Stochastic Processes:

Def: Let T be a countable set of time index. A function $X: S\times T \to \mathbb{R}$ is called a stochastic process.

$X(t,\omega)$ $\quad \omega\in S, t\in T$

→ As T is countable, $T=\{0,1,\dots\}$, we generally denote Stochastic process by $\{X_n : n\geq 0\}$

ex: No. of students getting AA each year.

ex: Price of gold.

ex: Min Temp of each day.

Def: The set of all possible values taken by stochastic process $\{X_n : n\geq 0\}$ is called State space

→ We will only consider almost countable state spaces & it will be denoted by $\{0,1,\dots\}$

→ $X_n = i \Rightarrow$ process is said to be state i at time n.

$$S \xrightarrow{p} I \xrightarrow{p} R \quad t$$

Susceptible Infected Recovery.

Markov chain

Def: A stochastic process $\{X_n : n\geq 0\}$ is said to be a Markov chain if

$$P(X_{n+1}=j \mid X_n=i, X_{n-1}=i_{n-1}, X_{n-2}=i_{n-2}, \dots, X_0=i_0) = P(X_{n+1}=j \mid X_n=i)$$

for all states $i_0, i_1, \dots, i_{n-1}, i$ & j & all $n\geq 0$.

0 1 . . . n n+1

Today

For a MC, the conditional distribution of X_{n+1} (future) given the past $X_0, \dots, X_{n-1}$ & present X_n depends only on present X_n and not $X_0, \dots, X_{n-1}$

→ Stocks, monarch butterfly migration.

2) $P(X_{n+1}=j \mid X_m=i_m, X_{m-1}=i_{m-1}, \dots, X_0=i_0) =$
$P(X_{n+1}=j \mid X_m=i_m)$.

3) $P(X_{n+1}=j \mid X_{n_1}=i_{n_1}, X_{n_2}=i_{n_2}, \dots, X_{n_k}=i_{n_k}) = P(X_{n+1}=j \mid X_{n_1}=i_{n_1})$
$n_k < n_{k-1} < \dots < n_2 < n_1 \leq n$

Time-homogenous Markov chain

Def: A MC $(X_n; n\geq 0)$ is said to be time homogenous if

$$P(X_{n+1}=j \mid X_n=i) = P(X_n=j \mid X_{n-1}=i) = p_{ij} \quad \forall n \geq 1.$$
$$= P(X_1=j \mid X_0=i)$$
$$= P(X_2=j \mid X_1=i)$$
$$= P(X_3=j \mid X_2=i)$$
$$= P(X_{11}=j \mid X_{10}=i)$$

$\Rightarrow \sum_{j=0}^{\infty} P_{ij} = 1.$ $\qquad P(A) = \sum_{i=1}^{\infty} P(A \cap B_i)$

$\Rightarrow P_{ij} \geq 0 \quad \forall i \,\&\, j.$

$\Rightarrow$ we only consider time-homogenous MC.

$\Rightarrow$ P_{ij} is called onestep transition probability from state i to state j.

$\Rightarrow$ matrix $P = ((P_{ij}))_{i,j\geq 0}$ is called one-step transition probability matrix.

ex: rain today $\rightarrow$ tomorrow rains $\Rightarrow \alpha = P(X_{n+1}=0 \mid X_n=0)$
doesn't rain $\rightarrow$ tomorrow rains $\Rightarrow \beta = P(X_{n+1}=0 \mid X_n=1)$
$X_n \rightarrow$ Rain on n^{th} day.
$\{X_n : n\geq 0\}$ is a MC (given) with
State space $\{0,1\}$

$$X_n = \begin{cases} 0 & \text{rain on } n^{th} \text{ day} \\ 1 & \text{not raining} \end{cases}$$

One-step transition probability matrix (TPM)

$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & \alpha & 1-\alpha \\ 1 & \beta & 1-\beta \end{array}$$

$\{X_{n+1}\}$ (columns), $\{X_n\}$ (rows)

row sum = 1

Example: Suppose chance of rain tomorrow depends on last two days.

$X_n = 0$ if rains on n^{th}
$= 1$ o.w.

