The Transportation Problem :

Let there be m supply stations $S_1, ..., S_m$ for a particular product and n destination stations $D_1, D_2, ..., D_n$ where the product is to be transported. Let c_{ij} be the cost of transportation of unit amount of the product from S_i to D_j. Let a_i be the available amount of the product at S_i and let d_j be the demand at D_j.
The problem is to find x_{ij}, $i = 1, 2, .., m$, $j = 1, 2, ..., n$, where x_{ij} is the amount of the product to be transported from S_i to D_j such that the demand at each D_j is met and the cost of transportation is minimum.
The problem is given by
$Min\ \sum_{i,j} c_{ij} x_{ij}$
subject to
$\sum_{j=1}^{n} x_{ij} \leq a_i$, $i = 1, 2, ..., m$
$\sum_{i=1}^{m} x_{ij} \geq d_j$, $j = 1, 2, ..., n$,
$x_{ij} \geq 0$ for $i = 1, 2, .., m, j = 1, 2, .., n$.
It is clear that for the transportation problem to be feasible $\sum_i a_i \geq \sum_j d_j$.
A transportation problem is said to be **balanced** if $\sum_i a_i = \sum_j d_j$.
In that case all the inequalities in the constraints should hold as equalities.
Hence a balanced transportation problem is given by,
$Min\ \sum_{i,j} c_{ij} x_{ij}$
subject to $\sum_{j=1}^{n} x_{ij} = a_i$, $i = 1, 2, ..., m$
$\sum_{i=1}^{m} x_{ij} = d_j$, $j = 1, 2, ..., n$, $x_{ij} \geq 0$ for $i = 1, 2, .., m, j = 1, 2, .., n$.
Note that since $\sum_i a_i = \sum_j d_j$ if $x = (x_{ij})_{mn \times 1}$ satisfies any $(m + n - 1)$ equations then it automatically satisfies all the $(m + n)$ equations.
That is, for any $r \in \{1, 2, .., m\}$
$\sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} - \sum_{i=1, i \neq r}^{m} \sum_{j=1}^{n} x_{ij} = \sum_j d_j - \sum_{i \neq r} a_i = a_r$.
Similarly for any $s \in \{1, 2, .., n\}$,
$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} - \sum_{j=1, j \neq s}^{n} \sum_{i=1}^{m} x_{ij} = \sum_i a_i - \sum_{j \neq s} d_j = b_s$.

We write the constraints of this problem as $A\mathbf{x} = \mathbf{b}$,
where

$$A_{(m+n) \times mn} = \begin{bmatrix} \underset{n}{\overbrace{111..11}} & \mathbf{0_n} & \mathbf{0_n} & ... & . & \mathbf{0_n} \\ \mathbf{0_n} & \underset{n}{\overbrace{111..11}} & \mathbf{0_n} & ... & . & \mathbf{0_n} \\ \mathbf{0_n} & \mathbf{0_n} & \underset{n}{\overbrace{111..11}} & \mathbf{0_n} & . & \mathbf{0_n} \\ . & . & . & . & ... & . \\ \mathbf{0_n} & . & . & ... & \mathbf{0_n} & \underset{n}{\overbrace{111..11}} \\ \overbrace{100...0} & \overbrace{100...0} & . & . & ... & \overbrace{100...0} \\ \overbrace{010...0} & \overbrace{010...0} & . & . & ... & \overbrace{010...0} \\ . & . & . & ... & ... & . \\ \overbrace{000..01} & \overbrace{000..01} & . & . & ... & \overbrace{000..01} \end{bmatrix}$$

and $\mathbf{b} = [a_1, a_2, .., a_m, d_1, d_2, ..., d_n]^T$ (the $\mathbf{0_n}$'s are row vectors with n components).
Since there are only $m + n - 1$ independent equations, rank(A)$= m + n - 1$.
It can alternatively be checked that the sum of the first m rows of A - sum of the last n rows of A gives the zero vector, since every column of A has exactly two nonzero 1's, one corresponding to a supply constraint and the other corresponding to a destination constraint. It can be easily checked that the rows of A after deleting any one row from A is LI.

