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- The above conditions are called the **FJ (Fritz John)** conditions and the point $(\mathbf{x}^*, \mathbf{u})$ (or $\mathbf{x}^*$) is called a **Fritz John**, or an **FJ**, point.

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  Hence $[2, 1]^T$ satisfies the KKT condition.

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- So **KKT** condition is **not** a necessary condition for a **local minimum**, although FJ conditions are **necessary conditions** for a **local minimum.**

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- $\nabla g_i(\mathbf{x}^*)$'s, $i \in I$ are **LD** at $\mathbf{x}^* = [0, 0]^T$ but $G_0 \neq \phi$.
- **Remark:** If $\nabla g_i(\mathbf{x}^*)$'s, $i \in I$ are **LI** then $G_0 \neq \phi$ but the converse is **not** true.

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- Check that $[0, 0]^T$ satisfies the KKT conditions as given in **(3)**, immediately after Theorem 7.
- Since f, g_1, g_2, g_3 is convex, where $g_2(x_1, x_2) = x_1 - x_2$ and $g_3(x_1, x_2) = -x_1 + x_2$,

- **Example 7:** Minimize $f(x_1, x_2) = x_1^3 - x_1 + x_2$,
  subject to
  $(x_1 - 1)^2 + 2x_2 \leq 1$,
  $x_1 - x_2 = 0$.
- Check that at $\mathbf{x}^* = [0, 0]^T$, the first constraint is binding.
- But $\nabla g_1(\mathbf{x}^*)) = 2[-1, 1]^T$, and $\nabla h_1(\mathbf{x}^*) = [1, -1]^T$.
- Since $\{\nabla g_1(\mathbf{x}^*)), \nabla h_1(\mathbf{x}^*)\}$ is **LD**, Theorem 11 is not applicable.
- Check that $[0, 0]^T$ satisfies the KKT conditions as given in **(3)**, immediately after Theorem 7.
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- $[0, 0]^T$ is the global minimum of f in the given feasible region.

- Consider the problem (P) of minimizing $f(x_1, x_2) = -x_1 x_2 + x_1^2 + 2x_2^2 - 2x_1 + e^{x_1+x_2}$ over R^2.

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    2. Is $[0, 0]^T$ a local minimum point for this problem? If not find $\mathbf{d} \in D_{\mathbf{x}^*}$ along which the function f will decrease.

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- Minimize $(x_1 - 2)^2 + (x_2 - 3)^2$
  subject to
  $x_1^2 + x_2^2 \leq 5.$
  $2x_1 + x_2 \leq 4.$
  $-x_1 \leq 0$
  $-x_2 \leq 0.$

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  3. Does there exist an $\mathbf{x}^* \in Fea(P)$ such that $G_{0,\mathbf{x}^*} = D_{\mathbf{x}^*}$?

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  2. $\mathbf{x}^*$ a global minimum of f over this feasible region.
  3. Does there exist an $\mathbf{x}^* \in Fea(P)$ such that $G_{0,\mathbf{x}^*} = D_{\mathbf{x}^*}$?
  4. Does there exist an $\mathbf{x}^* \in Fea(P)$ such that $G_{0,\mathbf{x}^*} \neq D_{\mathbf{x}^*}$?
  5. Does there exist an FJ point which is not a KKT point?

- Consider the following problem:
  Minimize $4x_1^2 - x_2^2 + 8x_1x_2$
  subject to
  $2x_1 + x_1^2 - x_2 \geq 0$
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  2. Find all points $\mathbf{x}^*$ in the feasible region at which $G_{0,\mathbf{x}^*} = \phi$.

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  1. Does $[0, 0]^T$ satisfy the first and the second order necessary conditions for a local minimum?
  2. Find all points $\mathbf{x}^*$ in the feasible region at which $G_{0,\mathbf{x}^*} = \phi$.
- If $\mathbf{x}^* \in S$ is such that for all $\mathbf{d} \in D_{\mathbf{x}^*} (\neq \phi)$, $\nabla f(\mathbf{x}^*)\mathbf{d} > 0$ then does it imply that $\mathbf{x}^*$ is a local minimizer of f?

- Consider the following problem:
  Minimize $-x_1^2 - 4x_1x_2 - x_2^2$
  subject to
  $x_2^2 + x_1^2 = 1.$
  1. If possible find an FJ point which is not a KKT point.

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  Minimize $-x_1^2 - 4x_1x_2 - x_2^2$
  subject to
  $x_2^2 + x_1^2 = 1.$
  1. If possible find an FJ point which is not a KKT point.
  2. If possible find a KKT point which is not an optimal point.

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  2. If possible find a KKT point which is not an optimal point.
  3. Are the first and the second order necessary conditions for a local minimum satisfied at the KKT point/s?

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  $x_2^2 + x_1^2 = 1.$
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  2. If possible find a KKT point which is not an optimal point.
  3. Are the first and the second order necessary conditions for a local minimum satisfied at the KKT point/s?
  4. If the objective function is changed to $2x_1^2 - x_1x_2 + x_2^2 - x_2$ and if $S = Fea(P) = \{(x_1, x_2) : x_2^2 + x_1^2 \leq 1\}$ then find all optimum solutions to this problem.

