Homework -3

1) ~~f(x,y)~~

$f(x,y) = -\log(1-x-y) - \log x - \log y.$

a) Gradient $g = \begin{bmatrix} \frac{\partial f}{\partial w_1} \\ \frac{\partial f}{\partial w_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$

$$\frac{\partial f}{\partial x} = -\frac{1}{1-x-y}(-1) - \frac{1}{x} = \frac{1}{1-x-y} - \frac{1}{x}$$

$$\frac{\partial f}{\partial y} = \frac{-1}{1-x-y}(-1) - \frac{1}{y} = \frac{1}{1-x-y} - \frac{1}{y}$$

$$g = \begin{bmatrix} \frac{1}{1-x-y} - \frac{1}{x} \\ \frac{1}{1-x-y} - \frac{1}{y} \end{bmatrix} = \frac{1}{1-x-y}\begin{bmatrix} \frac{x-1+x+y}{x} \\ \frac{y-1+x+y}{y} \end{bmatrix}$$

$$= \frac{1}{1-x-y}\begin{bmatrix} \frac{2x+y-1}{x} \\ \frac{x+2y-1}{y} \end{bmatrix}$$

Hessain:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y} \end{bmatrix}$$

$$\frac{\partial^2 f}{\partial x^2} = \cancel{\partial f} \frac{\partial}{\partial x}\left[(1-x-y)^{-1} - x^{-1} \right]$$

$$= -1(1-x-y)^{-2}(-1) + x^{-2}$$

$$= \frac{1}{(1-x-y)^2} + \frac{1}{x^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[(1-x-y)^{-1} - y^{-1} \right]$$

$$= \frac{1}{(1-x-y)^2} + \frac{1}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left[(1-x-y)^{-1} - x^{-1} \right] = \frac{1}{(1-x-y)^2}$$

$$H = \begin{bmatrix} \frac{1}{(1-x-y)^2} + \frac{1}{x^2} & \frac{1}{(1-x-y)^2} \\ \frac{1}{(1-x-y)^2} & \frac{1}{(1-x-y)^2} + \frac{1}{y^2} \end{bmatrix}$$

$$H = \frac{1}{(1-x-y)^2} \begin{bmatrix} \frac{x^2 + (1-x-y)^2}{x^2} & 1 \\ 1 & \frac{y^2 + (1-x-y)^2}{y^2} \end{bmatrix}$$

2) ~~e~~ $E(w) = \sum_{i=1}^{50} (y_i - (w_0 + w_1 x_i))^2$

$$g = \begin{bmatrix} \frac{\partial E}{\partial w_0} \\ \frac{\partial E}{\partial w_1} \end{bmatrix}$$

$$\frac{\partial E}{\partial w_0} = -2 \sum_{i=1}^{50} (y_i - (w_0 + w_1 x_i))$$

$$\frac{\partial E}{\partial w_1} = -2 \sum_{i=1}^{50} (y_i - (w_0 + w_1 x_i)) x_i$$

$$g_b = -2 \sum_{i=1}^{50} (y_i - (w_0 + w_1 x_i)) \begin{bmatrix} 1 \\ x_i \end{bmatrix}$$ where $i=1$ to 50

↳ batch learning

To update weights for every sample, we have to use

$$g_o = -2 (y_i - (w_0 + w_1 x_i)) \begin{bmatrix} 1 \\ x_i \end{bmatrix}$$

↳ online learning

In [391]:
```
import math
import sympy as sym
import random
import numpy as np
import matplotlib.pyplot as plt
import statistics
```

In [392]:
```
# a = [[1,2,3], [4,5,6]]
# a = np.asarray(a)
# 2*a
```

In [393]:
```
x,y,p = sym.symbols('x,y,p')
w=np.asarray([x,y])
```

In [394]:
```
# Domain

def pick():
    x = random.random()
    y = random.random()
    while x+y>=1 or x<=0 or y<=0:
        x = random.random()
        y = random.random()
    return x,y
# x,y  = pick()
# w = np.asarray([x,y])
```

In [395]:
```
# w[0]
```

In [396]:
```
log = math.log
f  = lambda p: -log(1-p[0]-p[1]) - log(p[0]) - log(p[1])

f2  = lambda p: -log(1-p[:,0]-p[:,1]) - log(p[:,0]) - log(p[:,1])
```

In [397]:
```
def der_x(w):
    return (1/(1-w[0]-w[1]) - (1/w[0]))

def der_y(w):
    return (1/(1-w[0]-w[1]) - (1/w[1]))

def der_x2(w):
    return ((1/(1-w[0]-w[1])**2) + (1/(w[0]**2)))

def der_y2(w):
    return ((1/(1-w[0]-w[1])**2) + (1/w[1]**2))

def der_xy(w):
    return (1/(1-w[0]-w[1])**2)

der_y2(w)
```

