delta)\,u_t} = 1-\alpha$$

Resolution in the Cobb-Douglas case II

- Integrating this last equation, we get

$$\frac{-1}{n+g+\delta}\ln\left[\frac{s-(n+g+\delta)\,u_t}{s-(n+g+\delta)\,u_0}\right]=(1-\alpha)\,t$$

and then

$$u_t=\frac{s-\left[s-(n+g+\delta)\,u_0\right]e^{-(n+g+\delta)(1-\alpha)t}}{n+g+\delta}.$$

- Using $\kappa_t = u_t^{\frac{1}{1-\alpha}}$ and the expression of κ^*, we then get

$$\kappa_t=\left\{\left(\kappa^*\right)^{1-\alpha}-\left[\left(\kappa^*\right)^{1-\alpha}-\kappa_0^{1-\alpha}\right]e^{-(n+g+\delta)(1-\alpha)t}\right\}^{\frac{1}{1-\alpha}},$$

which says that $\kappa_t^{1-\alpha}$ converges **exponentially**, at the rate $(n+g+\delta)(1-\alpha)$, to its steady-state value $\left(\kappa^*\right)^{1-\alpha}$.

Resolution in the Cobb-Douglas case III

- Let $y_t \equiv \frac{Y_t}{L_t}$ denote output per labor unit, which corresponds to per-capita GDP

- Using $y_t = A_t \kappa_t^{\alpha}$, we get

$$y_t = \left\{ (\kappa^*)^{1-\alpha} - \left[(\kappa^*)^{1-\alpha} - \kappa_0^{1-\alpha} \right] e^{-(n+g+\delta)(1-\alpha)t} \right\}^{\frac{\alpha}{1-\alpha}} A_0 e^{gt}$$

Positive implications

Long-term growth

- Let $G_t \equiv \frac{\dot{y}_t}{y_t}$ denote the growth rate of per-capita output

- We have $y_t = A_t f(\kappa_t)$, so

$$G_t = \ln \dot{y}_t = \ln \dot{A}_t + \ln \dot{f}(\kappa_t) = \frac{\dot{A}_t}{A_t} + \frac{f'(\kappa_t)\dot{\kappa}_t}{f(\kappa_t)} = g + \frac{f'(\kappa_t)\dot{\kappa}_t}{f(\kappa_t)}$$

- Since $\lim_{t \to +\infty} \frac{f'(\kappa_t)\dot{\kappa}_t}{f(\kappa_t)} = 0$, we get

$$\lim_{t \to +\infty} G_t = g,$$

that is to say that **the long-term growth rate is equal to the rate of technological progress**

The two sources of growth

- Let $k_t \equiv \frac{K_t}{L_t}$ denote the per-capita capital stock
- We have $y_t = F(k_t, A_t)$, so the two potential sources of growth of per-capita output y_t are
  - the increase in the per-capita capital stock k_t,
  - technological progress, that is to say the increase in productivity A_t
- In the short term, growth can come from these two factors
- In the long term, growth can come only from the second factor: without technological progress ($g = 0$), $k_t \to A_0\kappa^*$ when $t \to +\infty$, and there is no long-term growth

Long-term path of $\ln(y_t)$

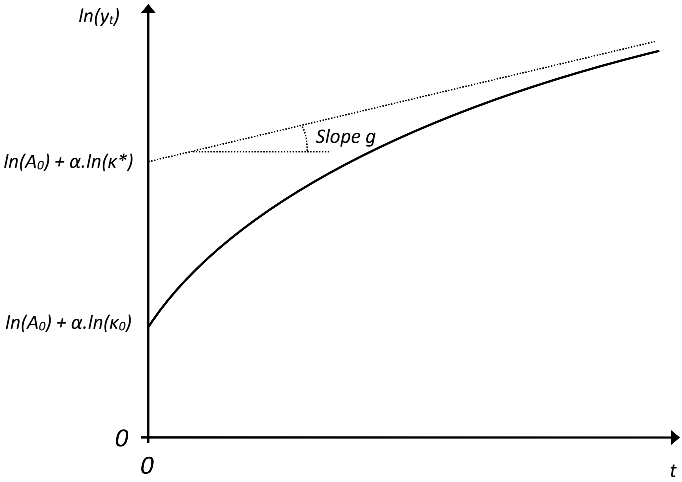
- Let $y_t^* \equiv A_t f(\kappa^*)$ denote the steady-state value of y_t.

