Fuzzy Sets and Fuzzy Logic

Fuzzy Logic

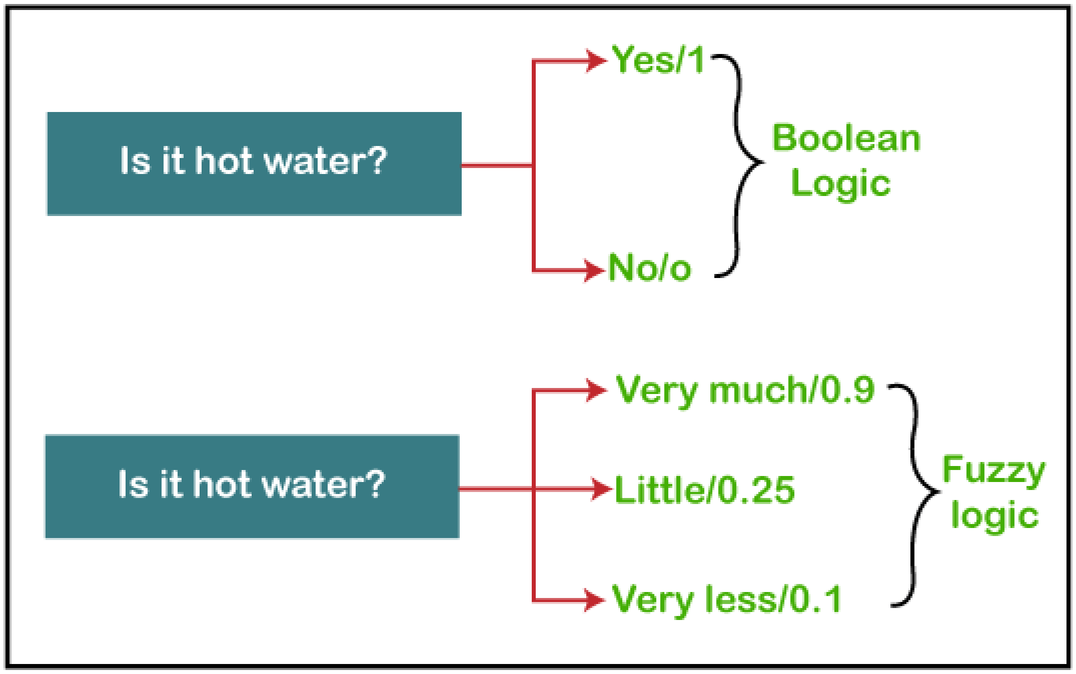
- Fuzzy logic is an approach to computing based on "degrees of truth" rather than the usual "true or false" (1 or 0) Boolean logic on which the modern computer is based.

- The idea of fuzzy logic was first advanced by Dr. Lotfi Zadeh of the University of California at Berkeley in the 1960s.

- In fuzzy mathematics, fuzzy logic is a form of many-valued logic in which the truth values of variables may be any real number between 0 and 1 both inclusive.

- Fuzzy logic has been used in numerous applications such as facial pattern recognition, air conditioners, washing machines, vacuum cleaners, antiskid braking systems, transmission systems, control of subway systems and unmanned helicopters, knowledge-based systems for multi objective optimization of power systems

Example of Fuzzy Logic as comparing to Boolean Logic



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- Fuzzy logic contains the multiple logical values and these values are the truth values of a variable or problem between 0 and 1. This concept was introduced by Lofti Zadeh in 1965 based on the Fuzzy Set Theory.
- In the Boolean system, only two possibilities (0 and 1) exist, where 1 denotes the absolute truth value and 0 denotes the absolute false value. But in the fuzzy system, there are multiple possibilities present between the 0 and 1, which are partially false and partially true.
- The Fuzzy logic can be implemented in systems such as micro-controllers, workstation-based or large network-based systems for achieving the definite output. It can also be implemented in both hardware or software.

Characteristics of Fuzzy Logic

Following are the characteristics of fuzzy logic:

1. This concept is flexible and we can easily understand and implement it.
2. It is used for helping the minimization of the logics created by the human.
3. It is the best method for finding the solution of those problems which are suitable for approximate or uncertain reasoning.
4. It always offers two values, which denote the two possible solutions for a problem and statement.

