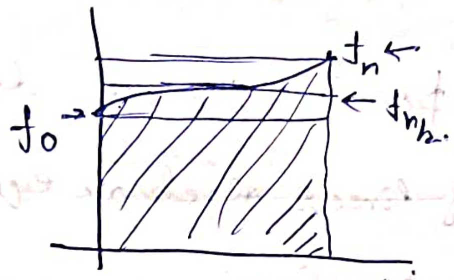
Numerical Integration (Quadrature).

Suppose you want to evaluate $\int_a^b f(x)dx = \sum_{i=0}^{n} w_i f_i$

Idea:- Replace $f(x)$ by polynomial interpolation

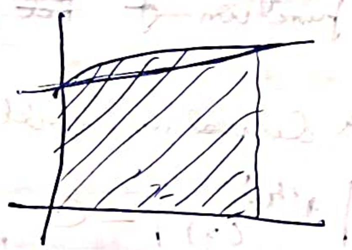
Eg:-

0 degree →



← will give 3 interpolating values

1 degree →



Numerical quadrature : Suppose you want to approximate the integral a to b. $\int_a^b f(x)\,dx$.

[finite interval (a, b)].

Most of such integrals cannot be evaluated explicitly.

∴ One has to seek some numerical approximation which is faster to evaluate rather than calculating the exact integral.

When we have tabulated values, i.e. (x_i, y_i), then we have to use the numerical quadrature to approximate the integral.

We can approximate as

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} w_i f_i \quad , \text{ where } w_i = \underline{\text{weights}}.$$

In other words, we replace the integrand $f(x)$ by the appropriate <u>interpolating polynomial</u>.

<u>20/2/24</u> Suppose $f(x)$ is sufficiently smooth on interval $[c,d]$, which contains $[a,b]$, $\left[\text{target}: \int_a^b f(x)\,dx\right]$,

then we can express $f(x) = p_k(x) + f[x_0, x_1, \ldots x_k, x]\,\psi_k(x)$

~~Then, the~~ $\left[\psi_k(x) = \prod_{j=0}^{k} (x - x_j)\right]$.

<u>Error</u> : $E(f) = \int_a^b f[x_0, x_1, \ldots x_k, x]\,\psi_k(x)\,dx.$

$$\left[I(f) = \int_a^b p_k(x)\,dx \right].$$

In some particular cases, the error can be simplified

If $\psi_k(x)$ is entirely of one sign on $[a,b]$,

then using MVT on the interval, you can express

$$E(f) = f[x_0, x_1, \ldots x_k, \xi] \int_a^b \psi_k(x)\,dx.$$

~~$\xi \in [c,d]$~~.

$\xi \in (a,b)$.

Suppose ~~f is (k+1) times~~ $f \in \mathcal{C}^{k+1}[a,b]$.

$$E(f) = \frac{f^{(k+1)}(\eta)}{(k+1)!} \int_a^b \psi_k(x)\,dx.$$

$\eta \in (c,d)$

Even if $\psi_k(x)$ may not be same sign everywhere, then also, we can do some simplifications.

$\boxed{\text{A particular case}}$ of this kind occurs when the integral of $\int_a^b \psi_k(x)\,dx = 0$. In such a case, we can use the following identity:-

$$f[x_0, \dots x_k, x] = f[x_0, x_1, \dots x_{k+1}]$$
$$+ f[x_0, \dots x_{k+1}, x](x - x_{k+1}).$$

Where x_{k+1} is an arbitrarily chosen pt.

$$E(f) = \int_a^b f[x_0 \dots x_k, x_{k+1}]\,\psi_k(x)\,dx.$$
$$+ \int_a^b f[x_0, x_1 \dots x_{k+1}, x]\,\underbrace{(x - x_{k+1})\psi_k(x)}_{\psi_{k+1}(x)}\,dx$$

Suppose, x_{k+1} is chosen such that $\psi_{k+1}(x) = (x - x_{k+1})\psi_k(x)$.

ψ_{k+1} is of one sign on $[a,b]$.

Suppose $f \in C^{k+2}[a,b]$, the error

$$E(f) = \frac{f^{(k+2)}(\eta)}{(k+2)!} \int_a^b \psi_{k+1}(x)\,dx \; : \; \eta \in (a,b)$$

~~$\eta \in (a,b)$.~~

~~Case 1~~ In case ~~[illegible]~~ $f^{(k+2)} = 0 \Rightarrow E(f) = 0$

$\Rightarrow$ This rule is exact upto $(k+1)$ Degree polynomial

missed

Case 1 :- $k=0$.

$$f(x) = \underbrace{f(x_0)}_{[p_0]} + f[x_0, x](x - x_0).$$

$$I(f) = I(p_0)$$

$$= \int_a^b f(x_0)\,dx = f(x_0)(b-a).$$

Say $x_0 = a \Rightarrow I(f) = f(a)[b-a]$. $\Leftarrow$ [Rectangle formula]

$\Rightarrow$ Then we have obtained the so called Rectangle rule, where we have taken $f_0 =$ left boundary.

$$[\psi_0(x) = x - x_0 = x - a].$$

$$\Rightarrow E(f) = E^R = f'(\eta) \int_a^b (x-a)\,dx$$

$$\Rightarrow \boxed{E^R = f'(\eta) \frac{(b-a)^2}{2}}$$ with $\eta \in [a,b]$,

$\Rightarrow$ Rectangle rule exact upto 0 Degree (constant) polyn.

Taking $x_0 = b$: Same (very similar thing).

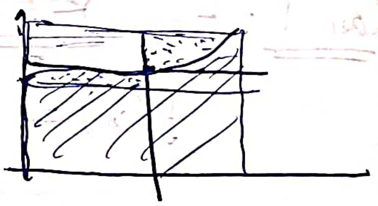
Suppose we take $x_0 = \frac{(a+b)}{2}$.

$\Rightarrow$ $\psi_0(x) = (x - x_0)$ fails to be of one sign on $[a,b]$

$$\int_a^b (x - x_0)\, dx = 0.$$

$$\psi_{k+1}(x) = \psi_1(x) = (x - x_0)(x - x_1) = (x - x_0)^2$$

(taking $x_1 = x_0$).



mid point

$$E^M = \frac{f''(\eta)(b-a)^3}{24}$$

$$I(f) = f\left(\tfrac{a+b}{2}\right)(b-a) := M.$$

$\boxed{\text{Mid point rule}}$

$k=1$

~~$f(x) = f(a)\, f(x_0$~~ $\quad$ Interval $= [a, b]$

$f(x) = p_1(x) = f(x_0) + f[x_0, x](x - x_0) + f[x_0, x_1, x]\,\psi_1(x)$.
$+ E(f)$.

$\psi_1(x) = (x - x_0)(x - x_1)$

If we choose $x_0 = a$, $x_1 = b$. $\Rightarrow$ $\psi_1 =$ one sign on $[a, b]$.

$$\therefore \quad E^{T}(f) = \frac{f''(\eta)}{2!} \int_a^b (x-a)(x-b)\,dx.$$

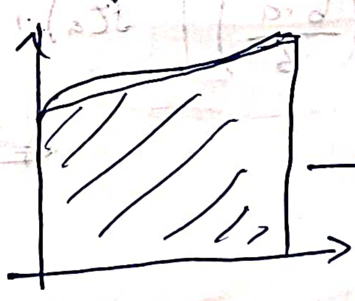
$$= \frac{-f''(\eta)}{12}(b-a)^3.$$

$$T = I(f) = \int_a^b f(a) + f[a,b](x-a)\,dx.$$

$$= \frac{1}{2}\left(f(a) + f(b)\right)(b-a).$$

$T = \frac{1}{2}\left(f(a) + f(b)\right)(b-a)$.

$E^{T}(f) = \frac{-f''(\eta)}{12}(b-a)^3$.



$\longrightarrow$ $\boxed{\text{trapezoidal rule}}$.

[If we approximate $f\left(\frac{a+b}{2}\right) \approx \frac{f(a) + f(b)}{2}$, we get the same formula]

$\underline{k=2}$

$$f(x) = p_2(x) + f[x_0, x_1, x_2, x]\,\psi_2(x).$$

[where $\psi_2(x) = (x-x_0)(x-x_1)(x-x_2)$].

Choose $x_0 = a$, $x_2 = b$, $x_1 = \frac{a+b}{2}$.

$$\Rightarrow \int_a^b \psi_2(x)\,dx = 0.$$

$$\Rightarrow E^{k=2}(f) = \frac{f^{(4)}(\eta)}{4!} \int_a^b \psi_3(x)\,dx.$$

[by taking $x_3 = x_1 = \frac{a+b}{2}$].

$\boxed{\text{Simpsons quadrature}}$

$$I(f) = I(p_2) = \int_a^b p_2(x)\,dx$$

$$= \left(\frac{b-a}{6}\right)\left[f(a) + 4 f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$\cancel{+ f(b)}]$.

Consider $k = 3$ (cubic polynomial).

$$f(x) = p_3(x) + f[x_0, x_1, x_2, x_3, x]\, \psi_3(x).$$

Take $x_0, x_1 = a, \quad x_2 = x_3 = b.$

$$\Rightarrow \psi_3(x) = (x-a)^2 \cdot (x-b)^2.$$

$$E(f) = \frac{1}{4!} f^{(4)}(\eta) \int_a^b (x-a)^2 (x-b)^2 \, dx.$$

$$= \frac{f^{(4)}(\eta)}{720} (b-a)^5.$$

and $I(f) = \int_a^b p_3(x)\, dx.$

$$p_3(x) = f(a) + f[a,a](x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2 (x-b).$$

$$\Rightarrow I(f) = f(a)(b-a) + f[a,a]\frac{(b-a)^2}{2} + f[a,a,b]\frac{(b-a)^3}{3}$$

$$+ f[a,a,b,b] \cdot \int_a^b (x-a)^2 \cdot (x-b)\, dx.$$

$\Downarrow$ Simplifying

$$\boxed{I(f) = \left(\frac{b-a}{2}\right)\left[f(a) + f(b)\right] + \frac{(b-a)^2}{12}\left[f'(a) - f'(b)\right].}$$

$= \boxed{\text{Corrected trapezoidal}}$ (CT).

In the end we are basically getting :-

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} w_i\, f(x_i).$$

Now, we consider the Lagrange Interpolating polynomial

$$p_n(x), \quad \underline{n \geq 1}, \quad h = \left(\frac{b-a}{n}\right).$$

$\underline{\underline{\text{at points}}}$: $\underline{\underline{x_j}} = a + jh \;;\; j = 0, 1, \ldots n.$

$$I(f) = I_n(f) = \int_a^b p_n(x)\,dx$$

$$= \int_a^b \sum_{j=0}^{n} l_{j,n}(x)\, f(x_j)\,dx$$

$$= \sum_{j=0}^{n} w_{j,n}\, f(x_j).$$

$$\left[w_{j,n} = \int_a^b l_{j,n}(x)\,dx \right].$$

$\underline{\underline{n=1}}$: $w_{0,1} = w_{1,1} = \left(\frac{b-a}{2}\right).$

$\underline{n=3}$; $w_0 = \int_a^b l_0(x)\,dx.$

$$= \int_a^b \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx.$$

$$= \int_0^3 \frac{(u-1)(u-2)(u-3)}{-6h^3}\, h^4\,du. = \frac{3h}{8}.$$

$$w_1 = \int_a^b l_1(x)\,dx.$$

u

$$= \int_a^b \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,dx.$$

$$= \int_0^3 \frac{(u)(u-2)(u-3)}{2h^3}\,h^4\,dx.$$

$$= \frac{9h}{8}.$$

$$\boxed{\Rightarrow I_n(f) = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right].}$$

Error

(i) for even 'n': $f(x)$ is $(n+2) \Rightarrow f \in C^{n+2}([a,b])$.

$$E(f) = I(f) - I_n(f) = C_n h^{n+3} f^{(n+2)}(\eta).$$

$$\text{where } C_n = \frac{1}{(n+2)!}\int_0^n u^2(u-1)\cdots(u-n)\,du.$$

(ii) for odd 'n': $f \in C^{(n+1)}([a,b])$.

$$E_n(f) = C_n\,h^{n+2} f^{(n+1)}(\eta)$$

$$\boxed{C_n = \frac{1}{(n+1)!}\int_0^n u(u-1)(u-2)\cdots(u-n)\,du.}$$

[Newton - Cotes formula]

Composite ~~Quad~~ Quadrature

Always, it is advisable to divide the interval $[a,b]$ into n-intervals of equal size

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b.$$

and apply the quadrature formulas in each interval seperately.

$$I(f) = \int_a^b f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx.$$

$\Rightarrow$ the idea is that the error will be less than that with the same method on the entire interval.

where $\int_{x_{i-1}}^{x_i} f(x)\,dx$ is approximated as $\int_{x_{i-1}}^{x_i} P_{i,*}(x)\,dx$.