- The path of $\ln(y_t) = \ln(A_0) + \ln[f(\kappa_t)] + gt$ has for asymptote, as $t \to +\infty$, the path of $\ln(y_t^*) = \ln(A_0) + \ln[f(\kappa^*)] + gt$, in the sense that

$$\lim_{t \to +\infty} [\ln(y_t) - \ln(y_t^*)] = 0.$$

- Therefore, the long-term path of $\ln(y_t)$ is a straight line that has
    - a y-intercept which depends positively on A_0, s,
    - a y-intercept which depends negatively on n, g, δ,
    - a slope which depends positively on g.

Graphical representation in the Cobb-Douglas case



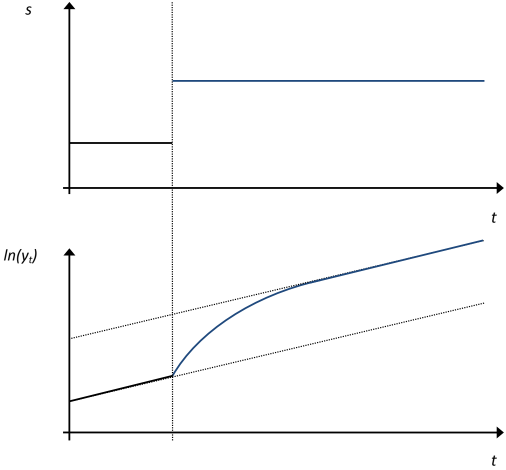
Effect of a discontinuous change in a parameter

- Following a discontinuous change in s, n, δ or g,
  - k_t remains a continuous function of time because it is a stock,
  - A_t remains a continuous function of time because it is a stock (if $g = g_0$ for $t < T$ and $g = g_1$ for $t \geq T$, then $A_t = A_0 e^{g_0 t}$ for $t \leq T$ and $A_t = A_T e^{g_1(t-T)}$ for $t \geq T$),
  - y_t remains a continuous function of time because $y_t = F(k_t, A_t)$

- Let $c_t \equiv \frac{C_t}{L_t}$ denote per-capita consumption

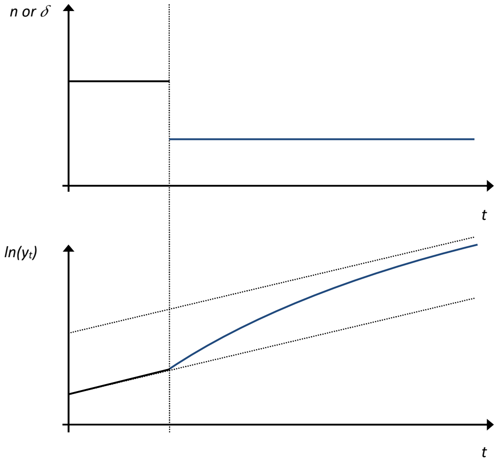
- We have $c_t = (1-s)y_t$, so
  - following a discontinuous change in n, δ or g, c_t remains continuous,
  - following a discontinuous change in s, c_t varies discontinuousy

