Tutorial 2

1. **Fall in the discount rate**. Since forward-looking behavior determines the optimal choice of consumption and investment by households, we must consider whether the change is expected or unexpected. If expected, households may adjust their behavior in advance. *If a change is expected, households may alter their behavior before the change occurs.*

   We here focus on the simple case where the change is unexpected. That is, households are optimizing given their belief that their discount rate will not change, and the economy is on the resulting balanced growth path. At a certain point, households suddenly realize that their preferences have changed and that they now discount future utility at a lower rate than before.

   Prior to the change, the economy is at the state of rest as shown by point A in Figure 1.

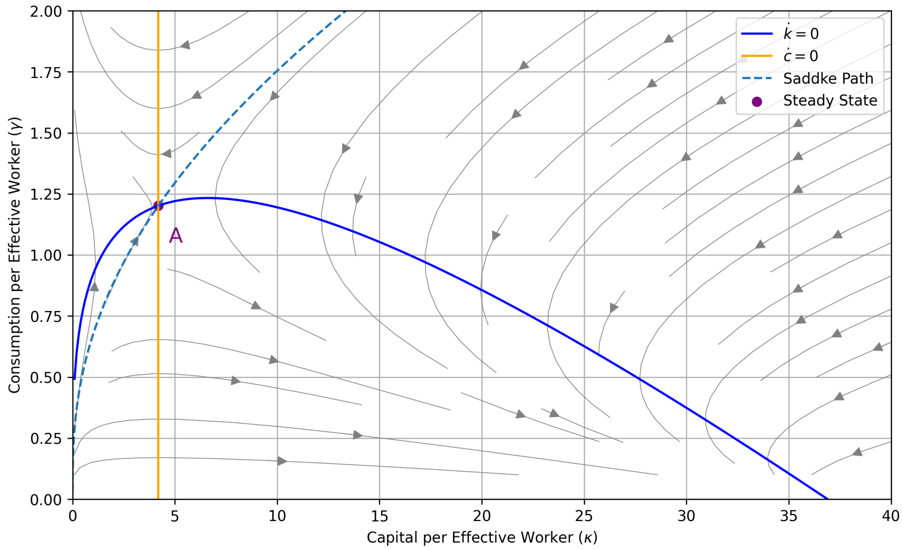

Figure 1: Phase Diagram

   Since κ_t evolves based on technology rather than preferences, ρ affects the growth rate of consumption per effective worker (γ_t) but not the capital accumulation equation. Thus, only the $\dot{\gamma}_t = 0$ locus is affected. Recall the Euler equation:

$$\frac{\dot{\gamma}_t}{\gamma_t} = \frac{1}{\theta}[f'(\kappa_t) - \delta - \rho - \theta g].$$

   Thus the value of κ_t where $\dot{\gamma}_t = 0$ equals zero is defined by $f'(\kappa^*) = \rho + \delta + \theta g$. Since $f''(\cdot)$ is negative, this means that the fall in ρ raises κ_1^*. The $\dot{\gamma}_t = 0$ line shifts to the right (red line in Figure 2). Why? If the discount rate increases, for consumption not to be changing, it must be that the interest rate is lower. For that to happen, the value of capital per effective worker in the steady state must be higher. At the time of the change in ρ, the value of κ the stock of capital per unit of effective labor is given by the history of the economy, and it cannot change discontinuously! The economy is state dependant, it depends on the value of the state variable. In particular, κ_t at the time of the change equals the value of κ_0^* on the old balanced growth path. In contrast, γ_t can jump at the time of the shock. Given our analysis of the dynamics of the economy, it is clear what occurs: at the time of the change, γ_t jumps down so that the economy is on the new saddle path (Point C). Thereafter, γ_t and κ_t rise gradually to their new balanced-growth-path values (arrows along the saddle path); these are higher than their values on the original balanced growth path (point B).

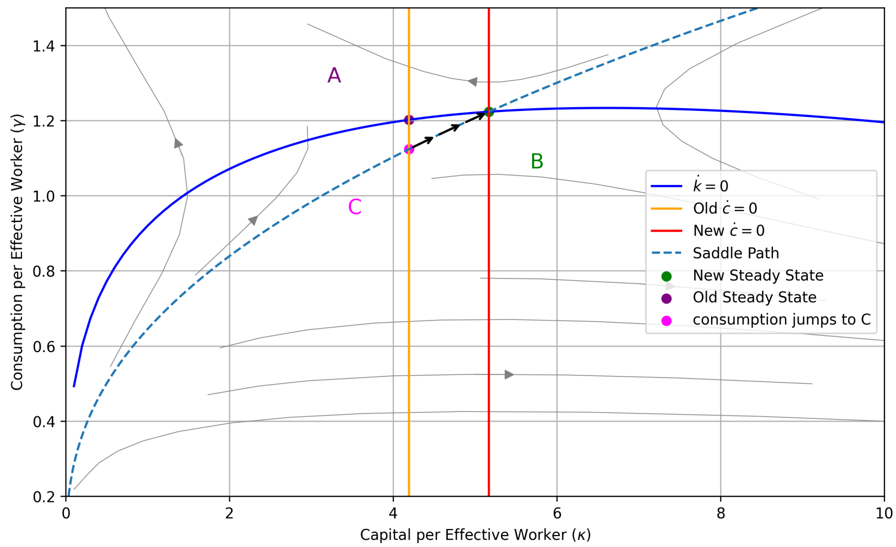

Figure 2: Phase Diagram - A fall in the discount rate

What is the economic intuition here? Households become more patient unexpectedly (or care more about their offspring), so they value future consumption more. This means that they would want to start saving more and consumption per effective units will fall on impact. Consumption will exhibit a discontinuous jump. Note that consumption is a jump/control variable. The increase in savings will lead to more investment and to capital accumulation in effective units. As capital in effective units starts to accumulate, there will be more resources, due to higher output, to allocate between consumption and investment (so both consumption and capital in effective units will be growing). The economy will eventually converge towards a steady state, where the level of capital is higher. As capital accumulates, the return to capital will gradually fall until $f'(\kappa_1^*) = \rho_1 + \delta + \theta g$.

