A is symmetric matrix. positive definite and diagonally dominant. ∴ System of linear algebraic eqns can be solved uniquely. A^{-1} exists matrix is invertible.

Thm: Error:- Let $f \in C^4[a,b]$ with the maximum

$$\max_{x\in[a,b]} |f^{(4)}(x)| = C.$$

If $s(x)$ is the unique cubic clamped spline interpolation to $f(x)$, w.r.t the nodes $a = x_0 < x_1 < x_2 \cdots < x_n = b$.

$$\max_{x\in[a,b]} |f(x) - s(x)| \le \frac{5C}{384} \max_{0\le j\le n-1} (x_{j+1} - x_j)^4$$

Numerical Integration.

Quadrature:

Numerical Quadrature: suppose you want to approximate the integral $\int_a^b f(x)\,dx$ with finite interval $[a,b]$. most of such integrals cannot be evaluated explicitly, and ∴ one has to seek some numerical approximation which is faster to evaluate rather than calculating the exact integral. when we have the tabulated values. (x_i, y_i) then we have to use the numerical quadrature to approximate the integral. We can approximate the integral $\int_a^b f(x)\,dx$ by $\sum_{i=0}^{n} w_i f(i)$, where $i = 0 \ldots n$. w_i are called the weights. In other words we replace the integrand $f(x)$ by appropriate interpolating polynomial.

20/2/24 Assume that the integrand $f(x)$ is sufficiently smooth on the interval on the interval ~~[...]~~ $[c,d]$ which ~~contains~~ contains the interval $[a,b]$ { the integral is $\int_a^b f(x)\,dx$. }. Then ~~$f(x) = p_n(x) + f[x_0, x_1, \ldots x_n$~~

$$f(x) = p_n(x_0) + f[x_0 \ldots x_n, x]\, \psi_n(x)$$

$$\psi_n(x) = \prod_{j=0}^{n} (x - x_j)$$

~~$I(f) = \int P_n(x)\,dx$~~

Error: $E(f) = \int_a^b f[x_0 \ldots x_k, x]\,\psi_k(x)\,dx$

In some particular cases the error can be simplified,

Suppose, $\psi_k(x)$ is of one sign on the interval (a,b). by using the mean value thm for integrals, we can express the errors as.

$$E(f) = f[x_0, \ldots x_k, \xi] \int_a^b \psi_k(x)\,dx \qquad \xi \in (a,b)$$

Suppose $f(x)$ is $(k+1)$ times cts diff on the interval (c,d), then.

$$E(f) = \frac{f^{(k+1)}(\eta)}{(k+1)!} \int_a^b \psi_k(x)\,dx. \qquad \eta \in (c,d).$$

Remark: Quadrature formula is exact upto polynomial of degree k.

Even if $\psi_k(x)$ is not of one sign, then also we can do some simplification a particular case of this kind occurs when the ~~integral of~~ $\int_a^b \psi_k(x)\,dx = 0$, in such a case we can use the following ~~the~~ identity.

$$f[x_0, x_1, \ldots x_k, x] = f[x_0, \ldots x_k, x_{k+1}] + f[x_0 \ldots x_{k+1}, x]\,(x - x_{k+1})$$

$$E(f) = \int_a^b f[x_0 \ldots x_k, x_{k+1}]\,\psi_k(x)\,dx$$

$$+ \int_a^b f[x_0 \ldots x_{k+1}, x]\,(x - x_{k+1})\,\psi_k(x)\,dx$$

$$= f[x_0 \ldots x_k, x_{k+1}] \int_a^b \psi_k(x)\,dx \rightarrow 0.$$

$$+ \int_a^b f[x_0 \ldots x_{k+1}, x]\,\psi_{k+1}(x)$$

Suppose x_{n+1} is chosen such that $\psi_{n+1}(x)$

$= (x - x_{n+1}) \psi_n(x)$

is of one sign on the interval $[a, b]$

then if f is $(n+2)$ times ctsly diff

then error will be. $\frac{f^{(n+2)}(\eta)}{(n+2)!} \int_a^b \psi_{n+1}(x)\,dx$.

$\eta \in [c, d]$.

Case 1: $n=0$

$f(x) = f(x_0) + f(x_0, x)\,\psi_0(x)$

$= f(x_0) + f[x_0, x](x - x_0)$.

$I(P_n) = f(x_0)(b-a)$

if $x_0 = a$ $\quad f(a)(b-a) \to$ rectangle rule

$R = I(f) = f(a)(b-a)$

$E(f) = E^R = \frac{f'(\eta)}{1!} \int_a^b (x-a)\,dx$ { E^R = error of rectangle rule.}

$= f'(\eta) \frac{(b-a)^2}{2}$.

we can take $x_0 = b$ also, $\psi(x)$ will be of same sign there also.

error remains same

suppose $x_0 = \frac{(a+b)}{2}$? $\quad \psi_0(x) = (x - x_0)$.

then $\int_a^b \psi_0(x)\,dx = 0$ $\quad \psi_1(x) = (x - x_1)(x - x_0)$

Take $x_1 = x_0$

$\psi_1(x) = (x - x_0)^2$

single sign

Error $= \frac{f''(\eta)}{2!} \int_a^b (x - x_0)^2\,dx$

$= \frac{f''(\eta)(b-a)^3}{24}$. ~~$\frac{(x-x_0)^3}{3} \to \frac{(b-a)^3}{8} \quad a-b$~~

$$I(f) = f\left(\frac{a+b}{2}\right)(b-a)$$

↑ called midpoint rule

Poly of deg 1: $n=1$

$$f(x) = P_1(x) = f(x_0) + f[x_0\, x_1](x-x_0) + f[x_0, x_1, x]\,\psi_1(x)$$

now $\psi_1(x) = (x-x_0)(x-x_1)$

Take $x_0 = a$, $x_1 = b$ then $\psi_1(x)$ will be one sign

$$\text{Error} = \frac{f''(\eta)}{2!} \times \int_a^b (x-a)(x-b)\, dx.$$

$$= f''(\eta) \int_a^b x^2 - (a+b)x + ab.$$

$$\frac{b^2-a^2}{2} - \cancel{(b^2-a^2)} + ab.$$

$$= \frac{f''(\eta)}{2} \cancel{\ast} \quad = -\frac{(b^2-a^2)}{2}.$$

$$= \frac{f''(\eta)}{12}(b-a)^3.$$

$$I(f) = \int_a^b f(x_0) + f[x_0, x_1](x-x_0)\, dx$$

$$= f(a)(b-a) + \frac{f(b)-f(a)}{b-a}\,\frac{(b-a)^2}{2}.$$

$$= \left(f(a) + \frac{f(b)-f(a)}{2}\right)$$

$$= \frac{[f(b)+f(a)]}{2}(b-a)$$

↑ Trapezoidal rule

$n=2$:

$f(x) = P_2(x) + f[x_0, x_1, x_2, x]\,\psi_2(x)$ for distinct nodal points x_0, x_1, x_2

$\therefore n=2$, $\psi_2(x) = (x-x_0)(x-x_1)(x-x_2)$ is not of same sign.

Choose ~~ane~~ $x_0 = a$, $x_1 = \frac{a+b}{2}$, $x_2 = b$

then $\int_a^b \psi_2(x)\, dx = 0$

Error will be $\frac{f^{4}(\eta)}{4!}\int_a^b \psi_3(x)\, dx$

Choose $x_3 = x$, then ψ_3 has same sign always.

$$I(f) = I(p_2) = \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right]$$

Simpson's rule

$$\int_a^b f(x) = \alpha f(a) + \beta f\left(\frac{a+b}{2}\right) + \gamma f(b)$$

take x_0, x_1, x_2 any 3 points to determine unknowns.

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$\underline{n=3}$

$$f(x) = p_3(x) + f[x_0, x_1, x_2, x_3, x]\,\psi_3(x)$$

$x_0 = x_1 = a,\quad x_2 = x_3 = b$

$$\psi_3(x) = (x-a)^2(x-b)^2.$$

$$E(f) = \text{Error} = \frac{1}{4!} f^{4}(\eta) \int_a^b (x-a)^2 (x-b)^2 dx.$$

$$= \frac{f^{4}(\eta)(b-a)^5}{720}$$

$$I(f) = \int_a^b p_3(x) = \cancel{f(x_0) + f[x_0, x_1][x}$$

$$f[a] + f[a,a](x-a) + f[a\,a\,b](x-a)^2 + f[a\,a\,b\,b](x-a)^2(x-b)$$

$$I(f) = f(a)(b-a) + f[a,a]\frac{(b-a)^2}{2} + f[a,a,b]\cdot\frac{(b-a)^3}{3}$$

$$+ f[a\,a\,b\,b]\ \ldots$$

$$I(f) = CI = \frac{(b-a)}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(a) - f'(b)]$$

Till now, we used Newton's divided differences, now we consider the Lagrange interpolating polynomial $p_n(x)$, for any $n \geq 1$

$h = \frac{(b-a)}{n}$, $\quad x_j = a + jh \quad j = 0, \ldots n$.

$$I(f) = \int_a^b f(x)\,dx = I_n(f) = \int_a^b p_n(x)\,dx.$$

$$I_n(x) = \int_a^b \sum_{j=0}^{n} l_{j,n}(x) f(x_j)\, dx = \sum_{j=0}^{n} w_{j,n}(x) f(x_j)$$

$$w_{j,n}(x) = \int_a^b l_{j,n}(x)\, dx.$$

for $n = 1$, $\quad l_0 = \dfrac{x - x_1}{(x_0 - x_1)}$ $\qquad l_1 = \dfrac{x - x_0}{x_1 - x_0}.$

$$w_0 = \int_a^b \frac{x-b}{(a-b)} = \frac{1}{2}\frac{(b-a)^2}{(b-a)} = \frac{1}{2}\left(b-a\right).$$

$$w_1 = \int_a^b \frac{x-a}{(b-a)} \cdot dx = \frac{(b-a)}{2}.$$

$$I_2(x) = \left(\frac{b-a}{2}\right) \times \{ f(a) + f(b) \}$$

$\uparrow$ Trapezoidal rule

Consider the case $n=3$

$$w_0 = \int_a^b l_0(x)\, dx = \int_{x_0}^{x_3} \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} \cdot dx.$$

$x = x_0 + uh \qquad 0 \le u \le 3.$

$$w_0 = -\frac{1}{6h^3} \int (u-1)(u-2)(u-3)\, h^4\, du$$

$$= -\frac{h}{6} \int_0^3 (u-1)(u-2)(u-3)\, du.$$

$$= \frac{3h}{8}.$$

$$w_1 = \int_{x_0}^{x_3} \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\, dx.$$

$$I_n(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$= \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right].$$

$\uparrow$ $3/8^{th}$ rule.