$P(X_{n+1}=0 \mid X_n=0, X_{n-1}=0) = 0.7$

$P(X_{n+1}=0 \mid X_n=0, X_{n-1}=1) = 0.5$

$P(X_{n+1}=0 \mid X_{n-1}=0, X_n=1) = 0.4$

$P(X_{n+1}=0 \mid X_n=1, X_{n-1}=1) = 0.2$

Define

$$Y_n = \begin{cases} 0 & \text{if it rained on } n^{th}, (n-1)^{th} \text{ day} \\ 1 & \text{if it rained on } n^{th} \text{ but not on } (n-1)^{th} \text{ day} \\ 2 & \text{if it rained on } (n-1)^{th} \text{ day; not on } n^{th} \text{ day} \\ 3 & \text{if it didn't rain on } n^{th}, (n-1)^{th} \text{ day.} \end{cases}$$

State space $= \{0,1,2,3\}$

$\{Y_n, n \geq 0\}$ is MC

Now transition probability Matrix.

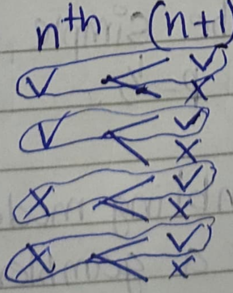
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.4 & 0 & 0.6 & 0 \\ 0.2 & 0 & 0.8 & 0 \end{bmatrix} \end{array} \quad \{X_{n+1}\}$$

$\{X_n\}$

	Rain $(n-1)^{th}$	n^{th}
0	✓	✓
1	✗	✓
2	✓	✗
3	✗	✗

Rain

n^{th} $(n+1)^{th}$



$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0.7 & 0 & 0.3 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0 & 0.2 & 0 & 0.8 \end{bmatrix} \end{array} \quad \{Y_{n+1}\}$$

$\{Y_n\}$

Example 6:

Consider a communication system that transmit the digits 0 & 1. Each digit must pass several stages, at each stage of which there will be p probability that it remains unchanged.

X_n be the digit entered at the nth stage

$$X_n = \begin{cases} 0 \\ 1 \end{cases}$$

$\{X_n ; n \geq 0\}$ is a 2-state MC. State space $= \{0,1\}$.

$P_{i,i+1} = 1-p = \cancel{ke}\ P_{i,i-1}$ $\quad i = 1,2,3, \ldots$

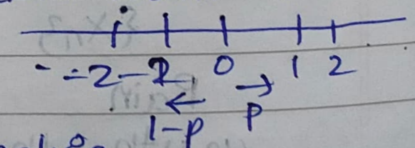
$$P(X_{n+1}=0 \mid X_n=0) = p = (P(X_{n+1}=1 \mid X_n=1))$$

TPM is

$$P = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$$

Example-7:

A MC whose state space is given by set of integer is said to be simple random walk if for some $0<p<1$. $P_{i,i+1} = p = 1 - P_{i,i-1}$.



It is said to be simple symmetric random walk if $p = \frac{1}{2}$.

Example-8: (Gambling model)

Consider a gambler who at each play of game either wins 1 with probability p or loss with probability 1-p. He quits if he is broke or N rupees attains.

$P_{0,0} = P_{N,N} = 1$ (absorbing states 0, N)

$P_{i,i+1} = P$

$P_{i,i-1} = 1-P$ Random walk $i = 1, 2, \ldots, N-1$

So, this is a finite simple random walks which with absorbing barriers at states 0 & N.

Chapman- Kolmogorov Equations :

Theorem: Consider MC having state space $\{0, 1, \ldots\}$ and one-step transition probability P_{ij} for $i,j = 0, 1, \cdots$ let us define

$$P_{ij}^{(n)} = P(X_n = j \mid X_0 = i) = P(X_{n+k} = j \mid X_k = i)$$

The chapman-Kolmogorov equations are given by

$$P_{ij}^{(m+n)} = \sum_{k=0}^{\infty} P_{ik}^{(m)} P_{kj}^{(n)} \quad \text{for all } m, n \geq 0 \text{ and all } i,j = 0, 1, \cdots$$

Proof:

$P(A) = \sum_{i=1}^{\infty} P(A \cap B_i)$

$$P_{ij}^{(m+n)} = P(X_{m+n} = j \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P(X_{m+n} = j, X_m = k \mid X_0 = i) \quad \text{(using Total Prob.)}$$

$$= \sum_{k=0}^{\infty} \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_0 = i)}$$

$$= \sum_{k=0}^{\infty} \frac{P(X_{m+n} = j, X_m = k, X_0 = i)}{P(X_m = k, X_0 = i)} \cdot \frac{P(X_0 = i, X_m = k)}{P(X_0 = i)}$$

$$= \sum_{k=0}^{\infty} P(X_{m+n} = j \mid X_m = k, X_0 = i)\, P(\cancel{X_0 = i}, X_m = k \mid X_0 = i)$$

$$= \sum_{k=0}^{\infty} P_{kj}^{(n)} P_{ik}^{(m)}$$

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$$\sum_{k=0}^{\infty} P_{ik}^{(m)} = 1$$

nstep transition Probability matrix by $P^{(n)}$, then

$$P^{(n+m)} = P^{(n)}P^{(m)} = P^{(m)}P^{(n)}$$

Example1:

Suppose that chance of rain Tomorrow is a Markov chain. Find probability that it will rain four days from today given it is raining today.