To see this let us assume that we have deleted the last destination constraint, $\sum_{i=1}^{m} x_{i(n-1)} = d_{n-1}$. Then each of the variables $x_{1,(n-1)}, \dots, x_{m,(n-1)}$ are now present in only the supply constraints, $1, 2, \dots, m$, respectively. So the columns corresponding to these variables has only one nonzero entry (which is a 1) in the positions (or rows) corresponding to the the supply constraints $1, \dots, m$, respectively, all the other entries being zero.
If the $(m+n-1)$ rows of A are LD, then there exists α_i's, $i = 1, \dots, m$ and β_i's, $i = m+1, \dots, m+n-1$ (atleast one of α_i's or β_i's should be nonzero) such that
$\sum_{i=1}^{m} \alpha_i \mathbf{a}_i{}^T + \sum_{i=m+1}^{m+n-1} \beta_i \mathbf{a}_i{}^T = \mathbf{0}_{1\times mn}$, (**)
where $\mathbf{a}_i^T$ denotes the ith row of A and WLOG we have considered the first m constraints to be the supply constraints.
But since columns $\tilde{\mathbf{a}}_{1,(n-1)}, \dots, \tilde{\mathbf{a}}_{m,(n-1)}$ of A (these are columns corresponding to the variables $x_{1,(n-1)}, \dots, x_{m,(n-1)}$) have exactly one nonzero entry each , so in order that
$\sum_{i=1}^{m} \alpha_i \mathbf{a}_i{}^T + \sum_{i=m+1}^{m+n-1} \beta_i \mathbf{a}_i{}^T = \mathbf{0}_{1\times mn}$,
check that each of $\alpha_1, \dots, \alpha_m$ has to be equal to be zero.
Then again each of the variables in the $(n-1)$ destinations, are present in exactly one destination constraint (after deleting the supply constraints from (**)), hence their columns will again have exactly one nonzero entry and by arguing similarly we get that each of the β_i's should be equal to 0.
Hence $rank(A) = m + n - 1$.

If we decide to remove the last equation then in $A\mathbf{x} = \mathbf{b}$, the dimension of A is $(m+n-1)\times mn$ and $\mathbf{b} = [a_1, a_2, .., a_m, d_1, d_2, ..., d_{n-1}]^T$.
Any basic feasible solution of this problem will have $m + n - 1$ basic variables and the order of any basis matrix say $\mathbf{B}$ is $(m + n - 1) \times (m + n - 1)$.

The basis matrix has a special structure which is discussed in the theorem below.

Theorem 1 : Let B be a basis matrix then:
1. There exists a row of B with exactly one nonzero entry (which is a 1).
2. The sub matrix obtained by deleting the corresponding row and column (containing the nonzero entry) from B will again be nonsingular and will have a row with a single nonzero entry.

Proof: Let us change the supply constraints to $\sum_{j=1}^{n} -x_{ij} = -a_i$, $i = 1, 2, ..., m$ (that is multiplying each of the supply constraints with (-1)).
We have to show that there is a row of B with exactly one nonzero entry.
Suppose not, then each row of B has at least 2 nonzero entries so the number of nonzero entries of B should be at least $2(m + n - 1)$. (1)
We know that any column of A has at most 2 nonzero entries. Since B has m+n-1 columns the total number of nonzero entries of B is at most $2(m + n - 1)$. (2)
From (1) and (2) we can conclude, the total number of nonzero entries of B is exactly equal to 2(m+n-1).
This implies, each column of B has exactly 2 nonzero entries, one +1 the other -1.
Hence sum of all the rows of B is $\mathbf{0_{m+n-1}}$.
That is the rows of B are linearly dependent, which is a contradiction.
Hence there exists a row of B with exactly one nonzero entry.
Let i be a row of B having exactly one nonzero entry and let the (i,j)th entry be nonzero. Consider the sub matrix of order $m + n - 2$ of B obtained by removing the i th row and the j th column from B.

Let us call it as B_1.
Since $|B_1| = |B|$ or $-|B|$, B_1 is nonsingular.
Also by just repeating the previous argument we can again conclude that there is a row of B_1 with exactly one nonzero entry. Hence the result.

Such matrices (such as B) are called triangular matrices, and because of this special structure of B it is easy to solve system of equations of the form $B\mathbf{x}_B = \mathbf{b}$ (which will give a basic solution of the transportation problem).

Exercise 1: If B is a square sub matrix of A having property 1 and 2 of theorem 1, then $|B| = \pm 1$.

Exercise 2: If D is any nonsingular submatrix of A then will D again have the same structure as B?

If the i th row of B has a single nonzero entry at the j th column, then one should start by assigning the value $x_{ij} = b_i$ (where b_i is either a_i or d_j).
Then remove the ith row and the j th column from B which will give us say the matrix B_1 and solve the system $B_1\mathbf{x}' = \mathbf{b}''$, where $\mathbf{x}'$ is obtained from $\mathbf{x}_B$ by removing the component x_{ij} and $\mathbf{b}'$ is obtained from $\mathbf{b}$ by removing the component b_i and changing the j th component from b_j to $b_j - b_i$.
Proceeding in this way one can solve the system of equations $B\mathbf{x}_B = \mathbf{b}$.
Note that any basic solution will be such that the basic variables will take values of the form, $\sum_i \alpha_i b_i$, where the α_i's are either 0,1 or -1.

Remark 1: Hence any basic feasible solution $\mathbf{x}$ of the transportation problem with supplies a_i, $i = 1, 2, ..., m$ and demand b_j, $j = 1, 2, ..., n$ has variables taking values of the form, $x_{ij} = \sum_i \alpha_i a_i + \sum_j \beta_j b_j$ where the α_i's and β_j's take values 0,1 or -1.