If we take rectangle rule:-

$$h = \frac{b-a}{N}, \quad I(f) \approx \left(\frac{b-a}{N}\right) \sum_{i=0}^{N-1} f(x_i).$$

$$E(R) = \sum_{i=0}^{N-1} \frac{f'(\eta_i)}{2!} h^2$$

$\boxed{\text{then, error = less than the original quadratures.}}$

If you use 'k' grid points in each Interval, exact upto k or k+1 degree (check)?

Using Hermite interpolation, using 'K' grid points, exact upto $2K+1$ degree $\longleftarrow$ Gaussian Quadrature.

Consider numerical quadrature of $\int_a^b f(x)\,dx = \int_a^b \omega(x)\, g(x)\, dx$.

$$I(g) \approx \underbrace{A_0\, g(x_0) + A_1\, g(x_1) \cdots A_K\, g(x_K)}$$

Note :- $\boxed{A_i \text{ are independent of function}}$

So far, we have taken $x_0, x_1, x_2 \cdots x_K$ are equally spaced.

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$$\text{Error} = g^{(k+1)}(\eta).$$

~~do~~ exact upto any polynomial of deg $\leq k$.

$$E(g) = I(g) - I(p_k).$$

$$= \int_a^b g[x_0, x_1, \dots x_k, x]\, \underbrace{w(x)\, \psi_k(x)}_{\text{integral} = 0}\, dx.$$

suppose $\Longrightarrow$ i.e. $\int_a^b \psi_k(x)\, w(x)\, dx = 0$

$\Rightarrow$ we know that

$$E(g) = \int_a^b g(x_0, x_1, \dots x_{k+1}, x]\, \psi_{k+1}(x)\, w(x)\, dx$$

now suppose $\int_a^b \psi_{k+1}(x)\, w(x)\, dx$, then, take another point x_{k+2}, then x_{k+3} . . . . .

$$E(g) = \int_a^b g(x_0, x_1 \dots x_{k+2}, x)\, \psi_{k+2}(x)\, w(x)\, dx$$

$$\underline{x_k, x_{k+1} \dots x_{k+1+m}}$$

$$\Downarrow$$

$$E(g) = I(g) - I(p_k)$$

$$= \int_a^b g[x_0, x_1 \dots x_{k+m+1}, x]\, \psi_{k+m+1}\, w(x)\, dx.$$

for several choices of $w(x)$, we can find $p_{k+1}(x)$ s.t.

$$\int_a^b p_{k+1}(x)\, q(x)\, w(x)\, dx = 0 \quad \text{where } \deg(q(x)) \leq k,$$

which tells that the polynomials are orthogonal wrt the weight $w(x)$.

We can express, the polynomial (orthogonal polynomial) $\phi_{k+1}(x)$ as

$$\phi_{k+1}(x) = \alpha_{k+1}(x-\xi_0)\dots(x-\xi_k),$$

where ξ_i's are $k+1$ distinct

points in the interval $[a,b]$ where $\phi_{k+1}(x)$ vanishes.

Hence, if we set $x_j = \xi_j$ for $j=0$ to k and let x_{k+j}

are arbitrary points in the interval $[a,b]$, where $j=1$ to $k+1$,

ie $m=k$.

$$\Rightarrow q(x) = \frac{(x-x_{k+1})\dots(x-x_{k+1+j})}{\alpha_{k+1}}$$

$\therefore$ The error $e_g = g[x_0, x_1 \ldots\ldots x_{k+m+1}, x] \psi_{2k+1}(x) \omega(x) dx$

In order to obtain the desired error, we can choose

$$x_{k+j} = \xi_{j-1}, \quad j = 1, 2 \ldots k+1$$

$$\psi_{2k+1}(x) = (x-x_0)(x-x_1) \ldots (x-x_{2k+1})$$

$$= (x-\xi_0)(x-\xi_1) \ldots (x-\xi_k)$$

$$= \left(\frac{P_{k+1}(x)}{\alpha_{k+1}} \right)^2$$

$\therefore$ By applying the MVT,

$$E(g) = g[x_0, x_1 \ldots x_{2k+1}, \eta] \int_a^b \left(\frac{P_{k+1}(x)}{\alpha_{k+1}} \right)^2 \omega(x) dx$$

$$= \frac{1}{(2k+2)!} g^{(2k+2)}(\eta) \frac{S_{k+1}}{\alpha_{k+1}^2} ; \quad S_{k+1} = \int_a^b P_{k+1}^2(x) \omega(x) dx.$$

To summarise the gaussian quadrature, we have to choose the nodal points are zeroes of the polynomial $P_{k+1}(x)$, which is orthogonal with the weight function $\omega(x)$ on $I = [a, b]$, which makes a gaussian quadrature is exact upto $(2k+1)$.

Assume this polynomial P_{k+1} as Legendre's polynomial

$$P_1(x) = x \quad , \quad \xi_0 = 0$$

$$P_2(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right), \quad \xi_0, \xi_1 = \pm \frac{1}{\sqrt{3}}$$

$$\cancel{P_3(x) = \frac{\pi}{2}(x^3), \ \xi_0,}$$

$$P_3(x) = \frac{5}{2}\left(x^3 - \frac{3}{5}x\right).$$

$$\xi_0 = -\sqrt{3/5}, \quad \xi_1 = 0, \quad \xi_2 = \sqrt{\frac{3}{5}}.$$

$$P_{n+1}(x) = x P_n(x) - \frac{n^2}{4n^2-1} P_{n-1}(x).$$

take $\int_a^b f(x)\,dx = \int_{-1}^{1} f(x(t))\,x'(t)\,dt.$

take $x(t) = \left(\frac{b-a}{2}\right)t + \frac{a+b}{2}.$

take $k=1$, then quadrature is exact

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k = 1

$x_0 = \xi_0 = -1/\sqrt{3}$, $x_1 = \xi_1 = 1/\sqrt{3}$.

$$\int_{-1}^{1} g(x)\,dx = A_0\, g\left(\frac{-1}{\sqrt{3}}\right) + A_1\, g\left(\frac{1}{\sqrt{3}}\right).$$

$$E = c\, g^{(4)}(\eta).$$

$$A_1 = \int_{-1}^{1} \frac{x-\left(-\frac{1}{\sqrt{3}}\right)}{\frac{1}{\sqrt{3}}-\left(-\frac{1}{\sqrt{3}}\right)}\,dx = 1 = A_0$$

$$\int_{-1}^{1} g(x)\,dx = g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

$$E = \frac{1}{135}\, g^{(iv)}(\eta).$$

K = n $\rightarrow$ $x_0 \quad x_1 \quad \cdots \quad x_n$.

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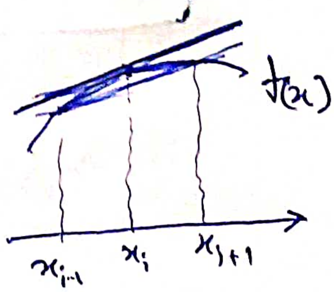
forward diff.

$$f'(x_i) = \frac{f_{i+1} - f_i}{h} \rightarrow O(h).$$

backward diff $\Leftarrow$
$$= \frac{f_i - f_{i-1}}{h} \Rightarrow O(h)$$

$$= \frac{f_{i+1} - f_{i-1}}{2h} \Rightarrow O(h^2).$$



$$f_{i+1} = f(x_{i+1}) = f(x_i) + h f'(x_i) + \frac{h^2}{2!} f''(\xi_i).$$

$$\frac{f(x_{i+1}) - f(x_i)}{h} - f'(x_i) = \frac{h}{2!} f''(\xi_i).$$

$$f_{i-1} = f(x_i) - h f'(x_i) + \frac{h^2}{2!} f''(x_i) - \frac{h^3 f_i'''}{3!}$$

$$+ \frac{h^4}{4!} f^{(4)}(\eta_i).$$

$$f_{i+1} = f(x_i) + h f'(x_i) + \frac{h^2}{2!} f''(x_i) + \frac{h^3 f'''(x_i)}{3!}$$

$$+ \frac{h^4}{4!} f^{(4)}(\xi_i).$$

~~$f(i+1)$~~

$$\frac{f_{i+1} - f_{i-1}}{2h} - f_i' = \frac{h^2}{3!} f'''(\eta_i) \Leftarrow O(h^2)$$

$$\boxed{\frac{f_{i+1} - f_{i-1}}{2h} = \text{average of forward and backward Difference.}}$$

Numerical differentiation

Suppose we are given the tabulated values, we want to determine the derivative of the data.

Eg:- Velocity from displacement,
or acceleration = 2^{nd} derivative of displacement.

~~Let $f(x) \in C'$~~

Let $f(x)$ be continuously differentiable function on interval $[c,d]$ that contains interval $[a,b]$.

divide $[c,d]$ into equally distributed k nodal points $x_0, x_1, x_2, \dots x_k$.

$\Rightarrow$ we can approximate $f(x)$ by a polynomial of degree k.

$$① - \quad f(x) = p_k(x) + f[x_0, x_1, \dots x_k, x] \psi_k(x).$$

furthermore, $\frac{d}{dx} f[x_0, x_1, \dots x_k, x]$

$$= f[x_0, x_1, \dots x_k, x, x].$$

$\Rightarrow$ differentiate ①: $f'(x) = p_k'(x) + \Big[f[x_0, x_1, \dots x_k, x, x] \psi_k(x)$
$+ f[x_0, x_1, \dots x_k, x] \psi_k'(x) \Big]$.

Let D denote the derivative of f at the point a [differential operator].

where $a \in [c, d]$ is some point.

$D(f) = p_k'(a)$, $E(f) = D(f) - D(p_k) =$

$f[x_0, x_1, \ldots x_k, a, a] \psi_k(a) + f[x_0, x_1, \ldots x_k, a] \psi_k'(a)$.

Since, the error is contained in two terms, by choosing $a = x_i \; \exists i$;

$\Rightarrow \psi_k(a) = 0$. $\Rightarrow$ first term will be dropped out.

Whereas, if you choose a such that $\psi_k'(a) = 0$, $\Rightarrow$ 2nd term will be dropped.

~~Case 1: $\psi_k(a) = 0 \to a =$~~

Suppose $\psi_k'(a) = q(a)$, where $q(x) = \dfrac{\psi_k(x)}{x - x_i} = (x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_k)$.

assume $a = x_i$, $\exists i$. $f \in C^{k+1}([c, d])$.

$$\Rightarrow E(f) = \frac{1}{(k+1)!} f^{(k+1)}(\eta) \prod_{\substack{j=0 \\ j \neq i}}^{k} (x - x_j).$$

Case 2. Suppose $\psi_k'(a) = q(a) = 0$.

