Post Quiz

20/2/23

Calculus on $\mathbb{R}^n$

always in this course, it is over $\mathbb{R}$

X – finite dim normed space over $\mathbb{R}$

Then all norm in X are equivalent i.e any 2 norms $\|\cdot\|_1$, $\|\cdot\|_2$ $\exists$ c_1 & c_2 (+ve real nos)

$\ni$ $c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1 \quad \forall\ x \in X$.

$x \in X$, $x = \sum_{i=1}^{n} \alpha_i x_i = \alpha_1 u_1 + \cdots + \alpha_n u_n$

$$\|x\|_1 = \sum_{i=1}^{n} |\alpha_i| \text{ (verify!)}$$

$$S = \{x \in X : \|x\|_1 = 1\}$$

Exercise:- S is compact

$$x_k = (\alpha_k^1 u_1 + \alpha_k^2 u_2 + \cdots + \alpha_k^n u_n)$$

Incomplete

$f : S \to \mathbb{R}$, $f(x) = \|x\|$

$$f\left(\frac{x}{\|x\|_1}\right) = \left\|\frac{x}{\|x\|_1}\right\|$$

As f is continuous, & S is compact, $f(S)$ is compact.

$$c_1 \leq f(x) \leq c_2$$

$$c_1 \leq \frac{\|x\|}{\|x\|_1} \leq c_2$$

$$c_1\|x\|_1 \leq \|x\| \leq c_2\|x\|_1$$

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$$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\,\|x\|_2$$

(The best possible constants c_1 & c_2)

H.W
$$\|x\|_q \leq \|x\|_p \quad \text{for } p<q$$

$$\|x\|_p \leq \|x\|_q \leq n^{\left(\frac{1}{q}-\frac{1}{p}\right)}\|x\|_p \quad \text{for } 1\leq q\leq p\leq \infty$$

$$\sum |x_i|^p \leq \left(\sum (|x_i|^p)^r\right)^{1/r}\left(n^{1/r'}\right)$$

$$= \left(\left(\sum |x_i|^q\right)^{1/q}\right)^p n^{\left(1-\frac{p}{q}\right)}$$

H.W P.T $\lim_{p\to\infty}\|x\|_p = \max_{i=1}^{n}\{|x|_i\}$

A map $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called linear map if

1. $T(x+y) = T.x + T.y \quad \forall\ x,y \in \mathbb{R}^n$
2. $T(\alpha x) = \alpha Tx \quad \forall\ \alpha\in\mathbb{R},\ x\in\mathbb{R}^n$

$$x = x_1e_1 + \cdots + x_ne_n$$

$$Tx = x_1Te_1 + \cdots + x_nTe_n$$

$$= \sum_{j=1}^{n} x_j\left(\sum_{i=1}^{m}\alpha_{ij}\cdot u_i\right)$$

$$= \sum_{i=1}^{m}\left(\sum_{j=1}^{n}x_j\alpha_{ij}\right)u_i$$

$Te_j = \sum_{i=1}^{m}\alpha_{ij}\,u_i$

$T \sim A_{m\times n}$

This is also the maximal set of orthonormal vectors, as it can represent any vector (i.e it is the basis)
(Ex:- $(1,0,0)$, $(0,1,0)$ cannot represent $x\in\mathbb{R}^3$ but $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ can)

showing that these are orthonormal

$\cancel{\langle x_i, x_j\rangle = 0}$

$\langle u_i, u_j\rangle = 0$ if $i\neq j$ $\|$ly $\langle x_i, x_j\rangle$
$= 1$ if $i=j$

$\therefore$ $\langle x_a u_b, x_c u_d\rangle = 1$ iff $(a,b)=(c,d)$
$= 0$ O.W

$\therefore$ They are orthonormal

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f:

$$T(\alpha X) = \alpha T(X)$$
$$T(X+Y) = T(X) + T(Y)$$

$$f:(a,b) \to \mathbb{R}$$
$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

a linear mapping

$$\lim_{h\to 0} \frac{f(c+h)-f(c)-\alpha h}{h} = 0 \quad h \to \alpha h$$

$T(x) = Ax$

$A = (a_{ij})$ $\quad a_{ij} = \langle Ae_j, u_i \rangle$

$L(\mathbb{R}^n, \mathbb{R}^m)$ = set of all linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Is L a vector space?

$$T(ax+y) = aT(x) + T(y)$$

H.W S.T $\dim(L(\mathbb{R}^n, \mathbb{R}^m)) = mn$.

From Rank Nullity Thm, $T: \mathbb{R}^m \to \mathbb{R}^n$ is invertible iff $m = n$.

TFEQ: $\quad T: \mathbb{R}^n \to \mathbb{R}^m$

1) T is invertible
2) T is 1-1
3) T is onto
4) $\ker T = \{0\}$

H.W Any map that preserves distance b/n all pts is called isometrica. S.T they are either rot^n or translati^n

$L(\mathbb{R}^n, \mathbb{R}^n) = M_n(\mathbb{R})$ is a metric space.

