Physics II: Electromagnetism

PH 102

Lecture-5

March-June 2022

Review

Line element $d\vec{r} = h_1\hat{e}_1 du_1 + h_2\hat{e}_2 du_2 + h_3\hat{e}_3 du_3$

Unit vectors:

$$\boxed{\begin{aligned}
\hat{e}_1 &\equiv \hat{r} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z} \\
\hat{e}_2 &\equiv \hat{\theta} = \frac{\frac{\partial\vec{r}}{\partial \theta}}{\left|\frac{\partial\vec{r}}{\partial \theta}\right|} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z} \\
\hat{e}_3 &\equiv \hat{\phi} = \frac{\frac{\partial\vec{r}}{\partial \phi}}{\left|\frac{\partial\vec{r}}{\partial \phi}\right|} = -\sin\phi\hat{x} + \cos\phi\hat{y}
\end{aligned}}$$

Spherical polar coordinate $h_1 = h_r = 1\,,\quad h_2 = h_\theta = r\,,\quad h_3 = h_\phi = r\sin\theta$

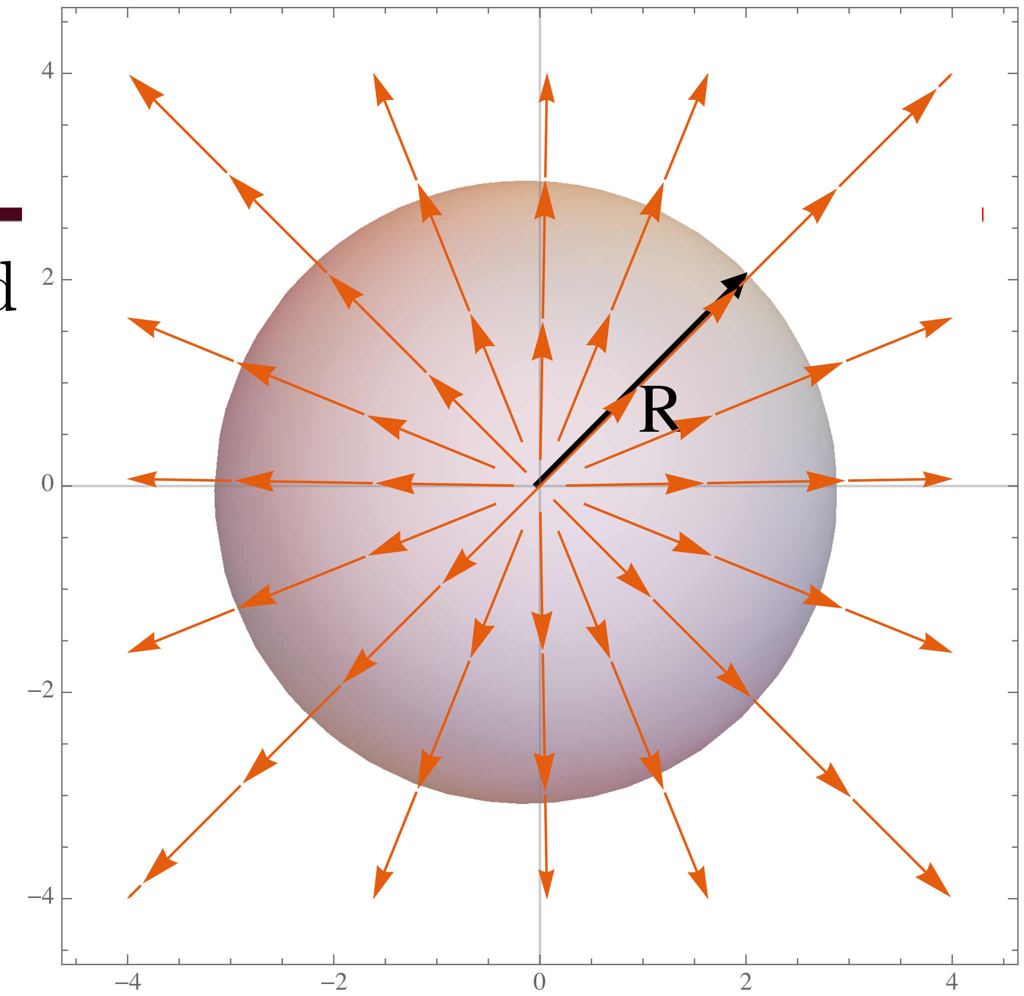
Cylindrical polar coordinate $h_1 = h_s = 1\,,\quad h_2 = h_\phi = s\,,\quad h_3 = h_z = 1$

Del operator in curvilinear coordinate system $\vec{\nabla} = \hat{e}_1\frac{1}{h_1}\frac{\partial}{\partial u_1} + \hat{e}_2\frac{1}{h_2}\frac{\partial}{\partial u_2} + \hat{e}_3\frac{1}{h_3}\frac{\partial}{\partial u_3}$

Symmetry of the problem decides what coordinate system to choose.

Divergence of $\vec{V} = \frac{\hat{r}}{r^2}$

At every direction, $\vec{V}$ is directed radially outward



The function has large positive divergence.