$\Rightarrow$ E 2nd will vanish.

Say k = odd number $\Rightarrow$ then we can achieve this by placing x_j's symmetrically about the point a.

$$x_{k-j} - a = a - x_j, \quad j = 0, 1, \ldots, (k-1)/2.$$

Now,

$$(x-x_j)(x-x_{k-j}) = (x-a+a-x_j)(x-a+a-x_{k-j})$$

$$= (x-a)^2 - (a-x_j)^2.$$

from this, $\psi_k(x) = \prod_{j=0}^{k} (x-x_j) = \prod_{j=0}^{(k-1)/2} [(x-a)^2 - (a-x_j)^2].$

derivative of $\psi_k(x)$ at a $= \psi'(a)$

$= \underline{\underline{0}}$

$\therefore$ Error, $E(f) = \frac{1}{(k+2)!} f^{(k+2)}(\eta) \prod_{j=0}^{(k-1)/2} [-(a-x_j)^2].$

If $k=0$, then, this is constant interpolant $\Rightarrow D(p_k) = 0.$

If $k=1$, $p_k(x) = \frac{f_1 - f_0}{x_1 - x_0}(x - x_0) + f_0.$

$= f[x_0, x_1]\, \psi_1(x) + f_0.$

$D(f) = D(p_k) = f[x_0, x_1].$

Since $k =$ odd $\Rightarrow$ at $a = \frac{x_0 + x_1}{2}$

$a = \frac{x_0 + x_1}{2}$. $x_0 = a - h$, $x_1 = a + h$. , $h = (x_1 - x_1)/2$

$$\Rightarrow f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

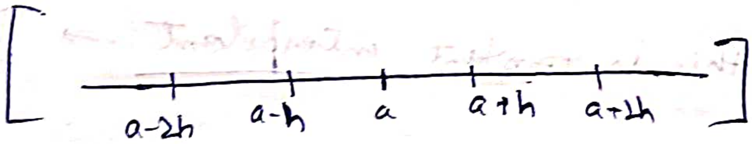
$$E(d) = \frac{h^2}{6} f'''(\eta) = O(h^2)$$

$\boxed{\text{Assumption: } f \in \mathcal{C}^3(I)}$

$$f''(a) = \alpha f(a+h) + \beta f(a-h) + \gamma f(a).$$

$$= \alpha f(a) + \beta f(a+h) + \gamma f(a+2h)$$

$$= \alpha f(a-2h) + \beta f(a-h) + \gamma f(a).$$



Solving diff. eqns.: (study).

- $y'' + 4y' - 5y = e^x$ → constant coefficients
- $e\, y'' + (x+1) y' - 5x^2 y = e^x$.
  ↳ variable coefficients

⇒ Always prescribe $x \in (a, b)$.

and limits: $y(a), y'(a)$ / $y(a), y'(b)$ / $y'(a), y'(b)$.

Generalæ eqn:

$$P(x)\, y'' + Q(x)\, y' + R(x)\, y = 0.$$ is linear.

Why Linear:- $Ly = P(x)y'' + Q(x)y' + R(x)y = 0.$

Operator $L = P(x)\dfrac{d^2}{dx^2} + Q(x)\dfrac{d}{dx} + R(x)\, I$

Another case: $u(x,t)$.: $u_t + a\, u_x = 0$.

Another $\Rightarrow$ $Pp + Qq = R \Rightarrow \dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

Order 1

Order 2

$u_t = u_{xx}$, $(x,t) \in (0,1)\times(0,T)$.

$\Rightarrow u(x,t) = X(x)\, T(t)$.

$u_{tt} = c^2 u_{xx}$. ← unsteady.

$(u_{tt} + u_{yy}) = 0$ ← steady.

[Air flow is an example of place where this is usefull.
Eg:- Concorde was stopped because of ↑ accidents, because nose shape].

⇒ Here, idea is to find solutions when exact analytic solution are provably impossible to find.

$$\text{IVP}\begin{cases} y'(t) = f(t,y), \quad \underbrace{t \in (a,b)}_{\hookrightarrow \text{domain of definition}} \\ \underbrace{y(a) = \alpha}_{\hookrightarrow \text{initial value}} \end{cases}$$

[$t \to$ independent variable
$y \to$ dependent variable]

$\Rightarrow$ Knowing when this problem will have a solution.

$\Rightarrow$ Called well-posedness :

A problem is well-posed if it admits solution st:

1. Existence
2. Uniqueness
3. Stability / Continuous dependence.

Condition for uniqueness:

Strong condition: f cont, f' cont.

Weak condition: $\underbrace{|f(x_1) - f(x_2)| \leq L\,|x_1 - x_2|}_{** \text{ Lipschitz condition } **}$.

$L \downarrow$ Lipschitz constant

(better because less constrained).

---

eg:- $y' = t + y$, $t \in (0,1)$.

$y(0) = \frac{1}{2}$.

$y' - y = t$, $\Rightarrow y' + P(x)\,y = Q(x)$

$$y' - e^{-t} y = e^{-t} t.$$

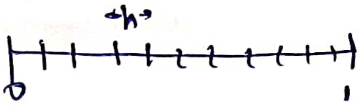
$$\Rightarrow (e^{-t} y) - y(0) = -e^{-x}(x+1)\Big|_0^t$$

$$\Rightarrow e^{-t} y - \frac{1}{2} = 1 - e^{-t}(t+1)$$

$$\Rightarrow \boxed{y(t) = \frac{3}{2} e^{t} - (t+1).}$$

$\boxed{\text{Euler's Method}}$.

Step 1: Discretize



Say problem:- $\begin{cases} y'(t) = f(t, y) \\ y(0) = \alpha \end{cases}$ $t \in [a, b)$.

$\Rightarrow$ 'N' points $\Rightarrow$ $h = \dfrac{(b-a)}{N}$.

$t_0 = a,\ t_1 = a+h,\ t_2 = a+2h \ldots\ t_n = a+nh.$

Step 2: Replacing derivative by finite difference.

$$(1)\ y' = \frac{y_{n+1} - y_n}{h} \to O(h)$$

$$y' = \frac{y_n - y_{n-1}}{h} \to O(h)$$

$$y' = \frac{y_{n+1} - y_{n-1}}{2h} \to O(h^2).$$

$\leftarrow$ any one can be taken.

Say we take forward difference. $\dfrac{y_{n+1} - y_n}{h} = f(t_n, y_n)$.

$\boxed{\text{difference Equation}}$ $\leftarrow$ $\Rightarrow$ $y_{n+1} = y_n + h f(t_n, y_n)$. $n = 0, 1, \ldots, N-1$.

$y_0 = \alpha$.

$\rightarrow$ $\boxed{\text{this is a recurrence relation}}$

Euler's Method Code :

Input : a, b, N, α

h = (b-a)/N.

t = a : h : b;

```
y(1) = α;
for i = 1:N
    y(i+1) = y(i) + h * f(t(i), y(i));
    exact(i) = y_exact(t, y);
end.
⇒ Error(i) = abs(y(i) - y_exact(i));
```

⇒ for each N, we will have table

ti	Approx	Exact	Error

$$\text{order} = \ln 2\left(\frac{E^N}{E^{2N}}\right) \qquad E^N = \text{error}_N = \boxed{\text{max error with } N.}$$

($\log_2$)

N	E^N	order.
20	↓	↓
40		
80		

3 plots:
- t vs y & y-exact (2 in 1).
- t vs errors.
- N vs order (Log Log plot).
- h vs E^N.

(or 4, if you want)

$\boxed{\begin{array}{c}\text{Implicit Euler} \\ \text{(or Backward Euler)}\end{array}}$

Backward: $\frac{y_n - y_{n-1}}{h} = f(t_n, y_n)$.

$$y_n = y_{n-1} + h \cdot f(t_n, y_n).$$

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}). \quad \text{(use)}$$

Max Iter $= 10^4$, $\delta = 10^{-8}$.

18/3/24 ~~Taylor~~

$$y(t_{i+1}) = y(t_i + h)$$
$$= y(t_i) + h\,y'(t_i) + \frac{h^2}{2!}\,y''(\xi_i)$$
$$= y(t_i) + h\,f(t_i, y(t_i)) + \frac{h^2}{2!}\,y''(\xi_i).$$

$$\xi_i \in [t_i, t_{i+1}).$$

$$y_{i+} = y_i + h\,f(t_i, y_i).$$

Truncation Error $= \dfrac{(\text{LHS} - \text{RHS})}{h} = O(h).$

① Consistency (TE)

If TE $\to 0$ as $h \to 0 \Rightarrow$ Consistent.

② Stability (Round-off).

If Round off error $= \boxed{\text{bounded}}$

$\Rightarrow$ Stable.

③ Convergence

$$\text{Error} = |\underbrace{y(t_i)}_{\text{exact}} - \underbrace{y_i}_{\text{approx soln.}}| \le C\,h^{p}$$

$p \to$ order.

If error $\to 0$ as $h \to 0$

Explicit Euler   we want Consistency + Stability ⟹ Convergence

$$w_{i+1} = w_i + h f(t_i, w_i)$$
$$w_0 = \alpha.$$

Lemma 1: $\forall\ x \geq -1,\ m > 0$, we have

$$0 \leq (1+x)^m \leq e^{mx}$$

$\because\ x \geq -1$, the taylor's expansion of exp(x) will be

$$e^x = 1 + x + \frac{x^2}{2!} e^{\xi}.$$
$$\exists\ \xi \in (0, x).$$

$$0 \leq (1+x) \leq e^x \Rightarrow \boxed{(1+x)^m \leq e^{mx}}$$

Lemma 2: Let s, t be two positive real numbers and $\{a_i\}$ be a sequence satisfying, $a_0 \geq -t/s$.
& $(i+1)^{th}$ entry $a_{i+1} \leq (1+s) a_i + t,\ i = 0, \dots k.$

then $a_{i+1} \leq \exp((i+1)s) \left[a_0 + t/s\right] - t/s.$

$$a_{i+1} \leq (1+s)(1+s) a_{i-1} + t, \quad i = 0, 1 \dots k.$$

~~$a_{i+1} \leq (1+s)^{i+1} a_0 + t \quad i = 0, 1 \dots k$~~

$$a_{i+1} \leq (1+s)^{i+1} a_0 + \left[(1+s)^{i} + \dots (1+s)^0\right] t.$$

$$\leq (1+s)^{i+1} a_0 + t \frac{\left[(1+s)^{i+1} - 1\right]}{s}$$

$$\leq e^{s(i+1)} \left[a_0 + \frac{t}{s}\right] - \frac{t}{s}.$$

[Replacing $(1+s)^{i+1}$ with $e^{s(i+1)}$ because $a_0 + t/s \geq 0$]

Thm: Convergence of explicit euler scheme.

Suppose f is continuous. $[y' = f(t,y)]$.

and satisfies the Lipschitz condition with constant L.

On the domain $D = \{(t,y) : a \le t \le b,\ -\infty < y < \infty\}$.

and $\exists\ M > 0$

st $|y''(t)| \le M$ $t \in [a,b]$.

Proof:-

Let $y(t)$ be the unique sol^n of IVP $\begin{cases} y'(t) = f(t,y) \\ y(a) = \alpha . \end{cases}$

$w_0, w_1, \dots w_N$

$w_{n+1} = w_n + h f(t_n, w_n)$.