Norm:- $\|T\| = \sup_{\|x\| \le 1} \|Tx\| < \infty$ on $L(\mathbb{R}^n; \mathbb{R}^m)$

Pf:- $x \in \mathbb{R}^n$, $\|x\| \le 1$, $|x_i| \le 1 \;\forall i$

$$\sum_{j=1}^{n} \|Te_j\| = M$$

$$\|Tx\| = \left\|\sum_{j=1}^{n} x_j Te_j\right\| \le \sum |x_j| \|Te_j\| \le M$$

H.W $\|T+S\| \le \|T\| + \|S\|$

$T: \mathbb{R}^n \to \mathbb{R}^m$

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Norm of L.T

$T: \mathbb{R}^n \to \mathbb{R}^m$

T is called bdd if $\exists\, k > 0$ s.t

$$\|Tx\|_{\mathbb{R}^m} \le k \|x\|_{\mathbb{R}^n} \;\forall\, x \in \mathbb{R}^n$$

Every L.T is bdd

$e_1, \ldots, e_n$ in $\mathbb{R}^n$, $x = \sum x_i e_i$, $x_i \in \mathbb{R}$

$$\|Tx\| = \|T(\textstyle\sum x_i e_i)\| = \|\textstyle\sum x_i Te_i\|$$

$$\le \sum |x_i| \|Te_i\|$$

$$\le \left(\sum \|Te_i\|^2\right)^{1/2} \left(\sum |x_i|^2\right)^{1/2}.$$

$$\boxed{\|Tx\| \le k\|x\|}$$

Remark:- As $\mathbb{R}^n, \mathbb{R}^m$ has finite dimensions, $\|Tx\|$ is bdd.

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1) $\inf\{k : \|Tx\| \le k\|x\| \ \forall\ x \in \mathbb{R}^n\} = \alpha$

2) $\sup\{\|Tx\| : \|x\| \le 1\} = \beta$

3) $\sup\{\|Tx\| : \|x\| = 1\} = \gamma$

$\gamma \le \beta$

$0 \ne x \in \mathbb{R}^n$, $\left\|\frac{x}{\|x\|}\right\| = 1 \Rightarrow \left\|T\left(\frac{x}{\|x\|}\right)\right\| \le \gamma$

$\Rightarrow \frac{1}{\|x\|}\|Tx\| \le \gamma$

$\|Tx\| \le \gamma\|x\|$

$\alpha \le \gamma$

$\|Tx\| \le \gamma$ if $\|x\| \le 1$

$\Rightarrow \beta \le \gamma$

T.S.T $\sup\{\|Tx\| : \|x\| <= 1\} \le \sup\{\|Tx\| : \|x\| = 1\}$

$\|Tx\| = \left\|\|x\| T\left(\frac{x}{\|x\|}\right)\right\| = \|x\| \left\|T\left(\frac{x}{\|x\|}\right)\right\|$

$\le \sup\{\|Tx\| : \|x\| = 1\}$

$\gamma = \sup_{\|x\|=1} \|Tx\| \le \alpha$ (from 1) since $\|Tx\| \le \alpha\|x\|$ & $\|x\| = 1$

4) $\sup\{\|Tx\| : \|x\| < 1\} = \delta$

H.W

$\|T\| = \sup_{\|x\| \le 1} \|Tx\|$ is norm $(X = L(\mathbb{R}^n, \mathbb{R}^m))$

$\therefore \|T\| \ge \|Tx\|$ $\forall$ $x \in$ domain of T

$T \in L(\mathbb{R}^n, \mathbb{R}^m)$, $S \in L(\mathbb{R}^m, \mathbb{R}^k)$

$S \circ T : \mathbb{R}^n \to \mathbb{R}^k$

$(S \circ T)(x) = S(Tx)$

$\|S \circ T\| \le \|S\|\,\|T\|$

Pf $\|S \circ T(x)\| = \|S(Tx)\|$

$\le \|S\|\,\|Tx\|$

$\le \|S\|\,\|T\|\,\|x\|$

H.W If T is linear, is T^{-1} linear?

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$T: \mathbb{R}^n \to \mathbb{R}^n$. The following are equivalent:- (TFAE)

1) T is 1-1 2) T onto 3) T bijection.
4) $\ker(T) = \{0\}$

If $\alpha > 0$, $\beta x \neq 0$, if
$\alpha \|x\| \leq \|\beta x\|$ it is called bdd below.

Every bdd linear transform^n is uniformly cts.
bcuz $\|Tx - Ty\| \leq \|T\| \, \|x - y\|$

$f: \underline{S^{n-1}} \to \mathbb{R}$ $f(x) = \|Tx\|$
$\alpha \leq \|Tx\| \leq \beta$
$\|x\| \leq 1$ $\alpha \|x\| \leq \|Tx\| \leq \beta \|x\|$

$L(\mathbb{R}^n)$ $n = 1$

$L(\mathbb{R}) \cong \mathbb{R}$

$GL(\mathbb{R}^n)$ = set of all invertible L.T (from $\mathbb{R}^n$ onto $\mathbb{R}^n$)

$n = 1$, $GL(\mathbb{R}) = (-\infty, 0) \cup (0, \infty)$
$A = \alpha$, $A^{-1} = \frac{1}{\alpha}$, $\frac{1}{\|A^{-1}\|} = |\alpha|$
$B(\alpha, |\alpha|)$

For some n,
$A \in GL(\mathbb{R}^n)$
$\alpha = \frac{1}{\|A^{-1}\|}$ $\|B - A\| < \alpha$
$\|x\| = \|A^{-1} A x\|$
$\frac{1}{\|A^{-1}\|} \|x\| \leq \|Ax\| \leq \|Ax - Bx\| + \|Bx\|$
$\leq \|A - B\| \|x\| + \|Bx\|$

$$\left(\frac{1}{\|A^{-1}\|} - \|A-B\|\right)\|x\| \leq \|Bx\|$$

$f: GL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R})$ $\quad f(A) = A^{-1}$

f is one-one, onto, invertible.