Out[397]:

$$\frac{1}{\left(-x-y+1\right)^2}+\frac{1}{y^2}$$

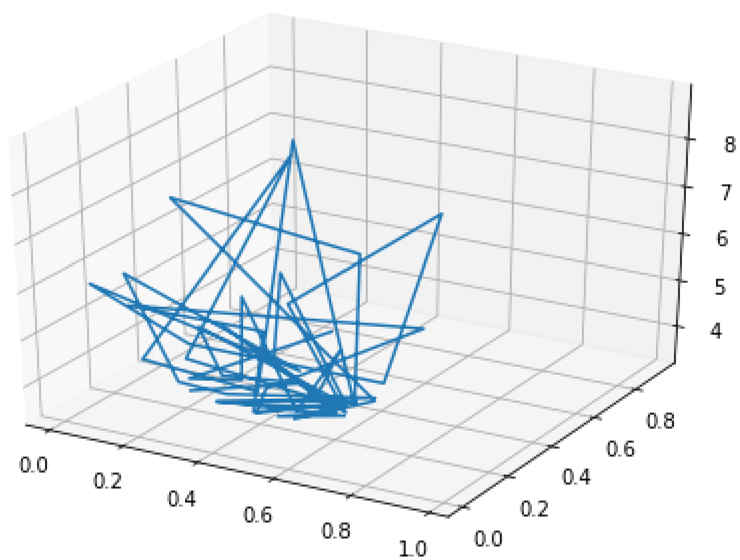
In [398]:
```
# Gradient
def g(w):
    return np.asarray([der_x(w), der_y(w)])
g(w)
```

Out[398]:
```
array([1/(-x - y + 1) - 1/x, 1/(-x - y + 1) - 1/y], dtype=object)
```

In [399]:
```
# Hessian
def h(w):
    return (1/2.302)*np.asarray([[der_x2(w), der_xy(w)], [der_xy(w), der_y2(w)]])
h(w)
```

Out[399]:
```
array([[0.434404865334492/(-x - y + 1)**2 + 0.434404865334492/x**2,
        0.434404865334492/(-x - y + 1)**2],
       [0.434404865334492/(-x - y + 1)**2,
        0.434404865334492/(-x - y + 1)**2 + 0.434404865334492/y**2]],
      dtype=object)
```

In [400]:
```
from mpl_toolkits.mplot3d import Axes3D

b =[]
d =[]
nu = []

for i in range(1,51):
    x,y = pick()
    b.append(x)
    d.append(y)
    w = np.asarray([x,y])
    nu.append(f(w))

fig = plt.figure()
ax = Axes3D(fig)
ax.plot(b, d, nu)
plt.show()
```



Q) 1) b)

In [447]:

```
def gd(w, f, lr, prec, g, prec_E):
    x = w[0]
    y = w[1]
    w_i = w
    print("x = " + str(x))
    print("y = " + str(y))
    print("For a randomly picked initial values of w= " + str(w) + ", the value of f(w) = " + str(f(w)))
    print('\n')
    step_size = np.asarray([1,1])
    i = 1
    fv = []
    p = []
    p1 =[]
    p2 =[]
    it =[]
    p1.append(x)
    p2.append(y)
    fv.append(f(w))

    it.append(i)
    s = 1
    E = 0
    while (step_size[0] > prec and step_size[1] > prec) and i < 100:
        prev_c = E
        prev = w
        w = w - lr*g(w)
        i += 1
        step_size = abs(prev- w)
        if w[0] <= 0 or w[1] <= 0 or w[0]+w[1] >=1:
            x,y = pick()
            w = np.asarray([x,y])
            i = 1
            fv = []
            p = []
            p1 =[]
            p2 =[]
            it =[]
            step_size = np.asarray([1,1])
            lr = lr - 0.02
            if lr <= 0:
                lr = lr+ 0.04
```

```
        E = f(w)
        s = abs(prev_c -E)
        it.append(i)
        p.append(w)
        p1.append(w[0])
        p2.append(w[1])
        fv.append(f(w))

        #print(w)

    print("Optimal w = " + str(w))
    print("For the optimal value of w= " + str(w) + ", the value of f(w) = " + str(f(w)))
    return i,p1, p2, fv, it, w, w_i
```

