MA 372 : Stochastic Calculus for Finance

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Exercises 3

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1. a) Prove that $\mathbb{E}[\exp\{iuW(t)\}] = \exp(-\frac{1}{2}u^2 t)$

   b) Deduce that $\mathbb{E}[W^4(t)] = 3t^2$ and more generally $\mathbb{E}[W^{2k}(t)] = \frac{(2k)!}{2^k k!}t^k,\ k \in \mathbb{N}$. Find $\mathbb{E}[W^6(t)]$.

   c) Compute the moment generating function of $(W(t_1), W(t_2), \cdots, W(t_m))$, i.e., find $\mathbb{E}[\exp\{u_1 W(t_1) + u_2 W(t_2) + \cdots + u_m W(t_m)\}]$.

   Hint: $u_1 W(t_1) + u_2 W(t_2) + \cdots + u_m W(t_m) = u_m(W(t_m) - W(t_{m-1}) + (u_{m-1} + u_m)(W(t_{m-1}) - W(t_{m-2})) + \cdots + (u_1 + u_2 + \cdots + u_m)W(t_1)$

2. Let (X, Y) be jointly normal with the density

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\left[-\frac{1}{2(1-\rho^2)}\left\{\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right\}\right]},$$

   where $\sigma_1 > 0, \sigma_2 > 0, |\rho| < 1$, and μ_1, μ_2 are real numbers. Define $W = Y - \frac{\rho\sigma_2}{\sigma_1}X$. Then show that X and W are independent. Find joint density function of X and W.

3. Let $W(t)$ be a Brownian motion and let $\mathcal{F}(t),\ t \geq 0$, be an associated filtration

   a) For $\mu \in \mathbb{R}$, consider the Brownian motion with drift μ:

$$X(t) = \mu t + W(t).$$

   Prove that $X(t),\ t \geq 0$ is a Markov process by showing that for any Borel-measurable function f, and for any $0 \leq s < t$,

$$\mathbb{E}[f(X(t))|\mathcal{F}(s)] = g(X(s)),$$

   where $g(x) = \int_{-\infty}^{\infty} f(y)p(\tau,x,y)dy$, $\tau = t-s$ and $p(\tau,x,y) = \frac{1}{\sqrt{2\pi\tau}}\exp\{-\frac{-(y-x-\mu\tau)^2}{2\tau}\}$

   b) For $\nu \in \mathbb{R}, \sigma > 0$, consider the geometric Brownian motion

$$S(t) = S(0)\exp\{\nu t + \sigma W(t)\}.$$

   Prove that $S(t),\ t \geq 0$ is a Markov process by showing that for any Borel-measurable function f, and for any $0 \leq s < t$,

$$\mathbb{E}[f(S(t))|\mathcal{F}(s)] = g(S(s)),$$

   where $g(x) = \int_{-\infty}^{\infty} f(y)p(\tau,x,y)dy$, $\tau = t-s$ and $p(\tau,x,y) = \frac{1}{\sigma y\sqrt{2\pi\tau}}\exp\{-\frac{-(\ln(y/x)-\nu\tau)^2}{2\sigma^2\tau}\}$

4. Let X_n be a symmetric random walk, that is

$$X_n = Y_1 + Y_2 + \cdots + Y_n,$$

where $Y_1, Y_2, \cdots$ is a sequence of independent identical distributed random variables such that $\mathbb{P}(Y_n = 1) = \mathbb{P}(Y_n = -1) = 1/2$.

a) Show that $X_n^2 - n$ is a martingale with respect to the filtration $\mathcal{F}_n = \sigma(Y_1, \cdots, Y_n)$.

b) Show that $Z_n = (-1)^n cos(\pi X_n)$ is a martingale with respect to the filtration $\mathcal{F}_n$

c) Show that $Z_n = (-1)^n cos(\pi(X_n + 100))$ is a martingale with respect to the filtration $\mathcal{F}_n$

d)Let τ be a stopping time with respect to the filtration $\mathcal{F}_n$. Then the stopped process X^τ is defined for $t \geq 0$ and $w \in \Omega$ by

$$X_n^\tau(w) := X_{\{n \bigwedge \tau(w)\}}(w).$$

Show that X^τ is adapted to the filtration $\mathcal{F}_n$.

e) Find $\mathbb{E}[(-1)^\tau]$, where τ the smallest n such that $|X_n| = 100$.

5. Show that $W^2(t) - t$ is a martingale with respect to the Brownian filtration.

6. Let $W(t)$ be a Brownian motion. Check whether the process $X(t) = 2W(t)+4t$ is a martingale with respect to Brownian. filtration.

7. Assume that $\lim_{t\to\infty} \frac{W(t)}{t} = 0$, a.s. Then prove that

$$Y(t) = \begin{cases} tW(1/t) & \text{if } 0 < t < \infty \\ 0 & \text{if } t = 0 \end{cases}$$

is a Brownian motion if $W(t)$ is.

8. Let $c > 0$. Show that $X(t) = \frac{1}{c}W(c^2 t); 0 \leq t < \infty$ is a Brownian motion if $W(t)$ is.

9. Show that for any fixed $T > 0$

$$X(t) = W(t + T) - W(T),\ t \geq 0$$

is a Brownian motion if $W(t)$ is.

10. Let Y be a real valued random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

$$\mathbb{E}[|Y|] < \infty.$$

Let $\mathcal{F}(t),\ t \geq 0$, be any filtration. Define

$$M(t) = \mathbb{E}[Y|\mathcal{F}(t)],\ t \geq 0.$$

Show that $M(t)$ is a martingale with respect to the filtration $\mathcal{F}(t),\ t \geq 0$.

11. Show that $W^3(t) - 3tW(t)$ is a martingale with respect to the Brownian filtration.

12. Show that if $M(t)$ is a martingale with respect to the filtration $\mathcal{F}(t),\ t \geq 0$, then
$$\mathbb{E}[M(t)] = \mathbb{E}[M(0)]$$
for all $t \geq 0$. Give an example of a stochastic process $M(t)$ satisfying
$$\mathbb{E}[M(t)] = \mathbb{E}[M(0)],\ \forall t \geq 0$$
and which is not a martingale with respect to its own filtration, (i.e., $\mathcal{F}(t) := \sigma\{M(s) | s \leq t\}$).

13. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{(X_n, \mathcal{F}_n) : n \geq 0\}$ is a supermartingale and $\mathbb{E}(X_n) = c \in \mathbb{R},\ \ \forall\, n$. Then show that $\{(X_n, \mathcal{F}_n) : n \geq 0\}$ is a martingale.

14. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\{(X_n, \mathcal{F}_n) : n \geq 0\}$ is a martingale such that $|X_n| \leq M$ $\mathbb{P}$-almost everywhere on Ω for all $n \geq 0$. We define $Y_n := \sum_{k=1}^{n} \frac{1}{k}(X_k - X_{k-1})$ for all n. Show that $\{(Y_n, \mathcal{F}_n) : n \geq 0\}$ is a martingale.

15. Let X_n be simple symmetric random walk, with $X_0 = 0$. Let $\tau = \inf\{n \geq 5 : X_{n+1} = X_n + 1\}$ be the first time after 4 which is just before the chain increases. Let $\rho = \tau + 1$
(a) Is τ a stopping time?
(b) Is ρ a stopping time?