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$\|A^{-1} - B^{-1}\|$

$= \|A^{-1}BB^{-1} - A^{-1}AB^{-1}\|$

$= \|A^{-1}(B-A)B^{-1}\|$

$\leq \|A^{-1}\|\,\|B^{-1}\|\,\|A-B\|$

Either show that $\underbrace{A_n^{-1} \longrightarrow A^{-1}}_{\text{seq criteria}}$ or continuity using $\epsilon - \delta$ then

—

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$

$x_0 \in \mathbb{R}^n \quad e_1, \ldots, e_n$

$$\lim_{t \to 0} \frac{f(x_0 + te_i) - f(x_0)}{t} = \frac{\partial f}{\partial x_i}(x_0)$$

$$f((x,y)) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & \text{o.w} \end{cases}$$

It has partial derivatives in all direxns but from seq criterion $\left((x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)\right)$, it is not cont.

$\cancel{\dfrac{1}{\frac{1}{m} + m}}$

$$f(x,y) = \begin{cases} \dfrac{x}{|y|}\sqrt{x^2+y^2}, & y \neq 0 \\ 0, & y = 0 \end{cases}$$

(Same $(x_n, y_n) = \left(\frac{1}{n}, \frac{1}{n}\right)$ shows it is disc)

Along $y = mx$ $\quad \dfrac{\partial f}{\partial x} = \dfrac{\frac{1}{m}\cancel{\sqrt{1+m^2}\,x\,h}}{\cancel{\sqrt{1+m^2}\,x\,h}}$

~~O.W~~ If $m = 0$, $\dfrac{\partial f}{\partial x} = 0$

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)-T(h)}{h} = 0$$

$f: G \subset \mathbb{R}^n \to \mathbb{R}^m$, $x_0 \in G$, if a $T \ni$

$$\lim_{h\to 0} \frac{\|f(x_0+h)-f(x_0)-T(h)\|}{\|h\|} \text{ exists \& } = 0,$$

then $f'(x_0) = T$.

$\phi: \mathbb{R}^n \xrightarrow{\text{linear}} \mathbb{R}$ $\quad \exists x_0 \in \mathbb{R}^n$

$\phi(x) = \langle x, x_0 \rangle$

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$f: E \subset \mathbb{R}^n \to \mathbb{R}^m$ — f is called diff at x_0 and $\boxed{f'(x_0) = T}$ (total derivative)

$x_0 \in E$, if $\exists\, T \in L(\mathbb{R}^n, \mathbb{R}^m) \ni$

$$\lim_{h\to 0} \frac{\|f(x_0+h)-f(x_0)-T(h)\|}{\|h\|} = 0$$

Can we say that 'f' is cts at x_0?

yes — $\lim_{h\to 0} f$

Prove that 'T' is unique i.e

if $$\lim_{h\to 0} \frac{\|f(x_0+h)-f(x_0)-T(h)\|}{\|h\|} = 0 \longrightarrow (1)$$

& $$\lim_{h\to 0} \frac{\|f(x_0+h)-f(x_0)-S(h)\|}{\|h\|} = 0 \longrightarrow (2) \Rightarrow T(x) = S(x)$$

Pf:- (1) − (2) & $\cancel{\|x\|}$ ~~$\|x+y\| - \|x\| \le \|y\|$~~

$\|x\| - \|y\| \le \|x-y\|$

& the next lemma

Lemma :- $B \in L(\mathbb{R}^n, \mathbb{R}^m) \ni \lim_{h \to 0} \frac{\|B(h)\|}{\|h\|} = 0 \rightarrow B \equiv 0$

Pf :- $\lim_{t \to 0} \frac{\|B(tx)\|}{\|tx\|} = 0$

$$\frac{\cancel{|t|}\,\|B(x)\|}{\cancel{|t|}\,\|x\|} = 0$$

$$r(h) = f(x_0+h) - f(x_0) - T(h) \in \mathbb{R}^m$$

$$\frac{\|r(h)\|}{\|h\|} \to 0 \quad \text{as } h \to 0$$

Chain rule :-

$f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ & $g : G \subset \mathbb{R}^m \to \mathbb{R}^k \ni f(E) \subset G$.
Let f & g are diff at $x_0 \in E$ & $y_0 = f(x_0) \in G$ respectively. Also, $f'(x_0) = A$ & $g'(y_0) = B$ then $F = g \circ f$ is diff at 'x_0' & $F'(x_0) = BA$

