NON–GENERIC COMPONENTS OF THE EMERTON–GEE STACK FOR GL_2

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Abstract. Let K be a finite unramified extension of $\mathbf{Q}_p$ with $p > 5$. We study the extremely non–generic irreducible components in the reduced part of the Emerton–Gee stack for GL_2. We show precisely which irreducible components are smooth or normal, and which have Gorenstein or Cohen–Macaulay normalizations, as well as determine the singular loci. We use our results to update expectations about the conjectural categorical p–adic Langlands correspondence.

1. Introduction

Let p be a fixed prime. Let K be a finite extension of $\mathbf{Q}_p$ with residue field k of degree f over $\mathbf{F}_p$, and absolute Galois group G_K. In [EG], Emerton and Gee studied the stack $\mathcal{X}_d$ of étale (φ, Γ)–modules of rank d defined over the formal spectrum of the ring of integers of a large finite extension of $\mathbf{Q}_p$ with residue field $\mathbf{F}$. As discussed in [EGH], $\mathcal{X}_d$ is expected to play the role of the stack of L–parameters in the so far conjectural categorical p–adic Langlands correspondence for $\mathrm{GL}_d(K)$.

By [EG, Thm. 1.2.1], $\mathcal{X}_d$ is a Noetherian formal algebraic stack and its underlying reduced substack $\mathcal{X}_{d,\mathrm{red}}$ is an algebraic stack of finite type over $\mathbf{F}$. The irreducible components of $\mathcal{X}_{d,\mathrm{red}}$ admit a natural labelling by Serre weights, which are the irreducible representations of $\mathrm{GL}_d(k)$ with coefficients in $\mathbf{F}$. Each Serre weight for $\mathrm{GL}_2(k)$ is described by (ordered) pairs of f–tuples of integers $\mathbf{m} = (m_j)_j$ and $\mathbf{n} = (n_j)_j$ with $n_j \in [0, p-1]$ for each j, and is correspondingly denoted $\sigma_{\mathbf{m},\mathbf{n}}$ (see Section 2.4 for details). We say that $\sigma_{\mathbf{m},\mathbf{n}}$ is non–Steinberg if $n_j < p-1$ for some j. Let $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ be the irreducible component of $\mathcal{X}_{2,\mathrm{red}}$ labelled by $\sigma_{\mathbf{m},\mathbf{n}}$. The following is our main theorem, upgrading the main result of [GKKSW].

Theorem 1.1 (Theorem 5.5)**.** *Let $p > 5$, K unramified over $\mathbf{Q}_p$, and $\sigma_{\mathbf{m},\mathbf{n}}$ a non–Steinberg Serre weight. Then the following are true:*

(i) The component $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ is not smooth if and only if either
  - *(a) $n_j = p-2$ for each $j \in \mathbf{Z}/f\mathbf{Z}$, or*
  - *(b) there exists a subset $\{i-k, \dots, i\} \subset \mathbf{Z}/f\mathbf{Z}$ with $n_{i-k} = 0$, $n_j = p-2$ whenever $j \in \{i-k, \dots, i\} \smallsetminus \{i-k, i\}$, and $n_i = p-1$.*

(ii) When (a) holds, $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ is not normal and its normalization admits a smooth–local cover by a Gorenstein and resolution–rational scheme. The non–normal locus on $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ has codimension f and its complement is smooth.

(iii) When (b) holds, $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ is normal but not Cohen–Macaulay. The non–Cohen–Macaulay locus on $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ has codimension ≥ 2 and its complement is smooth.

Taking $\overline{\mathbf{F}}$ to be an algebraic closure of $\mathbf{F}$, $\mathcal{X}_d(\overline{\mathbf{F}}) = \mathcal{X}_{d,\mathrm{red}}(\overline{\mathbf{F}})$ is the groupoid of continuous two–dimensional representations of G_K with coefficients in $\overline{\mathbf{F}}$. A study of the non–normal or non–Cohen–Macaulay loci in the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$ allows us to obtain the following two theorems, which can be viewed as partial generalizations of [San, Thm. 1].

Theorem 1.2 (Theorem 5.10). *Let $p > 3$, K unramified over $\mathbf{Q}_p$, and $\sigma_{\mathbf{m},\mathbf{n}}$ a non–Steinberg Serre weight. The versal ring at $\overline{\rho} \in \mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})(\overline{\mathbf{F}})$ is not normal if and only if $n_j = p-2$ for each j and, as a G_K–representation, $\overline{\rho}$ is of the form*

$$\left(\prod_{j\in\mathbf{Z}/f\mathbf{Z}} \omega_j^{(m_j-1)}\right) \otimes \begin{pmatrix} \mathrm{ur}_{\lambda'} & * \\ 0 & \mathrm{ur}_{\lambda''} \end{pmatrix}$$

where λ' and λ'' are arbitrary units in $\overline{\mathbf{F}}$, and

- *$*$ is the vanishing class if $\lambda' \neq \lambda''$, and*
- *$*$ lies in the 1–dimensional space of extension classes that vanish after restriction to the inertia subgroup if $\lambda' = \lambda''$.*

Theorem 1.3 (Theorem 5.11). *Let $p > 3$, K unramified over $\mathbf{Q}_p$, and $\sigma_{\mathbf{m},\mathbf{n}}$ a non–Steinberg Serre weight. Suppose there exists $i \in \mathbf{Z}/f\mathbf{Z}$ such that $n_{i+1} = 0$, $n_i = p-1$, and $n_j = p-2$ for $j \in \mathbf{Z}/f\mathbf{Z} \smallsetminus \{i, i+1\}$. The versal ring at $\overline{\rho} \in \mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})(\overline{\mathbf{F}})$ is not Cohen–Macaulay if and only if as a G_K–representation, $\overline{\rho}$ is of the form*

$$\left(\omega_{i+1}^{m_{i+1}} \otimes \prod_{j\neq i+1} \omega_j^{m_j-1}\right) \otimes \begin{pmatrix} \mathrm{ur}_{\lambda'} & * \\ 0 & \mathrm{ur}_{\lambda''} \end{pmatrix}$$

where λ' and λ'' are arbitrary units in $\overline{\mathbf{F}}$, and

- *$*$ is vanishing if $\lambda' \neq \lambda''$, and*
- *$*$ lies in the 1–dimensional space of extension classes that vanish after restriction to the inertia subgroup if $\lambda' = \lambda''$.*

Here, $\{\omega_j\}_j$ are choices of G_K–characters extending the f distinct niveau 1 fundamental characters of the inertia subgroup (see Section 2 for details), while $\mathrm{ur}_{\lambda'}$ and $\mathrm{ur}_{\lambda''}$ are the unramified characters mapping the geometric Frobenius to λ' and λ'' respectively. Note that the hypothesis in the statement of Theorem 1.3 does not encapsulate all normal non–Cohen–Macaulay irreducible components unless $f = 2$.

1.1. **Categorical p–adic Langlands.** In [EGH, Conj. 6.1.14], Emerton, Gee and Hellmann conjecture the existence of an exact fully faithful functor $\mathfrak{A}$ from a certain derived category of so–called smooth representations of $\mathrm{GL}_d(K)$ to a certain derived category of quasicoherent sheaves on $\mathcal{X}_d$, satisfying various properties. Without going into the details of what the appropriate categories are and the properties $\mathfrak{A}$ is expected to satisfy, we focus attention on a few consequences of their conjecture laid out in Sections 6 and 7 of *loc. cit.*, restricting to the case $d = 2$. Let $\sigma_{\mathbf{m},\mathbf{n}}$ be a non–Steinberg Serre weight viewed as a representation of $\mathrm{GL}_2(\mathcal{O}_K)$ via inflation, where $\mathcal{O}_K$ is the ring of integers of K. Let $\mathcal{L}(\sigma_{\mathbf{m},\mathbf{n}})$ denote the conjectural sheaf

$$\mathfrak{A}\left(\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O}_K)}^{\mathrm{GL}_2(K)} \sigma_{\mathbf{m},\mathbf{n}}\right).$$

Some of the expectations about $\mathcal{L}(\sigma_{\mathbf{m},\mathbf{n}})$ are as follows:

(1) It is a coherent sheaf concentrated in degree 0 by [EGH, Rmk. 6.1.26] and maximal Cohen–Macaulay on its support $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$. The latter essentially follows from exactness of $\mathfrak{A}$ and Cohen–Macaulay nature of sheaves associated to locally algebraic types (see [EGH, Rmk. 6.1.34]), along with compatibility with geometric Breuil–Mézard conjecture.
(2) Ignoring possible shifts of complexes, $\mathcal{L}(\sigma_{\mathbf{m},\mathbf{n}})$ is Grothendieck–Serre self–dual by [EGH, Rmk. 6.1.35]. Thus, when K is an unramified non–trivial extension of $\mathbf{Q}_p$, our Theorem 1.1 and Lemma 5.9 together imply that $\mathcal{L}(\sigma_{\mathbf{m},\mathbf{n}})$ is the pushforward of a self–dual maximal Cohen–Macaulay sheaf on the normalization of $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$.
(3) When K is an unramified non–trivial extension of $\mathbf{Q}_p$, $\mathcal{L}(\sigma_{\mathbf{m},\mathbf{n}})$ has rank 1 generically on $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$. This follows from combining the data on codimension of non–normal locus in potentially Barsotti–Tate deformation rings given in the proof of [LHMM, Thm. 4.6.10], conjecture about the rank of the generic fiber of sheaves corresponding to locally algebraic types in [EGH, Rmk. 6.1.34], and compatibility with geometric Breuil–Mézard conjecture. The key point is that one can always find a non–scalar tame inertial type τ such that $\sigma_{\mathbf{m},\mathbf{n}}$ appears in the Jordan–Holder decomposition of the $\mathrm{GL}_2(\mathcal{O}_K)$–representation associated to τ by inertial local Langlands, and such that the potentially Barsotti–Tate deformation rings of type τ are regular in codimension 1.

Motivated by these expectations, we obtain the following theorem, wherein the first part is a corollary of the statement about codimension of singular locus in Theorem 1.1.

Theorem 1.4 (Theorem 5.7)**.** *Let $p > 5$, K an unramified non–trivial extension of $\mathbf{Q}_p$, and $\sigma_{\mathbf{m},\mathbf{n}}$ a non–Steinberg Serre weight. Let $\iota : \mathcal{U} \hookrightarrow \mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ be the smooth open locus in $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$. Suppose $\mathcal{F}$ is a finite type maximal Cohen–Macaulay sheaf on $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ generically of rank 1. The following are true:*

(i) The sheaf $\mathcal{F}$ is isomorphic to the pushforward along ι of the invertible sheaf $\iota^*\mathcal{F}$ on $\mathcal{U}$.
(ii) If there does not exist i such that $(n_{i-1}, n_i) = (0, p-1)$, then $\mathcal{F}$ is the pushforward of a unique invertible sheaf on a smooth algebraic stack of Breuil–Kisin modules admitting a proper map onto $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$.

In the setup of Theorem 1.4 above and assuming a reasonable notion of a dualizing complex on an algebraic stack, note that if $\mathcal{F}$ is (Grothendieck–Serre) self–dual, then so is $\iota^*\mathcal{F}$. As we will see later in the proof, $\mathcal{U}$ is isomorphic to its preimage in the aforementioned stack of Breuil–Kisin modules. If the hypothesis in (ii) holds, then the codimension of the complement of the preimage of $\mathcal{U}$ turns out to be ≥ 2. Thus, by an argument involving the algebraic Hartog's Lemma on the dual of a line bundle on a smooth variety, $\mathcal{F}$ is seen to be the (non–derived) pushforward of a self–dual invertible sheaf on the stack of Breuil–Kisin modules we are considering. Hence, we expect that one can uniquely characterize $\mathcal{L}(\sigma_{\mathbf{m},\mathbf{n}})$ for all non–Steinberg $\sigma_{\mathbf{m},\mathbf{n}}$ as the pushforward of the unique self–dual invertible sheaf on $\mathcal{U}$ (equivalently, on an appropriate stack of Breuil–Kisin modules when the hypothesis in (ii) holds), and indeed use this characterization as a key ingredient towards constructing $\mathfrak{A}$ (*c.f.* [EGH, Sec. 7.6.10]).

1.2. **Strategy and outline.** We begin in Section 2 by setting up some standard notation and definitions. In Section 3, we review many of the constructions from [LHMM] that we use essentially. These constructions pertain to smooth–local charts on various closed substacks of $\mathcal{X}_{2,\mathrm{red}}$, as well as explicit auxiliary schemes through which maps from certain stacks of Breuil–Kisin modules to $\mathcal{X}_{2,\mathrm{red}}$ factor locally in the smooth topology. The constraints on p and the requirement for K to be unramified over $\mathbf{Q}_p$ appear in this section, most critically in the proof of Proposition 3.7. We use the results of [BBH$^+$] to impose "shape" conditions that cut out irreducible components in these charts, thus setting up smooth–local charts for the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$.

In Section 4, we undertake a detailed study of the geometry of these charts. We start off by analyzing the additional relations on the auxiliary schemes that come from imposing shape conditions. Next, in Section 4.2, we make the crucial observation that the charts for the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$ can be written as (typically non–trivial) products of varieties, with each factor in the product somewhat easier to study. Each factor admits a resolution of singularities by a smooth scheme, and we determine the locus where this resolution fails to be an isomorphism. The key argument is in Lemma 4.8. Next, by doing factor–wise computations, we obtain the cohomologies of the dualizing complex for the charts following the approach in [LHMM], which reduces the computations to those for the auxiliary schemes and Koszul resolutions.

Towards the end of this section, we describe the non–normal and non–Cohen–Macaulay loci as the vanishing loci of explicit ideals, as well as compute the dimensions of these loci. We also establish that the singular locus is precisely the non–normal/non–Cohen–Macaulay locus by noting that the smooth–local cover for an irreducible component of $\mathcal{X}_{2,\mathrm{red}}$ fails to be isomorphic to its resolution of singularities at precisely this locus.

Finally, we prove Theorems 5.5 and 5.7 in Section 5 by doing various combinatorial calculations and applying the results from Section 4 to the irreducible components of $\mathcal{X}_{2,\mathrm{red}}$ labelled by specific Serre weights. We also determine explicitly the Galois representations corresponding to finite type points of singular loci in the setup of Theorems 5.10 and 5.11.

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2. Notation and background

Fix a prime $p > 3$. Let K be a finite unramified extension of $\mathbf{Q}_p$ of degree f with ring of integers $\mathcal{O}_K$ and residue field k. Fix an algebraic closure $\overline{K}$ of K. For any algebraic extension L of K in $\overline{K}$, denote by G_L the group $\mathrm{Gal}(\overline{K}/L)$. Denote by I_L the inertia subgroup of G_L. Let $\pi' \in \overline{K}$ be a fixed $(p^f - 1)$-th root of p. Let K' be a tame extension of K obtained by attaching π'.

Let $\mathbf{F}$ be a finite extension of $\mathbf{F}_p$ that is the residue field of the ring of integers $\mathcal{O}$ of a finite field extension E of $\mathbf{Q}_p$ with uniformizer ϖ. Denote by $\overline{\mathbf{F}}$ a fixed algebraic closure of $\mathbf{F}$. We take $\mathbf{F}$ to be large enough so that all embeddings $k \hookrightarrow \overline{\mathbf{F}}$ factor through $\mathbf{F}$. Fix an embedding $\sigma_0 : k \hookrightarrow \mathbf{F}$ and let

$$\sigma_{f-j} := \sigma_0^{p^j}.$$

The map $j \mapsto \sigma_j$ induces an identification of sets

$$\mathbf{Z}/f\mathbf{Z} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{F}_p}(k, \mathbf{F}) = \mathrm{Hom}_{\mathbf{Z}_p}(\mathcal{O}_K, \mathcal{O}).$$

For each $j \in \mathbf{Z}/f\mathbf{Z}$, let $\omega_j : G_K \to \mathcal{O}^\times$ be the character given by

$$g \mapsto \sigma_j \left(\frac{g(\pi')}{\pi'} \right).$$

Abusing notation, we will denote the mod ϖ reduction of ω_j also by ω_j when it is clear that we are speaking of $\mathbf{F}$–coefficients. We will also denote the restriction $\omega_j|_{I_K}$ by ω_j when it is clear that we are speaking of I_K–representations. For λ a nonzero element of $\overline{\mathbf{F}}$, let

$$\mathrm{ur}_\lambda : G_K \to \overline{\mathbf{F}}^\times$$

be the unramified character mapping the geometric Frobenius element to λ.

2.1. **Tame inertial types.** A tame inertial type is the isomorphism class of a representation $\tau : I_K \to \mathrm{GL}_2(\mathcal{O})$ which has an open kernel, factors through the tame quotient of I_K, and extends to G_K. Such a representation is of the form $\tau \cong \eta_1 \oplus \eta_2$. We say that τ is a *principal series* tame type if both η_1 and η_2 extend to characters of G_K, cuspidal otherwise. It is *non–scalar* if $\eta_1 \neq \eta_2$. When τ is a principal series type, η_1 and η_2 factor through $I_K \twoheadrightarrow \mathrm{Gal}(K'/K)$, see for e.g. [Bre, Sec. 2]. In this article, it will suffice to restrict attention to principal series tame types, for which we now introduce notation from [LHMM].

Denote by $W = \{\mathrm{id}, w_0\}$ the Weyl group of GL_2 (defined over $\mathbf{Z}$). Here $w_0 = (1\ 2)$ is the longest element of the Weyl group. Let $B \subset \mathrm{GL}_2$ be the Borel subgroup of upper triangular matrices and $T \subset B$ the subgroup of diagonal matrices. We identify the group of its characters $X^*(T)$ with $\mathbf{Z}^2$ in the standard way. Let α denote the positive root of GL_2, and let $\langle\ ,\ \rangle : X^*(T) \times X_*(T) \to \mathbf{Z}$ be the duality pairing where $X_*(T)$ is the group of cocharacters of T. The Weyl group W acts naturally on $X^*(T)$. We extend this to a coordinate–wise action of $W^{\mathbf{Z}/f\mathbf{Z}}$ on $X^*(T)^{\mathbf{Z}/f\mathbf{Z}}$. Define

$$\widetilde{W} \stackrel{\text{def}}{=} X^*(T) \rtimes W$$

Denote by t_ν the image of $\nu \in X^*(T)$ under the obvious inclusion $X^*(T) \hookrightarrow \widetilde{W}$.

Suppose $\mu = (\mu_j) \in X^*(T)^{\mathbf{Z}/f\mathbf{Z}}$ and $s = (s_j)_j \in W^{\mathbf{Z}/f\mathbf{Z}}$ satisfying $s_0 s_1 \dots s_{f-1} = 1$. Let $\alpha_0 = \mu_0$ and $\alpha_j = s_{f-1}^{-1} s_{f-2}^{-1} \dots s_{f-j}^{-1}(\mu_{f-j})$ for $j \in \mathbf{Z}/f\mathbf{Z} \smallsetminus \{0\}$. For each $j \in \mathbf{Z}/f\mathbf{Z}$, let

$$\mathbf{a}^{(j)} = \sum_{i=0}^{f-1} \alpha_{-j+i} p^i \in X^*(T)$$

where α_l for $0 \leq l \leq f-1$ is to be interpreted as $\alpha_{l \bmod f}$. Viewing $\mathbf{a}^{(j)}$ as a cocharacter of the dual torus, we set

$$\tau(s, \mu) \stackrel{\text{def}}{=} \mathbf{a}^{(j)} \omega_j$$

for any $j \in \mathbf{Z}/f\mathbf{Z}$, since this definition does not depend on j. By [LHMM, Lem. 2.1.6], for any principal series τ, there exist μ and s such that $\langle \mu_j, \alpha^\vee \rangle \in [0, (p+1)/2]$ for each j, s_j is id whenever $\langle \mu_j, \alpha^\vee \rangle = 0$, and $\tau \cong \tau(s, \mu)$. By Lem. 2.1.8 in *loc.*

cit., whenever τ is non–scalar, there exists a unique $(s_{\mathrm{or},j})_j \in W^{\mathbf{Z}/f\mathbf{Z}}$ such that $\langle s_{\mathrm{or},j}^{-1}\left(\mathbf{a}^{(j)}\right), \alpha^\vee\rangle > 0$.

Remark 2.1. Since our application does not require cuspidal types, we don't include notation needed to describe them. However, we note that cuspidal types can also be described using the data of suitable s and μ where $s_0 s_1 \dots s_{f-1}$ equals w_0 (see for e.g. Section 2 in [LHMM]). Furthermore, everything in Sections 2, 3 and 4 can be generalized to include cuspidal types as well.

Definition 2.2. Let $\mu = (\mu_j)_j \in X^*(T)^{\mathbf{Z}/f\mathbf{Z}}$. We say μ is *small* if for each j, $\langle \mu_j, \alpha^\vee\rangle \in [0, (p+1)/2]$.

Definition 2.3. Define $\eta \in X^*(T)$ to be the element $(1,0)$. Abusing notation, we also let $\eta \in X^*(T)^{\mathbf{Z}/f\mathbf{Z}}$ be the element that is $(1,0)$ in each coordinate.

2.2. **Breuil–Kisin modules.** Let $\mathfrak{S} \stackrel{\text{def}}{=} W(k)[[u]]$, where $W(k)$ is the ring of Witt vectors of k. The ring $\mathfrak{S}_{K'}$ is equipped with a Frobenius endomorphism φ that extends the usual arithmetic Frobenius on $W(k)$ lifted from the p^f-power map on k, and maps u to u^p. It also admits an action of $\mathrm{Gal}(K'/K)$ extending the usual trivial action of $\mathrm{Gal}(K'/K)$ on $W(k)$, so that if $g \in \mathrm{Gal}(K'/K)$, then

$$g(u) = \frac{g(\pi')}{\pi'}u.$$

Let $E(u)$ denote the minimal polynomial of π' over $W(k)$. The subring $\mathfrak{S}^0$ of $\mathrm{Gal}(K'/K)$–invariants of $\mathfrak{S}$ is $W((k))[[v]]$ where

$$v := u^{p^f-1}.$$

For a $\mathcal{O}/\varpi^a$–algebra A where $a \geq 1$, let $\mathfrak{S}_A \stackrel{\text{def}}{=} (W(k) \otimes_{\mathbf{Z}_p} A)[[u]]$ and equip it with A–linear actions of φ and $\mathrm{Gal}(K'/K)$ extended naturally from the φ and $\mathrm{Gal}(K'/K)$ actions on $\mathfrak{S}$. The subring $\mathfrak{S}_A^0$ of $\mathrm{Gal}(K'/K)$–invariants of $\mathfrak{S}_A$ is $(W(k) \otimes_{\mathbf{Z}_p} A)[[v]]$. Let τ be a principal series type.

Definition 2.4. A Breuil–Kisin module $\mathfrak{M}$ of rank 2 with A–coefficients and descent data of type τ is a rank 2 projective $\mathfrak{S}_A$–module $\mathfrak{M}$ together with

- a φ–semilinear map $\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$ whose linearization is an isomorphism after inverting u, and
- a semilinear action of $\mathrm{Gal}(K'/K)$ on $\mathfrak{M}$ commuting with $\varphi_{\mathfrak{M}}$ such that Zariski-locally on $\operatorname{Spec} A$
$$\mathfrak{M} \otimes_{k,\sigma_j} A \mod u \cong \tau^\vee \otimes_{\mathcal{O}} A$$
as $\mathrm{Gal}(K'/K)$–representations.

We say that $\mathfrak{M}$ has *height at most h* if the cokernel of $\Phi_{\mathfrak{M}}$ is annihilated by $E(u)$.

Let $\mathfrak{M}$ be a Breuil–Kisin module of rank 2 with A–coefficients and descent data of type τ. For each $\sigma_j : W(k) \to \mathcal{O}$, there is a corresponding idempotent $\mathfrak{e}_j \in W(k) \otimes_{\mathbf{Z}_p} \mathcal{O}$ such that $x \otimes 1$ and $1 \otimes \sigma_j(x)$ have the same action on $\mathfrak{e}_j(W(k) \otimes_{\mathbf{Z}_p} \mathcal{O})$, a rank 1 $\mathcal{O}$–module. Set $\mathfrak{M}_j = \mathfrak{e}_j\mathfrak{M}$, a module over $A[[u]]$, and let

$$\Phi_{\mathfrak{M},j} : \varphi^*(\mathfrak{M}_{j-1}) \to \mathfrak{M}_j$$

be the map induced by $\Phi_{\mathfrak{M}}$. Each $\mathfrak{M}_j$ is Zariski locally on $\operatorname{Spec} A$ free as an $A[[u]]$ module by [BBH+, Lem. 2.3].