In [448]:

```
i,p1, p2, fv, it, w_opt, w_i = gd(w, f, 1, 0.00001, g, 0.01)
```

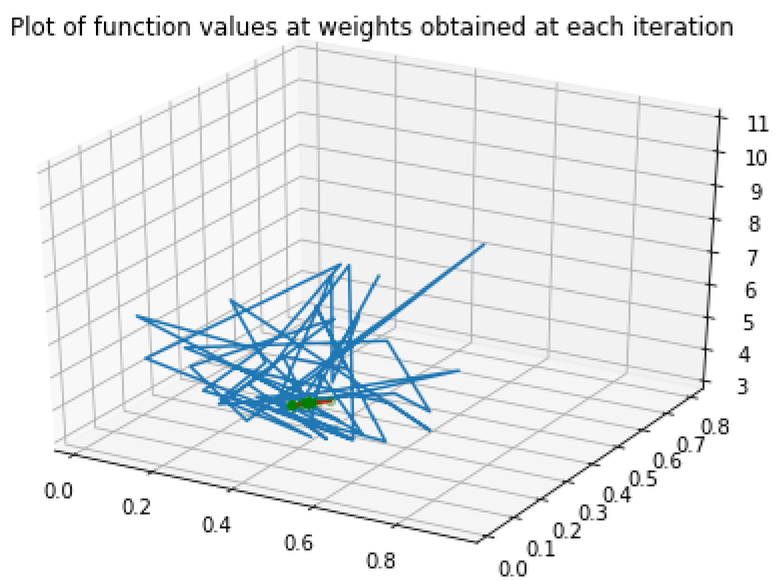
```
x = 0.3859154209875649
y = 0.026802878745336267
For a randomly picked initial values of w= [0.38591542 0.02680288], the value of f(w) = 5.103633707530626


Optimal w = [0.33333597 0.33333596]
For the optimal value of w= [0.33333597 0.33333596], the value of f(w) = 3.295836866191452
```

In [449]:

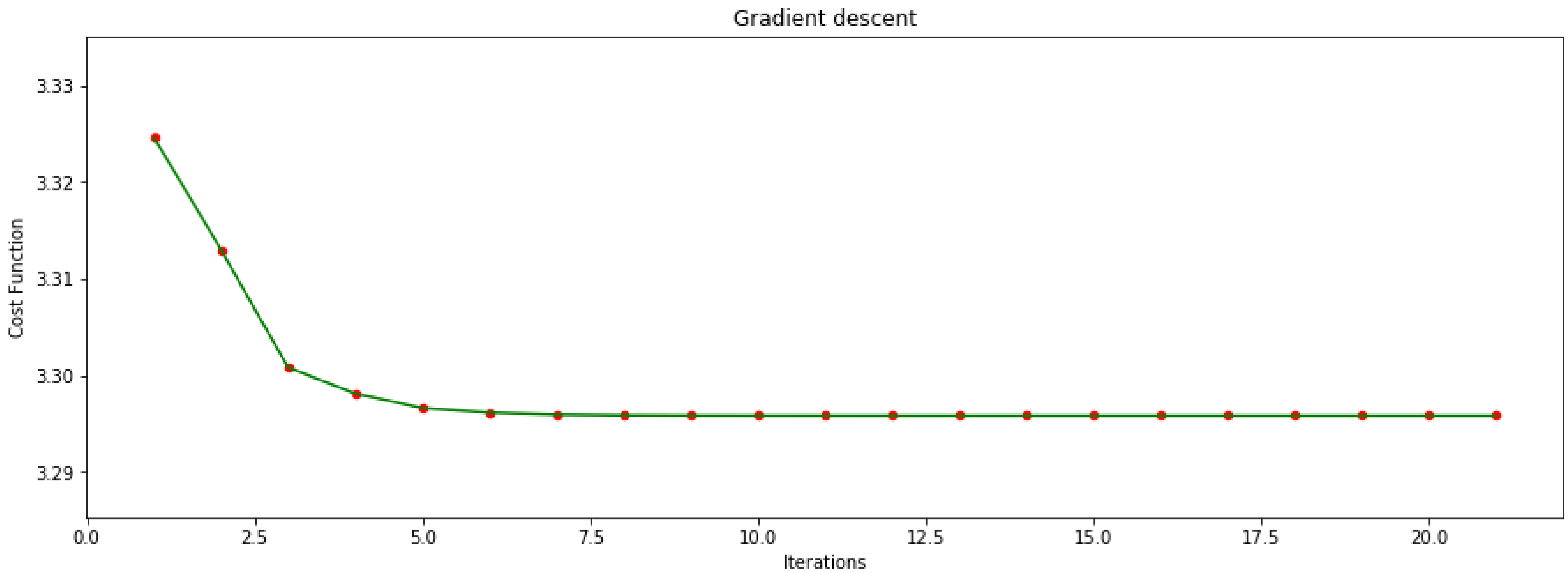
```
print(len(it))
```

```
21
```

In [450]:
```
plt.rcParams['agg.path.chunksize'] = 10000
b =[]
d =[]
nu = []

for i in range(1,51):
    x,y = pick()
    b.append(x)
    d.append(y)
    w = np.asarray([x,y])
    nu.append(f(w))

fig = plt.figure()
ax = Axes3D(fig)
ax.plot(b, d, nu)

ax.plot(p1, p2, fv, c='r')
ax.scatter(p1, p2, fv, c='g')
plt.title('Plot of function values at weights obtained at each iteration')
plt.show()
```