Pf :- TST $\lim_{h \to 0} \|F(x_0+h) -$

Let $r(h) = f(x_0+h) - f(x_0) - A(h)$
$s(k) = g(y_0+k) - g(y_0) - B(k)$

Given $\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = \lim_{h \to 0} \frac{\|s(k)\|}{\|k\|} = 0$

$$\frac{\|F(x_0+h) - F(x_0) - BAh\|}{\|h\|}$$

$$= \frac{\|g(y_0+k) - g(y_0) - BA(h)\|}{\|h\|}$$

$$= \frac{\|s(k) + B(k) - BA(h)\|}{\|h\|}$$

$$= \frac{\|B(k - A(h)) + s(k)\|}{\|h\|}$$

$$= \frac{\|B(r(h)) + s(k)\|}{\|h\|}$$

$$\leq \frac{\|B\| \, \|r(h)\|}{\|h\|} + \frac{\|s(k)\|}{\|h\|}$$

Ex:-

$f : \mathbb{R}^n \to \mathbb{R}^n$

1) $f(x) = C$

$f'(x) = 0$

2) $f(x) = x$

Claim:- $\dfrac{\|f(x+h) - f(x) - I(h)\|}{\|h\|} = 0$ $x + h - x - h$

3) $f(x) = \|x\| x$

Claim:- $\lim_{h \to 0} \dfrac{\| \|x+h\|(x+h) - \cancel{f(x)}\; x\|x\| - T(x) \|}{\|h\|} = 0$

$$f(x) = |x|\,x \quad (\mathbb{R} \to \mathbb{R})$$
$$\boxed{f'(0) = \lim_{h \to 0} \frac{h|h| - 0}{h} = 0}$$

1) $f(x) = (x_1, x_2^2)$ $f : \mathbb{R}^2 \to \mathbb{R}^2$

<u>MVT</u>

<u>Inverse f^n thm</u>

<u>Implicit f^n thm</u>

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$f: \mathbb{R}^2 \to \mathbb{R}^2$

1) $f((x,y)) = (y, x)$

~~2) $f: \mathbb{R}^2$~~ 2) $f(x,y) = (y^2, x^2)$

3) $f(x,y) = (x+y, y-x)$

$f(x,y) = (f_1(x,y), f_2(x,y))$

$f_{1,2}: \mathbb{R}^2 \to \mathbb{R}$ $\quad \frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y}, i = 1,2$

For 1), $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad \underbrace{Df(x) = A}_{\text{Derivative of } f}$

$$\frac{\|f(x+h) - f(x) - Ah\|}{\|h\|}$$

$$= (y+h_2, x+h_1) - (y,x) - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} h_2 \\ \end{pmatrix}$$

$f: E \subset \mathbb{R}^n \to \mathbb{R}^m$ $\quad x \in E, f(x) \in \mathbb{R}^m$

$$f(x) = (f_1(x), f_2(x), \ldots f_m(x))$$

$$= \sum_{i=1}^{m} \langle f(x), u_i \rangle . u_i$$

$f_i: E \subset \mathbb{R}^n \mapsto \mathbb{R}$

Quiz qn:- If f is cont $\Leftrightarrow$ ~~f is~~ all $f_i(x)$ are cont.

~~$\bar{u} = \bar{u}.\hat{i} + \bar{u}.\hat{j} + \bar{u}.\hat{k} = \sum u$~~

Assumption :- f is diff at x_0

$h = te_j$ ($t \in \mathbb{R}$ is small)

$$f(x_0+h) - f(x_0) = Ah + r(h)$$

$$\frac{f(x_0+te_j) - f(x_0)}{t} = A(e_j) + \frac{r(te_j)}{t}$$

$$\sum \lim_{t\to 0} \left(\frac{f_i(x_0+te_j) - f_i(x_0)}{t}\right) u_i = A(e_j)$$

$$\Rightarrow \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}(x_0)\, u_i = f'(x)e_j = Ae_j$$

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Ex :- $f : \mathbb{R}^2 \to \mathbb{R}^3$

$f(x,y) = (x, y, x+y)$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$f : E \subset \mathbb{R}^n \to \mathbb{R}$

$$\frac{f(x_0+tu) - f(x_0)}{t}$$

$\gamma(t) = x_0 + tu$

$\gamma : (-r, r) \to \mathbb{R}^n$ $\quad \phi : (-r,r) \to \mathbb{R}$

$\phi(t) = f(\gamma(t))$

$\phi'(t) = f'(\gamma(t))\gamma'(t)$

$\phi'(0) = f'(x_0)\gamma'(0) = f'(x_0).u = D_u f(x_0)$

$$\lim_{t\to 0} \frac{\phi(t) - \phi(0)}{t} = \phi'(0)$$

$$\lim_{t\to 0} \frac{f(x_0+tu) - f(u)}{t} = D_u f(x_0)$$

$f: E \subset \mathbb{R}^n \to \mathbb{R}^m$

$f(x) = (f_1(x), \ldots, f_m(x))$ — f is conti diff on E $\Leftrightarrow$ $\frac{\partial f_i}{\partial x_j}$ exists and is cont on E. ~~Moreover, $\sum_{i=1}^{n}$~~

$f: E \subset \mathbb{R}^n \to \mathbb{R}^m$

$f': E \subset \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^m)$

For given $\epsilon > 0$, $\exists\ \delta > 0$ s.t

if $\|x-y\| < \delta \Rightarrow \|f'(x) - f'(y)\| < \epsilon$

$f_i : E \subset \mathbb{R}^n \to \mathbb{R}$

$\frac{\partial f_i}{\partial x} : \mathbb{R}^n \to \mathbb{R}$

Moreover,

$$f'(x) = \left(\frac{\partial f_i}{\partial x_j}\right)_{ij} \text{ matrix}$$

f is ctr diff

$$\left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right|$$

$$= |\langle (f'(x) - f'(y)) e_j, u_i \rangle|$$

$$\le \|(f'(x) - f'(y)) e_j\| \|u_i\|$$

$$\le \|f'(x) - f'(y)\|$$

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MVT (Mean Value Thm)

$\phi : [a,b] \to \mathbb{R}$

1) ctr on $[a,b]$

2) diff on (a,b)

$$\exists\ c \in (a,b) \ni \phi'(c) = \frac{\phi(b) - \phi(a)}{b-a}$$

For $\phi : [a,b] \to \mathbb{R}^n$

$$\|\phi(b) - \phi(a)\| \le (b-a)\|\phi'(c)\|$$

$z = \phi(b) - \phi(a)$

$f(t) = \langle \phi(t), z \rangle$

f is cts & diff. Why?