B-NW

i) for even (n) assume that $f(x)$ is $n+2$ times cts diff. on the interval $[a,b]$ $\therefore f \in C^{n+2}([a,b])$

$$E(f) = I(f) - I_n(f) = C_n h^{n+3} f^{(n+2)}(\eta)$$

$$C_n = \frac{1}{(n+2)!} \int_0^n \mu^2(\mu-1)\ldots(\mu-n)\,d\mu.$$

ii) 'n' is odd. Assume f is $(n+1)$ times cts diff.

~~$E(f) = I f$~~

$$E(f) = C_n \cdot h^{n+2} f^{(n+1)}(\eta)$$

$$C_n = \frac{1}{(n+1)!} \int_0^n \mu(\mu-1)\ldots(\mu-n)\,d\mu.$$

↑

Newton - Cotes Formula

Composite Rule : Always it's advisable to divide the interval $[a,b]$ into n sub intervals. $a = x_0 < x_1 < \ldots x_{n-1} < x_n = b$, equidistant $[x_i, x_{i+1}]$ $i = 0$ to $n-1$ and apply all the quadrature formulae.

$$I(f) = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx.$$; Now one can replace the integrand $f(x)$ by polynomial of certain degree. $h = \frac{b-a}{N}$. $x_i = a + ih$.

$f_s = f(a+sh)$

$$\int_{x_{i-1}}^{x_i} f(x)dx = \int_{x_{i-1}}^{x_i} P_{i,k}(x)\,dx.$$

when $n = 0$, $f(x)$ is replaced by constant interpolant.

$$I(f) = \cancel{\frac{(b-a)}{}}\; h \times \left\{ \sum_{i=0}^{n-1} f(x_i) \right\}$$

Error will be $\sum_{i=0}^{n} \frac{f'(\eta_i)}{2} \cdot h^2$.

Here errors will be less, compared to original quadratures.

<u>Gaussian quadrature</u>

'n' deg polynomial integral exact upto $(2n+1)$ deg. { Use Hermite both f and f' }.

Consider the numerical quadrature of integral,

$$\int_a^b f(x)\,dx = \int_a^b w(x) g(x)\,dx.$$

$$I(g) \approx A_0 g(x_0) + A_1 g(x_1) + \dots A_n g(x_n)$$

where A_i's are the weights which are independent of the particular function $g(x)$, so far, we have, chosen the nodal points $x_0, \dots x_n$ all equispaced, and

---

<u>5/3/24</u> consider: $I(g) = \int_a g(x) w(x)\,dx$

$w(x)$ = non-negative integrable function.

$g(x) = \frac{f(x)}{w(x)}$ is smooth.

$w(x) = (x-a)^{\alpha}$.

Take $x_0, \dots x_n$ in (a,b)

$$g(x) = P_n(x) + g[x_0, \dots x_n, x]\,\psi_n(x).$$

$$I(g) = I(P_n) + \int_a^b g[x_0 \dots x_n, x]\,\psi_n(x)\,dx.$$

$$P_n(x) = \sum_{i=0}^{n} g(x_i)\,l_i(x)$$

$$I(P_n) = \int_a^b \sum w(x) g(x_i)\,l_i(x) = \sum g(x_i) \int_a^b w(x) l_i(x)\,dx$$

$$= A_0 g(x_0) + \dots A_n g(x_n).$$

In general, we know that, the error, in this quadrature will be $g^{(n+1)}$ or $g^{(n+2)}$ ... $\therefore$ exact upto poly of deg $\leq n$.

The error $E(g) = I(g) - I(p_n)$

$$= \int_a^b g[x_0, \cdots x_n, x] \underbrace{\psi_n(x)}_{w(x)} \, dx.$$

Consider $\int_a^b \psi_n(x) w(x) \, dx = 0.$

then we know that $E(g) = \int_a^b g[x_0, \cdots x_{n+1}, x] \psi_{n+1} w(x) \, dx$

now again consider $\int_a^b \psi_{n+1}(x) w(x) \, dx = 0$

$$E(g) = \int_a^b g[x_0, \cdots x_{n+2}, x] \psi_{n+2}(x) w(x) \, dx$$

$$\int_a^b \psi_n(x)(x - x_{n+1}) \cdots (x - x_{n+1+i}) w(x) \, dx = 0 \qquad i = 0, \cdots m-1$$

$$E(g) = I(g) - I(p_n) = \int_a^b g[x_0 \cdots x_{n+m+1}, x] \psi_{n+m+1}(x) w(x) \, dx$$

For several choices of $w(x)$, we can find a polynomial, $P_{n+1}(x)$

s.t $\int_a^b P_{n+1}(x) q(x) w(x) \, dx = 0$

where $q(x)$ is a polynomial of deg ~~atmost~~ almost n.

which tells that the polynomial are orthogonal to each other

w.r.t the weights $w(x)$

we can express the polynomial $P_{n+1}(x)$

$\alpha_{n+1}(x - \xi_0) \cdots (x - \xi_n)$ where ξ_i are $n+1$ distinct points in the interval $[a, b]$ which are roots of $P_{n+1}(x)$.

If we set $x_j = \xi_j$, $j = 0$ to n.

and x_{n+j} are arbitrary points in $[a, b]$ $j = 1$ to $n+1$ $(m = n)$

$$q(x) = \frac{(x - x_{n+1}) \cdots (x - x_{2n+1})}{\alpha_{n+1}} \qquad i = 0, \cdots m-1$$

$\therefore$ Error $= E(g) = \int g[x_0, \cdots x_{2n+1}, x] \psi_{2n+1}(x) w(x) \, dx.$

<u>Proof</u>. order to obtain the desired err.

$x_{n+j} = \xi_{j-1}, \; j = 1, \ldots n+1$

$$\psi_{2n+1}(x) = (x - x_0) \cdots (x - x_{2n+1})$$
$$= (x - \xi_0)^2 \cdots (x - \xi_n)^2$$
$$= \left[\frac{P_{n+1}(x)}{\alpha_{n+1}}\right]^2$$

$\psi_{2n+1} = (\;)^2 \therefore$ of same sign.

$\therefore$ By MVT Error $= g[x_0 \ldots x_{2n+1}, \xi] \int_a^b \left(\frac{1}{\alpha_{n+1}} P_{n+1}(x)\right)^2 w(x)\, dx$

$$= \frac{g^{(2n+2)}(\eta)}{(2n+2)!} \frac{S_{n+1}}{(\alpha_{n+1})^2}$$

when $S_{n+1} = \int_a^b P_{n+1}(x)^2 w(x)$

Note: To summarize choose nodal points $x_0, \ldots x_n$ & are zeros of ~~with~~ with itself poly P_{n+1} of deg $n+1$ orthogonal w.r.t the weight function. $w(x)$, $\{ q(x) = P_{n+1}(x) \}$. which tells the gaussian quadrature is exact for poly of deg upto. $(2n+1)$,

Legendre polynomial.

<u>Assume</u>.. Poly P_{n+1}. $\xi_0 = 0$

$w(x) = 1$; $P_1(x) = x$;

$P_2(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right)$ $\quad \xi_{0,1} = \pm\frac{1}{\sqrt{3}}$.

$P_3(x) = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right)$ $\quad \xi_0 = -\sqrt{\frac{3}{5}}, \xi_1 = 0, \xi_2 = \sqrt{3/5}$

$P_{n+1}(x) = x P_n x - \frac{n^2}{4n^2-1} P_{n-1}(x)$

6|9|22

$n=1$

$x_0 = \xi_0 = -\frac{1}{\sqrt{3}}$

$x_1 = \xi_1 = 1/\sqrt{3}$

$\int_{-1}^{1} g(x)\,dx = A_0 g(-1/\sqrt{3}) + A_1 g(1/\sqrt{3})$

$E = c g^{(iv)}(\eta)$.