Suppose $\tau \cong \tau(s,\mu)$ is non–scalar. For $j \in \mathbf{Z}/f\mathbf{Z}$, let $\mathbf{a}_0^{(j)}, \mathbf{a}_1^{(j)} \in \mathbf{Z}$ be such that $\mathbf{a}^{(j)} = (\mathbf{a}_0^{(j)}, \mathbf{a}_1^{(j)})$. By the argument in [GKKSW, Lem. 2.1.3] for e.g., Zariski locally on $\operatorname{Spec} A$ one can choose an ordered basis $\beta_j = (e_j, f_j)$ of $\mathfrak{M}_j$ so that $\mathrm{Gal}(K'/K)$ acts on e_j, f_j via

$$\omega_j^{-\mathbf{a}_0^{(j)}}, \omega_j^{-\mathbf{a}_1^{(j)}}$$

respectively. Following convention, we call a $\mathbf{Z}/f\mathbf{Z}$–tuple $\beta = (\beta_j)_j$ of such ordered bases an *eigenbasis* of $\mathfrak{M}$. Given an eigenbasis β of $\mathfrak{M}$, let $C^{(j)}_{\mathfrak{M},\beta}$ be the matrix of $\Phi_{\mathfrak{M},j}$ with respect to the bases $\varphi^*(\beta_{j-1})$ and β_j. For each $j \in \mathbf{Z}/f\mathbf{Z}$, set

$$A^{(j)}_{\mathfrak{M},\beta} \overset{\text{def}}{=} \mathrm{Ad}\left(s_{\mathrm{or},j}^{-1} u^{-\mathbf{a}^{(j)}}\right)(C^{(j)}_{\mathfrak{M},\beta}).$$

Here, the notation $\mathrm{Ad}\,A(B)$ means ABA^{-1} and if $\nu = (\nu_0,\nu_1) \in X^*(T)$ and $x \in \mathfrak{S}_A$, then x^ν is the diagonal matrix $\left(\begin{smallmatrix} x^{\nu_0} & 0 \\ 0 & x^{\nu_1}\end{smallmatrix}\right)$.

Proposition 2.5. [LLHLM, Prop. 5.1.8] *Let $\mathfrak{M}$ be a Breuil–Kisin module of rank 2 with A–coefficients and descent data of principal series non–scalar type $\tau \cong \tau(s,\mu)$. Let β_1, β_2 be two eigenbases of $\mathfrak{M}$ related via*

$$\beta_{2,j} D^{(j)} = \beta_{1,j}$$

with $D^{(j)} \in \mathrm{GL}_2(A[[u]])$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. Set

$$I^{(j)} \overset{def}{=} \mathrm{Ad}\left(s_{\mathrm{or},j}^{-1} u^{-\mathbf{a}^{(j)}}\right)\left(D^{(j)}\right).$$

Then $I^{(j)} \in \mathrm{GL}_2(A[[v]])$ is upper triangular mod v, and

$$A^{(j)}_{\mathfrak{M},\beta_2} = I^{(j)} A^{(j)}_{\mathfrak{M},\beta_1} \mathrm{Ad}\left(s_j^{-1} v^{\mu_j}\right)\left(\varphi\left(I^{(j-1)}\right)\right)^{-1}.$$

Furthermore, if $I^{(j)} \in \mathrm{GL}_2(A[[v]])$ upper triangular mod v for each $j \in \mathbf{Z}/f\mathbf{Z}$, then $\mathrm{Ad}\left(u^{\mathbf{a}^{(j)}} s_{\mathrm{or},j}\right)\left(I^{(j)}\right) = D^{(j)} \in \mathrm{GL}_2(A[[u]])$ and for any eigenbasis β, $\left(\beta^{(j)} D^{(j)}\right)_j$ is also an eigenbasis.

Definition 2.6. Following [LHMM], let $Y^{\eta,\tau}$ be the *fppf* stack over $\mathrm{Spf}(\mathcal{O})$ that assigns to a $\mathcal{O}/\varpi^a$–algebra A the groupoid of Breuil–Kisin modules of rank 2 with A–coefficients, descent data of type τ and height at most 1 satisfying the additional *determinant condition*

$$\det(\Phi_{\mathfrak{M}}) \in vA[[v]]^\times.$$

Let $Y^{\eta,\tau}_{\mathbf{F}}$ denote the special fiber of this stack.

Remark 2.7. By [BBH+, Prop. 2.7] and [CEGSb, Cor. 4.5.3(2)],

$$Y^{\eta,\tau}_{\mathbf{F}} = \mathcal{C}^{\tau^\vee,\mathrm{BT},1}$$

where the right hand side is the stack studied in [CEGSc].

We have the following description of the irreducible components of $Y^{\eta,\tau}_{\mathbf{F}}$.

Theorem 2.8. *There exists a bijective correspondence between $\{L,R\}^{\mathbf{Z}/f\mathbf{Z}}$ and the set of irreducible components of $Y^{\eta,\tau}_{\mathbf{F}}$ given in the following way: If*

$$S = (S_j)_j \in \{L,R\}^{\mathbf{Z}/f\mathbf{Z}},$$

then the corresponding irreducible component $Y^{\eta,\tau}_S$ is the closed substack of $Y^{\eta,\tau}_{\mathbf{F}}$ obtained by imposing the condition that if $S_j = L$ (resp. $S_j = R$), then $\mathfrak{M} \in Y^{\eta,\tau}_S(A)$

for a local $\mathbf{F}$*–algebra* A *if and only if* v *divides the top left (resp. bottom right) entry of* $A^{(j)}_{\mathfrak{M},\beta}$ *for some, equivalently any, choice of eigenbasis* β *of* $\mathfrak{M}$*.*

Proof. Immediate from [BBH+, Thm. 3.16]. □

2.3. **Étale φ-modules and Galois representations.** Let A be a $\mathcal{O}/\varpi^a$–algebra for some $a \geq 1$.

Definition 2.9. An étale φ–module $\mathcal{M}$ of rank 2 with A–coefficients is a rank 2 projective module over $\mathfrak{S}^0_A[1/v]$, together with a φ–semilinear map $\varphi_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ whose linearization $\Phi_{\mathcal{M}} : \varphi^*\mathcal{M} \to \mathcal{M}$ is an isomorphism.

Let $\mathcal{M}$ be an étale φ–module of rank 2 with A–coefficients. As in the setting of Breuil–Kisin modules, we can decompose $\mathcal{M} \cong \oplus_{j\in \mathbf{Z}/f\mathbf{Z}}\mathcal{M}_j$ by setting $\mathcal{M}_j := \mathfrak{e}_j\mathcal{M}$. The map $\Phi_{\mathcal{M}}$ induces maps $\Phi_{\mathcal{M},j} : \varphi^*\mathcal{M}_{j-1} \to \mathcal{M}_j$.

Definition 2.10. Let $\Phi\text{-Mod}^{\text{ét},2}_K$ denote the *fppf* stack over $\mathrm{Spf}(\mathcal{O})$ that assigns to a $\mathcal{O}/\varpi^a$–algebra A the groupoid of rank 2 étale φ–modules of rank 2 with A coefficients.

Define a map

$$\varepsilon_\tau : Y^{\eta,\tau} \to \Phi\text{-Mod}^{\text{ét},2}_K$$

by setting $\varepsilon_\tau(\mathfrak{M}) = \mathfrak{M}[1/u]^{\mathrm{Gal}(K'/K)}$.

Remark 2.11. The map ε_τ is proper by [CEGSb, Thm. 5.1.2] and by [BBH+, Thm. 4.5], the scheme–theoretic image $\mathcal{Z}^\tau$ of ε_τ is isomorphic to the Emerton–Gee stack of potentially Barsotti-Tate representations of type $\tau^\vee$, described in [EG, Defn. 4.8.8]. Denote this locus by $\mathcal{X}^{\tau^\vee,\mathrm{BT}}$. The scheme–theoretic image of $Y^{\eta,\tau}_{\mathbf{F}}$ is the reduced stack $\mathcal{Z}^{\tau,1}$. Denote by π the induced map $Y^{\eta,\tau}_{\mathbf{F}} \to \mathcal{Z}^{\tau,1}$.

Definition 2.12. For $S \in \{L, R\}^{\mathbf{Z}/f\mathbf{Z}}$, denote by $\mathcal{Z}^\tau_S$ the scheme–theoretic image of $Y^{\eta,\tau}_S$ under π.

Proposition 2.13. [LLHLM, Prop. 5.4.2] *Let* $\mathfrak{M} \in Y^{\eta,\tau}(A)$ *and* β *an eigenbasis of* $\mathfrak{M}$*. Let* $\tau = \tau(s, \mu)$*. Then there exists a basis* $\mathfrak{b}$ *for* $\varepsilon_\tau(\mathfrak{M})$ *such that the matrix of* $\Phi_{\mathcal{M},j}$ *with respect to* $\mathfrak{b}$ *is given by* $A^{(j)}_{\mathfrak{M},\beta}s_j^{-1}v^{\mu_j}$*.*

Finally, we describe how to assign a Galois representation to an étale φ–module. Fix a compatible sequence $\{\pi_n\}_n$ of p^n-th roots of p in $\overline{K}$, with $\pi^p_{n+1} = \pi_n$. Since $\gcd(e(K'/K), p) = 1$, $\{\pi_n\}^\infty_{n=0}$ determines a compatible sequence $\{\pi'_n\}^\infty_{n=0}$ of p^n-th roots of π' satisfying $(\pi'_n)^{e(K'/K)} = \pi_n$. Let

$$K_\infty := \cup_n K(\pi_n), \text{ and } \quad K'_\infty := \cup_n K'(\pi'_n).$$

By Fontaine's theory of the field of norms, if $|A| < \infty$, then there exists a fully faithful functor T from the étale φ–modules with A–coefficients to G_{K_∞}–representations with A–coefficients. To describe this functor, we first define $R := \lim_{x\mapsto x^p} \mathcal{O}_{\overline{K}}/p$. Let $\underline{\pi}' = (\pi'_n)_n \in R$ and let $[\underline{\pi}']$ be the canonical multiplicative lift of $\underline{\pi}'$ to the Witt vectors of R, $W(R)$. There exists a φ–equivariant inclusion $\mathfrak{S} \hookrightarrow W(R)$ over $W(k)$ that maps $u \to [\underline{\pi}']$. This embedding extends to inclusions

$$\mathcal{O}_{\mathcal{E}} \hookrightarrow W(\mathrm{Frac}(R))$$

and

$$\mathcal{E} \hookrightarrow W(\mathrm{Frac}(R))[1/p]$$

where $\mathcal{O}_{\mathcal{E}}$ is the p–adic completion of $\mathfrak{S}[1/u]$ and $\mathcal{E}$ is its ring of fractions. The ring $\mathcal{O}_{\mathcal{E}}$ is a discrete valuation ring with uniformizer p and residue field $k((u))$. Let $\mathcal{E}^{\mathrm{nr}}$ be the maximal unramified extension of $\mathcal{E}$ in $W(\mathrm{Frac}(R))[1/p]$ with ring of integers $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$ and residue field $k((u))^{\mathrm{sep}}$, a separable closure of $k((u))$. Let $\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}}$ be the p–adic completion of $\mathcal{O}_{\mathcal{E}^{\mathrm{nr}}}$.

Definition 2.14. Let $|A| < \infty$. If $\mathcal{M} \in \Phi\text{-}\mathrm{Mod}_K^{\text{ét},2}(A)$, set

$$T(\mathcal{M}) \stackrel{\text{def}}{=} \left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}^0[1/v]} \mathcal{M}\right)^{\varphi=1}$$

equipped with diagonal action of the group G_{K_∞}. We also define $T(\mathfrak{M})$ for $\mathfrak{M}$ a Breuil–Kisin module with A–coefficients and descent data by setting

$$T(\mathfrak{M}) \stackrel{\text{def}}{=} T\left(\varepsilon_\tau(\mathfrak{M})\right).$$

Using [EG, Thm. 2.4.1, 2.7.8] for e.g., we note that the right hand side above equals

$$\left(\mathcal{O}_{\widehat{\mathcal{E}^{\mathrm{nr}}}} \otimes_{\mathfrak{S}[1/u]} \mathfrak{M}[1/u]\right)^{\varphi=1},$$

which agrees with the definition of $T(\mathfrak{M})$ in [CEGSc, Defn. 2.2.3].

2.4. **Serre weights and the Emerton–Gee stack.** A Serre weight is an isomorphism class of an irreducible $\mathbf{F}$–representation of $\mathrm{GL}_2(k)$. Such representations are precisely those of the form

$$\sigma_{\mathbf{m},\mathbf{n}} := \bigotimes_{j\in\mathbf{Z}/f\mathbf{Z}} \left(\det{}^{m_j} \otimes \mathrm{Sym}^{n_j} k^2\right) \otimes_{k,\sigma_j} \mathbf{F}$$

where k^2 denotes the standard two–dimensional representation of $\mathrm{GL}_2(k)$ and $0 \le n_j \le p-1$ for each j. The representation is non–Steinberg if for some j, $n_j < p-1$.

If $\sigma_{\mathbf{m},\mathbf{n}}$ is a non–Steinberg Serre weight, then by the main result of [CEGSa], the corresponding irreducible component $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ of $\mathcal{X}_{2,\mathrm{red}}$ is the locus of mod p representations that admit crystalline lifts with Hodge–Tate weights $\{-n_j - m_j, -m_j+1\}$ in the j-th embedding. Here, we are normalizing Hodge–Tate weights so that all Hodge–Tate weights of the cyclotomic character are equal to -1.

3. SMOOTH–LOCAL CHARTS

3.1. **Loop groups and torsors over $Y_{\mathbf{F}}^{\eta,\tau}$ and $\mathcal{Z}^{\tau,1}$.** From now onwards, we will work entirely over $\mathbf{F}$–coefficients, although as described in detail in [LHMM], many of the descriptions below extend to $\mathcal{O}$–coefficients. Our primary objective in this subsection is to construct smooth–local affine charts on $\mathcal{Z}^{\tau,1}$ following closely various constructions in [LHMM, Sec. 3].

Let $\tau = \tau(s,\mu)$ be a fixed non–scalar principal series tame type with $\mu = (\mu_j)_j$ small. Assume that $p-2 > \max_j \langle \mu_j, \alpha^\vee\rangle$. This is true if $p > 3$ and $\max_j \langle \mu_j, \alpha^\vee\rangle \le 2$, or if $p > 5$. Let $\mu_j = (\mu_{j,0}, \mu_{j,1})$ for each j. Define a functor LG by setting for an $\mathbf{F}$–algebra A

$$LG(A) := \mathrm{GL}_2(A((v)))$$

as well as various subfunctors

$$\begin{aligned}
L^+\mathrm{G}(A) &:= \quad \mathrm{GL}_2(A[[v]]) \\
L^-\mathrm{G}(A) &:= \quad \mathrm{GL}_2(A[1/v]) \\
L^+\mathcal{G}(A) &:= \quad \{P \in L^+\mathrm{G}(A) \mid P \text{ is upper triangular mod } v\} \\
L^+_r\mathrm{G}(A) &:= \quad \{P \in L^+\mathcal{G}(A) \mid P \equiv \mathrm{id} \mod v^r\} \\
L^-_r\mathrm{G}(A) &:= \quad \{P \in L^-\mathrm{G}(A) \mid P \equiv \mathrm{id} \mod 1/v^r\} \\
\mathcal{A}(\eta)(A) &:= \quad \{P \in \mathrm{Mat}_2(A[[v]]) \mid P \text{ is upper triangular mod } v, \det P \in vA[[v]]^\times\}
\end{aligned}$$

where r is any positive integer in the definition of $L^+_r\mathrm{G}$ and $L^-_r\mathrm{G}$. Define

$$L\mathrm{G}^{\mathrm{bd},vv^\mu}(A) \subset L\mathrm{G}(A)^{\mathbf{Z}/f\mathbf{Z}}$$

to be the set of $(P_j)_j \in L\mathrm{G}(A)^{\mathbf{Z}/f\mathbf{Z}}$ satisfying, for each j,

- $P_j \in v^{\mu_{j,1}}\,\mathrm{Mat}_2(A[[v]])$, and
- $\det P_j \in v^{\mu_{j,0}+\mu_{j,1}+1}A[[v]]^\times$.

Remark 3.1. We are allowing nonzero values of $\mu_{j,1}$, and so, this definition of $L\mathrm{G}^{\mathrm{bd},vv^\mu}$ is slightly different from the mod p version of the one in [LHMM].

Define

$$L\mathrm{G}^\tau(A) := \{(W_j s_j^{-1} v^{\mu_j})_j \in L\mathrm{G}(A)^{\mathbf{Z}/f\mathbf{Z}} \mid W_j \in \mathcal{A}(\eta)(A) \text{ for each } j\} \subset L\mathrm{G}^{\mathrm{bd},vv^\mu}.$$

Definition 3.2. We define *shifted φ–conjugation* to be a left action of $L\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}$ (and its various subfunctors) on itself denoted by $\cdot_\varphi$ and given by setting

$$(P_j)_j \cdot_\varphi (Q_j)_j \overset{\mathrm{def}}{=} \left(P_j Q_j \varphi(P_{j-1})^{-1}\right).$$

Lemma 3.3. [LHMM, Lem. 3.2.1] *Let $\mathfrak{M} \in Y^{\eta,\tau}_{\mathbf{F}}(A)$ for an an $\mathbf{F}$–algebra A. Then, Zariski-locally on* $\operatorname{Spec} A$, *$\mathfrak{M}$ has an eigenbasis β. The assignment*

$$\mathfrak{M} \mapsto \left(A^{(j)}_{\mathfrak{M},\beta} s_j^{-1} v^{\mu_j}\right)_{j\in\mathbf{Z}/f\mathbf{Z}}$$

defines an isomorphism of algebraic stacks over $\mathbf{F}$

$$Y^{\eta,\tau}_{\mathbf{F}} \xrightarrow{\sim} \left[L\mathrm{G}^\tau \Big/_{\varphi} \left(L^+\mathcal{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\right] \tag{3.4}$$

and hence a morphism

$$Y^{\eta,\tau}_{\mathbf{F}} \to \left[L\mathrm{G}^{\mathrm{bd},vv^\mu} \Big/_{\varphi} \left(L^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\right]. \tag{3.5}$$

Define a morphism

$$\left[L\mathrm{G}^{\mathrm{bd},vv^\mu} \Big/_{\varphi} \left(L^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\right] \to \Phi\text{-}\mathrm{Mod}^{\text{ét},2}_K \tag{3.6}$$

by sending the class of $P = (P_j)_j$ to the étale φ–module $\iota(P)$ which is free of rank 2 and such that $\Phi_{\iota(P),j} : \varphi^*\left(\iota(P)_{j-1}\right) \to \iota(P)_j$ has matrix P_j in the standard basis.

Proposition 3.7. [LHMM, Prop. 3.2.4] *The map (3.6) is a closed immersion, and the map $\varepsilon_\tau|_{Y^{\eta,\tau}_{\mathbf{F}}}$ factors as*

$$Y^{\eta,\tau}_{\mathbf{F}} \xrightarrow{\pi} \mathcal{Z}^{\tau,1} \hookrightarrow \left[L\mathrm{G}^{\mathrm{bd},vv^\mu} \Big/_{\varphi} \left(L^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\right] \xhookrightarrow{(3.6)} \Phi\text{-}\mathrm{Mod}^{\acute{e}t,2}_K \tag{3.8}$$

where the composite of the first two arrows is the map (3.5).

Definition 3.9. Define a quotient sheaf

$$\mathrm{Gr}_1^{\mathrm{bd},vv^\mu} := \left[\left(L_1^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\Big\backslash L\mathrm{G}^{\mathrm{bd},vv^\mu}\right]$$

and its closed subsheaf

$$\mathrm{Gr}_1^\tau := \left[\left(L_1^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\Big\backslash L\mathrm{G}^\tau\right],$$

where the quotient is for action by left multiplication.

Remark 3.10. The sheaves $\mathrm{Gr}_1^{\mathrm{bd},vv^\mu}$ and Gr_1^τ are represented by finite type schemes. Indeed, the sheaf $\mathrm{Gr}_1^{\mathrm{bd},vv^\mu}$ is a torsor for an affine scheme over the quotient

$$\left[\left(L^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\Big\backslash L\mathrm{G}^{\mathrm{bd},vv^\mu}\right],$$

which in turn is a closed subscheme of a finite type scheme by the argument in [Zhu, Lem. 1.1.5]. Therefore, by [Sta, Tag 0245], $\mathrm{Gr}_1^{\mathrm{bd},vv^\mu}$ is representable.

Lemma 3.11. *There exists an isomorphism*

$$\mathrm{Gr}_1^{\mathrm{bd},vv^\mu} \cong \left[L\mathrm{G}^{\mathrm{bd},vv^\mu}\Big/_{\varphi}\left(L_1^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\right]$$

which induces an isomorphism

$$Y_{\mathbf{F}}^{\eta,\tau} \cong \left[\mathrm{Gr}_1^\tau\Big/_{\varphi}\left(B^{\mathbf{Z}/f\mathbf{Z}}\right)\right]$$

Proof. By [LHMM, Lem. 3.3.7]. □

Remark 3.12. The proofs of Proposition 3.7 and Lemma 3.11 critically use the assumption that $p-2 > \max_j\langle\mu_j,\alpha^\vee\rangle$.

Remark 3.13. If $(P_j)_j$ represents a point of $\mathrm{Gr}_1^{\mathrm{bd},vv^\mu}$ and $(g_j)_j$ of

$$\mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}} \cong \left(L^+\mathrm{G}/L_1^+\mathrm{G}\right)^{\mathbf{Z}/f\mathbf{Z}},$$

then we have

$$(g_j)_j \cdot_\varphi (P_j)_j = (g_jP_jg_{j-1}^{-1}).$$

Therefore, φ–action induces simply a *shifted conjugation* action of $\mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}}$ and its subgroups on $\mathrm{Gr}_1^{\mathrm{bd},vv^\mu}$.

Lemma 3.14. *The stack $Y_S^{\eta,\tau}$ is smooth.*

Proof. The inclusion

$$\mathrm{Ad}\begin{pmatrix}1 & \\ & v\end{pmatrix}\left(L_2^+\mathrm{G}\right) \subset L_1^+\mathrm{G}$$

allows the construction of a map

$$\prod_{j\in\mathbf{Z}/f\mathbf{Z}}\left(L_2^+\mathrm{G}\backslash L^+\mathrm{G}\right) \to \mathrm{Gr}_1^\tau \tag{3.15}$$

given in the following way: If $(A_j)_j \in L^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}$ represents an object on the left, then its image is the object represented by $(B_j)_j \in L\mathrm{G}^\tau$ where

$$B_j = \begin{cases} A_j \begin{pmatrix} v & \\ & 1 \end{pmatrix} & \text{if } S_j = L, \\ \begin{pmatrix} 1 & \\ & v \end{pmatrix} A_j & \text{if } S_j = R. \end{cases}$$

Suppose $\mathfrak{M}$ is a point of $Y_S^{\eta,\tau}$ admitting an eigenbasis β. By Theorem 2.8 and [BBH$^+$, (3.14)],

$$A^{(j)}_{\mathfrak{M},\beta} \in \begin{cases} L^+\mathrm{G} \begin{pmatrix} v & \\ & 1 \end{pmatrix} & \text{if } S_j = L, \\ \begin{pmatrix} 1 & \\ & v \end{pmatrix} L^+\mathrm{G} & \text{if } S_j = R. \end{cases}$$

Therefore, there exists a map

$$\prod_{j\in\mathbf{Z}/f\mathbf{Z}} \left(L_2^+\mathrm{G}\backslash L^+\mathrm{G}\right) \to Y_S^{\eta,\tau} \tag{3.16}$$

fitting into a commutative diagram

$$\begin{tikzcd} \prod_{j\in\mathbf{Z}/f\mathbf{Z}} \left(L_2^+\mathrm{G}\backslash L^+\mathrm{G}\right) \arrow[r, "(3.15)"] \arrow[d, "(3.16)"] & \mathrm{Gr}_1^\tau \arrow[d] \\ Y_S^{\eta,\tau} \arrow[r, hook, "\text{closed}"] & Y_{\mathbf{F}}^{\eta,\tau} \end{tikzcd}$$

where the right vertical arrow is induced from the second isomorphism in Lemma 3.11. The top horizontal and right vertical arrows are thus representable, and so the same is true for the map in (3.16). Since an eigenbasis always exists Zariski–locally, the map in (3.16) is surjective on points valued in local rings and therefore, is formally smooth. In particular, it is a smooth surjective map with smooth domain, and so, the codomain is also smooth. $\square$

Let $\widetilde{z} = (\widetilde{z}_j)_j \in \widetilde{W}^{\mathbf{Z}/f\mathbf{Z}}$. As described in [LHMM, Sec. 3.3], there exists an open immersion

$$\widetilde{U}(\widetilde{z}) := \left[\prod_{j\in\mathbf{Z}/f\mathbf{Z}} L^-\mathrm{G}\,\widetilde{z}_j\right] \cap \mathrm{Gr}_1^{\mathrm{bd},vv^\mu} \hookrightarrow \mathrm{Gr}_1^{\mathrm{bd},vv^\mu}.$$

The scheme $\widetilde{U}(\widetilde{z})$ has a natural structure as a product of schemes $\prod_j \widetilde{U}(\widetilde{z}_j)$.