Thus, just as with a permanent rise in the saving rate in the Solow model, the permanent fall in the discount rate produces temporary increases in the growth rates of capital per worker and output per worker. The key difference between these two experiments is that, when ρ falls, the fraction of output saved varies over time during the adjustment process, rather than remaining constant. The other difference is that the economy is dynamically efficient, the new steady state is below the golden rule level of capital in the Solow model. Can you show this?

2. **Government in the Ramsey-Cass-Koopmans model**

   (a) Starting with the instantaneous budget constraint in per-capita terms:

$$\dot{b}_t = w_t - \tilde{t}_t + (r_t - n)b_t - c_t.$$

   Rearranging terms and multiplying by $e^{-\int_0^t (r_\tau - n)d\tau}$ gives:

$$\left[\dot{b}_t - (r_t - n)b_t\right] e^{-\int_0^t (r_\tau - n)d\tau} = \left[w_t - \tilde{t}_t - c_t\right] e^{-\int_0^t (r_\tau - n)d\tau}.$$

We next integrate from 0 to T:

$$b_T e^{-\int_0^T (r_\tau - n)d\tau} - b_0 = \int_0^T (w_t - \tilde{t}_t - c_t) e^{-\int_0^t (r_\tau - n)d\tau} dt.$$

Taking the limit as $T \to +\infty$ and imposing the solvency constraint:

$$\int_0^{+\infty} c_t e^{-\int_0^t (r_\tau - n)d\tau} dt \leq b_0 + \int_0^{+\infty} (w_t - \tilde{t}_t) e^{-\int_0^t (r_\tau - n)d\tau} dt.$$

The inter-temporal budget consraint states that the present value of lifetime consumption (LHS of expression above) cannot exceed the sum of initial wealth (b_0) and the present value of net labor income (second expression of RHS). The solvency constraint is given by:

$$\lim_{T \to +\infty} \left[b_T e^{-\int_0^T (r_\tau - n)d\tau} \right] \geq 0,$$

which implies that the discounted value of per-capita net assets (b_T) cannot be negative as $T \to +\infty$.

The intertemporal budget constraint ensures that households can only spend as much as they can afford based on their initial wealth and the present value of their net labor income. It reflects the trade-off between present and future consumption. The solvency constraint prevents the household from accumulating unbounded debt. It guarantees that at any point in time, the household's wealth, discounted at the effective interest rate ($r_t - n$), remains non-negative in the long run. Together, these constraints ensure that household consumption and saving decisions are consistent with their resources over time.

(b) The household optimization problem is as follows: at time 0, for a given $(r_t, w_t, \tilde{g}_t, \tilde{t}_t)_{t \geq 0}$ and initial per-capita wealth b_0, the representative household maximizes its lifetime utility:

$$\max_{(c_t)_{t \geq 0}, (b_t)_{t > 0}} \int_0^{+\infty} e^{-(\rho - n)t} \left[u(c_t) + v(\tilde{g}_t) \right] dt,$$

subject to the constraints:

$$c_t \geq 0 \quad \forall t \geq 0,$$
$$\dot{b}_t = w_t - \tilde{t}_t + (r_t - n)b_t - c_t \quad \forall t \geq 0,$$
$$\lim_{t \to +\infty} \left[b_t e^{-\int_0^t (r_\tau - n)d\tau} \right] \geq 0.$$

The Hamiltonian associated with this optimization problem is:

$$H(c_t, b_t, \lambda_t, t) = e^{-(\rho - n)t}[u(c_t) + v(\tilde{g}_t)] + \lambda_t \left[w_t - \tilde{t}_t + (r_t - n)b_t - c_t \right],$$

where λ_t is the co-state variable associated with the evolution of wealth b_t, and it denotes the marginal utility of wealth. Using the Maximum Principle, the necessary conditions for an optimal solution are given by the following conditions. The first-order condition with respect to c_t:

$$\frac{\partial H}{\partial c_t} = e^{-(\rho - n)t} u'(c_t) - \lambda_t = 0 \quad \Longrightarrow \lambda_t = e^{-(\rho - n)t} u'(c_t).$$

The first order condition for with respect to b_t:

$$\dot{\lambda}_t = -\frac{\partial H}{\partial b_t} = -\lambda_t (r_t - n).$$

The first order condtion w.r.t. λ_t:

$$\dot{b}_t = w_t - \tilde{t}_t + (r_t - n)b_t - c_t.$$

And the transversality condition:

$$\lim_{t\to+\infty}\left[b_t e^{-\int_0^t (r_\tau - n)d\tau}\right] = 0.$$

It ensures that households do not accumulate excessive wealth at infinity (which would imply under-consuming) or end up with unbounded debt. Substituting $\lambda_t = e^{-(\rho-n)t}u'(c_t)$ into the co-state equation, we derive the Euler equation:

$$\frac{\dot{c}_t}{c_t} = \frac{r_t - \rho}{\theta},$$

where $\theta = -\frac{u''(c_t)c_t}{u'(c_t)}$ is the intertemporal elasticity of substitution in consumption. The Euler equation describes the inter-temporal trade-off in consumption and ensures optimal consumption smoothing over time. If $r_t > \rho$, households prefer future consumption over present consumption, so c_t grows over time. If $r_t < \rho$, households prefer present consumption, so c_t declines. If $r_t = \rho$, consumption remains constant.