But…

$$\vec{\nabla}.\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

$$\implies \int_{\mathcal{V}} (\vec{\nabla}.\vec{V})\, d\tau = 0$$

Suppose, we integrate over a sphere of radius R, entered at origin: the surface integral is

$$\oint \vec{V}.d\vec{a} = \int \left(\frac{1}{R^2}\hat{r}\right).(R^2 \sin\theta d\theta d\phi \hat{r}) = \left(\int_0^{\pi} \sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right) = 4\pi$$

But divergence theorem states that $\int_{\mathcal{V}}(\vec{\nabla}.\vec{V})d\tau = \int_{\mathcal{S}} \vec{V}.d\vec{a}$!

Is divergence theorem wrong??

Divergence of $\vec{V} = \frac{\hat{r}}{r^2}$

The source of the problem is the point r = 0, where the function blows up!

$$\vec{\nabla}.\vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0 \qquad \Longrightarrow \int_{\mathcal{V}}(\vec{\nabla}.\vec{V})\, d\tau = 0$$

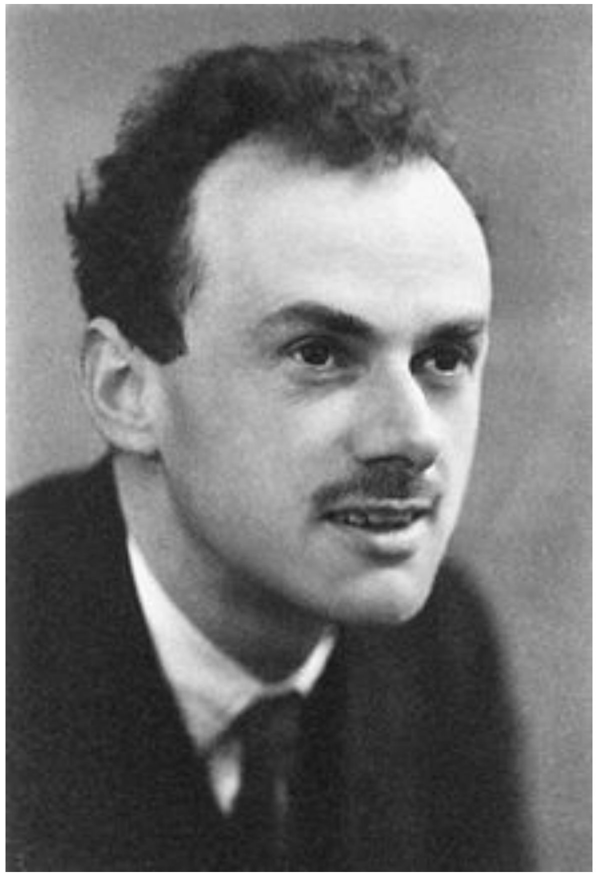
It is true that $\vec{\nabla}.\vec{V} = 0$ everywhere except at the origin. But, right at the origin the situation is more complicated.

Note that surface integral is independent of R; so if divergence theorem is right (and it is), we should expect $\int(\vec{\nabla}.\vec{V})d\tau = 4\pi$. The entire contribution must then be coming from the point $r = 0$.

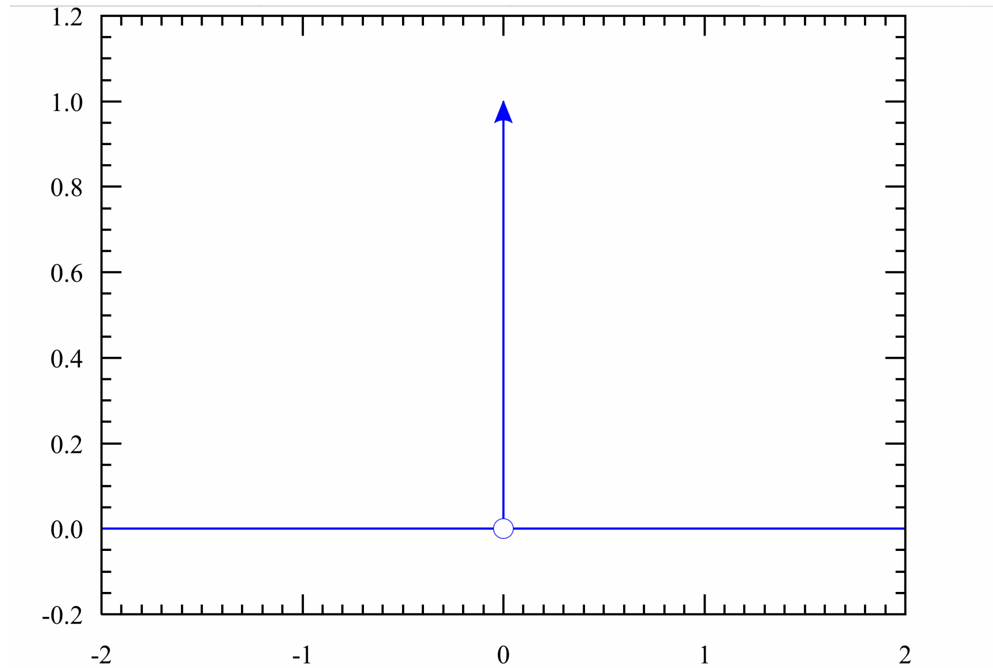
$\vec{\nabla}.\vec{V}$ has the bizarre property that it vanishes everywhere except at one point, and yet its integral over any volume containing that point is $4\pi \implies$ "No Ordinary Function".

