Quiz-1

Date: 23rd August, 2023 Time: 5::30pm -7:00pm Maximum marks: 15

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1. Consider the following problem (P):

$$\min\ 3x_1 - 2x_2$$

$$\begin{aligned} \text{subject to} \quad & -x_1 + 2x_2 \le 3 \\ & x_1 - 2x_2 \le 2 \\ & x_1 + x_2 \ge 1 \\ & x_1 \ge 0, x_2 \ge 0. \end{aligned}$$

   (a) Give a picture of Fea(P). Give all the extreme points and the **distinct** extreme directions of Fea(P). (No justification required).
   Soln: $[1,0]^T, [2,0]^T, [0,1]^T, [0,\frac{3}{2}]^T$ are the extreme points.
   $[2,1]^T$ or $\alpha[2,1]^T$, for any $\alpha > 0$ is the only **distinct** extreme direction.

   (b) **Check** whether (P) has an optimal solution. If yes, then give an optimal solution.
   Soln: Since $\mathbf{c}^T\mathbf{d} = [3,-2][2,1]^T = 4 > 0$, so (P) has an optimal solution and $[0,\frac{3}{2}]^T$ is the unique optimal solution.

   (c) If the objective function of (P) is written as $\min \mathbf{c}^T\mathbf{x}$, then **if possible** give a $\mathbf{c}'$ such that the LPP with the above feasible region and objective function, $\min \mathbf{c'}^T\mathbf{x}$, has infinitely many optimal solutions, but only one optimal extreme point.
   Soln: Multiple correct answers, for example you can take $\mathbf{c'}^T = [-1,2]^T$.

   (d) If Fea(P) is written as $A_{3\times 2}\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$, then by changing **exactly** one entry in $\mathbf{b}$, **if possible** give a $\mathbf{b}'$ such that for no $\mathbf{c} \in \mathbb{R}^2$,
   $\min \mathbf{c}^T\mathbf{x}$, subject to $A_{3\times 2}\mathbf{x} \le \mathbf{b}'$, $\mathbf{x} \ge \mathbf{0}$, has optimal solution ($A$ is unchanged).
   Soln: Multiple correct answers, for example you can take $\mathbf{b}' = [3,-4]^T$, then the new feasible region is the empty set. [3+2+1+1]

2. Let $\mathbf{x}_0$ be an optimal solution of the following problem (P):
$\min \mathbf{c}^T\mathbf{x}$, subject to $A_{3\times 4}\mathbf{x} \le \mathbf{b}$, $\mathbf{x} \ge \mathbf{0}$ (or $\tilde{A}\mathbf{x} \le \tilde{b}$ where $\tilde{A} = \begin{bmatrix} A \\ -I \end{bmatrix}$ and $\tilde{b} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$.
Check the correctness of the following statements with **brief** but **proper** justification.

   (a) If $\mathbf{c} = [1,-1,2,-3]^T$ then the second column of A has a positive entry.
   Soln: If every entry of the second column of A is non positive (or ≤ 0) then $\mathbf{d} = [0,1,0,0]^T$ will be a direction of $Fea(P)$ and $\mathbf{c}^T\mathbf{d} = -1 < 0$, which implies that (P) does not have an optimal solution, which contradicts that $\mathbf{x}_0$ is an optimal solution.

   (b) If $\mathbf{x}_0 = [1,2,1,3]^T$ then $\mathbf{x}_0$ is the **unique** optimal solution of (P).
   Soln: Since $\mathbf{x}_0 = [1,2,1,3]^T$ can lie on atmost three LI defining hyperplanes of $Fea(P)$ so $\mathbf{x}_0$ is not an extreme point. Since atleast one extreme point must be an optimal solution so any convex combination of that optimal extreme point and $\mathbf{x}_0$ is again optimal for (P), so (P) has infinitely many optimal solutions.

(c) If $\mathbf{x}^{'}$ lies on **exactly** k, Linearly Independent defining hyperplanes of Fea(P) and $\mathbf{d}$ is such that $\mathbf{x}^{'} + 2\mathbf{d} \in Fea(P)$, then $\mathbf{x}^{'} + \mathbf{d}$ cannot lie on $k+1$, Linearly Independent defining hyperplanes of Fea(P).
Soln: If $\mathbf{x}^{`} \in Fea(P)$ then the statement is True. Let $\mathbf{x}^{'} + \mathbf{d}$ lie on $k+1$, LI defining hyperplanes and a hyperplane H_0 with normal $\mathbf{a}_0$ on which $\mathbf{x}^{'}$ does not lie.
Then $\mathbf{a}_0^T\mathbf{x}^{'} < \tilde{b_0}$, and $\mathbf{a}_0^T(\mathbf{x}^{'} + \mathbf{d}) = \tilde{b_0}$ which implies $\mathbf{a}_0^T\mathbf{d} > 0$ and $\mathbf{a}_0^T(\mathbf{x}^{'} + 2\mathbf{d}) > \tilde{b_0}$ which is a contradiction.
However if $\mathbf{x}^{`}$ is not in $Fea(P)$ then the statement is False. There are many examples to justify the claim.
So whatever be your assumption, $\mathbf{x}^{`} \in Fea(P)$ or $\mathbf{x}^{`}$ not in $Fea(P)$, if you have argued correctly or have given the correct example to prove your point you will get **full** credit.