$A_1 = \int_{-1}^{1} \frac{x - (-1/\sqrt{3})}{(1/\sqrt{3} - (-1/\sqrt{3}))}\,dx = 1$

$A_0 = 1$

$\int_{-1}^{1} g(x)\,dx = g(-1/\sqrt{3}) + g(1/\sqrt{3})$

$E = \frac{1}{135} g^{(iv)}(\eta)$

$f'(x_i) = \frac{f_{i+1} - f_i}{h} + O(h) = \frac{f_i - f_{i-1}}{h} + O(h)$

$= \frac{f_{i+1} - f_{i-1}}{2h} = O(h^2)$

$f_{i+1} = f(x_{i+1}) = f(x_i) + hf'(x_i) + \frac{h^2}{2} f''(\xi)$

$\frac{f_{i+1} - f_i}{h} = f'(x_i) + \frac{h}{2} f''(\xi)$

$\Rightarrow \text{Error} = \left|\frac{f_{i+1} - f_i}{h} - f'(x_i)\right| = O(h)$

$$f_{i-1} = f_i - h f_i' + \frac{h^2}{2} f_i'' - \frac{h^3}{3} f_i''' + \frac{h^4}{4} f^{(iv)}(\xi_i)$$

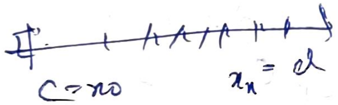
$$f_{i+1} = f_i + h f_i' + \frac{h^2}{2!} f_n'' + \frac{h^3}{3} f_i''' + \frac{h^4}{4} f^{iv} \xi_i$$

$$\frac{f_{i+1} - f_i}{2h} = f_i' + \frac{h^2}{3!} f'''_i(\eta)$$

$$\therefore \text{Error} = \underline{\underline{O(h^2)}}$$

Numerical differentiation

Suppose we are given the tabulated values and we want to determine the derivative of those data. For eg. if given the displacement at different time levels then we can calculate the velocity (f') or the acceleration (f''). Let $f(x)$ be a ctsly diff function on some interval $[c, d]$ which contains $[a, b]$.



$\because$ x_i's are distinct we can approximate $f(x)$ by poly of deg n.

$$f(x) = P_n(x) + f[x_0, \ldots x_n, x]\,\psi_n(x) \quad \text{—— (i)}$$

$$\frac{d}{dx} f[x_0, \ldots x_n, x] = f[x_0, \ldots x_n, x, x]$$

So, $$f'(x) = P_n'(x) + f[x_0, \ldots x_n, x, x]\,\psi_n(x) + f[x_0, \ldots x_n, x]\,\psi_n'(x)$$

Let D denote the derivative: $\cancel{D f(a)}$ $D(f) = f'(a)$, $a \in [c, d]$,

$$E(f) = D(f) - D(P_n)$$

$$= f[x_0, \ldots x_n, a, a]\,\psi_n(a) + f[x_0, \ldots x_n, a]\,\psi_n'(a)$$

$\because$ the ~~error~~ is containing 2 terms, by choosing $a = x_i$ for some i, then $\psi_n(x_i) = 0$, and 1st term vanishes, whereas if we choose a s.t $\psi_n'(a) = 0$, then 2nd term vanishes.

Suppose $\psi_n'(a) = q(a)$

$$q(x) = \frac{\psi_n(x)}{(x-x_i)}$$

$f \in C^{n+1}([c,d])$

Case (i), $a = x_i$ for some i

$$E(f) = \frac{1}{(n+1)!} f^{(n+1)}(\eta) \prod_{\substack{j=0 \\ j \neq i}}^{n} (x-x_j)$$

Case (ii) $\psi_n'(a) = q(a) = 0$, then, in the error, the 2nd term will vanish. Suppose n is an odd number. We can achieve this by placing nodal points symmetrically around a.

$$x_{n-j} - a = a - x_j, \quad j = 0 \ldots (n-1)/2$$

$$(x-x_j)(x-x_{n-j}) = (x-a+a-x_j)(x-a+a-x_{n-j}) = (x-a)^2 - (a-x_j)^2$$

$$\psi_n(x) = \prod_{j=0}^{n} (x-x_j) = \prod_{j=0}^{(n-1)/2} [(x-a)^2 - (a-x_j)^2]$$

$\cancel{\psi_n'(x) =}$

$$E(f) = \frac{1}{(n+2)!} f^{(n+2)}(\eta) \prod_{j=0}^{(n-1)/2} (-(a-x_j)^2)$$

$n = 0$, constant interpolant

deriv $= 0$. $\frac{d}{dx} P_n(x) = D(P_n) = 0$.

$n = 1$ Linear poly

$$P_n(x) = f(x_0) + f[x_0, x_1]\,\psi_1(x)$$

$$\cancel{\frac{d}{dx}} \quad D(f) = D(P_n) = f[x_0, x_1]$$

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(x_1)$

$f(a-h) = f(a) - hf'(a) - \cdots - \frac{h^3}{3!} f'''(x_2)$

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h}$$

$$E(f) = -\frac{h^2}{6} f'''(x)$$

$$f \in C^3(z) \qquad O(h^4)$$

$a = \frac{x_0 + x_1}{2}$

$x_0 = a - h$

$x_1 = a + h$

$h = \frac{x_1 - x_0}{2}$

$f''(a) = \alpha f(a+h) + \beta f(a-h) + \gamma f(a)$

$= \alpha f(a) + \beta f(a+h) + \gamma f(a+2h)$

$= \alpha f(a) + \beta f(a-h) + \gamma (a-2h)$

IVP. $\begin{cases} y'(t) = f(t,y), & t \in [a,b] \\ y(a) = \alpha \end{cases}$

well-posedness

1) Existence 2) Unique 3) Stability / continuous dependence

Perturbed problem

$z' = f(t,z) + \delta(t)$

$z(a) = \alpha + \epsilon$

if $|y(t) - z(t)| \le K(\epsilon)$

then we say ③ ~~consistence~~ condition is said ~~to~~ be satisfied.

if f - is cont. and

~~and $\frac{\partial f}{\partial y}$ is cont.~~

$|f(t,y_1) - f(t,y_2)| \le L(y_1 - y_2)$. $y_1, y_2 \in D$.

+ Lipschitz constant

then ① and ② are satisfied.

$\begin{cases} y' = t + y; & t \in (0,1) \\ y(0) = 1/2 \end{cases}$

$y' - y = t$

$e^{-x} = P(x)$

$IF = \int e^{P(x)\,dx} = e^{-x}$

Step 1: Discretize the domain.

$t_0 = a,\ t_1 = a+h,\ \ldots\ t_n = a + nh.$

Step 2: Replacing the ~~deviation~~ derivative by finite diff eqn.

i) $y' = \dfrac{y_{n+1} - y_n}{h}$ ii) $\dfrac{y_n - y_{n-1}}{h}$ iii) $\dfrac{y_{n+1} - y_{n-1}}{2h}$

fwd diff $\rightarrow$ Euler's Method.

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n).$$

$$y_{n+1} = y_n + h\, f(t_n, y_n) \qquad n = 0, 1, \ldots N-1$$

$$y_0 = \alpha$$

```
Input a, b, N, α.
h = (b-a)/N
t = a : h : b;
y(1) = α.
for i = 1 : N
y(i+1) = y(i) + h * f(t(i), y(i))
  exact(i) = y_exact(t, y).
  error(i) = abs(y(i) - y_exact(i)).
```

$$\text{order} = \ln 2 \left(\frac{E^N}{E^{2N}} \right)$$

E^N = max error by using N grid points.

t	Approx	Exact	Error

N	E^N	order

loglog (N, E^N)

t vs y anat   t vs y(i)   t vs errn.

$$y_{n+1}^{(k+1)} = y_n + hf(t_{n+1}, y_{n+1}^{(k)})$$

$$|y_n^{(n+1)} - y_n^{(n)}| < Tol = 10^{-8}$$

y(1) = α
for i = 1 : N.
  oy(1) = 0.5
  y(i+1) = y(i) + h f(ti, oy(i+1))
  if abs(oy(i+1) - y(i)) < Tol.

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Truncation error = TE
$= \frac{1}{h}(LHS - RHS)$

$$s_{i+1} = \frac{1}{h}\left[\left(y_i + hy_i' + \frac{h^2}{2}y_i'' \dots\right) - (y_i + hy_i')\right]$$

$$= \frac{h}{2!}y'' + \frac{h^2}{3!}y''' + \frac{h^3}{4!}y'''' + \dots$$

$$= O(h)$$

consistency

When Truncation error goes to 0 as $h \to 0$ we say the numerical scheme is Consistent

Stability ⇒ Related to round off error.
If round off error remain bounded we say scheme is stable

Convergence

~~stability~~ $|y(t_i) - y_i| \le c h^p$

as $h \to 0$

p = order.

If the error goes to 0 as $h \to 0$.

Ex: Explicit Euler.

$w_{i+1} = w_i + h f(t_i, w_i)$

$w_0 = \alpha$.

Consistency + stability ⇒ convergence {for any scheme}.

Lemma: 1 $\forall$ $x \ge -1$ and any ~~t~~ $m > 0$, we have $0 \le (1+x)^m \le \exp(mx)$

$\because$ $x \ge -1$ $1+x \ge 0$.

Taylor's expansion: ~~$e^{mx} = 1 + mx + \frac{(mx)^2}{2!} + \cdots$~~

$e^{x} = 1 + x + \frac{x^2}{2!} e^{\xi}$ for $\xi \in (0, x)$. if $x > 1$

$\because$ $1+x \ge 0$

$1+x \le e^{x}$

$\Rightarrow (1+x)^m \le e^{mx}$.

