Plan

- Directions of Fea(LPP)
- Extreme directions of Fea(LPP)
- Representation Theorem for Fea(LPP)
- Necessary and sufficient conditions for the existence of optimal solutions
- Optimal solutions in atleast one corner point

- $Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.

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- If $\mathbf{d}$ is a direction of a convex set S, then for all $\gamma > 0$, $\mathbf{x} + \alpha\mathbf{d} = \mathbf{x} + (\frac{\alpha}{\gamma})\gamma\mathbf{d} \in S$ for all $\alpha > 0$,

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- Directions $\mathbf{d}_1, \mathbf{d}_2$ of S are said to be distinct if $\mathbf{d}_1 \neq \gamma \mathbf{d}_2$ for any $\gamma > 0$
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  are all equal as directions of $Fea(LPP)$.
- Whereas $\mathbf{d}_1 = [1, 1]^T$, $\mathbf{d}_2 = [1, 0]^T$ give two distinct directions.

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- The set of all directions of $S = Fea(LPP)$ is a convex set.

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  $D' = \{\mathbf{d} \in \mathbb{R}^n : \mathbf{d} \geq \mathbf{0}, A\mathbf{d} \leq \mathbf{0}, \sum_i d_i = 1\}$ is a set of all distinct directions of S.

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- Also each $\mathbf{d} \in D$ is of the form $\mathbf{d} = \alpha\mathbf{d}'$ for some $\mathbf{d}' \in D'$ where $\alpha = \sum_i d_i (> 0)$.
- $D' =$
  $$\left\{ \mathbf{d} \in \mathbb{R}^n : \mathbf{d} \geq \mathbf{0}, \begin{bmatrix} & A & \\ 1 & 1, .., & 1 \\ -1 & -1, .., & -1 \end{bmatrix} \mathbf{d} \leq \begin{bmatrix} \mathbf{0} \\ 1 \\ -1 \end{bmatrix} \right\}.$$

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  The set D' now looks exactly like the feasible region of a LPP.

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- If $D \neq \phi$, then $Fea(LPP) = S(\neq \phi)$ must have atleast one extreme direction.
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- If $Fea(LPP) = S \neq \phi$ is unbounded then $D \neq \phi$ and S must have atleast one extreme direction.
- The number of distinct extreme directions of S is finite (why?).

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  $\{\mathbf{d} \in \mathbb{R}^n : -\mathbf{e}_j^T\mathbf{d} = \mathbf{0}\}$ for $j = 1, 2, \ldots, n$.
- If $\mathbf{d} \in D'$, is an extreme direction of S then it should lie on $(n-1)$ LI hyperplanes of the above mentioned $(m+n)$ hyperplanes, and the hyperplane

  $\{\mathbf{d} \in \mathbb{R}^n : [1, 1, \ldots, 1]\mathbf{d} = 1\}$,

  which gives a collection of n LI hyperplanes.

- **Exercise:** Check that if a $\mathbf{d} \in D$ lies on $(n-1)$ LI hyperplanes (out of the $(m+n)$ defining hyperplanes of D) given by $\{H_1, \ldots, H_{n-1}\}$, then $\{H, H_1, \ldots, H_{n-1}\}$ is LI where $H = \{\mathbf{d} \in \mathbb{R}^n : [1, 1, \ldots, 1]\mathbf{d} = 1\}$.

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- These are the only extreme directions of $S = Fea(LPP)$.

- **Theorem:**
  If $S = Fea(LPP) = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is nonempty, then S has atleast one extreme point.

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  For example take any single half space, or say a straight line in $\mathbb{R}^n$, which are polyhedral sets, but does not have any extreme point.
- The theorem works for $Fea(LPP)$ because of the non negativity constraints, that is because $Fea(LPP)$ is given a supply of n LI , defining hyperplanes.

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- **Exercise:** Is it possible to find a nonempty polyhedral set S, $S \subset \mathbb{R}^3$ which has three LI defining hyperplanes (not necessarily the nonnegativity constraints) but does not have any extreme point.
- **Definition:** Given S, a nonempty subset of $\mathbb{R}^n$, and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in S$, $\sum_{i=1}^{k} \lambda_i \mathbf{x}_i$, is called a convex combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, where $0 \leq \lambda_i \leq 1$ for all $i = 1, 2, \ldots, k$, and $\sum_{i=1}^{k} \lambda_i = 1$.

- All possible convex combinations of two distinct points gives a straight line segment with those two points as boundary points.

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- **Result:** Given $\phi \neq S \subset \mathbb{R}^n$, S is a convex set if and only if for all $k \in \mathbb{N}$,
  the convex combination of any k elements of S is again an element of S.

- **Theorem: (Representation Theorem)**
  If $S = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ is nonempty and if $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k$ are the extreme points of S and $\mathbf{d}_1, \mathbf{d}_2, .., \mathbf{d}_r$ are the distinct extreme directions of S (the set of directions is empty if S is bounded) then $\mathbf{x} \in S$ if and only if

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  and $\mu_j \geq 0$, for all $j = 1, 2, \ldots, r$.
- That is, $\mathbf{x} \in S \Leftrightarrow \mathbf{x}$ can be written as a convex combination of the extreme points of S plus a non negative linear combination of the extreme directions of S.

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  $x \geq 0, y \geq 0.$
- The extreme points are $[1,0]^T, [0,1]^T$ and $[0, \frac{1}{2}]^T$.
- Distinct extreme directions are $[1,0]^T, [1,1]^T$.
- $[0, \frac{3}{4}]^T \in Fea(LPP)$,
  $[0, \frac{3}{4}]^T = \frac{1}{2}[0, \frac{1}{2}]^T + \frac{1}{2}[0,1]^T$,
  $[\frac{3}{2}, 0]^T = 1[1,0]^T + \frac{1}{2}[1,0]^T$
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- **Observation 7:** Since D', the set of distinct directions of S (if it is nonempty) is a bounded set ($\mathbf{d} \geq \mathbf{0}$ and $\sum_{i=1}^{n} d_i = 1$),
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- If LPP(*) has an optimal solution then $\mathbf{c}^T\mathbf{d} \geq 0$ for all $\mathbf{d} \in D$.

- If $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all extreme directions $\mathbf{d}_j$ of the nonempty and unbounded feasible region S of a LPP, then it implies $\mathbf{c}^T\mathbf{d} \geq 0$ for all directions $\mathbf{d} \in D$, of the feasible region S.

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- **Observation 9:** From the representation theorem the **converse** follows that is if $S \neq \phi$ and $\mathbf{c}^T\mathbf{d}_j \geq 0$ for all $j = 1, 2, .., r$, then LPP(*) has an optimal solution, and atleast one optimal solution is attained at an extreme point of S.

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