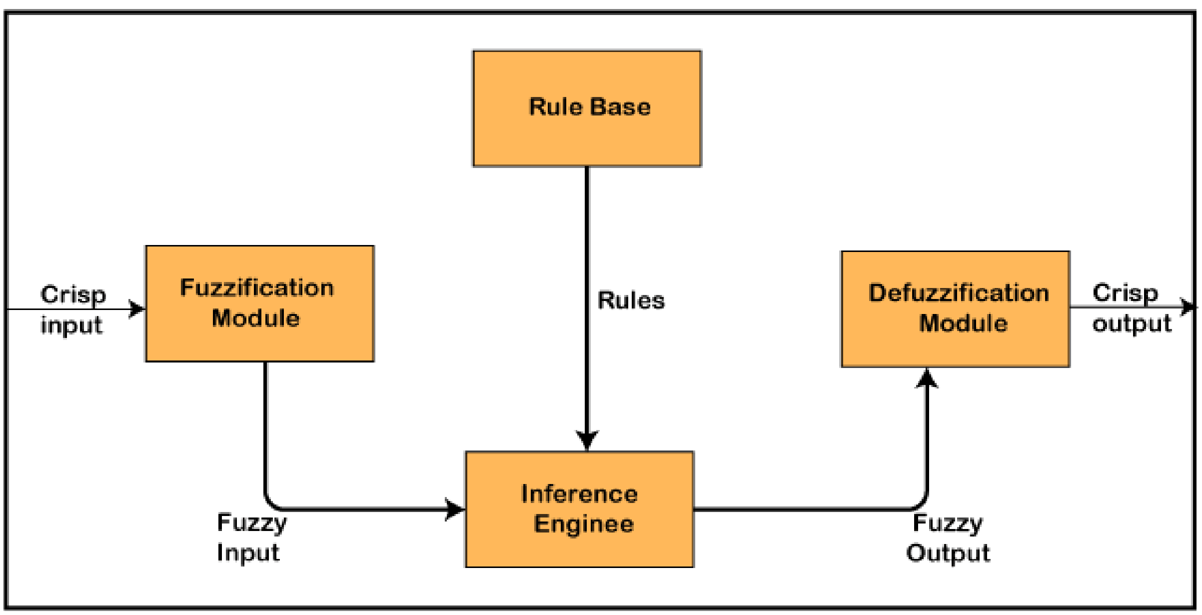
5. It allows users to build or create the functions which are non-linear of arbitrary complexity.
6. In fuzzy logic, everything is a matter of degree.
7. In the Fuzzy logic, any system which is logical can be easily fuzzified.
8. It is based on natural language processing.
9. It is also used by the quantitative analysts for improving their algorithm's execution.
10. It also allows users to integrate with the programming.

Architecture of a Fuzzy Logic System

In the architecture of the **Fuzzy Logic** system, each component plays an important role. The architecture consists of the different four components which are given below.

1. Rule Base
2. Fuzzification
3. Inference Engine
4. Defuzzification

Following diagram shows the architecture or process of a Fuzzy Logic system:



1. Rule Base

Rule Base is a component used for storing the set of rules and the If-Then conditions given by the experts are used for controlling the decision-making systems. There are so many updates that come in the Fuzzy theory recently, which offers effective methods for designing and tuning of fuzzy controllers. These updates or developments decreases the number of fuzzy set of rules.

2. Fuzzification

Fuzzification is a module or component for transforming the system inputs, i.e., it converts the crisp number into fuzzy steps. The crisp numbers are those inputs which are measured by the sensors and then fuzzification passed them into the control systems for further processing. This component divides the input signals into following five states in any Fuzzy Logic system:

- Large Positive (LP)
- Medium Positive (MP)
- Small (S)
- Medium Negative (MN)
- Large negative (LN)

3. Inference Engine

This component is a main component in any Fuzzy Logic system (FLS), because all the information is processed in the Inference Engine. It allows users to find the matching degree between the current fuzzy input and the rules. After the matching degree, this system determines which rule is to be added according to the given input field. When all rules are fired, then they are combined for developing the control actions.