Transportation Array: The mn variables x_{ij} can be arranged in an $m \times n$ array known as the $m \times n$ transportation array. In a transportation array each cell corresponds to a variable, that is the (i, j)th cell corresponds to x_{ij}. The m rows correspond to the m supply constraints, hence the sum of the variables in row i is given by a_i. Similarly the n columns correspond to the n demand constraints and the sum of the variables in column j is given by d_j.

Definition 1: A subset of cells of the transportation array is said to be linearly independent if the set of column vectors in the matrix A corresponding to the variables associated with the cells are linearly independent. Otherwise they are said to be linearly dependent.

Definition 2: A subset of (m+n-1) cells of the transportation array is said to be a basic set if they are linearly independent. The cells in a basic set are called basic cells.

Remark 2: Note that a basic set corresponds to a basic solution of the transportation problem, where the variables corresponding to the basic cells are basic variables and the rest are nonbasic variables.

Remark 3: Let $\mathcal{B}$ be a basic set of cells. If we consider the submatrix of $A_{(m+n-1)\times mn}$ obtained by taking the columns corresponding to the variables associated with the basic set $\mathcal{B}$, then the submatrix (call it B) will be a basis matrix, a square nonsingular matrix of dimension $m+n-1$.

By **Theorem 1**, there exists a row of B with exactly one nonzero entry.
Since we are now solving $Bx_B = [a_1, .., a_m, d_1, .., d_{n-1}]^T$ and each row of B corresponds to a constraint (supply or demand), there exists a constraint which has exactly one of the basic variables.
Since each row and column of the transportation array corresponds to a constraint, there exists a row or column of the transportation array which has exactly one cell from the basic set $\mathcal{B}$.
Also from Theorem 1 we get that if row i contains a single nonzero entry at (i, j) th position, then the submatrix obtained from B after deleting the i th row and the j th column from B again has the same property, that is there is a row of the submatrix with a single nonzero entry.
Hence if $\mathcal{B}$ be a basic set of cells and if the row or column having a single basic cell is struck off from the transportation array, then in the reduced (or remaining) array there will again be a row or column with a single basic cell.
Since every row and column of the array has at least one basic cell (why?), one can continue this process (of striking off rows/ columns) till all the rows and columns of the transportation array are struck off (or deleted).

Example 1: Consider the transportation problem with a_i and d_j as given below:

	$j=1$	2	3	4	5	6	a_i
$i=1$							7
2							17
3							5
4							24
d_j	15	10	9	3	8	8	

Let us first assume that cell $(2, 3)$ is a basic cell and then try to construct a basic feasible solution of the above problem.
Since the minimum of a_2 and d_3 is $d_3 = 9$, we take $x_{23} = 9$. Delete the third column and change a_2 from 17 to $a'_2 = 17 - 9 = 8$.
In the new array choose a basic cell say $(2, 4)$. Take $x_{24} = 3$ since $3 = min\{d_4 = 3, a'_2 = 8\}$.
Proceeding in this way we get the following basic feasible solution.

	$j=1$	2	3	4	5	6	a_i
$i=1$		[7]					7
2			[9]	[3]	[5]		17
3					[3]	[2]	5
4	[15]	[3]				[6]	24
d_j	15	10	9	3	8	8	

θ -loops
A collection of cells of the transportation array is said to form a θ- loop if it satisfies the following conditions.
1. Nonempty.
2. Every row and column of the transportation array either has 0 or 2 cells from this collection.
3. No proper subset of this collection satisfies both property 1 and property 2.

Consider the following examples.

	1	2	3	4
1	○			○
2	○	○		
3		○	○	
4			○	○

	1	2	3	4
1	○	○		
2	○	○		
3			○	○
4			○	○

and

	1	2	3	4
1	○	○		
2	○	○		
3			○	○
4			○	○

In the second and third example, the marked cells do not form a θ loop of the 4×4 transportation array, since it violates properties 1 and 2, respectively.
The first one however is a θ loop.

Theorem 4: The cells in a θ loop are linearly dependent.

Proof: Give the allocations $+\theta$ and $-\theta$ alternately to the cells in a θ loop and 0 to all the other cells in the array.
How to do this ? I have given the details below (**) which is optional and you may skip it if you are already convinced.
Then $\sum_{i,j} \alpha_{ij} \times$ (column of A corresponding to x_{ij}) $= \mathbf{0}$,
where $\alpha_{ij} = +\theta, -\theta, 0$ according to whether the cell (i,j) has been allotted $+\theta$ or $-\theta$ or 0.

(**)(Start with a cell with minimum row index among the cells of the θ-loop.
Say $i_1 = min_i\{(i,j) \in \theta - loop\}$.
Give allocation $+\theta$ to the cell say $(i_1, j_1) \in \theta$-loop.
Then there is a cell of the form $(i_1, j_2) \in \theta$-loop, $j_1 \neq j_2$, give it allocation $-\theta$.
The next cell is say $(i_2, j_2) \in \theta$-loop, $i_1 \neq i_2$, give it allocation θ.
This process will stop and will stop only when you get a cell of the form (i_k, j_1), the previous cell chosen by this process being (i_k, j_k) which gets an allocation $+\theta$.
Hence give the cell (i_k, j_1) the allocation $-\theta$).