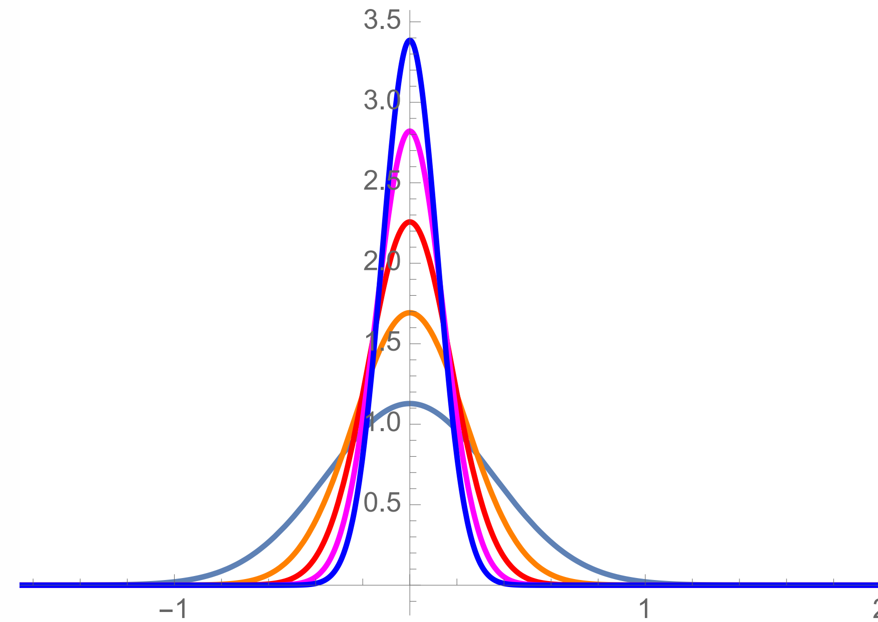
Dirac Delta Function

Dirac Delta Function



Paul Dirac





✦ Dirac delta function is a generalised function or distribution introduced by physicist **Paul Dirac**.

✦ Dirac Delta function is a function that is zero everywhere except one point and at that point it can be thought of as either undefined or as having an infinite value.

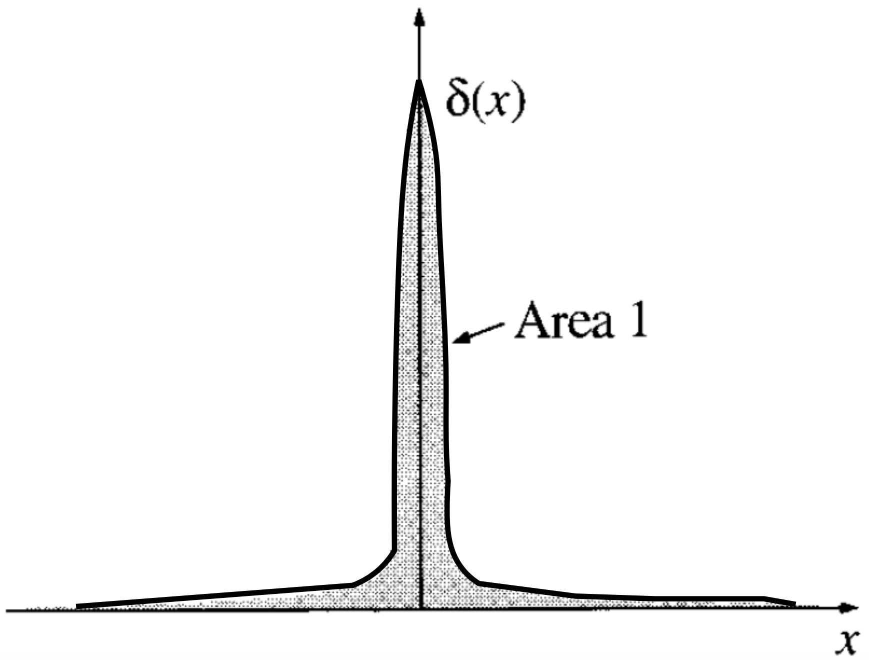
Dirac Delta Function

A real function δ on $\mathbb{R}$ is called Dirac Delta Function

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0; \\ \infty & \text{if } x = 0. \end{cases}$$

and

$$\int_{-\infty}^{\infty} \delta(x) dx = 1.$$



"Infinitely high, infinitesimally narrow spike with area 1"

This of course is a heuristic definition.

Not well defined at x=0, nor even continuous at x = 0.