4. Defuzzification

Defuzzification is a module or component, which takes the fuzzy set inputs generated by the **Inference Engine**, and then transforms them into a crisp value. It is the last step in the process of a fuzzy logic system. The crisp value is a type of value which is acceptable by the user. Various techniques are present to do this, but the user has to select the best one for reducing the errors.

Classical and Fuzzy Set Theory

To learn about classical and Fuzzy set theory, firstly you have to know about what is set.

Set

A set is a term, which is a collection of unordered or ordered elements. Following are the various examples of a set:

A set of all-natural numbers
A set of students in a class.
A set of all cities in a state.
A set of upper-case letters of the alphabet.

Classical Set

It is a type of set which collects the distinct objects in a group. The sets with the crisp boundaries are classical sets. In any set, each single entity is called an element or member of that set.

Mathematical Representation of Sets

Any set can be easily denoted in the following two different ways:

1. Roaster Form: This is also called as a tabular form. In this form, the set is represented in the following way:

Set_name = { element1, element2, element3, ..., element N}

The elements in the set are enclosed within the brackets and separated by the commas.

Following are the two examples which describes the set in Roaster or Tabular form:

Example 1:

Set of Natural Numbers: N={1, 2, 3, 4, 5, 6, 7, ...,n).

Example 2:

Set of Prime Numbers less than 50: X={2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47}.

Fuzzy Set

- The set theory of classical is the subset of Fuzzy set theory.
- Fuzzy logic is based on this theory, which is a generalisation of the classical theory of set (i.e., crisp set) introduced by Zadeh in 1965.
- A fuzzy set is a collection of values which exist between 0 and 1.
- Fuzzy sets are denoted or represented by the tilde (~) character.
- In the fuzzy set, the partial membership also exists. This theory released as an extension of classical set theory.

- This theory is denoted mathematically asA fuzzy set (Ã) is a pair of U and M, where U is the Universe of discourse and M is the membership function which takes on values in the interval [0, 1]. The universe of discourse (U) is also denoted by Ω or X.

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}$$

Operations on Fuzzy Set

Given Ã and B are the two fuzzy sets, and X be the universe of discourse with the following respective member functions:

The operations of Fuzzy set are as follows:

1. Union Operation: The union operation of a fuzzy set is defined by:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Example:

suppose A is a set which contains following elements:

A = {(X_1, 0.6), (X_2, 0.2), (X_3, 1), (X_4, 0.4)}

And, B is a set which contains following elements:

B = {(X_1, 0.1), (X_2, 0.8), (X_3, 0), (X_4, 0.9)}

then,

AUB = {(X_1, 0.6), (X_2, 0.8), (X_3, 1), (X_4, 0.9)}

Because, according to this operation

For X_1

$\mu_{A \cup B}(X_1) = \max(\mu_A(X_1), \mu_B(X_1))$

$\mu_{A \cup B}(X_1) = \max(0.6, 0.1)$

$\mu_{A \cup B}(X_1) = 0.6$

For X_2

$\mu_{A\cup B}(X_2) = \max(\mu_A(X_2), \mu_B(X_2))$

$\mu_{A\cup B}(X_2) = \max(0.2, 0.8)$

$\mu_{A\cup B}(X_2) = 0.8$

For X_3

$\mu_{A\cup B}(X_3) = \max(\mu_A(X_3), \mu_B(X_3))$

$\mu_{A\cup B}(X_3) = \max(1, 0)$

$\mu_{A\cup B}(X_3) = 1$

For X_4

$\mu_{A\cup B}(X_4) = \max(\mu_A(X_4), \mu_B(X_4))$

$\mu_{A\cup B}(X_4) = \max(0.4, 0.9)$

$\mu_{A\cup B}(X_4) = 0.9$

2. Intersection Operation:

The intersection operation of fuzzy set is defined by:

$$\mu_{A\cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Example:

suppose A is a set which contains following elements:

$A = \{(X_1, 0.3), (X_2, 0.7), (X_3, 0.5), (X_4, 0.1)\}$

And, B is a set which contains following elements:

$B = \{(X_1, 0.8), (X_2, 0.2), (X_3, 0.4), (X_4, 0.9)\}$

then,

$A\cap B = \{(X_1, 0.3), (X_2, 0.2), (X_3, 0.4), (X_4, 0.1)\}$

Because, according to this operation

For X_1

$\mu_{A\cap B}(X_1) = \min(\mu_A(X_1), \mu_B(X_1))$

$\mu_{A\cap B}(X_1) = \min(0.3, 0.8)$

$\mu_{A\cap B}(X_1) = 0.3$

For X_2

$\mu_{A\cap B}(X_2) = \min(\mu_A(X_2), \mu_B(X_2))$

$\mu_{A\cap B}(X_2) = \min(0.7, 0.2)$

$\mu_{A\cap B}(X_2) = 0.2$

For X_3

$\mu_{A\cap B}(X_3) = \min(\mu_A(X_3), \mu_B(X_3))$

$\mu_{A\cap B}(X_3) = \min(0.5, 0.4)$

$\mu_{A\cap B}(X_3) = 0.4$

For X_4

$\mu_{A\cap B}(X_4) = \min(\mu_A(X_4), \mu_B(X_4))$

$\mu_{A\cap B}(X_4) = \min(0.1, 0.9)$

$\mu_{A\cap B}(X_4) = 0.1$

3. Complement Operation:

The complement operation of fuzzy set is defined by:

$\mu_{\bar{A}}(x) = 1-\mu_A(x)$,

Example:

suppose A is a set which contains following elements:

$A = \{(X_1, 0.3), (X_2, 0.8), (X_3, 0.5), (X_4, 0.1)\}$

then,

$\bar{A}= \{(X_1, 0.7), (X_2, 0.2), (X_3, 0.5), (X_4, 0.9)\}$

Because, according to this operation

For X_1

$\mu_{\bar{A}}(X_1) = 1-\mu_A(X_1)$

$\mu_{\bar{A}}(X_1) = 1 - 0.3$

$\mu_{\bar{A}}(X_1) = 0.7$

For X_2

$\mu_{\bar{A}}(X_2) = 1-\mu_A(X_2)$

$\mu_{\bar{A}}(X_2) = 1 - 0.8$

$\mu_{\bar{A}}(X_2) = 0.2$

For X_3

$\mu_{\bar{A}}(X_3) = 1-\mu_A(X_3)$

$\mu_{\bar{A}}(X_3) = 1 - 0.5$

$\mu_{\bar{A}}(X_3) = 0.5$

For X_4

$\mu_{\bar{A}}(X_4) = 1-\mu_A(X_4)$

$\mu_{\bar{A}}(X_4) = 1 - 0.1$

$\mu_{\bar{A}}(X_4) = 0.9$

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Classical Set Theory	Fuzzy Set Theory
1. This theory is a class of those sets having sharp boundaries.	1. This theory is a class of those sets having un-sharp boundaries.
2. This set theory is defined by exact boundaries only 0 and 1.	2. This set theory is defined by ambiguous boundaries.
3. In this theory, there is no uncertainty about the boundary's location of a set.	3. In this theory, there always exists uncertainty about the boundary's location of a set.
4. This theory is widely used in the design of digital systems.	4. It is mainly used for fuzzy controllers.

Applications of Fuzzy Logic

Following are the different application areas where the Fuzzy Logic concept is widely used:

1. It is used in **Businesses** for decision-making support system.
2. It is used in **Automative systems** for controlling the traffic and speed.
3. Fuzzy logic systems also used in **Securities**.
4. It is also used in **microwave oven** for setting the lunes power and cooking strategy.
5. This technique is also used in the area of **modern control systems** such as expert systems.
6. **Finance** is also another application where this concept is used for predicting the stock market, and for managing the funds.
7. It is also used for controlling the brakes.
8. It is also used in the **industries of chemicals** for controlling the ph, and chemical distillation process.
9. It is also used in the **industries of manufacturing** for the optimization of milk and cheese production.

