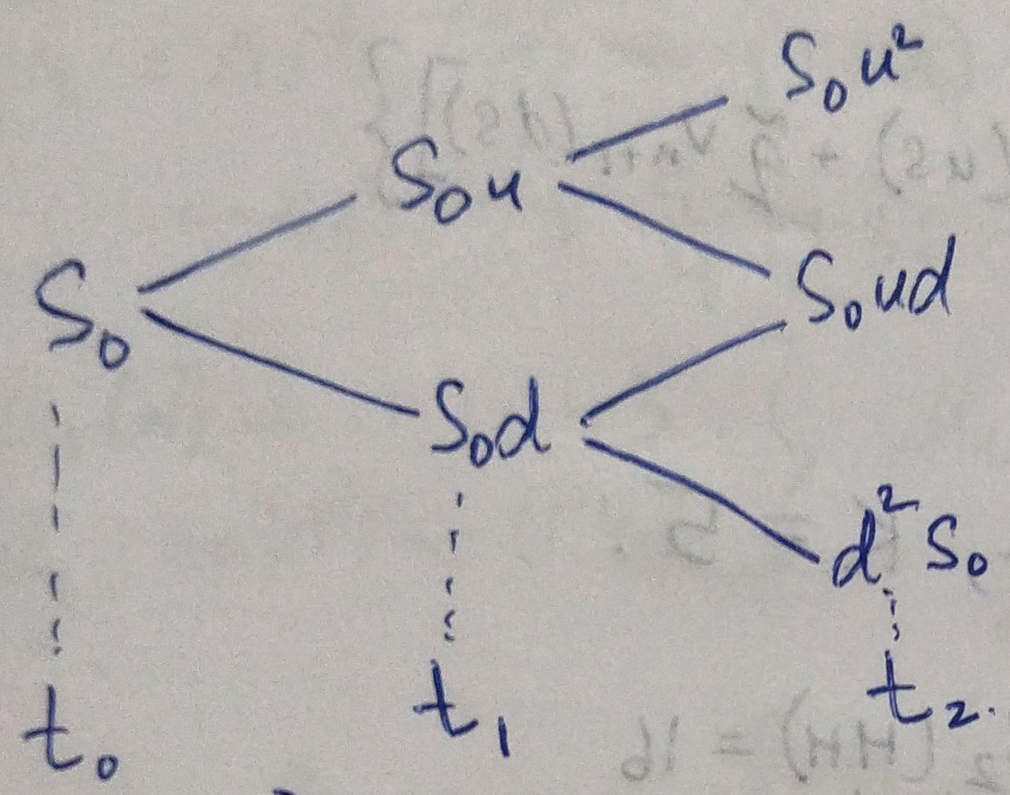
Straddles – $f(T) = |S(T) - K|$

American Derivatives

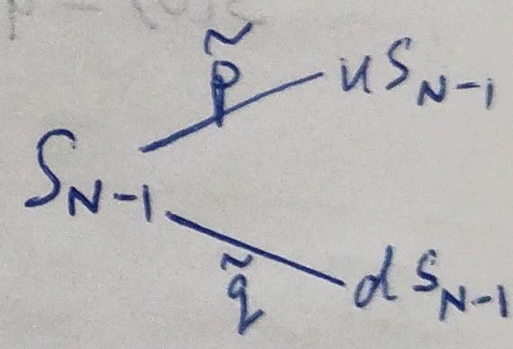
N-period binomial model with up factor u and down factor d, interest rate r.



No arbitrage condition:–

$$0 < d < 1 + r < u.$$

For a (european) derivative giving payoff $= g(S_N)$, then the price v_n at time $n \le N$ is given by



$$v_N(S_N) = g(S_N).$$

$$v_{N-1}(S_{N-1}) = \frac{\tilde{\mathbb{E}}_{N-1}(g(S_N))}{1+r}$$

$$= \frac{1}{1+r}\left[\tilde{p} \cdot g(uS_{N-1}) + \tilde{q} \cdot g(dS_{N-1})\right].$$

$$v_n(S) = \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(Su) + \tilde{q}\, v_{n+1}(Sd)\right].$$

where $\tilde{p} = \dfrac{1+r-d}{u-d}$, $\tilde{q} = 1 - \tilde{p}$.

Hedging $\Delta_n = \dfrac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)S_n}$

Suppose American derivative with payoff

$$g(S_n) \quad \forall \quad n \le N.$$

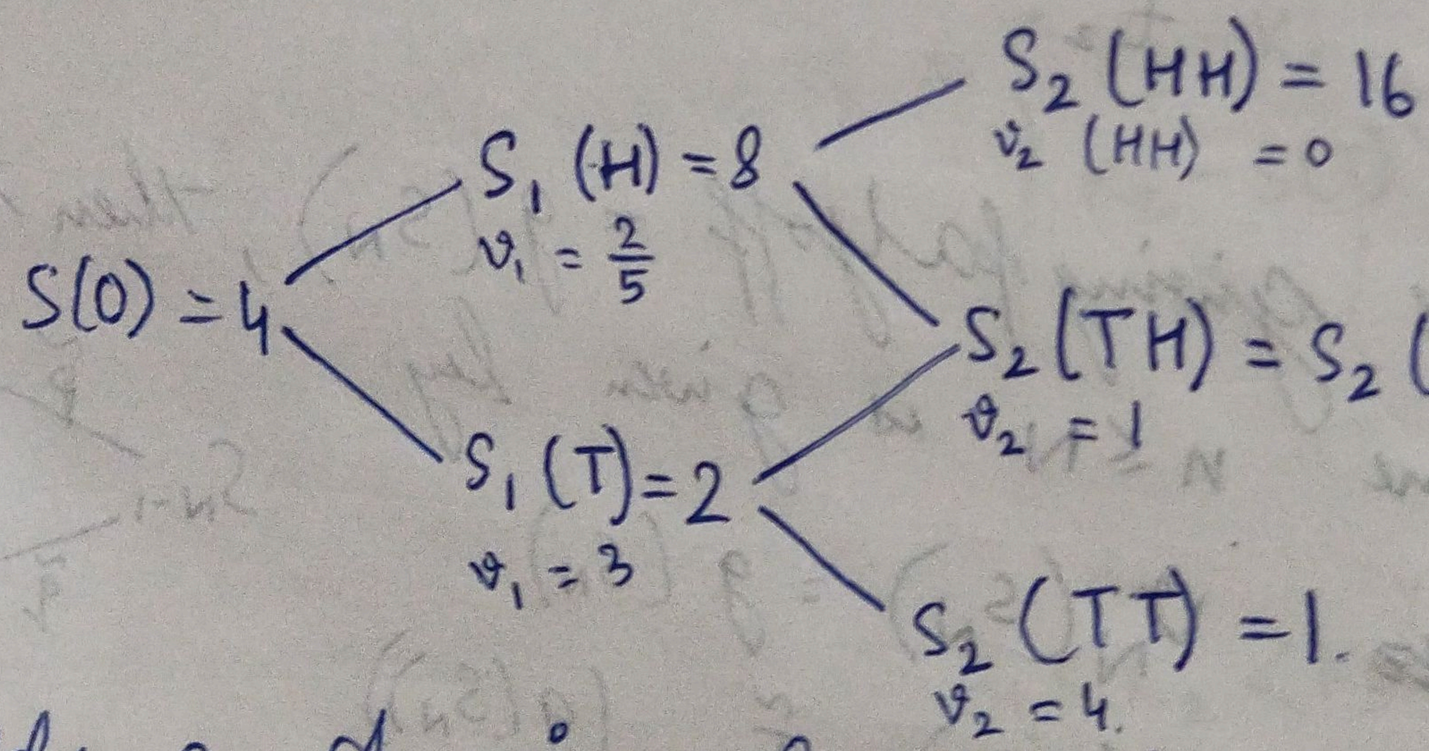
The price is given by v_n.