In [451]:
```
fig, ax = plt.subplots(figsize = (15,5))
plt.scatter(it,fv, s=20, c = 'r')
plt.plot(it,fv, c = 'g')
plt.title("Gradient descent")
plt.xlabel('Iterations')
plt.ylabel("Cost Function")
plt.show()
```



Q)1)c)

In [452]:
```
from numpy.linalg import inv
```

In [453]:
```
# Newtons Method
def nt(w, f, lr, prec, g, prec_E):
    x = w[0]
    y = w[1]
    print("x = " + str(x))
    print("y = " + str(y))
    print("For a randomly picked initial values of w= " + str(w) + ", the value of f(w) = " + str(f(w)))
    print('\n')
    step_size = np.asarray([1,1])
    fv = []
    p = []
    p1 =[]
    p2 =[]
    p1.append(x)
    p2.append(y)
    fv.append(f(w))
    lr= 1
    i = 1
    it =[]
    it.append(i)
    s = 1
    E = 0
    while (step_size[0] > prec and step_size[1] > prec) and i < 100:
        prev = w
        prev_c= E
        h_inv = inv(h(w))
        up = np.asarray(lr*(np.dot(h_inv, g(w))))
        w = w - up
        i += 1
        E = f(w)
        step_size = abs(prev- w)
        s = abs(prev_c - E)
        if w[0] <= 0 or w[1] <= 0 or w[0]+w[1] >=1:
            x,y = pick()
            w = np.asarray([x,y])
            i = 1
            fv = []
            p = []
            p1 =[]
            p2 =[]
            it =[]
            step_size = np.asarray([1,1])
```

```
                s = 1
                lr = lr - 0.02
                if lr <= 0:
                    lr = lr+ 0.04
            it.append(i)
            p.append(w)
            p1.append(w[0])
            p2.append(w[1])
            fv.append(f(w))

    print("Optimal w = " + str(w))
    print("For the optimal value of w= " + str(w) + ", the value of f(w) = " + str(f(w)))
    return i,p1, p2, fv, it, w
```

In [454]:

```
i,p1, p2, fv, it, wopt_nt = nt(w_i,f,1, 0.00001, g, 0.01)
```

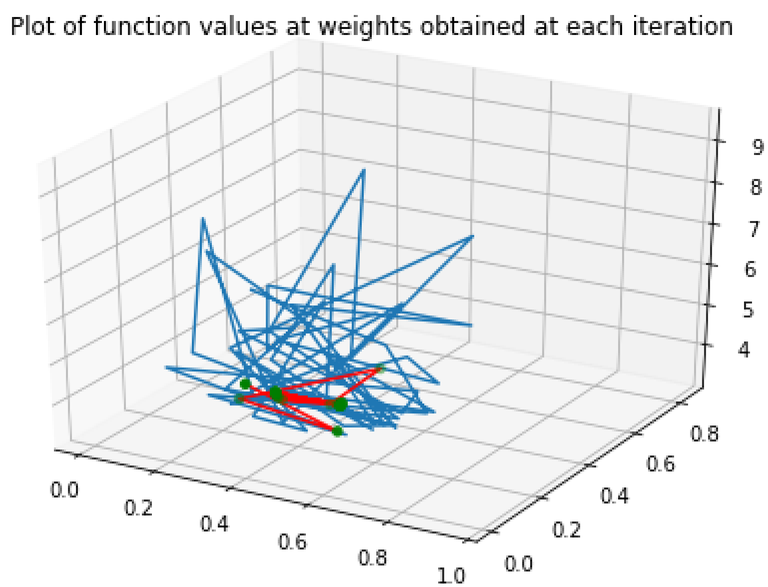
```
x = 0.38591542098756491
y = 0.026802878745336267
For a randomly picked initial values of w= [0.38591542 0.02680288], the value of f(w) = 5.103633707530626


Optimal w = [0.23941766 0.34857208]
For the optimal value of w= [0.23941766 0.34857208], the value of f(w) = 3.3701629951766705
```

In [455]:

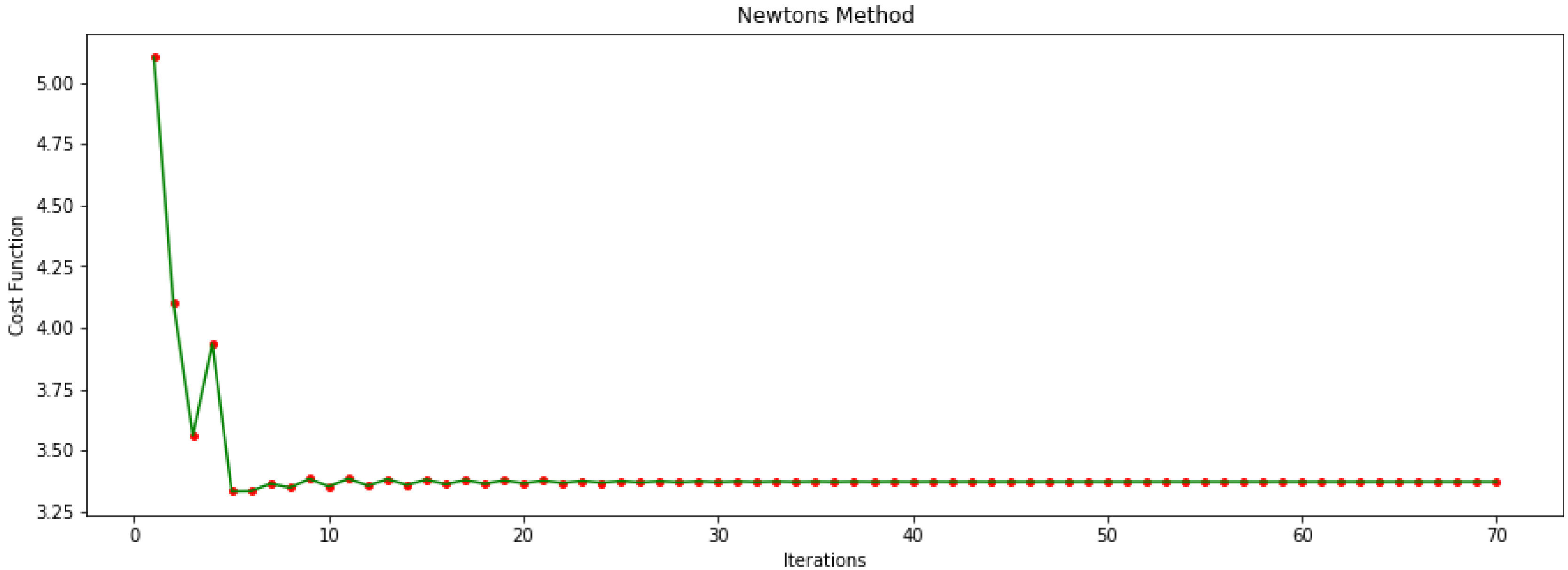
```
print(len(it))
```

```
70
```

In [456]:
```
plt.rcParams['agg.path.chunksize'] = 10000
b =[]
d =[]
nu = []

for i in range(1,51):
    x,y = pick()
    b.append(x)
    d.append(y)
    w = np.asarray([x,y])
    nu.append(f(w))

fig = plt.figure()
ax = Axes3D(fig)
ax.plot(b, d, nu)
ax.plot(p1, p2, fv, c='r')
ax.scatter(p1, p2, fv, c='g')
plt.title('Plot of function values at weights obtained at each iteration')
plt.show()
```



In [457]:
```
fig, ax = plt.subplots(figsize = (15,5))
plt.scatter(it,fv, s=15, c = 'r')
plt.plot(it,fv, c = 'g')
plt.title("Newtons Method")
plt.xlabel('Iterations')
plt.ylabel("Cost Function")
plt.show()
```



Q)1) d)

Based on my observations, the gradient descent took less number of iterations compared to Newtons method. And the minimum value of cost function obtained with gradient descent is smaller than that of Newtons method

Q) 2) b)

In [475]:
```
x = []
y = []
for i in range(1,51):
    x.append([1,i])
    y.append(i+np.random.uniform(-1,1))

#print(x)
x = np.transpose(x)
y = np.transpose(y)
x = np.asarray(x)
y = np.asarray(y).reshape(1,50)
#y.shape
```

Q)2)c)

In [476]:
```
# Linear Least Squares fit

def llsf(x,y):
    xt = np.transpose(x)
    t = inv(np.matmul(x, xt))
    xpseudo = np.matmul(xt,t)
    w = np.matmul(y,xpseudo)
    return w
```