$f'(c)(b-a) = f(b) - f(a)$

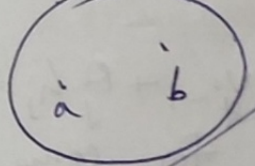
$\langle \phi(b) - \phi(a), z \rangle = \langle \phi'(c), z \rangle (b-a)$ Cauchy-Schwartz

$\|z\|^2 \le |b-a| \, \|\phi'(c)\| \, \|z\|$

$f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ & $f \in C'(E)$ $a, b \in E$ (E is convex)

If $\|f'(x)\| \le M$ $\forall x \in E$, then $\|f(b) - f(a)\| \le M|b-a|$

Pf:- $\gamma(t) = a + t(b-a)$; $\gamma : [0,1] \to \mathbb{R}^n$



$g(t) = f(\gamma(t))$ $\gamma(0) = a$, $\gamma(1) = b$

$\|f(b) - f(a)\| = \|g(1) - g(0)\| \le 1 \, \|f'(\gamma(c))\gamma'(c)\| = \|f'(\gamma(c))\| \, |b-a|$

Convexity for $\gamma(t)$ to belong to E

Cor:- $f : E \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ (E - convex open set)

If $f'(x) \equiv 0$ $\forall x \in E$ then f is a constant.

$f: E \subset \mathbb{R}^n \to \mathbb{R}^m$ f is diff at $x \in E$, then

$$f'(x) = \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{ij}$$

Converse :- If $D_j f_i = \frac{\partial f_i}{\partial x_j}(x)$ are cts on E for $i = 1, \ldots, m$ & $j = 1, \ldots, n$ then $f \in C'(E)$

Pf :- For $m = 1$, $h = (h_1, \ldots, h_n)$

Aim :- $\| f(x+h) - f(x) - \sum D_j f(x) h_j \| < \epsilon \|h\|$

$v_0 = 0$, $v_k = h_1 e_1 + \cdots + h_k e_k$, $k = 1, \ldots, n$

$\|v_k\| \leq h$

$$f(x+h) - f(x) = \sum_{j=1}^{n} \left(f(x+v_j) - f(x+v_{j-1}) \right)$$

From MVT,

$$\underbrace{f(x + v_{j-1} + h_j e_j)}_{v_k} - f(x + v_{j-1})$$

$$= \cancel{h_j D f(} h_j D_i f(x + v_{j-1} + \theta_j h_j e_j)$$ for some $\theta_j \in (0,1)$

$$|f(x+h) - f(x) - \sum D_j f(x) h_j|$$

$$= \left| \sum_{j=1}^{n} \left(D_j f(x + v_j + \theta_j h_j e_j) - D_j f(x) \right) h_j \right|$$

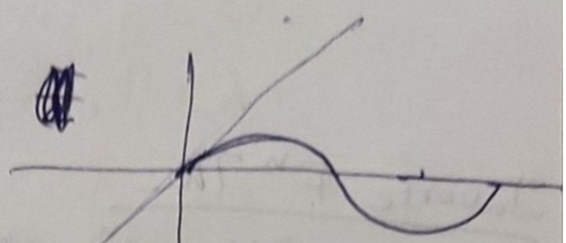
$$\leq \frac{1}{n} \sum \epsilon |h_j| \leq \epsilon \|h\|$$

f is diff at $x \in E$,

$$\| D_j f(x+h) - D_j f(x) \| < \epsilon/n, \quad \|h\| < \delta$$

$f : (X, d) \to (X, d)$ is called a contraxn if $\exists\, c < 1$
$\ni d(f(x), f(y)) \le c\, d(x, y)$

$x \in X$ is called fixed point
if $f(x) = x$.



A contraxn can't have two fixed pts. } only one or zero fixed pt

$$|f(x) - f(y)| \le c|x - y|$$
$$|x - y| \le c|x - y|$$

$f(x) = \frac{x}{2}$ $x \in (0, 1)$ has zero fixed pt.

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Thm :- Let $\phi : (X, d) \longmapsto (X, d)$ contraxn - If X is complete then ϕ has a unique fixed point
(i.e $\exists\, 1\ x \in X : \phi(x) = x$)

Pf :- $x_0 \in X$, $x_n = \phi(x_{n-1})$, $n = 1, 2, \ldots$

$$d(x_{n-1}, x_n) = d(\phi(x_n), \phi(x_{n-1})) \le c\, d(x_n, x_{n-1})$$
$$\le c^n d(x_1, x_0)$$

$c < 1$, $m > n$

$$d(x_m, x_n) \le d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots\ d(x_{n-1}, x_n)$$
$$\le (c^{m-1} + c^{m-2} + \cdots + c^n)\, d(x_1, x_0)$$
$$\le c^n(1 + c^2 + \cdots)\, d(x_1, x_0)$$
$$\le \frac{c^n}{1-c}\, d(x_1, x_0)$$

As $c \le 1$, $\Rightarrow \{x_n\}$ is a Cauchy seq in X.