Next, we construct the following commutative diagram for $S \in \{L, R\}^{\mathbf{Z}/f\mathbf{Z}}$ and $\widetilde{z} \in \widetilde{W}^{\mathbf{Z}/f\mathbf{Z}}$:

$$
\begin{tikzcd}
\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \arrow[rr] \arrow[dr, hook, "cl."] \arrow[dd, hook] & & \widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S \arrow[dr, hook, "cl."] \arrow[dd, hook] & & \\
& \widetilde{Y}^{\eta,\tau}(\widetilde{z}) \arrow[rr] \arrow[dd, hook] & & \widetilde{\mathcal{Z}}^{\tau,1}(\widetilde{z}) \arrow[r, hook, "cl."] \arrow[dd, hook] \arrow[ddr, phantom, "\square"] & \widetilde{U}(\widetilde{z}) \arrow[dd, hook] \\
\widetilde{Y}^{\eta,\tau}_S \arrow[rr, "\widetilde{\pi}_S"] \arrow[dr, hook, "cl."] \arrow[dd] & & \widetilde{\mathcal{Z}}^{\tau}_S \arrow[dr, hook, "cl."] \arrow[dd] & & \\
& \widetilde{Y}^{\eta,\tau} \arrow[rr, "\widetilde{\pi}"] \arrow[dd] & & \widetilde{\mathcal{Z}}^{\tau,1} \arrow[r, hook, "cl."] \arrow[dd] \arrow[ddr, phantom, "\square"] & \mathrm{Gr}_1^{\mathrm{bd},vv^{\mu}} \arrow[dd] \\
Y^{\eta,\tau}_S \arrow[rr, "\pi_S"] \arrow[dr, hook, "cl."] & & \mathcal{Z}^{\tau}_S \arrow[dr, hook, "cl."] & & \\
& Y^{\eta,\tau}_{\mathbf{F}} \arrow[rr, "\pi"] & & \mathcal{Z}^{\tau,1} \arrow[r, hook, "cl."] & \left[L\mathrm{G}^{\mathrm{bd},vv^{\mu}} \big/_{\varphi} \left(L^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\right]
\end{tikzcd}
\tag{3.17}
$$

where

- the bottom two arrows are respectively the first two arrows in (3.8);
- the vertical arrow in the bottom right

$$\mathrm{Gr}_1^{\mathrm{bd},vv^{\mu}} \to \left[L\mathrm{G}^{\mathrm{bd},vv^{\mu}} \Big/_{\varphi} \left(L^+\mathrm{G}^{\mathbf{Z}/f\mathbf{Z}}\right)\right]$$

  is the composition of the first isomorphism in Lemma 3.11 with the quotient under shifted conjugation by $\mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}}$;
- all hooked arrows annotated with *cl.* are closed immersions, while all those marked with a circle are open immersions;
- the schemes $\widetilde{\mathcal{Z}}^{\tau,1}$ and $\widetilde{\mathcal{Z}}^{\tau,1}$ are defined so that the right most squares (marked with a square symbol in the center) are pullback squares; and
- the stacks $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$, $\widetilde{Y}^{\eta,\tau}(\widetilde{z})$, $\widetilde{Y}^{\eta,\tau}_S$ and $\widetilde{Y}^{\eta,\tau}$, and the schemes $\widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S$ and $\widetilde{\mathcal{Z}}^{\tau}_S$, are defined so that the front, back and side faces (but not necessarily the top and bottom faces!) of the two cubes are pullback squares.

Remark 3.18. We make the following note for translation between the paper [LHMM] and this article. In the paper [LHMM], there are various pairs of different but related notions that are identified mod p, namely, $Y^{\mathrm{mod},\eta,\tau}$ and $Y^{\eta,\tau}$, $\widetilde{Y}^{\mathrm{mod},\eta,\tau}$ and $\widetilde{Y}^{\eta,\tau}$, and $\widetilde{Z}^{\mathrm{mod},\tau}$ and $\widetilde{\mathcal{Z}}^{\tau}$. Since we are working entirely mod p, we do not define Y^{mod} , $\widetilde{Y}^{\mathrm{mod},\eta,\tau}$ and $\widetilde{\mathcal{Z}}^{\mathrm{mod},\tau}$ at all.

Proposition 3.19. *(i) The stack $\widetilde{Y}^{\eta,\tau}$ is a scheme identifying with the closed subscheme of*

$$\mathrm{Gr}_1^{\mathrm{bd},vv^{\mu}} \times (B\backslash \mathrm{GL}_2)^{\mathbf{Z}/f\mathbf{Z}}$$

consisting of pairs $((P_j)_j,(l_j)_j)$ such that if $\widetilde{l}_j$ is a lift of l_j to GL_2 and $\widetilde{P}_j$ a lift of P_j to $L\mathrm{G}$, then

$$\left(\widetilde{l}_j\widetilde{P}_j\widetilde{l}_{j-1}^{-1}\right)_j \in L\mathrm{G}^{\tau}.$$

Equivalently, for each j,

$$\widetilde{l}_j\widetilde{P}_j\widetilde{l}_{j-1}^{-1}v^{-\mu_j}s_j \in \mathcal{A}(\eta).$$

(ii) For $\widetilde{z}=(\widetilde{z}_j)_j \in \widetilde{W}^{\mathbf{Z}/f\mathbf{Z}}$, the open subscheme $\widetilde{Y}^{\eta,\tau}(\widetilde{z}) \subset \widetilde{Y}^{\eta,\tau}$ identifies with the open subscheme consisting of those pairs $((P_j)_j,(l_j)_j)$ which, in addition to satisfying (1) above, have the property that for each j, P_j is represented (uniquely) by a matrix of the form

$$\kappa_j X_j \widetilde{z}_j$$

where κ_j is a point of GL_2 and X_j of $L_1^-\mathrm{G}$.

(iii) For $S=(S_j)_j \in \{L,R\}^{\mathbf{Z}/f\mathbf{Z}}$, the closed subscheme $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \subset \widetilde{Y}^{\eta,\tau}(\widetilde{z})$ further identifies with the subscheme consisting of those pairs

$$((P_j)_j,(l_j)_j)$$

which, in addition to satisfying (2) above, have the property that if $\widetilde{l}_j$ is any lift of l_j to GL_2 and $\widetilde{P}_j$ is a matrix lifting P_j, then

$$\widetilde{l}_j\widetilde{P}_j\widetilde{l}_{j-1}^{-1}v^{-\mu_j}s_j$$

has top left (resp. bottom right) entry divisible by v if $S_j=L$ (resp. $S_j=R$) for each j.

Proof. The first part of the statement is proven in [LHMM, Prop. 3.3.1]. The second part follows from the definition of $\widetilde{U}(\widetilde{z})$. The third part is immediate from Theorem 2.8. □

Lemma 3.20. *The proper maps $\widetilde{\pi}$ and $\widetilde{\pi}_S$ are scheme–theoretically dominant.*

Proof. Being smooth torsors over the reduced stacks $\mathcal{Z}^{\tau,1}$ and $\widetilde{\mathcal{Z}}_S^{\tau}$ respectively, $\widetilde{\mathcal{Z}}^{\tau,1}$ and $\widetilde{\mathcal{Z}}_S^{\tau}$ are reduced algebraic stacks. Being pullbacks of π and π_S respectively, the maps $\widetilde{\pi}$ and $\widetilde{\pi}_S$ are proper and surjective. The universal property of scheme–theoretic images finishes the proof. □

Lemma 3.21. *A Zariski open cover of $\widetilde{Y}^{\eta,\tau}$ (resp. $\widetilde{\mathcal{Z}}^{\tau}$) is given by the set of all $\widetilde{Y}^{\eta,\tau}(\widetilde{z})$ (resp. $\widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})$) such that $\widetilde{z}=(\widetilde{z}_j)_{j\in\mathbf{Z}/f\mathbf{Z}}$ with $\widetilde{z}_j=\widetilde{w}_js_j^{-1}v^{\mu_j}$ where $\widetilde{w}_j \in \{w_0t_\eta, t_{w_0(\eta)}\}$ for each j.*

Proof. The statement is true if $\widetilde{w}_j \in \{t_\eta, w_0t_\eta, t_{w_0,\eta}\}$ for each j by [LHMM, Lem. 3.3.5]. We claim that

$$\widetilde{U}(t_\eta s_j^{-1}v^{\mu_j})=\widetilde{U}(w_0t_\eta s_j^{-1}v^{\mu_j}).$$

Indeed, this is immediate because $L^-\mathrm{G}t_\eta s_j^{-1}v^{\mu_j}=L^-\mathrm{G}w_0t_\eta s_j^{-1}v^{\mu_j}$. □

3.2. **Auxiliary schemes.** Next, we construct certain auxiliary schemes through which the map $\widetilde{Y}_S^{\eta,\tau} \to \widetilde{\mathcal{Z}}_S^{\tau}$ factors and that will make it easier to study the geometry of $\widetilde{\mathcal{Z}}_S^{\tau}$ later. The constructions in this section follow closely those in [LHMM, Sec. 4.1] with some variants that allow a detailed study of various irreducible components of $\widetilde{\mathcal{Z}}^{\tau}$.

We fix $\tau=\tau(s,\mu)$ non–scalar principal series tame type with $\mu=(\mu_j)_j$ small; $\widetilde{z}=(\widetilde{z}_j)_j \in \widetilde{W}^{\mathbf{Z}/f\mathbf{Z}}$ with each $\widetilde{z}_j=\widetilde{w}_js_j^{-1}v^{\mu_j}$ for some $\widetilde{w}_j \in \{w_0t_\eta, t_{w_0(\eta)}\}$; and $S=(S_j)_j \in \{L,R\}^{\mathbf{Z}/f\mathbf{Z}}$.

Definition 3.22. Define $\widetilde{Ba}_j(\widetilde{z}_j)$ to be the closed $\mathbf{F}$–subscheme of $\mathbf{P}^1 \times \mathrm{GL}_2 \times L_1^- \mathrm{G} \times \mathbf{P}^1$ parameterizing tuples

$$(l_j, \kappa_j, X_j, r_j) \in \mathbf{P}^1 \times \mathrm{GL}_2 \times L_1^- \mathrm{G} \times \mathbf{P}^1$$

such that if $\widetilde{l}_j, \widetilde{r}_j$ are lifts to GL_2 of $l_j, r_j \in \mathbf{P}^1$ respectively (under the obvious map $\mathrm{GL}_2 \to B\backslash \mathrm{GL}_2 \cong \mathbf{P}^1$) and

$$W_j \stackrel{\text{def}}{=} \widetilde{l}_j \kappa_j X_j \widetilde{w}_j s_j^{-1} v^{\mu_j} \widetilde{r}_j^{-1} v^{-\mu_j} s_j, \tag{3.23}$$

then $W_j \in \mathcal{A}(\eta)$.

Define a closed subscheme $\widetilde{Ba}_j(\widetilde{z}_j)_{S_j}$ of $\widetilde{Ba}_j(\widetilde{z}_j)$ by further requiring that the top left entry of W_j is divisible by v if $S_j = L$ and the bottom right entry is divisible by v if $S_j = R$. Since μ_j is dominant, these criteria are independent of the choice of lifts $\widetilde{l}_j$ and $\widetilde{r}_j$.

Definition 3.24. Define $Ba_j(\widetilde{z}_j)$ to be the closed subscheme of $\widetilde{Ba}_j(\widetilde{z}_j)$ obtained by setting $\kappa_j = 1$. The scheme $Ba_j(\widetilde{z}_j)$ naturally admits a monomorphism into $\mathbf{P}^1 \times L_1^- \mathrm{G} \times \mathbf{P}^1$. Define a closed subscheme $Ba_j(\widetilde{z}_j)_{S_j}$ of $Ba_j(\widetilde{z}_j)$ as the fiber product

$$\widetilde{Ba}_j(\widetilde{z}_j)_{S_j} \times_{\widetilde{Ba}_j(\widetilde{z}_j)} Ba_j(\widetilde{z}_j).$$

We define a map

$$\widetilde{\mathrm{pr}}_j : \widetilde{Ba}_j(\widetilde{z}_j) \to \widetilde{U}(\widetilde{z}_j)$$

by mapping $(l_j, \kappa_j, X_j, r_j) \mapsto \kappa_j X_j \widetilde{z}_j$. Denote by pr_j the restriction of $\widetilde{\mathrm{pr}}_j$ to $Ba_j(\widetilde{z}_j)$. Denote by $p_j, q_j : Ba_j(\widetilde{z}_j) \to \mathbf{P}^1$ the obvious projections to first and last coordinates respectively.

Lemma 3.25. *The maps* $\widetilde{\mathrm{pr}}_j$ *and* pr_j *are projective.*

Proof. It suffices to prove the statement for $\widetilde{\mathrm{pr}}_j$. The map $\widetilde{\mathrm{pr}}_j$ sits in the following commutative diagram

$$\begin{tikzcd} \widetilde{Ba}_j(\widetilde{z}_j) \arrow[r, hook, "cl."] \arrow[d, "\widetilde{\mathrm{pr}}_j"] & \mathbf{P}^1 \times \mathrm{GL}_2 \times L_1^- \mathrm{G} \times \mathbf{P}^1 \arrow[d] \\ \widetilde{U}(\widetilde{z}_j) \arrow[r, hook] & L^- \mathrm{G} \end{tikzcd} \tag{3.26}$$

where the bottom arrow is the map sending $(P\widetilde{z}_j) \mapsto P$ and is a monomorphism; the rightmost projective surjection is given by mapping $(l_j, \kappa_j, X_j, r_j) \mapsto \kappa_j X_j$; and the top horizontal arrow is a closed immersion. Therefore, the composition of the top horizontal arrows with the right vertical arrow is projective, implying by cancellation that $\widetilde{\mathrm{pr}}_j$ is as well. $\square$

Define an isomorphism

$$\begin{aligned} \widetilde{Ba}_j(\widetilde{z}_j) &\xrightarrow{\sim} Ba_j(\widetilde{z}_j) \times \mathrm{GL}_2 \\ (l_j, \kappa_j, X_j, r_j) &\mapsto (l_j\kappa_j, X_j, r_j), \kappa_j. \end{aligned} \tag{3.27}$$

Here, $l_j \kappa_j$ is to be understood as the image of $\widetilde{l}_j \kappa_j$ in $B\backslash \mathrm{GL}_2 \cong \mathbf{P}^1$ for some lift $\widetilde{l}_j \in \mathrm{GL}_2$ of l_j. The isomorphism in (3.27) induces an isomorphism

$$\widetilde{Ba}_j(\widetilde{z}_j)_{S_j} \cong Ba_j(\widetilde{z}_j)_{S_j} \times \mathrm{GL}_2\,. \tag{3.28}$$

Lemma 3.29. *The scheme–theoretic image of $\widetilde{Ba}_j(\widetilde{z}_j)$ (resp. $\widetilde{Ba}_j(\widetilde{z}_j)_{S_j}$) under $\widetilde{\mathrm{pr}}_j$ is isomorphic to the product of GL_2 with the scheme–theoretic image of $Ba_j(\widetilde{z}_j)$ (resp. of $Ba_j(\widetilde{z}_j)_{S_j}$).*

Proof. Since $L^-\mathrm{G} \cong \mathrm{GL}_2 \times L_1^-\mathrm{G}$, pullback along the bottom arrow of (3.26) induces an isomorphism

$$\widetilde{U}(\widetilde{z}) \cong U(\widetilde{z}) \times \mathrm{GL}_2$$

for an appropriate closed subscheme $U(\widetilde{z}) \subset \widetilde{U}(\widetilde{z})$. Via (3.27) and the isomorphism above, the map $\widetilde{\mathrm{pr}}_j$ can be viewed as a map

$$Ba_j(\widetilde{z}_j) \times \mathrm{GL}_2 \to U(\widetilde{z}) \times \mathrm{GL}_2$$

given by sending $((l_j\kappa_j, X_j, r_j), \kappa_j) \mapsto (X_j\widetilde{z}_j, \kappa_j)$. Therefore $\widetilde{\mathrm{pr}}_j$ is the product of a map $\mathrm{pr}'_j : Ba_j(\widetilde{z}_j) \to U(\widetilde{z})$ and $\mathrm{id} : \mathrm{GL}_2 \to \mathrm{GL}_2$, implying that the scheme–theoretic image of $\widetilde{\mathrm{pr}}_j$ is isomorphic to a product of the scheme–theoretic image of pr'_j with GL_2. Finally, the observation that the scheme–theoretic image of pr'_j is isomorphic to that of

$$\mathrm{pr}_j : Ba_j(\widetilde{z}_j) \xrightarrow{\widetilde{\mathrm{pr}}'_j} U(\widetilde{z}) \xhookrightarrow{\text{closed}} \widetilde{U}(\widetilde{z})$$

finishes the proof. □

We let Z_j denote the scheme–theoretic image of $Ba_j(\widetilde{z}_j)_{S_j}$ under pr_j. Define

$$\widetilde{Ba}(\widetilde{z}) \stackrel{\text{def}}{=} \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \widetilde{Ba}_j(\widetilde{z}_j)$$

and let $\widetilde{Ba}(\widetilde{z})_S \subset \widetilde{Ba}(\widetilde{z})$ be the closed subscheme $\prod_{j\in\mathbf{Z}/f\mathbf{Z}} \widetilde{Ba}_j(\widetilde{z}_j)_{S_j}$.

$$\mathrm{pr}_{\widetilde{B}} : \widetilde{Ba}(\widetilde{z}) \longrightarrow \widetilde{U}(\widetilde{z})$$

by mapping $(l_j, \kappa_j, X_j, r_j)_j \mapsto (\kappa_j X_j \widetilde{z}_j)_j$. Denote by $\mathrm{Im}(\mathrm{pr}_{\widetilde{B}})$ the scheme–theoretic image of $\mathrm{pr}_{\widetilde{B}}$, admitting a closed immersion into $\widetilde{U}(\widetilde{z})$. By Lemma 3.29, the scheme–theoretic image of $\widetilde{Ba}(\widetilde{z})_S$ under $\mathrm{pr}_{\widetilde{B}}$ is isomorphic to

$$\prod_j (\mathrm{GL}_2 \times Z_j)\,,$$

which therefore admits a closed immersion into $\mathrm{Im}(\mathrm{pr}_{\widetilde{B}})$.

Lemma 3.30. *Setting $l_{j-1} = r_j$ for each $j \in \mathbf{Z}/f\mathbf{Z}$ cuts out closed subschemes $\widetilde{Y}^{\eta,\tau}(\widetilde{z}) \xrightarrow{\Delta} \widetilde{Ba}(\widetilde{z})$ and $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \xhookrightarrow{\Delta_S} \widetilde{Ba}(\widetilde{z})_S$.*

Proof. Obvious from the definitions of $\widetilde{Ba}(\widetilde{z})$ and $\widetilde{Ba}(\widetilde{z})_S$, and Proposition 3.19. □

We obtain the following commutative diagram

$$
\begin{tikzcd}
\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \arrow[rr, hook, "\Delta_S", "cl."'] \arrow[dr, two heads] \arrow[dd, hook, "cl."] & & \widetilde{Ba}(\widetilde{z})_S \arrow[dr, two heads] \arrow[dd, hook, "cl." near start] & & \\
& \widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S \arrow[rr, hook, "cl."' near end] \arrow[dd, hook, "cl." near start] & & \prod_j (\mathrm{GL}_2 \times Z_j) \arrow[dd, hook, "cl."] & \\
\widetilde{Y}^{\eta,\tau}(\widetilde{z}) \arrow[rr, hook, "\Delta" near end, "cl."' near end] \arrow[dr, two heads] & & \widetilde{Ba}(\widetilde{z}) \arrow[dr, two heads] & & \\
& \widetilde{\mathcal{Z}}^{\tau}(\widetilde{z}) \arrow[rr, hook, "cl."] & & \mathrm{Im}(\mathrm{pr}_{\widetilde{B}}) \arrow[r, hook, "cl."] & \widetilde{U}(\widetilde{z})
\end{tikzcd}
\tag{3.31}
$$

where

- except for the nodes $\widetilde{Ba}(\widetilde{z})_S$, $\widetilde{Ba}(\widetilde{z})$, $\prod_j (\mathrm{GL}_2 \times Z_j)$ and $\mathrm{Im}(\mathrm{pr}_{\widetilde{B}})$, this diagram is a subdiagram of (3.17);
- all two-headed arrows are proper and scheme–theoretically dominant; and
- all hooked arrows annotated with *cl.* are closed immersions.

4. Local geometry

We fix $\tau = \tau(s, \mu)$ non–scalar principal series tame type with $\mu = (\mu_j)_j$ small; $\widetilde{z} = (\widetilde{z}_j)_j \in \widetilde{W}^{\mathbf{Z}/f\mathbf{Z}}$ with each $\widetilde{z}_j \in \{w_0 t_\eta, t_{w_0(\eta)}\} s_j^{-1} v^{\mu_j}$; and $S = (S_j)_j \in \{L, R\}^{\mathbf{Z}/f\mathbf{Z}}$. Assume $p - 2 > \max_j \langle \mu_j, \alpha^\vee \rangle$.

The schemes $Ba_j(\widetilde{z}_j)$ are described in detail in [LHMM, Table 3] (in *loc. cit.*, everything is defined over $\mathcal{O}$ and comparing it to this article requires setting $p = 0$). We now add to those descriptions to further describe the closed subschemes $Ba_j(\widetilde{z}_j)_{S_j}$ explicitly.

Lemma 4.1. *For each j, the ideals cutting out $Ba_j(\widetilde{z}_j)_L$ and $Ba_j(\widetilde{z}_j)_R$ in $Ba_j(\widetilde{z}_j)$ as well as the scheme–theoretic images of $Ba_j(\widetilde{z}_j)_L$ and $Ba_j(\widetilde{z}_j)_R$ under pr_j are given by Tables 1 and 2 respectively. Here, $[x : y]$ are the coordinates of $l_j \kappa_j$, $[x' : y']$ are the coordinates of r_j, and the rest of the variables are in terms of* [LHMM, Table 3].

Proof. Let $(l_j \kappa_j, X_j, r_j)$ be a point of $Ba_j(\widetilde{z}_j)$ and let $\widetilde{l}_j$ and $\widetilde{r}_j$ be lifts to GL_2 of l_j and r_j respectively. Let

$$\widetilde{l}_j \kappa_j = \begin{pmatrix} u & z \\ x & y \end{pmatrix} \quad \text{and} \quad \widetilde{r}_j = \begin{pmatrix} t & -s \\ x' & y' \end{pmatrix},$$

where u, z, s, t can be freely chosen subject to the restriction that $\widetilde{l}_j, \widetilde{r}_j$ are invertible. Upon restricting to the distinguished opens $D(x, x'), D(y, x'), D(x, y'), D(y, y')$, we

$\langle\mu_j,\alpha^\vee\rangle$ \ $\widetilde{w}_j$		$w_0 t_\eta$	$t_{w_0(\eta)}$
	s_j		
>1	w_0	$I=(B)$ $Ba_j(\widetilde{z}_j)_L = \operatorname{Proj}\mathbf{F}[x,y]\times\operatorname{Spec}\mathbf{F}[C']$ $Z_j=\operatorname{Spec}\mathbf{F}[C']$	$I=(1)$ $Ba_j(\widetilde{z}_j)_L = Z_j = \varnothing$
	id	$I=(C)$ $Ba_j(\widetilde{z}_j)_L = \operatorname{Proj}\mathbf{F}[x,y]\times\operatorname{Spec}\mathbf{F}[C']$ $Z_j=\operatorname{Spec}\mathbf{F}[C']$	$I=(1)$ $Ba_j(\widetilde{z}_j)_L = Z_j = \varnothing$
$=1$	w_0	$I=(B,C,D)$ $Ba_j(\widetilde{z}_j)_L = \operatorname{Proj}\mathbf{F}[x,y]\times\operatorname{Proj}\mathbf{F}[x',y']$ $Z_j=\operatorname{Spec}\mathbf{F}$	$I=(1)$ $Ba_j(\widetilde{z}_j)_L = Z_j = \varnothing$
	id	$I=(C)$ $Ba_j(\widetilde{z}_j)_L = \operatorname{Proj}\mathbf{F}[x,y]\times\operatorname{Spec}\mathbf{F}[C']$ $Z_j=\operatorname{Spec}\mathbf{F}[C']$	$I=(1)$ $Ba_j(\widetilde{z}_j)_L = Z_j = \varnothing$
$=0$	id	$I=(x'-y'C)$ $Ba_j(\widetilde{z}_j)_L = \operatorname{Proj}\mathbf{F}[x,y]\times\operatorname{Spec}\mathbf{F}[C]$ $Z_j=\operatorname{Spec}\mathbf{F}[C]$	$I=(y'-x'B)$ $Ba_j(\widetilde{z}_j)_L = \operatorname{Proj}\mathbf{F}[x,y]\times\operatorname{Spec}\mathbf{F}[B]$ $Z_j=\operatorname{Spec}\mathbf{F}[B]$

Table 1. Ideals I cut out $Ba_j(\widetilde{z}_j)_L$ in $Ba_j(\widetilde{z}_j)$ and the scheme Z_j is the scheme–theoretic image of $Ba_j(\widetilde{z}_j)_L$ under pr_j. Different font colors correspond to different isomorphism classes of the tuple $(Ba_j(\widetilde{z}_j)_L, p_j|_{Ba_j(\widetilde{z}_j)_L}, q_j|_{Ba_j(\widetilde{z}_j)_L}, \mathrm{pr}_j|_{Ba_j(\widetilde{z}_j)_L})$.

will make the following choices for u, z, s, t:

$$\begin{aligned}
u&=0, &\quad z&=1, &\quad s&=1, &\quad t&=0 &\quad &\text{on } D(x,x'),\\
u&=1, & z&=0, & s&=1, & t&=0 & &\text{on } D(y,x'),\\
u&=0, & z&=1, & s&=0, & t&=1 & &\text{on } D(x,y'),\\
u&=1, & z&=0, & s&=0, & t&=1 & &\text{on } D(y,y').
\end{aligned}$$

We now deal with the different cases separately, taking the descriptions of

$$s_j w_j^{-1} X_j w_j s_j^{-1}$$

along with the relations the various variables appearing in X_j satisfy from [LHMM, Tables 2, 3]. In the following, let $k_j := \langle\mu_j,\alpha^\vee\rangle$.