Advantages of Fuzzy Logic

Fuzzy Logic has various advantages or benefits. Some of them are as follows:

1. The methodology of this concept works similarly as the human reasoning.
2. Any user can easily understand the structure of Fuzzy Logic.
3. It does not need a large memory, because the algorithms can be easily described with fewer data.
4. It is widely used in all fields of life and easily provides effective solutions to the problems which have high complexity.
5. This concept is based on the set theory of mathematics, so that's why it is simple.
6. It allows users for controlling the control machines and consumer products.
7. The development time of fuzzy logic is short as compared to conventional methods.
8. Due to its flexibility, any user can easily add and delete rules in the FLS system.

Disadvantages of Fuzzy Logic

Fuzzy Logic has various disadvantages or limitations. Some of them are as follows:

1. The run time of fuzzy logic systems is slow and takes a long time to produce outputs.
2. Users can understand it easily if they are simple.

3. The possibilities produced by the fuzzy logic system are not always accurate.
4. Many researchers give various ways for solving a given statement using this technique which leads to ambiguity.
5. Fuzzy logics are not suitable for those problems that require high accuracy.
6. The systems of a Fuzzy logic need a lot of testing for verification and validation.

Membership Function

- **The membership function** is a function which represents the graph of fuzzy sets, and allows users to quantify the linguistic term. It is a graph which is used for mapping each element of x to the value between 0 and 1.
- This function is also known as indicator or characteristics function.
- This function of Membership was introduced in the first papers of fuzzy set by **Zadeh**. For the Fuzzy set B, the membership function for X is defined as:

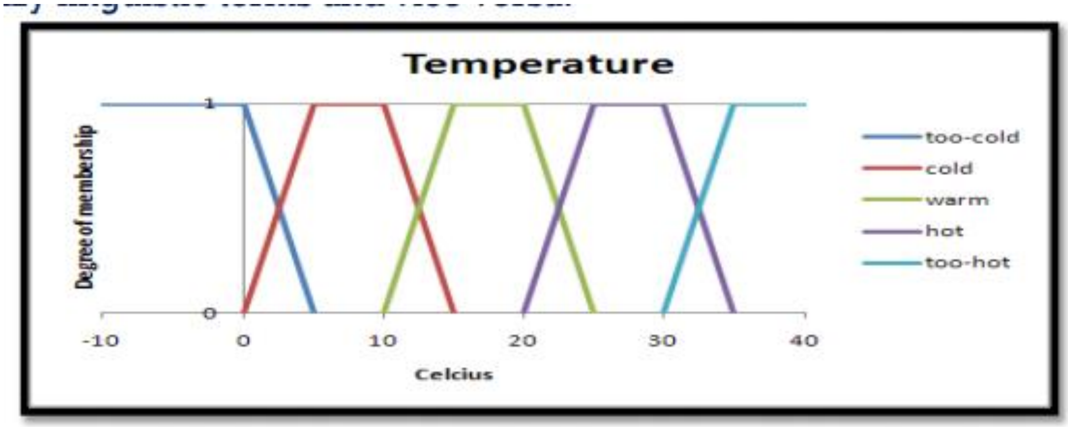
$$\mu B: X \rightarrow [0,1].$$

- In this function X, each element of set B is mapped to the value between 0 and 1. This is called a degree of membership or membership value.

Membership function (MF) - A function that specifies the degree to which a given input belongs to a set.

Degree of membership- The output of a membership function, this value is always limited to between 0 and 1. Also known as a membership value or membership grade.

Membership functions are used in the fuzzification and defuzzification• steps of a FLS (fuzzy logic system), to map the non-fuzzy input values to fuzzy linguistic terms and vice versa.

