MA 321 (Optimization)
Mid-semester Examination

Time: 2 pm - 4 pm   September 21, 2023   **Total marks:** 30

Notation: $\tilde{\mathbf{a}}_\mathbf{k}$ denotes the k-th column of A.

1. Consider the linear programming problem (P) of the form,

$$\text{Maximize } \mathbf{c}^T\mathbf{x}$$
$$\text{subject to } \mathbf{A}_{2\times 4}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0},$$

where $\tilde{\mathbf{a}}_1 = [1,1]^T, \tilde{\mathbf{a}}_2 = [1,-1]^T, \tilde{\mathbf{a}}_3 = [1,-2]^T,\ \tilde{\mathbf{a}}_4 = [-2,0]^T,\ \mathbf{b} = [4,-2]^T,\ \mathbf{c} = [2,3,-1,-6]^T$.

(a) Give all the entries of the simplex table (including the $c_j - z_j$ values) for the **BFS** corresponding to the basis $\{\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_3\}$.
Soln:

Solution:

$c_j - z_j$	0	5	0	-6	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{b}$
x_1	1	$\frac{1}{3}$	0	$-\frac{4}{3}$	2
x_3	0	$\frac{2}{3}$	1	$-\frac{2}{3}$	2

Calculations must be shown.

(b) Using the **simplex algorithm** either obtain the optimal table and the optimal solution or conclude that the problem does not have an optimal solution.
Soln: From the previous table since $c_2 - z_2 > 0$ and $c_4 - z_4 < 0$ and (P) is a maximization problem so x_2 is the entering variable. Since the $\min\{\frac{x_i}{u_{i2}} : u_{i2} > 0\} = \min\{6, 3\} = 3$, hence x_3 is the leaving variable.
The table is given below:

$c_j - z_j$	0	0	$-\frac{15}{2}$	-1	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{b}$
x_1	1	0	$-\frac{1}{2}$	-1	1
x_2	0	1	$\frac{3}{2}$	-1	3

Since $c_j - z_j \leq 0$ for all $j = 1,2,3,4$, the above table is optimal and the optimal solution is $[1,3,0,0]^T$.

(c) **If possible** give two distinct directions of Fea(P).
Soln: Since both the two tables has a non positive column so we get two directions from the formula $\begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$.

Two distinct directions obtained from above are $[4,0,3,4]^T$ and $[1,1,0,1]^T$.

[3.5 + 3.5 + 2]

Show the required calculations for the first table and give brief justification wherever applicable

2. Consider the following linear programming problem (P) given below:

$$\text{Maximize } c_1x_1 + c_2x_2 + c_3x_3$$

$$\begin{aligned} \text{subject to} \qquad & 6x_1 - 2x_2 + ex_3 \leq 3 \qquad (e > 0) \\ & 4x_1 + x_2 + x_3 \leq 6 \\ & x_1 \geq 1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \end{aligned}$$

where c_1, c_2, c_3, e are fixed constants.

(a) Give the Dual of (P) if $c_1 = -1, c_2 = -2, c_3 = 4$ and **check** whether it(Dual) has an optimal solution.
Soln: Since (P) can be written as:

$$\text{Maximize } c_1x_1 + c_2x_2 + c_3x_3$$

$$\begin{aligned} \text{subject to} \qquad & 6x_1 - 2x_2 + ex_3 \leq 3 \qquad (e > 0) \\ & 4x_1 + x_2 + x_3 \leq 6 \\ & -x_1 \leq -1 \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, \end{aligned}$$

The Dual of (P) is given by:

$$\text{Miniimize } 3y_1 + 6y_2 - y_3$$

$$\begin{aligned} \text{subject to} \qquad & 6y_1 + 4y_2 - y_3 \geq c_1 \qquad (e > 0) \\ & -2y_1 + y_2 \geq c_2 \\ & ey_1 + y_2 \geq c_3 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0. \end{aligned}$$

Since $[1, 2, 0]^T \in Fea(P) \neq \phi$ and $Fea(P)$ is bounded, (P) has an optimal solution hence the Dual of (P) also has an optimal solution(by duality theory).

(b) If $\mathbf{x}_0 = [1, 2, 0]^T$ is optimal for (P) and c_3 is changed to $c_3 - 1$, then **if possible check** whether $\mathbf{x}_0$ is optimal for the new problem $(P^{‘})$ (everything else of (P') is same as that of (P)).
Soln: $\mathbf{x}_0 \in Fea(P^{‘})$ since $Fea(P) = Fea(P^{‘})$. Let $\mathbf{y}_0$ be an optimal solution of the Dual of (P), then $\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{y}_0$ (by the fundamental theorem of duality). Since the feasible region of the Dual of $(P^{‘})$ is a superset of the feasible region of the Dual of (P), so $\mathbf{y}_0 \in Fea(\text{Dual } P^{‘})$. Since $\mathbf{c}^T\mathbf{x}_0 = \mathbf{b}^T\mathbf{y}_0$, so $\mathbf{x}_0$ is optimal for $(P^{‘})$ and $\mathbf{y}_0$ is optimal for the dual of $(P^{‘})$ (by duality theory).

(c) If $[y_1, y_2, y_3]^T = [r, s, t]^T$ $(r > 0, s > 0)$ is an optimal solution of the Dual of (P) and $c_3 \le c_2 \le c_1$ then **if possible obtain** an optimal solution of (P).

Soln: True.