(1) Let $k_j>1, (s_j,\widetilde{w}_j)=(w_0,w_0t_\eta)$. Then

$$X_j = w_j s_j^{-1}\begin{pmatrix}1+BC'v^{-k_j} & Bv^{-1}\\ C'v^{-k_j+1} & 1\end{pmatrix} s_j w_j = \begin{pmatrix}1+BC'v^{-k_j} & Bv^{-1}\\ C'v^{-k_j+1} & 1\end{pmatrix}$$

$\langle \mu_j, \alpha^\vee \rangle$ \ $\widetilde{w}_j$	s_j	$w_0 t_\eta$	$t_{w_0(\eta)}$
>1	w_0	$I = (x)$ $Ba_j(\widetilde{z}_j)_R = Z_j = \operatorname{Spec} \mathbf{F}[B, C']$	$I = (0)$ $Ba_j(\widetilde{z}_j)_R = Z_j = \operatorname{Spec} \mathbf{F}[C, C']$
	id	$I = (x)$ $Ba_j(\widetilde{z}_j)_R = Z_j = \operatorname{Spec} \mathbf{F}[C, C']$	$I = (0)$ $Ba_j(\widetilde{z}_j)_R = Z_j = \operatorname{Spec} \mathbf{F}[C, C']$
$=1$	w_0	$I = (xy' - yx')$ $Ba_j(\widetilde{z}_j)_R =$ $\operatorname{Spec} \mathbf{F}[B, C, D]$ $\times \operatorname{Proj} \mathbf{F}[x, y]/(Dx - Cy, Bx + Dy)$ $Z_j = \operatorname{Spec} \mathbf{F}[B, C, D]/(D^2 + BC)$	$I = (0)$ $Ba_j(\widetilde{z}_j)_R = Z_j = \operatorname{Spec} \mathbf{F}[C, C']$
	id	$I = (x)$ $Ba_j(\widetilde{z}_j)_R = Z_j = \operatorname{Spec} \mathbf{F}[C, C']$	$I = (0)$ $Ba_j(\widetilde{z}_j)_R =$ $\operatorname{Spec} \mathbf{F}[B, C, D]$ $\times \operatorname{Proj} \mathbf{F}[x, y]/(Dx - Cy, Bx + Dy)$ $Z_j = \operatorname{Spec} \mathbf{F}[B, C, D]/(D^2 + BC)$
$=0$	id	$I = (x)$ $Ba_j(\widetilde{z}_j)_R = \operatorname{Spec} \mathbf{F}[C] \times \operatorname{Proj} \mathbf{F}[x', y']$ $Z_j = \operatorname{Spec} \mathbf{F}[C]$	$I = (x)$ $Ba_j(\widetilde{z}_j)_R = \operatorname{Spec} \mathbf{F}[B] \times \operatorname{Proj} \mathbf{F}[x', y']$ $Z_j = \operatorname{Spec} \mathbf{F}[B]$

Table 2. Ideals I cut out $Ba_j(\widetilde{z}_j)_R$ in $Ba_j(\widetilde{z}_j)$ and the scheme Z_j is the scheme–theoretic image of $Ba_j(\widetilde{z}_j)_R$ under pr_j. Different font colors correspond to different isomorphism classes of the tuple $(Ba_j(\widetilde{z}_j)_R, p_j|_{Ba_j(\widetilde{z}_j)_R}, q_j|_{Ba_j(\widetilde{z}_j)_R}, \mathrm{pr}_j|_{Ba_j(\widetilde{z}_j)_R})$.

with the variables B, C' satisfying $x' - y'C' = xB = 0$. Thus, $Ba_j(\widetilde{z}_j) = D(y')$, $(s,t) = (0,1)$ and

$$(\det \widetilde{r}_j) W_j = \begin{pmatrix} u & z \\ x & y \end{pmatrix} \begin{pmatrix} 1 + BC'v^{-k_j} & Bv^{-1} \\ C'v^{-k_j+1} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} \begin{pmatrix} 1 & -x'v^{-k_j} \\ 0 & y' \end{pmatrix}.$$

This shows that $v \mid (W_j)_{1,1}$ if and only if $uB = 0$, and $v \mid (W_j)_{2,2}$ if and only if $xy' = 0$. Note that $uB = 0$ if and only if $B = 0$, since $u = 1$ on $D(y, y')$ and $B = 0$ on $D(x, y')$. We have $xy' = 0$ if and only if $x = 0$ since we are on $D(y')$.

(2) Let $k_j \geq 1, (s_j, \widetilde{w}_j) = (w_0, t_{w_0(\eta)})$. Then

$$X_j = w_j s_j^{-1} \begin{pmatrix} 1 & 0 \\ Cv^{-1} + C'v^{-k_j-1} & 1 \end{pmatrix} s_j w_j = \begin{pmatrix} 1 & Cv^{-1} + C'v^{-k_j-1} \\ 0 & 1 \end{pmatrix}$$

with the variables C, C' satisfying $x' - y'C' = x = 0$. Thus $Ba_j(\widetilde{z}_j) = D(y, y')$, $(u, z, s, t) = (1, 0, 0, 1)$ and

$$(\det \widetilde{r}_j)W_j = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 & Cv^{-1} + C'v^{-(k_j+1)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} 1 & -x'v^{-k_j} \\ 0 & y' \end{pmatrix}.$$

This shows that $v \nmid (W_j)_{1,1}$ and $v \mid (W_j)_{2,2}$.

(3) Let $k_j \geq 1, (s_j, \widetilde{w}_j) = (\mathrm{id}, w_0 t_\eta)$. Then

$$X_j = w_j s_j^{-1} \begin{pmatrix} 1 & 0 \\ Cv^{-1} + C'v^{-k_j-1} & 1 \end{pmatrix} s_j w_j = \begin{pmatrix} 1 & Cv^{-1} + C'v^{-k_j-1} \\ 0 & 1 \end{pmatrix}$$

with the variables C, C' satisfying $x' - y'C' = xC = 0$. Thus, $Ba_j(\widetilde{z}_j) = D(y')$, $(s, t) = (0, 1)$ and

$$(\det \widetilde{r}_j)W_j = \begin{pmatrix} u & z \\ x & y \end{pmatrix} \begin{pmatrix} 1 & Cv^{-1} + C'v^{-k_j-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} \begin{pmatrix} y' & 0 \\ -x'v^{-k_j} & 1 \end{pmatrix}.$$

This shows that $v \mid (W_j)_{1,1}$ if and only if $uy'C = 0$, and $v \mid (W_j)_{2,2}$ if and only if $x = 0$. The equality $uy'C = 0$ holds if and only if $uC = 0$ since we are on $D(y')$. Moreover, $uC = 0$ if and only if $C = 0$, since $u = 1$ on $D(y, y')$ and $C = 0$ on $D(x, y')$.

(4) Let $k_j > 1, (s_j, \widetilde{w}_j) = (\mathrm{id}, t_{w_0(\eta)})$. Then

$$X_j = w_j s_j^{-1} \begin{pmatrix} 1 + BC'v^{-k_j} & Bv^{-1} \\ C'v^{-k_j+1} & 1 \end{pmatrix} s_j w_j = \begin{pmatrix} 1 + BC'v^{-k_j} & Bv^{-1} \\ C'v^{-k_j+1} & 1 \end{pmatrix}$$

with the variables B, C' satisfying $x' - y'C' = x = 0$. Therefore, $Ba_j(\widetilde{z}_j) = D(y, y')$, $(u, z, s, t) = (1, 0, 0, 1)$ and

$$(\det \widetilde{r}_j)W_j = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} 1 + BC'v^{-k_j} & Bv^{-1} \\ C'v^{-k_j+1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} y' & 0 \\ -x'v^{-k_j} & 1 \end{pmatrix}$$

Hence, $v \nmid (W_j)_{1,1}$ and $v \mid (W_j)_{2,2}$.

(5) Let $k_j = 1, (s_j, \widetilde{w}_j) = (w_0, w_0 t_\eta)$. Then

$$X_j = w_j s_j^{-1} \begin{pmatrix} 1 - Dv^{-1} & Bv^{-1} \\ Cv^{-1} & 1 + Dv^{-1} \end{pmatrix} s_j w_j = \begin{pmatrix} 1 - Dv^{-1} & Bv^{-1} \\ Cv^{-1} & 1 + Dv^{-1} \end{pmatrix}$$

with the variables B, C, D satisfying $x'D - y'C = x'B + y'D = xD - yC = xB + yD = 0$. Thus,

$$(\det \widetilde{r}_j)W_j = \begin{pmatrix} u & z \\ x & y \end{pmatrix} \begin{pmatrix} 1 - Dv^{-1} & Bv^{-1} \\ Cv^{-1} & 1 + Dv^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} \begin{pmatrix} t & -x'v^{-1} \\ sv & y' \end{pmatrix}$$

This shows that $v \mid (W_j)_{1,1}$ if and only if $tuB + szC + (tz - su)D = 0$ and $v \mid (W_j)_{2,2}$ if and only if $xy' - yx' = 0$. Checking on each of the charts, we find that $tuB + szC + (tz - su)D = 0$ if and only if $B = C = D = 0$.

(6) Let $k_j = 1, (s_j, \widetilde{w}_j) = (\mathrm{id}, t_{w_0(\eta)})$. Then

$$X_j = w_j s_j^{-1} \begin{pmatrix} 1 - Dv^{-1} & Bv^{-1} \\ Cv^{-1} & 1 + Dv^{-1} \end{pmatrix} s_j w_j = \begin{pmatrix} 1 - Dv^{-1} & Bv^{-1} \\ Cv^{-1} & 1 + Dv^{-1} \end{pmatrix}$$

with the variables B, C, D satisfying $x'D - y'C = x'B + y'D = xy' - yx' = 0$. Thus,

$$(\det \widetilde{r}_j)W_j = \begin{pmatrix} u & z \\ x & y \end{pmatrix} \begin{pmatrix} 1 - Dv^{-1} & Bv^{-1} \\ Cv^{-1} & 1 + Dv^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} y' & sv \\ -x'v^{-1} & t \end{pmatrix}.$$

This shows that $v \mid (W_j)_{1,1}$ if and only if $uy' - zx' = 0$, and $v \mid (W_j)_{2,2}$ if and only if $txB + syC + (ty - sx)D = 0$. The relation $xy' - yx' = 0$ implies that $Ba_j(\widetilde{z}_j) = D(x, x') \cup D(y, y')$. Since $uy' - zx' = -x'$ on $D(x, x')$ and $uy' - zx' = y'$ on $D(y, y')$, $uy' - zx'$ vanishes nowhere on $Ba_j(\widetilde{z}_j)$. Further, since $txB + syC + (ty - sx)D$ equals $yC - xD$ on $D(x, x')$ and $xB + yD$ on $D(y, y')$, $txB + syC + (ty - sx)D$ vanishes on $Ba_j(\widetilde{z}_j)$.

(7) Let $k_j = 0, (s_j, \widetilde{w}_j) = (\mathrm{id}, w_0 t_\eta)$. Then

$$X_j = w_j s_j^{-1} \begin{pmatrix} 1 & 0 \\ Cv^{-1} & 1 \end{pmatrix} s_j w_j = \begin{pmatrix} 1 & Cv^{-1} \\ 0 & 1 \end{pmatrix}$$

with the variable C satisfying $xx' - xy'C = 0$. Thus,

$$(\det \widetilde{r}_j) W_j = \begin{pmatrix} u & z \\ x & y \end{pmatrix} \begin{pmatrix} 1 & Cv^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} \begin{pmatrix} y' & s \\ -x' & t \end{pmatrix}.$$

Hence, $v \mid (W_j)_{1,1}$ if and only if $u(x' - y'C) = 0$, and $v \mid (W_j)_{2,2}$ if and only if $x(t + sC) = 0$. Note that $u(x' - y'C) = 0$ if and only if $x' - y'C = 0$, since $u = 1$ on $D(y)$ and we already know that $x' - y'C = 0$ on $D(x)$. Further, $x(t + sC) = 0$ if and only if $x = 0$, since $x(t + sC) = x$ on $D(y')$, whereas on $D(x')$, x is a multiple of $x(t + sC) = xC$.

(8) Let $k_j = 0, (s_j, \widetilde{w}_j) = (\mathrm{id}, t_{w_0(\eta)})$. Then

$$X_j = w_j s_j^{-1} \begin{pmatrix} 1 & Bv^{-1} \\ 0 & 1 \end{pmatrix} s_j w_j = \begin{pmatrix} 1 & Bv^{-1} \\ 0 & 1 \end{pmatrix}$$

with the variable B satisfying $xx'B - xy' = 0$. Thus,

$$(\det \widetilde{r}_j) W_j = \begin{pmatrix} u & z \\ x & y \end{pmatrix} \begin{pmatrix} 1 & Bv^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} y' & s \\ -x' & t \end{pmatrix}.$$

Hence, $v \mid (W_j)_{1,1}$ if and only if $u(y' - x'B) = 0$, and $v \mid (W_j)_{2,2}$ if and only if $x(s + tB) = 0$. As in the previous case, $u(y' - x'B) = 0$ if and only if $y' - x'B = 0$, and $x(s + tB) = 0$ if and only if $x = 0$.

In order to compute Z_j, we note that if a reduced scheme Z fits into a commutative diagram

$$\begin{tikzcd} Ba_j(\widetilde{z}_j)_{S_j} \arrow[rr, "\mathrm{pr}_j"] \arrow[dr, two heads, "\text{surjection}"'] & & \widetilde{U}(\widetilde{z}) \\ & Z \arrow[ur, hook, "\text{monomorphism}"'] & \end{tikzcd},$$

then Lemma 3.25 implies that the map $Z \hookrightarrow \widetilde{U}(\widetilde{z})$ is a closed immersion, and so $Z_j = Z$. The descriptions of Z_j are now immediate except possibly the relations satisfied by B, C, D when $S_j = R$, $k_j = 1$ and $(s_j, \widetilde{w}_j) \in \{(w_0, w_0 t_\eta), (\mathrm{id}, t_{w_0(\eta)})$. The proof of Lemma 4.11(iii) shows that the only relations B, C, D satisfy with respect to each other are given by setting $(D^2 + BC)$ equal to 0. $\square$

4.1. **Classification of** $Ba_j(\widetilde{z}_j)_{S_j}$**.** We will henceforth restrict attention to the schemes $Ba_j(\widetilde{z}_j)_{S_j}$ for $S_j \in \{L, R\}$ and their scheme–theoretic images. To simplify notation, we will denote the maps $p_j|_{Ba_j(\widetilde{z}_j)_{S_j}}, q_j|_{Ba_j(\widetilde{z}_j)_{S_j}}, \mathrm{pr}_j|_{Ba_j(\widetilde{z}_j)_{S_j}}$ simply as p_j, q_j, pr_j respectively. Using Lemma 4.1, we find that when non–empty, $Ba_j(\widetilde{z}_j)_{S_j}, p_j, q_j, \mathrm{pr}_j$ admit one of the following descriptions up to isomorphism:

(1) The scheme $Ba_j(\widetilde{z}_j)_{S_j}$ is isomorphic to $\operatorname{Proj}\mathbf{F}[x,y] \times \operatorname{Proj}\mathbf{F}[x',y']$. The maps p_j and q_j are projections to the $[x:y]$ and $[x':y']$ coordinates respectively, while pr_j is a constant map. This happens when
$$S_j = L, \quad \widetilde{w}_j = w_0 t_\eta, \quad \text{and} \quad (\langle \mu_j, \alpha^\vee\rangle, s_j) = (1, w_0).$$
(2) The scheme $Ba_j(\widetilde{z}_j)_{S_j}$ is isomorphic to $\operatorname{Proj}\mathbf{F}[x,y] \times \mathbf{A}^1$. The map p_j is projection to the $[x:y]$ coordinate, the map q_j is projection to the $\mathbf{A}^1$ factor followed by inclusion into $\mathbf{P}^1$ given by mapping $C \mapsto [C:1]$, and the map pr_j is the projection to the $\mathbf{A}^1$ factor. This happens when
- $S_j = L, \quad \widetilde{w}_j = w_0 t_\eta, \quad$ and $\quad (\langle \mu_j, \alpha^\vee\rangle, s_j) \neq (1, w_0)$; or
- $S_j = L, \quad \widetilde{w}_j = t_{w_0(\eta)}, \quad$ and $\quad (\langle \mu_j, \alpha^\vee\rangle, s_j) = (0, \mathrm{id})$.

(3) The scheme $Ba_j(\widetilde{z}_j)_{S_j}$ is isomorphic to $\operatorname{Spec}\mathbf{F}[B,C,D] \times \operatorname{Proj}\mathbf{F}[x,y]/(Dx - Cy, Bx + Dy)$. The maps p_j and q_j are the same and given by projection to the $[x:y]$ coordinates, while pr_j extracts the variables B, C, D. This happens when
- $S_j = R, \quad \widetilde{w}_j = w_0 t_\eta, \quad$ and $\quad (\langle \mu_j, \alpha^\vee\rangle, s_j) = (1, w_0)$; or
- $S_j = R, \quad \widetilde{w}_j = t_{w_0(\eta)}, \quad$ and $\quad (\langle \mu_j, \alpha^\vee\rangle, s_j) = (1, \mathrm{id})$.

(4) The scheme $Ba_j(\widetilde{z}_j)_{S_j}$ is isomorphic to $\mathbf{A}^2$. The map p_j is a constant map, q_j is given by $(C, C') \mapsto [C':1]$, and pr_j extracts the variables C, C'. This happens when
- $S_j = R, \quad$ and $\langle \mu_j, \alpha^\vee\rangle > 1$; or
- $S_j = R, \quad \widetilde{w}_j = w_0 t_\eta, \quad$ and $\quad (\langle \mu_j, \alpha^\vee\rangle, s_j) = (1, \mathrm{id})$; or
- $S_j = R, \quad \widetilde{w}_j = t_{w_0(\eta)}, \quad$ and $\quad (\langle \mu_j, \alpha^\vee\rangle, s_j) = (1, w_0)$.

(5) The scheme $Ba_j(\widetilde{z}_j)_{S_j}$ is isomorphic to $\mathbf{A}^1 \times \operatorname{Proj}\mathbf{F}[x',y']$. The map p_j is constant, the map q_j is projection to $[x':y']$ coordinates, and the map pr_j is the projection to the $\mathbf{A}^1$ factor. This happens when
$$S_j = R, \quad \text{and} \quad (\langle \mu_j, \alpha^\vee\rangle, s_j) = (0, \mathrm{id}).$$

We use the above classification to define the class T_j of $Ba_j(\widetilde{z}_j)_{S_j}$. We will say that T_j equals $1,2,3,4$ or 5 if $Ba_j(\widetilde{z}_j)_{S_j}$ is of the form described in (1), (2), (3), (4) or (5) respectively.

4.2. **A different decomposition of $\widetilde{Ba}(\widetilde{z})_S$.** In order to make certain cohomological computations easier, we now consider $\widetilde{Ba}(\widetilde{z})_S$ as a product of schemes in a slightly different way and set up related notations. For each j with $T_j \neq 3$, define schemes $\widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{I}}_{S_j}$ and $\widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{II}}_{S_j}$ via the following isomorphism:

$$\begin{aligned} \widetilde{Ba}_j(\widetilde{z}_j)_{S_j} &\xrightarrow{\sim} \widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{I}}_{S_j} \times \widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{II}}_{S_j} \\ (l_j, \kappa_j, X_j, r_j) &\mapsto (l_j\kappa_j, \kappa_j), (X_j, r_j). \end{aligned} \tag{4.2}$$

The map $p_j : Ba_j(\widetilde{z}_j)_{S_j} \to \mathbf{P}^1$ (resp. q_j) induces a map $\widetilde{Ba}(\widetilde{z})_S \to \mathbf{P}^1$ via projection to the $Ba_j(\widetilde{z}_j)_{S_j}$–factor under the isomorphism in (3.28). We denote this map by p_j (resp. q_j) as well. Abusing notation further, via (4.2), we let p_j also denote the map $\widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{I}}_{S_j} \to \mathbf{P}^1$ and q_j the map $\widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{II}}_{S_j} \to \mathbf{P}^1$.

Definition 4.3. Given a class tuple $T = (T_j)_{j \in \mathbf{Z}/f\mathbf{Z}}$, let $\mathfrak{T}$ be the set of sequences $\mathfrak{t} = (i-k, i-k+1, \ldots, i)$ of elements in $\mathbf{Z}/f\mathbf{Z}$ where
- the length of the sequence, denoted $l(\mathfrak{t})$, is ≥ 2,
- $T_{i-k}, T_i \neq 3$, and
- whenever $l(\mathfrak{t}) \geq 3$, $T_{i-k+1} = \cdots = T_{i-1} = 3$.

It is evident from the definition that if $T_j \neq 3$ for some $j \in \mathbf{Z}/f\mathbf{Z}$, then there exists a sequence in $\mathfrak{T}$ that starts with j and a sequence that ends with j. Furthermore, whenever $\mathfrak{T} \neq \varnothing$, every $j \in \mathbf{Z}/f\mathbf{Z}$ shows up in at least one sequence in $\mathfrak{T}$. Note that $\mathfrak{T}$ depends on the data of $\widetilde{z}, s, \mu, S$ since it depends on the the class tuple.

Definition 4.4. Suppose $\mathfrak{t} = (i-k, \dots, i) \in \mathfrak{T}$.

(i) Define $B(\mathfrak{t})$ to be the scheme

$$\widetilde{Ba}_{i-k}(\widetilde{z}_{i-k})^{\mathrm{I}}_{S_{i-k}} \times \widetilde{Ba}_{i-k+1}(\widetilde{z}_{i-k+1})_{S_{i-k+1}} \times \cdots \times \widetilde{Ba}_{i-1}(\widetilde{z}_{i-1})_{S_{i-1}} \times \widetilde{Ba}_i(\widetilde{z}_i)^{\mathrm{II}}_{S_i}.$$

(ii) Define $Y(\mathfrak{t})$ to be the closed subscheme of $B(\mathfrak{t})$ obtained by setting $l_{j-1} = r_j$ for each $j \in \{i-k+1, \dots, i\}$. Denote by $\Delta(\mathfrak{t})$ the map $Y(\mathfrak{t}) \hookrightarrow B(\mathfrak{t})$.

(iii) Let

$$Z_{\widetilde{B}}(\mathfrak{t}) \stackrel{\mathrm{def}}{=} \mathrm{GL}_2 \times (Z_{i-k+1} \times \mathrm{GL}_2) \times \cdots \times (Z_{i-1} \times \mathrm{GL}_2) \times Z_i.$$

Recall that Z_j is the scheme–theoretic image of $Ba_j(\widetilde{z}_j)_{S_j}$ under pr_j, and by Lemma 3.29, $Z_j \times \mathrm{GL}_2$ is isomorphic to the the scheme–theoretic image of $\widetilde{Ba}_j(\widetilde{z}_j)_{S_j}$ under $\widetilde{\mathrm{pr}}_j$.

It is clear from the definitions that as long as $\mathfrak{T}$ is non–empty, there exist obvious isomorphisms

$$\widetilde{Ba}(\widetilde{z})_S \xrightarrow{\sim} \prod_{\mathfrak{t} \in \mathfrak{T}} B(\mathfrak{t}), \tag{4.5}$$

$$\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \xrightarrow{\sim} \prod_{\mathfrak{t} \in \mathfrak{T}} Y(\mathfrak{t}) \tag{4.6}$$

which induce an identification of the scheme–theoretic image of $\widetilde{Ba}(\widetilde{z})_S$ under $\mathrm{pr}_{\widetilde{B}}$, with the scheme $\prod_{\mathfrak{t} \in \mathfrak{T}} Z_{\widetilde{B}}(\mathfrak{t})$. Thus, the map $\mathrm{pr}_{\widetilde{B}}$ induces proper and scheme–theoretically dominant maps $\mathrm{pr}(\mathfrak{t}) : B(\mathfrak{t}) \to Z_{\widetilde{B}}(\mathfrak{t})$. The scheme–theoretic image of $\prod_{\mathfrak{t} \in \mathfrak{T}} Y(\mathfrak{t})$ under $(\mathrm{pr}(\mathfrak{t}))_{\mathfrak{t}}$ is isomorphic to $\widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S$.