Lemma 2: Let s, and t, be 2 +ve real no's. and $\{a_i\}$ be a sequence satisfying $a_0 \ge -t/s$

& $a_{i+1} \le (1+s) a_i + t, \quad i = 0, \dots k$

then $a_{i+1} \le \exp\left((i+1)s \left[a_0 + \frac{t}{s}\right] - \frac{t}{s}\right)$.

$a_0 + \frac{t}{s} \geq 0.$

$\therefore \left(a_0 + \frac{t}{s}\right) \le e^{x}.$

$$a_{i+1} \le (1+s) a_i + t$$
$$\le (1+s)\{(1+s) a_{i-1} + t\} + t$$
$$\le (1+s)^{i+1} a_0 + t\{1 + (1+s) + (1+s)^2 \dots\}$$
$$= (1+s)^{i+1} a_0 + \frac{t}{s}\left(\frac{(1+s)^{i+1} - 1}{}\right).$$
$$= (1+s)^{i+1} \times \left\{a_0 + \frac{t}{s}\right\} - \frac{t}{s}.$$

By lemma 1

$$a_i \le \exp((i+1)s)\left[a_0 + \frac{t}{s}\right] - \frac{t}{s}$$

Thm. : Convergence of Explicit Euler scheme.

Suppose, f is cts and satisfies the Lipschitz condition with constant L on $D = \{(t, y) : a \le t \le b, -\infty < y < \infty\}$

and $\exists$ a constant M s.t. $|y''(t)| \le M$, $t \in [a, b]$.

Let $y(t)$ be the unique soln of IVP $\begin{cases} y'(t) = f(t, y) \\ y(a) = \alpha \end{cases}$

$w_0, w_1, \dots w_N$

$w_{n+1} = w_n + h f(t_n, w_n) \quad n = 0, \dots N-1$

Thm

$$\text{Error} = |y(t_i) - w_i|$$

$$\leq \frac{hM}{2L}\left[\exp\left(L(t_i - a)\right) - 1\right]$$

$$= ch. \qquad \{c = \text{constant}\}.$$

$\therefore$ $\underline{\text{order} = 1}$

$$y(t_{i+1}) = y(t_i) + h\,y'(t_i) + \frac{h^2}{2!}\,y''(\xi_i)$$

$$w_{i+1} = w_i + h\,f(t_i, w_i)$$

$$y(t_{i+1}) - w_{i+1} = y(t_i) - w_i + h\left[f(t_i, y(t_i)) - f(t_i, w_i)\right] + \frac{h^2}{2}\,y''(\xi_i)$$

$$\leq |y(t_i) - w_i| + h\,|f(t_i, y_i) - f(t_i, w_i)| + \frac{h^2}{2}\,|y''(\xi_i)|.$$

$$= (y(t_i) - w_i) \times \{1 + hL\} + \frac{h^2}{2}M.$$

Now use prev Lemma $s = hL$

$$t = \frac{h^2}{2}M.$$

$$y(t_{i+1}) - w(i+1) \leq \exp\{(i+1)\,hL\} \times \left\{0 + \frac{hM}{2L}\right\} - \frac{hM}{2L}\}.$$

$$t_i = \underline{\underline{a + ih}}. \qquad h = \frac{b-a}{n}.$$

$$\therefore \exp\{((t_{i+1} - a)L\} \left(\frac{hM}{2L}\right) - \frac{hM}{2L}\}$$

$$= \frac{hM}{2L} \times \{\exp\{L(t_{i+1} - a)\} - 1\}$$

Here we assumed no round off error.

But in reality this doesn't happen

$\therefore$ If we consider the round off error then the scheme bounds.

$$w_0 = \alpha + \delta_0 .$$

$$w_{i+1} = w_i + hf(t_i, w_i) + \delta_{i+1} \qquad i = 0, \ldots N-1$$

Error will be here also.
error accumulates.

Assume,

$$|\delta_i| \leq \delta .$$

$$|y(t_i) - w_i| \leq \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left(\exp(L(t_i - a)) - 1\right)$$

$$+ |\delta_0| \exp(L(t_i - a))$$

$$\forall\ i = 0, \ldots N.$$

Prop.
$$\lim_{h \to 0} \left(\frac{hM}{2} + \frac{\delta}{h}\right) = \infty .$$

$$E(h) = \frac{hM}{2} + \frac{\delta}{h}$$

$$E'(h) = \frac{M}{2} - \frac{\delta}{h^2} = 0.$$

$$h^2 = \frac{2\delta}{M} \quad ; \quad h = \sqrt{\frac{2\delta}{M}}$$

if $h < \sqrt{\frac{2\delta}{M}}$, $E' < 0$, $\therefore E$ is dec

if $h > \sqrt{\frac{2\delta}{M}}$, E is inc.

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Solve IVP $\begin{cases} y'(t) = f(t,y), & t \in (a,b). \\ y(a) = \alpha. \end{cases}$

using numerical quadrature

$$\int_{t_n}^{t_{n+1}} y'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds.$$

if use rectangle rule. left limit

$$= f(t_n, y(t_n))\,h.$$

$$\therefore \boxed{w_{n+1} = w_n + h\,f(t_n, w_n)} \quad \leftarrow \text{explicit Euler.}$$

if we use right limit

$$w_{n+1} = w_n + h\,f(t_{n+1}, w_{n+1})$$

$\therefore$ Implicit Euler.

Take $\theta \in [0,1]$.

$$\therefore \int_{t_n}^{t_{n+1}} f(t, y(s))\,ds = \int_{t_n}^{t_{n+1}} (\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1}))\,ds$$

$$T.E = \frac{1}{h}(LHS - RHS).$$

	T.E.
$\theta = 0, 1,$	$O(h)$
$\theta = 1/2$	$O(h^2)$

Runge-Kutta methods (RK method)

The underlying idea of Runge-Kutta method is to take some weighted avg of slope of the tangent in order to increase or enhance the order of truncation error.

$$y(t_{n+1}) = y(t_n) + h\,\Phi(t_n, y(t_n), h).$$

$$y_n = \alpha.$$

where Φ is cts on all of its arguments, from t_n to t_{n+1}.

Starting from some point y_0.

make appropriate choice of Φ s.t order $= O(h^p)$

All schemes require only knowledge of previous point only

i.e t_{n+1} needs only t_n.

All these ~~are~~ are categorized as single-step methods.

Fwd, backwd are → single step.

Central diff → two-step.

multistep method can also be used.

{ Note R-K is single step }

R-stage R-K

$$y_{n+1} = y_n + \Phi(t_n, y_n, h)$$

$$\Phi(t, y, h) = \sum_{r=1}^{R} w_r K_r.$$

$$k_1 = hf(t_n, y_n)$$

$$k_r = h\left(f\left(t_n + hc_r, y_n + \sum_{s=1}^{r-1} a_{rs} k_s\right)\right) \quad r = 2, \dots R$$

$$c_r = \sum_{s=2}^{r-1} a_{rs} \quad r = 2, \dots R$$

We require w_r, c_r, and a_{rs}.

Consider $R = 2$, → Two stage Runge Kutta.

$$y_{n+1} = y_n + k_1 w_1 + w_2 k_2 \quad —(1)$$

$$(2)\begin{cases} K_1 = hf(t_n, y_n) \\ k_2 = hf(t_n + hc_2, y_n + a_{21} k_1) \end{cases}$$

We have 4 unknowns. w_1, w_2, c_2, a_{21}

In order to determine the values for the parameters, we use Taylor series expansion.

$$y(t_{n+1}) = y(t_n + h)$$
$$= y(t_n) + h\,y'(t_n) + \frac{h^2}{2!} y''(t_n) + \frac{h^3}{3!} y'''(t_n) + \cdots$$

$$y' = f(t, y)$$
$$y'' = f_t + f_y\, y' = f_t + f_y \cdot f.$$
$$y''' = f_{tt} + f_{ty} y' + f_{yy} (y')^2 + f_y\, y'' \; \cancel{f_{ty} y'^2}$$
$$= f_{tt} + f_{yy} f^2 + f_y\, y''$$
$$= f_{tt} + f_{yy} f^2 + f_y f_t + (f_y)^2 f.$$
$$= f_{tt} + f_{yy} f^2 + 2 f_{ty} f.$$

$$\therefore\; y(t_{n+1}) = y(t_n) + hf + \tfrac{1}{2} h^2 (f_t + f f_y) + O(h^3)$$

$$f(t_n + hc_2,\, y_n + a_{21} k_1) = f(t_n, y_n) + hc_2\, f_t(t_n, y_n) + a_{21}\, k_1\, f_y(t_n, y_n)$$

$$k_2 = h\,[\, f(t_n, y_n) + hc_2\, f_t(t_n, y_n) + a_{21}\, h\, \underbrace{f(t_n, y_n)}_{k_1}\, f_y(t_n, y_n)$$

$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2$$

$$y(t_{n+1}) = y(t_n) + h y'(t_n) + \frac{h^2}{2!} y''(t_n) + \frac{h^3}{3!} y'''(t_n)$$

$$= y(t_n) + w_1 h f + w_2 h\,[\, f(t_n, y_n) + h c_2 f_t + a_{21} h f f_y]$$

$$y(t_{n+1}) = y(t_n) + h f + \frac{h^2}{2} \{ f_t + f_y f \} + \frac{h^3}{3} \times [f_{tt} + f_{yy} f^2 + 2 f_y f_t + (f_y)^2 f + f f_{ty}]$$

=

<u>Comparing coefficients :-</u>

$w_1 + w_2 = 1$ → coeff of f

$c_2 w_2 = 1/2$ ⇒ coeff of f_t

$a_{21} \times w_2 = \frac{1}{2}$ → coeff of $f f_y$

∴ we have only 3 eqn we can choose c_2 as arbitrary. ≠ 0.