$$v_N(S_N) = \max\{g(S_N), 0\}.$$

$$v_{N-1}(S_N) = \max\{g(S_{N-1}), \frac{1}{(1+r)}[\tilde{p}\, v_N(uS_{N-1}) + \tilde{q}\, v_N(dS_{N-1})]\}$$

$$v_n(S_n) = \max\{g(S), \frac{1}{1+r}[\tilde{p}\, v_{n+1}(uS) + \tilde{q}\, v_{n+1}(dS)]\}.$$

<u>Example</u> – $u = 2, d = \frac{1}{2}, r = \frac{1}{4}, K = 5.$



$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{\frac{5}{4}-\frac{1}{2}}{2-\frac{1}{2}} = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

Consider an American put with $N=2$ and $g(s) = K - S$.

$$v_2(s) = \max\{5 - s, 0\}$$

$$v_n(s) = \max\{5 - s, \frac{1}{1+r}[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)]\}, \quad n = 0, 1.$$

$$v_2(16) = 0, \quad v_2(4) = 1, \quad v_2(1) = 4.$$

$$v_1(8) = \{\frac{1}{5/4}[\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 1], 0\} = \frac{4}{5}\cdot\frac{1}{2} = \frac{2}{5}.$$

$$v_1(2) = \{\frac{1}{5/4}[\tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\cdot 4], 3\} = 3, \quad v_0(4) = \max\{\frac{1}{5/4}[\tfrac{1}{2}\cdot\tfrac{2}{5} + \tfrac{1}{2}\cdot 3], 1\}$$

$$= 1.36$$

$$
\begin{tikzcd}[column sep=small]
 & v_1(8)=0.4 \arrow[dr] & v_2(16)=0 \\
v_0(4)=1.36 \arrow[ur] \arrow[dr] & & v_2(4)=1 \\
 & v_1(2)=3 \arrow[ur] \arrow[dr] & \\
 & & v_2(1)=4
\end{tikzcd}
$$

$3 = -3\Delta_0 + 1.7$

$3\Delta_0 = -1.3$

$\Delta_0 = -1.3/3$

$3 = 2\Delta_0(T) + \frac{5}{4}(\overset{0.34}{1.36} - 2\Delta_0)$

$3 = -\frac{1}{2}\Delta_0(T) + 1.7$ $\quad \frac{1}{2}\Delta_0(T) = -1.3$

$\Delta_0 = -2.6$

Hedging strategy –

$$0.4 = \Delta_0^{(H)} S_1(H) + (1+r)(X_0 - \Delta_0^{(H)} S_0)$$

$$= 8\Delta_0(H) + \frac{5}{4}(1.36 - 4\Delta_0^{(H)})$$

$\Rightarrow \Delta_0(H) = -0.43$ } Always comes same.

Similarly, $\Delta_0(T) = -0.43$ }

$$\Delta_n \text{ is always } = \frac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)S_n}$$

$$Y_n = \tilde{v}_n = \frac{1}{(1+r)^n} v_n$$

$$
\begin{tikzcd}[column sep=small]
 & Y_1(H)=0.32 \arrow[dr] & Y_2(HH)=0 \\
Y_0(4)=1.36 \arrow[ur] \arrow[dr] & & Y_2(HT)=Y_2(TH)=0.64 \\
 & Y_1(T)=2.4 \arrow[ur] \arrow[dr] & \\
 & & Y_2(TT)=2.56
\end{tikzcd}
$$

$\tilde{p} = \tilde{q} = \frac{1}{2}$.

$r = \frac{1}{4}$

If martingale, (Y_n)

$\tilde{\mathbb{E}}_n[Y_{n+1}] = Y_n$.

$\Rightarrow Y_1(T) = \frac{1}{2}(Y_2(TH) + Y_2(TT)) = \frac{1}{2}(2.56 + 0.64) = 1.6 \neq 2.4$.

Hence, not martingale but super martingale.

Theorem – Consider a N-period binomial model with

$$0<d<1+r<u, \quad \tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = 1-\tilde{p},$$

Let payoff function $g(S)$ be given and define

$v_N(S), v_{N-1}(S), \ldots, v_0(S)$ by

$$v_N(S) = \max\{g(S), 0\}.$$

$$v_n(S) = \max\left\{g(S), \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(uS) + \tilde{q}\, v_{n+1}(dS)\right]\right\}.$$

Define $\Delta_n = \dfrac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)\,S_n}$,

$$C_n = v_n(S_n) - \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(uS_n) + \tilde{q}\, v_{n+1}(dS_n)\right].$$

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n)$$

then,

$$X_n = V_n \;\&\; X_n \geq g(S_n) \;\; \forall n.$$

Stopping Time

A random variable $\tau : \Omega_N \longrightarrow \{0, 1, 2, \ldots, N, \infty\}$ satisfies if $\tau(\omega_1 \omega_2 \ldots \omega_n \omega_{n+1} \ldots \omega_N) = n$, then

$$\tau(\omega_1 \omega_2 \ldots \omega_n \omega'_{n+1} \ldots \omega'_N) = n \;\forall\; \omega'_{n+1} \ldots \omega'_N.$$

Our example :- (Stopping time when $g(S) > 0$).

$$\tau : \{HH, HT, TH, TT\} \longrightarrow \{1, 2, \infty\}.$$

$\tau(HH) = \infty$ $\quad$ $\tau(TH) = 1$

$\tau(HT) = 2$ $\quad$ $\tau(TT) = 1.$ stopping time ✓

Consider $\tau(HH) = 0$ $\quad \tau(TH) = 1$

$\tau(HT) = 0$ $\quad \tau(TT) = 2$ $\quad$ ... τ is not a stopping time.

$Y_{0\wedge\tau} = Y_0 = 1.36$

$Y_{1\wedge\tau} = Y_1$

$Y_{2\wedge\tau}(HH) = Y_2(HH) = 0$ $\quad Y_{2\wedge\tau}(TH) = Y_1(T) = 2.4$

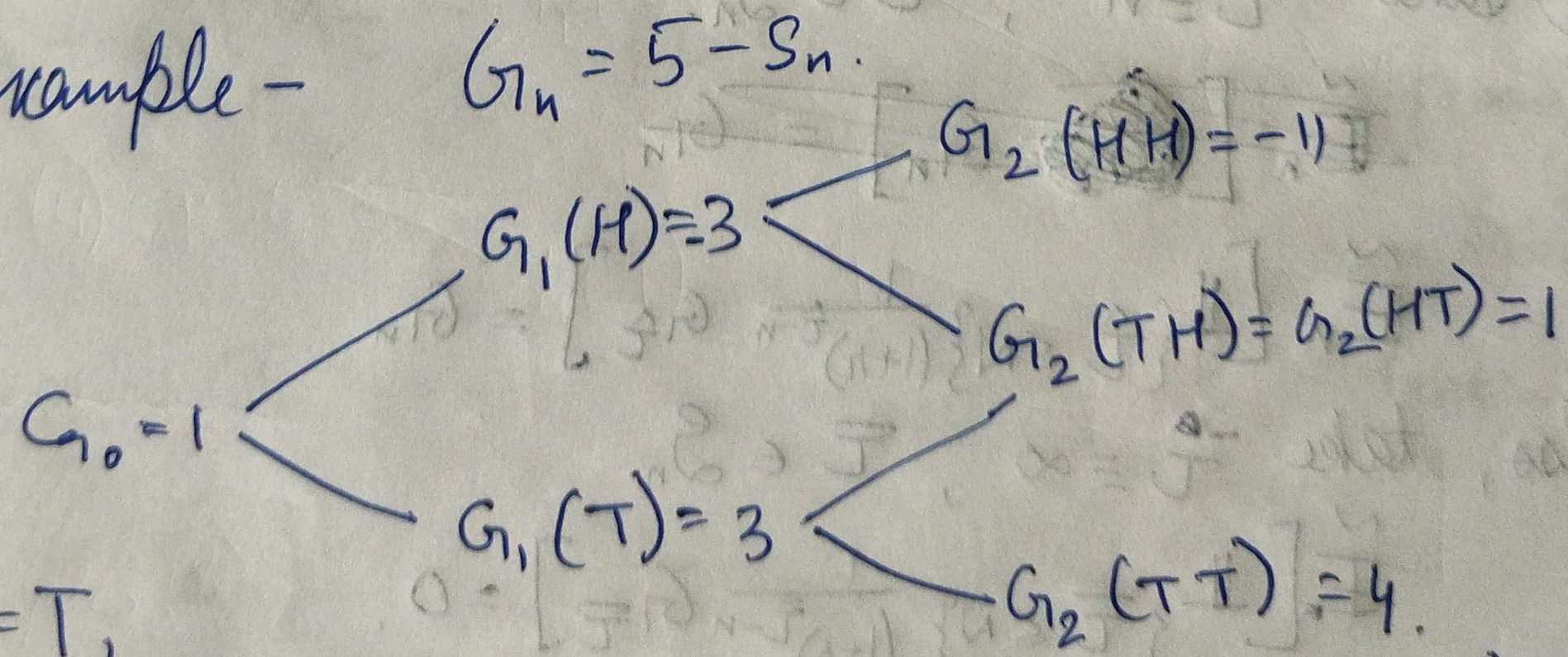
$Y_{2\wedge\tau}(HT) = Y_2(HT) = 0.64$ $\quad Y_{2\wedge\tau}(TT) = Y_1(T) = 2.4.$