Let $x_n \to x \Rightarrow \phi(x_n) \to \phi(x)$ (because ϕ is cont, in particular ϕ is uniformly cont
$\delta = c\epsilon$) $\llcorner$ in cont. def^n

As x_n is complete

Little incomplete

Inverse F^n Thm

Suppose $f: E \subset \mathbb{R}^n \mapsto \mathbb{R}^n$. $f - C'(E)$ — diff & derivative is cont.

for $a \in E$, $f'(a)$ exists and invertible.

Then

1) $\exists$ open sets U & V of $\mathbb{R}^n$ s.t $a \in U$, $b \in V$ and f is 1-1 on U, $f(U) = V$

2) If g is inverse of f, then $g \in C'(V)$ $g'(y) = (f'(x))^{-1}$

Pf:- $A = f'(a)$. Choose $\lambda = \dfrac{1}{2\|A^{-1}\|}$

$x \mapsto f'(x)$

λ $\exists \underset{\text{open ball}}{U} : \|f'(x) - A\| < \lambda$ $\forall x \in U$ (open)

For any $y \in \mathbb{R}^n$,

$$\phi(x) = x + A^{-1}(y - f(x))$$

$$\phi'(x) = I + A^{-1}(-f'(x))$$

$$= \cancel{AA^{-1}} A^{-1}(A - f'(x))$$

$$\|\phi'(x)\| \le \frac{1}{2\lambda} \lambda$$

$$\phi'(x) = \frac{1}{2} \quad \forall x \in U$$

$\Rightarrow x_1, x_2 \in U$, $\|\phi(x_1) - \phi(x_2)\| \le \frac{1}{2}\|x_1 - x_2\|$

$\Rightarrow \phi$ has at most one fixed pt in $U \Rightarrow y = f(x)$ for atmost one $x \in U$

$\Rightarrow \cancel{\phi} f$ is 1-1 on U

$\rightarrow V = f(U)$ assumption

$y_0 \in V \quad \exists x_0 \in U : f(x_0) = y_0$

Pf:- U is open $\Rightarrow \exists r > 0 : \overline{B(x_0, r)} \subset U$

Claim :- $B(y_0, \lambda r) \subset V$, $\|y - y_0\| < \lambda r$

$\|\phi(x_0) - x_0\| = \|A^{-1}(y - f(x_0))\|$

$\leq \|A^{-1}\| \|y - y_0\| < \frac{r}{2}$

$x_0 \in \overline{B(x_0, r)}$

$\|\phi(x) - x_0\| \leq \|\phi(x) - \phi(x_0)\| + \|\phi(x_0) - x_0\|$

$\leq \frac{1}{2}\|x - x_0\| + \frac{r}{2} \leq r$

$\Rightarrow \phi(x) \in \overline{B(x_0, r)}$

$\exists x \in \overline{B(x_0, r)} : \phi(x) = x \Rightarrow f(x) = y$

$y \in f(\overline{B(x_0, r)}) \subset f(U) = V$

~~Inverse fn thm~~

Thm $f : U \xrightarrow[\text{onto}]{1-1} V$, $g : V \to U : g(f(x)) = x \ \forall x \in U$

$y \in V$, $y + k \in V \quad \exists x \in U$ ~~a~~ $x + h \in U$ $f(x) = y$,

$f(x+h) = y + k$

$g(y+k) - g(y) - T(k)$

$= \cancel{x} + h - \cancel{x} - T(f(x+h) - f(x))$

$\phi(x) = x + A^{-1}(y - f(x))$

$\phi(x+h) = x + h + A^{-1}(y - f(x+h))$

$\phi(x+h) - \phi(x) = h - A^{-1}(k)$

$\|h - A^{-1}(k)\| = \|\phi(x+h) - \phi(x)\|$

$\leq \frac{1}{2}\|h\|$

~~$\|A^{-1}(k)\| \geq$~~ $\frac{\|h\|}{2} \leq \|A^{-1}(k)\|$

$\leq \|A^{-1}\| \|k\|$

$\leq \frac{1}{2\lambda}\|k\|$

$\|h - Tk\| \quad T = (f'(x))^{-1}$

$$= \|Tf'(x)h - T(f(x+h) - f(x))\|$$

$$\frac{\|g(y+k) - g(y) - Tk\|}{\|k\|} = \frac{\|h - T(k)\|}{\|k\|}$$

$$\le \|T\| \frac{\|f(x+h) - f(x) - f'(x)h\|}{\lambda \|h\|}$$

$$\longrightarrow 0 \text{ as } \|k\| \longrightarrow 0$$

$\Rightarrow g$ is diff & $g'(y) = (f'(x))^{-1}$

$g'(y)$ is cont as derivative is cont & $f: A \to A^{-1}$ is cont & composition of cont f^ns is cont.