Definition 4.7. For j with $T_j = 3$, let N_j be the ideal of $\Gamma(\widetilde{Ba}_j(\widetilde{z}_j)_{S_j})$ generated by the functions B, C, D. There exists an obvious inclusion

$$\Gamma(\widetilde{Ba}_j(\widetilde{z}_j)_{S_j}) \hookrightarrow \Gamma(\widetilde{Ba}(\widetilde{z})_S)$$

and if $\mathfrak{t} = (i-k, \dots, i) \in \mathfrak{T}$ and $j \in \{i-k, \dots, i\} \smallsetminus \{i-k, i\}$, then also an inclusion

$$\Gamma(\widetilde{Ba}_j(\widetilde{z}_j)_{S_j}) \hookrightarrow \Gamma(B(\mathfrak{t})).$$

Abusing notation, we let the ideals generated by the image of N_j under these inclusions also be denoted N_j.

Let N (resp. $N(\mathfrak{t})$) be the minimal ideal of $\Gamma(\widetilde{Ba}(\widetilde{z})_S)$ (resp. of $\Gamma(B(\mathfrak{t}))$) containing N_j for each j with $T_j = 3$ (resp. for each $j \in \{i-k, \dots, i\} \smallsetminus \{i-k, i\}$). We also let the image of $N(\mathfrak{t})$ under the obvious inclusion

$$\Gamma(B(\mathfrak{t})) \hookrightarrow \Gamma(\widetilde{Ba}(\widetilde{z})_S)$$

be denoted by $N(\mathfrak{t})$.

Lemma 4.8. *Let* $\mathfrak{t} = (i-k, \dots, i) \in \mathfrak{T}$.

(i) The map $\mathrm{pr}(\mathfrak{t})$ *induces an isomorphism of* $Y(\mathfrak{t})$ *with its scheme–theoretic image unless* $T_{i-k} \in \{1,2\}$ *and* $T_i \in \{1,5\}$.

(ii) When $T_{i-k} \in \{1,2\}$ *and* $T_i \in \{1,5\}$*, the dimension of* $Y(\mathfrak{t})$ *and its scheme–theoretic image under* $\mathrm{pr}(\mathfrak{t})$ *are the same if and only if* $l(\mathfrak{t}) > 2$*. When that happens,* $\mathrm{pr}(\mathfrak{t})$ *induces a birational map from* $Y(\mathfrak{t})$ *to its scheme–theoretic image.*

Proof. Since $\mathrm{pr}(\mathfrak{t})$ is proper, it suffices to show that it restricts to a monomorphism on $Y(\mathfrak{t})$ unless $T_{i-k} \in \{1,2\}$ and $T_i \in \{1,5\}$. Unpacking the definitions, we find that $\mathrm{pr}(\mathfrak{t})$ restricts to a monomorphism if and only if given the data of $\{X_j\}_{j=i-k+1}^{i}, \{\kappa_j\}_{j=i-k}^{i-1}$, the tuples $\{l_j\}_{j=i-k}^{i-1}$ and $\{r_j\}_{j=i-k+1}^{i}$ are uniquely determined after imposing the conditions $l_{j-1} = r_j$ for $j \in \{i-k+1, \dots, i\}$. We make the following observations:

- When $T_j = 1$, the data of X_j, κ_j imposes no constraints on l_j, r_j, which are unrelated to each other.
- When $T_j = 2$, the data of X_j, κ_j imposes no constraints on l_j, but completely determines r_j.
- When $T_j = 3$, X_j, κ_j do not completely determine l_j or r_j (except away from the vanishing locus of N_j), but l_j and r_j determine each other completely.
- When $T_j = 4$, κ_j determines l_j and X_j determines r_j.
- When $T_j = 5$, κ_j determines l_j but the data of X_j, κ_j imposes no constraints on r_j.

From these, the first part of the statement of the Lemma follows immediately. For the second part, suppose $\mathfrak{t} = (i-k, \dots, i)$ with $(T_{i-k}, T_i) \in \{(1,1),(1,5),(2,1),(2,5)\}$. If $l(\mathfrak{t}) = 2$, then since $l_{i-k} = r_i \in \mathbf{P}^1$ can take any value, $Y(\mathfrak{t})$ is a $\mathbf{P}^1$–torsor over its scheme–theoretic image. If $l(\mathfrak{t}) > 2$, then the third bullet above implies that away from $V(N(\mathfrak{t}))$, which is a non–empty open subscheme, $\mathrm{pr}(\mathfrak{t})$ restricts to a monomorphism on $Y(\mathfrak{t})$. $\square$

Definition 4.9. Let $\mathfrak{T}^* \subset \mathfrak{T}$ be the set of those $\mathfrak{t} = (i-k, \dots, i) \in \mathfrak{T}$ satisfying $T_{i-k} \in \{1,2\}$ and $T_i \in \{1,5\}$.

Lemma 4.10. *(Version of* [LHMM, Lem. 4.2.3]*) Let* $S \in \{L,R\}^{\mathbf{Z}/f\mathbf{Z}}$ *and* $\mathfrak{t} \in \mathfrak{T}$*.*

(i) The schemes $Ba_j(\widetilde{z}_j)_{S_j}$ *and* $B(\mathfrak{t})$ *are local complete intersections over* $\mathbf{F}$*. Whenever non–empty,* $Ba_j(\widetilde{z}_j)_{S_j}$ *has dimension* 2*.*

(ii) When $T_j = 3$*, the dualizing sheaf of* $Ba_j(\widetilde{z}_j)_{S_j}$ *is* $p_j^*\mathcal{O}(-1) \otimes q_j^*\mathcal{O}(-1)$*. If* $\mathfrak{t} = (i-k, \dots, i)$*, the dualizing sheaf of* $B(\mathfrak{t})$ *is*

$$p_{i-k}^*\mathcal{O}(-2) \otimes \left(\bigotimes_{j=i-k+1}^{i-1} p_j^*\mathcal{O}(-1) \otimes q_j^*\mathcal{O}(-1) \right) \otimes q_i^*\mathcal{O}(-2).$$

Proof. First, suppose $T_j = 3$. Let $\iota \colon Ba_j(\widetilde{z}_j)_{S_j} \hookrightarrow \mathbf{P}^1 \times \mathbf{A}^3$ be the embedding given by $([x:y],(B,C,D)) \mapsto ([x:y],(B,C,D))$. Set $s = x/y$. Below is an open cover of $\mathbf{P}^1 \times \mathbf{A}^3$ along with local generators of the ideal sheaf $\mathcal{I}$ whose vanishing locus gives $Ba_j(\widetilde{z}_j)_{S_j}$:

$$D(y) = \operatorname{Spec} \mathbf{F}[B,C,D,s], \qquad \mathcal{I}(D(y)) = (Ds - C, Bs + D);$$
$$D(x) = \operatorname{Spec} \mathbf{F}[B,C,D,s^{-1}], \qquad \mathcal{I}(D(x)) = (D - Cs^{-1}, B + Ds^{-1}).$$

The given generators clearly describe a regular sequence on each chart, and so the first statement holds in this case. The sheaf $\bigwedge^2 \iota^*\mathcal{I}$ is an invertible sheaf freely generated on $D(y) \cap Ba_j(\widetilde{z}_j)_{S_j}$ by $(Ds - C) \wedge (Bs + D)$, and on $D(x) \cap Ba_j(\widetilde{z}_j)_{S_j}$ by $(D - Cs^{-1}) \wedge (B + Ds^{-1})$. With these generators, the transition map from $D(y) \cap Ba_j(\widetilde{z}_j)_{S_j}$ to $D(x) \cap Ba_j(\widetilde{z}_j)_{S_j}$ is given by multiplication by s^2. Hence, $\wedge^2(\iota^*\mathcal{I})^\vee \cong \mathcal{O}(2) \cong p_j^*\mathcal{O}(1) \otimes q_j^*\mathcal{O}(1)$. Thus, we obtain that the dualizing sheaf of $Ba_j(\widetilde{z}_j)_{S_j}$ is $p_j^*\mathcal{O}(1) \otimes q_j^*\mathcal{O}(1) \otimes p_j^*\mathcal{O}(-2) \otimes q_j^*\mathcal{O}(-2) = p_j^*\mathcal{O}(-1) \otimes q_j^*\mathcal{O}(-1)$ as desired.

The rest of the first and second statements is now immediate. □

Lemma 4.11. *(Version of* [LHMM, Lem. 4.2.5]*) Let*

$$\mathcal{F} := p_j^*\mathcal{O}_{\mathbf{P}^1}(-1)^{\delta_j} \otimes_{\mathcal{O}_{Ba_j(\widetilde{z}_j)_{S_j}}} q_j^*\mathcal{O}_{\mathbf{P}^1}(-1)^{\epsilon_j}$$

be a sheaf on $Ba_j(\widetilde{z}_j)_{S_j}$, where $\epsilon_j, \delta_j \in \{0,1\}$. We have the following descriptions of cohomology groups associated to $\mathcal{F}$, as $\Gamma(Ba_j(\widetilde{z}_j)_{S_j})$–modules:

(i) If $T_j = 1$, then

$$R^i\Gamma\mathcal{F} = \begin{cases} \Gamma(Ba_j(\widetilde{z}_j)_{S_j}) & \text{if } i = 0, \epsilon_j = \delta_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

(ii) If $T_j = 2$, then

$$R^i\Gamma\mathcal{F} = \begin{cases} \Gamma(Ba_j(\widetilde{z}_j)_{S_j}) & \text{if } i = 0, \epsilon_j \in \{0,1\}, \delta_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

(iii) If $T_j = 3$, then

$R^i\Gamma\mathcal{F} =$

$$\begin{cases} \Gamma(Ba_j(\widetilde{z}_j)_{S_j}) & \text{if } i = 0, \delta_j + \epsilon_j \in \{0,2\}; \\ \Gamma(Ba_j(\widetilde{z}_j)_{S_j})[e_1, e_2]/(De_1 - Ce_2, Be_1 + De_2) & \text{if } i = 0, \delta_j + \epsilon_j = 1; \\ \Gamma(Ba_j(\widetilde{z}_j)_{S_j})/N_j & \text{if } i = 1, \delta_j + \epsilon_j = 2; \\ 0 & \text{otherwise.} \end{cases}$$

(iv) If $T_j = 4$, then

$$R^i\Gamma\mathcal{F} = \begin{cases} \Gamma(Ba_j(\widetilde{z}_j)_{S_j}) & \text{if } i = 0, \epsilon_j, \delta_j \in \{0,1\}; \\ 0 & \text{otherwise.} \end{cases}$$

(v) If $T_j = 5$, then

$$R^i\Gamma\mathcal{F} = \begin{cases} \Gamma(Ba_j(\widetilde{z}_j)_{S_j}) & \text{if } i = 0, \epsilon_j = 0, \delta_j \in \{0,1\}; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The proof for classes 1, 2, 4 and 5 follows from the following observations:

- $R\Gamma(\mathbf{A}^n, \mathcal{O})$ is concentrated in degree 0.
- $R\Gamma(\mathbf{P}^1, \mathcal{O}(-n))$ is zero if $n = 1$ and free of rank 1 over $\Gamma(\mathbf{P}^1, \mathcal{O}) = \mathbf{F}$ concentrated in degree 0 if $n = 0$.

In particular, by Künneth formula, we have:

- When $T_j = 1$, $R^i\Gamma\mathcal{F} = \oplus_{m+n=i} H^m(\mathbf{P}^1, \mathcal{O}(-\delta_j)) \otimes H^n(\mathbf{P}^1, \mathcal{O}(-\epsilon_j))$.
- When $T_j = 2$, $R^i\Gamma\mathcal{F} = \oplus_{m+n=i} H^m(\mathbf{P}^1, \mathcal{O}(-\delta_j)) \otimes H^n(\mathbf{A}^1, \mathcal{O})$.
- When $T_j = 4$, $R^i\Gamma\mathcal{F} = \oplus_{m+n=i} H^m(\mathbf{A}^1, \mathcal{O}) \otimes H^n(\mathbf{A}^1, \mathcal{O})$.
- When $T_j = 5$, $R^i\Gamma\mathcal{F} = \oplus_{m+n=i} H^m(\mathbf{A}^1, \mathcal{O}) \otimes H^n(\mathbf{P}^1, \mathcal{O}(-\epsilon_j))$.

Finally, suppose $T_j = 3$. Letting t denote x/y, $Ba_j(\widetilde{z}_j)_{S_j}$ admits an open cover by schemes $\operatorname{Spec}\mathbf{F}[t^{-1}, C]$ and $\operatorname{Spec}\mathbf{F}[t, B]$. Let $\iota_1 : \mathbf{F}[t^{-1}, C] \hookrightarrow \mathbf{F}[t^{\pm}, C]$ be the obvious inclusion, and let $\iota_2 : t^{\epsilon+\delta}\mathbf{F}[t, B] \hookrightarrow \mathbf{F}[t^{\pm}, C]$ be given by $t^n \mapsto t^n, B \mapsto -Ct^{-2}$. Therefore, $R\mathrm{pr}_{j*}\mathcal{F} \cong R\mathrm{pr}_{j*}(p_j^*\mathcal{O}_{\mathbf{P}^1}(-1)^{\delta_j+\epsilon_j})$ is computed by the Čech complex

$$\mathbf{F}[t^{-1}, C] \oplus t^{\delta_j+\epsilon_j}\mathbf{F}[t, B] \to \mathbf{F}[t^{\pm}, C]$$

where the differential maps $(f, g) \mapsto \iota_1(f) - \iota_2(g)$. This complex is quasi-isomorphic to the following complex of $\mathbf{F}[B, C, D]/(D^2 + BC)$ modules:

$$t^{\delta_j+\epsilon_j}\mathbf{F}[t, Ct^{-2}] \to \mathbf{F}[t^{\pm}, C]/\mathbf{F}[t^{-1}, C],$$

where B acts via multiplication by $-Ct^{-2}$, C acts via multiplication by C, and D acts via multiplication by Ct^{-1}.

When $\delta_j + \epsilon_j = 0$, the map is clearly surjective, while the kernel is free of rank 1 over $\mathbf{F}[B, C, D]/(D^2+BC)$, which thus gives the global functions. When $\delta_j+\epsilon_j = 1$, the map is again surjective while the kernel is generated by $\{C, Ct^{-1}\}$ which satisfy the relations $D \cdot C - C \cdot Ct^{-1} = 0$ and $B \cdot C + D \cdot Ct^{-1}$. Finally, when $\delta_j + \epsilon_j = 2$, a generator of the cokernel is given by the class of t while $B \cdot t, C \cdot t, D \cdot t$ give the zero class in the cokernel. In particular, the cokernel is isomorphic to $\mathbf{F}$ and supported at the vanishing locus of N_j. The kernel is free of rank 1 with generator C. □

Lemma 4.12. *The scheme–theoretic image of $\widetilde{Ba}(\widetilde{z})_S$ is a closed subscheme of $\widetilde{U}(\widetilde{z})$ that identifies naturally with*

$$\mathcal{S}pec\ \mathrm{pr}_{\widetilde{B}*}(\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}) \cong \operatorname{Spec}\Gamma(\widetilde{Ba}(\widetilde{z})_S).$$

Similarly, for $\mathfrak{t} \in \mathfrak{T}$, $Z_{\widetilde{B}}(\mathfrak{t})$ identifies naturally with

$$\mathcal{S}pec\ \mathrm{pr}(\mathfrak{t})_*\mathcal{O}_{B(\mathfrak{t})} \cong \operatorname{Spec}\Gamma(B(\mathfrak{t})).$$

Proof. The only thing to check is that the global functions on the scheme–theoretic images of $\widetilde{Ba}(\widetilde{z})_S$ and $B(\mathfrak{t})$ are the same as the global functions on $\widetilde{Ba}(\widetilde{z})_S$ and $B(\mathfrak{t})$ respectively. Tables 1 and 2 give descriptions of global functions on the scheme–theoretic image of $Ba_j(\widetilde{z}_j)_{S_j}$ under pr_j and Lemma 4.11 gives a description of $\Gamma(Ba_j(\widetilde{z}_j)_{S_j})$. The descriptions match and so, an application of Lemma 3.29 proves the desired statements. □

Lemma 4.13. *(Version of* [LHMM, Cor. 4.2.10]*) Consider the commutative diagram*

$$\begin{tikzcd} \widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \arrow[r, hook, "\Delta_S"] \arrow[dr, "\mathrm{pr}"'] & \widetilde{Ba}(\widetilde{z})_S \arrow[d, "\mathrm{pr}_{\widetilde{B}}"] \\ & \widetilde{U}(\widetilde{z}) \end{tikzcd}$$

The following are true:

(i) Whenever $i > 0$, $R^i\mathrm{pr}_{\widetilde{B}*}\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S} = 0$.
(ii) The map $\mathcal{O}_{\widetilde{U}(\widetilde{z})} \to \mathrm{pr}_{\widetilde{B}*}\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}$ is a surjection.
(iii) If $\mathcal{I}(\widetilde{z})$ is the ideal sheaf defining the closed immersion Δ_S, then
- $\operatorname{coker}(\mathcal{O}_{\widetilde{U}(\widetilde{z})} \to \mathrm{pr}_*\mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}) = R^1\mathrm{pr}_{\widetilde{B}*}\mathcal{I}(\widetilde{z})$, *and*
- $R^i\mathrm{pr}_*\mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} = R^{i+1}\mathrm{pr}_{\widetilde{B}*}\mathcal{I}(\widetilde{z})$ *for $i > 0$.*

Proof. Vanishing of $R^i\mathrm{pr}_{\widetilde{B}*}\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S} = 0$ if $i > 0$ follows from Lemma 4.11. The map $\mathcal{O}_{\widetilde{U}(\widetilde{z})} \to \mathrm{pr}_{\widetilde{B}*}\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}$ is a surjection by Lemma 4.12. Finally, the long exact sequence in cohomology obtained from the short exact sequence

$$0 \to \mathcal{I}(\widetilde{z}) \to \mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S} \to (\Delta_S)_*\mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} \to 0$$

of sheaves on $\widetilde{Ba}(\widetilde{z})_S$ implies the remaining statements. □

Lemma 4.14. *(Version of* [LHMM, Lem. 4.2.11]*) For each $j \in \mathbf{Z}/f\mathbf{Z}$, there exists a map*

$$\mathfrak{s}_j : q_j^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes_{\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}} p_{j-1}^*\mathcal{O}_{\mathbf{P}^1}(-1) \longrightarrow \mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S},$$

so that $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ is a complete intersection defined by the zero locus of $\{\mathfrak{s}_j\}_{j\in\mathbf{Z}/f\mathbf{Z}}$. If $\mathfrak{t} = (i-k,\dots,i) \in \mathfrak{T}$, $\mathfrak{s}_j$ for $j \in \{i-k+1,\dots,i\}$ can be viewed as a map

$$q_j^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes_{\mathcal{O}_{B(\mathfrak{t})}} p_{j-1}^*\mathcal{O}_{\mathbf{P}^1}(-1) \longrightarrow \mathcal{O}_{B(\mathfrak{t})}$$

and $Y(\mathfrak{t})$ is a complete intersection defined by the zero locus of $\mathfrak{s}_{i-k+1},\dots,\mathfrak{s}_i$.

Proof. Proof for $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ is identical to that of [LHMM, Lem. 4.2.11]. Since $\dim \widetilde{Y}^{\eta,\tau}(\widetilde{z})_S = \sum_{\mathfrak{t}\in\mathfrak{T}} \dim Y(\mathfrak{t})$ whenever $\mathfrak{T}$ is non–empty, $Y(\mathfrak{t})$ is forced to be a complete intersection as well. □

Corollary 4.15. *Let*

$$\mathcal{E} \overset{def}{=} \bigoplus_{j\in\mathbf{Z}/f\mathbf{Z}} \widetilde{q}_j^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes_{\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}} \widetilde{p}_{j-1}^*\mathcal{O}_{\mathbf{P}^1}(-1)$$

and for $\mathfrak{t} = (i-k,\dots,i) \in \mathfrak{T}$, let

$$\mathcal{E}(\mathfrak{t}) \overset{def}{=} \bigoplus_{j=i-k+1}^{i} q_j^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes_{\mathcal{O}_{B(\mathfrak{t})}} p_{j-1}^*\mathcal{O}_{\mathbf{P}^1}(-1).$$

The Koszul resolutions $\mathrm{Kos}_\bullet\left(\mathcal{E}, (\mathfrak{s}_j)_{j\in\mathbf{Z}/f\mathbf{Z}}\right)$ *and* $\mathrm{Kos}_\bullet\left(\mathcal{E}(\mathfrak{t}), (\mathfrak{s}_j)_{j=i-k+1}^{i}\right)$ *yield exact sequences*

$$0 \to \bigwedge^{f}\mathcal{E} \to \bigwedge^{f-1}\mathcal{E} \cdots \to \bigwedge^{1}\mathcal{E} \to \mathcal{I}(\widetilde{z}) \to 0, \ \textit{and}$$

$$0 \to \omega_{B(\mathfrak{t})} \to \omega_{B(\mathfrak{t})} \otimes \bigwedge^{1}\mathcal{E}(\mathfrak{t})^\vee \to \cdots \to \omega_{B(\mathfrak{t})} \otimes \bigwedge^{l(\mathfrak{t})-1}\mathcal{E}(\mathfrak{t})^\vee \to \Delta(\mathfrak{t})_*\omega_{Y(\mathfrak{t})} \to 0.$$

In particular, there exist two cohomological spectral sequences living in the second quadrant given by

$$E_1^{p,q} = R^q\Gamma\left(\bigwedge^{1-p}\mathcal{E}\right) \Longrightarrow R^{p+q}\Gamma\left(\mathcal{I}(\widetilde{z})\right), \ \textit{and}$$

$$E_1^{p,q} = R^q\Gamma\left(\omega_{B(\mathfrak{t})} \otimes \bigwedge^{l(\mathfrak{t})-1+p}\mathcal{E}(\mathfrak{t})^\vee\right) \Longrightarrow R^{p+q}\Gamma\left(\omega_{Y(\mathfrak{t})}\right).$$

Proof. The first exact sequence follows immediately from Lemma 4.14. For the second, note that for each $j \in \{i-k+1,\dots,i\}$, $q_j^*\mathcal{O}_{\mathbf{P}^1}(-1) \otimes_{\mathcal{O}_{B(\mathfrak{t})}} p_{j-1}^*\mathcal{O}_{\mathbf{P}^1}(-1)$ is the ideal sheaf of an effective Cartier divisor. Further, $Y(\mathfrak{t})$ is irreducible since $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ is, the latter because it is a $\mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}}$–torsor over the irreducible component $Y_S^{\eta,\tau}$ of $Y_{\mathbf{F}}^{\eta,\tau}$. Thus, [Kov, Prop. 6.10(iii)] gives the second exact sequence. □

Lemma 4.16. *The following are true:*

(i) *(Version of* [LHMM, Cor. 4.2.12]*) For $0 < a \leq f$, $b \geq a$, $R^b\Gamma\left(\bigwedge^a \mathcal{E}\right) \neq 0$ if and only if $a = b = f$ and $T_j = 3$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. In this special case,*

$$R^f\Gamma\left(\bigwedge^f \mathcal{E}\right) \cong \Gamma(\widetilde{Ba}(\widetilde{z})_S)/N \cong \mathbf{F}.$$

(ii) *Suppose $\mathfrak{t} = (i-k, \dots, i) \in \mathfrak{T}^*$ with $l(\mathfrak{t}) \geq 3$. Then*

$$R^b\Gamma\left(\omega_{B(\mathfrak{t})} \otimes \bigwedge^a \mathcal{E}(\mathfrak{t})^\vee\right) = \begin{cases} \neq 0 & \text{if } a = 0, b \in [2, l(\mathfrak{t})]; \\ \neq 0 & \text{if } a \in [1, l(\mathfrak{t}) - 3], b \in [2, l(\mathfrak{t}) - 1 - a]; \\ 0 & \text{otherwise.} \end{cases}$$

Further,

$$R^{l(\mathfrak{t})}\Gamma\left(\omega_{B(\mathfrak{t})}\right) = \Gamma(B(\mathfrak{t}))/N(\mathfrak{t}).$$

Proof. First, note that

$$\bigwedge^a \mathcal{E} = \bigoplus_{\underline{\epsilon}} \left(\bigotimes_{j \in \mathbf{Z}/f\mathbf{Z}} \widetilde{q}_j^* \mathcal{O}(-1)^{\epsilon_j} \otimes_{\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}} \widetilde{p}_j^* \mathcal{O}(-1)^{\epsilon_{j+1}} \right)$$

where the direct sum is over the set $\{\underline{\epsilon} \in \{0,1\}^{\mathbf{Z}/f\mathbf{Z}} \mid \sum \epsilon_j = a\}$. By Lemma 4.11,

$$R\Gamma\left(\widetilde{q}_j^* \mathcal{O}(-1)^{\epsilon_j} \otimes_{\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}} \widetilde{p}_j^* \mathcal{O}(-1)^{\epsilon_{j+1}}\right)$$

is concentrated in degree 0 unless $T_j = 3$ and $\epsilon_j = \epsilon_{j+1} = 1$, when it is concentrated in degrees 0 and 1. The first statement then follows from the Künneth formula.