Effect of a permanent increase in s



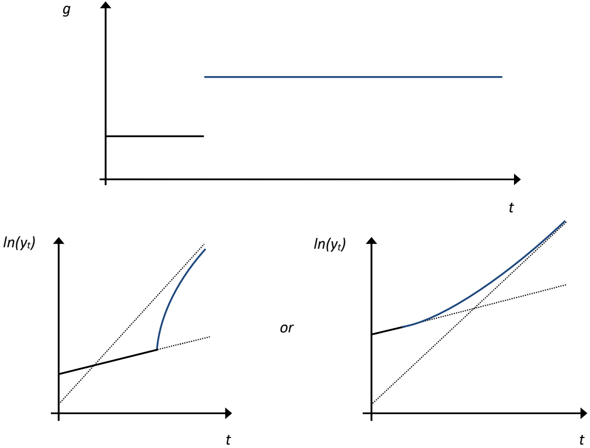
(The economy is assumed to be initially at the steady state)

Effect of a permanent decrease in n or δ



(The economy is assumed to be initially at the steady state. The speed of convergence of $\ln(y_t)$ to its new long-term path is lower than on page 41)

Effect of a permanent increase in g



(The economy is assumed to be initially at the steady state. The speed of convergence of $\ln(y_t)$ to its new long-term path is higher than on page 41)

Conditional convergence, not absolute convergence

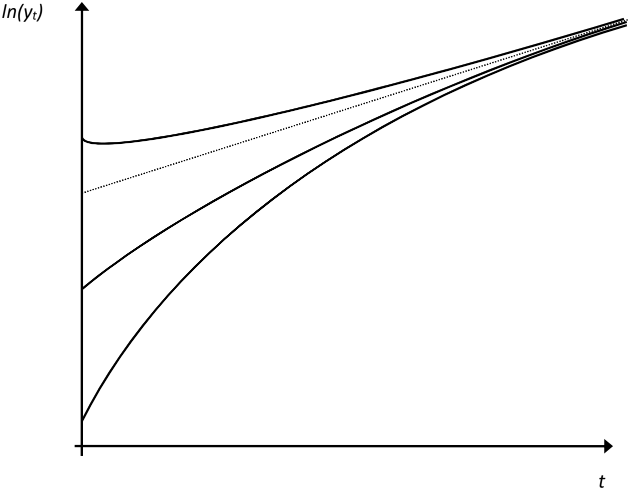
- **Conditional convergence** of per-capita output levels (in logarithm) across countries: long-term convergence of $\ln(y_t)$ across countries that have different y_0s but the same
  - technological parameters A_0, g, $f(.)$,
  - parameters governing the dynamics of capital and labor s, n, δ,

  because these countries have the same long-term path of $\ln(y_t)$

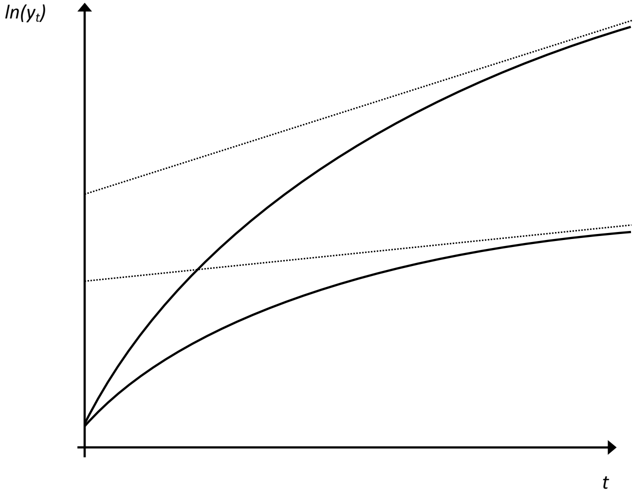
- **No absolute convergence**: no long-term convergence of $\ln(y_t)$ across countries that have different parameters A_0, g, $f(.)$, s, n, δ.

