PLAN : (Miscellaneous topics)

1) Review Rational subsets + a few definitions

2) Pushouts for Huber rgs exist in some situations

3) When is the presheaf on $\mathrm{Spa}(A, A^+)$ a sheaf?

4a) Cartier divisors on adic spaces

b) Analog of closed immersion for adic spaces

(ALL RINGS ARE HUBER & ALL MAPS ARE CONTINUOUS

Setup : A Huber, A_0 : rg of definition, I ideal of defn. $A^+ \subset A^0$ rg of integral elts

PRELUDE: Affinoid pre-adic space : $X = \mathrm{Spa}(A, A^+)$ where A, A^+ is a Huber pair

- As a set : $\{ x$, a cont. valuation on A s.t. $x(a) \leq 1 \ \forall a \in A^+ \}$

- Topological basis $\mathcal{B}$: Given by rational subsets $R\left(\frac{T}{s}\right)$, where T is a finite set, and $T \cdot A$ is open in A.

$$R\left(\frac{T}{s}\right) := \{ x : \quad x(t) \leq x(s) \neq 0 \}$$

- Presheaf: $\mathcal{O}_X\left(R\left(\frac{T}{s}\right)\right) = \left(A\left\langle \frac{T}{s} \right\rangle, A^+\left\langle \frac{T}{s}\right\rangle \right)$

$A\left(\frac{T}{s}\right)$ is A_s as a ring

Its rg of definition is $A_0\left[\frac{t}{s} \,\middle|\, t \in T\right]$, where A_0 is a rg of defn of A

The ideal of defn is $I A_0\left[\frac{t}{s} \,\middle|\, t \in T\right]$

(To make this a top rg, we do need $T \cdot A$ is open condition)

$A^+\left(\frac{T}{s}\right) :=$ Integral closure of $A^+\left[\frac{t}{s}\right]$

$\left(A\left\langle \frac{T}{s} \right\rangle, A^+\left\langle \frac{T}{s}\right\rangle \right)$ is the completion of $\left(A\left(\frac{T}{s}\right), A^+\left(\frac{T}{s}\right)\right)$

$$\mathcal{O}_X(U) := \varprojlim_{\substack{V \in \mathcal{B} \\ V \subseteq U}} \mathcal{O}_X(V)$$

Note : $\mathrm{Spa}\left(A\left\langle \frac{T}{s} \right\rangle, A^+\left\langle \frac{T}{s}\right\rangle \right)$ maps homeomorphically onto $R\left(\frac{T}{s}\right)$, under the map induced by $A \longrightarrow A\left\langle \frac{T}{s} \right\rangle$

$A\left\langle \frac{T}{s} \right\rangle$ is universal in the category of non-arch top rgs that are

1. complete
2. $A \xrightarrow{\varphi} B$ is continuous
3. $\varphi(s)$ is invertible
4. $\{\varphi(t_i)\varphi(s^{-1}) \mid t_i \in T\}$ is a power bdd set

Another description of $A\langle \frac{T}{s} \rangle$ when A is complete:

Consider the set of restricted power series $\hat{A}\langle X_i \mid t_i \in T \rangle$

$$\left\{ \sum_\nu a_\nu X^\nu : \forall \mathcal{U} \subset \hat{A},\ a_\nu \in \mathcal{U} \text{ for almost all } \nu \right\}$$

with topology given by sets of the form $\mathcal{U}\langle x \rangle$ (those with all coefs in $\mathcal{U}$)

Let I be the ideal generated by $\{(t_i - sX_i)\}$

Then $\hat{A}\langle X_t \rangle / \bar{I}$ is a complete rg having the same universal property as $A\langle \frac{T}{s} \rangle$.

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§ 5.1: Adic morphisms.

Defn: $A \xrightarrow{\varphi} B$ is adic if for one (and hence any) choice of rgs of defn $A_0 \subset A$, $B_0 \subset B$, s.t. $\varphi(A_0) \subset B_0$, and $I \subset A_0$ an ideal of definition, $\varphi(I)B_0$ is an ideal of defn for B_0.

Not obvious, but easy check

E.g. 1. Adic: $A \longrightarrow A\langle \frac{T}{s} \rangle$ for rational subsets.

2. Not adic: $\mathbb{Z}_p \longrightarrow \mathbb{Z}_p[[T]]$ (p,T)-adic topology

(Speak: rgs of defn are themselves. The ideal generated by p on the RHS does not contain T^n, but all powers of (p,T) contain some large powers of T.)