Given $\mathfrak{t}$ as in the second statement, Lemma 4.10 implies that

$$\omega_{B(\mathfrak{t})} \otimes \bigwedge^a \mathcal{E}(\mathfrak{t})^\vee \cong \bigoplus_{\underline{\epsilon}} \mathcal{K}(\underline{\epsilon})$$

where $\mathcal{K}(\underline{\epsilon})$ denotes the sheaf

$$p_{i-k}^* \mathcal{O}(-1)^{2-\epsilon_{i-k+1}} \otimes \left(\bigotimes_{j=i-k+1}^{i-1} q_j^* \mathcal{O}(-1)^{1-\epsilon_j} \otimes p_j^* \mathcal{O}(-1)^{1-\epsilon_{j+1}} \right) \otimes q_i^* \mathcal{O}(-1)^{2-\epsilon_i}$$

and the direct sum is over $\{\underline{\epsilon} \in \{0,1\}^{\mathbf{Z}/f\mathbf{Z}} \mid \sum_{j=i-k+1}^{i} \epsilon_j = a\}$. Since $T_i \in \{1,2\}$ and $T_{i+k} \in \{1,5\}$,

$$\widetilde{Ba}_i(\widetilde{z}_i)^{\mathrm{I}}_{S_i} \cong \mathbf{P}^1 \times \mathrm{GL}_2, \text{ and } \widetilde{Ba}_{i+k}(\widetilde{z}_{i+k})^{\mathrm{II}}_{S_{i+k}} \cong Z_{i+k} \times \mathbf{P}^1.$$

Therefore, $R\Gamma\left(\mathcal{K}(\underline{\epsilon})\right)$ is non–vanishing if and only if $\epsilon_{i-k+1} = \epsilon_i = 0$. In particular, $R\Gamma\left(\omega_{B(\mathfrak{t})} \otimes \bigwedge^a \mathcal{E}^\vee\right)$ is non–vanishing if and only if $a \leq l(\mathfrak{t}) - 3$. Assume now that $R\Gamma\left(\mathcal{K}(\underline{\epsilon})\right)$ is non–vanishing. Since the cohomologies of $p_{i-k}^*\mathcal{O}(-2)$ and $q_i^*\mathcal{O}(-2)$ are both concentrated in degree 1, $R^b\Gamma\left(\mathcal{K}(\underline{\epsilon})\right) \neq 0$ if and only if $b \geq 2$ and

$$R^{b-2}\Gamma\left(\bigotimes_{j=i-k+1}^{i-1} \widetilde{q}_j^* \mathcal{O}(-1)^{1-\epsilon_j} \otimes_{\mathcal{O}_{\widetilde{Ba}(\widetilde{z})_S}} \widetilde{p}_j^* \mathcal{O}(-1)^{1-\epsilon_{j+1}} \right) \neq 0.$$

By Lemma 4.11, this holds if and only if

$$\#\{j \in [i-k+1, i-1] \mid \epsilon_j = \epsilon_{j+1} = 0\} \geq b - 2 \geq 0. \tag{4.17}$$

If there exists $j \in \{i-k+1, \dots, i\} \setminus \{i-k+1, i\}$ such that $\epsilon_j = 1$, equivalently $a \geq 1$, then $\epsilon_{i-k+1} = 0$ implies that

$$\#\{j \in [i-k+1, i-1] \mid \epsilon_j = 0\} \geq b - 1.$$

In particular, $a + b \leq l(\mathfrak{t}) - 1$. In the other direction, if $a \geq 1$ and $a + b \leq l(\mathfrak{t}) - 1$, we can easily find some $\underline{\epsilon}$ so that (4.17) is satisfied, for instance by setting $\epsilon_{i-k+2} = \dots = \epsilon_{i-k+a+1} = 1$. If $\epsilon_j = 0$ for each $j \in [i-k+1, i-1]$, equivalently $a = 0$, then by (4.17), cohomology is non–zero in any degree $b \in [2, l(\mathfrak{t}) - 1]$.

Finally, the Künneth formula and Lemma 4.11 yield the desired explicit description of $R^{l(\mathfrak{t})}\Gamma\left(\omega_{B(\mathfrak{t})}\right)$. ☐

Corollary 4.18. *(i) If $i > 1$, $R^i\Gamma(\mathcal{I}(\widetilde{z})) = 0$. If $i = 1$, $R^i\Gamma(\mathcal{I}(\widetilde{z})) \neq 0$ if and only $T_j = 3$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. In this case, it is isomorphic to $\mathbf{F}$ and is supported at $V(N) \subset \operatorname{Spec}\Gamma(\widetilde{Ba}(\widetilde{z})_S)$.*

(ii) Suppose $\mathfrak{t} = (i-k, \dots, i) \in \mathfrak{T}^*$ and $l(\mathfrak{t}) \geq 3$. If $i > 1$, $R^i\Gamma(\omega_{Y(\mathfrak{t})}) = 0$. If $i \in \{0, 1\}$, $R^i\Gamma(\omega_{Y(\mathfrak{t})}) \neq 0$. The support of $R^0\Gamma(\omega_{Y(\mathfrak{t})})$ inside $\operatorname{Spec}\Gamma(B(\mathfrak{t})) \cong Z_{\widetilde{B}}(\mathfrak{t})$ is the entire scheme–theoretic image of $Y(\mathfrak{t})$ under $\mathrm{pr}(\mathfrak{t})$, while that of $R^1\Gamma(\omega_{Y(\mathfrak{t})})$ is $V(N(\mathfrak{t}))$.

Proof. The assertions about $R\Gamma(\mathcal{I}(\widetilde{z}))$ and $R^1\Gamma(\omega_{Y(\mathfrak{t})})$ follow immediately from Corollary 4.15 and Lemma 4.16. By the proof of Lemma 4.8(ii), the (affine) scheme–theoretic image of $Y(\mathfrak{t})$ is isomorphic to $Y(\mathfrak{t})$ away from $V(N(\mathfrak{t}))$. By Lemma 4.14, $Y(\mathfrak{t})$ is a local complete intersection and therefore Gorenstein. Hence, after restriction to the complement of $V(N(\mathfrak{t}))$, the dualizing sheaf on the scheme–theoretic image of $Y(\mathfrak{t})$ is nonzero and supported everywhere. Since it is obtained by restricting the quasicoherent sheaf on $Z_{\widetilde{B}}(\mathfrak{t})$ corresponding to the module $R^0\Gamma(\omega_{Y(\mathfrak{t})})$, we conclude that the support of $R^0\Gamma(\omega_{Y(\mathfrak{t})})$ includes a dense open of the reduced and irreducible scheme–theoretic image of $Y(\mathfrak{t})$. It follows that the support must be the entire scheme–theoretic image of $Y(\mathfrak{t})$. ☐

Proposition 4.19. *(Version of* [LHMM, Prop. 3.3.14]*) For $i > 0$*

$$R^i\mathrm{pr}_*\mathcal{O}_{\widetilde{Y}_S^{\eta,\tau}} = 0.$$

Proof. Since the codomain of $\mathrm{pr}_{\widetilde{B}}$ is affine, $R\mathrm{pr}_{\widetilde{B}*}\mathcal{I}(\widetilde{z})$ is the quasicoherent sheaf associated to $R\Gamma(\mathcal{I}(\widetilde{z}))$. The proof is then immediate from Lemma 4.13 and Corollary 4.18. ☐

Proposition 4.20. *Suppose $\dim\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S = \dim\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ and that $T_j \neq 3$ for some $j \in \mathbf{Z}/f\mathbf{Z}$. Then,*

$$R^b\mathrm{pr}_*\omega_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} \neq 0$$

if and only if $0 \leq b \leq |\mathfrak{T}^*|$. The support of $R^0\mathrm{pr}_*\omega_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}$ is all of $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$, while if $1 \leq b \leq |\mathfrak{T}^*|$,

$$R^b\mathrm{pr}_*\omega_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} \cong \bigoplus_{\{\mathfrak{t}_1,\dots,\mathfrak{t}_b\}\subset\mathfrak{T}^*} M(\{\mathfrak{t}_1,\dots,\mathfrak{t}_b\})$$

where $M(\{\mathfrak{t}_1,\dots,\mathfrak{t}_b\})$ is a $\Gamma(\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S)$ module with support $V(N(\mathfrak{t}_1))\cap\dots\cap V(N(\mathfrak{t}_b))$.

Proof. Since the codomain of pr is affine, $R\mathrm{pr}_*\omega_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}$ is the sheaf associated to the module $R\Gamma(\omega_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S})$, with support necessarily contained in the scheme–theoretic image $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ of $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$. By (4.6) and the Künneth formula,

$$R^n\Gamma\left(\omega_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}\right) = \bigoplus_{(m_\mathfrak{t})_\mathfrak{t}, \sum m_\mathfrak{t} = n} \left(\bigotimes_{\mathfrak{t}\in\mathfrak{T}} R^{m_\mathfrak{t}}\Gamma\left(\omega_{Y(\mathfrak{t})}\right)\right).$$

The desired result follows from the following two observations. First, when $\mathfrak{t} \in \mathfrak{T} \smallsetminus \mathfrak{T}^*$, Lemma 4.8 implies that $Y(\mathfrak{t})$ is isomorphic to its scheme–theoretic image under $\mathrm{pr}(\mathfrak{t})$ and is therefore affine. Therefore, $R\Gamma(\omega_{Y(\mathfrak{t})})$ is concentrated in degree 0. It is also non–zero and supported on all of $Y(\mathfrak{t})$ because $Y(\mathfrak{t})$ is Gorenstein by Lemma 4.14.

Second, when $\mathfrak{t} \in \mathfrak{T}^*$, Lemma 4.8(ii) implies that the hypothesis of Corollary 4.18(ii) is satisfied. Therefore, $R\Gamma(\omega_{Y(\mathfrak{t})})$ is concentrated in degrees 0 and 1 and is non–zero in both degrees with support as described in Corollary 4.18(ii). □

Proposition 4.21. *(Version of* [LHMM, Prop. 4.2.14]*) The relative dualizing sheaf of $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ over $\mathbf{F}$ is trivial if $T_j = 3$ for all $j \in \mathbf{Z}/f\mathbf{Z}$.*

Proof. By Lemma 4.14, $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \hookrightarrow \widetilde{Ba}(\widetilde{z})_S$ is a regular immersion with normal bundle

$$\bigoplus_{j\in\mathbf{Z}/f\mathbf{Z}} \left(p_{j-1}^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes q_j^*\mathcal{O}_{\mathbf{P}^1}(1)\right)$$

which has determinant

$$\bigotimes_{j\in\mathbf{Z}/f\mathbf{Z}} \left(p_j^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes q_j^*\mathcal{O}_{\mathbf{P}^1}(1)\right).$$

Thus, using Lemma 4.10, the dualizing sheaf of $\widetilde{Y}_S^{\eta,\tau}$ is trivial when $T_j = 3$ for all $j \in \mathbf{Z}/f\mathbf{Z}$. □

Let $\mathcal{Z}_S^{\tau,\mathrm{nm}}$ be the normalization of $\mathcal{Z}^\tau$, as described in [ABI, Appendix A]. Since $Y_S^{\eta,\tau}$ is smooth, the map $Y_S^{\eta,\tau} \to \mathcal{Z}_S^\tau$ factors through $\mathcal{Z}_S^{\tau,\mathrm{nm}}$ and we have the following Cartesian diagram

$$\begin{array}{ccccc} \widetilde{Y}_S^{\eta,\tau} & \longrightarrow & \widetilde{\mathcal{Z}}_S^{\tau,\mathrm{nm}} & \longrightarrow & \widetilde{\mathcal{Z}}_S^\tau \\ \downarrow & & \downarrow & & \downarrow \\ Y_S^{\eta,\tau} & \longrightarrow & \mathcal{Z}_S^{\tau,\mathrm{nm}} & \longrightarrow & \mathcal{Z}_S^\tau \end{array} \tag{4.22}$$

where the vertical arrows are $\mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}}$–torsors and the scheme $\widetilde{\mathcal{Z}}_S^{\tau,\mathrm{nm}}$ is the normalization of $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$. It admits an open cover by the normalizations $\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S$ of $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$.

Theorem 4.23. *Suppose $\widetilde{Y}_S^{\eta,\tau} \xrightarrow{\widetilde{\pi}_S} \widetilde{\mathcal{Z}}_S^\tau$ is birational. The following are true:*

(i) The scheme $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ is normal if and only if there exists $j \in \mathbf{Z}/f\mathbf{Z}$ with $T_j \neq 3$.

(ii) The scheme $\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S$ is Gorenstein and resolution–rational (c.f. [Kov, Defn. 9.1]*) whenever $\mathfrak{T}^* = \varnothing$. Otherwise, $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S = \widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S$ is not Cohen–Macaulay.*

Proof. By properness of $\widetilde{\pi}_S$, it admits a Stein factorization

$$\widetilde{Y}_S^{\eta,\tau} \xrightarrow{\widetilde{\pi}^{\mathrm{nm}}} \mathcal{S}pec\, \widetilde{\pi}_* \mathcal{O}_{\widetilde{Y}_S^{\eta,\tau}} \cong \widetilde{\mathcal{Z}}_S^{\tau,\mathrm{nm}} \to \widetilde{\mathcal{Z}}_S^{\tau} \tag{4.24}$$

where $\mathcal{S}pec\, (\widetilde{\pi}_S)_* \mathcal{O}_{\widetilde{Y}_S^{\eta,\tau}} \cong \widetilde{\mathcal{Z}}_S^{\tau,\mathrm{nm}}$ because $\widetilde{\pi}_S$ is birational with smooth domain. Pulling back along the open immersion

$$\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S \hookrightarrow \widetilde{\mathcal{Z}}_S^{\tau,\mathrm{nm}}$$

gives the Stein factorization of pr:

$$\mathrm{pr} : \widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \xrightarrow{\widetilde{\pi}^{\mathrm{nm}}} \mathcal{S}pec\, \mathrm{pr}_* \mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} \cong \widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S \to \widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S \overset{\iota}{\hookrightarrow} \widetilde{U}(\widetilde{z}).$$

The corresponding map on sheaves is

$$\mathcal{O}_{\widetilde{U}(\widetilde{z})} \twoheadrightarrow \iota_* \mathcal{O}_{\widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S} \to \mathrm{pr}_* \mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} \tag{4.25}$$

with cokernel $R^1 \mathrm{pr}_{\widetilde{B}*} \mathcal{I}(\widetilde{z})$ by Lemma 4.13(iii). Thus, the map $\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S \to \widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S$ fails to be an isomorphism if and only if $R^1 \mathrm{pr}_{\widetilde{B}*} \mathcal{I}(\widetilde{z}) \neq 0$. An application of Corollary 4.18 finishes the proof of the first part.

For the second statement, the proof proceeds in the same way as [LHMM, Thm. 4.6.6]. By Proposition 4.19,

$$R\mathrm{pr}_* \mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} = \mathcal{O}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}. \tag{4.26}$$

Let $\omega^{\bullet}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}$ be a dualizing complex of $\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S$. Then, by [Sta, Tag 0BZL],

$$\omega^{\bullet}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} := (\widetilde{\pi}^{\mathrm{nm}})^! \omega^{\bullet}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}$$

is a dualizing complex of $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$. We have

$$\begin{aligned} R\mathrm{pr}_* \omega^{\bullet}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S} &\cong R\mathrm{pr}_* R\mathcal{H}om_{\mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}}(\mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}, \omega^{\bullet}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}) \\ &\cong R\mathcal{H}om_{\mathcal{O}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}}(R\mathrm{pr}_* \mathcal{O}_{\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S}, \omega^{\bullet}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}) \\ &\cong R\mathcal{H}om_{\mathcal{O}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}}(\mathcal{O}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}, \omega^{\bullet}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}) \\ &\cong \omega^{\bullet}_{\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S}. \end{aligned}$$

The first and last isomorphisms follow from the fact that sheaf homomorphisms from the structure sheaf to any sheaf $\mathcal{F}$ are isomorphic to $\mathcal{F}$, the second from Grothendieck duality (see for e.g. [Sta, Tag 0AU3]) and the third from (4.26). If $\mathfrak{T}^* = \varnothing$, then there are two possibilities. First is that $\mathfrak{T} = \varnothing$, or equivalently $T_j = 3$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. By Lemma 4.8, the second possibility is that for each $\mathfrak{t} \in \mathfrak{T}$, $\mathrm{pr}(\mathfrak{t})$ induces an isomorphism of $Y(\mathfrak{t})$ with its scheme–theoretic image. In the first case, Propositions 4.21 and 4.19 imply that the dualizing complex of $\widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S$ is concentrated in one degree with trivial dualizing sheaf. In the second case, $\widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S$ is isomorphic to $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$, and in particular, smooth.

Finally, suppose $\mathfrak{T}^* \neq \varnothing$. By (i), $\widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S = \widetilde{\mathcal{Z}}^{\tau,\mathrm{nm}}(\widetilde{z})_S$ and by Proposition 4.20, it is not Cohen–Macaulay. $\square$

Corollary 4.27. *The versal ring at a point $\overline{\rho} \in \mathcal{Z}_S^{\tau}(\overline{\mathbf{F}})$ is not normal if and only if it admits a lift to a point of $\widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S$ for some $\widetilde{z} = (\widetilde{z}_j)$ with each $\widetilde{z}_j \in \{w_0 t_\eta, t_{w_0(\eta)}\} s_j^{-1} v^{\mu_j}$, such that*

- *$T_j = 3$ for each j, and*
- *the lift lies in $V(N) \subset \widetilde{\mathcal{Z}}^{\tau}(\widetilde{z})_S$.*

Proof. By the proof of Theorem 4.23, the versal ring at $\overline{\rho}$ is not normal if and only if for some $\widetilde{z} = (\widetilde{z}_j)$ with each $\widetilde{z}_j \in \{w_0 t_\eta, t_{w_0(\eta)}\} s_j^{-1} v^{\mu_j}$, $\overline{\rho}$ lifts to a point of $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ that lies in the support of $R^1\mathrm{pr}_{\widetilde{B}*}\mathcal{I}(\widetilde{z})$. The desired statement then follows from Corollary 4.18 along with the additional observation that being the cokernel of the map in (4.25), $R^1\mathrm{pr}_{\widetilde{B}*}\mathcal{I}(\widetilde{z})$ is necessarily supported in $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S \subset \widetilde{U}(\widetilde{z})$. □

Corollary 4.28. *The versal ring at a point $\overline{\rho} \in \mathcal{Z}^\tau_S(\overline{\mathbf{F}})$ is normal but not Cohen–Macaulay if and only if it admits a lift to a point of $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ for some $\widetilde{z} = (\widetilde{z}_j)$ with each $\widetilde{z}_j \in \{w_0 t_\eta, t_{w_0(\eta)}\} s_j^{-1} v^{\mu_j}$, such that*

- *$\mathfrak{T}^* \neq \varnothing$, and*
- *the lift lies in*

$$\bigcup_{\mathfrak{t} \in \mathfrak{T}^*} V(N(\mathfrak{t})) \subset \widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S.$$

Proof. Follows immediately from Theorem 4.23(ii) and Proposition 4.20. □

Corollary 4.29. *The stack $\mathcal{Z}^\tau_S$ is either normal or its non–normal locus is a closed substack of codimension f whose preimage in $Y^{\eta,\tau}_S$ also has codimension f. In the non–normal case, the complement of the non–normal locus is a smooth open substack isomorphic to its preimage in $Y^{\eta,\tau}_S$.*

Proof. Suppose $\widetilde{z}$ is such that $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ is not normal. By Corollary 4.27, $T_j = 3$ for each j and the non–normal locus is $V(N) \subset \widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S \subset \operatorname{Spec}\Gamma(\widetilde{Ba}(\widetilde{z})_S)$, which is cut out by the functions B, C for each j. These functions give a regular sequence of length $2f$ in $\Gamma(\widetilde{Ba}(\widetilde{z})_S)$, where the latter has dimension $6f$. Therefore, $V(N)$ has dimension $4f$ and its codimension in $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ is f.

The stability of $V(N)$ under shifted-conjugation by $\mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}}$ follows from direct computation and implies the descent to a closed substack of $\mathcal{Z}^\tau_S$. Precisely, the locus $V(N) \subset \operatorname{Spec}\Gamma(\widetilde{Ba}(\widetilde{z})_S) \subset \widetilde{U}(\widetilde{z})$ is characterized by tuples of matrices $(\kappa_j \widetilde{z}_j)_j \in \widetilde{U}(\widetilde{z})$ with

$$(\kappa_j)_j \in \mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}} \qquad \text{and} \qquad \widetilde{z}_j = \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix}$$

for each j. Such tuples are evidently stable under $\mathrm{GL}_2^{\mathbf{Z}/f\mathbf{Z}}$–action.

Next, we note from the explicit description in Section 4.1 that the preimage of $V(N)$ in $\widetilde{Ba}(\widetilde{z})_S$ is isomorphic to

$$\prod_{j \in \mathbf{Z}/f\mathbf{Z}} \operatorname{Proj}\mathbf{F}[x_j, y_j] \times \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \mathrm{GL}_2\,.$$

Let κ_j denote the universal point of the copy of GL_2 in the j-th factor. The preimage of $V(N)$ in $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ is cut out by setting

$$[x_j : y_j] = [x_{j-1} : y_{j-1}]\kappa_{j-1}^{-1}$$

for each j. Thus, we can describe this preimage exclusively in terms of $[x_0 : y_0]$ and $\{\kappa_j\}_j$, and get rid of the variables $[x_j : y_j]$ for $j \neq 0$. More precisely, setting $[x : y] := [x_0 : y_0]$ and

$$\kappa := \prod_{j=0}^{f-1} \kappa_j^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the preimage of $V(N)$ in $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ is isomorphic to the closed subscheme of $\operatorname{Proj}\mathbf{F}[x,y]\times\prod_j \mathrm{GL}_2$ obtained by setting $[x:y]=[x:y]\kappa=[ax+cy:bx+dy]$. This is the same as setting $bx^2+(d-a)xy-cy^2=0$, which cuts out a closed subscheme of pure dimension $4f$. Since the dimension of $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ is $5f$, the codimension of the preimage of $V(N)$ is f. Finally, the complement of $V(N)$ is isomorphic to its smooth preimage in $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ by precisely the same argument as in the proof of Lemma 4.8(ii). $\square$

Corollary 4.30. *Suppose $\mathcal{Z}_S^\tau$ is normal but not Cohen–Macaulay. Writing $\mathfrak{T}$ as $\mathfrak{T}(\widetilde{z})$ to indicate the dependence on $\widetilde{z}$, let*

$$d := \min_{\substack{\widetilde{z}\text{ s.t.}\\ \mathfrak{T}(\widetilde{z})^*\neq\varnothing}} \ \min_{\mathfrak{t}\in\mathfrak{T}(\widetilde{z})^*}(l(\mathfrak{t})-1).$$

Then the non–Cohen–Macaulay locus in $\mathcal{Z}_S^\tau$ is a closed substack of codimension d with preimage in $Y_S^{\eta,\tau}$ of codimension $d-1$. The complement of the non–Cohen–Macaulay locus is a smooth open substack isomorphic to its preimage in $Y_S^{\eta,\tau}$.