$P(X_{n+1}=0|X_n=0)=0.7$

$P(X_{n+1}=0|X_n=1)=0.4$

TPM

$$P = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Find

$P(X_4=0|X_0=0)$

↓ 4-step TPM is needed

$$P^{(4)} = P^4 = \begin{array}{c|cc} & 0 & \\ 0 & & \end{array}\begin{bmatrix} 0.57 & 0.42 \\ 0.57 & 0.43 \end{bmatrix}$$

Example 2:

Suppose that balls are succesively distributed among 8 urns. with each ball being equally likely to be put in any of the urns. What is probability that exactly 3 are occupied given 9 balls are distributed.

X_n = no. of urns occupied

$\{X_n : n \geq 0\}$ is a Markov chain ~~is Marko~~ with state space $\{0,1,\ldots,8\}$

TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{8} & \frac{7}{8} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{8} & \frac{6}{8} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{8} & \frac{5}{8} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{8} & \frac{4}{8} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{8} & \frac{3}{8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{6}{8} & \frac{2}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{8} & \frac{1}{8} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}.$$

To find $P^{(9)}_{0,3}$

$$P^{(9)}_{0,3} = \sum_{k=0}^{8} P^{(1)}_{0k} P^{(8)}_{k3} = P^{(8)}_{13} \qquad P^{(1)}_{01} = 1 \text{ else } 0$$

Here Markov chain is non-decreasing.
We will combine the states 4, 5, 6, 7, 8 in to one state
Call it state 4.

Define.

$$Y_n = \begin{cases} 1 & \text{exactly 1 urn occupied after n balls} \\ 2 & \text{exacty 2 urns} \\ 3 & \text{exactly 3 urns} \\ 4 & \text{exactly 4 urns.} \end{cases}$$

$P^{(8)}_{13}$ – denote the probability that exactly 3 urns are occupied after 8 balls are distributed given 1 urn is already occupied.

$MC = \{Y_n : n \geq 0\}$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/8 & 7/8 & 0 & 0 \\ 0 & 2/8 & 6/8 & 0 \\ 0 & 0 & 3/8 & 5/8 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \qquad P^{(8)}_{13} = P^{8}_{13}$$

$$P^4 = \begin{bmatrix} 0.0002 & 0.0256 & 0.2563 \\ & & \\ & & \\ & & \end{bmatrix}$$

Example 3: In a sequence of independent flips of a fair coin, let N denote the no. of flips until there is a run of 3 consecutive heads. Find.

$P(N \leq 8)$, $P(N=8)$

Define MC $\{X_n \mid n \geq 0\}$ State space $\{0,1,2,3\}$

$$\begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$H \to T$, $\to H$

$HH \to T$, $\to H$

To find $P_{03}^{(8)} = \frac{107}{256} = P(N \leq 8)$

$N \leq 8$ happens iff three consecutive heads happen before or at $N=8$, $P8$

H H H T T
4 5 6 7 8

$P(N=8) = P(N \leq 8) + P(N \leq 7)$

Fact:

Like a random variable is probablistically specified by it's distribution, stochastic process is specified by it's finite dimensional distributions i.e $P(X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n)$ $\forall n \geq 0$ & $\forall i_0, i_1, \ldots, i_n \in S \Rightarrow$ state space

Remark:

Let $P(X_0=i)=\mu_{i_0}$ for $i\in S$. Then

$$P(X_0=i_0, X_1=i_1, \dots, X_n=i_n) =$$
$$P(X_n=i_n \mid X_{n-1}=i_{n-1}, \dots, X_0=i_0)\, P(X_{n-1}=i_{n-1}, \dots, X_0=i_0)$$
$$= P(X_n=i_n \mid X_{n-1}=i_{n-1})\, P(X_{n-1}=i_{n-1} \mid X_{n-2}=i_{n-2}, \dots X_0=i_0)$$
$$P(X_{n-2}=i_{n-2}, \dots, X_0=i_0)$$
$$= P_{i_{n-1}i_n} P_{i_{n-2}i_{n-1}} \cdots P_{0,1}\, P(X_0=i_0)$$
$$= \left(\prod_{k=0}^{n-1} P_{i_k i_{k+1}}\right)\mu_{i_0}$$