Lemma: If A is Tate, then any $A \xrightarrow{\varphi} B$ is adic

Pf: Let $\pi \in A_0$ be a top. nilpotent unit. (can assume by taking powers) Then $\{\pi^n A_0\}$ give the topology on A_0. $\varphi(\pi)$ is a top nilpotent unit as φ is continuous. $\therefore B$ is Tate & for every rg of defn B_0 in B s.t. $\varphi(\pi) \in B_0$, $\{\varphi(\pi)^n B_0\}_n$ give the topology

PROPOSITION: A map $(A, A^+) \xrightarrow{\varphi} (B, B^+)$ of complete Huber rgs is adic if & only if $\mathrm{Spa}(B,B^+) \longrightarrow \mathrm{Spa}(A, A^+)$ carries analytic pts to analytic pts.

Pts with non open support.
$\Updownarrow$
(The support does not contain all top. nilpotent elts.)

Pf. (only one way)

$(\Rightarrow)$ let $x \in \mathrm{Spa}(B, B^+)^{an} \Rightarrow \mathrm{supp}\, x \not\supset \varphi(I) B_0$
$\exists b = \varphi(a)$, $a \in I$, s.t. $x(b) \neq 0$
$\Rightarrow (x \circ \varphi)(a) \neq 0$ & $x \circ \varphi \in \mathrm{Spa}(A, A^+)^{an}$

$(\Leftarrow)$ A little more involved. (needs completeness of B)

<u>Definition</u>: A Huber rg A is <u>analytic</u> if: the top. nilpotents generate the unit ideal. (e.g. Tate rgs)

$\Leftrightarrow$ All points in $\mathrm{Spa}(A, A^+)$ are analytic
$(\Rightarrow)$ as supp cannot be open, since cannot contain all top nilpotents
$(\Leftarrow)$ Else, let $p \supset$ top nilpotents. Consider the triv valuation on $\mathrm{frac}(A/p)$.

($\therefore$ just like Tate, maps from analytic rgs to complete rgs are adic)

<u>Definition</u>: A map $f : Y \longrightarrow X$ of pre-adic spaces is analytic if it carries analytic pts to analytic pts.

<u>Proposition</u> 5.1.5: (1) If $(A, A^+) \xrightarrow{\varphi} (B, B^+)$ is adic, then pullback along $\mathrm{Spa}(B, B^+) \longrightarrow \mathrm{Spa}(A, A^+)$ preserves rational subsets
$x \longmapsto x \circ \varphi$

Pf: First, if $T \cdot A$ is open for T a finite subset, $TA \supset I^n A_0$
$\Rightarrow \varphi(T) B \supset \varphi(I)^n B_0$
$\underset{\text{adic}}{\Rightarrow}$ $\varphi(T) B$ is open

$x \circ \varphi \in R\left(\frac{T}{s}\right) \in \mathrm{Spa}(A, A^+)$ $\iff$ $x \circ \varphi(t_i) \leq x \circ \varphi(s) \neq 0 \;\; \forall\, t_i \in T$

$\iff$ $x \in R\left(\frac{\varphi T}{s}\right)$

This is a rational set $\because$ $\varphi(T)B$ is open.

(2) (Existence of some pushouts in category of Huber pairs)

$$\begin{tikzcd}
(A, A^+) \arrow[r, "\text{adic}"] \arrow[d, "\text{adic}"'] & (B, B^+) \arrow[d] \\
(C, C^+) \arrow[r] & D, D^+ \arrow[ul, phantom, "\ulcorner"]
\end{tikzcd}$$

$D = B \otimes_A C$

D_0, rg of defn $= B_0 \otimes_{A_0} C_0$

Ideal of definition $= I(B_0 \otimes_{A_0} C_0)$

where I is the ideal of defn of A_0

$D^+ =$ int. closure of $B^+ \otimes_{A_+} C^+$ inside D

(Speak: If A, B, C were complete, we could complete D to get pushouts in the category of complete Huber pairs.)