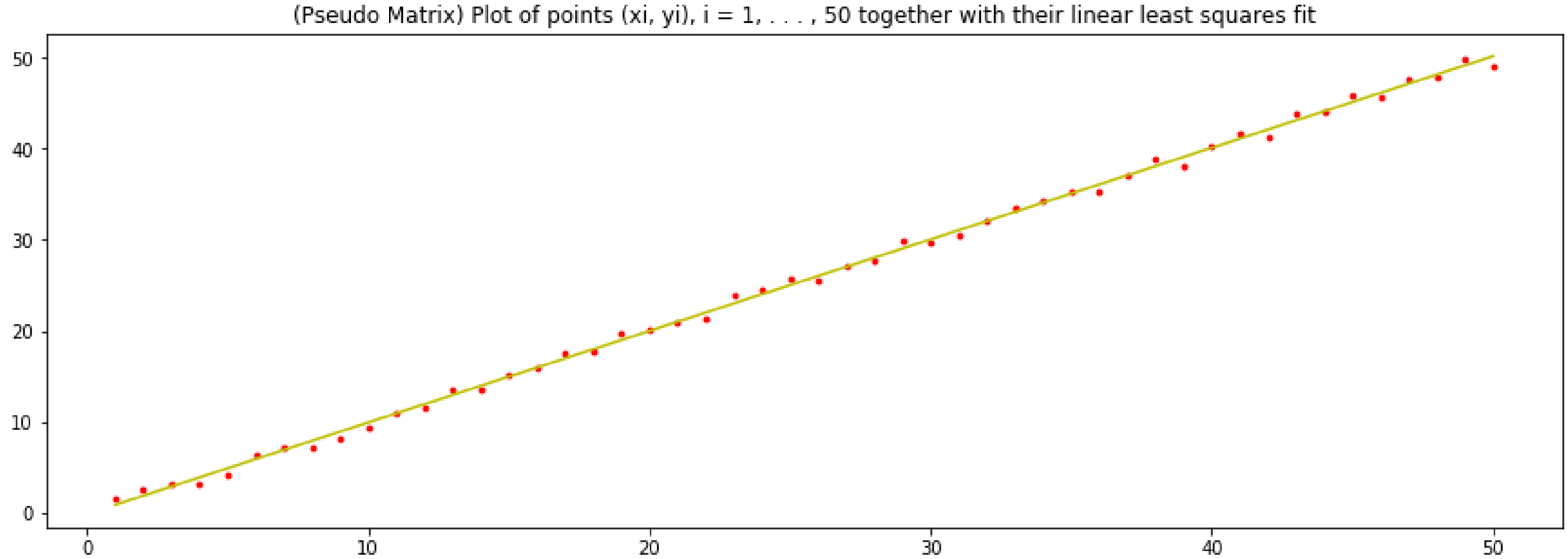
In [477]:
```
#x[:,2]
```

In [478]:
```
w_opt1 = llsf(x,y)
print("The optimal value of w using Linear Least squares fit: " + str(w_opt1))
```

```
The optimal value of w using Linear Least squares fit: [[-0.19031505  1.00825282]]
```

Q)2)d)

In [479]:
```
fig, ax = plt.subplots(figsize = (15,5))
plt.scatter(x[1,:],y, s = 8, c = 'r')
d1 = np.matmul(w_opt1, x)[0,:]
plt.plot(x[1,:],d1, c ='y')
plt.title('(Pseudo Matrix) Plot of points (xi, yi), i = 1, . . . , 50 together with their linear least square
s fit')
plt.show()
```



Q) 2) f)

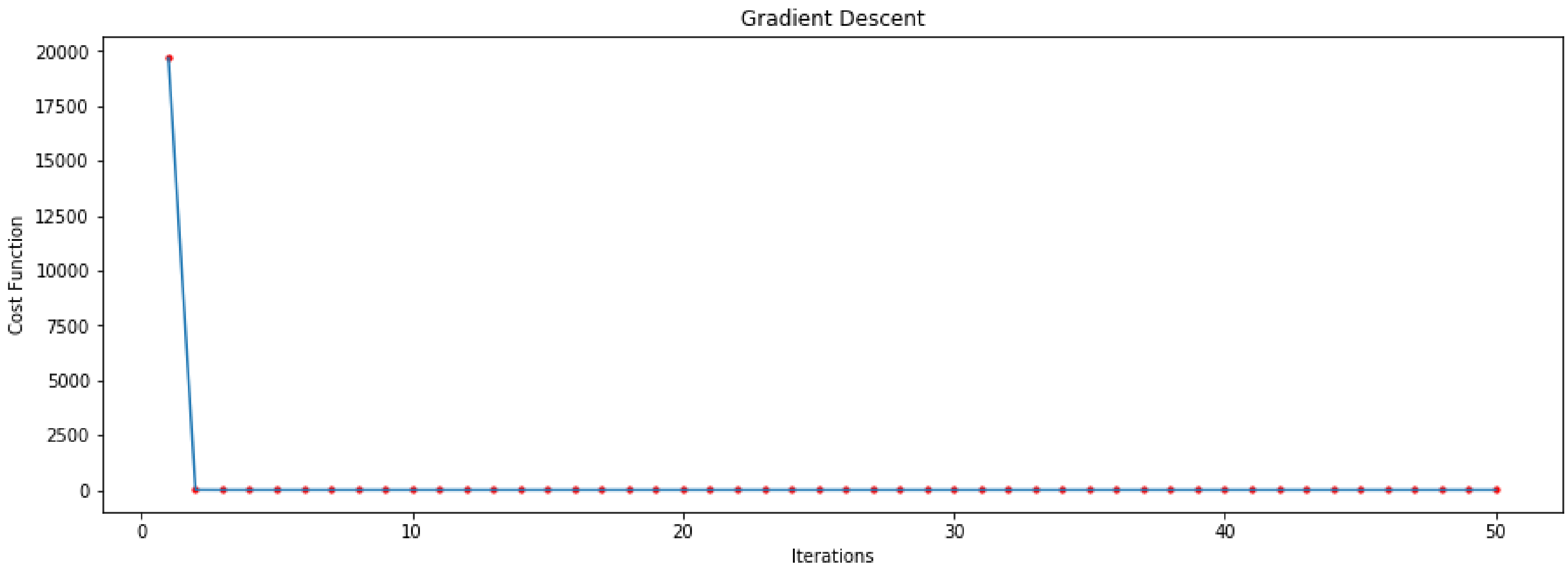
In [480]:
```
w0 = np.random.uniform(low = -1, high = 1)
w1 = np.random.uniform(low = -1, high = 1)
w = [w0, w1]
```