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$x = (x_1, x_2, \ldots, x_n)$

$y = (y_1, y_2, \ldots, y_m)$

$(x, y) \in \mathbb{R}^{n+m}$

$\rightarrow A \in L(\mathbb{R}^{n+m}, \mathbb{R}^n)$

$A : \mathbb{R}^{n+m} \longmapsto \mathbb{R}^n$

$h \in \mathbb{R}^n, \; k \in \mathbb{R}^m$ $\qquad A_x : \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$A(h, k) = A$ $\qquad f(x) = Ax$

$A_x(h) = A(h, 0)$ $\qquad f'(x) = A$

$A_y(k) = A(0, k)$

Ex:-

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A_x h = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h \\ 0 \end{bmatrix} \qquad A_y k = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 \\ k \end{bmatrix}$

$A_x(h) = A(h, 0)$

$A_y(k) = A(0, k)$

If A_x is invertible

$A_x h = 0 \iff h = 0$

$A(h,k) = 0$

$A_x h + A_y k = 0$

$h = -(A_x)^{-1} A_y(k)$

Given $k \in \mathbb{R}^m$, $\exists$ a unique $h \in \mathbb{R}^n \ni A(h,k) = 0$

Implicit f^n Thm

Let $f: E \subset \mathbb{R}^{n+m} \mapsto \mathbb{R}^n$, C' – map $\ni f(a,b) = 0$, for some $(a,b) \in E$

Put $A = f'(a,b)$ & A_x is invertible

Then $\exists$ open sets $U \subset \mathbb{R}^{n+m}$ & $W \subset \mathbb{R}^m \ni$

$(a,b) \in U$, $b \in W$.

For $y \in W$ $\exists$! a unique $x \in \mathbb{R}^n$: $(x,y) \in U$ & $f(x,y) = 0$

② $g: W \to \mathbb{R}^n$, $g(y) = x$, $f(g(y),y) = 0$

② g is C' – map 3) $g'(b) = \ldots$

Pf: $F: \mathbb{R}^{n+m} \mapsto \mathbb{R}^{n+m}$, $F(x,y) = (f(x,y), y)$

$F'(a,b)(h,k) = (f'(a,b)(h,k), k)$

$M = F(a+h, b+k) - F(a,b) - (f'(a,b)(h,k), k)$

$= (f(a+h, b+k), b+k) - (f(a,b), b) - ''$

$= (r(h,k), k)$ (where $r(h,k) = f(a+h, b+k) - f(a,b) - f'(a,b)(h,k)$)

$\frac{M}{\|h\|} = 0$

$f(x) = Ax$

$\begin{pmatrix} \cancel{a} & b \\ c & d \end{pmatrix} \begin{pmatrix} h \\ k \end{pmatrix} = \cancel{Ax}$

$F'(a,b)$ is invertible.

By IFT $\exists$ open sets U & V of $\mathbb{R}^{n+m}$ $\ni$

$(a,b) \in U$ & $F(a,b) = (0,b) \in V$

$F : U \xrightarrow[1-1]{\text{onto}} V$ $F(U) = V$

$W = \{ y \in \mathbb{R}^m : (0,y) \in V \}$

W is open (inverse image of open set is open)

($f : V \to W$ & $f((0,y)) = y$)

$G : V \mapsto U$ is C' – map $\mid$ $GF(x,y) = (x,y)$

$y \in W$ $\exists !$ $x \in \mathbb{R}^n$! $f(x,y) = 0$

$g : W \to \mathbb{R}^n$, $f(g(y),y) = 0$ $\forall$ $y \in W$, open $\mathbb{R}^m$

$GF(g(y),y) = (g(y),y)$ For $y \in W$

$G(0,y) = (g(y),y) \Rightarrow g$ is C' – map

$\Rightarrow$ $\phi : W \to \mathbb{R}^{n+m}$, $\phi(y) = (g(y),y)$

$f \circ \phi(y) = f(\phi(y)) = f(g(y),y) = 0$

$F : \mathbb{R}^{n+n} \to \mathbb{R}^{n+m}$, $F(x,y) = (f(x,y),y)$

$(f \circ \phi)'(b) = 0$

$f'(\phi(b))\,\phi'(b) = 0$

$\phi(b) = (g(b),b) = (a,b)$

$f'(a,b)\,\phi'(b) = 0$

$A\phi'(b) = 0$

$\phi'(b) = (g'(b),1)$

$O = A(\phi'(b)k)$

$\phi'(b)k = (g'(b)k, k)$

$= A_x(g'(b)k) + A_y(k)$

$g'(b) = -A_x^{-1}(A_y)$

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$$\underline{\int_a^b} f(x)\cdot dx = \sup_P L(P,f)$$

Lower integral — supremum over all partitions

$$\overline{\int_a^b} f(x)\cdot dx = \inf_P U(P,f)$$

Charecteristic f^n $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$

Ex of non integrable f^n:-

$$\chi_A(x) = \begin{cases} 1 & x \notin A \\ 0 & x \notin A \end{cases}$$

$$\int \chi_{[a,b]} = \lambda([a,b]) = m([a,b])$$

length of interval

$$A = [0,1] \setminus \mathbb{Q}$$

$$\int \chi_A = m(A)$$

$$C[0,1] = \{f : [0,1] \xrightarrow[cts]{} \mathbb{R}\}$$

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$[a,b]$, (a,b), $[a,b)$, $(a,b]$

$$\ell(I) = b - a$$

$$\{I_n\}_{n=1}^{\infty} \quad A \subset \cup I_n$$

Outer measure $\quad U(A) \leq \sum_{n=1}^{\infty} \ell(I_n)$

$$m^*(A) = \inf\left\{\sum_{n=1}^{\infty} \ell(I_n) : A \subset \bigcup_{n=1}^{\infty} I_n\right\}$$