Accessibility

Def: State j is said to be accesible from state i if there exists $n\geq 0$ such that $P_{ij}^{(n)}>0$, where

$$P_{ij}^{(0)} = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{ow} \end{cases}$$

If j is not accessible from i, then

$$P(\text{Ever be in } j \mid \text{starting from } i) = 0.$$
$$P\left(\bigcup_{n=0}^{\infty}\{X_n=j\} \mid X_0=i\right) \leq \sum_{n=0}^{\infty} P(X_n=j \mid X_0=i)=0$$

Communication

Def:

Two states i & j are said to communicate if i and j are accessible from each other, i.e, $\exists\, m\geq 0, \exists\, n\geq 0$ st $P_{ij}^{(n)}>0$ & $P_{ji}^{(m)}>0$

$i\to j$: j is accessible from i

$i\leftrightarrow j$: i and j communicate

Remark:

$i\leftrightarrow i$ (Reflexive) (Transitive) $i\leftrightarrow k,\ k\leftrightarrow j \Rightarrow i\leftrightarrow j$

$i\leftrightarrow j \Leftrightarrow j\leftrightarrow i$ (Symmetry)

Remark:

Thus communication is equivalence relation. Hence it partitions the state space into equivalence classes. $\longrightarrow$ [ ]

$$3. \quad P_{ij}^{(m+n)} = \sum_{l=0}^{\infty} P_{il}^{(m)} P_{lj}^{(n)} \geq P_{ik}^{(m)} P_{kj}^{(n)} > 0 \qquad i \to j$$

So $j \leftrightarrow i$

$$P_{ji}^{(m+n)} = \sum_{l=0}^{\infty} P_{jl}^{(m)} P_{li}^{(n)} \geq P_{jk}^{(m)} P_{kj}^{(n)} > 0$$

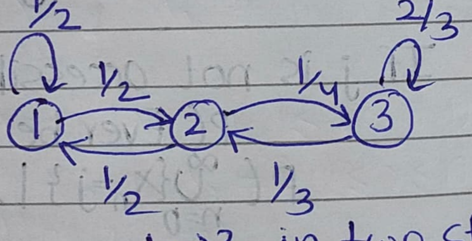
Irreducibility

Def:

A MC is said to be irreducible if all states communicate with each other. i.e there is single communicating class.

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & & & \\ 2 & & P_1 = & \\ 3 & & & \end{array}$$

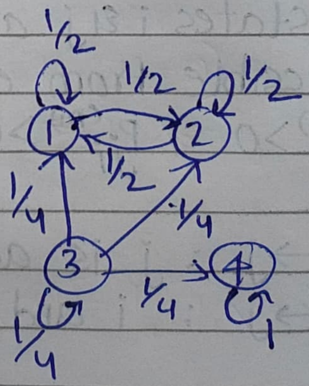
$$P_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/4 & 1/4 \\ 0 & 1/3 & 2/3 \end{bmatrix} \end{array}$$



$1 \to 3$ in two steps accesible

It is irreducible

$$P_2 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



Equivalence classes
$\{1,2\}$
$\{3\}$
$\{4\}$

Hitting Time

Def: For any $A \subset S$, the hitting time T_A is defined by

$$T_A = \inf\{n \geq 1 : X_n \in A\}$$
with the convention that $\inf \phi = \infty$

Remark: T_A is the first time after 0, when chain enters A.
Remark: T_A is also called first passage time.
Remark: $T_{\{i\}}$ will be denoted by T_i, $i \in S$.

Classification of states:

Def: A state i is called recurrent if $P(T_i < \infty | X_0 = i) = 1$
visit state i many times

Def: A State i is called Transient if $P(T_i < \infty | X_0 = i) < 1$
some probability assingned $T_i = \infty$

Remark: i is recurrent iff

$$f_{ii} = P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i) = 1.$$

$P(T_i = \infty | X_0 = i) = 1 - P(T_i < \infty | X_0 = i)$
i is recurrent iff $P(T_i = \infty | X_0 = i) = 0$.
i is transient iff $P(T_i = \infty | X_0 = i) > 0$.

if i is transient, T_i is not proper RV. T_i can take infinite time.
$E(T_i | X_0 = i) = \infty$.
else could be finite or infinite.