Aliter 1:
Let $(i_1, j_1) \in \theta$-loop.
Then the column in A corresponding to (i_1, j_1) has a 1 in the i_1 th supply constraint position and a 1 in the j_1 th destination constraint position.
Then there is a cell of the form $(i_1, j_2) \in \theta$-loop, $j_1 \neq j_2$.
So the vector $(+1)col(i_1, j_1) + (-1)col(i_1, j_2)$ has zero everywhere except in the j_1, j_2 th destination row position which is +1, -1, respectively,
where $col(i,j)$ gives the column in A corresponding to cell (i,j) or variable x_{ij}.
There exists a cell of the form $(i_2, j_2) \in \theta$-loop.

Then the vector, $(+1)col(i_1, j_1) + (-1)col(i_1, j_2) + (+1)col(i_2, j_2)$ has zero everywhere except in the j_1 th destination row position and i_2 supply row position which are both $+1$.
Hence after a finite number of steps we will get a cell of the form (i_k, j_1) such that the previous two cells obtained is of the form $(i_{k-1}, j_k), (i_k, j_k)$.
Check that
$col(i_1, j_1) - col(i_1, j_2) + col(i_2, j_2) - ... + col(i_k, j_k) - col(i_k, j_1) = \mathbf{0}$.
Hence the cells in the θ-loop are LD.

Aliter 2: (Given by students) Proof by contradiction.
Since a θ loop is a nonempty collection of cells, there exists a row or column of the array which has exactly two cells from the θ loop.
Let the rows which have cells from the θ loop be $i_1, i_2, .., i_r$ and the columns which have cells from the θ loop be $j_1, j_2, ..., j_s$. From the definition of a θ loop, each of these rows and columns should again have two elements from the θ loop.
If the cells in a θ loop are not LD they can be extended to a collection of $m + n - 1$ basic cells (since a collection of linearly independent columns of A (by deleting a row of A) in a vector space V (here $V = R^{m+n-1}$) can be extended to a basis of V by choosing vectors from the set of columns of A, since rank(A)=m+n-1).
Then there exists a row or column of the transportation array having exactly one of the $m + n - 1$ basic cells. Delete that row or column from the array. It is clear that continuing in this way one cannot strike off any of the rows $i_1, i_2, .., i_r$ and $j_1, j_2, ..., j_s$ (please refer to **Remark 3** and the subsequent discussion).
Hence contradiction.

Theorem 5 : If $\triangle$ is a nonempty collection of cells of the transportation array which contains no θ loop then it satisfies,
1. There exists a row or column of the array with exactly one cell from $\triangle$.
2. Every nonempty subset of $\triangle$ should satisfy property 1.

Proof: Note that if property 1 holds good then obviously 2 holds, since if $\triangle$ does not contain a θ-loop, then no subset of $\triangle$ can contain a θ-loop, hence 2 will hold good if 1 holds.
We attempt to give a proof by contradiction, hence suppose $\triangle$ is a nonempty collection of cells which contains no θ loop and it also does not satisfy property 1.
Let (i_1, j_1) be a cell in $\triangle$ with $i_1 = min\{i : (i, j) \in \triangle\}$.
Then there exists at least one more cell from $\triangle$ in the same row (of the array), otherwise it will satisfy property 1.
Let $(i_1, j_2) \in \triangle$, be that cell, hence $j_2 \neq j_1$.
Now there must exist one more cell from $\triangle$ in the column j_2. Let $(i_2, j_2) \in \triangle$, be that cell. Note that $i_2 \neq i_1$. Now there must exist one more cell from $\triangle$ in the row i_2. Let $(i_2, j_3) \in \triangle$, be that cell. Note that $j_3 \neq j_2$. Also $j_3 \neq j_1$ since otherwise $\{(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_1)\}$ will form a θ-loop which contradicts the hypothesis that $\triangle$ does not contain a θ-loop.
If we continue in this way then since the number of cells in $\triangle$ is finite one of two cases given below must necessarily occur.
Case 1: A row index gets repeated for the first time (after occurring twice initially for example here for the two cells $(i_1, j_1), (i_1, j_2)$ the index i_1 appears twice).
Continuing in the way as indicated in the previous paragraph if we are in the cell (i_k, j_{k+1}), $(k \geq 3)$, $i_1 \neq \ldots \neq i_k$ and $j_1 \neq \ldots \neq j_k \neq j_{k+1}$, then note that (from the construction, the way the cells are taken) the next cell will be (i_{k+1}, j_{k+1}). If $i_{k+1} \in \{i_1, \ldots, i_k\}$ and if $i_{k+1} = i_l$ then $l \leq k - 1$ and check that