$n = 0, 1, \dots N-1$

Then Error $= |y(t_i) - w_i|$,

$$\le \frac{hM}{2L}\left[\exp(L(t_i - a)) - 1\right]$$

$$\le ch.$$

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2!} y''(\xi_i).$$

$$w_{i+1} = w_i + h f(t_i, w_i).$$

$$y(t_{i+1}) - w_{i+1} = (y(t_i) - w_i) + h[f(t_i, y_i) - f(t_i, w_i)] + \frac{h^2}{2!} y''(\xi_i).$$

$$\Rightarrow |y(t_{i+1}) - w_{i+1}| \le |y(t_i) - w_i| + h\,|f(t_i, y_i) - f(t_i, w_i)| + \frac{h^2}{2!}|y''(\xi_i)|.$$

$$|y(t_{i+1}) - w_{i+1}| \le |y(t_i) - w_i|\,[1 + hL] + \frac{h^2}{2}M,$$

$$\text{Error} \le \left[\exp((i+1)hL) - 1\right]\left[\frac{hM}{2L}\right].$$

$$\Rightarrow \boxed{\text{Error} \le \frac{hM}{2L}\left[\exp(L(t_i - a)) - 1\right].}$$

In the previous thm, we assumed, all the calculations are free from round off errors. However, it is not true in practice.

$\therefore$ If we consider the round off error, then the scheme becomes $w_{i+1} = w_i + h\,f(t_i, w_i) + \underbrace{\delta_{i+1}}$. $i = 0, \dots N-1,$

Proof:- ~~you can~~

Assume $|\delta_i| < \delta$ (unit round off).

$$|y(t_i) - w_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)\left[\exp(L(t_i - a)) - 1\right] + |\delta_0|\exp(L(t_i - a))$$
$$\forall\ i = 0, 1, \dots N.$$

Proof (HW).

$$\lim_{h \to 0} \left(\frac{hM}{2} + \frac{\delta}{h} \right) = \infty.$$

$$E(h) = \frac{hM}{2} + \frac{\delta}{h}$$

$$\frac{M}{2} - \frac{\delta}{h^2} = 0 \Rightarrow h = \sqrt{\frac{2\delta}{M}}.$$

1. If $h < \sqrt{\frac{2\delta}{M}}$, $\Rightarrow E' = -\text{ve}$.

2. If $h > \sqrt{\frac{2\delta}{M}} \Rightarrow E' = +\text{ve}$.

19/3/24

$$y_{n+1} = Y_n + h\left[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\right]$$

$\theta = 0 \Rightarrow$ Implicit Euler. $\Big\}$ $o(h)$
$\theta = 1 \Rightarrow$ Explicit Euler.
$\theta = \frac{1}{2} \Rightarrow$ $\rightarrow \underrightarrow{o(h^2)}$.
TE.

$$TE = \frac{1}{h}(LHS - RHS).$$

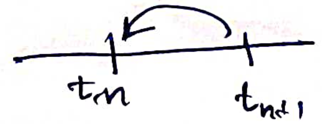
Runge-Kutta Method

- underlying idea of R-K method is to take weighted average of slope of tangent in order to increase/enhance the order of truncation error.

$$\begin{cases} y(t_{n+1}) = Y(t_n) + h\,\underbrace{\Phi(t_n, y(t_n), h)}_{\text{increment function: (check)}} \\ y_1 = \alpha. \end{cases}$$

Defn: Single-Step method.

at time t_{n+1}, only time t_n values are needed.



Using central difference → 2-Step / Multi-Step.

R-stage R-K

$$y_{n+1} = y_n + \Phi(t_n, y_n, h)$$

$$\Phi(t, y, h) = \sum_{r=1}^{R} w_r k_r.$$

$$k_1 = hf(t_n, y_n)$$

$$k_r = hf\left(t_n + hc_r, y_n + \sum_{s=1}^{r-1} a_{rs} k_s\right).$$

$$r = 2, \ldots, R.$$

$$c_r = \sum_{s=2}^{r-1} a_{rs}, \quad r = 2, \ldots, R$$

Take R=2:

$$y_{n+1} = y_n + w_1 k_1 + w_2 k_2. \quad (1)$$

$$(2) \begin{cases} k_1 = hf(t_n, y_n) \\ k_2 = hf(t_n + c_2 h, y_n + a_{21} K_1). \end{cases}$$

We have 4 unknowns $\rightarrow$ w_1, w_2, c_2, a_{21}.

In order to determine the values for the parameters, we use the taylor series expansion around t_n

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2} y''(t_n) \ldots\ldots$$

$$y'(t_n) = f(t_n, y_n).$$

$$y'' = f_t + f_y y' = f_t + f_y f.$$

$$\cancel{y''' = f_{tt} + f_t f_y + (f_y)^2 f + f_{yy} f}$$

$$y''' = f_{tt} + f_{yt} f + f_y f_t + f_{yy} f + (f_y)^2$$

$$\cancel{y(t_{n+1}) = y(t_n + h) =}$$

$$k_2 = h f(t_n + c_2 h,\ y_n + a_{21} k_1).$$

$$= h\Big[f(t_n, y_n) + c_2 h f_t(t_n, y_n)$$

$$+ a_{21} k_1 f_y(t_n, y_n)$$

$$+ \frac{(c_2 h)^2}{2} f_{tt}(t_n, y_n)$$

$$+ (c_2 h)(a_{21} k_1) f_{ty}(t_n, y_n)$$

$$+ \frac{(a_{21} k_1)^2}{2} f_{yy}(t_n, y_n)\Big].$$

$$\Rightarrow \left.\begin{aligned} w_1 + w_2 &= 1 \\ c_2 w_2 &= \frac{1}{2} \\ a_{21} w_2 &= \frac{1}{2} \end{aligned}\right] \text{ 3 equations for 4 unknowns}$$

$\because$ we have only 3 eqns $\Rightarrow$ we can chose c_2 arbitrarily.

$w_1 + w_2 = 1 \Rightarrow w_1 = 1 - w_2 = 1 - \frac{1}{2c_2}$.

$\rightarrow$ for different value of c_2, we get different families of soln

$$a_{21} = c_2$$

$$w_2 = \frac{1}{2c_2}$$

$$w_1 = 1 - \frac{1}{2c_2}$$

$$y_{n+1} = y_n + h f_n + \frac{h^2}{2}\left(f_t + f_n f_y\right)$$

$$+ \frac{c_2 h^3}{4}\left(f_{tt} + 2 f_n f_{ty} + f_n^2 f_{yy}\right).$$

<u>John Butchers' table</u>

$$\begin{array}{c|cc} c_2 & a_{21} & \\ \hline & w_1 & w_2 \end{array}$$

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

→ Improved tangent

$$\begin{array}{c|cc} 2/3 & 2/3 & \\ \hline & 1/4 & 3/4 \end{array}$$

→ Optimal method.

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

← Euler Cauchy.

3-stage R-K

$$\begin{array}{c|ccc} c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \hline & w_1 & w_2 & w_3 \end{array}$$

Nystrom Method.

$$\begin{array}{c|ccc} 2/3 & 2/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 2/8 & 3/8 & 3/8 \end{array}$$

$$\begin{array}{c|ccc} 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 4/6 & 1/6 \end{array}$$

Classical Method

$$\begin{array}{c|ccc} 1/2 & 1/2 & & \\ 3/4 & 0 & 3/4 & \\ \hline & 2/9 & 3/9 & 4/9 \end{array}$$

Nearly Optimal.

4-Stage R-K

$$\begin{array}{c|cccc} 1/2 & 1/2 & & & \\ 1/2 & 0 & 1/2 & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

Classical method.

$$\begin{array}{c|cccc} 1/3 & 1/3 & & & \\ 2/3 & -1/3 & 1 & & \\ 1 & 1 & -1 & 1 & \\ \hline & 1/8 & \frac{3}{8} & \frac{3}{8} & 1/8 \end{array}$$

Kutta Method.

$TE = \frac{1}{h}(LHS - RHS)$.

Stability

$$\begin{cases} y' = \lambda y, & \lambda < 0 \in \mathbb{R}. \\ y(0) = y_0 \end{cases}$$

Exact Soln: $y(t) = y_0 e^{\lambda t} \underbrace{\longrightarrow 0 \text{ as } t \to \infty}$.

this condition should ~~hold~~ hold for all of the R-K methods.

$$y_{n+1} = y_n + h f(t_n, y_n).$$

$$= y_n + h\lambda y_n.$$

$$\Rightarrow y_{n+1} = (1+\lambda h)^{n+1} y_0.$$

$$\boxed{y_n = (1+\lambda h)^n y_0.}$$ ~~will be~~

Say interval is $[0,1]$.

y_n converges if $(1+\lambda h)$ is st $|1+\lambda h| < 1$

$$\Rightarrow -1 < 1+\lambda h < 1$$

$$\Rightarrow h \in \left(0, \frac{-2}{\lambda}\right).$$

or

$$h \in \left[0, \frac{2}{|\lambda|}\right].$$

~~$\Rightarrow$ h must be reduced as n increases.~~

CONDITION for Stability :

$\Rightarrow$ the converge of any sol^n will happen only when $|1+\lambda h|<1$ or $h \in \left(0, \frac{2}{|\lambda|}\right)$.

(this can be checked, NEEDS TO BE CHECKED IN NEXT LAB)

Check in Lab That

i.e. as $n \to \infty$, $y_n \to 0$ only when $h \in \left(0, \frac{2}{|\lambda|}\right)$.

Case 2 :

$$y_{n+1} = y_n + h f(t_{n+1}, y_{n+1}).$$

$$= y_n + \lambda h y_{n+1}$$

$$\Rightarrow (1-\lambda h)\, y_{n+1} = y_n.$$

$$\Rightarrow \quad y_n = \frac{1}{(1-\lambda h)^n}\, y_0.$$

$1-\lambda h > 1 \quad \forall h>0$

$$\Rightarrow \quad y_n \to 0 \text{ as } n \to \infty.$$

$\therefore$ This method is ~~not~~ unconditionally stable, (no condition on h).

Case 3:

$$y_{n+1} = y_n + w_1 K_1 + w_2 K_2.$$

$$y_{n+1} = \left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right) y_n.$$

$$\Rightarrow \boxed{y_n = \left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right)^n y_0}$$

Condition for Stability: $y_n \to 0$ as $n \to \infty$.

$$\left|1 + \lambda h + \frac{(\lambda h)^2}{2}\right| < 1$$

$$\Rightarrow -2 < \lambda h + \frac{(\lambda h)^2}{2} < 0$$

$$-2 < \lambda h < 0$$

~~and~~

$$\Rightarrow \boxed{h \in \left(0, \frac{2}{|\lambda|}\right)}$$ same condition as in explicit euler.

This interval is called region of absolute Stability.

Order 3: $\left(0, \dfrac{2.51}{|\lambda|}\right)$,

Order 4: $\left(0, \dfrac{2.78}{|\lambda|}\right)$

$\leftarrow$ region of absolute Stability.

Original problem

$$y'(t) = f(t, y)$$

$$\Rightarrow \int_{t_n}^{t_{n+1}} y'(t)\,dt = \int_{t_n}^{t_{n+1}} f(t, y)\,dt$$

$$y_{n+1} - y_n =$$ ← we use only t_n to answer.

Multi-Step Scheme ⇐ why not use

to intepolate a polynomial for $f(t, y)$.