$w_2 = \frac{1}{2c_2}$; $a_{21} = c_2$.

$w_1 = 1 - w_2$.

$y_{n+1} = y_n + w_1 h f + w_2 h (f(t_n + c_2 h, y_n + a_{21} k_{11}))$

T.E =

$y_{n+1} = y_n + h f_n + \frac{h^2}{2}(f_t + f_n f_y)$

$+ \frac{c_2 h^3}{4} [f_{tt} + 2 f_n f_{ty} + f_n^2 f_{yy})$

$(LHS - RHS) = O(h^2)$

$T.E = \frac{1}{h}$

Butcher's table

$$\begin{array}{c|cc} c_2 & a_{21} & \\ \hline & \omega_1 & \omega_2 \end{array}$$

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$ ← Improved tangent.

$$\begin{array}{c|cc} 2/3 & 2/3 & \\ \hline & 1/4 & 3/4 \end{array}$$ ← ~~Improve~~ Optimal method.

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$ ← Euler-Cauchy.

$R = 3, 4.$

missed

1/4/23

$$y_n = c_1\beta_1^n + c_2\beta_2^n + \dots c_n\beta_n^n$$

$y' = \lambda y, \; t < 0.$ $\quad\quad$ $\beta_i \to$ Some value

$y(0) = y_0.$ $\quad\quad$ $|\beta_i| \leq 1$

$y(t) = y e^{\lambda t}$ $\quad\quad$ $\forall\, i = 1$

$$y_{n+1} - y_n = \frac{\lambda h}{24}\left[55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3}\right]$$

$$\beta^4 - \beta^3 - \frac{\lambda h}{24}\left[55\beta^3 - 59\beta^2 + 37\beta - 9\right] = 0.$$

$$\rho(\beta) + \lambda h\,\sigma(\beta) = 0.$$

$$\rho(\beta) = \beta^4 - \beta^3 = \beta^3(\beta - 1)$$

$$\sigma(\beta) = -\frac{1}{24} \times (\quad\quad).$$

as $h \to 0$. $\lambda h\,\sigma(\beta) \to 0.$ $\quad$ means all roots lie in $|z| \leq 1$ and root of modulus 1 is simple

$\rho(\beta) = 0$ when $\beta = 1$

$\therefore$ $\boxed{\text{Adam's Bashforth is strongly stable.}}$

Milne's scheme :-

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(f_{n+1} + 4f_n + f_{n-1}\right)$$

$y' = \lambda y.$ $\quad\quad$ $\sigma(\beta) = \frac{-1}{3\lambda}\left(\frac{f_{n+1} + 4f_n + f_{n-1}}{\beta}\right)$

$\cancel{\sigma(\beta) = -\frac{1}{3\lambda}(}$

$$\rho(\beta) = \beta^2 - 1$$

$$\beta = \pm 1$$

Round off error

$y_0 = \cancel{\alpha} \alpha \pm \epsilon$ ← Effect of Round-off error.

↑ should remain bounded; should not accumulate.

Error $= y(t) - y_n(t) = |y(t) - \tilde{y}_n + \tilde{y}_n - y_n|$

T.E = truncation error

R.E → Roundoff error.

$\leq \underbrace{|y(k) - \tilde{y}_n|}_{T.E} + \underbrace{|\tilde{y}_n - y_n|}_{R.E}$

$\leq C h^p$ $p > 0$.

↑ determined by taylor expansion.

C is called consistency constant.

For system of eqns. :-

$y_1' = f_1(t_1, t_2, y)$ ← system of first order ODE!

$y_2' = f_2(t_1, y_1, y_2)$ initial conditions $y_1(0) = \alpha$, $y_2(0) = \beta$.

$\bar{y}' = \bar{f}(t, \bar{y})$

$y''(t) + p(t)y'(t) + f(t)y(t) = 0$; $y(0) = \alpha$ ← Initial condition for y. $y'(0) = \beta$. ← Initial condition for v.

↑ 2nd order

Variables: v and y.

$y' - v = 0$

$v' + p(t)v + f(t)y = 0$. ← converted to first order

B.V.P

$u''(x) + p(x)u'(x) + f(x)u(x) = r(x)$ $x \in (0,1)$

$u(0) = \alpha$, $u(1) = \beta$

← will become initial condition for new variable

Assume $u'(0) = \epsilon$ and then solve using IVP methods,

If $|u_n - \underbrace{u(1)}_{\beta}| < Tol$ accept ϵ

This method is called Shooting Technique
↙
converting BVP into suitable IVP. with some initial condition $u'(0) = \epsilon$.

Merit of shooting technique, we can solve non-linear differential eqns. becoz. all the methods studied for IVP's can deal with non-linear.

$\boxed{\text{Finite Difference Schemes}}$

Step 1: Discretize the domain.

$0 = x_0 < x_1 \cdots < x_{n-1} < x_n < x_{n+1} = 1$

Step 2: Replace the derivatives by finite differences quotients.

$$u'(x) = \frac{u_{i+1} - u_i}{h} = \frac{u_i - u_{i-1}}{h} \Bigg\} O(h)$$

$$= \frac{u_{i+1} - u_{i-1}}{2h} \Big\} O(h^2)$$

$$u''(x) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \rightarrow O(h^2)$$

∴ We obtain 3 different difference schemes to solve the ~~BVB~~ BVP.

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + p(x)\frac{U_{i+1} - U_{i-1}}{2h} + q(x)U_i = r_i$$

$U_0 = \alpha \qquad U_{N+1} = \beta. \qquad 1 \le i \le N.$

~~collect coeffs of~~

$$\left(\frac{1}{h^2} + \frac{P_i}{2h}\right) U_{i+1} - \left(\frac{2}{h^2} - f(x_i)\right) U_i + \left(\frac{1}{h^2} - \frac{P(x_i)}{2h}\right) U_{i-1} = r_i$$

$i=1,$

$$\left(\frac{1}{h^2} + \frac{P_1}{2h}\right) U_2 - \left(\frac{2}{h^2} - f_1\right) U_1 + \cancel{\left(\frac{1}{h^2} - \frac{P_1}{2h}\right)\alpha}$$

$$= r_1 - \left(\frac{1}{h^2} - \frac{P_1}{2h}\right)\alpha$$

$i=2,$

$$(\quad) U_3 - (\quad) U_2 + (\quad) U_1 = \alpha,$$

$\vdots$

$i=2$ to N. we have 3 entries.

$i=N$.

$$\left(\frac{1}{h^2} + \frac{P_N}{2h}\right)\beta - \left(\frac{2}{h^2} - f_1\right) U_{N-1} + (\quad) U_{N-2} = r_N$$

$U_1 \dots U_N \rightarrow N$ unknowns.

N eqns.

+ Tridiagonal matrix $\rightarrow$ Sparse matrix

$\boxed{AU = r.}$ —— (i)

A is invertible, then we obtain unique soln.

2/4/24

$\boxed{U(0)=\alpha, \text{ and } U(1)=\beta} \rightarrow$ Dirichlet Boundary Condition.

Neumann B.C $- U'(0)=\alpha, \quad U'(1)=\beta$

Can be replaced by.

$$-\left(\frac{U_1 - U_0}{h}\right) = \alpha \qquad \frac{U_{N+1} - U_N}{h} = \beta.$$

$T \cdot E = O(h)$

$$T.E = -\frac{h^2}{24} \times \left[U^{(4)}(\xi_i) - 2p_i\, U^{(3)}(\eta_i) \right].$$

& need $U \in C^4(\Omega)$

In Dirichlet $O(h^2)$

In Neumann $O(h)$ → ∵ first order approximation of derivatives

Robin-type / mixed B.C.

$$U(0) - U'(0) = \alpha$$

$$U(1) + U'(1) = \beta.$$

$$V_0 - \frac{V_1 - V_0}{h} = \alpha.$$

$$V_{N+1} - \frac{V_{N+1} - V_N}{h} = \beta.$$

$$\left(1 + \frac{1}{h}\right) V_0 - \frac{1}{h} V_1 = \alpha$$

$$\frac{1}{h} V_N + \left(1 - \frac{1}{h}\right) V_{N+1} = \beta.$$

make matrix modifying basically

Inputs: a, b, N, α, β, @ p, q, r.

done in lab

```
A(1,1)
A(1,2)
b(1) =
for i = 2 : N-1
    A(i,i) =
    A(i,i-1) =
    :
    A(N+1, N) =
    A(N+1, N+1)
    b(N+1) =
```

$$-U''(x) + f(x, U) = 0 \qquad x \in (0,1).$$

$$U(0) = \alpha, \; U(1) = \beta.$$

Suppose $L = \frac{d^2}{dx^2} + p\frac{d}{dx}\, q\, I.$

$L(\nu_1 v_1 + \nu_2 v_2) = \nu_1 L(v_1) + \nu_2 L(v_2)$ → L is a linear operator.

Non-linear, Quasi-linear, Semilinear

$-u''(x) + p(x)\,u' + q(x)\,u = r(x)$

Depends if non-linearity occurs in which term.

if $p(x) \equiv 0$:

$-u''(x) + q(x)\,u = r(x)$

$Au = b$ A → is symmetric, p.d. unique soln.