Ex:- $A = \{a\}$, $m^*(A) = ?$

Pf:- $m^*(A) \leq 2\epsilon \quad A \subset (a-\epsilon, a+\epsilon)$

$m^*(A) \leq 0$

1) $m^*(A) \geq 0$ for any $A \subset \mathbb{R}$

2) $m^*(\phi) = 0$

3) $A = \{a_1, \ldots, a_n, \ldots\}$

$I_n = \left(a_n - \frac{\epsilon}{2^n}, a_n + \frac{\epsilon}{2^n}\right)$

$A \subset \bigcup I_n$

$$m^*(A) \leq \sum_{n=1}^{\infty} l(I_n)$$

$$= \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$$

$$m^*(A) = 0$$

4) $A \subset B \Rightarrow m^*(A) \leq m^*(B)$

$B \subset \bigcup I_n$, $A \subseteq \bigcup I_n \rightarrow m^*(A) \leq \sum l(I_n)$

$\cancel{m^*(A) \geq m^*(B)}$

$$m^*(A) \leq \inf\left\{\sum l(I_n) : B \subset I_n\right\}$$

$$= m^*(B)$$

5) Thm:- $m^*([a,b]) = b-a$

$[a,b] \leq \left(a - \epsilon/2, b + \epsilon/2\right)$

$m^*([a,b]) \leq b - a + \epsilon$

$\epsilon > 0$ $m^*([a,b]) \leq b-a$

Claim:- $(b-a) - \epsilon \leq m^*([a,b])$

$$m^*([a,b]) = \inf\left\{\sum l(I_n) : A \subset \bigcup I_n\right\}$$

$$\sum_{n=1}^{\infty} l(I_n) \leq m^*([a,b]) + \epsilon \text{ \& } [a,b] \subset \bigcup I_n$$

$$I_1 \cup \ldots \cup I_k \supset [a,b]$$

6) $m^*((a,b)) = b-a$

$\left[a-\frac{\epsilon}{4}, b+\frac{\epsilon}{4}\right] \subset (a,b)$

1) $m^*(\mathbb{R}) = \infty$

2) H.W $m^*(A \cup B) \leq m^*(A) + m^*(B)$

$A = [0,1] \cap \mathbb{Q}$

3) If $I_1, I_2, \ldots I_n$ covers A then
$l(I_1) + l(I_2) + \ldots + l(I_n) + \ldots \geq 1$

4) $m^*\{[0,1] \setminus \mathbb{Q}\} = 1$

$([0,1] \setminus \mathbb{Q}) \cup ([0,1] \cap \mathbb{Q}) = [0,1]$

$\underbrace{m^*([0,1])}_{1} \geq m^*([0,1] \setminus \mathbb{Q}) + \underbrace{m^*([0,1] \cap \mathbb{Q})}_{0 \text{ since countable}} \geq 1$

H.W

1) ~~$m^*(A \cup B$~~

2) $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m^*(A_n)$

3) $m^*(A+x) = m^*(A)$ where $A+x = \{a+x : a \in A\}$

Baire Category Thm :-

$\{G_n\}$ open dense subset of $\mathbb{R}^n$.

$G = \cap G_n$ is dense in $\mathbb{R}^n$.

O be any open subset $\mathbb{R}^n$, claim $G \cap O \neq \phi$

$x_1 \in O \cap G_1 \Rightarrow \exists\, r_1 < 1$, $\overline{B(x_1, r_1)} \subset O \cap G_1$

$x_2 \in B(x_1, r_1) \cap G_2 \Rightarrow \exists\, r_2 > 1$, $\overline{B(x_2, r_2)} \subset B(x_1, r_1) \cap G_2$

denseness was used :

$O \supset B(x_1$

G_δ - set

A subset G of $\mathbb{R}$ is called G_δ-set if G is countable intersexn of open subsets $\mathbb{R}$ i.e

$G = \cap G_n$

Ex :- 1) $\mathbb{Q}$ is a G_δ set

$\Rightarrow \underbrace{\mathbb{R} \setminus \mathbb{Q}}_{\text{is a } G_\delta \text{ set}} = \cup G_n$ where $\mathbb{Q} = \{r_n\}_{n=1}^{\infty}$, $G_n = \mathbb{R} - \{r_n\}$

F_σ - set

if $F = \bigcup_{n=1}^{\infty} F_n$, F_n - closed sets

$A \subseteq \mathbb{R}$ then for a given $\epsilon > 0$ $\exists$ open set 'O' $\ni$ $A \subset O$ & $m^*(O) \leq m^*(A) + \epsilon$ ~~$\exists \{I_n$~~

Pf :- $\exists \{I_n\} \ni A \subseteq \cup I_n = O$

$m^*(O)\epsilon \leq l(I_n) \leq m^*(A) + \epsilon$

$m^*(O) = m^*(\cup I_n) \leq \sum m^*(I_n) = \sum l(I_n)$

$\exists$ a G_δ-set $G \ni A \subset G$ & $m^*(G) = m^*(A)$

Pf: For each, $\frac{1}{n}$, $\exists\ O_n$ (open)

$$m^*(O_n) \leq m^*(A) + \frac{1}{n}$$

$$G = \bigcap O_n \Rightarrow m^*(G) \leq m^*(O_n) \leq m^*(A) + \frac{1}{n}$$

$$m^*(A) \leq m^*(G) \leq m^*(A) \qquad \forall n$$

$m^*: P(\mathbb{R}) \longmapsto [0, \infty]$ m^* is a sub additive set valued f^n.

$m^*(\bigcup A_n) \leq \sum m^*(A_n)$