Using $t_n, t_{n-1}, \dots, t_{n-m}$

Multi Step Scheme / Method

In the R-K Method, wee had to evaluate the slopes k_1, k_2, k_3, which are computationally expensive ~~in order to over~~

In order to overcome the computational cost, one can use the multistep method, which make use of interpolating integerant at more than one previous points

$$\int_{t_n}^{t_{n+1}} y'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds.$$

$$\rightarrow y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds. \quad (1)$$

In order to evaluate the integral, we evaluate a polynomial which interpolates $f(t, y)$ at $t_n, t_{n-1}, \dots, t_{n-m}$ ($m+1$ nodal points), by using Newton's backward difference formula (to make a polynomial of degree m)

$$\cancel{p_m(t) = \sum_{k=0}^{m} \binom{-\gamma+k}{k} \nabla^k f_n}$$

$$\cancel{\gamma = x_n - x}$$

We will use forward difference notation for backward difference,

$$\boxed{p_m(t) = \sum_{k=0}^{m} (-1)^k \binom{-s}{k} \Delta^k f_{n-k} \quad \text{where } s = (t - t_n)/h} \quad \text{— (2)}$$

$$y_{n+1} = y_n + h \int_0^1 \sum_{k=0}^{m} (-1)^k \binom{-s}{k} \Delta^k f_{n-k} \, ds.$$

$$= y_n + h \sum_{k=0}^{m} (-1)^k \int_0^1 \binom{-s}{k} \Delta^k f_{n-k} \, ds.$$

$$y_{n+1} = y_n + h \left[\gamma_0 f_n + \gamma_1 \Delta f_{n-1} \cdots \gamma_m \Delta f_{n-m} \right] \quad \text{(3)}$$

$$\gamma_k = (-1)^k \int_0^1 \binom{-s}{k} ds$$

$$\gamma_0 = 1, \quad \gamma_1 = \frac{1}{2}, \quad \gamma_2 = \frac{5}{12}, \quad \gamma_3 = \frac{3}{8}, \quad \gamma_4 = \frac{251}{720}$$

(1) is known as Adams - Bashforth formula.

$\underline{\underline{n=3}}$

t_{n-3}	y_{n-3}	f_{n-3}			
			Δf_{n-3}		
t_{n-2}	y_{n-2}	f_{n-2}		$\Delta^2 f_{n-3}$	
			Δf_{n-2}		$\Delta^3 f_{n-3}$
t_{n-1}	y_{n-1}	f_{n-1}		$\Delta^2 f_{n-2}$	
			Δf_{n-1}		
t_n	y_n	f_n			

Scheme:

$$y_{n+1} = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$

Error in Newton's BD : $h^4 f^{(4)}(\eta)\binom{-s}{4}$

$$E_{AB} = \frac{h}{h}\int_0^1 h^4 f^{(4)}(\eta)\binom{-s}{4}ds$$

$$= \frac{251}{720} h^5 y^{(5)}(\xi)$$

$$\Rightarrow TE = O(h^4) \quad \left(= \frac{E_{AB}}{h}\right).$$

Problem here:- It is not self-starting (because, we will first need the first 4 points).

1/4/24

$y_n = c_1 \beta_1^n + c_2 \beta_2^n \cdots c_k \beta_k^n$ → roots of characteristic equation.

$$y_{n+1} = y_n - \frac{\lambda h}{24}\left[55 y_n - 59 y_{n-1} + 37 y_{n-2} - 9 y_{n-3}\right].$$

⇒ Characteristic equation :

$$\beta^4 - \beta^3 - \frac{\lambda h}{24}\left[55\beta^3 - 59\beta^2 + 37\beta - 9\right] = 0.$$

$$\Rightarrow \rho(\beta) + \lambda h \, \sigma(\beta) = 0.$$

where $\rho(\beta) = \beta^4 - \beta^3 = \beta^3(\beta - 1)$.

$\sigma(\beta) = \ldots$

⇒ $\boxed{\text{This gives (IDK why) strongly stable.}}$

(multiple types of stability).

Q: $y_{n+1} = y_{n-1} + \frac{h}{3}\left(f_{n+1} + 4f_n + f_{n-1}\right)$.

$y' = \lambda y$. Stability ?

Ans:- Conditionally stable, depends on the value of λ.

Round off error
Stability

$\tilde{y}_n, y_n$

$\downarrow$ true $\quad \downarrow$ after round off.

$$y_{n+1} = y_n + h f(t_n, y_n)$$

$$\tilde{y}_{n+1} = \tilde{y}_n + h f(t_n, \tilde{y}_n).$$

$|y_n - \tilde{y}_n|$, $y_0 = \alpha \pm \varepsilon$.

$\downarrow$ round off error.

$$\text{Error} = |y(t_n) - y_n| = |y(t_n) - \tilde{y}_n + \tilde{y}_n - y_n|.$$

$$\leq \underbrace{|y(t_n) - \tilde{y}_n|}_{\text{Truncation error}} + \underbrace{|\tilde{y}_n - y_n|}_{\text{Round off error}}.$$

$$\leq C h^p, \; p > 0.$$

$C \downarrow$ (consistency constant) (taylor expansion's first non zero term).

$y \in C^\infty(\Omega)$.

System of first order ODEs

$$y_1' = f_1(t, y_1, y_2).$$

$$y_2' = f_2(t, y_1, y_2).$$

$$y_1(0) = \alpha, \quad y_2(0) = \beta.$$

$$\bar{y}' = \bar{F}(t, \bar{y}).$$

2nd order $\longrightarrow$ Can always be converted system of equations

$$\boxed{\begin{aligned} &y''(t) + p(t)\,y'(t) + q(t)y(t) = 0. \quad t \in [0,1). \\ &y(0) = \alpha \\ &y'(0) = \beta. \end{aligned}}$$

$\cancel{y' - v = 0}$ can be converted to $\downarrow$

$$\boxed{\begin{aligned} &y' - v = 0. \\ &v' + p(t)\,v + q(t)\,y = 0. \end{aligned}}$$

BVP.

$$u''(x) + p(x)\, u'(x) + q(x)\, u(x) = q(x)$$
$$(x \in (0,1)).$$

$u(0) = \alpha,\ u(1) = \beta.$

Here:- Shooting technique.

- Converting the BVP into suitable IVP with some initial condition $u'(0)$:

(if we have $u'(0)$, already have $u(0)$ $\Rightarrow$ we can solve as if it is an IVP).

$\Rightarrow$ start with $u'(0) = \varepsilon \pm \delta$.

$\Rightarrow$ change $u'(0)$ till the difference $|u_n - u(1)| <$ Tolerance.

$\Rightarrow$ choosing $u'(0)$ we are shooting.

Pros:- ① Solve non-linear DE. $\because$ all the methods studied for IVPs can be dealt with.

Cons:- ① Solve IVP for several values of δ.

Finite difference schemes

Step 1 :- Discretize the domain

$$0 = x_0 < x_1 < \cdots < x_{n-1} < x_n < x_{n+1} = 1.$$

Step 2 :- Replace the derivatives by finite difference equation

$$u'(x_i) = \frac{u_{i+1} - u_i}{h}$$

$$= \frac{u_i - u_{i-1}}{h} \qquad O(h).$$

$$= \frac{u_{i+1} - u_{i-1}}{2h} \qquad O(h^2)$$

$$u''(x_i) = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}.$$

$\Rightarrow$ 3 schemes to solve for

Eqn: $$\boxed{\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + p(x_i)\frac{u_{i+1} - u_{i-1}}{2h} + q_i u_i = r_i \qquad 1 \le i \le N. \qquad U_0 = \alpha,\ U_{N+1} = \beta.}$$

$$\left(\frac{1}{h^2}+\frac{p_i}{2h}\right)u_{i+1}-\left(\frac{2}{h^2}-q_i\right)u_i+\left(\frac{1}{h^2}-\frac{p_i}{2h}\right)u_{i-1}=r_i$$

$i=1 \Rightarrow$

$$\left(\frac{1}{h^2}+\frac{p_1}{2h}\right)u_2-\left(\frac{2}{h^2}-q_i\right)u_1=r_i-\left(\frac{1}{h^2}-\frac{p_i}{2h}\right)\alpha$$

$i=2$ to $N-1$

$$\Rightarrow (\ )\,u_{i+1}-(\ )\,u_i+(\ )\,u_{i-1}=r_i$$

$i=N$

$$\Rightarrow \left(\frac{1}{h^2}-\frac{p_N}{2h}\right)u_{N-1}-\left(\frac{2}{h^2}-q_N\right)u_N$$

$$=r_N-\left(\frac{1}{h^2}+\frac{p_N}{2h}\right)\beta$$

$\Rightarrow$ $\boxed{\text{N equations, N variables. } (U_1 \text{ to } U_N)}$

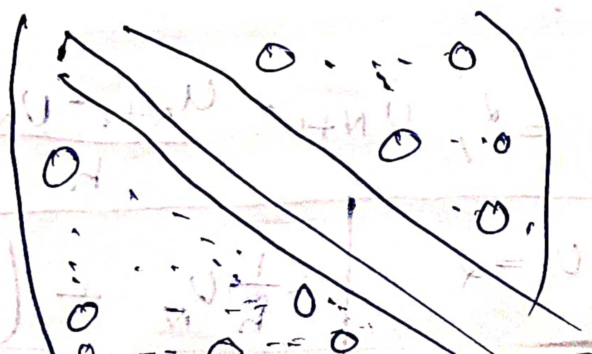
Here, if we write the system of equations as

$$AU=r.$$

$\downarrow$

$\boxed{\text{Sparse matrix.}}$

because $A \rightarrow$



$\boxed{\text{If } A \text{ is invertible then there is a unique solution}}$

2/4/24

~~The definite~~.

$u''(x) + p(x)\cdot u' + q(x)\, u = r(x),\ (0,1).$

Dirichlet Boundary Condition

$\Rightarrow u(0) = \alpha,\ u(1) = \beta.$

$TE = LHS - RHS \to \boxed{O(h^2)}$

$Au = b \qquad u_{n+1} = \beta. \quad \Rightarrow TE = \frac{-h^2}{24}\left(u^{(4)}(\xi_i) - 2p_i\, u^{(3)}(\gamma_i)\right).$

Neumann Boundary Condition

$\to -u'(0) = \alpha,\ u'(1) = \beta.$

$\left.\begin{array}{l} \left(\frac{1}{h}\right)U_0 - \left(\frac{1}{h}\right)U_1 = \alpha \Leftarrow \dfrac{U_1 - U_0}{h} = \alpha. \\ \dfrac{U_{N+1} - U_N}{h} = \beta \end{array}\right\} O(h). \Longrightarrow \boxed{O(h)}$

$\Downarrow$

$-\left(\frac{1}{h}\right) U_N + \left(\frac{1}{h}\right) U_{N+1} = \beta.$

Robin-type Mixed BC $\Rightarrow$

$u(0) - u'(0) = \alpha.$

$u(1) + u'(1) = \beta.$

?

$U_0 - \dfrac{U_1 - U_0}{h} = \alpha, \quad U_{N+1} - \dfrac{U_{N+1} - U_N}{h} = \beta.$

$\left(1+\frac{1}{h}\right) U_0 - \left(\frac{1}{h}\right) U_1 = \alpha$	$\frac{1}{h} U_N + \left(1 - \frac{1}{h}\right) U_{N+1} = \beta$

Inputs : a, b, N, α, β, @ p, q, r.

$A(1,1) =$

$A(1,2) =$

$b(1) =$

for $i = 2 : N-1$

~~$A(i,i) = A(i,i-1) = A(i,i+1) = b(i) =$~~

$A(i,i) =$

$A(i,i-1) =$

$A(i,i+1) =$

$b(i) =$

$A(N+1, N) =$

$A(N+1, N+1) =$

$b(N+1) =$

~~AUB~~ $AU = b$.

Solve using:

1. Gaussian.
2. Jacobi
3. Gauss sidal.
4. SOR.

Only one computation unlike shooting techn

$$-u''(x) + f(x,u) = 0, \quad x \in (0,1).$$

$$u(0) = \alpha, \; u(1) = \beta.$$

$$Lu = u''(x) + p(x)u'(x) + q(x)\,u = r(x).$$

↳ force.

