MA 372 : Stochastic Calculus for Finance

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Exercises 6

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1. Let $\mathcal{F}(t)$ be the filtration generated by Brownian motion $W(t)$. Find the martingale representation for the following martingales;

   a) $M(t) = \mathbb{E}[W^2(T)|\mathcal{F}(t)]$
   b) $M(t) = \mathbb{E}[W^3(T)|\mathcal{F}(t)]$
   c) $M(t) = \mathbb{E}[\exp\{\sigma W(T)\}|\mathcal{F}(t)]$

2. Let $r(t)$ and $\sigma(t)$ be non-random functions. Suppose $S(t)$ satisfies the following:

$$S(t) = S(0)\exp\Big\{\int_0^t \sigma(s)d\tilde{W}(s) - \frac{1}{2}\int_0^t (r(s) - \frac{1}{2}\sigma^2(s))ds\Big\}$$

where $\tilde{W}(t)$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. The price of an European call at time t, given by the risk-neutral valuation formula is

$$c(0, S(0)) = \tilde{\mathbb{E}}\Big[\exp\Big\{-\int_0^T r(s)ds\Big\}\Big(S(T) - K\Big)^+\Big]$$

Let

$$\begin{aligned} BSM(T, x, K, R, b) &= xN\Big(\frac{1}{b\sqrt{T}}[\log(\frac{x}{K}) + (R + \frac{b^2}{2})T]\Big) \\ &\quad - \exp\{-RT\}KN\Big(\frac{1}{b\sqrt{T}}[\log(\frac{x}{K}) + (R - \frac{b^2}{2})T]\Big) \end{aligned}$$

Show that

$$c(0, S(0)) = BSM(T, S(0), K, \frac{1}{T}\int_0^T r(t)dt, \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)dt})$$

3. Let $W(t),\ 0 \le t \le T$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t),\ 0 \le t \le T$ be a filtration for this Brownian motion. Consider a stock price process whose differential is

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t),\ \mu, \sigma \in \mathbb{R}, \sigma > 0.$$

(i) Write down the probability measure $\mathbb{Q}$ under which the discounted stock price $Y(t) = e^{-rt}S(t)$ is a martingale with respect to $\mathcal{F}(t)$.

(ii) Determine $d(e^{-rt}S(t))$ under the risk-neutral probability measure $\mathbb{Q}$.

4. Suppose the market has an arbitrage. So there ia a portfolio value process satisfying $X_1(0) = 0$ and $\mathbb{P}(X_1(T) \geq 0) = 1$, $\mathbb{P}(X_1(T) > 0) > 0$, for some positive time T.

   a) Show that if $X_2(0)$ is positive, then there exists a portfolio value process $X_2(t)$ satisfying at $X_2(0)$ and satisfying

   $$\mathbb{P}\Big(X_2(T) \geq \frac{X_2(0)}{D(T)}\Big) = 1 \text{ and } \mathbb{P}\Big(X_2(T) > \frac{X_2(0)}{D(T)}\Big) > 0$$

   b) Suppose that the market has a portfolio process $X_2(t)$ such that $X_2(0)$ is positive and the above holds. Then show that the model has a portfolio value process $X_1(t)$ which is an arbitrage.

5. Let $(W_1(t), W_2(t), W_3(t))$, $0 \leq t \leq T$ be a 3-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t)$, $0 \leq t \leq T$ be a filtration for this Brownian motion. Consider a financial market consisting of a risk-free asset $B(t)$ and three stocks (risky assets) $S_1(t), S_2(t)$ and $S_3(t)$, whose price at time t, $t > 0$ satisfy the following differentials:

   $$\begin{aligned}
   dB(t) &= 3\ B(t)\ dt,\ B(0) = 1 \\
   dS_1(t) &= S_1(t)\Big[4\ dt + dW_1(t) + dW_2(t) + dW_3(t)\Big] \\
   dS_2(t) &= S_2(t)\Big[(3+\alpha)\ dt + dW_1(t) + 2\ dW_2(t) + dW_3(t)\Big] \\
   dS_3(t) &= S_3(t)\Big[(3+\alpha^2)\ dt + 3\ dW_1(t) + 4\ dW_2(t) + \beta\ dW_3(t)\Big]
   \end{aligned}$$

   where α, β are positive constants.

   (a) When the above market is arbitrage free? (Find the conditions in terms of α, β)

   (b) When the above market is complete? (Find the conditions in terms of α, β)

   (c) When the above market has more then one risk-neutral probability measure? (Find the conditions in terms of α, β)

   (d) If $\alpha = 3$ and $\beta = 2$ then find the risk-neutral probability measure $\tilde{\mathbb{Q}}$ for the above market.