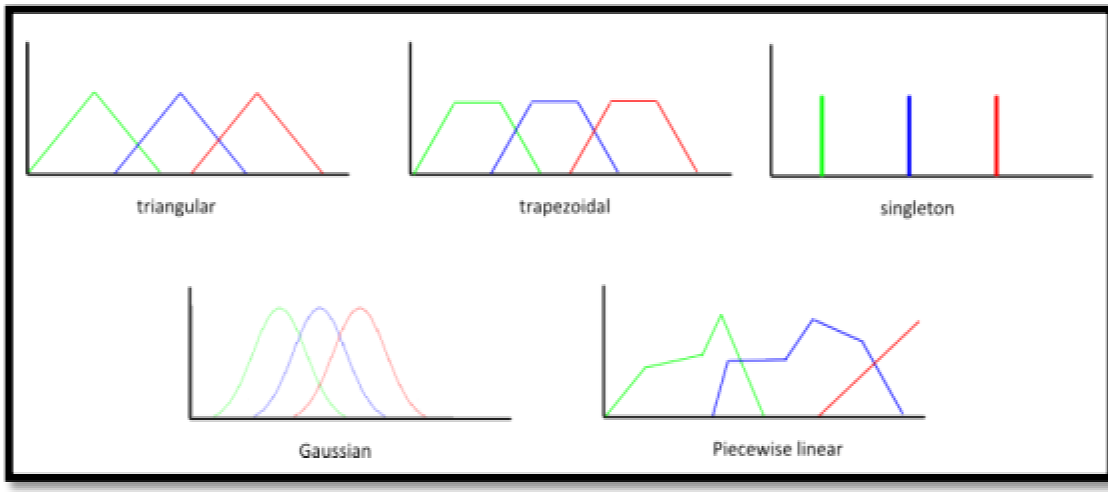


- Membership Functions for T (temperature) = **too-cold, cold, warm, hot, too-hot.**

Types of Membership Functions:

There are different forms of membership functions such as:

- o Triangular.
- o Trapezoidal.
- o Piecewise linear.
- o Gaussian.
- o Singleton.



Fuzzy singleton- A fuzzy set with a membership function that is unity at a one particular point and zero everywhere else.

Triangular MFs

A triangular MF is specified by three parameters {a, b, c} as follows:

$$\text{triangle}(x; a, b, c) = \begin{cases} 0, & x \leq a. \\ \frac{x-a}{b-a}, & a \leq x \leq b. \\ \frac{c-x}{c-b}, & b \leq x \leq c. \\ 0, & c \leq x. \end{cases}$$

By using min and max, we have an alternative expression for the preceding equation:

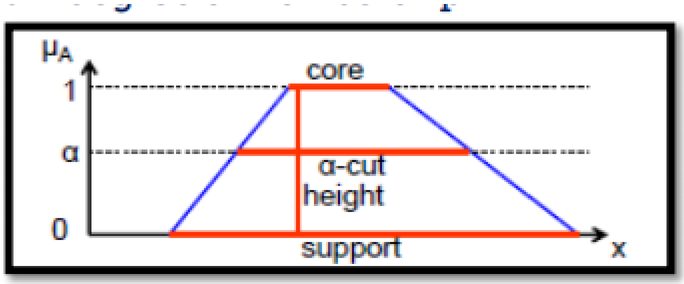
$$\text{triangle}(x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Fuzzy Membership Function: Basic Concepts

- **Support**: elements having non-zero degree of membership.
- **Core:** set with elements having degree of 1.
- ***α*-Cut**: set of elements with degree >= α.
- **Height**: maximum degree of membership.

Membership Functions in the Fuzzy Logic Toolbox

- A membership function (MF) is a curve that defines how each point in the input space is mapped to a membership value (or degree of membership) between 0 and 1. The input space is sometimes referred to as the universe of discourse.
- The only condition a membership function must really satisfy is that it must vary between 0 and 1.



Features of Membership Functions

We will now discuss the different features of Membership Functions.

Core

For any fuzzy set A˜, the core of a membership function is that region of universe that is characterize by full membership in the set. Hence, core consists of all those elements y of the universe of information such that,

$$\mu_{\tilde{A}}(y)=1$$

Support

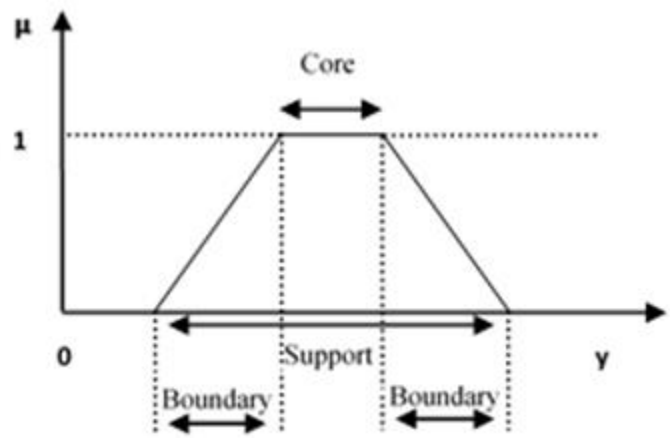
For any fuzzy set A˜, the support of a membership function is the region of universe that is characterize by a nonzero membership in the set. Hence core consists of all those elements y of the universe of information such that,