$\Rightarrow G = x^T A x - x^T b$.

$\Rightarrow \nabla G = 0$ if p.d matrix, minima can be found. {convex function}

For $-u''(x) + f(x,u) = 0$ if $f(x,u)$ is non-linear. we obtain system of non-linear eqns.

$$u_{xx} \left\{ -\frac{1}{h^2} u_{i-1} + \frac{2}{h^2} u_i - \frac{1}{h^2} u_{i+1} = f(x_i, u_i). \quad \text{— (i)} \right.$$

To solve system of non-linear algebraic eqns (i), we can use fixed point iteration, etc.

$u^{(0)}$ = initial approx.

$(x, u^{(m)})$

$f(x,u) = f(x, u^{(m)}) + (u - u^{(m)})\, f_u(x, u^{(m)})$ + higher order term.

$-u_{xx} + \left(f(x, u^{(m)}) + (u - u^{(m)})\, f_u(x, u^{(m)}) \right] = 0.$

System of linear ODE $u^{(m+1)}$

$\Rightarrow -u_{xx}^{(m+1)} + f_u(x, u^{(m)})\, u^{(m+1)} = -f(x, u^{(m)}) + f_u(x, u^{(m)})\, u^{(m)}$

to get $u_{xx}^{(m)}$ from u use the upper formula

$u(0) = \alpha, \quad u(1) = \beta.$ $m \geq 0$

first loop for n

inner loop our nodal points i.e n.

for $i = 1 : N+1$ & some initial value

old_U(i) = 0.5

end.

for $i = j : \max - 2t$.

A(1,1)

A(i,i)

U = A \ b.

if abs(old_U - U) < Tol $= 10^{-8}$

old_U = U.

Stability Analysis.

$$U''(x) + kU'(x) = 0$$

consider the 2nd order ODE subject to suitable B.C.s.

$$|k| \gg 1.$$

For $U'' \rightarrow$ we have ~~fwd, bckwd schemes D^+, D^-~~ central scheme D^0.

For U' we have fwd, bwd, central, D^+, D^-, D^0.

i) $$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + k\frac{U_{i+1} - U_{i-1}}{2h} = 0.$$

ii) $$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + k\frac{U_{i+1} - U_i}{h} = 0$$

iii) $$'' + k\frac{U_i - U_{i-1}}{h} = 0.$$

i) $\dfrac{\beta^2 - 2\beta + 1}{h^2} + \dfrac{k}{2h}(\beta^2 - 1) = 0.$

$\beta^2 \times \left\{ \dfrac{1}{h^2} + \dfrac{k}{2h} \right\} - \dfrac{2}{h^2}\beta + \left(\dfrac{1}{h^2} - \dfrac{k}{2h} \right) = 0.$

↑ get β value.

$$\beta = \frac{\frac{2}{h^2} \pm \sqrt{\frac{4}{h^4} - 4 \times \left\{ \frac{1}{h^4} - \frac{k^2}{4h^2} \right\}}}{2 \times \left\{ \frac{1}{h^2} + \frac{k}{2h} \right\}}$$

$$= \frac{\frac{2}{h^2} \pm \frac{k}{h}}{2 \times \left\{ \frac{1}{h^2} + \frac{k}{2h} \right\}}$$

$$= \underline{\underline{1}}, \quad \underline{\underline{\frac{2 - kh}{2 + kh}}}$$

$U_n = C_1 \beta_1^n + C_2 \beta_2^n$

$= C_1 + C_2 \left(\dfrac{2 - kh}{2 + kh} \right)^n.$

~~Suppose~~ we know actual soln is:

$U(x) = A + Be^{-kx}.$

Consider the case when k is +ve and the exact solution will go to 0 as $x \to \infty$.

~~$A = C_1$~~

for k +ve and too large

$\left(\dfrac{2 - kh}{2 + kh} \right)^n = (-1)^n.$ ← oscillation we need to avoid.

we need $2 - kh > 0$ so that inside quantity is +ve ~~and less than 1.~~

i.e $\boxed{h < 2/k}$

For case when $k < 0$, and too large or zero.

same restriction.

$h < \dfrac{2}{-k}$ i.e $h < \dfrac{2}{|k|}$

3/4/24

$u'' + ku' = 0$

for fwd difference

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + k\,\frac{v_{i+1} - v_i}{h} = 0.$$

roots $\beta_1 = 1$, $\beta_2 = \frac{1}{1+kh}$.

$$v_n = c_1(1)^n + c_2\left(\frac{1}{1+kh}\right)^n, \quad k>0,\ h>0.$$

Padé approx.

for $k>0$, no problem $n \to \infty$, $v_n \to c_1$
∴ No restriction

if $k<0$.
we need
$1 + kh > 0$
$\Rightarrow \boxed{h > \frac{1}{|k|}}$

for Backward differ.

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + k\left(\frac{v_i - v_{i-1}}{h}\right) = 0.$$

$\beta_1 = 1$, $\beta_2 = 1 - kh$,

$$v_n = c_1 + c_2(1-kh)^n.$$

if $k<0 \to$ no restriction.

if $k>0$, $h < \frac{1}{k}$.

called Upwind schemes ∴ → if $k>0$; fwd, $k<0 \to$ backward

but central diff is difference of both ∴ restricted

it goes against the flow.

One has to be careful while approximating the ~~conductive~~ ~~term U^2,~~ convective term U^2.

$AU_{tt} + BU_{tx} + CU_{xx} = 0$

if $B^2 - 4AC > 0$ hyper

$= 0$ para

< 0 ellipse

Unsteady time dependent

$U_{tt} = c^2 U_{xx}$ ← hyperbolic

$U_t - U_{xx} = 0$ ← Parabolic

$U_{tt} + a^2 U_{xx}$ ← Elliptic.

Steady state → $\Delta U = U_{xx} + U_{yy} = 0$ $f(x,y)$

$U_t \cdot (U_{xx} + U_{yy}) = 0$.

↳ Parabolic PDE.

$U_t - U_{xx} = 0$; $(x,t) \in (0,1) \times (0,T)$ } IBVP.

$t_0 \to U(x,0) = \phi(x)$

$U(0,t) = f_1(t)$, $U(1,t) = f_2(t)$

Boundary value

$U(x,t) = X(x)\,T(t)$

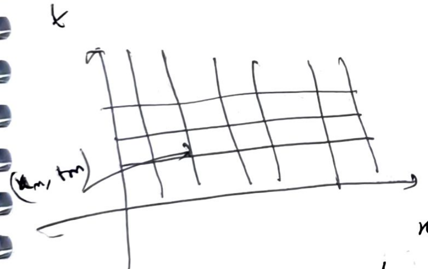
$U_m^n = U(x_m, t_n)$

Step 1: Discretize the domain

~~domain~~

$\Delta t = k = \frac{1}{N}$

$\Delta x = h = \frac{1}{M}$.



Step 2: Derivatives by Finite difference

$$U_t\Big|_{x_m, t_n} = \left\{ \begin{array}{l} \dfrac{U_m^{n+1} - U_m^n}{k} \\ \dfrac{U_m^n - U_m^{n-1}}{k} \\ \dfrac{U_m^{n+1} - U_n^{n-1}}{2k} \end{array} \right.$$

$U_m^0 = \phi(x_m)$

$U_0^n = f_1(t_n)$

$U_M^n = f_2(t_n)$

$$u_{xx}\Big|_{(x_m, t_n)} \approx \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}$$

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}$$

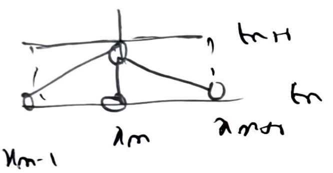
$$\Rightarrow \quad U_m^{n+1} = \lambda U_{m-1}^n + (1-2\lambda)U_m^n + \lambda U_{m+1}^n$$

$$0 \le n \le N-1$$
$$1 \le m \le M-1$$

$\boxed{n=0}$

↳ Obtain from boundary.

then use this



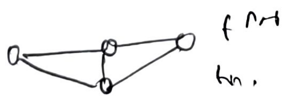
This is an explicit scheme

FTCS → Forward time - Central space.

$$\cancel{\frac{U_m^{n} - U_m^{n-1}}{k} = \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}}$$

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2}$$

$$\Rightarrow \quad -\lambda U_{m-1}^{n+1} + (1+2\lambda)U_m^{n+1} - \lambda U_{m+1}^{n+1} = U_m^n \quad \text{—(i)}$$



we have to solve system of liner algebraic eqns.

This is implicit euler, central scheme

(BTCS) → Backward time - Central space

b/oz of implicit nature, one has to solve system of linear algebraic eqns. at each time level. This is computationally expensive compared to FTCS.

Truncation error of this scheme $= \frac{1}{a}$ (LHS - RHS).

Calculate T.E.

If we average

$$U_m^{n+1} - \cancel{\frac{U_m^{n+1}}{2k}} = \cancel{U_{m+1}^n - 2U_m^n + U_{m-1}^n} \quad \cancel{\frac{}{h^2}}$$

$$\lambda U_{m+1}^n - \cancel{2\lambda U_m^n} \quad U_m^n + (1+\lambda) U_{m-1}$$

$$\Rightarrow U_m^{n+1} = \quad (1-2\lambda) U_m^n + \lambda U_{m+1}^n$$

$\frac{1}{2}$ (BE's + FRS)

$$U_m^{n+1} = \quad (\quad U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}) = .$$

$$U_m^{n+1} - \lambda (U_{m+1}^{n+1}$$

$$-\frac{\lambda}{2} U_{m-1}^{n+1} + (1+\lambda) U_m^{n+1} - \frac{\lambda}{2} U_{m+1}^{n+1}$$

$$= \frac{\lambda}{2} U_{m-1}^n + (1-\lambda) U_m^n + \frac{\lambda}{2} U_{m+1}^n$$

∴ This is an implicit scheme.