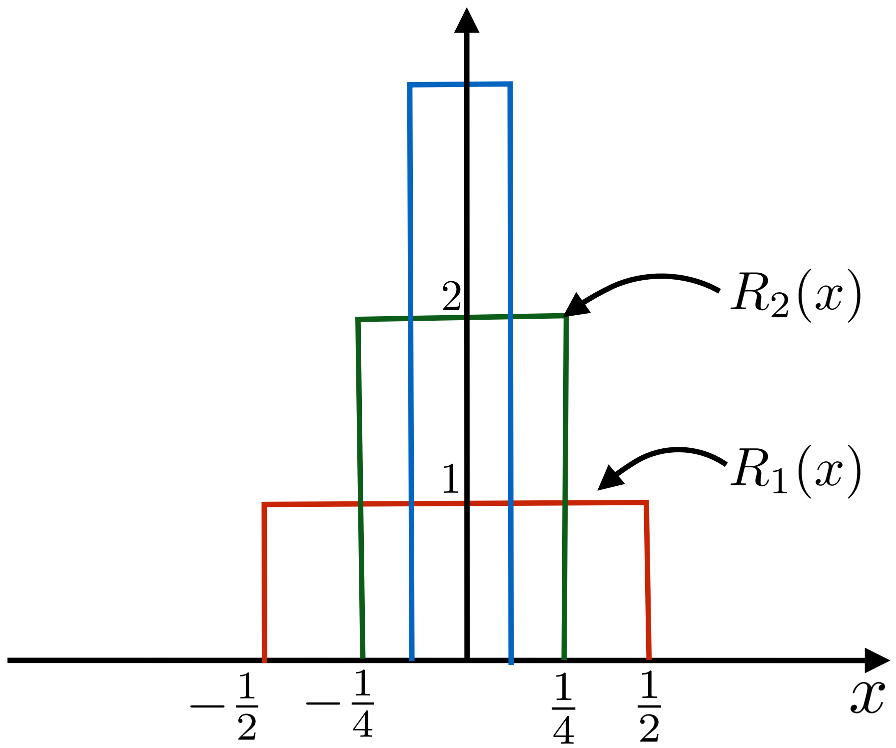
In a traditional sense, it is not a function and can be called a generalised function or a distribution.

Then, how to "see" them?

The best way to look at a delta function is as a limit of a sequence of functions. We give a few such examples:

★ We can have a sequence of function as

$$R_n(x) = \begin{cases} 0 & \text{if } x \leq -\frac{1}{2^n}; \\ 2^{n-1} & \text{if } -\frac{1}{2^n} < x < \frac{1}{2^n}; \\ 0 & \text{if } x > \frac{1}{2^n}. \end{cases}$$



For a fixed n, it represents a rectangle of height n and width between $-\frac{1}{2^n}$ to $\frac{1}{2^n}$. As $n \to \infty$, width decreases but height increases in such a proportion that the area always remains 1. So, as $n \to \infty$, $R_n \to \delta$.

Some representations of Dirac delta function

Integral representation :

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ik(y-x)}\, dk = \delta(y-x)$$

Fourier integral representation

Limiting case ($\epsilon \to 0$):

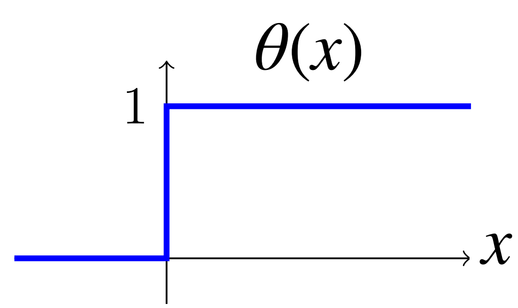
$$\delta(t) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-t^2/2\sigma^2}$$

Limit of Gaussian function

$$\delta(t) = \lim_{\epsilon\to 0} \frac{1}{\pi}\frac{\epsilon}{t^2+\epsilon^2}.$$

Limit of Lorentz function

Heaviside step function $\theta(x)$



$$\int_{-\infty}^{x} \delta(y)\, dy = \theta(x)$$

$$\frac{d}{dx}\theta(x-a) = \delta(x-a)$$

Dirac Delta Function: Properties

⋆ For a continuous function $f(x)$,

$$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$$

This means that for a continuous function $f(x)$, the product $f(x)\delta(x)$ is zero everywhere except at $x = 0$. It follows: $f(x)\delta(x) = f(0)\delta(x)$.

⋆ Translation: $\delta(x-a) = \left\{ \begin{array}{ll} 0 & \text{if } x \neq a \\ \infty & \text{if } x = a \end{array} \right\}$ with $\int_{\infty}^{\infty} \delta(x-a)dx = 1$

Therefore the first property tells us $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$

⋆ Although δ itself is not a legitimate function, integrals over δ are perfectly acceptable. In fact two expressions involving delta functions (say, $D_1(x)$ and $D_2(x)$) are called equal if $\int_{-\infty}^{\infty} f(x)D_1(x)dx = \int_{-\infty}^{\infty} f(x)D_2(x)dx$, for all $f(x)$.

Dirac Delta Function: Properties

Scaling : $\delta(kx) = \frac{1}{|k|}\delta(x)$, where k is any constant.

