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Lump payments of dividends:-

$0 = t_0 < t_1 < t_2 \ldots < t_n < T = t_{n+1}$

t_i - dividend paying date

at time t_i, the dividend paid is $a_i S(t_i -)$,

a_i is $\mathcal{F}(t_i)$ - mble and $a_i \in [0,1]$

The price of the stock is given by

$$S(t_i) = (1-a_i) S(t_i -)$$

$$dS(t) = \alpha(t) S(t) dt + \sigma(t) S(t) . dW(t) \text{ for } t_i \le t < t_{i+1}$$

If ur portfolio process is $\Delta(t)$ then

$$dX(t) = \Delta(t) dS(t) + r(t) [X(t) - \Delta(t) S(t)] . dt$$

$$= \Delta(t) [\alpha(t) S(t) . dt + \sigma(t) S(t) . dW(t)] + R(t) [X(t) - \Delta(t) S(t)] . dt$$

$$= R(t) X(t) dt + \Delta(t) S(t) (\alpha(t) - R(t)) dt + \sigma(t) \Delta(t) S(t) . dW(t)$$

$$= R(t) X(t) . dt + \Delta(t) S(t) \sigma(t) \left[\frac{\alpha(t) - R(t)}{\sigma(t)} . dt + dW(t) \right]$$

$$= R(t) X(t) . dt + \Delta(t) S(t) \sigma(t) (\Theta(t) . dt + dW(t))$$

where $\Theta(t) = \dfrac{\alpha(t) - R(t)}{\sigma(t)}$

Change the measure to $\tilde{\mathbb{P}}$ so that under $\tilde{P}$

$d\tilde{W}(t) = dW(t) + \Theta(t) . dt$ is a B.M.

By risk neutral pricing formula, $D(t) X(t) = \tilde{\mathbb{E}} [X(T) D(T) | \mathcal{F}(t)]$

Consider α, R, σ, a_i's are const.

$$= \tilde{\mathbb{E}}\left[(S(T)-K)^{+} D(T) \,\middle|\, \mathcal{F}(t) \right]$$

$$dS(t) = \alpha S(t)\,dt + \sigma S(t)\cdot dW(t) \quad \ldots \quad t_i \le t < t_{i+1}$$

$$= R S(t)\cdot dt + \sigma S(t)\cdot d\tilde{W}(t)$$

$$S(t_{i+1}) = S(t_i)\exp\left\{ \sigma\left(\tilde{W}(t_{i+1}) - \tilde{W}(t_i)\right) + \left(R - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) \right\}$$

$$S(t_{i+1}) = (1 - a_{i+1})\left(S(t_{i+1}^{-})\right)$$

$$= (1 - a_{i+1})S(t_i)\exp\left\{ \sigma\left(\tilde{W}(t_{i+1}) - \tilde{W}(t_i)\right) + \left(R - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) \right\}$$

$$\frac{S(t_{i+1})}{S(t_i)} = (1 - a_{i+1})\exp\left\{ \sigma\left(\tilde{W}(t_{i+1}) - \tilde{W}(t_i)\right) + \left(R - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) \right\}$$

$$\frac{S(T)}{S(0)} = \prod_{i=0}^{n} \frac{S(t_{i+1})}{S(t_i)}$$

$$= \prod_{i=0}^{n} (1 - a_{i+1})\exp\left\{ \sigma\tilde{W}(T) + \left(R - \frac{\sigma^2}{2}\right)T \right\}$$

$$S(T) = S(0)\prod_{i=0}^{n} (1 - a_{i+1})\exp\left\{ \sigma\tilde{W}(T) + \left(R - \frac{\sigma^2}{2}\right)T \right\}$$

Price of call at time 0 is

$$C(0, S(0)) = S(0)\prod_{i=0}^{n}(1 - a_{i+1})N(d_+) - e^{-rT}K N(d_-)$$

where

$$d_{\pm} = \frac{1}{\sigma\sqrt{T}}\left[\log\left(\frac{S(0)}{K}\right) + \sum_{i=0}^{n}\log(1 - a_{i+1}) + \left(r \pm \frac{\sigma^2}{2}\right)T \right]$$

Stochastic diff eq^n

~~$dY(t) = \alpha(t)Y(t).dt + \sigma(t).Y(t).dW(t)$~~

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$$\begin{cases} dX(t) = b(t,X(t)).dt + \sigma(t,X(t)).dW(t) \quad \text{---} (*) \\ X(0) = x \end{cases}$$

$$X(t) = X(0) + \int_0^t b(s,X(s)).ds + \int_0^t \sigma(s,X(s)).dW(s) \rightarrow (4)$$

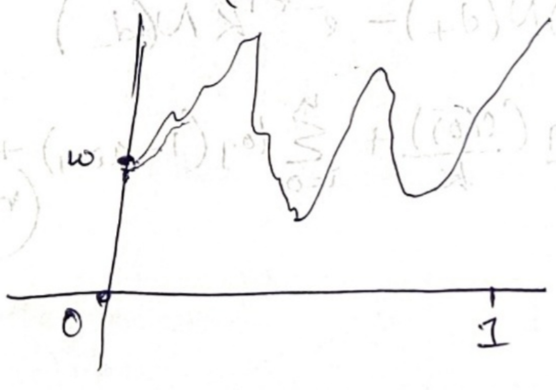
Let (Ω, F, P) be a probability space & $\{W(t) : t \in [0,T]\}$ be a B-M with filtrat^n $F(t) = \sigma\{W(s), 0 \le s \le t\}$

Def^n:- A sol^n of the eq^n is a cont stochastic process $X(t)$, $t \in [0,T]$ with the following properties