- For a nonlinear programming problem (P) of the form,

$$\text{Minimize } f(\mathbf{x})$$
$$\text{subject to } g_i(\mathbf{x}) \leq 0, \text{ for } i = 1, \ldots, m, \quad \mathbf{x} \in \mathbb{R}^n,$$

where all the g_i's and f are continuously differentiable throughout $\mathbb{R}^n$, check the correctness of the following statements with proper justification.

1. If $\mathbf{x}^* \in Fea(P)$ is a KKT point of the above problem then $-\nabla f(\mathbf{x}^*)$ lies in the cone generated by $\nabla g_i(\mathbf{x}^*), i \in I$, where I gives the indices of the binding constraints (given by g_i's) at $\mathbf{x}^*$.

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2. If $g_2 = -g_3$ then all points $\mathbf{x} \in Fea(P)$ are FJ (Fritz John) points.

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3. If $\nabla f(\mathbf{x}^*) = 0$ for some $\mathbf{x}^* \in Fea(P)$ then $\mathbf{x}^*$ is a KKT point.

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3. If $\nabla f(\mathbf{x}^*) = 0$ for some $\mathbf{x}^* \in Fea(P)$ then $\mathbf{x}^*$ is a KKT point.
4. If $\mathbf{x}^*$ is an FJ point and there is a solution to the FJ conditions at $\mathbf{x}^*$ with $u_0 = 0$ (u_0 is the coefficient of $\nabla f(\mathbf{x}^*)$ in the FJ conditions), then $\mathbf{x}^*$ is not a KKT point.

- If $\mathbf{x}^*$ is a KKT point then it is also an FJ point.

- If $\mathbf{x}^*$ is a KKT point then it is also an FJ point.
- If $\mathbf{x}^*$ is an FJ point and $\nabla g_i(\mathbf{x}^*)$'s are LD, then $G_{0,\mathbf{x}^*} = \phi$ (or in other words there exists a solution to the FJ conditions with $u_0 = 0$).

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- If $\nabla g_i(\mathbf{x}^*)$'s are LI, then $G_{0,\mathbf{x}^*} \neq \phi$.

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- If $G_{0,\mathbf{x}^*} = \phi$ then $\nabla g_i(\mathbf{x}^*)$'s are LD.
- If $\mathbf{x}^*$ is not an interior point of the feasible region S of (P) and $F_{0,\mathbf{x}^*} = \phi$ then $\mathbf{x}^*$ is a local minimizer.

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- If $\mathbf{x}^*$ is not an interior point of the feasible region S of (P) and $F_{0,\mathbf{x}^*} = \phi$ then $\mathbf{x}^*$ is a local minimizer.
- If all the g_i's and f are convex functions and $\mathbf{x}^*$ is such that $F_{0,\mathbf{x}^*} \cap G_{0,\mathbf{x}^*} = \phi$, then $\mathbf{x}^*$ is a global minimum of f in S.

- If $\mathbf{x}^*$ is a KKT point then it is also an FJ point.
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- If $\nabla g_i(\mathbf{x}^*)$'s are LI, then $G_{0,\mathbf{x}^*} \neq \phi$.
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- If $\mathbf{x}^*$ is not an interior point of the feasible region S of (P) and $F_{0,\mathbf{x}^*} = \phi$ then $\mathbf{x}^*$ is a local minimizer.
- If all the g_i's and f are convex functions and $\mathbf{x}^*$ is such that $F_{0,\mathbf{x}^*} \cap G_{0,\mathbf{x}^*} = \phi$, then $\mathbf{x}^*$ is a global minimum of f in S.
- If $\mathbf{x}^*$ is a local minima of (P) with $F_{0,\mathbf{x}^*} \neq \phi$ but $G_{0,\mathbf{x}^*} = \phi$ then $\mathbf{x}^*$ is not a KKT point.

- There exists a (P) such that for all $\mathbf{x}^* \in S$, $G_{0,\mathbf{x}^*} = D_{\mathbf{x}^*} \neq \phi$.

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- There exists a (P) with $Fea(P) \neq \phi$ such that $G_{0,\mathbf{x}^*} = \phi$ for all $\mathbf{x}^* \in S$.
- There exists a (P) with $Fea(P) \neq \phi$ such that $D_{\mathbf{x}^*} = \phi$ for all $\mathbf{x}^* \in S$.
- Give examples of nonconstant functions on $\mathbb{R}^n$ which are both convex and concave and those which are neither convex nor concave.

- For a linear programming problem (P) of the form,
  Minimize $\mathbf{c}^T\mathbf{x}$ subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
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- Consider the linear programming problem.
  Minimize $2x_1 - 3x_2$
  subject to $x_1 + 2x_2 \leq 3$
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  $x_1 \geq 0, x_2 \geq 0$. Find the KKT conditions for this problem at a local minimum point of this problem.
  Solve the KKT conditions for u_i's. Hence find an optimal solution of the dual.

- For a linear programming problem (P) of the form,
  Minimize $\mathbf{c}^T\mathbf{x}$ subject to $A_{m\times n}\mathbf{x} \leq \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$,
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  Minimize $2x_1 - 3x_2$
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  Solve the KKT conditions for u_i's. Hence find an optimal solution of the dual.
- Minimize $-x_1$
  subject to
  $-(1-x_1)^3 + x_2 = 0$
  $-(1-x_1)^3 - x_2 = 0.$

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  $-(1 - x_1)^3 - x_2 = 0.$
- Check whether $[1, 0]^T$ is a KKT point of the above problem. How many feasible points does this problem have? What is your conclusion?