$\{(i_l, j_{l+1}), (i_{l+1}, j_{l+1}), (i_{l+1}, j_{l+2}), \ldots, (i_k, j_{k+1}), (i_l, j_{k+1})\}$ forms a θ-loop, which contradicts the hypothesis that $\triangle$ does not contain a θ-loop.
Case 2: A column index gets repeated for the first time (after occurring twice or the column index j_1 of the first cell is repeated for the first time).
Continuing in the above way if we are in the cell (i_k, j_k), $(k \geq 3)$, $i_1 \neq \ldots \neq i_k$ and $j_1 \neq \ldots \neq j_k$, then note that (from the construction, the way the cells are taken) the next cell will be (i_k, j_{k+1}). If $j_{k+1} \in \{j_1, \ldots, j_k\}$ and if $j_{k+1} = j_t$ then $t \leq k-1$ and check that
$\{(i_t, j_t), (i_t, j_{t+1}), (i_{t+1}, j_{t+1}), \ldots, (i_k, j_k), (i_k, j_t)\}$ forms a θ-loop, which contradicts the hypothesis that $\triangle$ does not contain a θ-loop.

Theorem 6 : If $\triangle \neq \phi$ is a collection of cells of the transportation array which contains no θ loop as a subset, then $\triangle$ is linearly independent.
Proof. If $\triangle \neq \phi$ is not LI then there exists a nonzero linear combination of the columns corresponding to the variables associated with cells in $\triangle$ which gives the zero vector.
That is there exists α_{ij} not all zeros, such that
$\sum_{(i,j)\in\triangle} \alpha_{ij} \times$ (columns corresponding to x_{ij} in A) $= 0$. (**)
Since the columns corresponding to x_{ij} in A has a 1 at the row corresponding to the i th supply constraint and a 1 in the row corresponding to the j th demand constraint, from (**) we get
$\sum_{i,(i,j)\in\triangle} \alpha_{ij} = 0$ for all $j = 1, .., n$ and
$\sum_{j,(i,j)\in\triangle} \alpha_{ij} = 0$ for all $i = 1, .., m$.
If we consider the collection of cells corresponding to nonzero α_{ij}'s (a subset of $\triangle$), then this set of cells do not satisfy the condition that there exists a row or column of the transportation array having exactly one cell from this set.
This contradicts that $\triangle$ is a collection of cells which contains no θ loop.
Alternatively :(suggested by a student Siddharth)
If $\triangle \neq \phi$ is a collection of cells of the transportation array which contains no θ-loop, then by **theorem 5** it satisfies
1. There exists a row or column of the array with exactly one cell from $\triangle$.
2. Every nonempty subset of $\triangle$ should satisfy property 1.

If $\mathbf{B}$ is the matrix whose columns are those columns of A which correspond to the cells in $\triangle$ then $\mathbf{B}$ is an $(m+n-1) \times k$ sub matrix of $A_{(m+n-1)\times mn}$, where the number of cells in $\triangle$ is k. From property 1, it satisfies the property that there exists a row of $\mathbf{B}$ with exactly one nonzero entry (which is a 1). Further if we delete that row and the corresponding column (that is eliminating the variable) with the 1 from $\mathbf{B}$ then the reduced sub matrix $\mathbf{B}_1$ (if it is not the zero matrix) again has the same property. This is because of property 2. Continue this till all the variables (or cells) are eliminated. Since at every stage we have eliminated exactly one variable and deleted exactly one constraint, after the $(k-1)$th stage we will be left with exactly one **nonzero** row with a single nonzero 1 and one column (that is one variable left to be eliminated) and some zero rows (that is the submatrix after the $(k-1)$-th stage, will be a column vector of the form $\mathbf{e}_i$). Hence if $i_1, \ldots, i_k$ (note that there will be k such rows) be the rows of $\mathbf{B}$ which gave the single nonzero 1's in all k stages of elimination taken together, then the sub matrix $\mathbf{B}'$ of $\mathbf{B}$ with these rows $(i_1, \ldots, i_k)$ and all the k columns of $\mathbf{B}$ will be nonsingular with determinant $+1$ or -1 (since then $\mathbf{B}'$ will satisfy property 1 and 2 mentioned in Theorem 1). Hence all the columns of $\mathbf{B}$ are LI (or $\mathbf{rank}(\mathbf{B}) = k$), or the cells in $\triangle$ are LI.
(Note that I have used the result that given a matrix A with k rows or k columns, then rank $(A) = k$ if and only if there exists a $k \times k$ square sub matrix of A which is nonsingular (nonzero determinant).)

Corollary 6: So from the previous theorems we can conclude that a subset of cells $\triangle$ of the transportation array is linearly independent if and only if it contains no θ-loop.