1) $X(t)$ is adapted to the filtrat^n $F(t)$, $t \in [0,T]$
2) $P(X(0) = x) = 1$
3) $\int_0^T |b(s,X(s))|.ds < \infty$ & $E\int_0^T (\sigma(s,X(s)))^2 ds < \infty$
4) It satisfies (4)

Def^n:- The SDE above is said to have a unique sol^n if X & $\tilde{X}$ are 2 sol^n's then

$$P(X(t) = \tilde{X}(t) : 0 \le t < T) = 1$$



P.T $P(w : X(t) = \tilde{X}(t)) = 1 \ \forall t \Rightarrow P(X(t) = \tilde{X}(t) : 0 \le t \le T) = 1$

Thm :- Suppose that the coeff $b(t,x)$, $\sigma(t,x)$ satisfy

$$|b(t,x)-b(t,y)| + |\sigma(t,x)-\sigma(t,y)| \le k|x-y|$$

$$|b(t,x)| + |\sigma(t,x)| \le k(1+|x|)$$

for some $+^{ve}$ const k then the SDE * has a unique sol^n and $\mathbb{E}\int_0^T |X(t)|^2 dt < \infty$ (If derivative is not bounded, it is not globally lipschitz)

(not Lipschitz on $\mathbb{R}$ but on a compact set)

$dX(t) = X^2(t).\,dt$

$X(0) = 1$

$X(t) = \dfrac{1}{1-t}$ (This is not a well defined sol^n) at $t=1$

$dX(t) = 3X^{2/3}$ $\quad X(t) = \begin{cases} 0 & \text{for } t \le a \\ (t-a)^3 & \text{for } t > a \end{cases}$ (Not unique)

$X(0) = 0$

Gronwall's inequality :- Let $f(\cdot)$ be a cont $f^n \ni$ $f(t) \le c + k\int_0^t f(s).ds$ for $0 \le t \le T$, where c is a const and k is a $+^{ve}$ const then $f(t) \le ce^{kt}$, $t \in [0,T]$

Pf:- $\quad g(t) = c + k\int_0^t f(s).ds \quad t \in [0,T]$

By fund thm,

$$g'(t) = kf(t) \quad \forall\ t \in [0,T]$$

$$g(t) = c + k\int_0^t f(s).ds \ge f(t)$$

$$g'(t) \le kg(t)$$

$$\Rightarrow g'(t) - kg(t) \le 0$$

$$e^{-kt}\left(g'(t) - kg(t)\right) \le 0$$

$$d\left(e^{-kt}g(t)\right) \le 0$$

$$e^{-kt}g(t) \text{ is } \downarrow$$

$$e^{-kt}g(t) \le g(0) = c$$

$$f(t) \le g(t) \le ce^{kt}$$

$$d\left(e^{-kt}\left(c + k\int_0^t f(s).ds\right)\right) = e^{-kt} \times kf(t) - k\left(c + k\int_0^t f(s).ds\right).e^{-kt} \le 0$$

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Uniqueness pf:- Suppose $X_1(t)$, $X_2(t)$ two sol^{n}s of the SDE

$$X_1(t) = x + \int_0^t b(s, X_1(s)).ds + \int_0^t \sigma(s, X_1(s)).dW(s)$$

$$X_2(t) = x + \int_0^t b(s, X_2(s)).ds + \int_0^t \sigma(s, X_2(s)).dW(s)$$

$$\mathbb{P}(X_1(t) = X_2(t) : t \in [0,T]) = 1$$

$$X_1(t) - X_2(t) = \int_0^t b(s, X_1(s)) - b(s, X_2(s)).ds + \int_0^t (\sigma(s, X_1(s)) - \sigma(s, X_2(s))).dW(s)$$

$$(a+b)^2 \leq 2(a^2+b^2)$$

$$\mathbb{E}|X_1(t) - X_2(t)|^2 \leq 2\Big\{\underbrace{\mathbb{E}\Big|\int_0^t (b(s, X_1(s)) - b(s, X_2(s))ds\Big|^2}_{I} + \underbrace{\mathbb{E}\Big|\int_0^t (\sigma(s, X_1(s)) - \sigma(s, X_2(s))dW(s)\Big|^2}_{II}$$

II ← Ito isometry

$$\leq 2\Big\{\mathbb{E}\Big[\int_0^t |b(s, X_1(s)) - b(s, X_2(s))|.ds\Big]^2 + 2\,\mathbb{E}\int_0^t |\sigma(s, X_1(s)) - \sigma(s, X_2(s))|^2.ds$$

$\boxed{\text{Holder inequality on } I \quad \int fg \leq \int f \int g \quad g=1}$

Use Fubini's the 1st and then Holder's inequality

$$\mathbb{E}|X_1(t) - X_2(t)|^2 \leq 2(t+1)K^2\, \mathbb{E}\int_0^t |X_1(s) - X_2(s)|^2.ds$$