(c) The government's instantaneous budget constraint is given by:

$$\dot{D}_t = r_t D_t + \tilde{G}_t - \tilde{T}_t,$$

where D_t is public debt, r_t is the interest rate, $\tilde{G}_t$ is government spending, and $\tilde{T}_t$ taxes. Rewriting in per-capita terms, with $d_t = \frac{D_t}{L_t}$, $\tilde{g}_t = \frac{\tilde{G}_t}{L_t}$, and $\tilde{t}_t = \frac{\tilde{T}_t}{L_t}$, and considering population grows exponentially at rate n, we obtain:

$$\dot{d}_t = (r_t - n)d_t + \tilde{g}_t - \tilde{t}_t.$$

Rearranging and multiplying by the exponential term $e^{-\int_0^t (r_\tau - n)d\tau}$, we get:

$$\left[\dot{d}_t - (r_t - n)d_t\right] e^{-\int_0^t (r_\tau - n)d\tau} = \left(\tilde{g}_t - \tilde{t}_t\right) e^{-\int_0^t (r_\tau - n)d\tau}.$$

Integrating both sides from $t = 0$ to $t = T$, we have:

$$d_T e^{-\int_0^T (r_\tau - n)d\tau} - d_0 = \int_0^T \left(\tilde{g}_t - \tilde{t}_t\right) e^{-\int_0^t (r_\tau - n)d\tau} dt.$$

Taking the limit as $T \to +\infty$, the government's intertemporal budget constraint becomes:

$$d_0 \leq \int_0^{+\infty} (\tilde{t}_t - \tilde{g}_t) e^{-\int_0^t (r_\tau - n)d\tau} dt,$$

where d_0 is the initial level of debt, and the right-hand side represents the actualized value at time 0 of future primary surpluses (tax revenue net of government spending).
The condition for equality is given by the government's solvency constraint:

$$\lim_{T\to+\infty}\left[d_T e^{-\int_0^T (r_\tau - n)d\tau}\right] \leq 0.$$

The intertemporal budget constraint imposes that the initial level of public debt d_0 cannot exceed the present value of future primary surpluses $\int_0^{+\infty} (\tilde{t}_t - \tilde{g}_t) e^{-\int_0^t (r_\tau - n)d\tau} dt$. This means that the government must finance its debt either through taxation or by reducing public spending in the long term. The solvency constraint ensures that the present value of public debt in the long term is non-positive:

$$\lim_{t \to +\infty} \left[d_t e^{-\int_0^t (r_\tau - n) d\tau} \right] \leq 0.$$

This condition rules out unsustainable fiscal practices such as borrowing indefinitely to repay past debt (a "Ponzi scheme"). It implies that in the long run, public debt cannot grow faster than the interest rate r_t. The solvency constraint excludes the possibility of financing the debt solely through continuous new borrowing, as this would require future generations to bear an ever-increasing debt burden. In summary, the intertemporal budget constraint and the solvency condition together ensure the sustainability of public finances, requiring the government to align its long-term spending and revenue policies to maintain fiscal stability. The government must eventually run primary surpluses (higher taxes than spending) to satisfy solvency.

(d) Market-clearing conditions ensure that at equilibrium, all resources are fully allocated and there are no excess supplies or demands in any market. The goods-market-clearing condition, derived from Walras' law, is given by:

$$\dot{K}_t = Y_t - C_t - \tilde{G}_t - \delta K_t,$$

where Y_t is total output, C_t is private consumption, G_t is public consumption, and δK_t represents capital depreciation. Finally, the other endogenous variables in the economy, such as output and public consumption, are determined residually using the equilibrium paths for κ_t and γ_t alongside the other equilibrium conditions. The asset market clearing condition:

$$B_t = \tilde{B}_t + D_t,$$

where B_t is total assets, $\tilde{B}_t$ private assets, D_t public assets held domestically. Total financial assets in the economy are composed of private savings and government debt.

(e) The equilibrium paths for the key variables $\kappa_t \equiv \frac{k_t}{A_t}$ (capital per efficiency unit of labor) and $\gamma_t \equiv \frac{c_t}{A_t}$ (consumption per efficiency unit of labor) are characterized by the following differential equations, initial condition, and transversality condition:

$$\dot{\kappa}_t = f(\kappa_t) - \gamma_t - \chi_t - (n + g + \delta) \kappa_t,$$

where $\chi_t \equiv \frac{g_t}{A_t}$ (public consumption per efficiency unit of labor), n is the population growth rate.

$$\frac{\dot{\gamma}_t}{\gamma_t} = \frac{1}{\theta} \left[f'(\kappa_t) - \delta - \rho - \theta g \right],$$

where ρ is the subjective discount rate, and θ is the inverse of the intertemporal elasticity of substitution. The initial value of the capital per effective unit of labor is,

$$\kappa_0 = \frac{K_0}{A_0 L_0},$$

where K_0 is the initial aggregate capital stock, A_0 is the initial level of technology, and L_0 is the initial labor force.

$$\lim_{t \to +\infty} \left\{ \kappa_t e^{-\int_0^t \left[f'(\kappa_\tau) - (n+g+\delta) \right] d\tau} \right\} = 0,$$

which ensures the transversality condition is satisfied. In addition, for simplicity, we focus on values of d_0, $(g_t)_{t \geq 0}$, and $(t_t)_{t \geq 0}$ such that the government's intertemporal budget constraint is binding:

$$d_0 = \int_0^{+\infty} (\tilde{t}_t - g_t) e^{-\int_0^t (r_\tau - n) d\tau} dt.$$