Note : Delta function has meaning only under integral sign

Proof: Chose an arbitrary test function $f(x)$ and consider the integral:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx$$

Let $y \equiv kx$, so that $x = y/k$ and $dx = dy/k$. If $k > 0$, the integration limits are unchanged but if $k < 0$, the $x = \infty$ implies $y = -\infty$, and vice versa. Restoring the proper order of the limits:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \pm\int_{-\infty}^{\infty} f(y/k)\delta(y)\frac{dy}{k} = \pm\frac{1}{k}f(0) = \frac{1}{|k|}f(0)$$

Therefore, under the integral sign, $\delta(kx)$ serves the same purpose as $(1/|k|)\delta(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx) = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right].$$

Infact, this property tells us $\delta(-x) = \delta(x)$.

Dirac Delta Function: in three dimensions

Generalize in 3-D: $\delta^3(\vec{r}) = \delta(x)\delta(y)\delta(z)$

3-D Dirac Delta is zero everywhere except at origin (0,0,0), with its volume integral being 1

$$\int_{\text{all space}} \delta^3(\vec{r})d\tau = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

Generalizing $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(z)$ in 3-D:

$$\int_{\text{all space}} f(\vec{r})\delta^3(\vec{r}-\vec{r_0})d\tau = f(\vec{r_0})$$

Dirac Delta Function: in three dimensions

Let us get back to the divergence paradox :

Recall that $\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 0$, if $\vec{r} \neq 0$.

The one and only point where divergence is non-zero is origin.

But do we know the value of the divergence at origin?

Assume that it is $k\delta^3(\vec{r})$

Divergence theorem $\Longrightarrow \int_{\mathcal{V}} \left(\vec{\nabla}.\frac{\hat{r}}{r^2}\right) d\tau = \oint_{\mathcal{S}} \frac{\hat{r}}{r^2}.d\vec{a}$

4π

$$\Longrightarrow k\int_{\mathcal{V}} \delta^3(\vec{r})d\tau = 4\pi \quad \Longrightarrow k = 4\pi$$

$$\boxed{\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\vec{r})}$$

Few examples:

1. Evaluate $\int_0^3 x^3 \delta(x-2)dx$.

   The delta function picks out the value of x^3 at the point $x = 2$, so the integral is $2^3 = 8$. Note however, if the upper limit had been 1 (instead of being 3), the answer would be 0, because the spike would then be outside the domain of integration.

2. Evaluate $\int_2^6 (3x^2 - 2x - 1)\delta(x-3)dx$.

   Recall that $\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$. Here $f(x) = (3x^2 - 2x - 1)$, $a = 3$ and it lies between the limits of the integration. Therefore $\int_2^6 (3x^2 - 2x - 1)\delta(x-3)dx = f(3) = 20$.

3. Evaluate $\int_{-2}^{2} (2x+3)\delta(3x)dx$.

   Change variable $x = t/3$. Then $\displaystyle\int_{-2}^{2} (2x+3)\delta(3x)dx = \int_{-\frac{2}{3}}^{\frac{2}{3}} \left(2\frac{t}{3} + 3\right)\delta(t)\frac{dt}{3} = 1$

   Alternatively, you can use $\delta(3x) = \delta(x)/3$ and proceed accordingly.

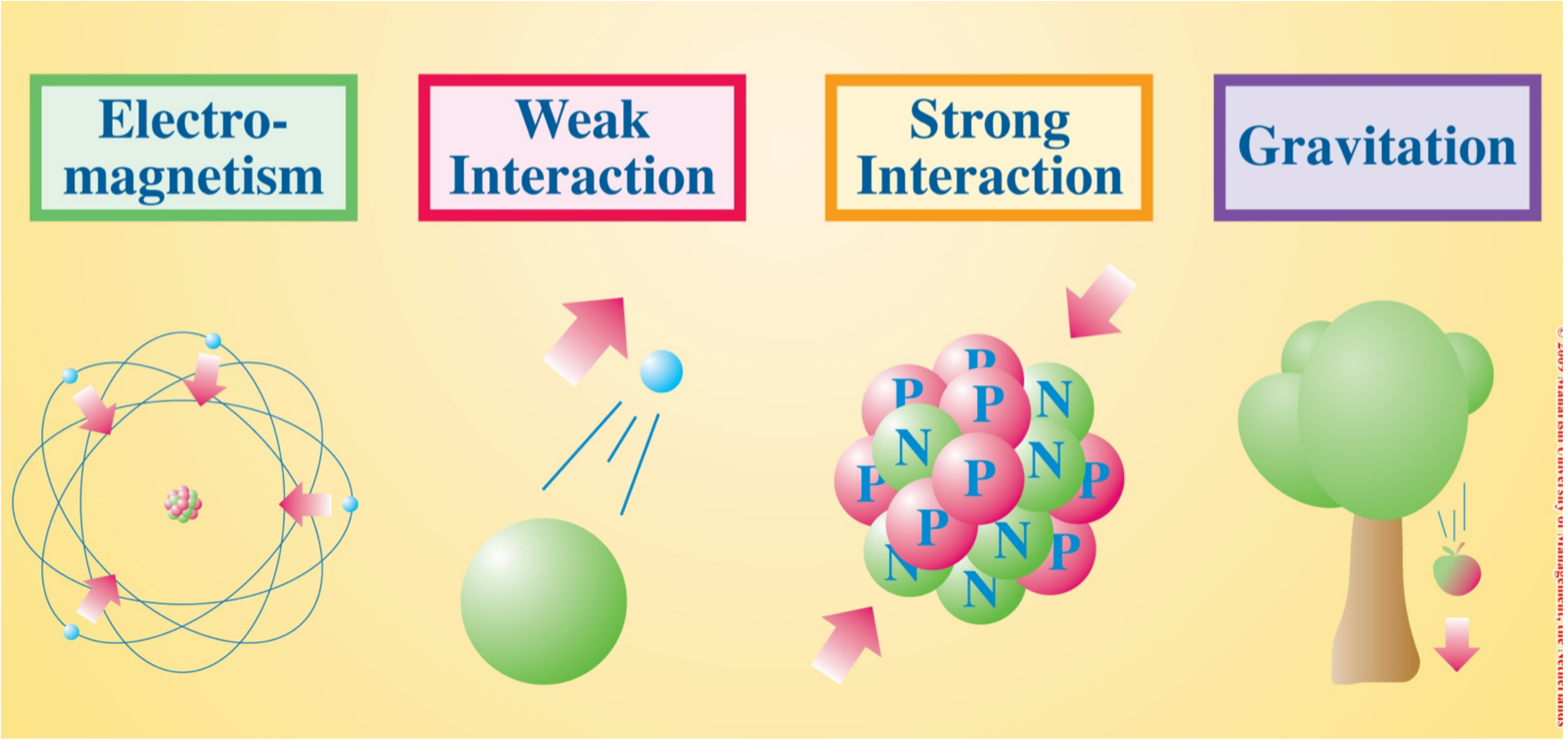
4. Evaluate $J = \int_{\mathcal{V}} (r^2 + 2)\vec{\nabla}.\left(\frac{\hat{r}}{r^2}\right) d\tau$. Here $\mathcal{V}$ is a sphere of radius R centred at origin.