Since the dual of (P) has an optimal solution, (P) has an optimal solution by the fundamental theorem of duality. Since $r > 0$ and $s > 0$ in the given optimal solution of the dual of (P), by the complementary slackness property the first two defining constraints of Fea(P) must be satisfied as an equality by any optimal solution $\mathbf{x}_0$ of (P).

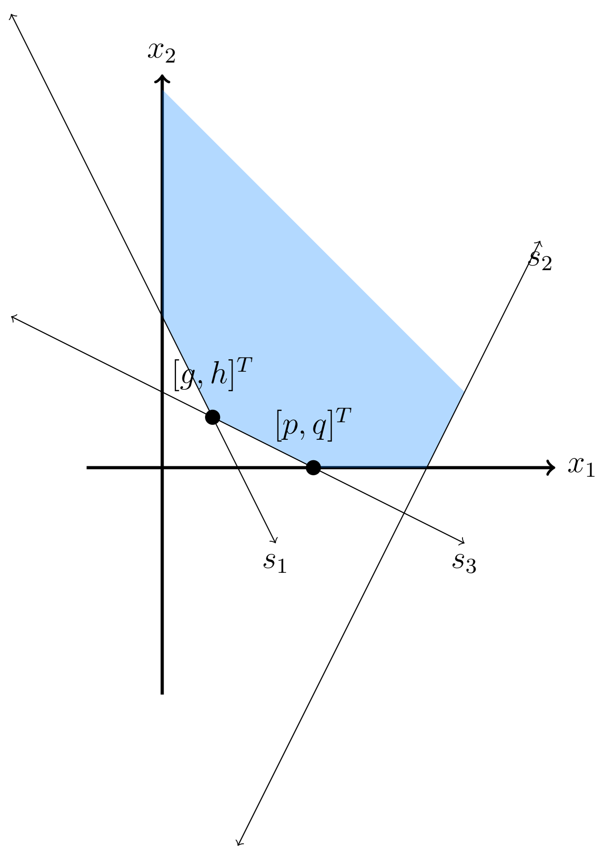
Since $c_3 \le c_2 \le c_1$ and $e > 0$, any feasible solution (hence optimal solution) of the Dual of (P) must satisfy the third constraint as a strict inequality hence by the complementary slackness property any optimal solution of (P) must have $x_3 = 0$. By solving the first two inequalities defining $Fea(P)$ as an equation for x_1, x_2 we get $x_1 = \frac{15}{14}, x_2 = \frac{12}{7}$. The optimal solution is $[\frac{15}{14}, \frac{12}{7}, 0]^T$.

(**All the parts in the above question are independent**). **[2.5 + 2.5 + 3]**

3. Consider a linear programming problem (P) of the form,

$$\text{Minimize } \mathbf{c}^T\mathbf{x}, \quad \text{subject to } A_{2\times 3}\mathbf{x} \le \mathbf{b},\ \mathbf{x} \ge \mathbf{0}.$$

with the shaded feasible region given below and the hyperplanes corresponding to each of the added variables s_i, marked accordingly:



(a) Give the basic and the non basic variables of the **BFS** corresponding to $[p, q]^T$.

Soln: From the picture $s_3 = 0$ and $x_2 = 0$ and x_1, s_1, s_2 are all strictly greater than zero. Hence basic variables are x_1, s_1, s_2, and the non basic variables are s_3 and x_2.

(b) Give the sign of every entry ($+$, $-$ or 0) in the **non basic columns** (excluding the $c_j - z_j$ values) of the simplex table corresponding to $[p,q]^T$ (**BFS** of part (a)).

Soln:

(i)

	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{e}}_3$	
x_1	$+$	$-$	
s_1	$+$	$-$	
s_2	$-$	$+$	

Brief justification has to be provided for the above otherwise no credit is given.

(c) If $[p,q]^T$ is optimal for (P) then in the corresponding simplex table should all the $c_j - z_j$ values, necessarily be non negative?

Soln:

True. Let the BFS corresponding to $[p,q]^T$ be $\mathbf{x}_0$. Suppose for some j, $c_j - z_j < 0$, for some nonbasic variable in the simplex table corresponding to $[p,q]^T$. If we try to make that nonbasic variable positive then the cost associated with the new feasible solution $\mathbf{x}'$ is

$\mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x}_0 + x_j'(c_j - z_j)$, (**)

where $0 \le x_j' \le \frac{x_r}{u_{rj}}$, and $\frac{x_r}{u_{rj}}$ is the minimum ratio. Note that all the BFS of (P) are non degenerate since every extreme point lies at the point of intersection of exactly two LI hyperplanes, so for any BFS of (P), all the basic variables are strictly positive. Hence the minimum ration $\frac{x_r}{u_{rj}} > 0$, corresponding to $\mathbf{x}_0$. From (**) it follows, $\mathbf{c}^T\mathbf{x}' < \mathbf{c}^T\mathbf{x}_0$, which contradicts that $\mathbf{x}_0$ is optimal for (P).

(d) If $[p,q]^T$ is an optimal solution of (P) then does the Dual of (P) have a unique optimal solution?

Soln:

Since $s_1 > 0, s_2 > 0$ (from the picture), for the corresponding optimal BFS of (P), any optimal solution of the Dual satisfies, $y_1 = 0$, $y_2 = 0$ by the complementary slackness property.

Since $x_1 = p > 0$ in the optimal solution of (P) by complementary slackness the constraint in the Dual corresponding to x_1 has to be satisfied as an equality by any optimal solution of the Dual. That determines y_3 uniquely since in each of the constraints of the Dual (follows from the picture) the coefficient of y_3 is > 0. $[\mathbf{2+5+3+2}]$

Brief and precise justification is required for each of the above parts

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