Def: A recurrent state i is called null recurrent if $E(T_i|X_0=i)=\infty$ and postive recurrent if $E(T_i|X_0=i)<\infty$.

Theorem: {Strong Markov property} For any state i, and any intial distribution $\mu=\{\mu_i\}$ and any $k<\infty$ and any states $i_1, i_2, \dots, i_k$.

$$P_\mu(X_{T_i+j}=i_j, j=1,2,\dots,k, T_i<\infty) = P_\mu(T_i<\infty) P(X_j=i_j, j=1,2,\dots,k | X_0=i)$$

Proof:

Let $n \in \mathbb{N}$

$P_\mu(X_{T_i+j}=i_j, 1\le j\le k, T_i=n)$

$= P_\mu(X_{n+j}=i_j, 1\le j\le k, X_n=i, X_r\neq i, 1\le r\le n-1)$

$= P_\mu(X_{n+j}=i_j, 1\le j\le k | X_n=i, X_r\neq i, 1\le r\le n-1) P(X_n=i, X_r\neq i, 1\le r\le n-1)$

$= P_\mu(X_j=i_j, 1\le j\le k | X_0=i) P(T_i=n)$

Taking sum over n both sides.

$$\Rightarrow \sum_{n=1}^{\infty} P(X_{T_i+j}=i_j, j\in\{1,2,\dots,k\}, T_i=n) = \sum_{n=1}^{\infty} P(X_j=i_j, 1\le j\le k | X_0=i) P_\mu(T_i=n)$$

$$\Rightarrow P_\mu(X_{T_i+j}=i_j, 1\le j\le k, T_i<\infty) = P_\mu(T_i<\infty) P(X_j=i_j, 1\le j\le k | X_0=i)$$

Def: Let i be a state. Define $T_i^{(0)}=0$ and for $k\geq 0$.

$$T_i^{(k+1)} = \begin{cases} \inf\{n : n > T_i^{(k)}, X_n = i\} & \text{if } T_i^{(k)} < \infty \\ \infty & \text{o.w.} \end{cases}$$

Theorem: Let i be recurrent. Then for all $k\geq 0$.

$$P(T_i^{(k)} < \infty \mid X_0 = i) = 1.$$

Proof:

We will use induction. Since in recurrent state, the above is true for $k=1$. We assume that it is true for $k=n$. We have to prove that it is true for $k=n+1$.

$$P(T_i^{(n+1)} < \infty \mid X_0 = i) = \sum_{m=n}^{\infty} P(T_i^{(n+1)} < \infty, T_i^{(n)} = m \mid X_0 = i)$$

$$= \sum_{m=n}^{\infty} P(T_i^{(n+1)} < \infty \mid T_i^{(n)} = m, X_0 = i)\, P(T_i^{(n)} = m \mid X_0 = i)$$

Time homogenous

$$= \sum_{m=n}^{\infty} 1 \times P(T_i^{(n)} = m \mid X_0 = i)\, P(T_i^{(1)} < \infty \mid X_0 = i)$$

$= \cancel{1 \text{ (Since}}$

$$= P(T_i^{(n)} < \infty \mid X_0 = i) = 1$$

Cycles:

Def: Let

$\eta_r = \{X_j : T_i^{(r)} \leq j < T_i^{(r+1)}, T_i^{(r+1)} - T_i^{(r)}\}$ for $r = 0, 1, 2, \ldots$ The η_r's are called cycles or excursions.

$$\eta_0 = \{X_0, X_1, \ldots, X_{T_i^{(1)}-1}, T_i^{(1)}\}$$

$$\eta_1 = \{X_{T_i^{(1)}}, X_{T_i^{(1)}+1}, \ldots, X_{T_i^{(2)}-1}, T_i^{(2)} - T_i^{(1)}\}$$

Theorem:

Let i be a recurrent state. Under $X_0 = i$, the sequence $\{\eta_r\}_{r=0}^{\infty}$ are i.i.d as random vector with random number of components, i.e., for any $k \in \mathbb{N}$

$$P(\eta_r = (x_{r_0}, x_{r_1}, \ldots, x_{r_{j_r}}, j_r), r = 0, 1, \ldots, k \mid X_0 = i)$$
$$= \prod_{r=0}^{k} P(\eta_r = (x_{r_0}, x_{r_1}, \ldots, x_{r_{j_r}}, j_r \mid X_0 = i)$$

for any states $x_{r_0}, x_{r_1}, \ldots, x_{r_{j_r}}$ and time $j_r, r = 0, 1, \ldots, k$

