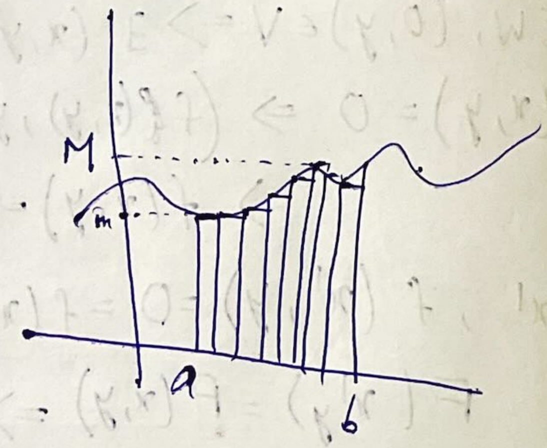
Measure Theory

Measure

$[a,b]\,,\,[a,b)\,,\,(a,b)\,,\,(a,b] \longrightarrow b-a.$

$\{x\}\,,\,\{x_1,\dots,x_n\}\,,\,\mathbb{N},\,\mathbb{Z},\,\mathbb{Q} \longrightarrow 0.$

Integration



$m \leqslant f(x) \leqslant M$

$m(b-a) \leqslant \int_a^b f(x)\,dx \leqslant M(b-a)$

$x_0 = a < x_1 < x_2 < \dots < x_n = b$

$\uparrow$
$P.$

$I_i = [x_{i-1}, x_i]$

$\sup_{x\in I_i} f(x) = M_i$

$\inf_{x\in I_i} f(x) = m_i$

$$\underbrace{\sum \ell(I_i)\, m_i}_{L(P,f)} \leqslant \int \leqslant \underbrace{\sum \ell(I_i)\, M_i}_{U(P,f)}$$

$$\underline{\int_a^b} f(x)\,dx := \sup_P L(P,f)$$

$$\overline{\int_a^b} f(x)\,dx := \inf U(P,f).$$

$f(x) = \begin{cases} 1 & , x \notin \mathbb{Q} \\ 0 & , x \in \mathbb{Q}. \end{cases}$

$\chi_A(x) = \begin{cases} 1 & , x \in A \\ 0 & ; x \notin A. \end{cases}$

$\boxed{\int_0^1 f(x) = m(A) \quad , A = [0,1] \setminus \mathbb{Q}.}$

$\int \chi_A = m(A)$

$\int \chi_{[a,b]} = m([a,b])$
$= \ell([a,b]).$

$\{I_n\}_{n=1}^{\infty}$, $A \subset \bigcup_n I_n \longrightarrow I_n = (a_n, b_n)$.

$U(A) \leq \sum_n \ell(I_n)$

$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} \ell(I_n) : A \subset \bigcup_{n=1}^{\infty} I_n \right\}$. $\boxed{m^*(A) \geq 0 \ \forall \ A \subset \mathbb{R}}$

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eg:- ① $\{a\} = A$, $m^*(A) \leq 2\varepsilon \ \forall \ \varepsilon > 0$.

$\therefore m^*(A) \leq 0$.

$\Rightarrow m^*(A) = 0$. $\boxed{m^*(\phi) = 0}$.

② $A = \{a_1, a_2, \ldots, a_n, \ldots\}$. $\longleftarrow$ Countable set.

Take $I_n = \left(a_n - \frac{\varepsilon}{2^{n+1}}, a_n + \frac{\varepsilon}{2^{n+1}}\right)$.

$\therefore A \subset \bigcup I_n$.

$\ell(I_n) = \frac{\varepsilon}{2^n}$.

$\therefore \sum_{n=1}^{\infty} \ell(I_n) = \varepsilon$.

$\therefore m^*(A) \leq \varepsilon \ \forall \ \varepsilon > 0 \Rightarrow m^*(A) = 0$.

P.T. $A \subset B \Rightarrow m^*(A) \leq m^*(B)$.

$B \subset \bigcup I_n$, $A \subseteq \bigcup I_n \Rightarrow m^*(A) \leq \sum \ell(I_n)$

$\therefore m^*(A) \leq m^*(B)$.

Theorem:- $m^*(I) = \ell(I)$ — not open.

$I = [a,b] \subset \left(a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right)$.

$\Rightarrow m^*[a,b] \leq b - a + \varepsilon \ \forall \ \varepsilon > 0$

$\therefore m^*[a,b] \leq b - a$.

Claim :- $(b-a) - \varepsilon \leqslant m^*([a,b])$.

$$m^*([a,b]) = \inf\{\sum \ell(I_n) : A \subset \cup I_n\}.$$

$\exists \{I_n\} \ni \sum \ell(I_n) < m^*([a,b]) + \varepsilon$ and $[a,b] \subset \cup I_n$.