$$\mu_{\tilde{A}}(y)>0$$

Boundary

For any fuzzy set A˜, the boundary of a membership function is the region of universe that is characterized by a nonzero but incomplete membership in the set. Hence, core consists of all those elements y of the universe of information such that,

$$1>\mu A˜(y)>0$$



Features of Membership Function

Problem No.1 **The elements in two sets A and Bare given as A={2,4} and B= {a,b,c) Find the various Cartesian products of these two sets.** **Find AXB=? BXA=? AXA=? BXB=?**
Solution:.'I'he various Cartesian products of these two given sets are A X B = {(2, a). (2. b), (2, c), (4, a), (4, b), (4, c)} B x A = {(a, 2), (a, 4), (b, 2), (b, 4), (c, 2), (c, 4)) Ax A = A2 = { ((2,2), (2,4). (4,2), (4.4)} BXB = B2 = {(a,a),(a,b),(a,c),(b,b),(b,a),(b,c),(c,c),(c,a),(c,b)};

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Note

- A fuzzy set A is characterised by a generalised characteristic function A : X !
- [0; 1], called membership function of A and de¯ned over a universe of discourse X.

- This universe of discourse in a concrete case has to be chosen according to the speci¯c situation of this case.

- Obviously, for each usual, i.e. crisp set M its usual characteristic function

$$\mu_{\mathcal{M}} = \chi_{\mathcal{M}}$$

- such a membership function. Therefore we consider crisp sets as special cases of fuzzy sets, viz. those ones with only 0 and 1 as membership degrees. Fuzzy sets A;B are equal if they have the same membership functions:

$$A = B \quad \Leftrightarrow \quad \mu_A(x) = \mu_B(x) \quad \text{for all } x \in \mathcal{X}.$$

- We use IF(X) to denote the class of all fuzzy subsets of the universe of discourse X.
- If one intends to describe some fixed fuzzy set A over the universe of discourse X then one defines its membership function μ_A either by giving some formula to describe μ_A , or by a table of the values, or by a picture of the graph of μ_A .
- Using a table for this purpose is especially useful for finite universes of discourse X. E.g. for X0 = {a1; a2; : : : ; a6} the table
- C : a1 a2 a3 a4 a5 a6
  0:3 0:7 0:9 0:6 0 0:2
- describes a fuzzy set C with μC(a2) = 0:7 and μC(a6) = 0:2. In the case that one has a natural ordering for the elements of the universe X.

$$\boldsymbol{m}_A = (\mu_A(x_1), \ldots, \mu_A(x_n))$$

Sometimes for C one also uses a representation in form of a "sum":

$$C = 0.3/a_1 + 0.7/a_2 + 0.9/a_3 + 0.6/a_4 + 0/a_5 + 0.2/a_6.$$

Here the term $0/a_5$ may also be deleted. But, we will not use this representation of a fuzzy set as a sum in the present book. For finite universes of discourse $\mathcal{X}$ and $A \in \mathbb{F}(\mathcal{X})$ this representation is also written as

$$A = \sum_{x \in \mathcal{X}} \mu_A(x) \,/\, x \tag{2.5}$$

and for infinite universes $\mathcal{X}$ as

$$A = \int_{x \in \mathcal{X}} \mu_A(x) \,/\, x. \tag{2.6}$$

In all these cases the sign "/" does not mean a slanting fraction line. Also the signs $\sum$ and $\int$ are neither denoting a true sum nor a true integral but have only symbolic meaning.