$$J = \int_{\mathcal{V}} (r^2 + 2) 4\pi\delta^3(\vec{r})d\tau = 4\pi(0+2) = 8\pi$$

Electrostatics

Fundamental forces in nature

- Gravitational
- **Electromagnetic**
- Weak
- Strong



We will study the nature of electromagnetic forces in this course

https://wordpress.com/

Concept of Electrostatic

The primary goal of this course is to understand set of four equations known as the Maxwell's equations

$$\begin{aligned}
\vec{\nabla}.\vec{E} &= \frac{\rho}{\epsilon_0} \\
\vec{\nabla}\times\vec{E} &= -\frac{\partial\vec{B}}{\partial t} \\
\vec{\nabla}.\vec{B} &= 0 \\
\vec{\nabla}\times\vec{B} &= \mu_0\vec{J}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}
\end{aligned}$$

- Situations described by these equations can be extremely complicated and to start with we will simplify life by assuming that nothing depends on time - "**static** case"

$$\begin{aligned}
\vec{\nabla}.\vec{E} &= \frac{\rho}{\epsilon_0} \\
\vec{\nabla}\times\vec{E} &= 0
\end{aligned}$$

Electrostatics

$$\begin{aligned}
\vec{\nabla}.\vec{B} &= 0 \\
\vec{\nabla}\times\vec{B} &= \mu_0\vec{J}
\end{aligned}$$

Magnetostatics

- Electricity and magnetism are distinct phenomena so long as charges and currents are static. Independence of E and B does not appear until there are charges or currents. Only when there are sufficiently rapid changes in the charges and currents with time, will E and B depend on each other!

Some important points to note before we start

- ✓ Each particle in the Universe carries with it a number of properties. They determine how the particle interacts with each of the four forces. For the force of gravity, this property is mass. **For the force of electromagnetism, the property is called electric charge.**

- ✓ For the purposes of this course, we can think of electric charge as a real number, $q \in R$. Importantly, charge can be positive or negative. It can also be zero, in which case the particle is unaffected by the force of electromagnetism.

- ✓ The SI unit of charge is the Coulomb, denoted by C. At a fundamental level, Nature provides us with a better unit of charge. This follows from the fact that charge is quantised: the charge of any particle is an integer multiple of the charge carried by the electron: $e = 1.60217657 \times 10{-}19$ C. i.e. $q = n\, e$.