Rmk: $\mathrm{Spa}(D, D^+) = \mathrm{Spa}(B, B^+) \times_{\mathrm{Spa}(A, A^+)} \mathrm{Spa}(C, C^+)$

(Speak: Not sure why this is true. Do know why things are adic as opposed to preadic)

Eg:
No pushout:

$$\begin{tikzcd}
(\mathbb{Z}_p, \mathbb{Z}_p) \arrow[r] \arrow[d] & (\mathbb{Q}_p, \mathbb{Z}_p) \arrow[d] \\
(\mathbb{Z}_p[[T]], \mathbb{Z}_p[[T]]) \arrow[r] & \left(\mathbb{Q}_p\langle T, \frac{T^n}{p}\rangle, \mathbb{Z}_p\langle T, \frac{T^n}{p}\rangle\right)
\end{tikzcd}$$

cont. top. checked

If (D, D^+) was a pushout, $T \in D$ would be top nilpotent as the map from $\mathbb{Z}_p[[T]]$ is continuous, & $p \in D^\times$ as $p \in \mathbb{Q}_p^\times$. $\therefore$ as D^+ is open $T^m \in pD^+$ for $m \geq N$
$\Rightarrow \frac{T^m}{p} \in D^+$

But then D^+ cannot admit a map to $\mathbb{Z}_p\langle T, \frac{T^{m+1}}{p}\rangle$

(as p is not invertible here & we could not map even upon taking int closure)

5.2 Analytic adic spaces: (Rgs will be analytic)

Recall: A is sheafy if $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is a sheaf of top. rgs for all A^+.

(Speak: In this section we will talk about when rgs are sheafy)

We say that A is uniform if A° is bdd

the rg of power bdd elts.

Th 5.21 (Berkovich) For A analytic, uniform:

$$A \longrightarrow \prod_{x \in \mathrm{Spa}(A,A^+)} K(x)$$

is a homeomorphism onto image, where $K(x)$ is the completed residue field.

Therefore, $A^\circ = \{f \in A \mid f \in \mathcal{O}_{K(x)}, x \in X\}$

rg of integers

Remark: 1) If $x: A \longrightarrow \Gamma_x \cup \{0\}$ is a valuation, $\exists a \in A$, s.t. $a^n \to 0$, s.t. $a \notin \mathrm{supp}\, x$, as top. nilpotents generate unit ideal. If x is continuous, $x(a)^n \to 0$ & $\therefore$ a is a topological nilpotent in the topology induced by x $\therefore$ x is a microbial valuation, & top on x is induced by a rk 1 valuation

2) $\mathcal{O}_{K(x)}$ = Rg of power bdd elements in $K(x)$

Remark: The Theorem also follows from Huber's result $A^+ = \{f \in A \mid x(f) \leq 1, x \in X\}$

Stronger sees all vals

(They claim: I am not sure how)

Corollary: Let $\tilde{\mathcal{O}}_X$ be the sheafification of $\mathcal{O}_X$. If A is uniform, then $A \longrightarrow H^0(X, \tilde{\mathcal{O}}_X)$ is injective.

Pf: We have, $A \longrightarrow H^0(X, \tilde{\mathcal{O}}_X) \longrightarrow \prod_x K(x)$,

where the composite is injective

Note: $\mathrm{Spa}(A, A^+) = X = \mathrm{Spa}(\hat{A}, \hat{A^+})$

$\therefore$ We have. $\mathcal{O}_X(X) = \hat{A} \hookrightarrow H^0(X, \tilde{\mathcal{O}}_X)$. If this was a sheaf, we would have bijectivity.

(For sheafyness, we strengthen the uniformity condition)

<u>Definition</u>: A complete analytic Huber pair (A, A^+) is stably uniform if $\mathcal{O}_X(U)$ is uniform for all rational subsets $U \subset X = \mathrm{Spa}(A, A^+)$

<u>Th</u>: If the complete analytic Huber pair (A, A^+) is stably uniform, then it is sheafy.

(Moreover, sheafyness w/o other assumptions implies)

<u>Th</u>: If the complete analytic Huber pair (A, A^+) is sheafy then $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

Strategy of Pf:

Apparently, some combinatorial & inductive arguments allow a reduction of the problem to computing $H^i(X, \mathcal{O}_X)$ for a simple Laurent covering $X = U \cup V$ where $U = \{ |f| \le 1 \}$ & $V = \{ 1 \le |f| \}$