In [481]:
```
def lms(w,x,y, prec, lr):
    epoch = []
    error = []
    it = 1
    step_size = np.asarray([1,1])
    while (step_size[0] > prec and step_size[1] > prec) and it < 51:

        prev = w
        E = 0
        for i in range(0,50):
            d = np.dot(w, x[:,i])
            w = w + np.dot((lr*(y[0][i] - d)), x[:,i])
            E = E + np.power((y[0][i] - d),2)
        step_size = abs(np.asarray(prev)- np.asarray(w))
        error.append(E)
        epoch.append(it)
        it += 1
    print('Final error: ' + str(E))
    print('final weight: ' + str(w))
    return w, error, epoch
```

In [482]:
```
w_opt2, error, epoch = lms(w,x,y, 0.00000001, 0.0001)
```

```
Final error: 18.53220900308202
final weight: [-0.40554382  1.0102493 ]
```

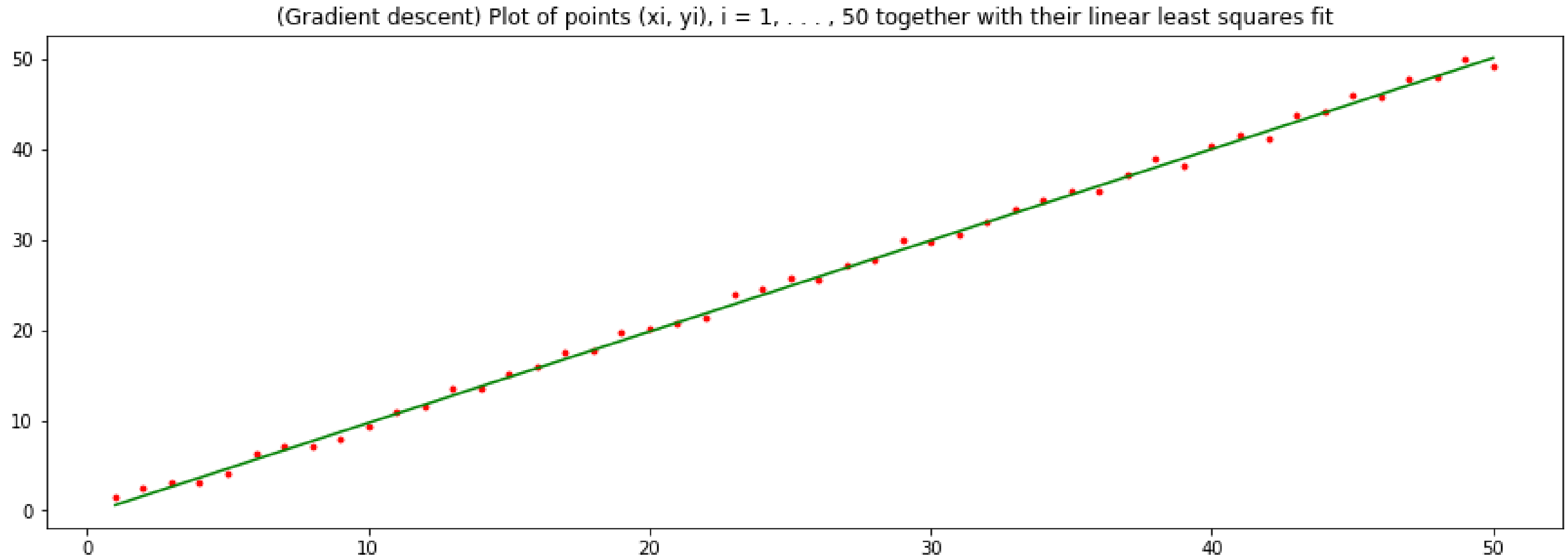
In [483]:
```
fig, ax = plt.subplots(figsize = (15,5))
plt.scatter(epoch,error, s=10, c = 'r')
plt.plot(epoch,error)
plt.title("Gradient Descent")
plt.xlabel('Iterations')
plt.ylabel("Cost Function")
plt.show()
```



In [484]:
```
d2 =[]
for i in range(0,50):
    d2.append(np.matmul(w_opt2,x[:,i]))
```

In [485]:
```
fig, ax = plt.subplots(figsize = (15,5))
plt.scatter(x[1,:],y, s = 8, c = 'r')
plt.plot(x[1,:],d2, c = 'g')
plt.title('(Gradient descent) Plot of points (xi, yi), i = 1, . . . , 50 together with their linear least squ
ares fit')
```