$\mathcal{S}_n = \{\tau\text{-stopping time} : \tau : \Omega_N \to \{n, n+1, \dots, N, \infty\}\}$

Consider an american derivative with intrinsic value at n is G_n.

<u>Define</u> :- $V_n = \max_{\tau \in \mathcal{S}_n} \tilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau - n}} G_\tau\right]$, $n = 0, 1, \dots, N.$

Our example - $G_n = 5 - S_n.$



If $\omega_1 = T$,

possible τ's -

$\tau_2(TT) = 2$, $\tau_2(TH) = \infty$ $\quad \tau_2(TT) = \infty$, $\tau_2(TH) = 2$

$\tau_2(TT) = 2$, $\tau_2(TH) = 2$ $\quad \tau_2(TT) = 1 = \tau_2(TH).$

$\tau_2(TT) = \infty$, $\tau_2(TH) = \infty$

For $\tau(TH) = 1 = \tau(TT)$

$$v_1^{\tau}(T) = \tilde{\mathbb{E}}_1\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-1}}\right]$$

$$= \tilde{\mathbb{E}}_1\left[G_1(T)\right]$$

$$= 3.$$

General American Derivatives

Theorem – If $v_n = \max_{\tau \in S_n} \tilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau\right]$

then, ① $v_n \geq \max\{G_n, 0\}$ $\forall n$

② $\frac{1}{(1+r)^n} v_n$ is a supermartingale

③ If Y_n is another process satisfying ① and ②

then $v_n \leq Y_n$ $\forall n$.

Proof :- ① Take $\hat{\tau} \equiv n$, $\hat{\tau} \in S_n$.

$$\cancel{\tilde{\mathbb{E}}_n\left[G_n\right] = G_n}$$

$$\tilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\hat{\tau} \le N\}} \frac{1}{(1+r)^{\hat{\tau}-n}} G_{\hat{\tau}}\right] = G_n$$

Also, take $\bar{\tau} \equiv \infty$, $\bar{\tau} \in S_n$.

$$\tilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\bar{\tau} \le N\}} \frac{1}{(1+r)^{\bar{\tau}-n}} G_{\bar{\tau}}\right] = 0.$$

$\therefore v_n \geq G_n$ and $v_n \geq 0$

$\Rightarrow v_n \geq \max\{G_n, 0\}$.

② Let n be given and suppose τ^* attains the maximum in the definition of V_{n+1}.

$$V_{n+1} = \tilde{\mathbb{E}}_{n+1}\left[\mathbb{1}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n-1}} G_{\tau^*}\right].$$

$$\tau^* \in \mathcal{S}_{n+1} \Rightarrow \tau^* \in \mathcal{S}_n.$$

$$\therefore V_n \ge \tilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*}\right].$$

$$= \tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_{n+1}\left[\mathbb{1}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*}\right]\right].$$

$$\cancel{\tilde{\mathbb{E}}_n[V_{n+1}]}. = \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)} V_{n+1}\right].$$

$$\therefore \boxed{\frac{1}{(1+r)^n} V_n \ge \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{n+1}} V_{n+1}\right]}$$

③ Suppose Y_n is another process satisfying ① and ②,

$$Y_k \ge \max\{G_k, 0\}. \quad \forall k.$$

$$\mathbb{1}_{\{\tau \le N\}} G_\tau \le \mathbb{1}_{\{\tau \le N\}} \max\{G_\tau, 0\} \le \mathbb{1}_{\{\tau \le N\}} \max\{G_{\tau\wedge N}, 0\} + \mathbb{1}_{\{\tau > N\}} \max\{G_{\tau\wedge N}, 0\}$$

$$= \max\{G_{\tau\wedge N}, 0\}.$$

$$\le Y_{\tau\wedge N}.$$

$$\therefore \mathbb{1}_{\{\tau \le N\}} G_\tau \le Y_{\tau\wedge N} \quad \forall \tau.$$

$\tau \in \mathcal{S}_n$.

$$\tilde{\mathbb{E}}_n\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau}} G_\tau\right].$$

$$\le \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{\tau\wedge N}} Y_{\tau\wedge N}\right]$$

$$\le \frac{1}{(1+r)^{\tau\wedge n}} Y_{\tau\wedge n} \quad \text{(from ②)}.$$

$$= \frac{1}{(1+r)^n} Y_n \quad \text{(Because } \tau \in \mathcal{S}_n \Rightarrow \tau \ge n\text{)}.$$

$$\Rightarrow \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau\right] \le Y_n \quad \forall \tau.$$

$$\therefore \boxed{V_n \le Y_n}$$

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* 1 Lec Miss

Stopping Time

Definition – A random variable τ taking values in $[0,\infty]$ is called a stopping time if

$$\{\tau \le t\} \in \mathcal{F}(t) \ \forall t.$$

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$$

Define $\tau_m = \min\{t \ge 0 : S(t) = m\}$.

If $S(t)$ never reaches level m, then we interpret $\tau_m = \infty$.

Ex – $X(t)$ is a martingale (sub/super) then $X(t \wedge \tau_m)$ is also a martingale (sub/super).

$$X(t \wedge \tau_m) = \begin{cases} X(t), & t \le \tau_m \\ X(\tau_m), & t > \tau_m. \end{cases}$$

Perpetual put –

The underlying assest price is given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$$

where $\tilde{W}(t)$ is a BM under risk-neutral $\tilde{\mathbb{P}}$.

$$S(t) = S(0)\,e^{\sigma \tilde{W}(t) + (r - \frac{\sigma^2}{2})t}$$

The american perpetual put pay (intrinsic value) (No "T").

$K - S(t)$ if it is exercised at time t.

Def^n - Let $\mathcal{S}$ be the set of all stopping time

$$v_*(x) = \max_{\tau \in \mathcal{S}} \mathbb{E}\left[(K - S(\tau))\, e^{-r\tau}\right], \quad S(0) = x.$$

If $\tau = \infty$, we interpret $e^{-r\tau}$ as 0.

Lemma - Let $\tilde{W}(t)$ be a BM under $\tilde{\mathbb{P}}$. Let μ be a real number and let m be a positive number.