The government's solvency constraint then becomes:

$$\lim_{t\to+\infty}\left[d_t e^{-\int_0^t (r_\tau - n)d\tau}\right] = 0.$$

Using $b_t = \tilde{b}_t + d_t$, the transversality condition

$$\lim_{t\to+\infty}\left[b_t e^{-\int_0^t (r_\tau - n)d\tau}\right] = 0 \quad \text{can be rewritten as:} \quad \lim_{t\to+\infty}\left[\tilde{b}_t e^{-\int_0^t (r_\tau - n)d\tau}\right] = 0.$$

We then get, in the same way as we we did in the CKR without a government,

$$\lim_{t\to+\infty}\left\{\kappa_t e^{-\int_0^t \left[f'(\kappa_\tau)-(n+g+\delta)\right]d\tau}\right\} = 0,$$

where $\kappa_t \equiv \frac{k_t}{A_t}$ (with $k_t \equiv \frac{K_t}{L_t}$), δ is depreciation rate and $f(x) \equiv F(x,1)$ for any $x \geq 0$, F being the production function. Using the government's instantaneous budget constraint $\dot{d}_t = (r_t - n)d_t + \tilde{g}_t - \tilde{t}_t$, we can rewrite households' instantaneous budget constraint $\dot{b}_t = w_t - \tilde{t}_t + (r_t - n)b_t - c_t$ as

$$\dot{\tilde{b}}_t = w_t - \tilde{g}_t + (r_t - n)\tilde{b}_t - c_t.$$

Using $\tilde{b}_t = \kappa_t A_t$, substituting $r_t = f'(\kappa_t) - \delta$ and $w_t = A_t[f(\kappa_t) - \kappa_t f_t'(\kappa_t)]$, and dividing by A_t, we get the following expression

$$\dot{\kappa}_t = f(\kappa_t) - \gamma_t - \chi_t - (n+g+\delta)\,\kappa_t$$

where $\gamma_t \equiv \frac{c_t}{A_t} = \frac{C_t}{A_t L_t}$ and $\chi_t \equiv \frac{\tilde{g}_t}{A_t} = \frac{\tilde{G}_t}{A_t L_t}$.

This differential equation implies the goods-market-clearing condition: $\dot{K}_t = Y_t - C_t - \tilde{G}_t - \delta K_t$. As in the CKR model without government, the Euler equation can be rewritten as

$$\frac{\dot{\gamma}_t}{\gamma_t} = \frac{1}{\theta}\left[f'(\kappa_t) - \delta - \rho - \theta g\right].$$

(f) Here, $(\kappa_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$ are therefore determined by two differential equations, one initial condition and one terminal condition:

$$\dot{\kappa}_t = f(\kappa_t) - \gamma_t - \chi_t - (n+g+\delta)\,\kappa_t,$$
$$\frac{\dot{\gamma}_t}{\gamma_t} = \frac{1}{\theta}\left[f'(\kappa_t) - \delta - \rho - \theta g\right],$$
$$\kappa_0 = \frac{K_0}{A_0 L_0},$$
$$\lim_{t\to+\infty}\left\{\kappa_t e^{-\int_0^t \left[f'(\kappa_\tau)-(n+g+\delta)\right]d\tau}\right\} = 0.$$

The other endogenous variables are residually determined, from $(\kappa_t)_{t\geq 0}$ and $(\gamma_t)_{t\geq 0}$, using the other equilibrium conditions.

(g) We assume temporarily that $\forall t \geq 0$, (i) $\chi_t = \chi > 0$ and (ii) households expect that $\forall \tau \geq t$, $\chi_\tau = \chi$. We then show that κ_t and γ_t are constant at the **steady state** ($\equiv$ situation in which κ_0 is such that, in equilibrium, all quantities are non-zero and grow at constant rates).

Replacing $\dot{\kappa}_t$ with 0 in the differential equation in $\dot{\kappa}_t$, we get

$$\gamma_t = f(\kappa_t) - (n+g+\delta)\,\kappa_t - \chi,$$

which corresponds to a **bell-shaped curve** in the plane (κ_t, γ_t) of Figure 3. Replacing $\dot{\gamma}_t$ with 0, we get

$$f'(\kappa_t) = \delta + \rho + \theta g,$$

which corresponds to a **vertical straight line** in the plane (κ_t, γ_t). The **intersection point** of this curve and this straight line corresponds to the steady-state value of (κ_t, γ_t), denoted by (κ^*, γ^*). The only difference is that the bell-shaped curve has shifted downwads. At the steady state, there is **no dynamic inefficiency due to capital over-accumulation**: $\kappa^* < \kappa_{gr}$. The level of steady state capital is the same but the level of private consumption is lower than the case where there is no government. There exists a unique path, called "**saddle path**", along which (κ_t, γ_t) can converge to (κ^*, γ^*). We show that the unique equilibrium path of (κ_t, γ_t) for a given κ_0 is the saddle path.