$$= 2(t+1)K^2 \int_0^t \mathbb{E}|X_1(s) - X_2(s)|^2.ds$$

Let $g(t) = \mathbb{E}\{X_1(t) - X_2(t)\}^2$

Then $g(t) \leq 2(t+1)K^2 \int_0^t g(s).ds$

By Gronwall's inequality,

$0 \leq g(t) \leq 0 \quad \forall \; t \in [0,T]$

$\Rightarrow g(t) = 0 \quad \forall \; t \in [0,T]$

$\mathbb{E}|X_1(t) - X_2(t)|^2 = 0 \quad \forall \; t \in [0,T]$

$\Rightarrow \mathbb{P}(X_1(t) = X_2(t)) = 1 \quad \forall \quad t \in [0, T]$

$\Rightarrow \mathbb{P}(X_1(t) = X_2(t), \ t \in \mathbb{Q} \cap [0,T]) = 1$

$\mathbb{Q}$ – set of rat'nal nos

$$dS(t) = \mu(t) S(t) \cdot dt + \sigma(t) S(t) \cdot dW(t)$$

$$S(0) = S_0$$

$$|\mu(t)| + |\sigma(t)| \leq K$$

$$b(t, x) = \mu(t)\, x$$

$$\sigma(t, x) = \sigma(t)\, x$$

$f(x) = \ln x$

$$d(\ln S(t)) = \frac{1}{S(t)} \cdot dS(t) - \frac{1}{2S^2(t)} \cdot dS(t) \cdot dS(t)$$

$$d(\ln S(t)) = \frac{1}{S(t)} \left(\mu(t) S(t) \cdot dt + \sigma(t) S(t)\, dW(t) \right) - \frac{1}{2S^2(t)} \left[\sigma^2(t) \cdot S^2(t) \cdot dt \right]$$

$$= \left(\mu(t) - \frac{1}{2}\sigma^2(t) \right) \cdot dt + \sigma(t) \cdot dW(t)$$

$$S(t) = S(0) \exp\left\{ \int_0^t \left(\mu(s) - \frac{1}{2}\sigma^2(s) \right) \cdot ds + \int_0^t \sigma(s) \cdot dW(s) \right\}$$

(Bernt Oksendal SDE)

$$\frac{dX(t)}{dt} = f(t)X(t) + g(t)$$

$$I.F = e^{-\int_0^t f(s).ds} = h(t)$$

$$h(t).\frac{dX(t)}{dt} - f(t).h(t)X(t) = g(t).h(t)$$

$$d\big(h(t)X(t)\big) = g(t)h(t)$$

$$h(t)X(t) - x = \int_0^t g(s).h(s).ds$$

$$X(t) = \frac{x}{h(t)} + \frac{\int_0^t g(s).h(s).ds}{h(t)}$$

Linear Stochastic DE

$$dX(t) = \big(\phi(t)X(t) + \theta(t)\big).dW(t) + \big(f(t)X(t) + g(t)\big).dt$$

$$X(0) = x$$

$$I.F \quad H(t) = e^{-Y(t)}$$

$$Y(t) = \int_0^t \phi(s).dW(s) + \int_0^t f(s)\,ds - \frac{1}{2}\int_0^t \phi^2(s).ds$$

$$d\big(H(t)X(t)\big) = H(t)\,dX(t) + X(t)\,dH(t) + dX(t)\,dH(t)$$

$$dH(t) = -e^{-Y(t)}.dY(t) + \frac{1}{2}e^{-Y(t)}.dY(t).dY(t)$$

$$= -H(t)f(t).dt - H(t)\phi(t).dW(t) + H(t)\phi^2(t).dt$$

$$dX(t).dH(t) = -H(t)\phi(t)\big(\phi(t)X(t) + \theta(t)\big).dt$$

$$d\big(X(t)H(t)\big) = H(t)\big[\theta(t).dW(t) + g(t).dt - \theta(t)\phi(t)\,dt\big]$$

$$X(t) = xe^{Y(t)} + \int_0^t e^{Y(t)-Y(s)}\theta(s).dW(s) + \int_0^t e^{Y(t)-Y(s)}\big(g(s) - \theta(s)\phi(s)\big).ds$$

Example :- $dX(t) = \mu X(t) dt + \sigma dW(t)$

$$X(0) = x_0$$

$$f(t) = \mu,\ g(t) = 0,\ \phi(t) = 0,\ \theta(t) = \sigma$$

$$Y = \mu t$$

$$X(t) = x_0 e^{\mu t} + \int_0^t e^{\mu(t-s)} \sigma\, dW(s)$$

(OR)

$$dX(t) - \mu X(t) \cdot dt = \sigma dW(t)$$

$$d(e^{-\mu t} X(t)) = \sigma e^{-\mu t} dW(t)$$

$$X(t) e^{-\mu t} - x_0 = \int_0^t \sigma e^{-\mu s} dW(s)$$

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Give an example where $P(X_t = Y_t) = 1\ \forall t \in [0,T]$ is true but $P(X_t = Y_t, t \in [0,T]) = 1$ is false.

Consider the SDE

$$dX(t) = b(t, X(t)) \cdot dt + \sigma(t, X(t)) \cdot dW(t) \longrightarrow (1)$$

Let $h(y)$ be a mble f^n.

$$g(t,x) = \mathbb{E}\left[h(X(T)) \mid X(t) = x\right]$$

$$:= \mathbb{E}^{t,x}\left[h(X(T))\right]$$

Thm :- Assume that $\mathbb{E}^{t,x}\left[h(X(T))\right] < \infty\ \forall t\ \forall x$ where $X(\ )$ is the sol^n of the SDE (1). Then for $t \in [0,T]$

$$\mathbb{E}\left[h(X(T)) \mid \mathcal{F}(t)\right] = g(t, X(t))$$

Corollary :- The sol^n of SDE's are Markov processes.