Number of visits:

For any state i, let N_i be the number of visits to state i. Then

a) if i is recurrent $\Rightarrow P(N_i = \infty \mid X_0 = i) = 1$

b) i is transient $\Rightarrow P(N_i = n \mid X_0 = i) = f_{ii}^{n}(1 - f_{ii})$ for $n = 0, 1, 2, \ldots$

where $f_{ii} = P(T_i < \infty \mid X_0 = i)$ is probability of returning to i starting from i. Thus.

$$N_i \mid X_0 = i \sim Geo(1 - f_{ii})$$

Corollary:

1. A state i is called recurrent iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} = \infty$

2. transient iff $\sum_{n=1}^{\infty} P_{ii}^{(n)} < \infty$.

Define $\delta_{X_n, i} = \begin{cases} 1 & \text{if } X_n = i \\ 0 & \text{ow} \end{cases}$

$$N_i = \sum_{n=1}^{\infty} \delta_{X_n, i}$$

$$E(N_i | X_0 = i) = E\left(\sum_{n=1}^{\infty} \delta_{X_n, i} \,|\, X_0 = i\right)$$

$$= \sum_{n=1}^{\infty} E(\delta_{X_n, i} | X_0 = i) = \sum_{n=1}^{\infty} p_{ii}^{(n)}$$

if i is recurrent, then

$$P(T_i < \infty | X_0 = i) = 1$$

$$E(N_i | X_0 = i) = \infty = \sum_{n=1}^{\infty} p_{ii}^{(n)}$$

if i is transient, then

$$E(N_i | X_0 = i) = \frac{f_{ii}}{1 - f_{ii}} < \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$$

Let $\sum_{n=1}^{\infty} p_{ii}^{(n)} = \infty$, if possible, let i be transient state

$\Rightarrow \sum_{n=1}^{\infty} p_{ii}^{(n)} < \infty$ is a contradiction

Hence, i is recurrent

<u>Some Theorems:</u>

<u>Theorem</u>. If the state space is finite, then at least one state must be recurrent.

let $S = \{0, 1, 2, \dots, k\}$, $k < \infty$

Since,

$$n = \sum_{i \in S} \sum_{j=1}^{n} \delta_{X_j, i} \qquad \delta_{X_j, i} = \begin{cases} 1 & \text{if } X_j = i \\ 0 & \text{ow.} \end{cases}$$

There exist a state i_0, such that as $n \to \infty$, $\sum_{j=1}^{n} \delta_{X_j, i} \to \infty$ with Positive probability. That implies i_0 is recurrent state

$$P(N_i = \infty | X_0 = i) = 1.$$

Theorem: $i \leftrightarrow j$ Then 1. If i is recurrent, j is recurrent
2. If i is transient, j is transient

$i \leftrightarrow j$ $\exists$ $m \geq 0, n \geq 0$ such that $P_{ij}^{(m)} > 0$ & $P_{ji}^{(n)} > 0$

Now,

$$P_{ij}^{(m+k+n)} > P_{ij}^{(m)} P_{jj}^{(n)} \sum_{k=1}^{\infty} P_{jj}^{(k)}$$

$\sum_{k=1}^{\infty} P_{ii}^{(k)} < \infty$ for i being transient

if i is transient
$\Rightarrow \sum_{k=1}^{\infty} P_{ii}^{(m+k+n)} < \infty$

$\Rightarrow \sum_{k=1}^{\infty} P_{jj}^{(k)} < \infty \Rightarrow j$ is transient.

Similarly

$$\sum_{k=1}^{\infty} P_{jj}^{(n+k+m)} > P_{ji}^{(n)} P_{ij}^{(m)} \sum_{k=1}^{\infty} P_{ii}^{(k)}$$

i is recurrent, $\sum_{k=1}^{\infty} P_{ii}^{(k)} = \infty \Rightarrow \sum_{k=1}^{\infty} P_{jj}^{(n)} = \infty$

$\Rightarrow j$ is recurrent.

