# MA 372 : Stochastic Calculus for Finance

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Exercises 4

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1. We shall call $f(t), t \in [0,T]$ a simple process if there is a finite sequence of numbers $0 = t_0 < t_1 < \cdots < t_n = T$ and square integrable random variables $\eta_0, \eta_1, \cdots, \eta_{n-1}$ such that $f(t,w) = \sum_{j=0}^{n-1} \eta_j(w)\mathbb{I}_{[t_j,t_{j+1})}(t)$, where $\eta_j$ is $\mathcal{F}_{t_j}$-measurable. The set of simple processes will be denoted by $M^2_{step}([0,T]\times\Omega)$

   a) Show that $M^2_{step}([0,T]\times\Omega)$ is a vector space, that is, $af+bg \in M^2_{step}([0,T]\times\Omega)$ for any $f,g \in M^2_{step}([0,T]\times\Omega)$ and $a,b \in \mathbb{R}$.

   b) Show that $I : M^2_{step}([0,T]\times\Omega) \to L^2$ is a linear map, i.e., for any $f,g \in M^2_{step}([0,T]\times\Omega)$ and $a,b \in \mathbb{R}$
   $$I(af+bg) = aI(f) + bI(g).$$

   c) For any $f,g \in M^2_{step}([0,T]\times\Omega)$
   $$E\Big[I(f)I(g)\Big] = E\Big[\int_0^T f(t)g(t)dt\Big]$$

2. Check whether the following processes $X(t)$ are martingale with respect to Brownian filtration

   a) $X(t) = W(t) + 4t$ b) $X(t) = W^2(t)$
   c) $X(t) = t^2W(t) - 2\int_0^t sW(s)ds$

3. Use Ito's formula to prove that the following stochastic process are martingale with respect to Brownian filtration

   a) $X(t) = e^{\frac{t}{2}}\cos W(t)$ b) $X(t) = e^{\frac{t}{2}}\sin W(t)$
   c) $X(t) = e^{W(t)-\frac{t}{2}}$ d) $X(t) = (W(t)+t)e^{-W(t)-\frac{t}{2}}$

4. Define $\beta_k(t) = \mathbb{E}[W^k(t)]$; $k = 0,1,2,\cdots$; $t \geq 0$
   Use Ito's formula to prove that
   $$\beta_k(t) = \frac{1}{2}k(k-1)\int_0^t \beta_{k-2}(s)ds; \quad k \geq 2$$

   a) Deduce that $\mathbb{E}[W^4(t)] = 3t^2$ and find $\mathbb{E}[W^6(t)]$.
   b) Show that $\mathbb{E}[W^{2k+1}(t)] = 0$ and $\mathbb{E}[W^{2k}(t)] = \frac{(2k)!t^k}{2^k k!}$

5. For $c, \alpha$ constants, define
   $$X(t) = e^{ct+\alpha W(t)}.$$
   Prove that
   $$dX(t) = (c + \frac{1}{2}\alpha^2)X(t)dt + \alpha X(t)dW(t)$$

6. Let $\Pi = \{t_0, t_1, \cdots, t_n\}$ be a partition of $[0, T]$ with $0 = t_0 < t_1 < \cdots < t_n = T$. For $\alpha \in [0, 1]$, consider the sum

$$S_\alpha(\Pi) = \sum_{j=0}^{n-1} \Big[(1-\alpha)W(t_j) + \alpha W(t_{j+1})\Big](W(t_{j+1}) - W(t_j)).$$

Evaluate the limit $\lim_{||\Pi|| \to 0} S_\alpha(\Pi)$ (in $L^2$), where $\|\pi\| = \max_{j=1,2,\cdots,n}(t_j - t_{j-1})$.

7. If $f(t, x) = e^{t/2}(\sin x + \cos x)$, then check whether the process $f(t, W(t))$ is a martingale with respect to Brownian filtration.

8. Suppose $f, g$ are square integrable, deterministic (non random) functions and that there exist constants $C, D$ such that

$$C + \int_0^T f(s)dW(s) = D + \int_0^T g(s)dW(s), \ a.e. w \in \Omega.$$

(i) What is the relationship between $C$ and $D$?
(ii) What is the relationship between $f(t)$ and $g(t)$?

9. If $f(t, x) = x^5 - 10tx^3 + 15t^2x$, then check whether the process $f(t, W(t))$ is a martingale with respect to Brownian filtration.

10. Suppose that $\{W(t); t \geq 0\}$ is a standard Brownian motion with $W(0) = 0$. Determine an expression for

$$\int_0^t \sin(W(s))dW(s)$$

that does not involve Ito integrals.

11. Suppose $f(t)$ is a deterministic function. Let $X(t) = X(0) + \int_0^t f(s)dW(s)$. Determine an expression for

$$\int_0^t f(s)X(s)dW(s)$$

that does not involve Ito integrals.

12. Let

$$X(t) = \int_0^t W(s)\ ds$$

for $t \geq 0$. Find the mean, variance and distribution function of the random variable X(2).