Theorem 7: If $\mathcal{B}$ is a collection of $m+n-1$ basic cells of the transportation array and $(p,q) \notin \mathcal{B}$, then $\mathcal{B} \cup \{(p,q)\}$ contains one and only one θ-loop and this loop includes the cell (p,q).
Proof: Since the rank of the coefficient matrix A is $m+n-1$ any collection of $m+n$ cells are linearly dependent. From the previous result we get $\mathcal{B} \cup \{(p,q)\}$ contains at least one θ-loop. Also that loop should include the cell (p,q), since the other cells are LI.
Suppose there were two θ-loops in $\mathcal{B} \cup \{(p,q)\}$ containing the cell (p,q), say θ_1-loop and θ_2-loop, where $\theta_1 \neq \theta_2$. Since the associated cells are LD we would get two different nontrivial linear combinations of the columns of A corresponding to the associated variables of $\mathcal{B} \cup \{(p,q)\}$ giving the zero vector.
That is there exists α_{ij}'s not all zeros such that
$\sum_{(i,j)\in\theta_1} \alpha_{ij} \times$ (columns corresponding to x_{ij} in $A) = 0$. (*)
Note that the α_{ij}'s can be chosen to be either 1 or -1 (why?).
Also there exists β_{ij}'s not all zeros such that
$\sum_{(i,j)\in\theta_2} \beta_{ij} \times$ (columns corresponding to x_{ij} in $A) = 0$. (**)
Similarly the β_{ij}'s can be chosen to be either 1 or -1.
Since (p,q) belongs to both θ_1 and θ_2 loops it implies that the column in A corresponding to variable (p,q) can be written as two different linear combinations of elements of a basis of R^{m+n} (or R^{m+n-1} when the last row is removed), which is a contradiction.

How to get the optimal solution from a given basic feasible solution:
Let $\mathbf{x} = (x_{ij})$ be the initial basic feasible solution. Only x_{ij} 's corresponding to the basic cells (m+n-1) can be positive, the rest are all nonbasic variables, taking the value zero.
Note that the dual of the transportation problem is given by
$Max \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$
subject to,
$u_i + v_j \leq c_{ij}$ for all $i = 1,..,m$, $j = 1,..,n$.
Using duality theory, we know that if we can get a feasible solution of a dual which satisfies the **complementary slackness property** with a feasible solution of the primal, then both the solutions are optimal for the primal and the dual respectively.
(Feasible solutions $\mathbf{x}$, $\mathbf{y}$ of the primal and the dual respectively are said to satisfy **complementary slackness property** if the following is satisfied:
whenever $y_i > 0$, $(A\mathbf{x})_i = b_i$,
whenever $x_i > 0$, $(A^T\mathbf{y})_i = c_i$.)

Step 1: Just like in simplex method, for the basic cells corresponding to $\mathbf{x}$ assuming $c_{ij} = u_i + v_j$ we try to solve this set of $m+n-1$ equations for u_i and v_j. But since there are $m+n-1$ equations and $m+n$, u_i, v_j's we can fix the value of any one of the variables and solve for the others.
Since any one of the $(m+n)$ equations of the transportation problem can be removed, one can take the corresponding variable of the dual say $v_n = 0$ and can consider that variable as absent from the equations $c_{ij} = u_i + v_j$.
Note that this set of equations is obtained from $\mathbf{y}^T B = \mathbf{c}_B^T$, where $\mathbf{y}^T = [u_1, ..., u_m, v_1, ..., v_{n-1}]$.
We have $m+n-1$ equations and $m+n-1$ unknowns, which can be easily solved by back substitution.

Step 2: Check if this $\mathbf{y}$ is feasible for the dual, that is if $u_i + v_j \leq c_{ij}$ for all the nonbasic cells. If yes, then stop.
The corresponding basic feasible solution is then optimal for the primal.
If not, then go to Step 3.

Step 3: Find the θ-loop in $\mathcal{B} \cup \{(p,q)\}$, where the cell (p,q) is such that $c_{p,q} - u_p - v_q = min\{c_{ij} - u_i - v_j : c_{ij} - u_i - v_j < 0\}$.
The existence and uniqueness of this loop is guaranteed by **Theorem 7**.

Step 4: Assign value $+\theta$ to cell (p,q) and alternately assign $+\theta$ and $-\theta$ to all the cells in the θ-loop, so that sum of the allocations in each row and column($+\theta$ and $-\theta$ allocations) add up to zero.
Take $+\theta = min\{x_{ij} \in \theta\text{-loop} \ : \ \text{cell } (i,j) \text{ is assigned value } -\theta\}$ and find the new basic feasible solution say x' where x'_{ij} is either equal to x_{ij}, $x_{ij} + \theta$ or $x_{ij} - \theta$.
Note that now (p,q) is a basic cell.
Also if $x_{rs} = min\{x_{ij} \in \theta\text{-loop} \ : (i,j) \text{ is assigned value } -\theta\}$, then the variable x_{rs} becomes a nonbasic variable in $\mathbf{x}'$. If there is a tie for this minimum value, choose one amongst them as the leaving variable (or cell) arbitrarily such that you again have $(m+n-1)$ basic cells in the next iteration.