Theorem: let i be recurrent and $i \to j$. Then
$f_{ij} = P(T_j < \infty \mid X_0 = i) = 1$,
$f_{ji} = P(T_i < \infty \mid X_0 = j) = 1$ and.
and j is recurrent
(Not true if i is transient)

Theorem: Suppose that $\{X_n\}$ is irreducible and recurrent. Then $\forall\, i \in S$, $P_\mu(T_i < \infty) = 1$ for any initial distribution μ.

$$P_\mu(T_i < \infty) = \sum_{j \in S} P_\mu(T_i < \infty \mid X_0 = j) P_\mu(X_0 = j)$$

$$= \sum_{j \in S} P_\mu(X_0 = j) = 1.$$

example 1:

$$\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

finite State space, atleast one recurrent, irreducible, so all recurrent.

2. $\Bigg[$

$P_{ii}^{(n)} = \frac{1}{2} \;\; \forall n,\; i = 1, 2, 3, 4$
Now if 5 is recurrent,
$5 \to 1$
But ~~$5 \not\to 1$~~ $1 \not\to 5$
$\to$ 5 is transient

example 3: Consider a simple random walk: $S = \{0, \pm 1, \pm 2, \ldots$
$P_{i,i+1} = p = 1 - P_{i,i+1}$

1) The chain is irreducible
2) If $p \neq \frac{1}{2}$, the state 0 is transient
3) If $p = \frac{1}{2}$, the state 0 is recurrent.

Now, $P_{00}^{(2n)} = \binom{2n}{n} p^n (1-p)^n$ and $P_{00}^{(2n-1)} = 0 \quad \forall n \in \mathbb{N}$.

$$= \frac{(2n)!}{n!\, n!} p^n (1-p)^n$$

$$\sim \frac{\sqrt{2\pi}\,(2n)^{2n+1/2} e^{-2n}}{2\pi\, n^{2n+1} e^{-2n}} p^n (1-p^n)$$

$$\sim \frac{2^{2n+1/2}}{\sqrt{2\pi}\sqrt{n}} p^n (1-p)^n$$

$$= \frac{4^n p^n (1-p)^n}{\sqrt{n\pi}}$$

$$P_{00}^{(2n)} \sim \frac{1}{\sqrt{n\pi}} \quad (\text{if } p = 1/2)$$

$$\Rightarrow \sum_{n=1}^{\infty} P_{00}^{(n)} = \sum_{n=1}^{\infty} P_{00}^{(2n)} \sim \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ divivges.}$$

let $p \neq 1/2$ and define $x_n = \frac{4^n p^n (1-p)^n}{\sqrt{n\pi}}$

using ratio test,

$$\frac{x_{n+1}}{x_n} = \frac{4^{n+1} p^{n+1} (1-p)^{n+1}}{\sqrt{(n+1)\pi}} \cdot \frac{\sqrt{n\pi}}{4^n p^n (1-p)^n}$$

$$= 4p(1-p)\sqrt{\frac{n}{n+1}}$$

$$\rightarrow 4p(1-p) < 1 \text{ for } p \neq 1/2$$

So, $\sum x_n$ converges if $p \neq 1/2$.

So, State 0 is transient.

Period:

Def: The period of a state i is defined by the greatest common divisor of all $n \geq 1$ for which $P_{ii}^{(n)} > 0$ i.e.

$$d(i) = \begin{cases} \gcd\{n \geq 1 : P_{ii}^{(n)} > 0\} & \text{if } \{n \geq 1 : P_{ii}^{(n)} > 0\} \neq \phi \\ 0 & \text{ow.} \end{cases}$$

example: $S = \{0, \pm 1, \pm 2, \ldots\}$ $P_{i,i+1} = a$, $P_{i,i-1} = b$, $P_{ii} = c$, where

If $c > 0$, $P_{ii}^{(1)} > 0 \Rightarrow d(i) = 1$

if $c = 0$, $P_{ii}^{(n)} > 0 \Rightarrow d(i) = 2$.

Def: for $i \in S$, $N_n(i) = \#\{t : 0 \leq t \leq n, X_t = i\}$ is no. of visits to state i during $\{0, 1, 2, \ldots, n\}$

Def:

for $i \in S$, $L_n(i) = \frac{N_n(i)}{n+1}$. Then $\{L_n(i) : i \in S\}$ is called empirical distribution at time n.