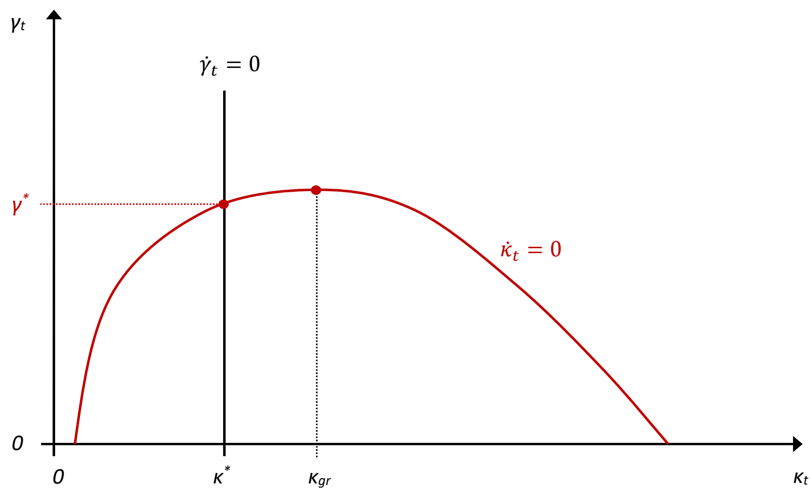

Figure 3: Phase diagram with government spending

(h) Note that lump-sum taxes do not appear in any of the four equilibrium conditions on κ_t and γ_t. So, in equilibrium, no endogenous variable (except public debt) depends on the lump-sum taxes. In other words, **the effect of public expenditures on the economy does not depend on the way they are financed** (current lump-sum tax or current borrowing reimbursed with a future lump-sum tax). This result is called "**Ricardian equivalence**" was first stated by Ricardo (1817), and later formalized by Barro (1974).[1]

This result is due to the fact that the way public expenditures are financed does not affect households' intertemporal budget constraint. Using the intertemporal budget constraint of the government,

$$d_0 = \int_0^{+\infty} (\tilde{t}_t - \tilde{g}_t) e^{-\int_0^t (r_\tau - n) d\tau} dt,$$

we can rewrite households' intertemporal budget constraint

$$\int_0^{+\infty} c_t e^{-\int_0^t (r_\tau - n) d\tau} dt \leq b_0 + \int_0^{+\infty} (w_t - \tilde{t}_t) e^{-\int_0^t (r_\tau - n) d\tau} dt$$

as

$$\int_0^{+\infty} c_t e^{-\int_0^t (r_\tau - n) d\tau} dt \leq b_0 + \int_0^{+\infty} (w_t - \tilde{g}_t) e^{-\int_0^t (r_\tau - n) d\tau} dt.$$

[1] David Ricardo was English economist, born in 1772 in London, deceased in 1823 in Gatcombe Park and Robert J. Barro is an American economist, born in 1944 in New York, professor at Harvard University since 1987.

Thus, the way public expenditures are financed (current or future tax) does not affect households' choices because it does not affect the actualized value of their future after-tax incomes. Consider a given amount of public expenditures at a given time. In the case (denoted by A) in which the government finances these expenditures with current taxes, households reduce their current consumption to pay this tax. In the alternative case (denoted by B) in which the government borrows to finance these expenditures, households also reduce their current consumption, in order to save in anticipation of the future taxes.

Because the rate of return of households' savings is equal to the interest rate at which the government borrows, households save in Case B an amount equal to the tax that they pay in Case A. As a consequence, households' current consumption is the same in the two cases.[2,3]

(i) We assume now that χ_t can vary over time, these variations *can be expected, or not expected*, by households, at each time, households are unaware that they may be surprised, at a later time, by the value of χ_t or by the announcement of its future path, at each time at which they are surprised by the current value of χ_t or by the announcement of its future path, households solve their new optimization problem and change their current and expected future behavior accordingly, the path of taxes over time adjusts to variations in χ_t in such a way that the government's intertemporal budget constraint always remains binding.

κ_t is a **stock**, and hence a **continuous** function of time (except following "earthquake shocks", destroying part of the capital stock). γ_t and $\dot{\gamma}_t$ are **flows**, and hence potentially **discontinuous** functions of time. γ_t and $\dot{\gamma}_t$ can be discontinuous only at the times at which households are surprised by the current value of χ_t or by the announcement of its future path. The reason is that when there is no surprise about the path of χ_t (even when this path is discontinuous), it is optimal for households to smooth γ_t and $\dot{\gamma}_t$ over time, as the differential equation in $\dot{\gamma}_t$ implies.

We now assume that there exists a time t_0 such that $t_0 > 0$ and

- $\forall t \in [0; t_0[$, (i) $\chi_t = \chi$, (ii) households expect that $\forall \tau \geq t$, $\chi_\tau = \chi$, and the economy is at the steady state,
- the government credibly announces at t_0 that $\forall t \geq t_0$, $\chi_t = \chi' > \chi$,
- the government conducts, from t_0, the fiscal policy announced at t_0.

From t_0, χ_t is constant over time and there are no more surprises, so the economy is on its new saddle path (lower than the old saddle path). So, the economy jumps at t_0 from the old steady state (Point A) to the new one (Point B) and remains at the new one thereafter (as shown in Figure 4). Consumption γ_t falls at t_0 by $\chi' - \chi$ because the permanent increase in χ_t reduces at t_0 the actualized value of households' future after-tax incomes by $(\chi' - \chi) \int_{t_0}^{+\infty} e^{-\int_{t_0}^{t} (r_\tau - n) d\tau} dt$.

[2] The literature is not very conclusive about the empirical validity of the Ricardian equivalence (Seater, 1993). Several issues could explain a lack of empirical validity:

i. different generations and no bequest or altruism between generations,

ii. distortive taxes,

iii. households' non-optimizing behavior,

iv. households' liquidity constraints.

[3] Ricardian equivalence nonetheless remains a useful benchmark to analyze the effects of the way public expenditures are financed. In the context of fiscal consolidation in the euro area after the 2008-2009 crisis, the empirical validity of the Ricardian equivalence was an important issue in the debates − for instance, in Trichet (2010):

"*The concern is, however, that in the short run the deficit reductions − although unavoidable in the long run − have negative effects on aggregate demand. The economy, it is sometimes argued, is at present too fragile and thus consolidation efforts should be postponed or even new fiscal stimulus measures added.*

As I pointed out recently, I am sceptical about this line of argument. Indeed, the strict Ricardian view may provide a more reasonable central estimate of the likely effects of consolidation. For a given expenditure, a shift from borrowing to taxation should have no real demand effects as it simply replaces future tax burden with current one."