Step 5: Go to Step 1.

Remark 4: If x_{pq} is a nonbasic variable in a BFS and if the column corresponding to this variable in the corresponding simplex table be denoted by $\mathbf{u_{pq}}$, then the $\mathbf{k}$ th component of this column, $u_{\mathbf{k},pq} = -1, 1,$ or 0 depending on whether the $\mathbf{k}$ th basic variable gets the allocation θ, $-\theta$ or is not there in the θ-loop containing the cell (p,q) in $\mathcal{B} \cup \{(p,q)\}$.
Hence if (p,q) is the entering variable of the new basis then according to the minimum ratio rule given by the simplex algorithm, the leaving variable is (r,s) if
$x_{rs} = min\{x_{ij} \in \theta\text{-loop} : \text{ cell } (i,j) \text{ is assigned value } -\theta\}$.

Example: Consider the following transportation problem (P) with c_{ij}'s, a_i's (40,30,30) and d_j's (30,50,20) as given below:

2	5	1	40
1	4	5	30
1	5	3	30
30	50	20	

Check whether the initial basic feasible solution $\mathbf{x}_0$ with basic cells
$\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$, is optimal for (P) (by taking $v_2 = 0$, where v_2 is the dual variable corresponding to the second demand constraint).
Also find the optimal solution.
Solution: The BFS with $\mathcal{B} = \{(1,1),(1,2),(2,2),(2,3),(3,2)\}$ as the basic cells is given by $x_{11} = 30, x_{12} = 10, x_{22} = 10, x_{23} = 20, x_{32} = 30$ as the values of the basic variables (note that it is a nondegenerate BFS) and all the other variables (nonbasic variables) $x_{13}, x_{21}, x_{31}, x_{33}$ take the value 0.
To check the optimality of the above BFS we need to calculate the $c_{ij} - u_i - v_j$ values for all the nonbasic cells (for the basic cells $c_{ij} - u_i - v_j$ values are equal to 0) by taking any one of the u_i's or v_j's equal to 0 and solving for the other u_i's and v_j's from the equations $c_{ij} - u_i - v_j = 0$ for the basic cells. If all the $c_{ij} - u_i - v_j$ values are nonnegative then the above table is optimal.
The following table shows the $c_{ij} - u_i - v_j$ values against each cell, where we have taken $v_2 = 0$ (you can take any one of u_i, v_j values to be equal to 0 whichever one you like, you can check that the $c_{ij} - u_i - v_j$ values will be same as the one given below) for easier calculations and the rest of the u_i, v_j values are obtained by solving the equations given by $c_{ij} - u_i - v_j = 0$ for the basic cells,

that is by solving the 5 equations given below:
$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
$c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$
$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
$c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
$c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
(Check that $u_1 = 5, v_1 = -3, u_2 = 4, v_3 = 1, u_3 = 5$) and hence check that
$c_{13} - u_1 - v_3 = 1 - 5 - 1 = -5, c_{21} - u_2 - v_1 = 1 - 4 - (-3) = 0$, $c_{31} - u_3 - v_1 = 1 - 5 - (-3) = -1$,
$c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3$.

0	0	-5	40
0	0	0	30
-1	0	-3	30
30	50	20	

Since all the $c_{ij} - u_i - v_j$ values are not nonnegative, the above table is not optimal.
The most negative value of $c_{ij} - u_i - v_j$ is in cell $(1, 3)$, so this will be the entering variable in the basis of the new basic feasible solution.
Consider the unique θ- loop in $\mathcal{B} \cup \{(1, 3)\}$ which is given by $\{(1, 2), (2, 2), (2, 3), (1, 3)\}$.
Since $(1, 3)$ is the entering variable, so if we give $+\theta$ allocation to cell $(1, 3)$ (or value of $x_{13} = +\theta$) then $x_{12} = 10 - \theta$, $x_{22} = 10 + \theta$, $x_{23} = 20 - \theta$ (since the new BFS must satisfy all the supply and the demand constraints so the total amount of allocation in row i must be equal to a_i and the total allocation in column j must be equal to d_j), so the maximum value of θ is equal to 10 since x_{12} has to be nonnegative in the new BFS.
So we enter x_{13} in the basis of the new BFS and it takes the value 10 and x_{12} leaves the basis.
So now $\mathcal{B} = \{(1, 1), (1, 3), (2, 2), (2, 3), (3, 2)\}$ and the values of the basic variables are given by: $x_{11} = 30, x_{13} = 10, x_{22} = 20, x_{23} = 10, x_{32} = 30$ as the values of the basic variables (note that it is a nondegenerate BFS) and the all the other variables (nonbasic variables) $x_{13}, x_{21}, x_{31}, x_{33}$ take the value 0.
Now if we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
$c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
$c_{23} - u_2 - v_3 = 0$, where $c_{23} = 5$
$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
$c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
Check that $v_1 = 2, v_2 = 0, v_3 = 1, u_2 = 4, u_3 = 5$ and hence check that $c_{21} - u_2 - v_1 = 1 - 4 - 2 = -5, c_{12} - u_1 - v_2 = 5 - 0 - 0 = 5$, $c_{31} - u_3 - v_1 = 1 - 5 - 2 = -6$, $c_{33} - u_3 - v_3 = 3 - 5 - 1 = -3$.
The following table gives the $c_{ij} - u_i - v_j$ values for the above BFS with
$\mathcal{B} = \{(1, 1), (1, 3), (2, 3), (2, 2), (3, 2)\}$.