Thm :- (Feynman-Kac) Consider the SDE

$$dX(t) = b(t, X(t)) \cdot dt + \sigma(t, X(t)) \cdot dW(t)$$

Let $h(y)$ be a Borel mble f^n. Define

$$g(t,x) = \mathbb{E}^{t,x}\left[h(X(T))\right]$$

Then $g(t,x)$ satisfies the PDE

$$g_t(t,x) + b(t,x)\, g_x(t,x) + \frac{1}{2}\sigma^2(t,x)\, g_{xx}(t,x) = 0$$

and the terminal condit^n

$$g(T,x) = h(x)$$

$$\cancel{g(T,x) = \mathbb{E}[h(X(T}$$

Pf:- Terminal condit^n is obv:-

$$g(T,x) = \mathbb{E}[h(X(T))\,|\,X(T)=x] = h(x)$$

$$g(t,X(t)) = \mathbb{E}[h(X(T))\,|\,\mathcal{F}(t)] \quad \text{is a martingale}$$

Let $0 \le s \le t \le T$. Then

$$g(s,X(s)) = \mathbb{E}[h(X(T))\,|\,\mathcal{F}(s)]$$

T.S.T $\quad \mathbb{E}[g(t,X(t))\,|\,\mathcal{F}(s)] = g(s,X(s))$

g is smooth enough to apply Ito's lemma.

$$dg(t,X(t)) = g_t(t,X(t))\,dt + g_x(t,X(t))\,dX(t) + \frac{1}{2}g_{xx}(t,X(t))\,dX(t)\,dX(t)$$

$$= g_t(t,X(t))\,dt + g_x(t,X(t))[b(t,X(t))\,dt + \sigma(t,X(t))\,dW(t)] + \frac{1}{2}g_{xx}(t,X(t))\,\sigma^2(t,X(t))\,dt$$

As g is a martingale, dt term $= 0$.

Thm:- Discounted Feyman Kac

$$\cancel{g(t,x)}\; f(t,x) = \mathbb{E}^{t,x}[e^{-r(T-t)}h(X(T))]$$

then $f_t(t,x) + b(t,x) f_x(t,x) + \frac{1}{2}\sigma^2(t,x) f_{xx}(t,x) = r f(t,x)$

Pf:- $\quad f(t,x) = e^{-r(T-t)} g(t,x)$

$$\mathbb{E}[f(t,X(t))\,|\,\mathcal{F}(s)] = \mathbb{E}[e^{-r(T-t)}h(X(T))\,|\,\mathcal{F}(s)] \neq f(t,X(s))$$

$$e^{-rt} f(\cancel{x} t, X(t)) = \mathbb{E}\left[e^{-r(T-t)} h(X(T)) \mid \mathcal{F}(t)\right]$$
$$= \mathbb{E}\left[e^{-rT} h(X(T)) \mid \mathcal{F}(t)\right]$$

$0 \le s \le t \le T$

$$\mathbb{E}\left[e^{-rt} f(t, X(t)) \mid \mathcal{F}(s)\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[e^{-rT} h(X(T)) \mid \mathcal{F}(t)\right] \mid \mathcal{F}(s)\right]$$
$$= \mathbb{E}\left[e^{-rT} h(X(T)) \mid \mathcal{F}(s)\right]$$
$$= e^{-rs} f(t, X(s))$$

$\Rightarrow e^{-rt} f(t, X(t))$ is a martingale.

$$\rightarrow d\left(e^{-rt} f(t, X(t))\right) = -re^{-rt} f(t, X(t)) \cdot dt + e^{-rt} df(t, X(t)) + \underbrace{d(e^{-rt})\, df(t, X(t))}_{=0}$$

Ito product rule

$$= -re^{-rt} f(t, X(t)) \cdot dt + e^{-rt}\left[f_t(t, X(t)) \cdot dt + f_x(t, X(t)) \cdot dX(t) + \frac{1}{2} f_{xx}(t, X(t)) \cdot dX(t) \cdot dX(t)\right]$$

$$= e^{-rt}\left[-rf(t, X(t)) + f_t(t, X(t)) + b(t, X(t)) f_x(t, \quad \right]$$

$=$

Ex:

$$C(t,x) = \mathbb{E}^{\tilde{\mathbb{P}}^{t,x}}\left[e^{-r(T-t)} (S(T)-K)^+\right]$$

$$dS(t) = rS(t) \cdot dt + \sigma S(t) \cdot d\tilde{W}(t)$$

$b(t,x) = \underline{\underline{rx}}$ since risk neutral measure

$\sigma(t,x) = \sigma x$

The PDE for $C(t,x)$ is

$$C_t(t,x) + rx C_x(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) = rC(t,x)$$
$$C(T,x) = (x-K)^+$$

Multidimensional Feyuman-Kac Thm:-

Let $W(t)=(W_1(t), W_2(t))$ be a 2-D B.M.

$$dX_1(t) = \beta_1(t, X_1(t), X_2(t))\,dt + \gamma_{11}(t, X_1(t), X_2(t))\,dW_1(t) + \gamma_{12}(t, X_1(t), X_2(t))\,dW_2(t)$$

$$dX_2(t) = \beta_2(t, X_1(t), X_2(t))\cdot dt + \gamma_{21}(t, X_1(t), X_2(t))\cdot dW_1(t) + \gamma_{22}(t, X_1(t), X_2(t))\cdot dW_2(t)$$

Let $h(x_1, x_2)$ be a Borel m'ble f^n

$$g(t, x_1, x_2) = \mathbb{E}\left[h(X_1(t), X_2(t)) \mid X_1(t)=x_1, X_2(t)=x_2\right] = \mathbb{E}^{t,x_1,x_2}\left[h(X_1(T), X_2(T))\right]$$

$$f(t, x_1, x_2) = \mathbb{E}^{t,x_1,x_2}\left[e^{-r(T-t)} h(X_1(T), X_2(T))\right]$$