If not linear

Fully nonlinear

Quasilinear

Semilinear.

$\phi(x) = 0.$

$$Lu \equiv u'' + q(x)\,u = r(x).$$

$Au = b. \Rightarrow$ quadratic form

$$G = x^T A x - x^T b.$$

$\nabla G = 0$ gives $AX = b.$

If A is positive definite

$\Rightarrow$ $\cancel{G\ is}$ $X^T A X$ is convex??

$\Rightarrow$ $\nabla G = 0$ can be found easily using methods like ~~grad~~ steepest descent.

~~Jacobi, Gauss seidal,~~

Static: Jacobi, Gauss seidal, SO (check).

dynamic: Steepest descent, gradient descent etc.

inverted

$(x, u^{(m)})$.

$$f(x,u) \approx f(x,u^{(m)}) + (u-u^{(m)}) \, f_u(x,u^{(m)})$$

+ higher order terms. (non linear) (terms dropped)

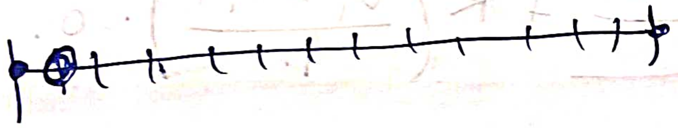
$u^{(0)}$ – initial approximation.

$$-u_{xx} + \left[f(x,u^{(m)}) + (u-u^{(m)}) \, f_u(x,u^{(m)}) \right] = 0.$$

System of linear ~~equations~~ DEs $\Rightarrow$ $\boxed{\text{solve using } Au=b.}$

$$\Rightarrow -u_{xx}^{(m+1)} - f_u(x,u^{(m)}) \, u^{(m+1)} = -f(x,u^{(m)}) + f_u(x,u^{(m)}) \, u^{(m)}$$

$u(0)=\alpha, \; u(1)=\beta$.



(Here, you are iterating multiple times because of truncation).

$\Rightarrow$ you can solve when $\boxed{f(x,u) \text{ is non linear}}$ using either the shooting technique, or by truncating taylor series and making linear (linearizing semi linear).

$$y'' + p(x,u,u') = 0,$$

$\longrightarrow$ Solve this ~~using~~ by linearizing the quasi ln

Stability Analysis

$$u''(x) + K u'(x) = 0.$$

$$|K| >> 1.$$

$u''(x) \to D^{+}D^{-}$ (single scheme).

$u'(x) \Rightarrow \underset{\text{forward}}{D^{+}}, \underset{\text{backward}}{D^{-}}, D^{0}$

(i) $$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + K\frac{U_{i+1} - U_{i-1}}{2h} = 0.$$

(ii) $$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + K\left(\frac{U_{i+1} - U_i}{h}\right) = 0.$$

(iii) $$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} + K\left(\frac{U_i - U_{i-1}}{h}\right) = 0.$$

(i) $\to$ $$\left(\frac{1}{h^2} + \frac{K}{2h}\right)U_{i+1} - \frac{2}{h^2}U_i + \left(\frac{1}{h^2} - \frac{K}{2h}\right)U_{i-1} = 0.$$

$$\Rightarrow \left(\frac{1}{h^2} + \frac{K}{2h}\right)\beta^2 - \frac{2}{h^2}\beta + \left(\frac{1}{h^2} - \frac{K}{2h}\right) = 0.$$

$$(2 + Kh)\beta^2 - 4\beta + (2 - Kh) = 0.$$

$$\Rightarrow \frac{4 \pm 4Kh}{2(2 + Kh)} = 1, \frac{2 - Kh}{2 + Kh}$$

$$\Rightarrow u_n = C_1 \beta_1^n + C_2 \beta_2^n$$
$$= C_1 + C_2 \left(\frac{2-kh}{2+kh}\right)^n .$$

Exact solution $\Rightarrow \boxed{u(x) = A + Be^{-kx}}$

Case (i) $k \to +ve$ $\Rightarrow$ $e^{-kx} \to 0$ as $x \to \infty$.

$2 - kh > 0 \Rightarrow \boxed{h < \frac{2}{k}}$.

$\longrightarrow$ if k is large, and $h \geq \frac{2}{k}$,

$\Rightarrow \frac{2-kh}{2+kh} \approx -1 \Rightarrow \underbrace{C_1 + C_2(-1)^n}_{\text{oscillation}}$.

$\Rightarrow$ for large k, h has to be very small in value.

Case (ii) $k \to -ve$ $\Rightarrow$ $e^{-kx} \to \infty$ as $x \to \infty$.

$2 + kh > 0$, $\boxed{h < \frac{2}{|k|}}$.

if ~~not~~ $h > \frac{2}{|k|}$ $\Rightarrow$ denominator negative.

$\Rightarrow \frac{2-kh}{2+kh} < -1 \Rightarrow \underbrace{C_1 + C_2(-1)^n}_{\text{spurious oscillation that diverge.}}$

3/4/24

$$u'' + k u' = 0$$

Central diff $\rightarrow$ Stability condition is $h < \frac{2}{|k|}$

Forward diff :- D^+

$$\beta = 1, \frac{1}{1+kh}.$$

$$U_n = c_1 (1)^n + c_2 \left(\frac{1}{1+kh} \right)^n. \leftarrow$$ These approximation are called Pade' Apprx.

$\boxed{\text{No restriction}} \leftarrow k > 0, h > 0$

If $k < 0, h > 0$ $\quad 1 + kh > 0 \Rightarrow h < \frac{1}{|k|}$

Backward diff :- D^-

$$\beta = 1, 1 - kh.$$

$$U_n = C_1 (1)^n + C_2 (1 - kh)^n.$$

If $k < 0 \implies$ No restriction.

$k > 0 \implies 1 - kh > 0 \Rightarrow h < \frac{1}{k}.$

Here, why is D_0 restricted in both, because average of D_+, D_-.

This flow equation, there is some concept of against or along flow here.

When going along flow, resistance $= 0$.

against flow, resistance $\neq 0$.

One has to be careful when approximating the convection term U'.

It depends on convecting coefficient

$$u_i' = \begin{cases} D^+ u_i = \dfrac{U_{i+1} - U_i}{h} & : K > 0. \\ D^- U_i = \dfrac{U_i - U_{i-1}}{h} & , K < 0. \end{cases}$$

Unsteady
$$P_p + Q_f = R.$$

time dependent

$u_t + a u_x = 0$ ← hyperbolic.

$u_t - u_{xx} = 0$ ← parabolic.

$u_{tt} = c^2 u_{xx}$ ← hyperbolic.

Steady state ← $\Delta u = u_{xx} + u_{yy} = 0$ ← elliptic.

time independent.

$u_t - (u_{xx} + u_{yy}) = 0$ → time dependent.

Parabolic PDEs

$$u_t - u_{xx} = 0 \quad , \quad (x,t) \in (0,1) \times (0,T)$$

$t=0 \leftarrow u(x,0) = \phi(x)$.

boundary $\leftarrow u(0,t) = f_1(t)$, $u(1,t) = f_2(t)$.

IBVP

(check) If homogeneous boundary conditions $\Rightarrow$ Seperable

$u(x,t) = X(x)T(t)$.

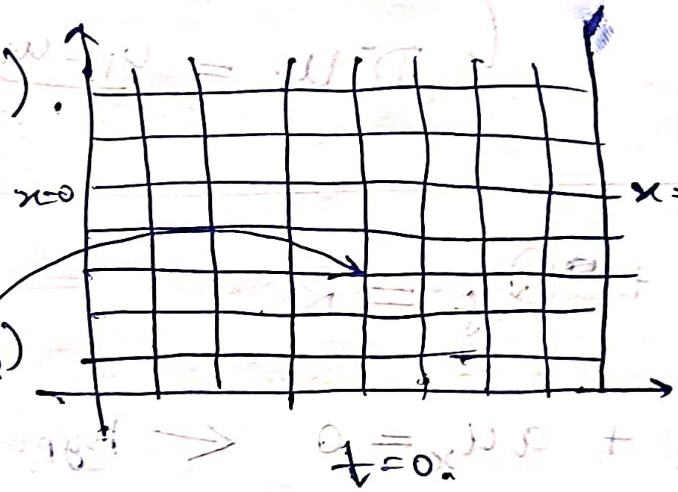
$$u_m^n = u(x_m, t_n).$$

IC: $u_m^0 = \phi(x_m)$.

BC: $u_0^n = f_1(t_n)$

$u_M^n = f_2(t_n)$.

(x_m, t_n)



Step 1: Discretization of the domain.

$$\Delta t = k = 1/N.$$

$$\Delta x = h = 1/M.$$

Step: Derivatives using finite diff.

$$u_t \Big|_{(x_m, t_n)} = \begin{cases} \dfrac{u_m^{n+1} - u_m^n}{k} \rightarrow D^+ \\[2ex] \dfrac{u_m^n - u_m^{n-1}}{k} \rightarrow D^- \\[2ex] \dfrac{u_m^{n+1} - u_m^{n-1}}{2k} \rightarrow D^0 \end{cases}$$

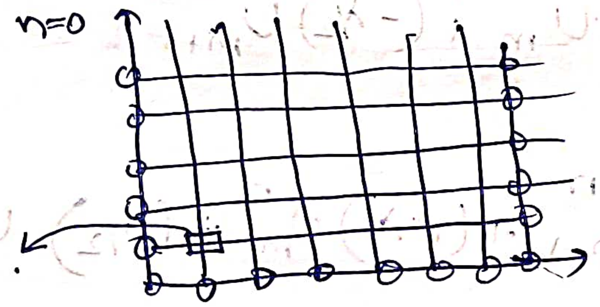
$$u_{xx}\Big|_{(x_m,t_n)} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}.$$

$$\frac{u_m^{n+1} - u_m^n}{k} - \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} = 0.$$

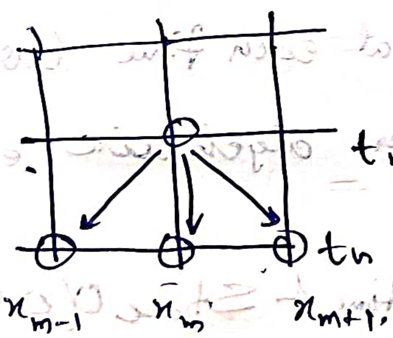
D^+: $u_m^{n+1} = \lambda u_{m-1}^n + (1-2\lambda) u_m^n + \lambda u_{m+1}^n$

$(\lambda = k/h^2)$.

Here → $\begin{bmatrix} 0 \le n \le N-1 \\ 1 \le m \le M-1. \end{bmatrix}$.



(Richardson scheme).



⇒ $\boxed{\text{Explicit scheme}}$ (Explicit Euler for time, Central space).

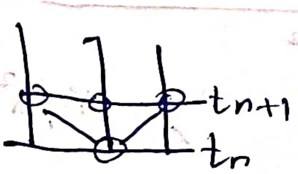
This scheme is $\boxed{\text{FTCS}}$

(Forward in Time, Central in Space)

D^-: $$\frac{u_m^{n+1} - u_m^n}{k} - \frac{u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}}{h^2} = 0.$$

$$-\lambda u_{m-1}^{n+1} + (1+2\lambda) u_m^{n+1} - \lambda u_{m+1}^{n+1} = u_m^n.$$

$\begin{bmatrix} 0 \le n \le N-1 \\ 1 \le m \le M-1. \end{bmatrix}$



⇒ (Implicit Euler Time, central space) ⇒ $\boxed{\text{BTCS}}$

Because of Implicit nature, one has to solve system of linear algebraic equations at each time level. This is computationally expensive compared to FTCS.