$I_1, \ldots, I_k \ni I_1 \cup I_2 \cup \ldots \cup I_k \supset [a,b]$.

$$b-a \leqslant \ell(I_1) + \ell(I_2) + \ldots + \ell(I_k) \leqslant \sum \ell(I_n) \leqslant m^*([a,b]) + \varepsilon$$

$\therefore m^*[a,b] \geqslant (b-a) - \varepsilon$.

$\therefore m^*[a,b] \geqslant b-a$.

For open interval,

$$b-a-\frac{\varepsilon}{2} \leqslant m^*((a,b)) \leqslant b-a.$$

$\hookrightarrow [a+\frac{\varepsilon}{4}, b-\frac{\varepsilon}{4}] \subset (a,b)$

$$b-a-\frac{\varepsilon}{2} \leqslant m^*((a,b)) \quad \forall\ \varepsilon > 0.$$

$\therefore m^*((a,b)) = b-a$.

$m^*(A \cup B) \leqslant m^*(A) + m^*(B)$.

$A = [0,1] \setminus Q$

$B = [0,1] \cap Q$

$\ast\ 1 \leqslant 0 + m^*(A) \leqslant 1$.

$\therefore \underline{m^*(A) = 1}$.

H/W

① $m^*(A \cup B) \leqslant m^*(A) + m^*(B)$

② $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} m^*(A_n)$

③ $m^*(A+x) = m^*(A)$

G_δ-set

A subset G of $\mathbb{R}$ is called G_δ-set if G is countable intersection of open subsets in $\mathbb{R}$ ie, $G = \bigcap G_n$, G_n's open.

eg:- $\mathbb{R} \setminus \mathbb{Q}$.

F_σ-set

if F is ~~union~~ countable union of closed subsets of $\mathbb{R}$.

$A \subseteq \mathbb{R}$, then $\forall \varepsilon > 0$, $\exists$ open set 'O' $\ni$ $A \subset O$

and $m^*(O) \leqslant m^*(A) + \varepsilon$.

$\sum l(I_n)$

$\exists \{I_n\} \ni A \subseteq \bigcup I_n = O$

?. $\boxed{m^*(O) \leq} \sum l(I_n) \leqslant m^*(A) + \varepsilon$.

$m^*(O) = m^*(\bigcup I_n)$

$\leq \sum m^*(I_n)$

$= \sum l(I_n)$.

$m^*(A) \leq \sum l(I_n)$

$m^*(O)$

$\exists$ a G_δ set $G \ni$

$A \subset G$ & $m^*(G) = m^*(A)$

$\forall \varepsilon = \frac{1}{n}$, $\exists O_n$ (open) $\ni$

$m^*(O_n) \leqslant m^*(A) + \frac{1}{n}$.

$G = \bigcap O_n \Rightarrow m^*(G) \leqslant m^*(O_n) \leqslant m^*(A) + \frac{1}{n} \ \forall n$

$\therefore \ m^*(A) \leqslant m^*(G) \leqslant m^*(A)$.

$m^*: P(\mathbb{R}) \longrightarrow [0, \infty]$.

$m^*(\cup A_n) \leq \sum m^*(A_n)$.

$\mathcal{B}(\mathbb{R}) =$ ~~set of all~~ Borel algebra

$G \in \mathcal{B}(\mathbb{R})$.

$G = A \cup G \setminus A$. (which is not always)

if ~~$m^*(G)$~~ $=$ ~~$m^*(A)$~~ $+ m^*(G \setminus A)$

$m^*(G \setminus A) = 0 \Rightarrow \Leftarrow$.

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*2 classes

Theorem:- If $P \neq \emptyset$, perfect set in $\mathbb{R}^n$, then P is uncountable

Proof:- $P = P' \neq \emptyset$ and let us assume P is countable.

$P = \{x_1, x_2, \ldots\}$.

$a_1 = x_1$

$a_2 = x_2, \quad B(a_2, \varepsilon_1), \quad \varepsilon_1 < |x_1 - x_2|$.

$a_3 \neq x_2, \quad \varepsilon_2 < \min\{|a_3 - x_3|, \varepsilon_1/2\}, \quad B(a_3, \varepsilon_2)$.

$B_1, \ldots, B_n$ is a sequence of balls.

$B_n = B(a_n, \varepsilon_{n-1})$

$\cap \overline{B}_n = \{a_0\}$

P - is closed $\Rightarrow a_0 \in P$ and $a_0 \in P'$

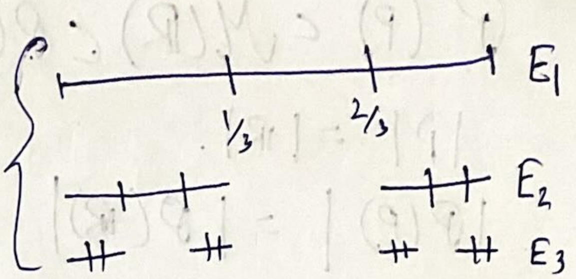
but $a_0 \neq x_n \; \forall n \quad \Rightarrow \Leftarrow$.