Proof. Suppose $\widetilde{z}$ is such that $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ is not Cohen–Macaulay. Equivalently, by Corollary 4.28, $\mathfrak{T}(\widetilde{z})^*\neq\varnothing$. Let $\mathfrak{t}=(i-k,\dots,i)\in\mathfrak{T}(\widetilde{z})^*$. We note that if $T_j\in\{1,2\}$, $\dim\widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{I}}_{S_j}=\dim\mathrm{GL}_2+1$; if $T_j=3$, $\dim\widetilde{Ba}_j(\widetilde{z}_j)_{S_j}=\dim(Z_j\times\mathrm{GL}_2)$; and if $T_j\in\{1,5\}$, $\dim\widetilde{Ba}_j(\widetilde{z}_j)^{\mathrm{II}}_{S_j}=\dim Z_j+1$. Therefore,

$$\dim Z_{\widetilde{B}}(\mathfrak{t})=\dim B(\mathfrak{t})-2.$$

By Lemma 4.8, the scheme–theoretic image of $Y(\mathfrak{t})$ under $\mathrm{pr}(\mathfrak{t})$ has the same dimension as that of $Y(\mathfrak{t})$ and using Lemma 4.14, we find that

$$\dim Y(\mathfrak{t})=\dim B(\mathfrak{t})-(l(\mathfrak{t})-1).$$

Since $V(N(\mathfrak{t}))$ is cut out by the functions B,C for each $j\in\{i-k,\dots,i\}\smallsetminus\{i-k,i\}$ and these functions give a regular sequence of length $2(l(\mathfrak{t})-2)$ in $\Gamma(Z_{\widetilde{B}}(\mathfrak{t}))$,

$$\dim V(N(\mathfrak{t}))=\dim B(\mathfrak{t})-2-2(l(\mathfrak{t})-2).$$

Thus, the codimension of $V(N(\mathfrak{t}))$ in the scheme–theoretic image of $Y(\mathfrak{t})$ is $l(\mathfrak{t})-1$. Viewing $N(\mathfrak{t})$ as an ideal in $\Gamma(\widetilde{Ba}(\widetilde{z})_S)$ instead, the same is thus true for the codimension of $V(N(\mathfrak{t}))$ in $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$. The fiber in $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ over each point of $V(N(\mathfrak{t}))$ is directly seen to be isomorphic to $\mathbf{P}^1$ and so, the codimension of the preimage of $V(N(\mathfrak{t}))$ in $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ is $l(\mathfrak{t})-2$.

The closed scheme $V(N(\mathfrak{t}))$ descends to a closed substack of $\mathcal{Z}_S^\tau$ of codimension $l(\mathfrak{t})-1$ by the same argument as in Corollary 4.29 and the complement of

$$\bigcup_{\substack{\widetilde{z}\text{ s.t.}\\ \mathfrak{T}(\widetilde{z})^*\neq\varnothing}} \ \bigcup_{\mathfrak{t}\in\mathfrak{T}(\widetilde{z})^*} V(N(\mathfrak{t}))$$

is isomorphic to its preimage by the proof of Lemma 4.8(ii). The desired statements follow immediately. $\square$

5. Combinatorics of Serre weights, tame types and shapes

Let $\tau=\eta_1\oplus\eta_2$ be a tame inertial type with implicit, fixed ordering of the two characters (η_1,η_2). The paper [CEGSc] indexes irreducible components of the moduli stack of Breuil–Kisin modules with descent data of tame type $\tau^\vee$ by subsets $J\subset\mathbf{Z}/f\mathbf{Z}$. The irreducible component indexed by J is denoted $\mathcal{C}^\tau(J)$. In the following Lemma, we set up a dictionary between the notation of [CEGSc] and this article.

Lemma 5.1. *Let $\tau^\vee = \eta_1 \oplus \eta_2$ be a non–scalar principal series tame inertial type. Suppose $\eta_1\eta_2^{-1} = \prod_{j\in\mathbf{Z}/f\mathbf{Z}} \omega_j^{\gamma_j}$ with $\gamma_j \in [0, p-1]$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. The following are true:*

(i) If $\gamma_j \le (p+1)/2$ for each $j \in J$, then $\tau = \tau(s,\mu)$ where $s_j = s_{\mathrm{or},j} = \mathrm{id}$ and $\langle \mu_j, \alpha^\vee\rangle = \gamma_j$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. Furthermore,

$$C^{\tau^\vee}(\mathbf{Z}/f\mathbf{Z}) = Y_S^{\eta,\tau} \text{ where } S = R^{\mathbf{Z}/f\mathbf{Z}}.$$

(ii) Let M be the set of maximal subsets $\{i-k,\dots,i\} \subset \mathbf{Z}/f\mathbf{Z}$ satisfying
  (a) $\gamma_{i-k} < (p+1)/2$,
  (b) $\gamma_{i-k+1},\dots,\gamma_i \ge (p+1)/2$ if $k \ne 1$, and
  (c) $\gamma_i > (p+1)/2$.

  Here, maximality is in the sense that the subset is not properly contained in another subset satisfying the three conditions. Suppose either that $0 \in \mathbf{Z}/f\mathbf{Z}$ is not contained in any such subset, or $\gamma_0 < (p+1)/2$. Then $\tau = \tau(s,\mu)$ where if $A = \{i-k,\dots,i\} \in M$, then $(\langle\mu_j,\alpha^\vee\rangle, s_j, s_{\mathrm{or},j})$ is

$$\begin{aligned} &(\gamma_j+1, w_0, w_0) && \text{if } j = i-k,\\ &(p-1-\gamma_j, \mathrm{id}, w_0) && \text{if } j \in A \smallsetminus \{i-k, i\},\\ &(p-\gamma_j, w_0, \mathrm{id}) && \text{if } j = i. \end{aligned}$$

  On the other hand, if j is not in any subset contained in M, then

$$(\langle\mu_j,\alpha^\vee\rangle, s_j, s_{\mathrm{or},j}) = (\gamma_j, \mathrm{id}, \mathrm{id}).$$

  Furthermore, $C^{\tau^\vee}(\mathbf{Z}/f\mathbf{Z}) = Y_S^{\eta,\tau}$, where $S = (S_j)_{j\in\mathbf{Z}/f\mathbf{Z}}$ is as follows: If $A = \{i-k,\dots,i\} \in M$, then $S_j = L$ for $j \in A \smallsetminus \{i\}$ while $S_i = R$. If j is not in any subset contained in M, then $S_j = R$.

Proof. We first make some general observations for $\tau^\vee \cong \eta_1 \oplus \eta_2$. Suppose

$$\tau \cong \chi \otimes \left(\prod_{j\in\mathbf{Z}/f\mathbf{Z}} \omega_j^{a_j} \oplus \prod_{j\in\mathbf{Z}/f\mathbf{Z}} \omega_j^{b_j} \right)$$

for some character $\chi = \prod_j \omega_j^{c_j}$ such that $a_0 > b_0$ and for each j, $\min\{a_j, b_j\} = 0$ and $\max\{a_j,b_j\} \in [0,(p+1)/2]$. Let

$$\mu_j := (\max\{a_j,b_j\} + c_j, c_j).$$

One verifies immediately that $\tau = \tau(s,\mu)$ where $s_{f-1}, s_{f-2},\dots,s_0$ are determined (successively and) uniquely by the following constraints:

$$\begin{aligned} &s_{f-1}^{-1}s_{f-2}^{-1}\dots s_{f-j}^{-1}(\mu_{f-j}) = (a_{f-j}+c_{f-j}, b_{f-j}+c_{f-j}) \text{ for } 1 \le j \le f-1,\\ &s_j = \mathrm{id} \text{ whenever } \mu_j = (c_j,c_j), \text{ and}\\ &s_0 s_1 \dots s_{f-1} = \mathrm{id}\,. \end{aligned}$$

The permutation $s_{\mathrm{or},j}$ is id if and only if for some $k \in [1,f]$ (taking indices in $\mathbf{Z}$ instead of $\mathbf{Z}/f\mathbf{Z}$), $a_{j+i} = b_{j+i}$ for each $i \in [1,k-1]$ and $a_{j+k} > b_{j+k}$.

A criterion for a Breuil–Kisin module $\mathfrak{M}$ with A coefficients to be a point of $\mathcal{C}^{\tau^\vee}(J)$ is given in [BBH+, Cor. 3.17] and is as follows: For each j, consider an ordered basis (Zariski locally on A) $\{e_j, f_j\}$ of $\mathfrak{M}_j$ with $I(K'/K)$ acting via η_1 on e_j

and via η_2 on f_j. Consider the matrix of the Frobenius map $\Phi_{\mathfrak{M},j} : \varphi^*\mathfrak{M}_{j-1} \to \mathfrak{M}_j$ with respect to the basis $\{1 \otimes e_{j-1}, 1 \otimes f_{j-1}\}$ of the domain and $\{e_j, f_j\}$ of the codomain. Then $\mathfrak{M} \in \mathcal{C}^{\tau^\vee}(J)(A)$ if and only if v divides the top left entry (resp. bottom right entry) of the matrix for $\Phi_{\mathfrak{M},j}$ whenever $j \in J$ (resp. $j \notin J$). Thus, we find that $Y_S^{\eta,\tau} = C^{\tau^\vee}(J)$ where $j \in J$ if and only if

$$S_j = \begin{cases} L & \text{if } s_{\mathrm{or},j} = \mathrm{id} \qquad \text{and } \eta_1^{-1} = \chi \otimes \prod_j \omega_j^{a_j}, \\ R & \text{if } s_{\mathrm{or},j} = w_0 \qquad \text{and } \eta_1^{-1} = \chi \otimes \prod_j \omega_j^{a_j}, \\ R & \text{if } s_{\mathrm{or},j} = \mathrm{id} \qquad \text{and } \eta_1^{-1} = \chi \otimes \prod_j \omega_j^{b_j}, \\ L & \text{if } s_{\mathrm{or},j} = w_0 \qquad \text{and } \eta_1^{-1} = \chi \otimes \prod_j \omega_j^{b_j}. \end{cases}$$

Now, we verify parts (i) and (ii) separately. First, suppose $\gamma_j \leq (p+1)/2$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. Therefore,

$$\tau^\vee \cong \eta_2 \otimes \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{\gamma_j} \otimes \left(1 \oplus \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{-\gamma_j}\right)$$

and

$$\tau \cong \eta_2^{-1} \otimes \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{-\gamma_j} \otimes \left(1 \oplus \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{\gamma_j}\right).$$

Thus, we can take $\chi = \eta_2^{-1} \otimes \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{-\gamma_j}$, $(a_j, b_j) = (\gamma_j, 0)$ for each $j \in \mathbf{Z}/f\mathbf{Z}$, and $\eta_1^{-1} = \chi \otimes \prod_j \omega_j^{b_j}$. The algorithm above shows immediately that $(\mu, s, s_{\mathrm{or}})$ and $\mathcal{C}^{\tau^\vee}(\mathbf{Z}/f\mathbf{Z})$ are as in the statement of the first part of the Lemma.

Next, suppose $(\gamma_j)_{j \in \mathbf{Z}/f\mathbf{Z}}$ satisfies the hypothesis in the second part. If $A = \{i-k, \ldots, i\} \in M$, let

$$A^{\mathrm{i}} \stackrel{\mathrm{def}}{=} \{i-k\}, \qquad A^{\mathrm{o}} \stackrel{\mathrm{def}}{=} \{i\}, \qquad A^* \stackrel{\mathrm{def}}{=} A \smallsetminus \{i-k\},$$

and

$$M^{\mathrm{i}} \stackrel{\mathrm{def}}{=} \bigcup_{A \in M} A^{\mathrm{i}}, \qquad M^{\mathrm{o}} \stackrel{\mathrm{def}}{=} \bigcup_{A \in M} A^{\mathrm{o}}, \qquad M^* \stackrel{\mathrm{def}}{=} \bigcup_{A \in M} A^*.$$

By definition,

- if $j \notin M^* \cup M^{\mathrm{i}}$, then $\gamma_j \leq (p+1)/2$;
- if $j \in M^{\mathrm{i}}$, then $\gamma_j + 1 \leq (p+1)/2$;
- if $j \in M^* \smallsetminus M^{\mathrm{o}}$, then $p - 1 - \gamma_j < (p+1)/2$;
- if $j \in M^{\mathrm{o}}$, then $p - \gamma_j < (p+1)/2$.

Setting

$$\chi^{-1} = \eta_2 \otimes \prod_{j \notin M^*} \omega_j^{\gamma_j} \otimes \prod_{j \in M^* \smallsetminus M^{\mathrm{o}}} \omega_j^{p-1} \otimes \prod_{j \in M^{\mathrm{o}}} \omega_j^{p}, \tag{5.2}$$

we have $\tau^\vee \otimes \chi \cong$

$$\left(\prod_{j \in M^* \smallsetminus M^{\mathrm{o}}} \omega_j^{-p+1+\gamma_j} \otimes \prod_{j \in M^{\mathrm{o}}} \omega_j^{-p+\gamma_j}\right) \oplus \left(\prod_{j \notin M^* \cup M^{\mathrm{i}}} \omega_j^{-\gamma_j} \otimes \prod_{j \in M^{\mathrm{i}}} \omega_j^{-\gamma_j - 1}\right)$$

and therefore, $\tau \otimes \chi^{-1} \cong$

$$\left(\prod_{j \in M^* \smallsetminus M^{\mathrm{o}}} \omega_j^{p-1-\gamma_j} \otimes \prod_{j \in M^{\mathrm{o}}} \omega_j^{p-\gamma_j}\right) \oplus \left(\prod_{j \notin M^* \cup M^{\mathrm{i}}} \omega_j^{\gamma_j} \otimes \prod_{j \in M^{\mathrm{i}}} \omega_j^{\gamma_j+1}\right).$$

Since $0 \notin M^*$ by hypothesis,

$$\begin{aligned} a_j &= \gamma_j, & b_j &= 0 & &\text{for } j \notin M^* \cup M^{\mathrm{i}}, \\ a_j &= \gamma_j + 1, & b_j &= 0 & &\text{for } j \in M^{\mathrm{i}}, \\ a_j &= 0, & b_j &= p-1-\gamma_j & &\text{for } j \notin M^* \cup M^{\mathrm{i}}, \\ a_j &= 0, & b_j &= p-\gamma_j & &\text{for } j \in M^{\mathrm{i}}. \end{aligned}$$

In particular, $\eta_1^{-1} = \chi \otimes \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{b_j}$. Following through the steps of the algorithm to compute s_j, $s_{\mathrm{or},j}$ and S_j immediately produces the desired statement. $\square$

Definition 5.3. A sequence of integers $(a_1, \dots, a_n)$ is said to satisfy $\heartsuit$ if $a_1 = p-1$, $a_2 = \cdots = a_{n-1} = 1$ and $a_n = 0$.

Lemma 5.4. *Given s and μ, let $\widetilde{z} \in \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \{w_0 t_\eta, t_{w_0(\eta)}\} s_j^{-1} v^{\mu_j}$. Assume $p-2 > \max_j \langle \mu_j, \alpha^\vee \rangle$.*

(i) If s, μ, S are as Lemma 5.1(i), the class tuple $T = (T_j)_j$ satisfies $T_j \in \{3,4\}$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. Furthermore, if $\langle \mu_j, \alpha^\vee \rangle = 1$ for each j, then for each $T \in \{3,4\}^{\mathbf{Z}/f\mathbf{Z}}$, there exists $\widetilde{z}$ so that the class tuple associated to $\widetilde{Ba}(\widetilde{z})_S$ is T.

(ii) If s, μ, S are as in Lemma 5.1(ii), then $T = (T_j)_j$ satisfies the following:

- *If $A = \{i-k, \dots, i\} \in M$, then*

$$\begin{aligned} &T_j \in \{1,2\} & &\text{if } j = i-k, \\ &T_j = 2 & &\text{if } j \in A \smallsetminus \{i-k, i\}, \text{ and} \\ &T_j \in \{3,4\} & &\text{if } j = i. \end{aligned}$$

- *If j is not in any subset contained in M, then $T_j \in \{3,4,5\}$.*

Furthermore, there exists $\widetilde{z}$ so that the set $\mathfrak{T}^*$ (associated to the data of $\widetilde{z}, s, \mu, S$) is non–empty if and only if $(\gamma_j)_{j \in \mathbf{Z}/f\mathbf{Z}}$ contains some subsequence $(\gamma_{i-k}, \dots, \gamma_i)$ satisfying $\heartsuit$. The number of such subsequences is $\geq |\mathfrak{T}^*|$. Suitable $\widetilde{z}$ can be chosen so that equality holds, and the lowest value of $l(\mathfrak{t}) - 1$ attained over $\mathfrak{t} \in \mathfrak{T}^*$ as $\widetilde{z}$ varies is the smallest length of a sequence $(\gamma_{i-k}, \dots, \gamma_i)$ satisfying $\heartsuit$.

Proof. The proof of (i) and the first half of (ii) follows from simple comparison of (μ, s, S) in the statements of Lemma 5.1 with the classification of $\widetilde{Ba}_j(\widetilde{z}_j)_{S_j}$ in Section 4.1 and Tables 1 and 2.

For the remaining statements in the setting of Lemma 5.1(ii), we first make the following observation: If $\mathfrak{t} = (i-1, i, \dots, i+l) \in \mathfrak{T}^*$, then by definition of $\mathfrak{T}^*$ and the constraints on T, $i \in M^{\mathrm{o}}$; $T_j = 3$ for each $j \in \{i, \dots, i+l-1\}$; and either $i+l \in M^{\mathrm{i}}$ and $T_{i+l} = 1$, or $i+l \notin M^* \cup M^{\mathrm{i}}$ and $T_{i+l} = 5$.

By comparison with Section 4.1 and Tables 1 and 2, $T_i = 3$ implies $\gamma_i = p-1$; $T_j = 3$ for $j \in \{i, \dots, i+l-1\} \smallsetminus \{i\}$ implies $\gamma_j = 1$; and $T_{i+l} \in \{1,5\}$ implies $\gamma_{i+l} = 0$. On the other hand, if the sequence $(\gamma_i, \dots, \gamma_{i+l})$ satisfies $\heartsuit$, then Tables 1 and 2 demonstrate the existence of suitable $(\widetilde{z}_i, \dots, \widetilde{z}_{i+l})$ so that we get $\mathfrak{t} = (i-1, i, \dots, i+l) \in \mathfrak{T}^*$. Note that $l(\mathfrak{t}) - 1$ is the length of the sequence $(\gamma_i, \dots, \gamma_{i+l})$. $\square$

Now, we are ready to prove the main theorem of this section, which provides an upgrade of [GKKSW, Thm. 5.0.1].

Theorem 5.5. *Let $p > 5$ and let $\sigma = \sigma_{\mathbf{m},\mathbf{n}}$ be a non–Steinberg Serre weight. The following are true:*

(i) The component $\mathcal{X}(\sigma)$ is not smooth if and only if one of the following two holds:
  (a) For each $j \in \mathbf{Z}/f\mathbf{Z}$, $n_j = p-2$.
  (b) There exists a subset $\{i-k, \dots, i\} \subset \mathbf{Z}/f\mathbf{Z}$ with $n_{i-k} = 0$, $n_j = p-2$ if $j \in \{i-k, \dots, i\} \setminus \{i-k, i\}$, and $n_i = p-1$.

(ii) When (a) holds, $\mathcal{X}(\sigma)$ is not normal and its normalization admits a smooth–local cover by a Gorenstein and resolution–rational scheme. The non–normal locus on $\mathcal{X}(\sigma)$ has codimension f and its complement is smooth.

(iii) When (b) holds, $\mathcal{X}(\sigma)$ is normal but not Cohen–Macaulay. The non–Cohen–Macaulay locus on $\mathcal{X}(\sigma)$ has codimension ≥ 2 and its complement is smooth.

Proof. When neither (a) nor (b) holds, this is the main result of [GKKSW] when $n_j \neq 0$ for some $j \in \mathbf{Z}/f\mathbf{Z}$ and follows, for e.g., from the Appendix of loc. cit. when $n_j = 0$ for each $j \in \mathbf{Z}/f\mathbf{Z}$.

For the rest of the cases, [GKKSW, Prop. 4.1.2] shows that $\mathcal{X}(\sigma)$ is the scheme–theoretic image of $\mathcal{C}^{\tau^\vee}(\mathbf{Z}/f\mathbf{Z})$ for a principal series $\tau^\vee = \eta_1 \oplus \eta_2$ with

$$\eta_1\eta_2^{-1} = \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{p-1-n_j}.$$

Thus, $\mathcal{X}(\sigma)$ is the scheme–theoretic image of $Y_S^{\eta,\tau}$ where, when (a) holds, s, μ, S are as in Lemma 5.1(i) and $\langle \mu_j, \alpha^\vee \rangle = 1$ for each j. When (b) holds, then s, μ, S are as in Lemma 5.1(ii) (after possibly relabelling $0 \in \mathbf{Z}/f\mathbf{Z}$ so as to guarantee that $0 \in M^\circ$.)

Hence, Lemma 5.4 and Theorem 4.23 imply that $\mathcal{X}(\sigma) = \mathcal{Z}_S^\tau$ is not normal when (a) holds, and its normalization $\mathcal{Z}^{\tau,\mathrm{nm}}$ admits a smooth–local cover by the Gorenstein and resolution–rational scheme $\widetilde{\mathcal{Z}}_S^{\tau,\mathrm{nm}}$. On the other hand, when (b) holds, it is normal and admits a smooth–local cover by the non–Cohen–Macaulay scheme $\widetilde{\mathcal{Z}}_S^\tau = \widetilde{\mathcal{Z}}_S^{\tau,\mathrm{nm}}$. The statements about the codimension of the non–normal locus when (a) holds and the non–CM locus when (b) holds follow from Corollaries 4.29 and 4.30 respectively, which also prove the the smoothness of the complements of these loci. $\square$

Remark 5.6. Let $p = 5$ and let $\sigma_{\mathbf{m},\mathbf{n}}$ be non–Steinberg. If $n_j = p-2$ for each $j \in \mathbf{Z}/f\mathbf{Z}$, then $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ is not normal and its normalization admits a smooth–local cover by a Gorenstein and resolution–rational scheme. If there exists $i \in \mathbf{Z}/f\mathbf{Z}$ such that $n_{i+1} = 0$, $n_j = p-2$ if $j \notin \{i, i+1\}$, and $n_i = p-1$, then $\mathcal{X}(\sigma_{\mathbf{m},\mathbf{n}})$ is normal but not Cohen–Macaulay. These assertions are proved in the same fashion as Theorem 5.5, noting in both cases that $\max_j \langle \mu_j, \alpha^\vee \rangle = 1$ allows for $p = 5$.

By considering the codimension of the singular locus, we also obtain the following result.

Theorem 5.7. *Let $p > 5$, $f > 1$, $\sigma = \sigma_{\mathbf{m},\mathbf{n}}$ a non–Steinberg Serre weight, and $\iota : \mathcal{U} \hookrightarrow \mathcal{X}(\sigma)$ the smooth open locus in $\mathcal{X}(\sigma)$. Suppose $\mathcal{F}$ is a finite type maximal Cohen–Macaulay sheaf on $\mathcal{X}(\sigma)$ generically of rank 1. The following are true:*

(i) The sheaf $\mathcal{F}$ is isomorphic to the pushforward along ι of the invertible sheaf $\iota^*\mathcal{F}$ on $\mathcal{U}$.

(ii) If there does not exist i such that $(n_{i-1}, n_i) = (0, p-1)$, then $\mathcal{F}$ is the pushforward under π_S of a unique invertible sheaf on $Y_S^{\eta,\tau}$ for some suitable τ, S.

Proof. When $\mathcal{X}(\sigma)$ is smooth, (i) is trivial and (ii) follows from [GKKSW, Prop. 4.2.1] when $n_j \neq 0$ for some $j \in \mathbf{Z}/f\mathbf{Z}$ and from the Appendix of loc. cit. when $n_j = 0$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. Otherwise, fix τ and S such that $\mathcal{X}(\sigma) = \mathcal{Z}_S^\tau$. By Theorem 5.5, the singular locus on $\mathcal{Z}_S^\tau$ has codimension ≥ 2. Therefore, Lemma 5.9 below shows that $\mathcal{F}$ is obtained as the pushforward of a maximal Cohen–Macaulay sheaf on the normalization of $\mathcal{Z}_S^\tau$. Thus, after replacing $\mathcal{Z}_S^\tau$ by its normalization if necessary, we may assume that $\mathcal{Z}_S^\tau$ is normal.

Denote by F the pullback of $\mathcal{F}$ to $\widetilde{\mathcal{Z}}_S^\tau$ and let $j : U \hookrightarrow \widetilde{\mathcal{Z}}_S^\tau$ be the pullback of ι along the map $\widetilde{\mathcal{Z}}_S^\tau \to \mathcal{Z}_S^\tau$. Applying [HK, Thm. 3.5] to the structure map $\widetilde{\mathcal{Z}}_S^\tau \to \operatorname{Spec} \mathbf{F}$, we infer the existence of an isomorphism

$$F \xrightarrow{\sim} j_* j^* F.$$

Next, let $j_Y : U \hookrightarrow \widetilde{Y}_S^{\eta,\tau}$ be the pullback of j along $\widetilde{\pi}_S$. Note that we get an isomorphism between U and its pullback along $\widetilde{\pi}_S$ from Corollaries 4.29 and 4.30. We thus have

$$j = \widetilde{\pi}_S \circ j_Y.$$

Since U is smooth, the Auslander-Buchsbaum formula implies that j^*F is invertible. Therefore, there exists a Weil divisor $D \subset \widetilde{Y}_S^{\eta,\tau}$ such that $j^*F \cong j_Y^*\mathcal{O}(D)$, where $\mathcal{O}(D)$ is the (invertible) sheaf associated to D. By the algebraic Hartog's Lemma, $\mathcal{O}(D)$ is the unique invertible sheaf restricting to j^*F on $j_Y(U)$. Additionally,

$$(j_Y)_* j^* F \cong \mathcal{O}(D).$$

Thus, when the hypotheses in both (i) and (ii) hold,

$$j_* j^* F \cong (\widetilde{\pi}_S)_* (j_Y)_* j^* F \cong (\widetilde{\pi}_S)_* \mathcal{O}(D).$$

Descent data on F corresponding to the sheaf $\mathcal{F}$ restricts to descent data on j^*F. Since $j(U)$ is a dense open subscheme of $\widetilde{\mathcal{Z}}_S^\tau$ and F has no embedded primes being maximal Cohen–Macaulay, descent data on j^*F uniquely extends to descent data on j_*j^*F. When the complement of $j_Y(U)$ has codimension 2, the descent data on j^*F also extends uniquely to descent data on $\mathcal{O}(D)$. Therefore, by descending, we finish the proofs of parts (i) and (ii). □

Remark 5.8. To be precise, the pushforward functors considered in Theorem 5.7 are the quasicoherent pushforwards defined in [Sta, Tag 077A].