(d) If $\mathbf{d}(\neq \mathbf{0})$ is such that for all $\mathbf{x} \in Fea(P)$ there exists $\alpha_x > 0$ (depending on $\mathbf{x}$) such that $\mathbf{x} + \alpha_x\mathbf{d} \in Fea(P)$, then Fea(P) is unbounded.
Soln: If Fea(P) is bounded then exists an $\alpha > 0$ such that $\mathbf{x}_0 + \alpha\mathbf{d}$ does not belong to $Fea(P)$. Let $\gamma = \max\{\alpha > 0 : \mathbf{x}_0 + \alpha\mathbf{d} \in Fea(P)\}$, then due to the given condition, $\gamma > 0$ and $\mathbf{x}_0 + \gamma\mathbf{d} \in Fea(P)$.
For $\mathbf{x} = \mathbf{x}_0 + \gamma\mathbf{d}$ there exists no $\alpha_x > 0$ such that $\mathbf{x} + \alpha_x\mathbf{d} \in Fea(P)$, which is a contradiction.

(e) **(Bonus question)** If $\mathbf{d}_0(\neq \mathbf{0})$ is such that $\mathbf{x}_0 + \alpha\mathbf{d}_0$ is optimal for all $\alpha \geq 0$, then there exists $\tilde{\mathbf{a}}_{i_1}^T, \tilde{\mathbf{a}}_{i_2}^T, \tilde{\mathbf{a}}_{i_3}^T$ (rows of $\tilde{A}$), and $\beta_1, \beta_2, \beta_3$, real numbers such that $\mathbf{c} = \beta_1\tilde{\mathbf{a}}_{i_1} + \beta_2\tilde{\mathbf{a}}_{i_2} + \beta_3\tilde{\mathbf{a}}_{i_3}$.
Soln: Since $\mathbf{c}^T(\mathbf{x}_0 + \alpha\mathbf{d}_0)$ is equal to the optimal value for all $\alpha \geq 0$, so $\mathbf{c}^T\mathbf{d}_0 = 0$.
Since $\mathbf{x}_0 + \alpha\mathbf{d}_0 \in Fea(P)$ for all $\alpha \geq 0$ so $\mathbf{d}_0$ is a direction of $Fea(P)$ and $Fea(P)$ is unbounded.
So $\mathbf{d}_0$ can be written as a non negative linear combination of the extreme directions $\mathbf{d}_j$, $j = 1, \ldots, k$ of $Fea(P)$.
Let $\mathbf{d}_0 = \sum_j \beta_j\mathbf{d}_j$, where $\beta_j \geq 0$, for all $j = 1, \ldots, k$ and $\sum_j \beta_j > 0$ **(**)**.
Since (P) has an optimal solution, $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all $j = 1, \ldots, k$. Since $\mathbf{c}^T\mathbf{d}_0 = 0$, $\mathbf{c}^T\mathbf{d}_j = 0$ if $\beta_j > 0$ in **(**)**.
WLOG let $\mathbf{c}^T\mathbf{d}_1 = 0$. Since $\mathbf{d}_1$ is an extreme direction, it is orthogonal to $4-1 = 3$ LI rows of $\tilde{A}$.
Let those rows be $\tilde{\mathbf{a}}_{i_1}^T, \tilde{\mathbf{a}}_{i_2}^T, \tilde{\mathbf{a}}_{i_3}^T$.
If $\{\mathbf{c}, \tilde{\mathbf{a}}_{i_1}, \tilde{\mathbf{a}}_{i_2}, \tilde{\mathbf{a}}_{i_3}\}$ is LI then $\mathbf{d}_1 \in \mathbb{R}^4$ must be the zero vector (done in class) which is a contradiction, hence $\{\mathbf{c}, \tilde{\mathbf{a}}_{i_1}, \tilde{\mathbf{a}}_{i_2}, \tilde{\mathbf{a}}_{i_3}\}$ is LD.
Since $\{\tilde{\mathbf{a}}_{i_1}, \tilde{\mathbf{a}}_{i_2}, \tilde{\mathbf{a}}_{i_3}\}$ is LI, so there exists $\beta_1, \beta_2, \beta_3$, real numbers such that
$\mathbf{c} = \beta_1\tilde{\mathbf{a}}_{i_1} + \beta_2\tilde{\mathbf{a}}_{i_2} + \beta_3\tilde{\mathbf{a}}_{i_3}$.

(**All parts in the above questions are independent**)

[2+2+2+2+5]