At each level we have to solve system of. ~~time to~~ liner algebraic eqns.

$Ax = b$ { ∵ RHS is known}

Here we get $O(\Delta t^2 + \Delta x^2)$

Crank-Nicolson scheme

Earlier schemes had $O(\Delta t + \Delta x^2)$

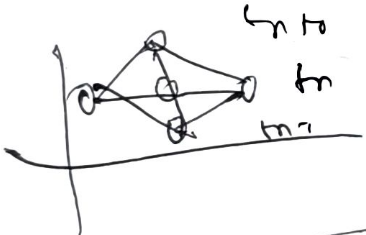
All are
Two level schemes : $n+1$ th requires n th only

↑ one step

Utor Scheme

Use central difference : →

$$U_m^{n+1} = U_m^{n-1} + \delta U_{m-1}^{n} - 2\delta U_m^{n} + \delta U_{m+1}^{n}$$



Elliptic BVP's.

$\Delta u = f(x,y)$, Ω ← domain

boundary condition → $u(x,y) = g(x,y) =$ $\partial\Omega$ ← boundary $= \Gamma$.

Step 1 : discretization of domain.

$h = (b-a)/n$

$k = (d-c)/m$ $\quad h = k,\ m = n.$

Step 2: Replacing the derivatives by difference quotients.

$$\frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2} + \frac{U_{ij+1} - 2U_{ij} + U_{ij-1}}{h^2} = f_{ij}$$

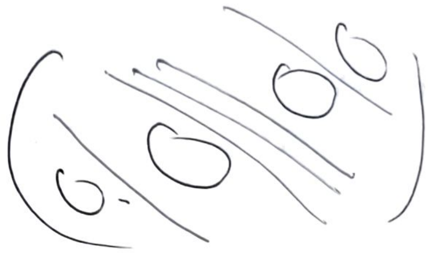
$1 \le i,j \le m.$

$$U_{i+1,j} - 4U_{ij} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1} = h^2 f_{ij}$$

for $i,j = 1:5$

$N = 4.$

~~Replace~~ $p = (i-1)(N+1) + j.$



Represent U_{ij} by U_p.

$$AU = F$$

A = stiffness matrix.

5 non-zero diagonal entries.

To solve this 25 variables

∴ 25×25 matrix.

A = symmetric, +ve definite, diagonally dominant, irreducible

diagonally dominant means $|a_{ii}| \geq \sum_{j=1,\, j\neq i} |a_{ij}|$

9/4/24

$$U_{i\,j\pm 1} = U_{ij} \pm h\frac{\partial u}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial y^2} \pm \frac{h^3}{3!}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial y^4}$$

we need $u \in C^4(\Omega)$

~~$U_{i\pm1 j} =$~~

$$\frac{U_{i+1 j} - 2U_{ij} + U_{i-1 j}}{h^2} + \left(\frac{U_{ij+1} - 2U_{ij} + U_{ij-1}}{h^2}\right) = f_{ij}$$

$$\Rightarrow U_{ij}^{(l+1)} = \frac{1}{4}\left(U_{i+1 j}^{(l)} + U_{i-1 j}^{(l+1)} + U_{ij+1}^{(l)} + U_{ij-1}^{(l+1)}\right) - \frac{h^2}{4} f_{ij}$$

Poisson solvers

Jacobi-Iteration

changed to $l+1$, then Gauss Seidel.

Stopping criteria:-

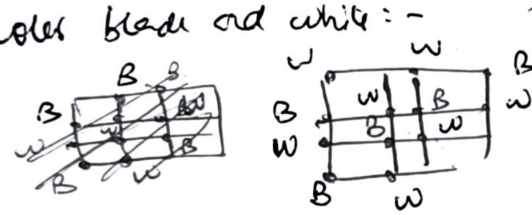
$$\left| U_{ij}^{(l+1)} - U_{ij}^{(l)} \right| < Tol = 10^{-8}.$$

(Any $f_{ij} > 0$)

$U_{ij} <$ Avg of neighbours. $\therefore$ ~~max~~ max attained at boundary.

called maximum principle.

Color black and white:-



Iterate over black first then white and so on.

well-suited for parallel computing.

For any mesh function V_{ij} defined on Ω^h discrete domain

the following maximum principle holds true.

$$LU \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y).$$

{Discretized approximation}

$L^h V_{ij} \geq 0 \quad \forall\, i,j$

max over boundary $V_{ij} \geq$ max over Ω^h ~~interior~~ V_{ij}

Proof: Assume the contrary; i.e. maximum is at some interior point.

(i_0, j_0). $\quad V_{i_0 j_0} = \max_{\Omega^h} V_{ij}$ and $V_{i_0,j_0} > \max_{\Gamma^h} V_{ij}$

Given that {Γ^h denotes boundary of domain}
$L^h v_{ij} \geq 0$.

$V_{i_0,j_0} \leq \frac{1}{4} \times$ Neighboring.

but $V_{i_0,j_0} > = \max$ of neighboring.

$\therefore V_{i_0,j_0} =$ all neighbors.

if any neighbor on boundary $\to$ contradiction else repeat process till boundary.

if max ~~[illegible]~~ exists in the interior then function remains constant all over the domain. Similarly we can show the

<u>minimum principle</u>

if $L^h v_{ij} \leq 0$, then $\min_{\Gamma^h} v_{ij} \leq \max_{\Omega^h} v_{ij}$.

<u>Corollary</u>: Uniqueness of soln. – ~~[illegible]~~ $L^h v_{ij} = f_{ij}$ then

Let v_{ij} and v_{ij} be two soln

$v_{ij} = v_{ij}$

consider $w_{ij} = v_{ij} - v_{ij}$ $\quad L w_{ij} = 0, \geq 0$

at boundary $v_{ij} = v_{ij} = g_{ij}$ $\quad \therefore w_{ij} = 0$

Applying both max and min principle. everywhere value ≤ 0 and ≥ 0

$\Rightarrow$ <u>value $= 0$</u>

~~Upper bound for~~

The discrete solution V_{ij} satisfies, the following bound.
$\max_{\Omega^h} |V_{ij}| \leq \max_{\Gamma^h} |V_{ij}| + \frac{1}{2} \max_{\Omega^h} |L^h v_{ij}|$
mod Ω^h is interior.

$$\Theta_{ij} = (ih)^2/2$$

$$\max_{\Omega^h} |\Theta_{ij}| = \frac{1}{2} \qquad \{ \text{ if } \Omega = (0,1)^2$$

$$L^h \Theta_{ij} = \frac{\frac{(i+1)^2h^2}{2} - \frac{2i^2}{h^2} + \frac{(i-1)^2h^2}{2}}{h^2}$$

$$+ \quad ($$

$$= 1.$$

$M = \max_{\Omega^h} |L^h v_{ij}|$ and define two mesh function w_{ij}^+ and w_{ij}^- s.t $w_{ij}^{\pm} = \pm v_{ij} + M\Theta_{ij}$

$$L^h w_{ij}^{\pm} = L^h \pm v_{ij} + M L^h (\Theta_{ij})$$

$$= \cancel{L^h \pm v_{ij} \pm M} \quad \underbrace{\pm M}_{\text{max value}} + M$$

$$\geq 0.$$

$$\max_{\Omega^h} w_{ij}^{\pm} \leq \max_{\Gamma^h} w_{ij}^{\pm}$$

$\pm$

Missed lecture → 11/4/23

16/9/23.

✓ Hyperbolic PDE → difficult to solve.

$v_t + a v_x = 0$

To study the stability, we DFT ← Discrete fourier transform

We know that the fourier transform of any real ~~fut~~ valued function $v(x)$ is:

$$\hat{v}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} v(x)\, dx. \quad \text{—} (1)$$

Inverse:

$$v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{v}(\omega)\, d\omega. \quad \text{—} (2)$$

The fourier inverse formula ~~transform~~ expresses that the function as a superposition of the waves, given by $e^{i\omega x}$, with different amplitude ~~amplitude code~~ $(\hat{v}(\omega))$. In a similar way we can define the fourier transform of any mesh / grid function?

$h\mathbb{Z} = \{ hm : m \in \mathbb{Z}$ integers.

Suppose v is a grid function defined on integers.