0	5	0	40
-5	0	0	30
-6	0	-3	30
30	50	20	

So now the entering variable for the new BFS is x_{31}, since $c_{31} - u_3 - v_1$ value is the most negative for this cell among all $c_{ij} - u_i - v_j$ values.

Consider the unique θ- loop in $\mathcal{B} \cup (3, 1)$ which is given by $\{(3, 1), (3, 2), (2, 2), (2, 3), (1, 3), (1, 1)\}$. Since $(3, 1)$ is the entering variable, so if we give $+\theta$ allocation to cell $(3, 1)$ (or value of $x_{31} = +\theta$

) then $x_{11} = 30 - \theta$, $x_{13} = 10 + \theta$, $x_{23} = 10 - \theta$, $x_{22} = 20 + \theta$, $x_{32} = 30 - \theta$, so the maximum value of θ is equal to 10 since x_{23} has to be nonnegative in the new BFS.
Hence the entering variable for the new BFS is x_{31} and $x_{31} = 10$ and x_{23} is the leaving variable and the new BFS is given by $x_{11} = 20, x_{13} = 20, x_{22} = 30, x_{31} = 10, x_{32} = 20$ (the basic variables) and all the other (nonbasic) variables taking the value 0.
The basic set of cells is given by $\mathcal{B} = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$.
To check for optimality we need to again calculate the $c_{ij} - u_i - v_j$ values for this BFS. Now if we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
$c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
$c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
$c_{32} - u_3 - v_2 = 0$, where $c_{32} = 5$.
Check that $v_1 = 2, v_2 = 6, v_3 = 1, u_2 = -2, u_3 = -1$ and hence check that $c_{23} - u_2 - v_3 = 5 - (-2) - 1 = 6, c_{21} - u_2 - v_1 = 1 - (-2) - 2 = 1$, $c_{12} - u_1 - v_2 = 5 - 0 - 6 = -1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3$.
The following table gives the $c_{ij} - u_i - v_j$ values for the above BFS with
$\mathcal{B} = \{(1,1),(1,3),(2,2),(3,1),(3,2)\}$.

0	−1	0	40
1	0	6	30
0	0	3	30
30	50	20	

So now the entering variable is x_{12} (note that it had left the basis before and is now re entering the basis) The θ-loop is given by $\{(3,1),(3,2),(1,2),(1,1)\}$.
Since $(1,2)$ is the entering cell in the basic set of cells, so if we give $+\theta$ allocation to cell $(1,2)$ (or value of $x_{12} = +\theta$) then $x_{11} = 20 - \theta$, $x_{31} = 10 + \theta$, $x_{32} = 20 - \theta$, so the maximum value of θ is equal to 20 since x_{11}, x_{32} has to be nonnegative in the new BFS and make any one of the variables x_{11} or x_{32} leave the basis. Let x_{32} leave the basis.
To check for optimality we need to again calculate the $c_{ij} - u_i - v_j$ values for this BFS. Now if we take $u_1 = 0$, then solving for $c_{ij} - u_i - v_j = 0$ for the basic cells, that is by solving the 5 equations given below:
$c_{11} - u_1 - v_1 = 0$, where $c_{11} = 2$
$c_{13} - u_1 - v_3 = 0$, where $c_{13} = 1$
$c_{22} - u_2 - v_2 = 0$, where $c_{22} = 4$
$c_{31} - u_3 - v_1 = 0$, where $c_{31} = 1$
$c_{12} - u_1 - v_2 = 0$, where $c_{12} = 5$.
Check that $v_1 = 2, v_2 = 5, v_3 = 1, u_2 = -1, u_3 = -1$ and hence check that $c_{23} - u_2 - v_3 = 5 - (-1) - 1 = 5, c_{21} - u_2 - v_1 = 1 - (-1) - 2 = 0$, $c_{32} - u_3 - v_2 = 5 - (-1) - 5 = 1$, $c_{33} - u_3 - v_3 = 3 - (-1) - 1 = 3$.

Since all the $c_{ij} - u_i - v_j$ values are nonnegative the above BFS is optimal and the optimal value is given by:
$c_{11}x_{11} + c_{12}x_{12} + c_{13}x_{13} + c_{22}x_{22} + c_{31}x_{31} = 2 \times 0 + 5 \times 20 + 1 \times 20 + 4 \times 30 + 1 \times 30 = 270$.