Set $X(t) = \mu t + \tilde{W}(t)$.

$$\tau_m = \min\{t \geq 0 : X(t) = m\}.$$

If $X(t)$ never reaches m, then $\tau_m = \infty$.

$$\tilde{\mathbb{E}}\left[e^{-\lambda \tau_m}\right] = e^{-m\left(-\mu + \sqrt{\mu^2 + 2\lambda}\right)} \quad \forall \lambda > 0.$$

$$e^{-\lambda \tau_m} = 0, \text{ if } \tau_m = \infty.$$

Set a level L

If $S(0) = x < L$,

$$v_L(x) = (K - x).$$

If $S(0) = x > L$,

$$\tau_L = \min\{t \geq 0 : S(t) = L\}.$$

$$v_L(x) = \tilde{\mathbb{E}}\left[(K - S(\tau_L))\, e^{-r\tau_L}\right].$$

$$= (K - L)\, \tilde{\mathbb{E}}\left[e^{-r\tau_L}\right].$$

$$S(t) = x e^{\{\sigma \tilde{W}(t) + (r - \frac{1}{2}\sigma^2)t\}}.$$

$$S(\tau_L) = L.$$

$$\Leftrightarrow x e^{\sigma \tilde{W}(\tau_L) + (r - \frac{1}{2}\sigma^2)\tau_L} = L.$$

$$\Leftrightarrow \sigma \tilde{W}(\tau_L) + \left(r - \frac{1}{2}\sigma^2\right)\tau_L = \ln(x/L).$$

$$\Leftrightarrow -\tilde{W}(\tau_L) - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)\tau_L = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right).$$

Here, $\mu = -\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$, $\lambda = r$, $m = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right)$.

$$\mu^2 + 2\lambda = \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 + 2r.$$

$$= \frac{1}{\sigma^2}\left(r + \frac{1}{2}\sigma^2\right)^2.$$

$$\Rightarrow -\mu + \sqrt{\mu^2 + 2\lambda} = \frac{2r}{\sigma}.$$

$$\tilde{\mathbb{E}}\left[e^{-r\tau_L}\right] = e^{-\frac{1}{\sigma}\ln\left(\frac{x}{L}\right)\cdot\frac{2r}{\sigma}} = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}.$$

$$\therefore v_L(x) = \begin{cases} (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}, & x > L \\ (K-x), & x \le L. \end{cases}$$

Using $\frac{\partial v_L(x)}{\partial L} = 0$ (for maximum),

$$g(L) := (K-L)\,L^{2r/\sigma^2}$$

$$g'(L) = -L^{2r/\sigma^2} + \frac{2r}{\sigma^2}(K-L)\,L^{\frac{2r}{\sigma^2}-1} = 0.$$

$$\Rightarrow \cancel{L^{2r/\sigma^2}} = \frac{2r}{\sigma^2}\,\frac{(K-L)\,\cancel{L^{\frac{2r}{\sigma^2}}}}{L}$$

$$\Rightarrow L = \frac{2r}{\sigma^2}(K-L)$$

$$\Rightarrow L^* = \frac{\frac{2r}{\sigma^2}K}{1 + \frac{2r}{\sigma^2}} = \frac{2rK}{\sigma^2 + 2r}.$$

$$\therefore v_{L^*}(x) = \begin{cases} (K-x), & \text{if } x \le L^* \\ (K-L^*)\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}, & x > L^*. \end{cases}$$

$$v'_{L^*}(x) = \begin{cases} -1, & x \le L^* \\ -(K-L^*)\frac{2r}{x\sigma^2}\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}, & x \ge L^*. \end{cases}$$

$v'_{L^*}(x)$ is cts at $x = L^*$.

$$v''_{L^*}(x) = \begin{cases} 0, & x < L^* \\ (K-L^*)\frac{2r(2r+\sigma^2)}{\sigma^4 x^2}\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}, & x > L^*. \end{cases}$$

For $x > L^*$,

$$r v_{L^*}(x) - r x v'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L^*}(x)$$

$$= \left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}} (K-L^*)\left[r + \frac{2r^2}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right] = 0$$

If $0 \le x < L^*$,

$$r v_{L^*}(x) - r x v'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L^*}(x)$$

$$= r(K-x) + rx = rK.$$

$v_{L^*}(x)$ satisfies the linear complementarity conditions:

$$v(x) \ge (K-x)^+ \quad \forall x \ge 0 \quad \text{——— (1)}$$

$$r v(x) - r x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \ge 0 \quad \forall x \quad \text{——— (2)}$$

and $\forall x \ge 0$, equality holds in exactly one of either (1) or (2) ——— (3)

Theorem:- $\tau_{L^*} = \min\{t \geq 0 : S(t) = L^*\}$.

Then $e^{-rt} v_{L^*}(S(t))$ is a supermartingale and $e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale.

Proof:- $d\left(e^{-rt} v_{L^*}(S(t))\right)$

$$= -re^{-rt} v_{L^*}(S(t))\,dt + e^{-rt}\, d\,v_{L^*}(S(t))$$

$$= -re^{-rt} v_{L^*}(S(t))\,dt + e^{-rt}\left[v'_{L^*}(S(t))\,dS(t) + \tfrac{1}{2} v''_{L^*}(S(t))\,dS(t)\,dS(t)\right]$$

$$\cancel{= -re^{-rt} v_{L^*}(S(t))\,dt +}$$

$$= e^{-rt}\left[-r\,v_{L^*}(S(t)) + r\,S(t)\,v'_{L^*}(S(t)) + \tfrac{1}{2}\sigma^2 S^2(t)\,v''_{L^*}(S(t))\right]dt$$

$$+ e^{-rt} v'_{L^*}(S(t))\,\sigma\,S(t)\,d\tilde{W}(t).$$

$$= -e^{-rt}\, rK\,\mathbb{1}_{\{S(t)<L^*\}}\,dt + e^{-rt}\sigma S(t)\,d\tilde{W}(t)\,v'_{L^*}(S(t)).$$

$\Rightarrow e^{-rt} v_{L^*}(S(t))$ is a supermartingale.

$$\int_0^{u\wedge\tau_{L^*}} d\left(e^{-rt} v_{L^*}(S(t))\right) = -\int_0^{u\wedge\tau_{L^*}} e^{-rt}\, rK\,\mathbb{1}_{\{S(t)<L^*\}}\,dt + \int_0^{u\wedge\tau_{L^*}} e^{-rt}\sigma S(t)\,v'_{L^*}(S(t))\,d\tilde{W}(t)$$

$$\Rightarrow e^{-r(u\wedge\tau_{L^*})} v_{L^*}(S(u\wedge\tau_{L^*})) - v_{L^*}(S(0)) = \int_0^{u\wedge\tau_{L^*}} e^{-rt}\sigma S(t)\,v'_{L^*}(S(t))\,d\tilde{W}(t).$$

$\therefore e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale.

Corollary -

$$v_{L^*}(x) = \max_{\tau \in \mathcal{S}} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right], \quad S(0) = x.$$

Proof :-

$e^{-rt} v_{L^*}(S(t))$ is a supermartingale.

$$v_{L^*}(x) = v_{L^*}(S(0)) \geq \tilde{\mathbb{E}}\left[e^{-rt} v_{L^*}(S(t))\right].$$

Let $\tau \in \mathcal{S}$, then $e^{-r(t\wedge\tau)} v_{L^*}(S(t\wedge\tau))$ is also supermartingale

$$v_{L^*}(x) \geq \tilde{\mathbb{E}}\left[e^{-r(t\wedge\tau)} v_{L^*}(S(t\wedge\tau))\right].$$

$t \to \infty$, By DCT (v_{L^*} is bdd).

$$v_{L^*}(x) \geq \tilde{\mathbb{E}}\left[e^{-r\tau} v_{L^*} S(\tau)\right]$$

$$\geq \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right] \quad (\text{by } ①).$$

$$v_{L^*}(x) \geq \max_{\tau \in \mathcal{S}} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

Now, $e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale.

$$\therefore v_{L^*}(x) = \tilde{\mathbb{E}}\left[e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))\right].$$

$t \to \infty$, by DCT,

$$v_{L^*}(x) = \tilde{\mathbb{E}}\left[e^{-r\tau_{L^*}} v_{L^*}(S(\tau_{L^*}))\right].$$

$$= \tilde{\mathbb{E}}\left[e^{-r\tau_{L^*}}(K - S(\tau_{L^*}))\right] \quad (\text{by } ①).$$

$$\therefore v_{L^*}(x) \leq \max_{\tau \in \mathcal{S}} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

$$\therefore v_{L^*}(x) = \max_{\tau \in \mathcal{S}} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

Linear Complementarity conditions —

$$v(x) \geq (K-x)^+ \quad \forall\, x \text{ ——— (a)}$$

$$rv(x) - rxv'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0 \quad \forall\, x \geq 0 \text{ — (b)}$$

and $\forall\, x > 0$, equality holds in (a) or (b) — (c)

$$d\left(e^{-rt} v_{L^*}(S(t))\right) = -e^{-rt} rK \mathbb{1}_{\{S(t) < L^*\}} dt$$

$$+ e^{-rt} \sigma S(t) v'_{L^*}(S(t))\, d\tilde{W}(t)$$

Corollary - Consider an agent with initial capital $X(0) = v_{L^*}(S(0))$. Suppose agent uses the portfolio $\Delta(t) = v'_{L^*}(S(t))$ and consumed cash at the rate $C(t) = rK\mathbb{1}_{\{S(t)<L^*\}}$.

