$a, b \in \mathbb{R}$, f - m'ble f^n.

$E_{a,b} = \{x : a < f(x) \leq b\}$. m'ble.

$a\chi_{E_{a,b}}(x) \leq f(x) \; \forall \; x \in E_{a,b}$.

$f \geq 0$, f is m'ble.

For $n \in \mathbb{N}$, $k = 1, 2, \ldots, n2^n$.

$$E_{n,k} = \left\{x : \frac{k-1}{2^n} < f(x) \leq \frac{k}{2^n}\right\}.$$

$$F_n = \{x : f(x) > n\}.$$

$$\sum_{k=1}^{n2^n} \frac{k-1}{2^n} \chi_{E_{n,k}} + n\chi_{F_n} =: \phi_n(x).$$

$\phi_n \leq f$

$\phi_n \uparrow$ with $\uparrow$ in n.

$\phi_n \leq \phi_{n+1}$ and $\phi_n \uparrow f$.

$$f(x) - \phi_n(x) \leq \frac{k}{2^n} - \frac{k-1}{2^n} = \frac{1}{2^n} \to 0 \; (\text{if } f(x) < n)$$

Monotone Convergence Theorem :- (MCT)

Suppose $\{f_n\}$ is a seq. of non-neg. m'ble f^ns $\ni$ $f_n \leq f_{n+1}$. If $\lim f_n = f \; (= \sup_n f_n)$, then $\lim \int f_n = \int f = \int \lim f_n$.

$f = 0_{a.e.} \Rightarrow \int f = 0$ (trivial)

$E_n = \{x : f(x) > \frac{1}{n}\}$.

$E = \{x : f(x) > 0\} = \bigcup E_n$.

$\frac{1}{n}\chi_{E_n}(x) \leq f(x)$ $\quad m(E_n) = 0 \; \forall n$.

$f \geqslant f_n$

$\Rightarrow \int f_n \leqslant \int f.$

$\lim\limits_{n\to\infty} \int f_n \leqslant \int f.$

Let ϕ be a simple f^n s.t $\phi \leqslant f$.

$0<\alpha<1,$

$E_n = \{ x : f_n(x) > \alpha\phi(x) \}.$

$\int f_n \geqslant \int_{E_n} f_n \geqslant \alpha \int_{E_n} \phi(x).$

$\Rightarrow \lim \int f_n \geqslant \alpha \int_{E_n} \phi$

$\Rightarrow \lim \int f_n \geq \lim \alpha \int_{E_n} \phi = \alpha \int \phi.$

$\Rightarrow \lim \int f_n \geq \sup\limits_{\phi \leqslant f} \int \phi = \int f.$

ϕ - simple f^n.

$E_n \subseteq E_{n+1}, \ldots \quad , \cup E_n = E$

$\int_E \phi = \lim\limits_{n\to\infty} \int_{E_n} \phi \quad (?)$

P.T. $\int (f+g) = \int f + \int g.$

$\phi_n \uparrow f \; , \; \psi_n \uparrow g.$

$\Rightarrow \phi_n + \psi_n \uparrow f+g.$

$\int (\phi_n + \psi_n) = \int \phi_n + \int \psi_n.$

$\lim \int \phi_n = \int f, \; \lim \int \psi_n = \int g.$

$\lim \int (\phi_n + \psi_n) = \int \lim (\phi_n + \psi_n) = \int (f+g)$

$\int \sum\limits_{n=1}^{\infty} f_n = \boxed{\sum\limits_{n=1}^{\infty} \int f_n}.$

$F_N = \sum_{n=1}^{N} f_n$.

$\Rightarrow \int F_N = \sum_{n=1}^{N} \int f_n$

$\Rightarrow \lim \int F_N = \lim \sum_{n=1}^{N} \int f_n$

$\Rightarrow \int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$.

<u>Fatou's theorem</u> :- $\{f_n\}$ is seq of non-neg m'ble f^n.

$\int \liminf f_n \leq \liminf \int f_n$.

<u>Proof</u>:- $F_k = \inf_{n \geq k} f_n$, $\lim_k F_k = \liminf f_n$.

$F_n \leq f_j \;\forall j \geq n$

$\int F_n \leq \int f_j$

$\Rightarrow \int F_n \leq \inf_{j \geq n} \int f_j$

$\Rightarrow \lim \int F_n \leq \liminf \int f_n$ — ①

By MCT, $\lim \int F_n = \int \lim F_n = \int \liminf f_n$.

$\Rightarrow \int \liminf f_n \leq \liminf \int f_n$.

<u>Fatou's Lemma</u> :- $f_n \geq 0$, $f_n \to f$.

$\int \liminf f_n \leq \liminf \int f_n$.

eg:- $f_n = n \chi_{(0,\frac{1}{n})}$.