Then

$$g_t(t,x_1,x_2) + \beta_1(t,x_1,x_2)\, g_{x_1}(t,x_1,x_2) + \beta_2(t,x_1,x_2)\, g_{x_2}(t,x_1,x_2) + \left[\frac{1}{2}(\gamma_{11}^2+\gamma_{12}^2) g_{x_1x_1} + \frac{1}{2}(\gamma_{21}^2+\gamma_{22}^2) g_{x_2x_2} + (\gamma_{11}\gamma_{21}+\gamma_{12}\gamma_{22}) g_{x_1x_2}\right](t,x_1,x_2) = 0$$

$$\left[f_t + \beta_1 f_{x_1} + \beta_2 f_{x_2} + \frac{1}{2}(\gamma_{11}^2+\gamma_{12}^2) f_{x_1x_1} + \frac{1}{2}(\gamma_{21}^2+\gamma_{22}^2) f_{x_2x_2} + (\gamma_{11}\gamma_{21}+\gamma_{12}\gamma_{22}) f_{x_1x_2}\right](t,x_1,x_2) = r f(t,x_1,x_2)$$

$$g(t,x_1,x_2) = \mathbb{E}\left[h(X_1(T), X_2(T)) \mid X_1(t)=x_1, X_2(t)=x_2\right] = \mathbb{E}^{t,x_1,x_2}\left[h(X_1(T), X_2(T))\right]$$

$$f(t,x_1,x_2) = \mathbb{E}^{t,x_1,x_2}\left[e^{-r(T-t)} h(X_1(T), X_2(T))\right]$$

Vasicek model

$$dR(t) = (\alpha - \beta R(t))dt + \sigma\, dW(t), \quad R(0) = r_0 \quad \alpha, \beta, \sigma \text{ are const}$$

~~R(t)=~~ $H(t) = e^{\beta t}$. Let $X(t) = \int_0^t e^{\beta s}\, dW(s)$

$$R(t) = r_0 e^{-\beta t} + \frac{\alpha}{\beta}(1-e^{-\beta t}) + \sigma e^{-\beta t}\int_0^t e^{\beta s}\, dW(s)$$

$$\Leftrightarrow \quad f(t,x) = e^{-\beta t} r_0 + \frac{\alpha}{\beta}(1-e^{-\beta t}) + \sigma e^{-\beta t} x$$

$$R(t) \sim N\left(r_0 e^{-\beta t} + \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right), \frac{\sigma^2}{2\beta}\left(1-e^{-2\beta t}\right)\right)$$

$$X(t) = x_0 + \int_0^t \alpha X(s)\,.\,ds + \int_0^t \sigma\,.\,dW(s)$$

$$\mathbb{E}[X(t)] = x_0 + \alpha\int_0^t \mathbb{E}[X(s)]\,.\,ds$$

Let $g(t) = \mathbb{E}[X(t)]$

$$g(t) = x_0 + \alpha\int_0^t g(s)\,.\,ds$$

$$\frac{d}{dt}g(t) = \alpha g(t)$$

$$g(t) = x_0 e^{\alpha t}$$

$$Var(X(t)) = \mathbb{E}[X^2(t)] - (\mathbb{E}[X(t)])^2$$

$$d(X^2(t)) = 2X(t)\,.\,dX(t) + \frac{1}{2}(2\,dX(t)\,.\,dX(t))$$

$$= 2X(t)[\alpha X(t)\,.\,dt + \sigma\,dW(t)] + \sigma^2\,.\,dt$$

$$d(X^2(t)) = (2\alpha(X(t))^2 + \sigma^2)\,.\,dt + 2X(t)\sigma\,dW(t)$$

$$X^2(t) = x_0^2 + \int_0^t (2\alpha X^2(s) + \sigma^2)\,.\,ds + \sigma\int_0^t 2X(s)\,.\,dW(s)$$

$$\mathbb{E}[X^2(t)] = x_0^2 + 2\alpha\int_0^t \mathbb{E}[X^2(s)]\,.\,ds + \sigma^2 t$$

Let $f(t) = \mathbb{E}[X^2(t)]$

Then $$f(t) = x_0^2 + 2\alpha\int_0^t f(s)\,.\,ds + \sigma^2 t$$

$$\frac{d}{dt}f(t) = 2\alpha f(t) + \sigma^2 \quad \& \quad f(0) = x^2$$

$$\cancel{\frac{1}{2\alpha}\ln\left(\frac{2\alpha f(t) + \sigma^2}{2\alpha x_0^2}\right) = t}$$

$$\cancel{f(t) = e^{2\alpha t} * 2\alpha x_0^2 - \frac{\sigma^2}{2\alpha}}$$

$$f(t) = \frac{\sigma^2}{2\alpha}\left(e^{2\alpha t} - 1\right) + x_0^2 e^{2\alpha t}$$

$$\frac{du}{dt} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} - K(x)u$$

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Gaussian process:-

Def:- A Gaussian process $X(t), t \geq 0$ is a stochastic process such that for arbitrary time pts

$0 < t_1 < t_2 < t_3 \ldots < t_n$ the r.vs $X(t_1), X(t_2), \ldots, X(t_n)$ are jointly normally distributed.