Then, $X(t) = v_{L^*}(S(t))$ $\forall\, t$ and $X(t) \geq (K-x)^+$.

Proof - $dX(t) = \Delta(t)\,dS(t) + r(X(t) - \Delta(t)S(t))dt - C(t)dt$.

$$d\left(e^{-rt}X(t)\right) = -re^{-rt}X(t)dt + e^{-rt}dX(t).$$

$$= e^{-rt}\left[\Delta(t)dS(t) - r\Delta(t)S(t)dt - C(t)dt\right].$$

$$= e^{-rt}\left[\sigma S(t)\Delta(t)d\tilde{W}(t) - C(t)dt\right].$$

$$= \sigma e^{-rt} v'_{L^*}(S(t))S(t)\,d\tilde{W}(t) - rK\mathbb{1}_{\{S(t)<L^*\}}dt.$$

$$= d\left(e^{-rt} v_{L^*}(S(t))\right).$$

$$\therefore \cancel{e^{-rt}} X(t) = \cancel{e^{-rt}} v_{L^*}(S(t)) \Rightarrow \boxed{X(t) = v_{L^*}(S(t))}.$$

Stopping set $\mathcal{I} = \{x \geq 0 : v_{L^*}(x) = (K-x)^+\}$.

Continuation set $\mathcal{C} = \{x \geq 0 : v_{L^*}(x) > (K-x)^+\}$.

Let $V(t)$ be the value process of the American perpetual put ~~process~~ option, then

① $V(t) \geq (K - S(t))^+ \quad \forall\, t \geq 0$

② $e^{-rt} V(t)$ is a supermartingale

③ $\exists$ a stopping time τ^* such that

$$V(0) = \tilde{\mathbb{E}}\left[e^{-r\tau^*}(K - S(\tau^*))^+\right].$$

<u>American put</u> :- $[0, T]$

Underlying asset - $dS(t) = rS(t)dt + \sigma S(t) d\tilde{W}(t)$.

<u>Def^n</u> :- Let $0 \leq t \leq T$ and $x \geq 0$ be given, $S(t) = x$.

$$\mathcal{F}_u(t) = \sigma\{S(v) : t \leq v \leq u\}.$$

$\mathcal{S}_{t,T}$ $\rightarrow$ set of all stopping times for $\mathcal{F}_u(t)$ $t \leq u \leq T$, taking values in $[t, T]$ or ∞.

$$v_{t,x} = \max_{\tau \in \mathcal{S}_{t,T}} \tilde{\mathbb{E}}\left[e^{-r(\tau - t)}(K - S(\tau)) \mid S(t) = x\right].$$

$$\tau = \infty \Rightarrow e^{-r(\tau - t)}(K - S(\tau)) = 0.$$

Linear complementarity conditions–

$$v(t,x) \geq (K-x)^+ \quad \forall\ x \geq 0,\ t \in [0,T] \quad \text{— (a)}$$

$$rv(t,x) - v_t(t,x) - rx\,v_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0$$
$$\forall\ x \geq 0,\ t \in [0,T] \quad \text{— (b)}$$

$\forall\ t \in [0,T]$, $x \geq 0$ equality holds either in (a) or (b).

$$\mathcal{I} = \{(t,x) : v(t,x) = (K-x)^+\}$$

$$\mathcal{C} = \{(t,x) : v(t,x) > (K-x)^+\}.$$

$$\tau^* = \min\{u \in [t,T] : v(u, S(u)) = (K - S(u))^+\}.$$

<u>Theorem</u>:– $e^{-rt} v(t, S(t))$ is a supermartingale $t \leq u \leq T$. and the stopped process $e^{-r(u \wedge \tau^*)} v(u \wedge \tau^*, S(u \wedge \tau^*))$ is a martingale.

<u>Corollary</u>– Consider an agent with initial capital $X(0) = v(0, S(0))$. Suppose this agent uses the portfolio process: $\Delta(u) = v_x(u, S(u))$ and consume at rate

$C(u) = rK \mathbb{1}_{\{S(u) \leq L(T-u)\}}$. Then $X(u) = v(u, S(u))\ \forall\ u \geq 0$ and $X(u) \geq (K - S(u))^+$.

1 Lec Miss

Thomas Bjork

Continuous time Portfolio Theory

S_t^i = price of 1 unit of ith asset at time t

h_t^i = No. of units of ith asset at t.

h_t = portfolio process $(h_t^1, h_t^2, \dots, h_t^N)$.

V_t^h = value of portfolio h_t at t.

$$V_t^h = h_t S_t.$$

$$dV_t^h = h_t dS_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

Def^n :- For given portfolio h, corresponding relative portfolio or portfolio weights w is given by

$$W_t^i = \frac{h_t^i S_t^i}{V_t^h}, \quad i=1,2,\dots,N$$

$$\sum_{i=1}^{N} W_t^i = 1.$$

Lemma :- A portfolio is self financing if

$$dV_t^h = \sum_{i=1}^{N} h_t^i dS_t^i$$

$$= V_t^h \sum \frac{W_t^i dS_t^i}{S_t^i}$$

Def^n:- Consider a financial market with vector price process. A contingent claim with date of maturity T also called T-claim is any RV $\mathcal{X} \in \mathcal{F}_t^S$.

A contingent claim is called simple if $\mathcal{X} = \phi(S(T))$.

The function ϕ is called the contract f^n.

Eg:- European call $\rightarrow \max\{x-K, 0\} = \phi(x)$.

Notation:- $\Pi(t, \mathcal{X})$ for price process of the contingent claim $\mathcal{X}$.

$$\Pi(T, \mathcal{X}) = \mathcal{X} = \phi(S(T)) \text{ (for simple claim)}$$

Def^n:- An arbitrage ~~process~~ possibility on a financial market is an self-financing portfolio $h \ni$

$$V^h(0) = 0, \quad \mathbb{P}(V^h(T) \geq 0) = 1, \quad \mathbb{P}(V^h(T) > 0) > 0.$$

Property 1:- Suppose $\exists$ a self-financing portfolio $h \ni$

$$dV^h(t) = K(t) V^h(t)\, dt$$

where K is adapted. Then $K(t) = r(t) \ \forall t$.

$$dB(t) = rB(t)dt$$
$$dS(t) = S(t)\mu(t, S(t))\, dt + S(t)\sigma(t, S(t))\, dW(t).$$

Take $\mathcal{X} = \phi(S(T))$

We assume that the price process is given by

$$\Pi(t, \mathcal{X}) = F(t, S(t)) :- \Pi(t).$$

$$d\Pi(t) = dF(t, S(t)) = F_t(t, S(t))\,dt + F_x(t, S(t))\,dS(t)$$
$$+ \frac{1}{2} F_{xx}(t, S(t))\,dS(t)\,dS(t).$$
$$= \left(F_t + \mu S F_x + \frac{1}{2} S^2 \sigma^2 F_{xx}\right)^{(t, S(t))} dt + S\sigma F_x^{(t, S(t))} dW(t).$$

$$d\Pi(t) = \mu_\Pi(t)\,\Pi(t)\,dt + \sigma_\Pi(t)\,\Pi(t)\,dW(t),$$

where $\mu_\Pi(t) = \dfrac{F_t + \mu S F_x + \frac{1}{2} S^2 \sigma^2 F_{xx}}{F}$.