HW Calculate the truncation error of FTCS & BTCS.

$\hookrightarrow O(\Delta t, \Delta x^2)$

New scheme: $\frac{1}{2}$(FTCS + BTCS) $\longrightarrow$ Crank Nikolson scheme

~~$\left(\frac{1}{2K}\right) U_{m-1}^{n+1} + (0) U_m^{n+1} +$~~

$$\left(-\lambda/2\right) U_{m-1}^{n+1} + \left(\tfrac{1}{2}+\lambda\right) U_m^{n+1} + \left(-\lambda/2\right) U_{m+1}^{n+1}$$

$$= \left(\lambda/2\right) U_{m-1}^{n} + (1+\lambda) U_m^{n} + \left(\lambda/2\right) U_{m+1}^{n}$$

This is an implicit ~~sh~~ scheme at each time level, we have to solve system of linear algebraic eqⁿs (tridiagonal matrix).

Advantage:- Second order in both time & space $O(\Delta t^2 + \Delta x^2)$

Note:- These are all $\boxed{\text{2-level schemes}}$.

---

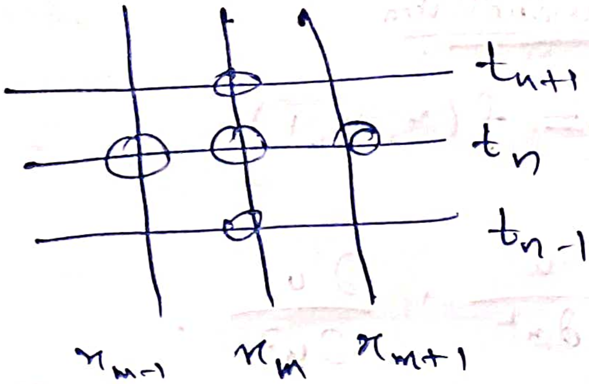
w/. central diff.

$$\frac{U_m^{n+1} - U_m^{n-1}}{K} - \frac{U_{m+1}^{n} - 2U_m^{n} + U_{m-1}^{n}}{h^2} = 0.$$

$$\lambda = K/h^2.$$

$$U_m^{n+1} = U_m^{n-1} + \lambda U_{m-1}^{n} - 2\lambda U_m^{n} + \lambda U_{m+1}^{n}$$

$\hookrightarrow$ Three level scheme.



t_{n+1}
t_n
t_{n-1}
x_{m-1}
x_m
x_{m+1}

8/4/24. 2D Heat conduction

$$u_t - \Delta u = f(x, y, t).$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

$$= u_{xx} + u_{yy}.$$

Poisson Eqn in 2e

Elliptic boundary value problems

$$\Delta u = f(x, y) \leftarrow \Omega = \text{domain}.$$

$$u(x, y) = g(x, y) \text{ on } \partial\Omega = \Gamma. \text{ (boundary)}.$$

Step 1:

Discretization of the domain. call $u(x_i, y_j) = u_{ij}$.

$$h = (b-a)/N$$

$$k = (d-c)/M$$

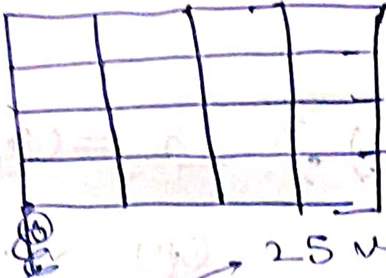
Step 2:- Replacing the derivatives by difference quotient.

$$\frac{u_{i+1,j} - 2u_{ij} + u_{i-1,j}}{h^2} + \frac{u_{i\,j+1} - 2u_{ij} + u_{ij-1}}{h^2}$$

$$= f_{ij} \quad 1 \le i, j \le M.$$

$$u_{i+1,j} - 4u_{ij} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} = h^2 f_{ij}$$

Say unit square:
$N=4$.

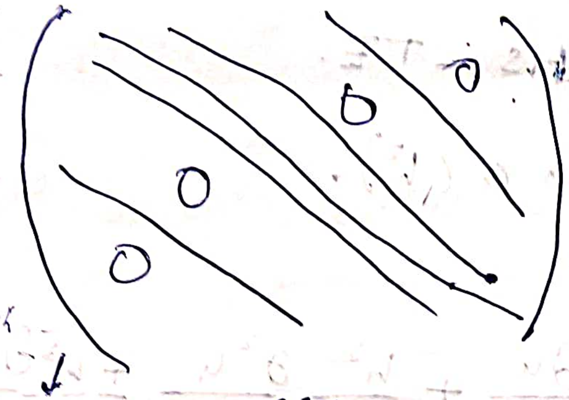


25 variables.

represent u_{ij} as U_p where $p = (j-1)(N+1)+i$
for $i,j = 1$ to 5.

make a matrix to solve ESLE of U_p.

$\Rightarrow$ 25×25 $\rightarrow$ (or 9×9).



$\rightarrow$ Symmetric.

$\rightarrow$ 5 diagonals.

$\Rightarrow$ $\boxed{\text{Sparse matrix}}$

Only non negative will be
$A(p,p)$, $A(p,p-1)$, $A(p,p+1)$,
$A(p,p-N)$, $A(p,p+N)$.

The matrix A is diagonally dominant, irreducible.

(means $|a_{ii}| \geq \sum_j |a_{ij}|$).

[Is it +ve/-ve semi definite?]

---

if 9×9 $\Rightarrow$ not taking known quantities
if 25×25 $\rightarrow$ taking known quantities, adding 1 to the row if known.

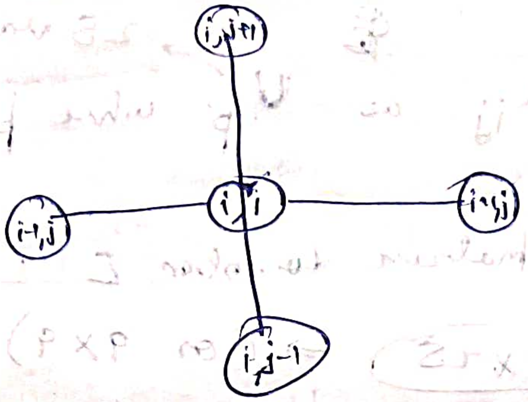
Mixed boundary conditions may be given.

like $u(x,y) = g(x,y)$ on Γ_1

or something
$u_x(x,y) = g_1(x,y)$ on Γ_2,
$u_y(x,b) = g_2(x,y)$ on Γ_3

9/4/24

$\Delta u = f(x,y)$, $\Omega = (0,1)^2$, $TE = O(h^2)$.



LHS - RHS = TE.

$u'' = f \longrightarrow O(h^2)$.

$$u_{ij\pm1} = u_{ij} \pm h \frac{\partial u}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial y^2} \pm \frac{h^3}{3!}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial y^4} + \cdots$$

$\Rightarrow$ taking $u \in C^4(\Omega)$ $\Rightarrow$ 3^{rd} & 1^{st} order cancel

2^{nd} = target

$\Rightarrow 4^{th}$ = error.

$\Longrightarrow$ $\boxed{\text{Error} = O(h^2).}$

$\underline{AU = F}$, $u_{ij} = \frac{1}{4}(u_{ij+1} + u_{ij-1} + u_{i+1j} + u_{i-1j}) - \frac{h^2}{4} f_{ij}$.

$\Downarrow$

$\boxed{\text{Poisson solvers}}$ $u_{ij}^{(n+1)} = \frac{1}{4}\left(u_{ij+1}^{(n)} + u_{ij-1}^{(n)} + u_{i+1j}^{(n)} + u_{i-1j}^{(n)}\right) - \frac{h^2}{4} f_{ij}$

(Jacobi iteration)

(gauss cydel). $u_{ij}^{(n+1)} = \frac{1}{4}\left(u_{ij+1}^{(n+1)} + u_{ij-1}^{(n+1)} + u_{i+1j}^{(n+1)} + u_{i-1j}^{(n)}\right) - \frac{h^2}{4} f_{ij}$

Stopping criteria

$$\left| u_{ij}^{(l+1)} - u_{ij}^{(l)} \right| < TOL = 10^{-8}$$

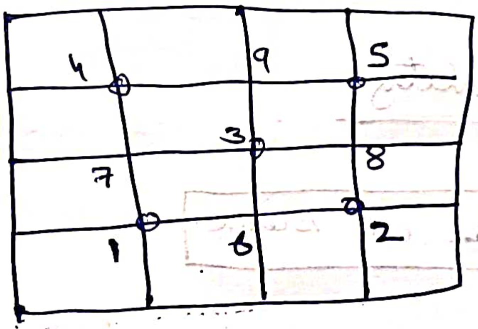
Maximum principle : If $f \geq 0 \Rightarrow$ $\boxed{\text{maximum is attained at boundary.}}$

Red / Black ordering (for parallel computing)

if I colour the nodes red / black such that black only rely only on red, red only on black.

$\Rightarrow$ $\boxed{\text{Chess board.}}$

Eg:-



Ω^h (domain Ω discretized by h).

Maximum Principle

$$L^h u_{ij} = \left(\frac{u_{i+1j} - 2u_{ij} + u_{ij+1}}{h^2} \right) \cdots = f_{ij}$$

if $L^h v_{ij} \geq 0 \;\forall ij$

$$\Rightarrow \boxed{\max_{\Gamma^h} v_{ij} \geq \max_{\Omega^h} v_{ij}}$$

P.T.O.

for proof.

Proof:- Assume contrary.

Max at interior (i_0, j_0), $v_{i_0,j_0} = \max_{\Omega^h} v_{ij}$

$$v_{i_0 j_0} > \max_{\Gamma^h} v_{ij}$$

given $L^h v_{ij} \geq 0 \; \forall \, i,j$.

$$v_{i_0 j_0} \leq \frac{1}{4}\left(v_{(i_0+1),j_0} + v_{i_0-1,j_0} + v_{i_0,j_0+1} + v_{i_0,j_0-1}\right).$$

$$v_{i_0,j_0} \geq \max\{v_{i_0+1,j_0}, v_{i_0-1,j_0}, v_{i_0,j_0+1}, v_{i_0,j_0-1}\}.$$

$\Rightarrow$ Equality

$\Rightarrow$ $\boxed{\text{All values are same.}}$

If any of the 4 neighbouring points on boundary. $\Rightarrow\!\Leftarrow$.

otherwise, repeat until we have a boundary point $\Rightarrow\!\Leftarrow$.

Thm: If max or minimum exists on interior $\Rightarrow$ function remain constant. Ill^y, we can show minimum principle. $(L^h v_{ij} \leq 0)$.

Corrolary:- (uniqueness) Let u_{ij}, v_{ij} be two solutions of $L^h u_{ij} = f_{ij}$

then $u_{ij} \equiv v_{ij}$. $\left[\text{Problem} = \begin{array}{ll} Lu = f & : \Omega \\ u = g & : \Gamma = \partial\Omega \end{array}\right]$

Proof :- $w_{ij} = u_{ij} - v_{ij}$.

$$L^h w_{ij} = L^h u_{ij} - L^h v_{ij} = f_{ij} - f_{ij} = 0 \; : \Omega^h.$$

Ill^y. $w_{ij} = 0 : \Gamma^h$.

$$\max_{\Gamma^h} w_{ij} \geq \max_{\Omega^h} w_{ij} \geq \min_{\Omega^h} w_{ij} \geq \min_{\Gamma^h} w_{ij}.$$

$$\therefore w_{ij} = 0 \ \forall \ i,j.$$

L^h here is a discrete operator.

(in terms of lin. alg., it is a matrix).