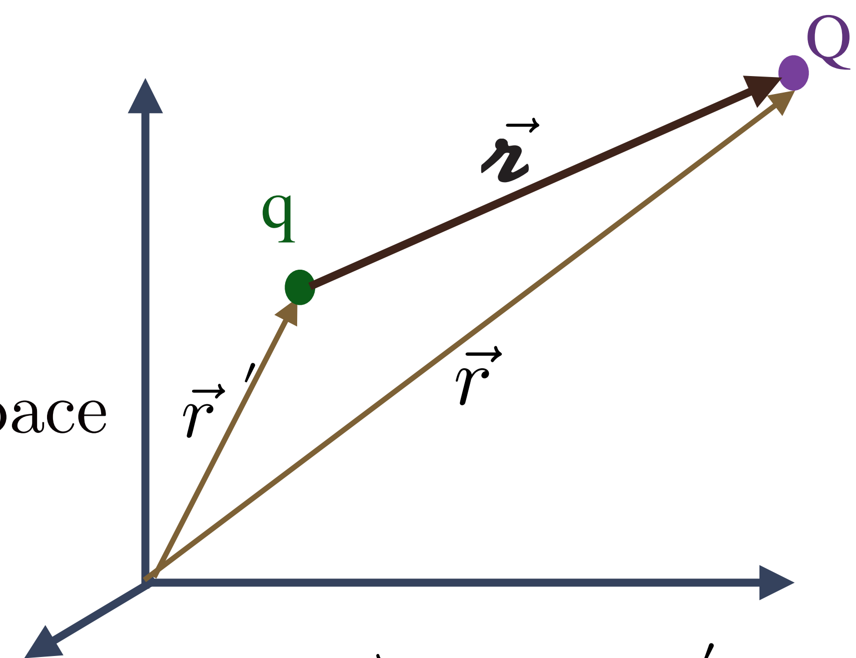
Electrostatics: Coulomb's law

Suppose we have a point charge q is at rest. Then what is the force on a test charge $\vec{\mathscr{r}}$

Coulomb's law

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}}$$ (based on experiments)

$\epsilon_0 = 8.85 \times 10^{-12}\ \frac{C^2}{N\ m^2}$ is called permittivity of free space



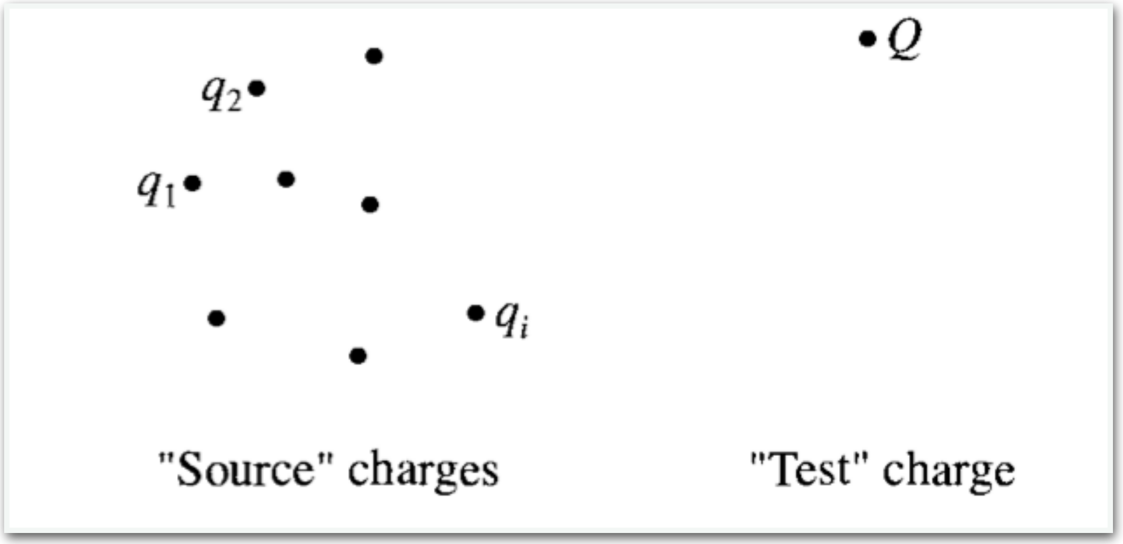
$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$
(separation vector)

- ✦ Between two charges at rest, force is directly proportional to the product of the charges
- ✦ Force is inversely proportional to the square of the distance between.
- ✦ The force is along the straight line from one charge to another

What happens when there are many point charges?

Electrostatics: Principle of Superposition

If we have electric charges, say $q_1, q_2, q_3, \ldots$ as the source charges as shown below



Then the force exerted by these charges on another charge, Q can be calculated by the principle of superposition.

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \ldots$$

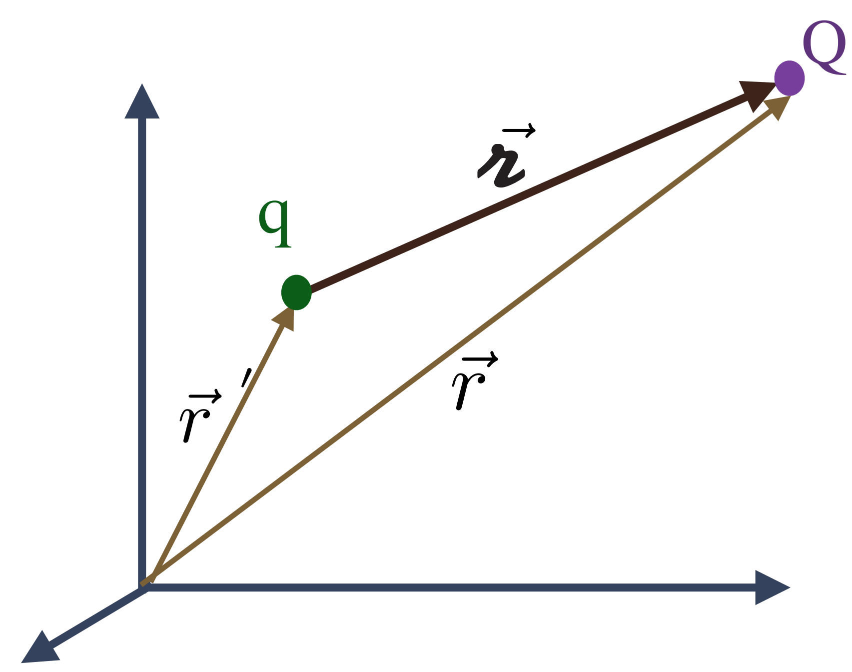
where, F_1: force on Q due to q_1

F_2: force on Q due to q_2 and so on

The force on any charge is the vector sum of the Coulomb forces from each of the other charges present.