and $\sigma_\Pi(t) = \dfrac{S \sigma F_x}{F}$

Consider a portfolio (u_S, u_Π)
asset claim

$$dV(t) = V(t)\left[\frac{u_S\, dS(t)}{S(t)} + \frac{u_\Pi\, d\Pi(t)}{\Pi(t)}\right].$$

$$= V(t)\left[u_S \mu + u_\Pi \mu_\Pi\right] dt + V(t)\left[u_S \sigma + u_\Pi \sigma_\Pi\right] dW(t)$$

$$u_S + u_\Pi = 1.$$

Let $\begin{cases} u_S \sigma + u_\Pi \sigma_\Pi = 0 \\ u_S + u_\Pi = 1 \end{cases} \Rightarrow u_S = \dfrac{\sigma_\Pi}{\sigma_\Pi - \sigma}, \quad u_\Pi = \dfrac{-\sigma}{\sigma_\Pi - \sigma}.$

$$dV(t) = V(t)\left[u_S \mu + u_\Pi \mu_\Pi\right] dt$$

$$u_S \mu + u_\Pi \mu_\Pi = r.$$

$$u_S = \frac{\sigma_\Pi}{\sigma_\Pi - \sigma} = \frac{\cancel{\sigma} S F_x / F}{\cancel{\sigma} S F_x / F - \cancel{\sigma}} = \frac{S F_x}{S F_x - F}.$$

$$u_\Pi = \frac{-\sigma}{\sigma_\Pi - \sigma} = \frac{-F}{S F_x - F}.$$

$$\frac{SF_x}{SF_x - F}\mu + \frac{-F}{SF_x - F}\left[\frac{F_t + \mu S F_x + \frac{1}{2}\sigma^2 S^2 F_{xx}}{F}\right] = r$$

$$\Rightarrow \boxed{F_t + rSF_x + \frac{1}{2}\sigma^2 S^2 F_{xx} = rF}$$

$$\Pi(T, \phi) = \phi(S(T))$$
$$F(T, x) = \phi(x)$$

Meta-theorem

Let M denote the number of underlying traded assets in the model (excluding the risk-free asset) and let R denote the number of random sources. Then,

1. The model is arbitrage free iff $M \leq R$
2. The model is complete iff $M \geq R$
3. The model is complete & arbitrage free iff $M = R$.

In BSM model, $M = 1, R = 1$.

$\therefore$ Model is complete and arbitrage free.

Incomplete Market

$$\begin{cases} dX(t) = \mu(t, X(t))\,dt + \sigma(t, X(t))\,dW(t) \\ dB(t) = rB(t)\,dt \end{cases}$$

Simple Claim – $\mathcal{X} = \phi(X(T))$.

eg:- $\phi(x) = \begin{cases} 50{,}000, & x < 0 \\ 0, & x \geq 0. \end{cases}$ $\longrightarrow$ cannot be hedged as there is nothing to hedge with.

$$\boxed{M = 0,\ R = 1}$$

$$\mathcal{X} = \phi(X(T)) : F(t, X(t))$$

$$\mathcal{Y} = r(X(T)) : \underbrace{G(t, X(t))}_{\text{Price process}}.$$

$$dF(t, X(t)) = \cancel{F_t(t, X(t))dt}\ \mu_F F dt + \sigma_F F dW(t).$$

$$dG(t, X(t)) = \mu_G G dt + \sigma_G G dW(t).$$

where, $\mu_F = \dfrac{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}}{F}$, $\sigma_F = \dfrac{\sigma F_x}{F}$.

and

$$\mu_G = \frac{G_t + \mu G_x + \frac{1}{2}\sigma^2 G_{xx}}{G}, \quad \sigma_G = \frac{\sigma G_x}{G}.$$

Now, from a self financing portfolio based on F and G, (u_F and u_G)

$$dV = V\left(u_F \frac{dF}{F} + u_G \frac{dG}{G}\right).$$

$$= V\left(u_F \mu_F dt + \sigma_F u_F dW(t) + \mu_G u_G dt + \sigma_G u_G dW(t)\right).$$

In order to make this portfolio locally risk-free,

$$\begin{cases} \sigma_F u_F + \sigma_G u_G = 0, & (\because dW(t) \text{ term is made } 0) \\ u_F + u_G = 1. \end{cases}$$

$$\Rightarrow u_F = \frac{-\sigma_G}{\sigma_F - \sigma_G}, \quad u_G = \frac{\sigma_F}{\sigma_F - \sigma_G}.$$

dt term must be r.

$$\Rightarrow dV(t) = V(t)[\underbrace{\mu_F u_F + \mu_G u_G}_{r}]dt$$

$$\Rightarrow \mu_F u_F + \mu_G u_G = r.$$

$$\Rightarrow -\mu_F \sigma_G + \mu_G \sigma_F = r(\sigma_F - \sigma_G).$$

$$\Rightarrow \boxed{\frac{\mu_F - r}{\sigma_F} = \frac{\mu_G - r}{\sigma_G}}$$ (same market price of risk).

~~Property~~ Proposition - Assume that the market is arbitrage free. Then $\exists$ a universal process $\lambda(t)$ such that

$$\frac{\mu_F - r}{\sigma_F} = \lambda(t) \text{ for any choice of the derivative } F.$$

$$\frac{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx} - rF}{\sigma F_x} = \lambda.$$

$$\Rightarrow \begin{cases} F_t + (\mu - \lambda\sigma) F_x + \frac{1}{2}\sigma^2 F_{xx} = rF \\ F(T, x) = \phi(x) \end{cases}$$

Proposition - Assuming the market is arbitrage-free, the price f^n $F(t,x)$ of the T-claim $\phi(X(T))$ solves the following equation:

$$\begin{cases} F_t(t,x) + \{\mu(t,x) - \boxed{\lambda(t,x)}\sigma(t,x)\} F_x(t,x) + \frac{1}{2}\sigma^2(t,x) F_{xx}(t,x) = rF \\ F(T,x) = \phi(x) \end{cases}$$

$$\begin{cases} F_t + (\mu - \lambda\sigma) F_x + \frac{1}{2}\sigma^2 F_{xx} = rF \\ F(T,x) = \phi(x) \end{cases},$$

$$\lambda(t,x) = \frac{\mu_G - r}{\sigma_G}.$$

<u>Proposition</u>:- Assuming the market is arbitrage free, the pricing formula $F(t,x)$ of the T-claim $\phi(X(T))$ is given by

$$F(t,x) = e^{-r(T-t)} \tilde{\mathbb{E}}[\phi(X(T)) \mid X(t) = x]$$

The dynamics of X under the risk-neutral measure $\tilde{\mathbb{P}}$ is given by:

$$dX(t) = (\mu - \lambda\sigma)\,dt + \sigma\, d\tilde{W}(t).$$

$$\tilde{W}(t) = W(t) + \int_0^t \lambda\, ds$$

$$\tilde{\mathbb{P}}(A) = \int_A Z\, d\mathbb{P}(\omega), \quad Z(T) = Z = e^{-\int_0^T \lambda(u)\,dW(u) - \frac{1}{2}\int_0^T \lambda^2(u)\,du}.$$

---

given μ, σ, $\phi^i(X(T))$ $i = 1, 2, \ldots, n$.

$\lambda = \lambda(t, x, \beta)$ (where β is parameter).

$$\begin{cases} F_t^i + (\mu - \lambda(t,x,\beta)\sigma) F_x^i(t,x) + \frac{1}{2}\sigma^2 F_{xx}^i = rF^i \\ F^i(T,x) = \phi^i(x) \end{cases}$$

gives $F^i(t,x,\beta)$. which approximates $\Pi^i(t)$ ← real life price in market.