$$\forall x \in \mathcal{X} : \ \mu_\emptyset(x) = 0, \tag{2.7}$$

and has the *universal set* X over $\mathcal{X}$ characterised by the membership function

$$\forall x \in \mathcal{X} : \ \mu_X(x) = 1. \tag{2.8}$$

Sometimes for any $\alpha \in [0,1]$ one also considers the α-*universal set* $X^{[\alpha]}$, characterised by

$$\forall x \in \mathcal{X} : \ \mu_{X^{[\alpha]}}(x) = \alpha. \tag{2.9}$$

Operations with Fuzzy Sets

For the basic set algebraic operations Zadeh (1965) has already given such extensions. For the *union* $A \cup B$ of fuzzy sets A, B he considered the definition

$$\begin{aligned} &C := A \cup B : \\ &\quad \mu_C(x) =_{\text{def}} \max\{\mu_A(x), \mu_B(x)\} \quad \text{for all } x \in \mathcal{X}, \end{aligned} \tag{2.70}$$

for the *intersection* $A \cap B$ of fuzzy sets A, B the definition

$$\begin{aligned} &D := A \cap B : \\ &\quad \mu_D(x) =_{\text{def}} \min\{\mu_A(x), \mu_B(x)\} \quad \text{for all } x \in \mathcal{X}, \end{aligned} \tag{2.71}$$

and for the *complement* A^c of a fuzzy set A (relative to the universe of discourse $\mathcal{X}$) the definition

$$K := A^c : \quad \mu_K(x) =_{\text{def}} 1 - \mu_A(x) \quad \text{for all } x \in \mathcal{X}. \tag{2.72}$$

$$A \cup B = B \cup A, \qquad \text{(commutativity)} \tag{2.75}$$
$$A \cup (B \cup C) = (A \cup B) \cup C, \qquad \text{(associativity)} \tag{2.76}$$
$$A \cup A = A, \qquad \text{(idempotency)} \tag{2.77}$$
$$A \subseteqq B \Rightarrow A \cup C \subseteqq B \cup C, \qquad \text{(monotonicity)} \tag{2.78}$$

as well as

$$A \cup \emptyset = A, \qquad A \cup X = X. \tag{2.79}$$

Rules (2.75) to (2.78) hold true in the same way also for the intersection of fuzzy sets. Instead of (2.79) of course one then has

$$A \cap \emptyset = \emptyset, \qquad A \cap X = A. \tag{2.80}$$

As in traditional set algebra also for fuzzy sets two kinds of distributivity hold true for the operations (2.70), (2.71):

$$\begin{aligned} A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\ A \cap (B \cup C) &= (A \cap B) \cup (A \cap C). \end{aligned} \tag{2.81}$$

Another basic operation for sets is the cartesian product. The corresponding notion for fuzzy sets is that of a *fuzzy cartesian product* which for fuzzy sets $A, B \in I\!F(\mathcal{X})$ is the fuzzy set $A \otimes B \in I\!F(\mathcal{X} \times \mathcal{X})$ with membership function defined for all $a, b \in \mathcal{X}$ by

$$\begin{aligned} &C := A \otimes B : \\ &\qquad \mu_C((a,b)) =_{\text{def}} \min\{\mu_A(a), \mu_B(b)\}. \end{aligned} \tag{2.91}$$

Here (a, b) is the – usual – ordered pair of a, b. By the way, for definition (2.91) it is completely unimportant to take A, B as fuzzy subsets of the same universe of discourse: for $A \in I\!F(\mathcal{X})$ and $B \in I\!F(\mathcal{Y})$ one can use the same definition for $A \otimes B$ as in (2.91) and simply gets $A \otimes B \in I\!F(\mathcal{X} \times \mathcal{Y})$.

Immediately a series of laws results. For any fuzzy sets A, B, C one e.g. has

$$A \otimes (B \cup C) = (A \otimes B) \cup (A \otimes C), \tag{2.92}$$
$$A \otimes (B \cap C) = (A \otimes B) \cap (A \otimes C), \tag{2.93}$$
$$A \otimes (B \otimes C) = (A \otimes B) \otimes C, \tag{2.94}$$
$$A \otimes \emptyset = \emptyset \otimes A = \emptyset \tag{2.95}$$