(3) $$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} v_m. \qquad \xi \in [-\pi, \pi]$$

$$\hat{v}(-\pi) = \hat{v}(\pi)$$

(4) $$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi}\, \hat{v}(\xi)\, d\xi$$

In finite difference, it is natural to start with some grid function v_m and formula (4) gives the transform. Let h be ~~steps~~ the step size. (uniform). Then formula (3) and (4) can be defined

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{m=\infty} e^{-imh\xi}\, v_m h \quad \text{—} (5)$$

$\xi \in [-\pi/h, \pi/h]$

$$V_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi_1} \hat{v}(\xi_1)\, d\xi_1 \quad -(6)$$

diff. $v(x)$ w.r.t x & formula (2).

$$\frac{\partial v(x)}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega\, e^{i\omega x}\, \hat{v}(\omega)\, d\omega \quad -(7)$$

From this we can conclude that the fourier transform of $\frac{\partial v}{\partial x}$ is

$\underline{i\omega \hat{v}(\omega)}$. i.e $\widehat{\left(\frac{\partial v(x)}{\partial x}\right)} = i\omega \hat{v}(\omega)$ -8

This relation shows that under the fourier transform the operation of the differentiation is converted to operation of multiplication

Consider the first order hyperbolic PDE,

$U_t + aU_x = 0.$

Here we apply the transform only for the spatial variable x,

$\hat{v}_t(t,\omega)$ $\hat{U}_t = -ia\omega \hat{U}$. this is

an ODE in time. The soln of ODE $\hat{U}(t,\omega) = e^{-ia\omega t}\, \hat{v}(0,\omega)$

$\boxed{\text{PDE, converted to ODE}}$

Now make inverse transform to get original soln of PDE.

Consider the following difference scheme.

FTBS

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_m^n - U_{m-1}^n}{h} = 0$$

$$U_m^{n+1} = (1-a\lambda)U_m^n + a\lambda U_{m-1}^n \quad -(9)$$

$$\lambda = \frac{k}{h}.$$

Keep n constant, applying fourier inversion formula given in (6) we get,

$$U_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi_1} \hat{U}^n(\xi_1)\, d\xi_1 \qquad \text{— (10) —}$$

Using (10) in (9),

$$\frac{1}{\sqrt{2\pi}} \Big\{ (1-a\lambda) \int_{-\pi/h}^{\pi/h} e^{imh\xi_1} \hat{U}^n(\xi_1)\, d\xi_1 + a\lambda \int_{-\pi/h}^{\pi/h} e^{i(m-1)h\xi_1} \hat{U}^n(\xi_1)\, d\xi_1 \Big\}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} e^{imh\xi_1} \hat{U}^{n+1}(\xi_1)\, d\xi_1 = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi_1} \Big[(1-a\lambda) + a\lambda e^{-ih\xi_1} \Big] \hat{U}^n(\xi_1)\, d\xi_1 \Big\}$$

The integrands are same (∵ the fourier transform is unique.)

$$\Rightarrow \hat{U}^{n+1}(\xi_1) = \Big((1-a\lambda) + a\lambda e^{-ih\xi_1} \Big) \hat{U}^n(\xi_1) \qquad \text{— (11).}$$

$$= g(h\xi_1)\, \hat{U}^n(\xi_1)$$

The discrete fourier transform helps in converting the 2 different indices in difference eqn to one.

This formula shows that advancing the solution of the scheme by 1 time step, is equivalent to multiplying the fourier transform of the solution by the amplification factor. $g(h\xi_1)$: this is called amplification factor because its magnitude is the amount that the amplitude of each freq. in the solution given by $\hat{U}^n(\xi_1)$ is amplified in advancing the soln by 1 step. From this recurrence relation we can obtain $\hat{U}^n(\xi_1) = (g(h\xi_1))^n \hat{U}^0(\xi_1)$

Now, we have to evaluate the magnitude of amplification factor:

say $h\xi_1 = \theta$.

$g(\theta) = (1-a\lambda) + a\lambda e^{-i\theta}$.

$\cancel{g(\theta)\,R^2\theta} = (1-a\lambda) + a\lambda(\cos\theta + i\sin\theta)$.

magnitude $g(\theta))^2 = \left((1-a\lambda) + a\lambda\cos\theta\right)^2$

$+ \quad (a\lambda)^2 \sin^2\theta$

$(1-a\lambda)^2 + a^2\lambda^2\cos^2\theta + 2a\lambda\cos\theta(1-a\lambda)$
$+ a^2\lambda^2\sin^2\theta$.

$= 1 + a^2\lambda^2 + a^2\lambda^2 + 2a\lambda\cos\theta$
$- 2a^2\lambda^2\cos\theta$.

$\sigma < 1$

When $a^2\lambda^2 + a\lambda\cos\theta - a^2\lambda^2\cos\theta < 0$.

$\Rightarrow a\lambda + \cos\theta - a\lambda\cos\theta < 0$.

$\Rightarrow a\lambda \times + \cos\theta\{1-a\lambda\} < 0$.

$\Rightarrow \cos\theta = \dfrac{a\lambda}{a\lambda-1}$

$\therefore$ Need, $a\lambda > 0$, $a\lambda - 1 < 0$

$\lambda = \underline{(0, 1/a)}$ or $a\lambda < 0$ $a\lambda - 1 > 0$.

Missed class $U_t + a u_x = 0$.

$u_m^{n+1} - u_m^n \over k$ $+ \dfrac{u_{m+1}^n - u_m^n}{h}$.

$\cancel{\text{If}}\ R = -1$

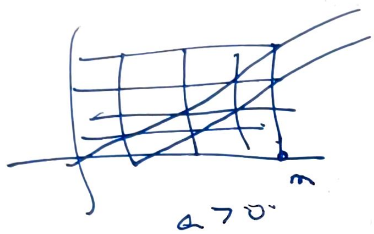
$R = \dfrac{ak}{h}$, $\quad U_m^{n+1} = U_m^n - R(u_{m+1}^n - u_m^n)$

$\Rightarrow U_m^{n+1} = U_{m+1}^n$

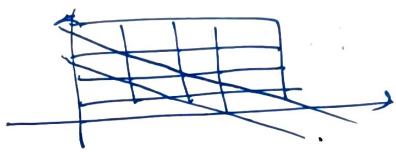
or if R lies between -1 to 0, then we are atleast ~~starting~~ reaching in the correct direction

Similarly for the $a > 0$ case,

$\boxed{0 < \lambda \le 1/2}$ ← for stability.
↑ CFL ← Courant – Friedrichs – Lewy. condition.

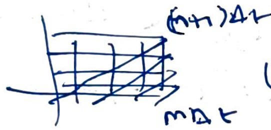


$a > 0$



$a < 0$.

FTBS.



$$\frac{U_m^{n+1} - U_m^n}{t} \quad \frac{}{x}$$

$U_m^{n+1} = U_m^n - R\,(U_m^n - U_{m-1}^n)$

Solution At point $(m\Delta x, (n+1)\Delta t)$ depends on the value of the 2 mesh points ~~$(m\Delta x, n\Delta t)$~~ and $((m-1)\Delta x, m\Delta x, n\Delta t)$

continue this process down to the zero time level, depends on the points $((m-n-1)\Delta x, 0) \ldots (m\Delta x, 0.)$

Analytical domain $x - at = ct$ should be a subset of numerical domain. If we change Δx and Δt, then as long as the ratio R and the point $(m\Delta x, (n+1)\Delta t)$, still ~~remain~~ depend on the points starting from ~~$(m-n-1)\Delta x, m\Delta x, 0$~~ $((m-n-1)\Delta x, 0), \ldots (m\Delta x, 0)$

Which says that the

numerical domain of dependence of this point $(m\Delta x, (n+1)\Delta t)$ for the difference scheme

FTBS. $D_n = [(m-n-1)\Delta x, \ldots\ldots, m\Delta x]$

It should be noted that all of the points that the solution at this point $(m\Delta x, (n+1)\Delta t)$, depends on for all possible refinement satisfying $R = \frac{ak}{h}$ for a fixed. are contained in the interval D_n.

$$U_m^{n+1} = \alpha U_m^n + \beta U_{m-1}^n$$

↑ we can generalize the difference scheme of this form. $\boxed{\text{for FTBS } \alpha = 1-R,\ \beta = R}$

The domain of dependence remains same D_n.

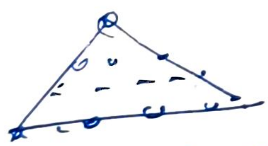
- For FTFS, $U_m^{n+1} = U_m^n - R(U_{m+1}^n - U_m^n)$

$$U_m^{n+1} = \alpha U_m^n + \beta U_{m+1}^n$$

$\cancel{D_n = (0, \ldots (n+1)\Delta x)}$ $(m\Delta x, \ldots (m+n+1)\Delta x)$?

Can take central scheme also.

$$U_m^{n+1} = \alpha U_{m-1}^n + \beta U_m^n + \gamma U_m^n$$



Thus, we can ~~observe~~ observe that the numerical domain of dependence does not contain much info about a particular difference scheme. The numerical domain of dependence only knows where ~~a token~~ the difference scheme reaches.

CFL condition : A p.d.e and an associated difference scheme is said to satisfy CFL ~~scheme~~ condition if the analytic domain of dependence ~~be~~ D_a : $(x - at)$ is contained in the numerical domain of dependence D_n. The CFL condition is only necessary not sufficient for convergence.

consider ~~say~~ $u_t + a u_x = 0$.

and analytic soln $u(x,t) = f(x - at)$

for some (x, t).

Let $x - at = x_0$

Let $(x, t) = (m\Delta x, (n+1)\Delta t)$, Then $x_0 = x - at$

$= m\Delta x - a(n+1)\Delta t$

$= (m - R(n+1))\Delta x$

$\left(R = a\dfrac{\Delta t}{\Delta x}\right)$

We can notice that if x_0 lies in the interval

$x_0 \in ((m-n-1)\Delta x, m\Delta x)$

~~false~~ iff $R \in [0, 1]$

Which says analytical domain of PDE is contained in numerical domain iff $R \in (0, 1]$

CFL is necessary for stability of FTBS.