Now,
$$U = R\left(\frac{f}{1}\right) \qquad \mathcal{O}_X(U) = A\langle T\rangle / \overline{(T-f)}$$
$$V = R\left(\frac{1}{f}\right) \qquad \mathcal{O}_X(V) = A\langle S\rangle / \overline{(Sf-1)}$$
$$U \cap V = R\left(\frac{\{1, f^2\}}{f}\right) \longrightarrow \mathcal{O}_X(U\cap V) = A\left\langle \frac{\{1, f^2\}}{f} \right\rangle$$
$$\| $$
$$A\left\langle \frac{1}{f}, \frac{f}{1} \right\rangle$$
$$\|$$
$$A\langle S, T\rangle / \overline{(Sf-1, T-f)}$$
$$\|$$
$$A\langle T, T^{-1}\rangle / \overline{(T-f)}$$

WTS that the Cech Complex for this covering is exact. It is:

$$0 \longrightarrow A \longrightarrow A\langle T\rangle/\overline{T-f} \oplus A\langle S\rangle/\overline{(Sf-1)} \xrightarrow{\alpha} A\langle T, T^{-1}\rangle/\overline{T-f} \longrightarrow 0$$
$$a \longmapsto a + a$$

Lemma: 1) α is surjective. ✓ $T \mapsto T$, $S \mapsto -T^{-1}$

2) If A is uniform, $(T-f) \subset A\langle T\rangle$, etc. are closed

Now, if A is normed, the top. on $A\langle T\rangle$ is given by:
$$\left|\sum a_\nu T^\nu\right|_{A\langle T\rangle} = \sup |a_\nu|_A$$

By $A \hookrightarrow \prod K(x)$, top on A is given by $|a|_A = \sup\limits_{\substack{x \in X \\ \text{rk } 1 \text{ pts}}} |a|_{K(x)}$

$$\therefore \quad |f|_{A\langle T \rangle} = \sup_{\substack{x \in X \\ \text{rk } 1}} |f|_{K(x)\langle T \rangle}$$

$|\cdot|_{K(x)\langle T \rangle}$ is the Gauss norm & multiplicative (as opposed to sub-multiplicative which is normally the condition for Banach algebra)

$$\therefore \quad \forall\, g \in A\langle T \rangle, \quad |(T-f)g|_{A\langle T \rangle} = |T-f|_{A\langle T \rangle} \cdot |g|_{A\langle T \rangle} \geq |g|_{A\langle T \rangle}$$

as T has coef 1

$\therefore$ $\{(T-f)g_n\}$ Cauchy $\Rightarrow$ g_n Cauchy

$g_n \to g \Rightarrow (T-f)g_n \to (T-f)g$

$\therefore$ $(T-f)$ is closed.

Same for the others.

3. <u>Kernel of α is A</u>

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & (T-f)A\langle T \rangle \oplus (ST-f)A\langle S \rangle \arrow[r, two heads] \arrow[d] & (T-f)(A\langle T, T^{-1} \rangle) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, hook] \arrow[d, equal] & A\langle T \rangle \oplus A\langle S \rangle \arrow[r, two heads] \arrow[d] & A\langle T, T^{-1} \rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r, "\varepsilon"] & \dfrac{A\langle T \rangle}{T-f} \oplus \dfrac{A\langle S \rangle}{ST-f} \arrow[r, two heads] \arrow[d] & \dfrac{A\langle T, T^{-1} \rangle}{T-f} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

because RHS $= (T-f)A\langle T \rangle + (1 - fT^{-1})A\langle T^{-1} \rangle$

Columns are exact. Top two rows are exact

$\Rightarrow$ ε is exact

Th on vector bundles :

Let (A, A^+) be a sheafy analytic Huber pair and $X = \mathrm{Spa}(A, A^+)$

We have a categorical equivalence :

$\{$ finite projective A-mods $\}$

$\longleftrightarrow$ $\{$ for $X = \cup U_i$ $\quad U_i = \mathrm{Spa}(A_i, A_i^+)$ $\quad U_{ij} := U_i \cap U_j$, $U_{ij} = \mathrm{Spa}(A_{ij}, A_{ij}^+)$

(M_i, β_{ij}) : M_i finite projective A_i mods

$\beta_{ij} : M_i \otimes_{A_i} A_{ij} \xrightarrow{\sim} M_j \otimes_{A_j} A_{ij}$

satisfying the cocycle condition $\}$

data of

locally finite free $\mathcal{O}_X$ - modules

§ 5.3 : Cartier divisors

Definition : An adic (speak: preadic w/ $\mathcal{O}_X$ a sheaf) space is uniform if $\forall$ open affinoid $U = \mathrm{Spa}(R, R^+) \subset X$, the Huber rg R is uniform.