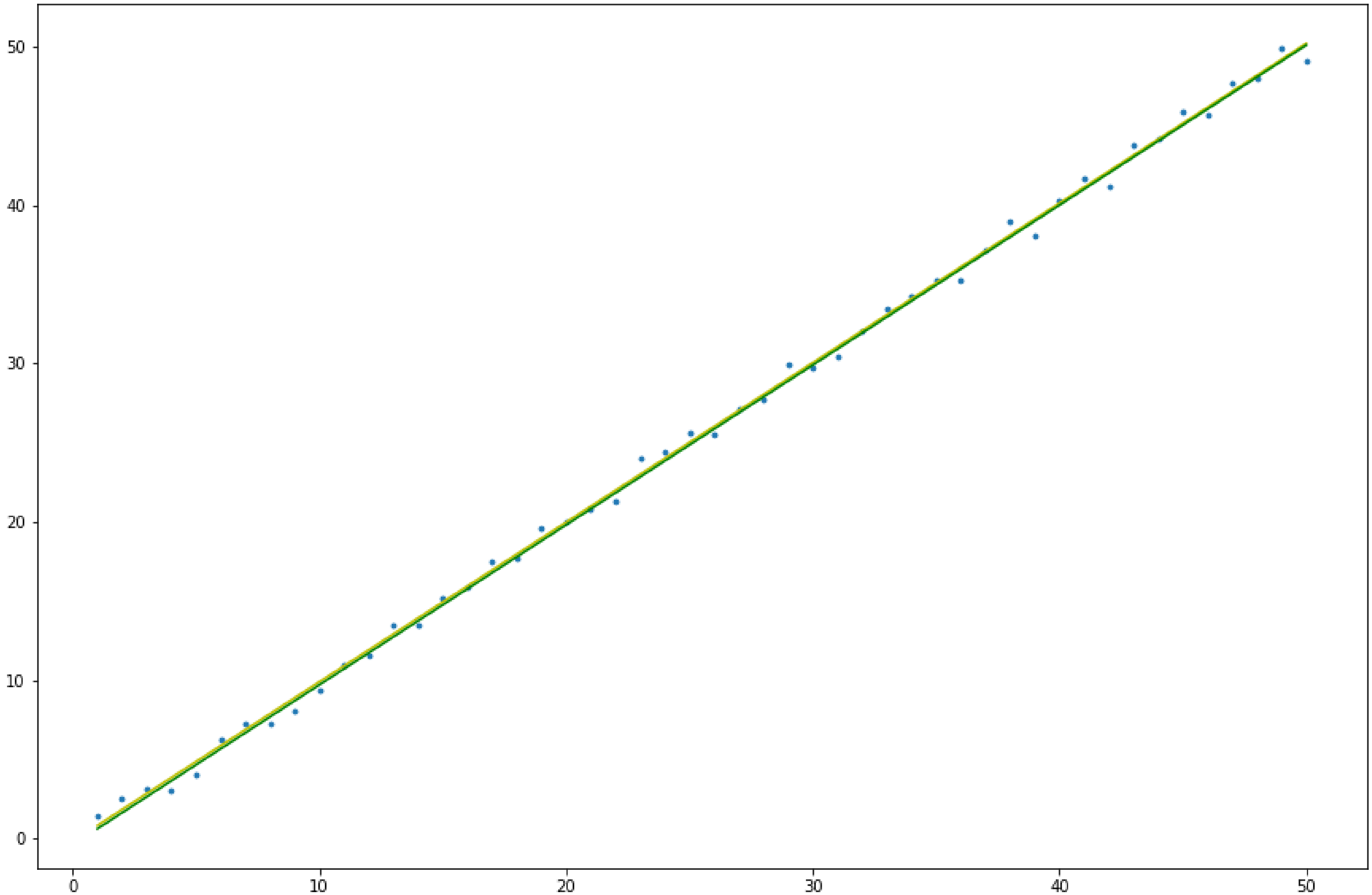
Out[485]:
```
Text(0.5, 1.0, '(Gradient descent) Plot of points (xi, yi), i = 1, . . . , 50 together with their linear leas
t squares fit')
```



Comparision between fits obtained using pseudo matix and gradient descent

In [486]:
```
fig, ax = plt.subplots(figsize = (15,10))
plt.scatter(x[1,:],y, s = 8)
plt.yscale('linear')
# Pseudo Matrix plot
# Yellow line
plt.plot(x[1,:],d1, c ='y')

# Gradient plot
# Green Line
plt.plot(x[1,:],d2, c = 'g')
```

Out[486]: [<matplotlib.lines.Line2D at 0x1df7d6d9208>]



The plots of fits obtained from pseudo inverse and gradient descent are almost similar (Yellow plot is of Pseudo Matrix and Green is of Gradient Descent.

In [ ]:

```
```