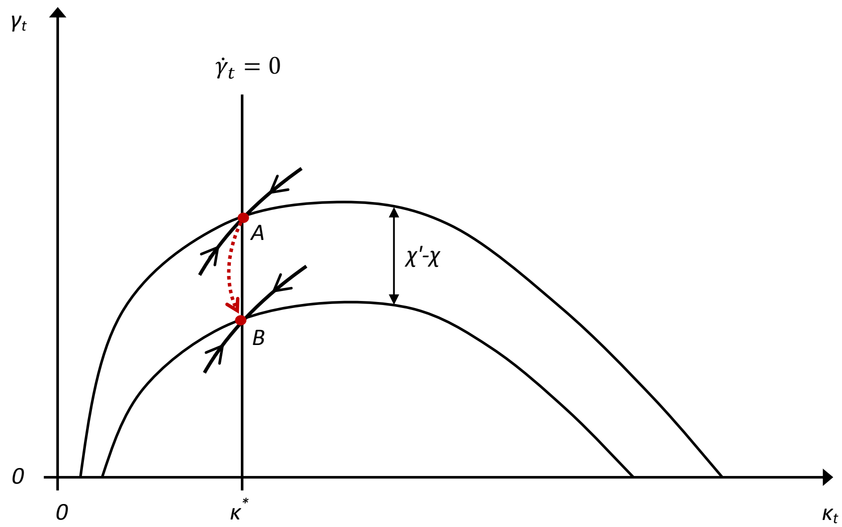

Figure 4: Phase diagrame - a permanent raise in government spending

(j) We now assume that there exist t_0 and t_1 such that $0 < t_0 < t_1$ and

- $\forall t \in [0; t_0]$, (i) $\chi_t = \chi$, (ii) households expect that $\forall \tau \geq t$, $\chi_\tau = \chi$, and (iii) the economy is at the corresponding steady state,
- the government credibly announces at t_0 that (i) $\forall t \in [t_0; t_1[$, $\chi_t = \chi' > \chi$, and (ii) $\forall t \geq t_1$, $\chi_t = \chi$,
- the government conducts, from t_0, the fiscal policy announced at t_0.

From t_1, χ_t is constant over time, equal to its old value, and there are no more surprises, so the economy is on its old saddle path. From t_0 to t_1, the possible paths are those that

- start from a point C on the vertical straight line,
- go northwestwards if C is above B,
- remain at B if C coincides with B,
- go southeastwards if C is below B.

For the economy to be on its old saddle path at t_1, C must lie between A and B.

Figure 6 illustrates the economy jumps from A to C at t_0, then moves from C to D between t_0 and t_1, then moves from D to A between t_1 and $+\infty$. Consumption γ_t falls at t_0 by less than $\chi' - \chi$ because the temporary increase in χ_t reduces at t_0 the actualized value of households' future after-tax incomes by less than $(\chi' - \chi) \int_{t_0}^{+\infty} e^{-\int_{t_0}^{t} (r_\tau - n) d\tau} dt$. From t_0 to t_1, γ_t increases, despite the high level of χ_t, thanks to a larger and larger decrease in κ_t ($\ddot{\kappa}_t < 0$).

From t_1, γ_t increases as κ_t recovers. This path is preferable to the path that jumps from A to B at t_0, remains at B between t_0 and t_1, and then jumps back from B to A at t_1, because it smooths γ_t and $\dot{\gamma}_t$ over time from t_0.

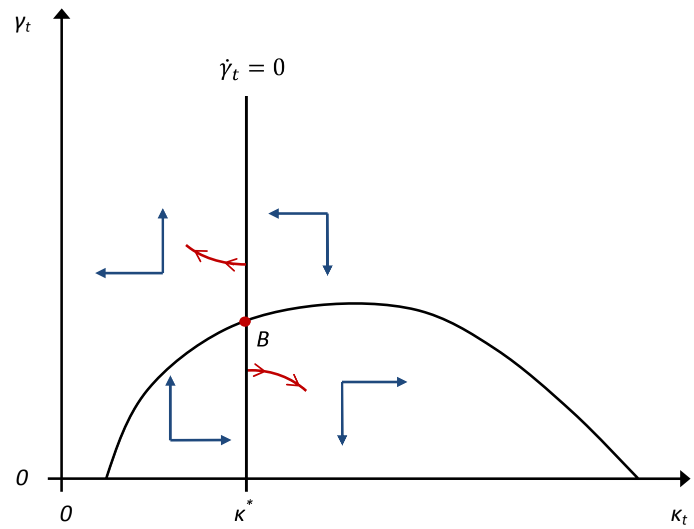

Figure 5: Phase diagrame - a temporary and unexpected increase in government spending