A r.vs $\bar{X} = (X_1, X_2, \ldots, X_n)$ is jointly normal if it has joint density

$$f_X(\bar{X}) = \frac{1}{\sqrt{(2\pi)^n |C|}} \exp\{(\bar{X}-\mu)^T C^{-1} (\bar{X}-\mu)^t\}$$

$\bar{X} = (X_1, X_2, \ldots, X_n)$, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, C – cov matrix

We denote mean of $X(t)$ by $m(t)$ & cov of $X(t)$ & $X(s)$ by $C(t,s)$

$$m(t) = \mathbb{E}\{X(t)\}, \quad C(t,s) = \mathbb{E}[(X(t)-m(t))(X(s)-m(s))]$$

Ex:- ① $W(t), t \geq 0$ B.M is a Gaussian process

$(W(t_1), W(t_2)-W(t_1), W(t_3)-W(t_2), \ldots,)$ is jointly normal as each of them are independent of one ~~and othe~~ another.

If $\bar{X} = (X_1, X_2, \ldots, X_n)$ is jointly normal & $Y = A\bar{X}^t$ is also a jointly normal vector $A = (a_{ij})$

Here $A = \begin{bmatrix} 1 & 0 & \cdots & \\ 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & \cdots \\ & & & \end{bmatrix}$

$$m(t) = \mathbb{E}[W(t)] = 0$$

$s < t$ $\quad C(s,t) = \mathbb{E}[W(t)W(s)] = \mathbb{E}[W(s)(W(t)-W(s)) + (W(s))^2]$

$$= s \wedge t = \min(s,t)$$

Ex:- $I(t)=\int_0^t \Delta(s)\,dW(s)$ where $\Delta(t)$ is non random. Then $I(t), t\geq 0$ is a Gaussian process.

$$\left(I(t) \sim N\left(0, \int_0^t \Delta^2(s)\,ds\right)\right)$$

Pf:-

$$I(t_1), I(t_2)-I(t_1), \ldots, I(t_n)-I(t_{n-1})$$

For a fixed u, set

$$M_u(t)=\exp\left\{u\int_0^t \Delta(s)\,dW(s)-\frac{1}{2}u^2\int_0^t \Delta^2(s)\,ds\right\}$$

Ex:- $M_u(t)$ is a martingale

H.W. T.S.T $dM_u(t)=\_\_\_\,dW(t)$

$$M_{u_2}(t_1)=\mathbb{E}\left[M_{u_2}(t_2)\mid \mathcal{F}(t_1)\right]$$

$$\mathbb{E}\left[e^{u_1 I(t_1)+u_2(I(t_2)-I(t_1))}\right]$$

$$\frac{M_{u_1}(t_1)}{M_{u_2}(t_2)}\cdot M_{u_2}(t_1)=\frac{M_{u_1}(t_1)}{M_{u_2}(t_1)}\mathbb{E}\left[M_{u_2}(t_2)\mid \mathcal{F}(t_1)\right]$$

$$M_{u_1}(t_1)=\mathbb{E}\left[M_{u_2}(t_2)\frac{M_{u_1}(t_1)}{M_{u_2}(t_1)}\,\middle|\,\mathcal{F}(t_1)\right]$$

$$=\mathbb{E}\left[\exp\left\{u_1 I(t_1)+u_2(I(t_2)-I(t_1))\right\}\exp\left\{-\frac{1}{2}u_1^2\int_0^{t_1}\Delta^2(s)\,ds-\frac{1}{2}u_2^2\int_{t_1}^{t_2}\Delta^2(s)\,ds\right\}\,\middle|\,\mathcal{F}(t_1)\right]$$

Expect^n on both sides

$$1=\mathbb{E}\left[\exp\left\{u_1 I(t_1)+u_2(I(t_2)-I(t_1))\right\}\exp\left\{-\frac{1}{2}u_1^2\int_0^{t}\Delta^2(s)\,ds-\frac{1}{2}u_2^2\int_{t_1}^{t_2}\Delta^2(s)\,ds\right\}\right]$$

$$= \exp\left\{\frac{1}{2}u_1^2\int_0^{t_1}\Delta^2(s)\,ds\right\}\exp\left\{\frac{1}{2}u_2^2\int_{t_1}^{t_2}\Delta^2(s)\,ds\right\}$$

$$= \mathbb{E}\left[e^{u_1 I(t_1)}\right]\mathbb{E}\left[e^{u_2(I(t_2)-I(t_1))}\right]$$

For $I(t)=\int_0^t \Delta(s)\,dW(s)$

$s<t$ $\quad m(t)=\mathbb{E}[I(t)]=0$

$$C(s,t)=\mathbb{E}[I(t)I(s)]$$

$$=\mathbb{E}\{I(s)(I(t)-I(s))+I^2(s)\}$$

$$= 0\times 0+\cancel{\mathbb{E}}\int_0^s\Delta^2(u)\,du = \int_0^{t\wedge s}\Delta^2(u)\,du$$

Brownian Bridge

~~Let~~ Def^n: Let $W(t)$ be a B.M. Fix $T>0$, we define the Brownian Bridge from 0 to 0 (i.e $X(0)=0$ & $X(T)=0$) to the process $X(t)=W(t)-\frac{t}{T}W(T)$, $t\in[0,T]$

(Note that $X(t)$ is not adapted to the same filtrat^n as $W(t)$)

This is a Gaussian process.

Brownanian Bridge:

Defn: let $W(t)$ be a B.M. Fix $T>0$, we define the Brownarian bridge from 0 to 0 on $[0,T]$ to be the process $X(t) = W(t) - t/T\, W(T)$, $t \in [0,T]$

This is a guassian process.