$0 < \alpha < 1$,

$\frac{1}{n} < \alpha \quad \forall n \geq n_0$

$n \chi_{(0,\frac{1}{n})}(\alpha) = 0$

$f_n(\alpha) = 0 \;\forall n \geq n_0 \Rightarrow f_n(x) \to 0 \;\forall x \in \mathbb{R}$.

$\int f_n = 1 \not\to \int 0$.

eg:- $f_n = \begin{cases} \chi_{(0,1)} & , n \text{ is odd} \\ \chi_{(1,2)} & , n \text{ is even.} \end{cases}$

$\liminf f_n(x) = 0 \quad \forall x \in \mathbb{R}$.

but $\int f_n = 1 \quad \forall n \in \mathbb{N}$.

$\int \liminf f_n = 0$ but $\liminf \int f_n = 1$

$\leq$

$f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$.

$f(x) = f^+(x) - f^-(x)$

$|f(x)| = f^+(x) + f^-(x)$.

Assume, $\int f^+ < \infty$ and $\int f^- < \infty$

and $\int f = \int f^+ - \int f^-$

$\int |f| = \int f^+ + \int f^-$.

For any m'ble f^n.

Properties of integral:-

1. $a \in \mathbb{R} \Rightarrow af$
2. $f, g \Rightarrow f \pm g$
3. $f \leq g \Rightarrow \int f \leq \int g$
4. $f = 0$ a.e. $\Rightarrow \int f = 0$

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eg:- $f : [0,1] \to \mathbb{R}$.

$f(x) = \begin{cases} 0 & , x \in \mathbb{Q} \\ n & , x \in \mathbb{Q}^c. \text{ no. of zeroes appearing just after decimal point} \end{cases}$

$g(x) = n$, if $\frac{1}{10^{n+1}} < x \leq \frac{1}{10^n}$.

$g = f$ a.e.

$$\int g = \sum_{n=1}^{\infty} \frac{9n}{10^{n+1}}$$

Lebesgue Dominated Convergence Theorem (DCT):-

Let, $\{f_n\}$ be a seq. of m'ble f^n and $f_n \longrightarrow f$ a.e. If $\exists$ an int. f^n g $\ni$ $|f_n(x)| \leq g(x)$ a.e. then $\lim \int f_n = \int f = \left(\lim \int f_n\right)$.

Proof:- $|f_n(x)| \leq g(x) \Rightarrow \int f_n^+$ and $\int f_n^- < \infty$

$\Rightarrow |f(x)| \leq g(x) \Rightarrow \int f^+, \int f^- < \infty$

$\Rightarrow f_n$'s and f are int. f^n.

$F_n = g + f_n \geq 0$, $\int \liminf F_n \leq \liminf \int F_n$.

$\Rightarrow \int \liminf (g+f_n) \leq \liminf \int (g+f_n)$

$\Rightarrow \cancel{\int g} + \int \liminf f_n \leq \cancel{\int g} + \liminf \int f_n$.

$\Rightarrow \int \liminf f_n \leq \liminf \int f_n$.

$\Rightarrow \int f \leq \liminf \int f_n$ — (1)

$G_n = g - f_n \geq 0$, $\int \liminf G_n \leq \liminf \int G_n$

$\Rightarrow \int \liminf (g - f_n) \leq \liminf \int (g - f_n)$

$\Rightarrow \limsup f_n \leq \int f$ — (2)

$\therefore \limsup f_n = \liminf f_n = \lim f_n = f$.

$f : [a,b] \longrightarrow \mathbb{R}$, bdd, R-int

$\forall \varepsilon > 0$, $\exists$ Partition $P \ni$ $U(P,f) - L(P,f) < \varepsilon$.

Specifically,

For $\varepsilon = \frac{1}{n}$,

$$P_n = \{a_0 \leq a_1 < a_2 < \dots < a_{P_n} = b\}$$

$$M_i = \sup_{x \in I_i} (f(x))$$

$$m_i = \inf_{x \in I_i} f(x)$$

$$U(P_n, f) - L(P_n, f) < \frac{1}{n}.$$

$$\sum_{i=1}^{P_n} M_i \, l(I_i) = \int_a^b M_i \chi_{I_i} = \int \phi_n = U(P_n, f)$$

$$\psi_n = \sum_{m_i} \chi_{I_i} \quad , \int \psi_n = L(P_n, f).$$

$$P_1 \subset P_2 \subset \dots$$

$$\phi = \inf \phi_n \quad ; \quad \psi = \sup \psi_n.$$

$\cancel{\phi(x) \leq f(x) \leq \psi(x)}$.

$$\psi(x) \leq f(x) \leq \phi(x)$$

Let, $E = \{x \in [a,b] : \phi(x) = \psi(x)\}$

P.T. $m(E) = 0$.