Cantor set

$P = \bigcap_{n=1}^{\infty} E_n$

↳ Cantor set.

closed.



① E_n is the union of 2^n intervals of length 3^{-n}.

② $E_1 \supset E_2 \supset E_3 \supset \cdots$

③ E_n – compact.

Theorem:- P is non empty compact and perfect set of $\mathbb{R}$.

Proof:- Let, $x \in P$. Claim $x \in P'$.

$x \in E_n \; \forall n$.

For large enough n, $x \in I_n \subset I$.

$I_n = [a_n, b_n]$.

x is limit point.

$x \in P'$.

$\therefore \underline{P = P'}$

$m(P) \leq m(E_n)$

$= \frac{2^n}{3^n} \; \forall n.$

$\Rightarrow m(P) = 0.$

P is measureable.

Let, E be any subset of P.

$\therefore m(E) = 0 \Rightarrow E$ is measureable.

$\mathcal{P}(P) \subset \mathcal{M}(\mathbb{R}) \subset \mathcal{P}(\mathbb{R})$.

$|P| = |\mathbb{R}|$.

$|\mathcal{P}(P)| = |\mathcal{P}(\mathbb{R})|$

$\therefore |\mathcal{M}(\mathbb{R})| = |\mathcal{P}(\mathbb{R})|$

$F \subset E \subset G$

$m(E \setminus F) = m(G \setminus E) = 0$

$|\mathbb{R}| = |\mathcal{B}(\mathbb{R})| < |\mathcal{M}(\mathbb{R})| = |\mathcal{P}(\mathbb{R})|$. ***

Theorem:- FAE

① E is m'able set

② $\forall \varepsilon > 0$, $\exists$ an open set $O \ni m^*(O \setminus E) < \varepsilon$ $(O \supset E)$.

③ $\exists$ G_δ-set $G \ni m(G \setminus E) = 0$ $(G \supset E)$.

④ $\forall \varepsilon > 0$, $\exists$ a closed set $F \subset E \ni m(E \setminus F) < \varepsilon$.

⑤ $\exists F_\sigma$-set $A \subset E \ni m(E \setminus A) = 0$

① $\Rightarrow$ ②

Case I:- $m(E) < \infty$

$\exists$ an open set $O \ni E \subset O$

$m^*(O) < m^*(E) + \varepsilon$

$m^*(O \setminus E) = m^*(O) - m^*(E) < \varepsilon$.

Case II:- $m(E) = \infty$, $E_n = E \cap [-n, n]$.

$m(E_n) < \infty$.

$\exists O_n \supset E_n : m(O_n \setminus E_n) < \varepsilon/2^n$.

$O = \cup O_n$, $E = \cup E_n$.

$O \setminus E \subset \cup (O_n \setminus E_n)$ H/W

$m(O \setminus E) \leq \sum m(O_n \setminus E_n) < \sum \varepsilon/2^n = \varepsilon$.

(2) => (3)

$\varepsilon = \frac{1}{n}$, $\exists$ open set $O_n \ni m^*(O_n \setminus E) < \frac{1}{n}$.

$G = \cap O_n$, $G \setminus E \subset O_n \setminus E$

$m^*(G \setminus E) \leq m^*(O_n \setminus E) < \frac{1}{n} \; \forall n$.

$\Rightarrow m^*(G \setminus E) = 0$.

(3) => (1)

$\exists$ G_δ set - $G \ni m(G \setminus E) = 0$ and $E \subset G$.

$m(G \setminus E) = 0$

G is measureable, $G \setminus E$ is m'able.

$\Rightarrow G \setminus (G \setminus E)$ m'able

$\Rightarrow E$ - m'able.

E-m'able $\Leftrightarrow \exists A \subset \mathcal{B}(\mathbb{R}) \ni A \supset E$ and $m(A \setminus E) = 0$

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(1) => (4)

$\forall \varepsilon > 0$, $\exists O \ni E^c \subset O$, $m^*(O \setminus E^c) < \varepsilon$.

$O \setminus E^c = E \setminus F$ where $F = O^c$.

$\therefore m^*(E \setminus F) < \varepsilon$.

Similarly (5)

Q:) $\varepsilon > 0$, does $\exists$ an open dense set of $\mathbb{R} \ni m(G) \leq \varepsilon$.