Lemma 5.9. *Let M be a finitely generated maximal Cohen–Macaulay module over a noetherian integral domain A that is regular in codimension 1. Then M is obtained by restriction of scalars from a maximal Cohen–Macaulay module defined over the normalization of A.*

Proof. Being maximal Cohen–Macaulay on an integral domain, the only associated prime of M is the zero ideal. Hence, M is torsion-free and admits an injection

$$M \hookrightarrow M \otimes_A \mathrm{Frac}(A).$$

We will show that M equals the finitely generated A–submodule $M' \subset M \otimes_A \mathrm{Frac}(A)$ generated by the action of the normalization of A on M. We can assume A is local with maximal ideal $\mathfrak{m}$. We prove the statement by induction on the dimension of A. The case when $\dim A \in \{0,1\}$ is immediate since A is normal in that case. Now, suppose $\dim A \geq 2$. By construction, M' is torsion-free and hence, has depth ≥ 1. By induction, if M'/M is non–zero, then its support is $\{\mathfrak{m}\}$ and in particular, its depth is 0. Consider the short exact sequence

$$0 \to M \to M' \to M'/M \to 0$$

and the long exact sequence obtained from it by applying the functor $\mathrm{Hom}_A(A/\mathfrak{m}, \_)$. Since depth $M' \geq 1$, nonzero M'/M implies that depth $M = 1$, a contradiction. $\square$

Theorem 5.10. *Let $p > 3$ and let $\sigma = \sigma_{\mathbf{m},\mathbf{n}}$ be a non–Steinberg Serre weight. The versal ring at $\overline{\rho} \in \mathcal{X}(\sigma)(\overline{\mathbf{F}})$ is not normal if and only if $n_j = p-2$ for each j, and as a G_K–representation, $\overline{\rho}$ is of the form*

$$\left(\prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{(m_j-1)}\right) \otimes \begin{pmatrix} \mathrm{ur}_{\lambda'} & * \\ 0 & \mathrm{ur}_{\lambda''} \end{pmatrix}$$

where λ' and λ'' are arbitrary units in $\overline{\mathbf{F}}$, and

- *$*$ is vanishing if $\lambda' \neq \lambda''$, and*
- *$*$ lies in the 1–dimensional space of extension classes that vanish after restriction to I_K if $\lambda' = \lambda''$.*

Proof. As noted in the proof of Theorem 5.5, $\mathcal{X}(\sigma)$ is the scheme–theoretic image of $\mathcal{C}^{\tau^\vee}(\mathbf{Z}/f\mathbf{Z}) = Y_S^{\eta,\tau}$, where $\tau^\vee \cong \eta_1 \oplus \eta_2$ with

$$\eta_1 = \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{m_j} \text{ and } \qquad \eta_2 = \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{m_j-1},$$

and s, μ, S are as in Lemma 5.1 (i). Therefore $\mu_j = (-m_j+1, -m_j)$ for each $j \in \mathbf{Z}/f\mathbf{Z}$.

By Corollary 4.27, the versal ring at $\overline{\rho}$ is not normal if and only if it lifts to a point in the vanishing locus of N in $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ for the unique (by Table 2)

$$\widetilde{z} \in \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \{w_0 t_\eta, t_{w_0(\eta)}\} s_j^{-1} v^{\mu_j}$$

such that the class tuple $(T_j)_j$ associated to $\widetilde{Ba}(\widetilde{z})_S$ satisfies $T_j = 3$ for each $j \in \mathbf{Z}/f\mathbf{Z}$. By the proof of Lemma 4.1, $V(N)$ is the image of the closed locus in $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ cut out by setting $X_j = \mathrm{id}$ for each j.

Let $\mathfrak{M}$ be a Breuil–Kisin module with $\overline{\mathbf{F}}$–coefficients in the image of this locus under the map $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \to Y_S^{\eta,\tau}$. The matrix $A^{(j)}_{\mathfrak{M},\beta}$ is given by W_j described in (3.23). By [LHMM, Table 3], $l_j \kappa_j = r_j$ for each j, or equivalently, $\widetilde{l}_j \kappa_j \widetilde{r}_j^{-1} \in B(\overline{\mathbf{F}})$. Therefore,

$$A^{(j)}_{\mathfrak{M},\beta} \in B(\overline{\mathbf{F}}) \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix}.$$

On the other hand, one verifies immediately that for any $(b_j)_j \in B(\overline{\mathbf{F}})^{\mathbf{Z}/f\mathbf{Z}}$,

$$([0:1], b_j, \mathrm{id}, [0:1])_j \in \widetilde{Ba}(\widetilde{z})_S$$

gives a point of $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ in the fiber over $V(N)$. Equivalently, the finite type points in the non–normal locus are precisely the étale φ–modules admitting Breuil–Kisin models (with descent data of type τ) $\mathfrak{M}$ with

$$A^{(j)}_{\mathfrak{M},\beta} = \begin{pmatrix} \lambda'_j & x_j v \\ 0 & \lambda''_j v \end{pmatrix}$$

where λ'_j, λ''_j are invertible while x_j is an arbitrary scalar for each $j \in \mathbf{Z}/f\mathbf{Z}$. Following the classification of rank 1 Breuil–Kisin modules with descent data in [CEGSc, Lem. 4.1.1], such Breuil–Kisin modules are certain extensions of $\mathfrak{M}'' \stackrel{\text{def}}{=} \mathfrak{M}(r'', (\lambda''_j)_j, c'')$ by $\mathfrak{M}' \stackrel{\text{def}}{=} \mathfrak{M}(r', (\lambda'_j)_j, c')$, where for each $j \in \mathbf{Z}/f\mathbf{Z}$, $r''_j = p^f - 1$, $r'_j = 0$, $c''_j = \sum_{i \in \mathbf{Z}/f\mathbf{Z}} p^i m_{j-i}$, $c'_j = \sum_{i\in\mathbf{Z}/f\mathbf{Z}} p^i(m_{j-i} - 1)$. We claim that the extension class is 0 whenever $\prod_j \lambda'_j \neq \prod_j \lambda''_j$. Otherwise, the space of extensions is 1–dimensional. To see this, we simplify the matrices $A^{(j)}_{\mathfrak{M},\beta}$ in the following way: First, by scaling the elements of β appropriately, we can assume that $\lambda'_j = \lambda''_j = 1$ for all $j \neq 0$. Let

$$g_j = \begin{pmatrix} 1 & \alpha_j \\ 0 & 1 \end{pmatrix} \in B(\overline{\mathbf{F}}).$$

The matrix $g_j A^{(j)}_{\mathfrak{M},\beta}\left(\mathrm{Ad}\ s_j^{-1} v^{\mu_j}(g_{j-1}^{-1})\right)$ is given by

$$\begin{pmatrix} \lambda'_j & v(x_j - \alpha_{j-1}\lambda'_j + \alpha_j\lambda''_j) \\ 0 & \lambda''_j v \end{pmatrix}.$$

When $f = 1$, if $\lambda'_0 \neq \lambda''_0$, take $\alpha_0 = x_0/(\lambda''_0 - \lambda'_0)$ to kill x_0. On the other hand, if $\lambda'_0 = \lambda''_0$, then no choice of α_0 can alter x_0. When $f \geq 2$, the extension class is seen to be vanishing if we can find $(\alpha_j)_j$ so that

$$\underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 1 \\ \lambda''_0 & 0 & 0 & 0 & \dots & 0 & 0 & -\lambda'_0 \end{bmatrix}}_{C} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{f-3} \\ \alpha_{f-2} \\ \alpha_{f-1} \end{bmatrix} = - \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{f-2} \\ x_{f-1} \\ x_0 \end{bmatrix}.$$

Using cofactor expansion along the bottom row, we see that the determinant is $\pm(\lambda'_0 - \lambda''_0)$. Therefore, the extension class is vanishing when $\lambda'_0 \neq \lambda''_0$. Otherwise, we can choose any α_0 and make successive choices of $\alpha_1, \dots, \alpha_{f-1}$ so that $x_j - \alpha_{j-1}\lambda'_j + \alpha_j\lambda''_j$ is 0 for each $j \in \{1, \dots, f-1\}$.

We need not consider any other types of change–of–basis matrices $(g_j)_j \in L^+\mathcal{G}^{\mathbf{Z}/f\mathbf{Z}}$ because if each x_j has to be killed, then using that $\langle \mu_j, \alpha^\vee \rangle \leq p - 2$, one checks immediately that it has to be killed by the constant part of $(g_j)_j$. Further, diagonal matrices cannot kill x_j. Following [CEGSc, Defn. 4.2.4], we additionally note that $(\mathfrak{M}'', \mathfrak{M}')$ has refined profile $(\mathbf{Z}/f\mathbf{Z}, r'')$. Therefore, by [CEGSc, Prop. 5.1.8], the extension class of $\mathfrak{M}[1/u]$ is non–vanishing if and only if that of $\mathfrak{M}$ is non–vanishing.

Computing the G_K–representations associated to $\mathfrak{M}'$ and $\mathfrak{M}''$ using [CEGSc, Lem. 4.1.4], we get the desired statement, except for the precise characterization of the one-dimensional extension class when $\lambda'_0 = \lambda''_0$. (Note that the formula actually gives G_{K_∞}–representations for K_∞ a wildly ramified extension of K, but by [CEGSc, Prop. 2.2.6], this is enough.)

Finally, to characterize the 1–dimensional extension class that survives when $\lambda'_0 = \lambda''_0$, let $\mathcal{M} = \varepsilon_\tau(\mathfrak{M})$. By Proposition 2.13, for each j, there exists a basis of $\mathcal{M}_j$ with respect to which the matrix of $\varphi^{(j)}_{\mathcal{M}} : \mathcal{M}_{j-1} \to \mathcal{M}_j$ is given by

$$A^{(j)}_{\mathfrak{M},\beta} s_j^{-1} v^{\mu_j} = A^{(j)}_{\mathfrak{M},\beta} \begin{pmatrix} v & 0 \\ 0 & 1 \end{pmatrix} v^{-m_j}.$$

We claim that the G_{K_∞}–representation $T(\mathcal{M})$ splits after restriction to G_{L_∞}, where L is the unramified extension of K of degree p with residue field $l = \mathbf{F}_{p^{pf}}$ and $L_\infty = LK_\infty$. Indeed, one can check easily that the étale φ–module $\mathcal{M}_L$ corresponding to $T(\mathcal{M})|_{L_\infty}$ is given by $l((v)) \otimes_{k((v))} \mathcal{M}$. Thus, for $j \in \mathbf{Z}/pf\mathbf{Z}$, the matrix F_j for the Frobenius map $\varphi^{(j)}_{\mathcal{M}_L} : (\mathcal{M}_L)_{j-1} \to (\mathcal{M}_L)_j$ with respect to the obvious basis is given by

$$\begin{aligned} &\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v^{1-m_{j \bmod f}} && \text{if } j \not\equiv 0 \mod f, \\ &\begin{pmatrix} \lambda'_0 & x_0 \\ 0 & \lambda'_0 \end{pmatrix} v^{1-m_{j \bmod f}} && \text{if } j \equiv 0 \mod f. \end{aligned}$$

Let

$$g_j = \begin{pmatrix} 1 & -d_j x_0/\lambda'_0 \\ 0 & 1 \end{pmatrix}$$

where $d_j \in [0, p-1]$ is such that $j - d_j f \in \{0, 1, \ldots, f-1\} \bmod pf$. Then the change–of–basis that maps F_j to $g_j F_j g_{j-1}^{-1}$ exhibits $\mathcal{M}_L$ as a split extension of two rank one étale φ–modules, as desired. Therefore, ρ is split after restriction to G_L, and in particular to I_L. By [DDR, Cor. 3.2], the space of extensions that vanish after restriction to $I_K = I_L$ is 1–dimensional. Thus, by comparison of dimension, we get the desired description of the extension class that survives when $\lambda'_0 = \lambda''_0$. $\square$

Theorem 5.11. *Let $p > 3$ and let $\sigma = \sigma_{\mathbf{m},\mathbf{n}}$ be a Serre weight with $n_0 = p-1$, $n_1 = 0$, and $n_j = p-2$ for $j \notin \{0,1\}$. The versal ring at $\overline{\rho} \in \mathcal{X}(\sigma)(\overline{\mathbf{F}})$ is not Cohen–Macaulay if and only if as a G_K–representation, $\overline{\rho}$ is of the form*

$$\left(\omega_1^{m_1} \otimes \prod_{j \neq 1} \omega_j^{m_j - 1} \right) \otimes \begin{pmatrix} \mathrm{ur}_{\lambda'} & * \\ 0 & \mathrm{ur}_{\lambda''} \end{pmatrix}$$

where λ and λ'' are arbitrary units in $\overline{\mathbf{F}}$, and

- *$*$ is vanishing if $\lambda' \neq \lambda''$, and*
- *$*$ lies in the 1–dimensional space of extension classes that vanish after restriction to I_K if $\lambda' = \lambda''$.*

Proof. As noted in the proof of Theorem 5.5, $\mathcal{X}(\sigma)$ is the scheme–theoretic image of $Y_S^{\eta,\tau}$ where, by Lemma 5.1(ii), $(\langle \mu_j, \alpha^\vee \rangle, s_j, s_{\mathrm{or},j}, S_j)$ equals

$$\begin{array}{ll} (1, w_0, w_0, L) & \text{if } j = 0, \\ (1, w_0, \mathrm{id}, R) & \text{if } j = 1, \\ (1, \mathrm{id}, \mathrm{id}, R) & \text{if } j \notin \{0,1\} \end{array}$$

and by [GKKSW, Prop. 4.1.2], $\tau^\vee \cong \eta_1 \oplus \eta_2$ where

$$\eta_1 = \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{m_j}, \qquad \eta_2 = \prod_{j \in \mathbf{Z}/f\mathbf{Z}} \omega_j^{m_j+n_j}.$$

Thus, $\mu_j = (c_j+1, c_j)$ for each j where, by (5.2), $c_j = -m_j$ if $j \neq 1$ and $c_1 = -m_1-1$.

By Corollary 4.28 and the proof of Lemma 5.4 we are looking for $\overline{\rho}$ admitting a lift to $\widetilde{\mathcal{Z}}^\tau(\widetilde{z})_S$ for some $\widetilde{z} = (\widetilde{w}_j s_j^{-1} v^{\mu_j})_j$ with $\widetilde{w}_j \in \{w_0 t_\eta, t_{w_0(\eta)}\}$ such that the associated type class tuple $(T_j)_j$ satisfies $T_1 = \cdots = T_{f-1} = 3$ and $T_0 = 1$, so that the lift lies in the vanishing locus of $N(\mathfrak{t})$ where $\mathfrak{t} = (0, 1, \ldots, f-1, 0) \in \mathfrak{T}^*$. By comparing s, μ to Tables 1 and 2, we find that $\widetilde{w}_0 = \widetilde{w}_1 = w_0 t_\eta$ while $\widetilde{w}_j = t_{w_0(\eta)}$ for $j \notin \{0,1\}$.

The proof of Lemma 4.1 shows that $V(N(\mathfrak{t}))$ is the image of the closed locus in $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ cut out by setting $X_j = \mathrm{id}$ for each $j \neq 0$, and that X_0 is necessarily id. Let $\mathfrak{M}$ be a Breuil–Kisin module with $\overline{\mathbf{F}}$–coefficients in the image of this locus under the map $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S \to Y_S^{\eta,\tau}$. The matrix $A^{(j)}_{\mathfrak{M},\beta}$ is given by W_j described in (3.23). By [LHMM, Table 3], $l_j \kappa_j = r_j$, equivalently $\widetilde{l}_j \kappa_j \widetilde{r}_j^{-1} \in B(\overline{\mathbf{F}})$, for each $j \neq 0$. Therefore, there exist $b_j \in B(\overline{\mathbf{F}})$ for $j \neq 0$ and $b_0 \in \mathrm{GL}_2(\overline{\mathbf{F}})$ such that

$$A^{(j)}_{\mathfrak{M},\beta} = \begin{cases} b_0 \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} & \text{if } j = 0, \\ b_1 \begin{pmatrix} 0 & 1 \\ v & 0 \end{pmatrix} & \text{if } j = 1, \\ b_j \begin{pmatrix} 1 & 0 \\ 0 & v \end{pmatrix} & \text{if } j \notin \{0,1\}. \end{cases} \tag{5.12}$$

On the other hand, one verifies immediately that for any $(b_j)_j$ with $b_j \in B(\overline{\mathbf{F}})$ when $j \neq 0$ and $b_0 \in \mathrm{GL}_2(\overline{\mathbf{F}})$,

$$([0:1], b_j, \mathrm{id}, [0:1])_j \in \widetilde{Ba}(\widetilde{z})_S$$

gives a point of $\widetilde{Y}^{\eta,\tau}(\widetilde{z})_S$ in the fiber over $V(N(\mathfrak{t}))$. In other words, the finite type points in the non–CM locus are precisely the étale φ–modules admitting Breuil–Kisin models (with descent data of type τ) $\mathfrak{M}$ with $A^{(j)}_{\mathfrak{M},\beta}$ satisfying (5.12) for each j. Let

$$b_0 = \begin{pmatrix} U & V \\ W & Z \end{pmatrix} \in \mathrm{GL}_2(\overline{\mathbf{F}})$$

and for each $j \neq 0$,

$$b_j = \begin{pmatrix} \lambda'_j & x'_j \\ 0 & \lambda''_j \end{pmatrix}.$$

Therefore,

$$C^{(j)}_{\mathfrak{M},\beta} = \begin{cases} \begin{pmatrix} W & Zu^{p^f-1-l_0} \\ Uu^{l_0} & Vv \end{pmatrix} & \text{if } j=0, \\ \begin{pmatrix} x_1 v & \lambda'_1 u^{l_1} \\ \lambda''_1 u^{p^f-1-l_1} & 0 \end{pmatrix} & \text{if } j=1, \\ \begin{pmatrix} \lambda'_j & x'_j u^{l_j} \\ 0 & \lambda''_j v \end{pmatrix} & \text{if } j \notin \{0,1\}, \end{cases} \tag{5.13}$$

for suitable integers $l_j \in (0, p^f-1)$. Specifically, $l_0 = p^{f-1} - \sum_{i=0}^{f-2} p^i$ and $l_1 = -1 + \sum_{i=1}^{f-1} p^i$. Let $\beta_j = (e_j, f_j)$ (note that it follows from the proof of Lemma 5.1 that β_j is ordered with respect to the ordering (η_2, η_1) of the eigenvalues).

Suppose first that $b_0 \in B(\overline{\mathbf{F}})$, equivalently $W=0$. We observe from the matrices that $f_0+e_1+\cdots+e_{f-1}$ gives a free generator of a sub-Breuil Kisin module $\mathfrak{M}'$ of $\mathfrak{M}$ and $e_0+f_1+\cdots+f_{f-1}$ lifts a basis of the quotient $\mathfrak{M}''$. Thus, in the sense of [CEGSc, Sec. 4.2], $\mathfrak{M}$ is an extension of $\mathfrak{M}'' \cong \mathfrak{M}(r'', (\lambda''_j)_j, c'')$ by $\mathfrak{M}' \cong \mathfrak{M}(r', (\lambda'_j)_j, c')$ where $\lambda'_0 = U$, $\lambda''_0 = Z$,

$$\begin{aligned} r''_j &= p^f - 1 - l_0, & r'_j &= l_0 & &\text{if } j=0, \\ r''_j &= p^f - 1 - l_1, & r'_j &= l_1 & &\text{if } j=1, \\ r''_j &= p^f - 1, & r'_j &= 0 & &\text{if } j \notin \{0,1\}, \end{aligned}$$

and $\{c'_j\}_j, \{c''_j\}_j$ satisfy

$$\omega_j^{c''_j} = \eta_1, \quad \omega_j^{c'_j} = \eta_2 \text{ when } j \neq 0,$$

while $\omega_0^{c''_0} = \eta_2$, $\omega_0^{c'_0} = \eta_1$. Note that $(\mathfrak{M}'', \mathfrak{M}')$ has refined profile $(\mathbf{Z}/f\mathbf{Z} \setminus \{0\}, r'')$. Using exactly the same arguments as in the proof of Theorem 5.10, we find that the Galois representations associated to such $\mathfrak{M}[1/u]$ are precisely those of the form described in the statement of the theorem.

Next, suppose $W \in \overline{\mathbf{F}}^\times$. Consider a change of basis on $\mathfrak{M}$ that transforms $b_j = A^{(j)}_{\mathfrak{M},\beta} \widetilde{w}_j^{-1}$ to $b'_j = g_j A^{(j)}_{\mathfrak{M},\beta} \left(\mathrm{Ad}\, s_j^{-1} v^{\mu_j}(g_{j-1}^{-1})\right) \widetilde{w}_j^{-1}$ with $g_j \in B(\overline{\mathbf{F}})$ for each j. We claim that by doing a careful change of basis, we can arrange for $V = x_1 = \cdots = x_{f-1} = 0$. First, one sees immediately that by taking $g_j = \mathrm{id}$ for $j \notin \{f-1, 0\}$ and suitable $g_{f-1}, g_0 \in B(\overline{\mathbf{F}})$, we can arrange $V=0$. Similarly, it is easy to see that after doing a change of basis with $g_{f-1} = g_0 = \mathrm{id}$ and suitable $g_j \in B(\overline{\mathbf{F}})$ for each $j \notin \{0, f-1\}$, we can take $x_j = 0$ whenever $j \notin \{0,1\}$ and further, $\lambda'_j = \lambda''_j = 1$ when $j \neq 0$. Next, take

$$g_j = \begin{pmatrix} 1 & \alpha_j \\ 0 & 1 \end{pmatrix}$$

where $\alpha_0 \in \overline{\mathbf{F}}$ satisfies

$$W\alpha_0^2 + (U - Z - Wx_1)\alpha_0 - Ux_1 = 0,$$

and $\alpha_1 = \cdots = \alpha_{f-1} = \alpha_0 - x_1$. Thus

$$b'_j = \begin{cases} \begin{pmatrix} U + \alpha_0 W & -U\alpha_{f-1} - W\alpha_0\alpha_{f-1} + Z\alpha_0 \\ W & Z \end{pmatrix} & \text{if } j = 0, \\ \begin{pmatrix} 1 & x_1 - \alpha_0 + \alpha_1 \\ 0 & 1 \end{pmatrix} & \text{if } j = 1, \\ \begin{pmatrix} 1 & -\alpha_{j-1} + \alpha_j \\ 0 & 1 \end{pmatrix} & \text{if } j \notin \{0, 1\}. \end{cases}$$

By choice of α_j, the top right entry of b'_j vanishes for each j.

Now, assume $V = x_1 = \cdots = x_{f-1} = 0$ and let $\mathcal{M} = \varepsilon_\tau(\mathfrak{M})$. By Proposition 2.13, one can choose a basis for each j so that the matrix of $\varphi_{\mathcal{M}}^{(j)} : \mathcal{M}_{j-1} \to \mathcal{M}_j$ is given by $A_{\mathfrak{M},\beta}^{(j)} s_j^{-1} v^\mu$. We have

$$A_{\mathfrak{M},\beta}^{(j)} s_j^{-1} v^\mu = \begin{cases} \begin{pmatrix} U & 0 \\ W & Z \end{pmatrix} v^{-m_0+1} & \text{if } j = 0, \\ \begin{pmatrix} \lambda'_j & 0 \\ 0 & \lambda''_j \end{pmatrix} v^{-m_1} & \text{if } j = 1, \\ \begin{pmatrix} \lambda'_j & 0 \\ 0 & \lambda''_j \end{pmatrix} v^{-m_j+1} & \text{if } j \notin \{0, 1\}. \end{cases}$$

After a change of basis involving permuting the two basis elements of $\mathcal{M}_j$ for each j, we find that $\mathcal{M}$ could also have been obtained via the first case, that is, when we assumed $b_0 \in B(\overline{\mathbf{F}})$. Therefore, we get no new representations upon assuming that W is a unit. This finishes the proof. □

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