Note : If X is an analytic adic space, then X is uniform $\Leftrightarrow$ X is covered by open affinoids $U_i = \mathrm{Spa}(R_i, R_i^+)$ where R_i is stably uniform.

Definition : Let X be uniform analytic adic space. A Cartier divisor on X is an ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$ that is locally free of rank 1. The support of a Cartier divisor is the supp of $\mathcal{O}_X / \mathcal{I}$

Prop: Let (R, R^+) be a stably uniform Tate-Huber pair, & let $X = \mathrm{Spa}(R, R^+)$.

The map $I \longmapsto \mathcal{I} = I \cdot \mathcal{O}_X$ induces a bijective correspondence b/w invertible ideals $I \subset R$ s.t. the vanishing locus of I in X is nowhere dense & Cartier divisors on X.

Therefore, If $\mathcal{I} \subset \mathcal{O}_X$ is a Cartier divisor, then its support $Z \subset X$ is a nowhere dense closed subset of X.

Pf: By th earlier, any Cartier divisor is of the form $I \otimes \mathcal{O}_X$ for an ideal $I \subset R$

The notes claim that I must be invertible (i.e. generally locally free sheaf of rk 1 in $\mathrm{Spec}\, R$)

(So all Cartier divisors come from invertible ideal sheaves but not all may give a Cartier divisor. We'll see which ones do.

Specifically, an ideal sheaf may become 0 on certain open sets if the local generator becomes a zero divisor)

WTS: Cartier divisor

$\checkmark \Leftrightarrow$ Locally, $I\mathcal{O}_X(U)$ is generated by a nonzero divisor. In other words, $I \otimes_R \mathcal{O}_X \longrightarrow \mathcal{O}_X$ is injective as map of sheaves

$? \Leftrightarrow$ I invertible w/ nowhere dense vanishing locus

By localization, WMA that $I = (f)$, with Z as the vanishing locus $= \{x : \mathrm{supp}\, x \ni f\} = \mathrm{supp}^{-1}(V(f))$ $\therefore$ closed

$(\Rightarrow)$ If Z contains an open subset $U = \mathrm{Spa}(A, A^+)$, $f = 0$ on U as $A \hookrightarrow \prod_{x \in U} K(x)$ (& f & 0 map to the same pt) since X is uniform

$(\Leftarrow)$ Assume $\exists V$ on which $fg = 0$ for $g \in B$
$\quad V = \mathrm{Spa}(B, B^+)$

$S = \{g = 0\} = \mathrm{supp}^{-1}(V(g))$ is closed in V
& contains $V - Z$ (If $x(f) \neq 0 \Leftrightarrow x \in V - Z$ then $x(fg) = 0 \Rightarrow x(g) = 0$)

$\Rightarrow$ $V \setminus S$ is an open subset of Z

$\Rightarrow$ $V - S = \phi$ as Z is nowhere dense

$\Rightarrow$ $S = V \Rightarrow g = 0$ by uniformity as before

$\therefore$ f is not a nzd.

<u>Proposition</u> : Let X be a uniform adic space and $\mathcal{J} \subset \mathcal{O}_X$ a Cartier divisor w/ supp Z & $j: \mathcal{U} = X - Z \hookrightarrow X$

There are injective maps of sheaves:

$$\mathcal{O}_X \hookrightarrow \varinjlim_n \mathcal{J}^{\otimes -n} \to j_* \mathcal{O}_{\mathcal{U}}$$

(Note: As $|f(x)| \neq 0 \ \forall x \in \mathcal{U}$, f is a unit on $\mathcal{U}$)

<u>Pf</u>: WMA $X = \mathrm{Spa}(R, R^+)$ & $\mathcal{J} = f\mathcal{O}_X$ for some nzd $f \in R$ whose vanishing locus is nowhere dense & check on global sections.

$\varinjlim_n \mathcal{J}^{\otimes -n}$ Let $\mathcal{J}^{-1}(X) = R[\frac{1}{f}]$ ↙ module gen by $\frac{1}{f}$

$$\mathcal{J}^{-1}(X) \longrightarrow \underbrace{\mathcal{J}^{-1}(X) \otimes \mathcal{J}^{-1}(X)}_{\cong R[\frac{1}{f^2}]} \longrightarrow \cdots$$

$$\frac{a}{f} \longmapsto \frac{a}{f} \otimes \frac{f}{f}$$

$$\varinjlim_n (\mathcal{J}^{-1}(X))^{\otimes n} = R\left[1, \frac{1}{f}, \frac{1}{f^2}, \frac{1}{f^3}, \cdots\right]$$

$$R \longrightarrow R[f^{-1}] \longrightarrow H^0(U, \mathcal{O}_U)$$

The maps are injective if $R \longrightarrow H^0(U, \mathcal{O}_U)$ is.