- An economy grows all the more rapidly as it is far away from its own long-term path, not all the more rapidly as it is poor

Example of conditional convergence

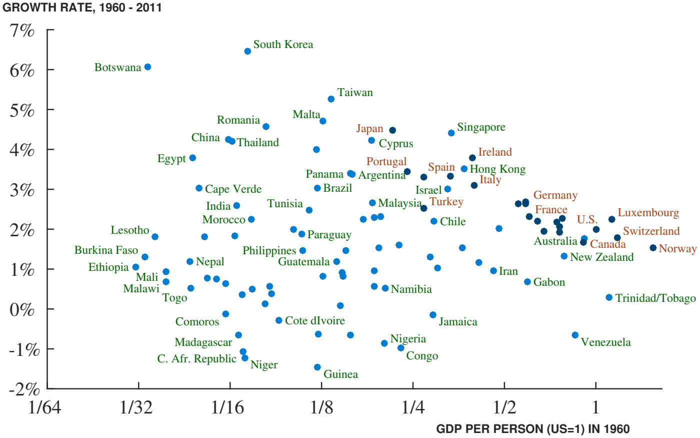


Example of divergence



In the data, no sign of absolute convergence...

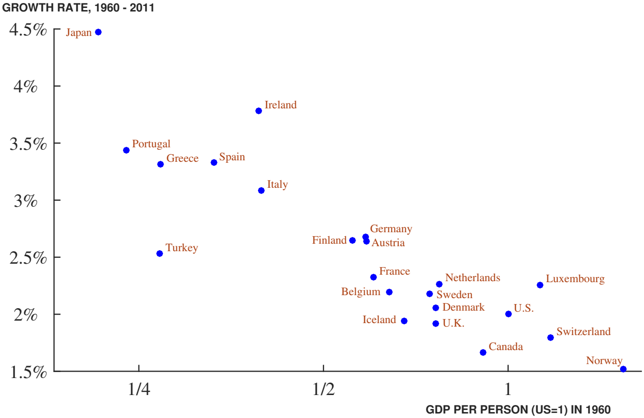
No convergence of per-capita GDPs within a group of heterogeneous countries



Source: Jones (2015). "Growth rate, 1960-2011": average annual growth rate from 1960 to 2011

...but some signs of conditional convergence

Convergence of per-capita GDPs within a sub-group of homogeneous countries (the OECD countries)



Source: Jones (2015). "Growth rate, 1960-2011": average annual growth rate from 1960 to 2011

Conditional-convergence tests

- The empirical literature that tests conditional convergence usually estimates, on panel data, an equation of type

$$\frac{1}{T}\ln\left(\frac{y_{i,t+T}}{y_{i,t}}\right)=\beta_0+\beta_1\ln\left(y_{i,t}\right)+\beta_2X_{i,t}+u_{i,t},$$

  where $X_{i,t}$ is a vector of control variables including s_i, n_i, δ_i (assuming that countries have access to the same technology)

- The conditional-convergence hypothesis then corresponds to $\beta_1<0$ and is usually not rejected by the data

Normative implications

Golden rule of capital accumulation I

- Steady-state per-capita consumption is equal to

$$(1-s)\, y_t^* = (1-s)\, A_t f(\kappa^*).$$

- It is positive and goes to 0 as $s \to 0$ and as $s \to 1$
- As a consequence, it is maximal for a value $s_{gr} \in]0; 1[$ of s
- Using $sf(\kappa^*) = (n+g+\delta)\,\kappa^*$, we can rewrite it as