The discrete sol^n v_{ij} satisfies the following bound:

$$\max_{\Omega^h} |v_{ij}| \leq \max_{\Gamma^h} |v_{ij}| + \frac{1}{2} \max_{\Omega^h} |L^h v_{ij}|$$

Take: $\phi_{ij} = (ih)^2/2$. $\max_{\Gamma^h} |\phi_{ij}| = \frac{1}{2}$.

$$L^h \phi_{ij} = 1$$

$$M = \max_{\Omega^h} |L^h V_{ij}|.$$

$$w_{ij}^+, w_{ij}^- : w_{ij}^{\pm} = \pm v_{ij} + M \phi_{ij}.$$

$$\Rightarrow L^h w_{ij}^{\pm} \geq 0. \Rightarrow \max_{\Omega^h} w_{ij}^{\pm} \leq \max_{\Gamma^h} w_{ij}^{\pm}$$

** Missed 1 lecture **

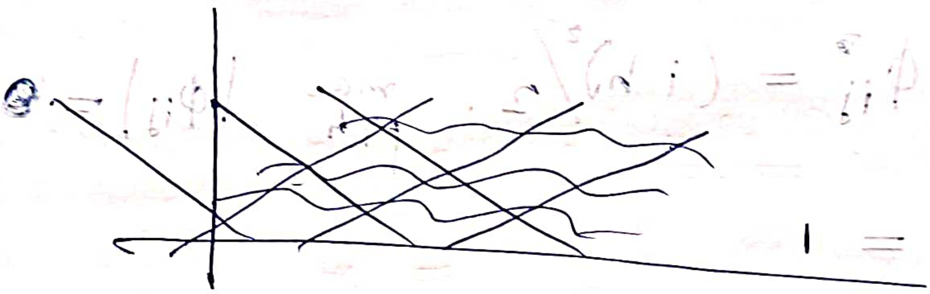
16/4/24

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_{m+1}^n - U_m^n}{h} = 0.$$

Diff. eqn: $u_t + a u_x = g(x,t)\; x \in \mathbb{R}$.

$\therefore u(x,0) = f(x)$.

$u(x,t) = f(x - at) \rightarrow$ wave equation.



FTFS

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_{m+1}^n - U_m^n}{h} = g(x_m, t_n).$$

FTCS

BTFS, BTBS, BTCS

CTFS, CTBS, CTCS

$$TE = \frac{1}{k}(LHS - RHS).$$

Stability (FFT, DFT)

We know that Fourier transform of $\mathbb{R}$ valued function $u(x)$ is given by

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} u(x)\, dx \quad - (1) \text{ (continuous)}$$

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega)\, d\omega \quad - (2) \text{ inverse}).$$

The fourier transform inversion formula expresses that the function as a superposition of the waves given by the exponential with different amplitude $\hat{u}$.

In a similar way, we can define the fourier transform for any mesh function/grid function.

$$h\mathbb{Z} = \{hm : m \in \mathbb{Z}\}.$$

Suppose v is a grid function defined on integers m

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} v_m \quad - (3)$$

$\hookrightarrow \xi = [-\pi, \pi]$.

$$\hat{v}(-\pi) = \hat{v}(\pi)$$

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi} \hat{v}(\xi)\, d\xi$$

In finite difference, it is natural to start with some grid function v_m

$$(4) - v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi} \hat{v}(\xi)\, d\xi.$$

Let ~~each~~ $h =$ uniform step size (equally spaced grid is not necessary).

(S) - $\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{m=\infty} e^{-imh\xi} v_m h.$

$\xi \in [-\pi/h, \pi/h].$

(I) - $v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \hat{v}(\xi) d\xi.$

If we differentiate our inversion formula (I),

$$\frac{\partial u}{\partial x}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\omega . e^{i\omega x} \hat{u}(\omega) d\omega.$$

$\Rightarrow$ fourier transform of $\frac{\partial u(x)}{\partial x}$ is given by

$$\widehat{\left(\frac{\partial u(x)}{dx}\right)} = i\omega \, \hat{u}(\omega).$$

This relation shows that under the fourier transform, the operation of differentiation is turned into the operation of multiplication.

Consider the first order hyperbolic PDE.

$$u_t + a u_x = 0$$

Here, we apply the transform only to the spacial variable x, then we obtain $\hat{u}(t,\omega)$.

$$⑦ - \quad \hat{u}_t = -ia\omega\hat{u}. \longrightarrow \text{ODE in time.}$$

$\Rightarrow$ Solution is

$$⑧ - \quad \hat{u}(t,\omega) = e^{-ia\omega t}\hat{v}_0(\omega).$$

Consider the following recurrence scheme (finite difference scheme).

$$\frac{u_m^{n+1} - u_m^{n}}{k} + h\,\frac{v_m^{n} - v_{m-1}^{n}}{h} = 0.$$

$$v_m^{n+1} = (1 - a\lambda)\, v_m^{n} + a\lambda\, v_{m-1}^{n}, \quad \lambda = k/h. \quad - ⑨.$$

~~Using the~~

Using the discrete Fourier transform (in ⑥), we get

$$v_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi}\, \hat{u}^n(\xi)\, d\xi. \quad - ⑩.$$

* For all the terms in equation ⑨, you get v_m^{n+1}.

$$\frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi}\, \hat{u}^{n+1}(\xi)\, d\xi = v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{\pi/h}^{\pi/h} e^{imh\xi} \left[(1 - a\lambda) + a\lambda e^{-ih\xi}\right] \hat{u}^m(\xi)\, d\xi.$$

Since, $\boxed{\text{fourier transform is unique}}$, we deduce that the integrant of LHS is same as RHS.

$$\hat{v}^{n+1}(\xi) = \left[(1-a\lambda) + a\lambda e^{-ih\xi}\right]\hat{u}^{n}(\xi),$$

$$= g(h\xi)\,\hat{u}^{n}(\xi). \quad -(11)$$

The discrete fourier transform helps in converting the two different indices in the difference equation, into one index.

This formula shows that advancing the solution of the difference scheme by 1 step is equivalent to multiplying the fourier transform of the solution by the $\boxed{\text{amplification factor}}$.

This is called amplification factor because its amplitude is the amount by which the amplitude of each solution in given the $\hat{v}_n(\xi)$

From this recurrence relation, we can obtain:

$$\hat{u}^{n}(\xi) = (g(h\xi))^{n}\,\hat{u}^{0}(\xi).$$

$$\Rightarrow g(\theta) = (1-a\lambda) + a\lambda e^{-i\theta}.$$

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$|g(\theta)| < 1$: one-step $\Rightarrow \exists\, k, \theta, k, h$

$|g(\theta, k, h)| \leq 1 + Kh \quad \forall\, k, K \in \Omega$.

g is independent of k & h $\Rightarrow |g(\theta)| \leq 1$.

$\left[0, \frac{1}{a}\right]$: $|g(\theta)| = 1$.

$u_t = u_{xx}$.

FTCS: (conditionally stable).

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}.$$

$$\Rightarrow U_m^{n+1} = \lambda U_{m+1}^n + (1-2\lambda) U_m^n + \lambda U_{m-1}^n.$$

$$\lambda = \frac{k}{h^2}$$

$$U_m^n = g^n e^{im\theta}.$$

$$g^{n+1} e^{im\theta} = \lambda g^n e^{i(m+1)\theta} + (1-2\lambda) g^n e^{im\theta} + \lambda g^n e^{i(m-1)\theta}$$

$$\Rightarrow g(\theta) = \lambda e^{i\theta} + (1-2\lambda) + \lambda e^{-i\theta} = 1 - 4\lambda \sin^2\theta/2.$$

$$\Rightarrow |g(\theta)| \leq 1 \Rightarrow \lambda \in \left(0, \tfrac{1}{2}\right].$$

$$\Rightarrow \boxed{\frac{k}{h^2} \leq \frac{1}{2} \text{ for stability.}}$$

conditionally (because implicit).

BTCS: (should be stable). $\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2}$

$$g(\theta) = \frac{1}{-\lambda e^{i\theta} + (1+2\lambda) - \lambda e^{-i\theta}} = \frac{1}{1 + 4\lambda \sin^2\theta/2}$$

Crank nicholson: unconditionally stable (implicit).

Average of these two equations:

$$\frac{U_m^{n+1}-U_m^n}{k} = \frac{1}{2}\left(\frac{U_{m+1}^{n+1}-2U_m^{n+1}+U_{m-1}^{n+1}}{h^2} + \frac{U_{m+1}^{n}+2U_m^{n}+U_{m-1}^{n}}{h^2}\right)$$

$$\Rightarrow g(\theta) = \frac{1-2\lambda\sin^2\theta/2}{1+2\lambda\sin^2\theta/2} \Rightarrow \text{unconditionally stable.}$$

HW: Study the stability of FTCS, BTCS for 2D parabolic schemes.

FTFS $U_t + aU_x = 0$

$$\frac{U_m^{n+1}-U_m^n}{k} + a\,\frac{U_{m+1}^n - U_m^n}{h} = 0$$

$$R = \frac{ak}{h} \quad \begin{matrix}\leftarrow \Delta t\\ \leftarrow \Delta h\end{matrix}$$

$$g(\theta) = (1+R) - Re^{i\theta}.$$

Case 1: a: +ve	Case 2: a: −ve
$\boxed{\text{always unstable}}$	

$|R| \le 1 \Rightarrow$ difference scheme is stable.

$$|g(\theta)| = (1+2R\sin^2\theta/2)^2 + (R\sin\theta)^2.$$

Here, you have to use an alternate way to study the stability, for that determine the max & min of $|g^2(\theta)|$. for $\theta \in [-\pi, \pi)$.

$$|g(0)| = 1, \quad |g(\pm\pi)| = |1+2R|.$$

$$\Rightarrow \quad -1 \le R \le 0.$$

So

Case 1: if $a < 0 \Rightarrow R < 0$.

$\Rightarrow$ scheme is conditionally stable.

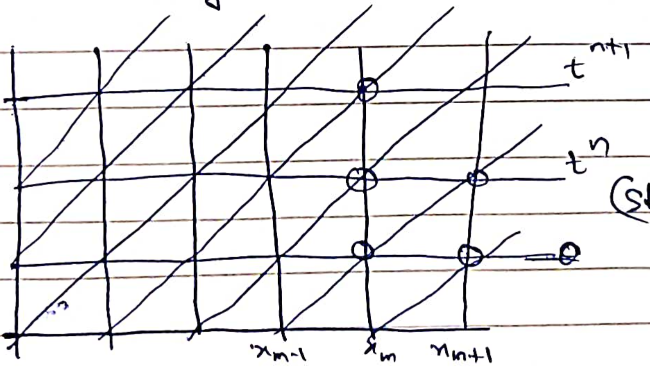
$$\boxed{R \ge -1}$$

Case 2: if $a > 0 \Rightarrow R > 0$.

$\Rightarrow$ Scheme is unconditionally unstable.

if $a = 0 \Rightarrow$ Just an ODE.

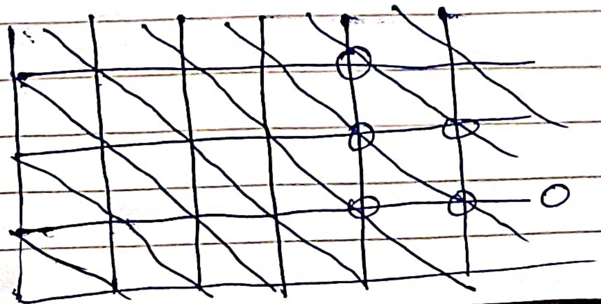
Characteristic:



$a > 0$

(Stencil dependency going away from wave).

Characteristic:



$a < 0$

(stencil dependency with the wave).