Electric field

Let us write the force on Q due to q as follows:



$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}} = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\mathscr{r}}Q \equiv \vec{E}(\vec{r})Q$$

- $\vec{E}(\vec{r})$ is called the electric field of the source charge q.

- The field is a function of the position $\vec{r}$, since the separation vector $\vec{\mathscr{r}}$ depend on the location of the charge Q (field point).

- While it takes two charges to feel a force, it takes only one charge to produce a field. A charge at the origin produces the field $\vec{E}(\vec{r}) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$ at point $\vec{r}$.

- The field due to charge q is non-zero everywhere, not just where there is another charge to feel the field.

This leads us to an important conclusion that electric force is a long range force like gravitational force.

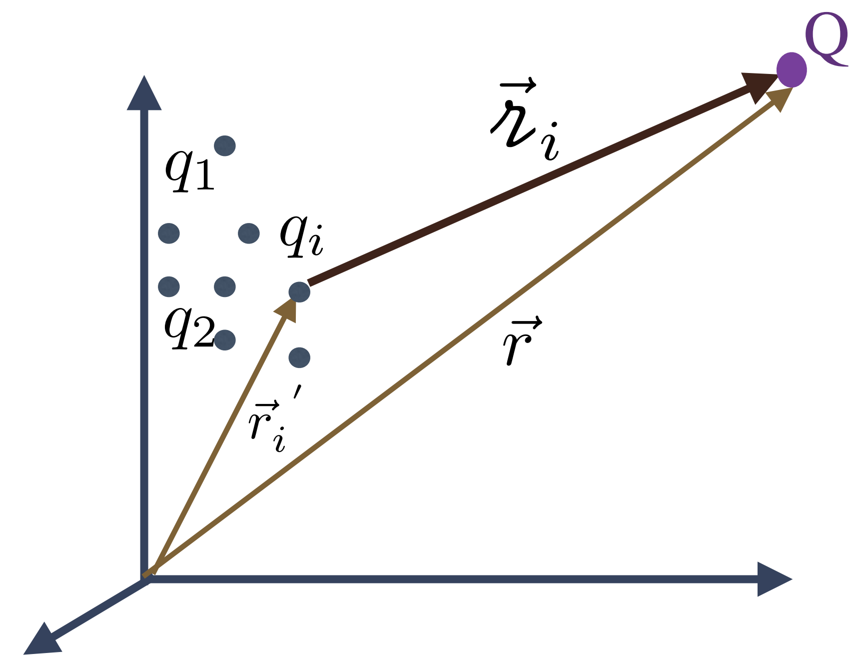
Electric field

• If there are many charges,invoke the superposition principle: the field at some $\vec{r}$ due to many charges will be the (vector) sum of the fields due to each one.

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots = \frac{1}{4\pi\epsilon_0}\left(\frac{q_1 Q}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2 Q}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \cdots\right)$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{q_1}{\mathscr{r}_1^2}\hat{\mathscr{r}}_1 + \frac{q_2}{\mathscr{r}_2^2}\hat{\mathscr{r}}_2 + \cdots\right)$$

$$= Q\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{\mathscr{r}_i^2}\hat{\mathscr{r}}_i \quad = Q\vec{E}(\vec{r})$$



• Note that $\vec{E}(\vec{r})$, which is a vector quantity, depends on the location of the field point and is determined by the configurations of the source charges q_i.

• To measure a field is easier: put a known test charge q at $\vec{r}$, equate the force it experiences to $q\vec{E}$. If $q = 1C$, the force and $\vec{E}$ are numerically equal but dimensionally different. That is why “field is the force on a unit charge”.

Summary

A real function δ on $\mathbb{R}$ is called Dirac Delta Function if

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0; \\ \infty & \text{if } x = 0. \end{cases} \qquad \boxed{\int_{-\infty}^{\infty} \delta(x)dx = 1.}$$

Properties

$$\boxed{\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)} \qquad \boxed{\int_{-\infty}^{\infty} f(x)\delta(kx) = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|k|}\delta(x)\right].}$$

- **Coulomb's law**

$$\boxed{\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\mathscr{r}}} \qquad \epsilon_0 = 8.85 \times 10^{-12}\ \frac{C^2}{N\ m^2}$$

- **Principle of Superposition**

Total force on charge q_1 due to presence of q_2 and q_3 is given by

$$\underbrace{\vec{F} = \vec{F}_{12}}_{\text{Force on } q_1 \text{ due to } q_2} + \underbrace{\vec{F}_{13}}_{\text{Force on } q_1 \text{ due to } q_3}$$

- If there are many charges, the field at some position $\vec{r}$ due to many charges will be the vector sum of fields due to each one.

Thank You