$$A_t \left[f(\kappa^*) - (n+g+\delta)\,\kappa^* \right]$$

Golden rule of capital accumulation II

- As a consequence, s_{gr} is the unique value of s such that

$$f'(\kappa^*) = n + g + \delta$$

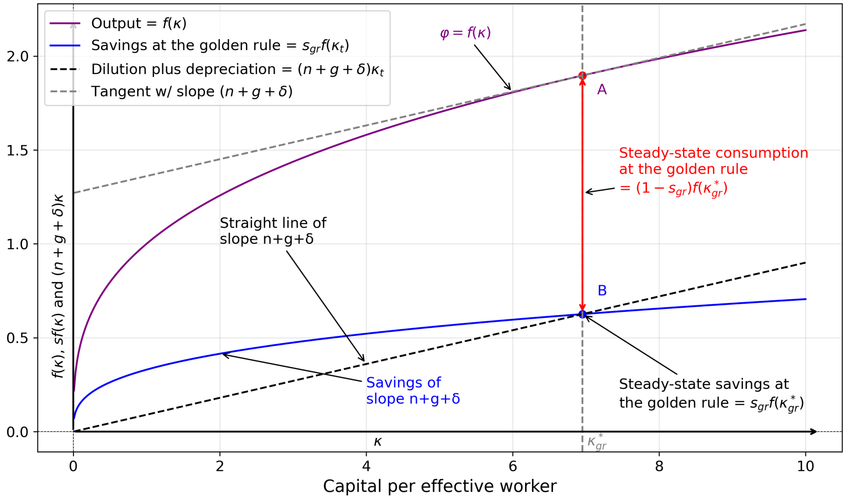
  (i.e. such that the marginal productivity of capital per effective-labor unit is equal to the sum of the capital depreciation and dilution rates)

- This last equation is called the "**golden rule of capital accumulation**" (Phelps, 1966)

- **Edmund S. Phelps**: American economist, born in 1933 in Evanston, professor at Columbia University since 1971, laureate of the Sveriges Riksbank's prize in economic sciences in memory of Alfred Nobel in 2006 "*for his analysis of intertemporal tradeoffs in macroeconomic policy*"

- On the next page, we first determine Point A by using the golden rule of capital accumulation, and then we deduce Points B and C; the segment AB represents the maximal vertical distance between the production curve and the dilution-plus-depreciation straight line

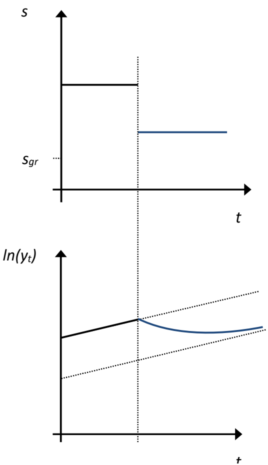
Golden rule of capital accumulation III



When $s > s_{gr}$ I

- **When** $s > s_{gr}$, a decrease in s towards s_{gr} would increase per-capita consumption $(1-s)\,y_t$ at all times:
  - in the long term (by definition of s_{gr}),
  - in the short term (as the increase in $1-s$ would outweigh the decrease in y_t)
- In this case, **there is dynamic inefficiency** ($\equiv$ situation in which one could increase per-capita consumption at all times), **due to capital over-accumulation**
- To the extent that agents' welfare depends positively on their consumption in the short and long terms, this reduction of s towards s_{gr} is desirable

When $s > s_{gr}$ II

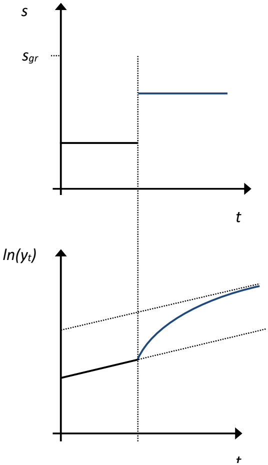


(The economy is assumed to be initially at the steady state)

When $s < s_{gr}$ I

- **When** $s < s_{gr}$, an increase in s towards s_{gr}
  - would increase per-capita consumption in the long term (by definition of s_{gr}),
  - would reduce it in the short term (as the decrease in $1 - s$ would outweigh the increase in y_t)
- In this case, **there is no dynamic inefficiency**
- To assess the desirability of this increase in s towards s_{gr}, we need to weight the utility of consumption in the short term and the utility of consumption in the long term (which is done in Chapter 2)

When $s < s_{gr}$ II



(The economy is assumed to be initially at the steady state)