If $g \in R$ maps to 0, then vanishing locus of g is a closed subset $\supset U$ $\Rightarrow$ it is all of X as $U^c = Z$ is nowhere dense.

By uniformity, $g = 0$.

Definition: In the above situation, a function $f \in \mathcal{O}_U(U)$ is meromorphic along the Cartier divisor $\mathcal{I} \subset \mathcal{O}_X$ if it lifts (necessarily uniquely) to $H^0(X, \varinjlim_n \mathcal{I}^{\otimes -n})$

If $\mathcal{I} \subset \mathcal{O}_X$ is a Cartier divisor, one can form $\mathcal{O}_X / \mathcal{I}$.

Definition: Let X be a uniform analytic adic space. A Cartier divisor $\mathcal{I} \subset \mathcal{O}_X$ on X with support Z is closed if the triple $(Z, \mathcal{O}_X/\mathcal{I}, (v_x)_{x \in Z})$ is an adic space.

(Supposed to evoke a closed immersion)

Prop: Let X be uniform, analytic adic space. A cartier divisor $\mathcal{I} \subset \mathcal{O}_X$ is closed $\Leftrightarrow$ $\mathcal{I}(U) \hookrightarrow \mathcal{O}_X(U)$ has closed image $\forall$ open affinoid $U \subset X$.

In that case, for all open affinoid $U = \mathrm{Spa}(R, R^+) \subset X$, the intersection $U \cap Z = \mathrm{Spa}(S, S^+)$ is an affinoid adic space, where $S = R/I$ & S^+ is the int. closure of R^+ in S

Pf : ($\Rightarrow$) Closed cartier divisor $\Rightarrow$ $(\mathcal{O}_X/\mathcal{J})(U)$ is complete & separated

$\Rightarrow$ $\mathcal{J}(U) \hookrightarrow \mathcal{O}_X(U)$ is closed

($\Leftarrow$) Enough to check that $(Z, \mathcal{O}_X/\mathcal{J}, (v_x)_{x \in Z})$ is an adic space locally.

$\therefore$ Assume $X = \mathrm{Spa}(A, A^+)$. $\mathcal{J}(X) = I$ closed in A

$B = A/I$ is a complete Huber rg & $A \longrightarrow B$ is adic

Let B^+ be the integral closure of A^+ in B

Then $Z = \mathrm{Spa}(B, B^+) \xrightarrow{f} X = \mathrm{Spa}(A, A^+)$ is a closed immersion with image $= \mathrm{supp}^{-1} V(I)$.

(both carry subspace topology from $\mathrm{Spv}\, A$, $\therefore$ homeomorphism onto image)

For $U = R\left(\frac{T}{s}\right)$ in $\mathrm{Spa}(A, A^+)$

$f^{-1}U = R\left(\frac{T}{s}\right)$ in $\mathrm{Spa}(B, B^+)$

$$\begin{tikzcd}
\mathcal{O}_Z(f^{-1}U) = A/I\left(\frac{T}{s}\right)^\wedge & & A/I \arrow[ll] & A \arrow[l] \arrow[dl] \\
 & & \mathcal{O}_X(U) \arrow[ull, dashed, "\exists!"'] \arrow[dll] & \\
\mathcal{O}_X(U)/I\mathcal{O}_X(U) \arrow[uu, dotted] & & &
\end{tikzcd}$$

$\therefore$ we have natural maps

$\mathcal{O}_X/\mathcal{J} \longrightarrow \mathcal{O}_Z$

(exists as the image of $\mathrm{Spa}(A/I(\frac{T}{s}))$ lands inside $U \cap Z \subset U$

$$\begin{tikzcd}
\mathcal{O}_X(U)/\mathcal{J}_X(U) & A/I \arrow[l] \arrow[d] & A \arrow[l] \\
 & \mathcal{O}_Z(U \cap Z) \arrow[ul, dotted, "\exists!"] &
\end{tikzcd}$$

(exists as image of Spa $\mathcal{O}_X(U)/\mathcal{J}_X(U)$ lands inside $U \cap Z$)

$\therefore$ $\mathcal{O}_X/\mathcal{J} \cong \mathcal{O}_Z$