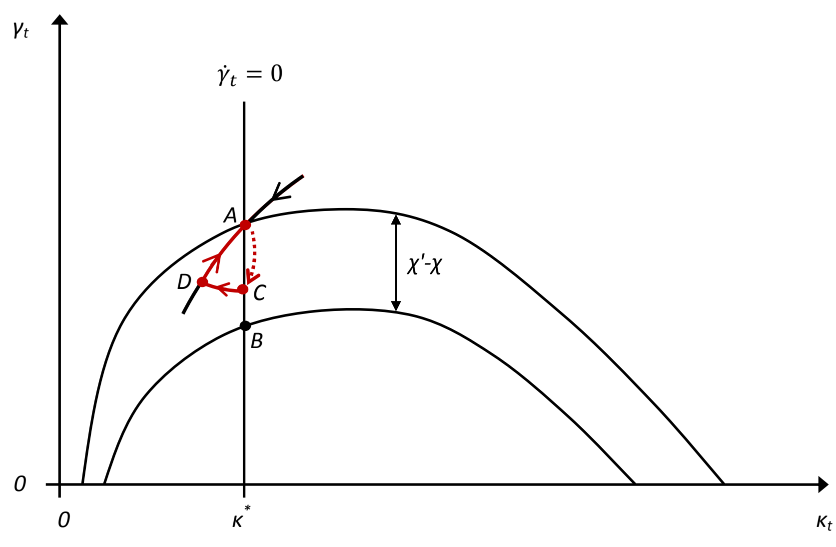

Figure 6: Phase diagrame - a temporary and unexpected raise in government spending 2

3. (a) The first condition states that the stock of capital and the flows of consumption and non-renewable resources must be positive or zero. The second condition is the resource constraint of goods (in the centralized framework) or the condition of market equilibrium of goods (in the decentralized framework). The third condition states that the amount of non-renewable resources used between the dates 0 and $+\infty$ must not exceed the stock of non-renewable resources available on the date 0.

(b) Denote M the minimum in question. If $S_0 < M$, then the path would not be feasible because it would violate the third condition previously stated. If $S_0 > M$, then we could, at least for one date t, increase R_t and therefore the production Y_t, and use this additional production to increase C_t without modifying $(K_t)_{t\geq 0}$, which would make the path not dynamically efficient. Therefore, $S_0 = M$.

(c) The optimization problem of the previous question falls into the general case seen in class. The Hamiltonian is:

$$H(R_t, K_t, \lambda_t, t) \equiv R_t + \lambda_t \left[F(K_t, R_t, L) - C_t^* - \delta K_t\right].$$

The first-order condition on the control variable and the evolution condition of the state co-variable are:

$$0 = 1 + \lambda_t F_R(K_t, R_t, L),$$

$$\dot{\lambda}_t = -\lambda_t \left[F_K(K_t, R_t, L) - \delta\right].$$

We can easily deduce the Hotelling rule.

(d) In the decentralized and competitive framework, F_K represents the rental price (expressed in terms of goods) of a unit of capital, and F_R the selling price (expressed in terms of goods) of a unit of non-renewable resources. The household wishing to save a unit of good at date t has at least two options:

i. Rent it out as capital to firms, which makes the household earn $1 - \delta dt + F_K dt$ at date $t + dt$.
ii. Buy $\frac{1}{F_R(t)}$ non-renewable resource units at date t and sell them at the price $F_R(t+dt)$ at date $t + dt$, which earns them $\frac{F_R(t+dt)}{F_R(t)} = 1 + \frac{\dot{F}_R}{F_R} dt$ at date $t + dt$.

In equilibrium, the household must be indifferent between these two options. Hence, the Hotelling rule: the rate of return on property titles on the non-renewable resource is equal to the rate of return of property titles on capital.

(e) If the path chosen by the planner was not dynamically efficient, then the planner could, at least at one date $t \geq 0$, increase the consumption C_t without decreasing it on any other date. This would increase the value taken by the intertemporal utility function (which depends positively on consumption). Therefore, the path chosen by the planner would not be optimal, which is contradictory.

(f) Let us denote by $(R_t^*)_{t\geq 0}$ the path of R_t chosen by the planner. The remaining optimization problem for this planner is, for a given K_0, to choose $(K_t)_{t>0}$ and $(C_t)_{t\geq 0}$ in order to maximize the intertemporal utility under the constraints:

$$\forall t \geq 0,\ K_t \geq 0,\ C_t \geq 0,$$

$$\forall t \geq 0,\ \dot{K}_t = K_t^\alpha (R_t^*)^\beta - C_t.$$

This optimization problem falls under the general case seen in class. The Hamiltonian is:

$$H(C_t, K_t, \lambda_t, t) \equiv \frac{C_t^{1-\theta} - 1}{1-\theta} + \lambda_t \left[K_t^\alpha (R_t^*)^\beta - C_t\right].$$

The first-order condition on the control variable and the evolution condition of the state co-variable are:

$$0 = C_t^{-\theta} - \lambda_t,$$
$$\dot{\lambda}_t = \lambda_t \left[\rho - \alpha K_t^{\alpha-1}(R_t^*)^{\beta}\right].$$

We can easily deduce the Euler equation.

(g) Using the Hotelling rule, we get

$$\dot{x}_t = x_t^{\alpha}$$

and therefore

$$x_t = \left(x_0^{1-\alpha} + (1-\alpha)t\right)^{\frac{1}{1-\alpha}}.$$

Using Euler's equation, we then obtain

$$\frac{\dot{C}_t}{C_t} = \frac{\alpha}{\left(x_0^{1-\alpha} + (1-\alpha)t\right)} - \frac{\rho}{\theta},$$

and then

$$C_t = C_0 e^{-\frac{\rho}{\theta}t}\left(\frac{x_0^{1-\alpha} + (1-\alpha)t}{x_0^{1-\alpha}}\right)^{\frac{\alpha}{(1-\alpha)\theta}}.$$

Hence,

$$\lim_{t\to+\infty} C_t = 0.$$

This situation may legitimately seem unfair to future generations, who will (asymptotically) have an infinitely small consumption and an infinitely large marginal utility of consumption, simply because they are born later.

(h) The conditions to apply the first welfare theorem are satisfied, so the competitive equilibrium is socially optimal.