Appendix

Limited resources

- Natural resources, pollution, and environmental considerations are absent from standard Solow-Swan model
    - Natural resources are limited
    - Malthus (1798) $\Rightarrow$ fixed supply of land constrains economic growth

- What are the environmental limitations to long-run growth?
    - Factors with *well-defined property rights* - natural resources and land
    - Those that do not - pollution free air and water
    - Property rights $\Rightarrow$ markets provide a signal as to how goods should be used. If a resource is limited, it will have a high price today
    - No property rights $\Rightarrow$ the good has an *externality*, firms pollute without compensating the people they harm
        - Strong case for government intervention

$\hookrightarrow$ Chapters 4 will explicitly model a damage function

Production function

- Extending the Cobb-Douglas to include non-renewable resources, R_t, and land, T_t,

$$Y_t = K_t^{\alpha} R_t^{\beta} T_t^{\gamma} (A_t L_t)^{1-\alpha-\beta-\gamma},\ \alpha > 0,\ \beta > 0,\ \gamma > 0,\ \alpha + \beta + \gamma < 1$$

- Dynamics of capital, labor and effective units of labor are as before
- The amount of land is fixed,

$$\dot{T}_t = 0$$

- Resources eventually must decline

$$\dot{R}_t = -bR_t, \text{ with } b > 0$$

- By assumption, A_t, L_t, R_t all grow at a constant rate, and T_t is fixed

Growth with limited resources

- Taking the log of Y_t and differentiating w.r.t. time yields

$$\begin{aligned}\frac{\dot{Y}_t}{Y_t} &= \alpha\frac{\dot{K}_t}{K_t}+\beta\frac{\dot{R}_t}{R_t}+\gamma\frac{\dot{T}_t}{T_t}+(1-\alpha-\beta-\gamma)\left(\frac{\dot{A}_t}{A_t}+\frac{\dot{L}_t}{L_t}\right)\\ g_Y &= \alpha g_K-\beta b+(1-\alpha-\beta-\gamma)(n+g)\end{aligned}$$

- For balanced growth, K and Y must growth at constant rate $g_Y = g_K$. So,

$$g_Y = g_K = \frac{(1-\alpha-\beta-\gamma)(n+g)-\beta b}{1-\alpha} < (n+g)$$

$\Rightarrow$ Fixed factors of production and non-renewable resources slowdown economic growth in the long-term

Conclusion

Main predictions of the model

- In the long term, growth comes only from technological progress
- The effect of capital accumulation on growth disappears in the long term because of the decreasing marginal productivity of capital
- There is conditional convergence of per-capita output levels (in logarithm) across countries
- There is dynamic inefficiency, due to capital over-accumulation, when the saving rate exceeds its golden-rule value

Two limitations of the model

- **The saving rate s is exogenous.** If it were endogenous, then
  - could we still have dynamic inefficiency?
  - what role should a policy affecting the saving rate play?

  $\hookrightarrow$ Chapter 2 endogenizes the saving rate

- **The rate of technological progress g is exogenous.** If it were endogenous, then
  - could some policies affect it?
  - what role should they play?

  $\hookrightarrow$ Chapters 4 and 5 ("endogenous-growth theories") endogenize the rate of technological progress.

Appendix

Proof that $z \mapsto f(z)/z$ is strictly decreasing

- Function f is strictly concave, and hence such that any arc is above its chord.

- In particular, $\forall y \in \mathbb{R}^+ \setminus \{0\}$, $\forall \lambda \in]0,1[$,

$$f(\lambda y) = f[(1-\lambda)0 + \lambda y] > (1-\lambda)f(0) + \lambda f(y) = \lambda f(y).$$

- Setting $\lambda = \frac{x}{y}$ with $0 < x < y$, we then get: $\forall (x,y) \in (\mathbb{R}^+ \setminus \{0\})^2$, if $x < y$ then $\frac{f(x)}{x} > \frac{f(y)}{y}$.

- Function $z \mapsto f(z)/z$ is therefore strictly decreasing.