Def^n – A zero coupon bond with maturity date T also called T-bond is a contract which guarantees the holder Re 1 to be paid on the date T. The price at time t is denoted by ~~$p(p,T)$~~ $p(t,T)$.

Assumptions – ① $\exists$ a market for T-bonds for $T > 0$.

② $p(t,t) = 1$

③ For each t, the bond price $p(t,T)$ is ~~cont~~ differentiable wrt T.

Payoff of deterministic process is risk free. All bonds here are risk-free.

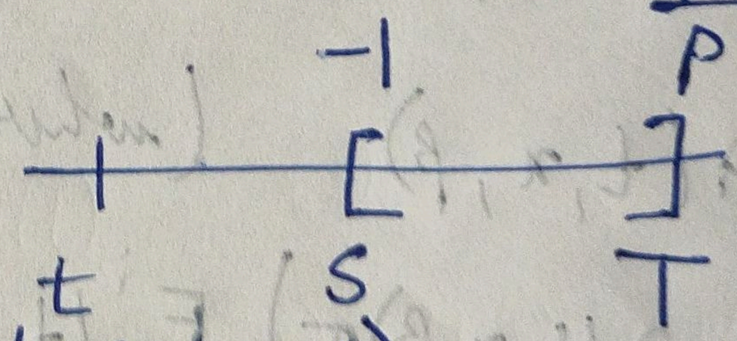
For fixed t,

$p(t,T)$ is a differentiable f^n on T.

The graph of this f^n is called "the bond price curve at time t" or "the term structure at t" (Plot wrt T)

Forward interest rate

Calculate interest rate for $[s,T]$ at time t.



$\frac{p(t,s)}{p(t,T)}$.

→ At time t, sell one s-bond $(p(t,s))$.

→ At time t, buy $\frac{p(t,s)}{p(t,T)}$ T-bonds.

→ Investment at t is 0.

The simple forward rate (or LIBOR rate) is the solution to the equation $1 + L(T-S) = \frac{p(t,S)}{p(t,T)}$.

The continuous compounded forward rate R is the solution to the equation $e^{R(T-S)} = \frac{p(t,S)}{p(t,T)}$.

Def^n:- ① The simple forward rate for $[S,T]$ at time t referred as LIBOR forward rate is defined as

$$L(t;S,T) = -\frac{p(t,T) - p(t,S)}{(T-S)\,p(t,T)}.$$

② The simple spot rate for $[S,T]$, referred to as LIBOR spot rate is defined as

$$L(S,T) = -\frac{p(S,T) - 1}{(T-S)\,p(S,T)}$$

③ The continuously compounded forward rate for $[S,T]$ is defined as

$$R(t;S,T) = -\frac{\log(p(t,T)) - \log p(t,S)}{(T-S)}$$

④ The continuously compounded spot rate for $[S,T]$ is defined as

$$R(S,T) = -\frac{\log p(S,T)}{(T-S)}.$$

⑤ The instantaneous forward rate with maturity at time T, contracted at t is defined as:

$$f(t,T) = -\frac{\partial}{\partial T}\log p(t,T).$$

⑥ The instantaneous short rate at time t is defined as

$$r(t) = f(t,t).$$

Money Market Account

$$B(t) = e^{\int_0^t r(s)\,ds} \iff \begin{cases} dB(t) = B(t)\,r(t)\,dt \\ B(0) = 1 \end{cases}$$

Lemma:- For $t \le s \le T$, we have

$$p(t,T) = p(t,s)\, e^{-\int_s^T f(t,u)\,du}.$$

& in particular,

$$p(t,T) = e^{-\int_t^T f(t,u)\,du} \quad (\text{put } s=t).$$

Proof:- $f(t,u) = -\dfrac{\partial \log p(t,u)}{\partial u}$.

$$\Rightarrow \int_s^T f(t,u)\,du = -\int_s^T \partial \log p(t,u).$$

$$\Rightarrow -\left[\log p(t,T) - \log p(t,s)\right] = \int_s^T f(t,u)\,du.$$

$$\Rightarrow p(t,T) = p(t,s)\, e^{-\int_s^T f(t,u)\,du}.$$

Assume that

$$dr(t) = a(t)\,dt + b(t)\,dW(t) \quad \text{— ①}$$

$$dp(t,T) = p(t,T)\,m(t,T)\,dt + p(t,T)\,v(t,T)\,dW(t) \quad \text{— ②}$$

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t). \quad \text{— ③}$$

Property – ① If $p(t,T)$ satisfies ②, then the forward rate dynamics we have

$$df(t,T) = \alpha(t,T)\,dt + \sigma(t,T)\,dW(t), \text{ where}$$

$$\begin{cases} \alpha(t,T) = v_T(t,T)\,v(t,T) - m_T(t,T) \\ \sigma(t,T) = -v_T(t,T). \end{cases}$$

② If $f(t,T)$ satisfies ③, then the short rate satisfies

~~dynam~~ $$dr(t) = a(t)\,dt + b(t)\,dW(t), \text{ where}$$

$$\begin{cases} a(t) = f_T(t,t) + \alpha(t,t) \\ b(t) = \sigma(t,t) \end{cases}$$

③ If $f(t,T)$ satisfies ③, then $p(t,T)$ satisfies

$$dp(t,T) = p(t,T)\left\{ r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2 \right\} dt + p(t,T)\,S(t,T)\,dW(t), \text{ where}$$

$$A(t,T) = -\int_t^T \alpha(t,s)\,ds$$

$$S(t,T) = -\int_t^T \sigma(t,s)\,ds.$$

Proof:– $$d\log p(t,T) = \frac{1}{p(t,T)}\,dp(t,T) - \frac{1}{2}\frac{1}{p^2(t,T)}\,dp(t,T)\,dp(t,T).$$

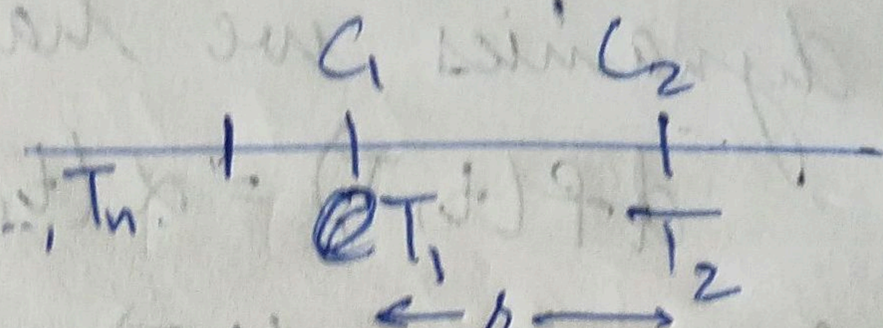
$$= m(t,T)\,dt + v(t,T)\,dW(t) - \frac{1}{2}v^2(t,T)\,dt$$

$$\Rightarrow \log p(t,T) - \log p(0,T) = \int_0^t m(u,T)\,du + \int_0^t v(u,T)\,dW(u) - \frac{1}{2}\int_0^t v^2(u,T)\,du.$$

$$\Rightarrow -f(t,T) + f(0,T) = \int_0^t m_T(u,T)\,du + \int_0^t v_T(u,T)\,dW(u) - \int_0^t v(u,T)\,v_T(u,T)\,du.$$

~~Let~~ Hence, α and σ are found.

Fixed coupon bonds



- Fix a number of dates $T_0, T_1, \dots, T_n$
- At T_i, the owner of the bond recieves C_i = coupon.
- At times T_n, the owner of " " face values of the bond K.

Then the price at $t < T_1$,

$$p(t) = C_1 p(t, T_1) + C_2 p(t, T_2) + \dots + C_n p(t, T_n) + K p(t, T_n)$$

$$= K p(t, T_n) + \sum_{i=1}^{n} C_i p(t, T_i)$$

Yield:- (T- bond) Yield $\to y$.

$$p(t, T) = 1 \cdot e^{-y(T-t)}$$

Def^n:- yield, $y(t, T) = \dfrac{-\log p(t, T)}{(T-t)}$

At fixed t, T vs $y(t, T)$ is called yield curve.