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$X(t) = W(t) - t/T\, W(T)$, $0 \le t \le T$

where $W(t)$ is a B.M

$X(0) = 0 = X(T)$

$X(t)$ is a Gaussian process

$0 < t_1 < t_2 \cdots < t_n < T$

$X(t_1), X(t_2), \ldots \ldots X(t_n)$ – jointly normal

To prove that $X(t)$ is a Guassian, we need to show this

$X(t_1) = W(t_1) - t_1/T\, W(T)$, $X(t_2) = W(t_2) - \frac{t_2}{T} W(T) \ldots$

$\ldots X(t_n) = W(t_n) - \frac{t_n}{T} W(T)$

... jointly normal

$x(t_1) = W(t_1) - t_1/_T\, W(T), \quad x(t_2) = W(t_2) - \frac{t_2}{T} W(T) \ldots$

$\ldots\ x(t_n) = W(t_n) - \frac{t_n}{T} W(T)$

$W(t_1), W(t_2), \ldots, W(T)$ - jointly normal

$\Rightarrow x(t_1), \ldots, x(T)$ - jointly normal

$$m(t) = \mathbb{E}(X(t)) = 0$$

$$C(s,t) = \mathbb{E}\left[X(t)X(s)\right] = \mathbb{E}\left[\left(W(t) - t/_T\, W(T)\right)\left(W(s) - \frac{s}{T} W(T)\right)\right]$$

$$= \mathbb{E}\left[W(t)W(s) - \frac{t}{T} W(T)W(s) - \frac{s}{T} W(t)W(T) + \frac{st}{T^2} W^2(T)\right]$$

$$= t \wedge s - \frac{ts}{T} - \frac{ts}{T} + \frac{st}{T^2} \cdot T$$

$$= t \wedge s - \frac{st}{T}$$

$t \wedge s = \min(s,t)$

<u>Defn</u>: Let $W(t)$ be a B.M. Fix $T > 0$, $a, b \in \mathbb{R}$. Define the Brownian Bridge from a to b on $[0,T]$ to be the process

[0,T] to be the process

$$X^{a\to b}(t) = a + \frac{(b-a)}{T}t + X(t), \quad t \in [0,T]$$

where $X(t)$ is a Brownanian bridge from 0 to 0 $\quad X^{a\to b}(0) = a, \quad X^{a\to b}(T) = b$

$$M^{a\to b}(t) = a + \frac{(b-a)}{T}t \quad (\text{since } E[X(t)] = 0)$$

$$C^{a\to b}(s,t) = E\left[\left(X^{a\to b}(t) - M^{a\to b}(t)\right)\left(X^{a\to b}(s) - M^{a\to b}(s)\right)\right]$$

$$= E[X(t)X(s)]$$

(same Covariance as Brownanian bridge)

$$= t \wedge s - \frac{st}{T}$$

suppose $Y(t)$ is Gaussian, $f(t)$ - some deterministic func not random, $Y(t)f(t) \to$ Gaussian

$\longrightarrow \; I(t) = \int_0^t \Delta(s)\, dW(u)$

$c(t,t) = t - \frac{t^2}{T} = \frac{t(T-t)}{T}$

$c^I(t,t) = \int_0^t \Delta^2(s)\, ds$

$Y(t) = (T-t) \int_0^t \frac{1}{T-u}\, dW(u), \quad 0 \le t < T$

$I(t) = \int_0^t \frac{1}{T-u}\, dW(u)$

$0 < t_1 < t_2 \; \dots\dots \; t_n < T$

$Y(t_1) = \underbrace{(T-t_1)}_{\text{deterministic}} \overbrace{I(t_1)}^{\text{gaussian}}, \; \dots\dots, \; Y(t_n) = (T-t_n)\, I(t_n)$

$Y(t_1), \dots\dots, Y(t_n)$ – jointly normal

$M^I(t) = \mathbb{E}[I(t)] = 0, \quad M^Y(t) = \mathbb{E}[Y(t)] = 0$

$c^I(s,t) = \int_0^{t \wedge s} \frac{1}{(T-u)^2}\, du$

$$C^I(s,t) = \int_0^{t\wedge s} \frac{1}{(T-u)^2}\, du$$

$$= \frac{1}{T-(s\wedge t)} - \frac{1}{T}, \quad t, s \in [0,T)$$

$0 \le s \le t < T$

$$C^Y(s,t) = \mathbb{E}\left[(T-s)\, I(s)\, (T-t)\, I(t)\right]$$

$$= (T-s)(T-t)\, \mathbb{E}\left[I(s)\, I(t)\right]$$

$$= (T-s)(T-t)\left[\frac{1}{T-s} - \frac{1}{T}\right]$$

$$= \frac{(T-t)\, s}{T} = s - \frac{st}{T}$$

$C^Y(s,t) = s \wedge t - \frac{st}{T}$ for $s,t \in [0,T)$

$C^Y(t,t) = t - \frac{t^2}{T} = \frac{t(T-t)}{T}$

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$$C^Y(s,t) = s \wedge t - \frac{st}{T} \quad \text{for } s,t \in [0,T]$$

$$C^Y(t,t) = t - \frac{t^2}{T} = \frac{t(T-t)}{T}$$

as $t \to T$, $C^Y(t,t) \to 0$

$$\rightarrow \quad Y(t) = \begin{cases} (T-t)\displaystyle\int_0^t \frac{1}{T-u}\, dW(u), & 0 \le t < T \\ 0, & t = T \end{cases}$$