Theorem: for fixed state $i \in S$. Then

1) i is transient iff $\sum_{k=0}^{\infty} P_{ii}^{(k)} < \infty$

2) i is null recurrent iff $\sum_{k=0}^{\infty} P_{ii}^{(k)} = \infty$ and $\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} P_{ii}^{(k)} = 0$

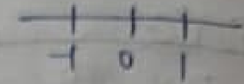
$$E(L_n(i) | X_0 = i) = \frac{1}{n+1} \sum_{k=0}^{n} P_{ii}^{(k)}$$

$$\delta_{X_k,i} = \begin{cases} 1 & \text{if } X_k = i \\ 0 & \text{ow} \end{cases}$$

$$\therefore N_n(i) = \sum_{k=0}^{n} \delta_{X_k,i}$$

$$E(N_n(i) | X_0 = i) = \sum_{k=0}^{n} E(\delta_{X_k,i} | X_0 = i) = \sum_{k=0}^{n} P(X_k = i | X_0 = i)$$

$$= \sum_{k=0}^{n} P_{ii}^{(k)}$$



for SRW,

$$P_{00}^{(2n)} \sim \frac{1}{\sqrt{n\pi}} \quad \text{and} \quad P_{00}^{(2n-1)} = 0$$

That means $P_{00}^{(n)} \to 0$ as $n \to \infty$. So

$$\frac{1}{n+1} \sum_{k=0}^{n} P_{ii}^{(k)} \to 0 \text{ as } n \to \infty,$$ So 0 is null recurrent.

Stationary distribution

Def: A vector $\{\pi_i\}_{i \in S}$ is called a stationary distribution for a MC with transition probability matrix $P = ((P_{ij}))$ if $\pi_i \geq 0 \ \forall i \in S$, $\sum_{i \in S} \pi_i = 1$ and $\sum_{j \in S} \pi_j P_{ji} = \pi_i \ \forall i \in S$.

$\{\pi_1, \pi_2, \ldots, \pi_n\} \to$ stationary distribution

Remark:

$$\underline{\pi} P = \underline{\pi}$$

$$(\pi_1 \pi_2 \ldots \pi_k) \begin{pmatrix} P_{11} & \cdots & P_{1k} \\ P_{21} & \cdots & P_{2k} \\ & & \\ P_{n1} & \cdots & P_{kk} \end{pmatrix} = (\pi_1 \pi_2 \ldots \pi_k)$$

Thus a stationary distribution is a left eigen vector corresponding eigen value 1 and $\underline{\pi}\, \underline{1} = 1$

Remark:

Let $\{\pi_i\}_{i \in S}$ be a stationary distribution of a MC $\{X_n\}$ and intial distribution is same as stationary distribution. Then

$$P(X_n = i) = \pi \text{ if } P(X_0 = i) = \pi_i \ \forall i \in S.$$

for $n=1$

$$P(X_1=i) = \sum_{j\in S} P(X_1=i|X_0=j)\ P(X_0=j)$$

$$= \sum_{j\in S} P_{ji}\ \pi_j = \pi_i$$

Assume true for k.

for $n=k+1$

$$P(X_{k+1}=i) = \sum_{j\in S} P(X_{k+1}=i|X_k=j)\ P(X_k=j)$$

$$= \sum_{j\in S} P_{ji}\ \pi_j = \pi_i$$

<u>Remark</u>: If S is finite, Stationary distribution exists.

<u>Theorem</u>: let $\{X_n\}$ be a MC having stationary distribution π, then $\pi_i>0 \Rightarrow$ positive recurrent i.

<u>Theorem</u>: let $\{X_n\}$ be a MC, then a stationary distribution exists iff there is atleast one positive recurrent state.

<u>Corollary</u>: Simple Random walk does not admit a stationary distribution.

SRW,

$$P_{i,i+1} = p$$
$$P_{i,i-1} = 1-p$$

if $p=\frac{1}{2}$, state 0 is null recurrent

if $p\neq\frac{1}{2}$, State 0 is transient.

Now SRW is irreducible,

So, either all states are null recurerent or transien

So, no positive recurrent states, hence no stationary distribution.

Corollary:

A finite state MC has atleast one positive recurrent state

Corollary:

In general, stationary distribution may not exist. If exist, it may not be unique.

ex: Consider MC with $P_{i-1,i}=1$ $\forall\ i=1,2,\dots$

If possible MC has stationary distribution

$$\sum_{j\in S} \pi_j P_{ji} = \pi_i$$

$$\Rightarrow \quad \pi_{i-1} = \pi_i$$

all π_i are constant $\Rightarrow \sum_{i\in S} \pi_i = \infty$