Suppose $r(t) = \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$

$\mathcal{X} = \phi(r(T)) = \{p(t, T) : t \geq 0\}$

Consider 2 bonds with maturity T and S.

$$p(t, T) = F^T(t, r(t))$$

$$p(t, S) = F^S(t, r(t))$$

$$dF^T = F_t^T dt + F_r^T dr(t) + \frac{1}{2} F_{rr}^T(t, r(t)) dr(t) dr(t)$$

$$= F^T \alpha^T dt + F^T \sigma^T dW(t)$$

$$\alpha^T = \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T}, \quad \sigma^T = \frac{\sigma F_r^T}{F^T}.$$

Portfolio with weights (u_T, u_S)

$$\Rightarrow \frac{\alpha^S(t) - r(t)}{\sigma^S(t)} = \frac{\alpha^T(t) - r(t)}{\sigma^T(t)} =: \lambda(t).$$

<u>Proposition</u> – (Term structure equation)

The bond price F^T will satisfy the equation

$$\begin{cases} F_t^T + \{\mu - \lambda\sigma\} F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F(T, r) = 1 \end{cases}$$

<u>Proposition</u> – (Risk neutral valuation).

Bond price given by the formula:

$$p(t,T) = \mathbb{E}^Q\left[e^{-\int_t^T r(s)ds} \,\middle|\, r(t) = r\right].$$

dynamics of r is given by

$$\begin{cases} dr(s) = \{\mu - \lambda\sigma\} ds + \sigma dW^Q(s) \\ r(t) = r \end{cases}$$

$$dr(t) = \cancel{\mu dt +} \mu(t, r(t))dt + \sigma(t, r(t))dW(t)$$

$$X = \phi(r(T))$$

changes boundary condition to $F^T(T, r) = \phi(r)$.

and valuation to $p(t,T) = \mathbb{E}^Q\left[e^{-\int_t^T r(s)ds} \phi(r(T)) \,\middle|\, r(t) = r\right]$.

Term structure equation

$$F_t + \{\mu - \sigma\lambda\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0$$

$$F(T, r) =$$

$$F(t, r; T) = p(t, T)$$

Dynamics of $r(t)$

$$dr(t) =$$

① Vasicek model →

$$dr(t) = (b - a\,r(t))\,dt + \sigma\,dW(t), \quad a > 0$$

② CIR →

$$dr(t) = (b - a\,r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t), \quad a > 0$$

③ Dothan →

$$dr(t) = ar\,dt + \sigma r\,dW(t).$$

④ Black Dorman Zy → $dr(t) = \theta(t) r(t)\,dt + \sigma(t) r(t)\,dW(t).$

⑤ Ho-Lee → $dr(t) = \theta(t)\,dt + \sigma\,dW(t)$

⑥ Hull-White → $dr(t) = (\theta(t) - a\,r(t))\,dt + \sigma\,dW(t), \quad a(t) > 0$

⑦ " " → $dr(t) = (\theta(t) - a\,r(t))\,dt + \sigma\sqrt{r(t)}\,dW(t), \quad a(t) > 0.$

Def^n:- If the term structure $\{p(t,T) : 0 \le t < T,\ T > 0\}$ has the form $p(t,T) = F(t, r(t); T)$ and $F(t, r, T) = e^{A(t,T) - B(t,T) r}$ where A and B are deterministic, the model is said to posses Afine term

structure (ATS).

$$dr(t) = \mu(t, r(t))\,dt + \sigma(t, r(t))\,dW(t) \qquad \text{(Q-model)}$$

$$\begin{cases} F_t + \mu(t, r(t))F_r + \frac{1}{2}\sigma^2 F_{rr} - rF = 0. \\ F(T, r) = 1. \end{cases}$$

Assume that $F(t, r(t), T) = e^{A(t,T) - B(t,T)r}$.

$$\Rightarrow \left[A_t(t,T) - B_t(t,T)\,r\right]F + \mu F(-B(t,T)) + \frac{1}{2}\sigma^2(B(t,T))^2 F - rF = 0$$

$$A(T,T) = 0 = B(T,T).$$

Assume that $\mu(t, r) = \alpha(t)\,r + \beta(t)$

$$\sigma^2(t, r) = \gamma(t)\,r + \delta(t).$$

$$\Rightarrow A_t(t,T) - (B_t(t,T) + 1)\,r - \mu B(t,T) + \frac{1}{2}\sigma^2(B(t,T))^2 = 0$$

$$\Rightarrow A_t(t,T) - (B_t(t,T) + 1)\,r - (\alpha(t)r + \beta(t))B(t,T) + \frac{1}{2}B^2(t,T)(\gamma(t)r + \delta(t)) = 0$$

$$\Rightarrow \begin{cases} A_t(t,T) - (B_t(t,T) + 1 + \alpha(t)B(t,T) - \frac{1}{2}B^2(t,T)\gamma(t))\,r \\ \quad - \beta(t)B(t,T) + \frac{1}{2}B^2(t,T)\,\delta(t) = 0. \\ A(T,T) = 0 = B(T,T) \end{cases}$$

$$\begin{cases} B_t(t,T) + \alpha B(t,T) - \frac{1}{2}\gamma(t)B^2(t,T) = -1. \\ A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T) \\ A(T,T) = 0 = B(T,T) \end{cases}$$

Proposition - Assume that μ and σ are

$$\mu(t, r) = \alpha(t) r + \beta(t)$$
$$\sigma^2(t, r) = \gamma(t) r + \delta(t)$$

Then the model admits an ATS where A and B satisfy the system of equations.

① Vasicek model →

$$dr(t) = (b - a r(t))dt + \sigma dW(t).$$

Here, $\alpha(t) = -a$, $\beta(t) = b$, $\gamma(t) = 0$, $\delta(t) = \sigma^2$.

$$B_t(t, T) - aB(t, T) = -1, \quad B(T, T) = 0$$

$$\Rightarrow B(t, T) = \frac{1}{a}\left[1 - e^{-a(T-t)}\right].$$

and $$A(t, T) = \frac{\sigma^2}{2}\int_t^T B^2(s, T)\,ds - b\int_t^T B(s, T)\,ds.$$

② Ho-Lee model →

$$dr(t) = \theta(t)dt + \sigma dW(t)$$

$\alpha(t) = 0$, $\beta(t) = \theta(t)$, $\sigma^2 = \delta(t)$, $\gamma(t) = 0$.

$$B_t(t, T) = -1, \quad B(T, T) = 0$$

$$\Rightarrow B(t, T) = T - t.$$

$$A(t, T) = \int_t^T \theta(s)(s - T)\,ds + \frac{\sigma^2}{2}\frac{(T-t)^3}{3}.$$

③ CIR model →

$$dr(t) = (b - a r(t))dt + \sigma\sqrt{r(t)}\,dW(t)$$

Here, $\alpha(t) = -a$, $\beta(t) = b$, $\gamma(t) = \sigma^2$, $\delta(t) = 0$

$$\begin{cases} B_t(t, T) - aB(t, T) - \frac{1}{2}\sigma^2 B^2(t, T) = -1 \\ B(T, T) = 0 \end{cases} \qquad \begin{cases} A_t(t, T) = bB(t, T) \\ A(T, T) = 0 \end{cases}$$

$$F(t,r) = A_0(T-t)e^{-B(T-t)r}.$$

$$B(x) = \frac{2(e^{\gamma x}-1)}{(\gamma+a)(e^{\gamma x}-1)+2\gamma}.$$

$2ab/\sigma^2$

$$A_0(x) = \left[\frac{2\gamma e^{(a+\gamma)\left(\frac{x}{2}\right)}}{(\gamma+a)(e^{\gamma x}-1)+2\gamma}\right].$$