$Y(t)$ is a continuous Gaussian process on $[0,T]$

has mean & covariance funcⁿ:

$$m^Y(t) = 0, \quad C^Y(t,s) = t \wedge s - \frac{st}{T}$$

$\underline{7/2/24}$

$X^{a\to b}(t)$ - Brownian bridge from a to b

$0 = t_0 < t_1 < \dots < t_n < T$

$X^{a\to b}(t_1), \dots, X^{a\to b}(t_n)$

$$m^{a\to b}(t) = a + \frac{(b-a)t}{T} = \frac{(T-t)a}{T} + \frac{bt}{T}$$

$$C^{a\to b}(t,s) = s \wedge t - \frac{st}{T}$$

For $0 \le s \le t < T$,

$$C^{a\to b}(t,s) = s - \frac{st}{T} = \frac{s(T-t)}{T}$$

Set $T_j = T - t_j$ and $T_0 = T$

$$Z_j = \frac{X^{a\to b}(t_j)}{T_j} - \frac{X^{a\to b}(t_{j-1})}{T_{j-1}}$$

$X^{a\to b}(t_1) \dots X^{a\to b}(t_n)$ are jointly normal

$\Rightarrow Z_1, \dots, Z_n$ are jointly normal

$$\mathbb{E}[Z_j] = \frac{1}{T_j}\mathbb{E}[X^{a\to b}(t_j)] - \frac{1}{T_{j-1}}\mathbb{E}[X^{a\to b}(t_{j-1})]$$

$$= a\left(\frac{1}{T_j} - \frac{1}{T_{j-1}}\right) + \frac{b-a}{T}\left(\frac{t_j}{T_j} - \frac{t_{j-1}}{T_{j-1}}\right)$$

$$= a$$

$$\Rightarrow \mathrm{Var}(Z_j) = \frac{1}{T_j^2} C^{a\to b}(t_j, t_j) + \frac{1}{T_{j-1}^2} C^{a\to b}(t_{j-1}, t_{j-1}) - \frac{2}{T_j T_{j-1}} C^{a\to b}(t_j, t_{j-1})$$

$$= \frac{t_j - t_{j-1}}{T_j T_{j-1}}$$

$$\mathrm{Cov}(Z_i, Z_j) = \frac{1}{T_i T_j} C^{a\to b}(t_i, t_j) - \frac{1}{T_i T_{j-1}} C^{a\to b}(t_i, t_{j-1})$$

$$- \frac{1}{T_{j-1} T_j} C^{a\to b}(t_{i-1}, t_j)$$

$$+ \frac{1}{T_{i-1} T_j} C^{a\to b}(t_{i-1}, t_{j-1})$$

$$= 0$$

$Z_1, Z_2, \ldots, Z_n$ are independent (as $Z_1, Z_2, \ldots, Z_n$ are jointly normal and $\mathrm{Cov}(Z_i, Z_j) = 0$ for $i \neq j$)

$$f_{Z_1 \ldots Z_n}(z_1 \ldots z_n) = \exp\left\{ \frac{-1}{2} \sum_{j=1}^{n} \frac{\left(z_j - \frac{b(t_j - t_{j-1})}{T_j T_{j-1}} \right)^2}{\left(\frac{t_j - t_{j-1}}{T_j T_{j-1}} \right)} \right\} \times$$

$$z_j = \frac{x_j}{T_j} - \frac{x_{j-1}}{T_{j-1}} \qquad \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi \left(\frac{t_j - t_{j-1}}{T_j T_{j-1}} \right)}}$$

$$\frac{\partial z_j}{\partial x_j} = \frac{1}{T_j} \qquad J = \begin{bmatrix} \frac{1}{T_1} & 0 & \cdots & 0 \\ \frac{-1}{T_1} & \frac{1}{T_2} & \cdots & 0 \\ & & \ddots & \\ 0 & \cdots & & \frac{1}{T_n} \end{bmatrix}$$

$$\frac{\partial z_j}{\partial x_{j-1}} = \frac{-1}{T_{j-1}}$$

$$\therefore f_{X^{a\to b}(t_1) \ldots X^{a\to b}(t_n)}(x_1, \ldots, x_n) = \sqrt{\frac{T}{T - t_n}} \prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}}$$

$$\exp\left\{ \frac{-1}{2} \sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}} - \frac{(b - x_n)^2}{2(T - t_n)} - \frac{(b-a)^2}{2T} \right\}$$

$$= \frac{p(T - t_n, x_n, b)}{p(T, a, b)} \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j)$$

where $p(\tau, x, y) = \frac{1}{\sqrt{2\pi\tau}} \exp\left\{ \frac{-(y-x)^2}{2\tau} \right\}$

The joint density of $W(t_1), \ldots, W(t_n), W(T)$

$$f_{W(t_1)\ldots W(T)}(x_1 \ldots x_n, b) = p(T-t_n, x_n, b) \prod_{j=1}^{n} p(t_j - t_{j-1}, x_{j-1}, x_j)$$

where $W(0) = a$

$$p(t_1 - t_0, a, x_1) = p(t_1, a, x_1)$$

$$p(t_2 - t_1, x, x_2) =$$

Corrolary :- The density of $M^{a\to b}(T)$ is

$$f_{M^{a\to b}(T)}(y) = \frac{2(2y-b-a)}{T} e^{-\frac{2}{T}(y-a)(y-b)}$$

$$y > \max\{a, b\}$$

Pf :- $M(t) = \max\limits_{0\le s\le t} W(s)$

The condit^nal distribut^n of $M(t)$ given $W(t) = w$ is

$$f_{M(t)/W(t)}(m/w) = \frac{2(2m-w)}{t} \exp\left\{-\frac{2m(m-w)}{t}\right\} \quad w \le m, m \ge 0$$

∴ Define $M^{0\to w}(T) = \max\limits_{0\le t\le T} X^{0\to w}(t)$

$$f_{M^{0\to w}(T)}(m) = f_{\frac{M(T)}{W(T)}}($$