(i) The conditions to apply the first welfare theorem are not satisfied because there is an externality: a particular household m does not take into account the negative effect of exploitating non-renewable resources on the well-being of other households (via its effect on climate change), or even its effect on its own well-being (because this effect is negligible, because the household is atomistic). The planner internalizes the externality. The competitive equilibrium is therefore socially sub-optimal. A tax on the use of non-renewable resources, redistributed as a lump sum to households and such that the private cost of using these resources is equal to its social cost (the latter including the loss in well-being due to climate change), would be an optimal economic policy in this context.

(j) Since the capital depreciation is null, if this path were not dynamically efficient, then we would have:

$$S_0 > M,$$

where M is defined in question 3.(b). We could then, at all dates $t \geq 0$, increase R_t and thus the production Y_t to increase the level of consumption C_t while keeping it constant over time, without modifying $(K_t)_{t\geq 0}$. The path would therefore not be the one chosen by the Rawlsian planner.[4]

[4] A Rawlsian planner prioritizes improving the well-being of the least advantaged members of society. Instead of maximizing total welfare (as a utilitarian planner would), the goal is to maximize the minimum utility level in society. Justice as Fairness - Inspired by Rawls' "A Theory of Justice", the planner ensures that social and economic inequalities are only justified if they benefit the worst-off. Policies are designed as if decision-makers do not know their future position in society, ensuring fairness and impartiality. In an economic growth model, a Rawlsian planner would allocate resources to ensure the poorest individuals gain the most, even if this means sacrificing overall efficiency. In tax policy, a Rawlsian planner would design progressive taxation to redistribute income toward the disadvantaged.

(k) We can verify that the indicated path is indeed the unique solution to the system made up of the two differential equations (Hotelling's rule and the equation $\dot{K}_t = F(K_t, R_t, L) - C_t$), the initial condition on the capital stock, the constraint of saturated nonrenewable resources (i.e. the third condition of question 3.(a) with an equality and not a strict inequality), the condition of constancy of consumption over time $C_t = C$, and the condition of linearity of K_t in t.

The flow of non-renewable resources R_t must decrease in time towards zero to verify the non-renewable resources constraint (the third condition from question 3.(a)). Production must not decrease towards zero in order to maintain a constant level of consumption through time; therefore, the capital stock must increase towards infinity. The level of constant consumption depends, of course, positively on K_0 and S_0.

We consider an economy with a Cobb-Douglas production function using capital (K) and a non-renewable resource (R):

$$Y_t = K_t^\alpha R_t^\beta,$$

where α is the capital share and β is the resource share, with $\alpha + \beta < 1$ to ensure decreasing returns in reproducible inputs. Under Hartwick's rule, all resource rents are reinvested in capital to sustain constant consumption. The Hotelling rule states that the shadow price of the resource grows at the rate of interest r:

$$\frac{\dot{F}_R}{F_R} = F_K.$$

Since consumption is constant (C), the resource extraction path must adjust accordingly. Assuming a functional form for R_t:

$$R_t = \lambda K_t^{-\frac{\alpha}{\beta}},$$

where λ is a constant to be determined. With zero capital depreciation, capital evolves as:

$$\dot{K}_t = Y_t - C = K_t^\alpha R_t^\beta - C.$$

We assume a linear capital path:

$$K_t = K_0 + \mathcal{A}t.$$

Differentiating gives:

$$\dot{K}_t = \mathcal{A}.$$

Equating with the accumulation equation:

$$\mathcal{A} = K_t^\alpha R_t^\beta - C.$$

Substituting $R_t = \lambda K_t^{-\frac{\alpha}{\beta}}$:

$$\mathcal{A} = K_t^\alpha (\lambda K_t^{-\frac{\alpha}{\beta}})^\beta - C.$$

Simplifying:

$$\mathcal{A} = \lambda^\beta K_t^{\alpha-\alpha} - C = \lambda^\beta - C.$$

Setting $\mathcal{A} = \frac{\beta C}{1-\beta}$, we find:

$$\lambda^\beta = \frac{C}{1-\beta}.$$

This gives

$$(1-\beta) K_t^\alpha R_t^\beta = C.$$

Thus, the resource extraction path is:

$$R_t = \left(\frac{C}{1-\beta}\right)^{\frac{1}{\beta}} \left(K_0 + \frac{\beta C}{1-\beta} t\right)^{-\frac{\alpha}{\beta}}.$$

The total stock of the resource is S_0, so integrating R_t over time:

$$S_0 = \int_0^\infty R_t dt.$$

Substituting R_t and solving the integral yields:

$$C = (1-\beta)\left[(\alpha-\beta)S_0\right]^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}.$$

Thus, the Rawlsian planner follows the capital and resource paths:

$$K_t = K_0 + \frac{\beta C}{1-\beta} t,$$

$$R_t = \left(\frac{C}{1-\beta}\right)^{\frac{1}{\beta}} \left(K_0 + \frac{\beta C}{1-\beta} t\right)^{-\frac{\alpha}{\beta}},$$

with consumption level:

$$C = (1-\beta)\left[(\alpha-\beta)S_0\right]^{\frac{\beta}{1-\beta}} K_0^{\frac{\alpha-\beta}{1-\beta}}.$$

(l) It is easy to verify that the indicated path verifies this rule. This rule stipulates that in order not to harm future generations, the rent derived from the exploitation of non-renewable resources must be re-invested (i.e. the price $F_R(K_t, R_t, L)$ multiplied by the quantity $R_t\, dt$) in the form of physical capital ($\dot{K}_t\, dt$).

It is easy to verify that

$$\dot{K}_t = K_t^\alpha R_t^\beta - C = K_t^\alpha R_t^\beta - (1-\beta) K_t^\alpha R_t^\beta = \beta K_t^\alpha R_t^\beta = F_R(K_t, R_t, L) R_t.$$