A:) $\mathbb{Q}$ is dense in $\mathbb{R}$, $m(\mathbb{Q}) = 0$

$\forall \varepsilon > 0$,

$\exists G$ (open) $\ni \mathbb{Q} \subset G$, $m(G \setminus \mathbb{Q}) < \varepsilon$.

$\Rightarrow m(G) = m(G \setminus \mathbb{Q}) < \varepsilon$.

$\mathbb{Q} = \{r_n\}$, $G = \bigcup_{n \in \mathbb{N}} \left(r_n - \frac{\varepsilon}{2^n}, r_n + \frac{\varepsilon}{2^n} \right)$

$[0,1]$

$x \sim y$, iff $x - y \in \mathbb{Q} \cap [-1,1] = \mathbb{Q}_1 = \{r_n\}$.

$[0,1] = \dot{\bigcup} E_\alpha \leftarrow \alpha^{th}$ equivalence class.

$V \to$ set containing exactly one point from each E_α.

$\forall \alpha$, E_α is countable.

V is uncountable.

if V is m'able $\Rightarrow V_n$'s are m'able, $m(V) = m(V_n)$.

$V_n := V + r_n = \{x + r_n : x \in V\}$.

$\boxed{V_n \cap V_m \neq \emptyset \Rightarrow n = m}$

$m^*(V) = m^*(V_n)$.

$[0,1] \subseteq \bigcup_n V_n \subset [-1,2]$.

$x \in [0,1] \Rightarrow x \in \dot{\bigcup} E_\alpha$, $x \in E_\alpha$ for some α.

$x \sim x_\alpha$, $x_\alpha \in V$

$x = x_\alpha + r_n \in V_n$.

$m^*([0,1]) \leq m^*(\bigcup_n V_n) \leq m^*([-1,2])$

$\Rightarrow 1 \leq m^*(\bigcup_n V_n) \leq 3$.

$1 \leq m(V) + m(V) + \ldots \leq 3$ ↯

$m^*(\bigcup V_n) < \sum_{n=1}^{\infty} m^*(V_n)$ (equality ↯).

Measureable Function

Idea –

$f : [a,b] \longrightarrow \mathbb{R}$.

$I_1, I_2, \dots, I_n \ni I_1 \cup I_2 \cup \dots \cup I_n = [a,b]$

$m_i = \inf_{x \in I_i} f(x)$.

$L(P,f) = \int m_i \chi_{I_i} = \sum m_i l(I_i) = \sum m_i m(I_i)$.

$\sum \alpha_i \chi_{E_i} = f(x)$

$\int f(x) = \int \alpha_i \chi_{E_i} \overset{?}{=} \alpha_i \int \chi_{E_i} = \boxed{\sum \alpha_i m(E_i)}$.

$\phi_n \longrightarrow f$

$\int f = \lim_{n \to \infty} \int \phi_n$

$\{a_n\}$, $\limsup a_n = \inf_k \left(\sup_{n \geq k} a_n \right)$.

$A_k = \sup_{n \geq k} a_n = \{a_k, a_{k+1}, \dots\}$

$B_k = \inf_{n \geq k} a_n = \{a_k, a_{k+1}, \dots\}$.

$\limsup a_n = \inf_{k \geq 1} A_k = \alpha$

$\lim \cancel{\sup} \inf a_n = \sup_{k \geq 1} B_k = \beta$.

$\{E_n\} \longrightarrow$ seq. of sets in $\mathbb{R}$.

$\limsup E_n := \bigcap_{k \geq 1} F_k$, $F_k = \bigcup_{n \geq k} E_n$

$\liminf E_n := \bigcup_{k \geq 1} G_k$, $G_k = \bigcap_{n \geq k} G_k$.

$\{E_n\}$ — set of m'ble sets in $\mathbb{R}$.

① If $E_1 \subset E_2 \subset \ldots$ then
$$m\left(\bigcup E_n\right) = \lim \left(m(E_n)\right).$$

② If $E_1 \supset E_2 \supset E_3 \supset \ldots$ and $m(E_1) < \infty$ then
$$m\left(\bigcap E_n\right) = \lim \left(m(E_n)\right).$$

→ not true if $m(E_n)$ is ∞.

Eg:- $E_n = [n, \infty)$.

$m(E_n) = \infty$

$\bigcap E_n = \phi$

$m(\bigcap E_n) = 0$ $\not=$

Proof: ① $F_1 = E_1$, $F_2 = E_2 \setminus E_1$, $F_n = E_n \setminus E_{n+1}$

F_n's are disjoint m'ble sets.

① $E_N = \bigcup_{n=1}^{N} F_n$ $\left(\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n\right)$.

$$\begin{aligned} m\left(\bigcup E_n\right) &= m\left(\bigcup F_n\right) \\ &= \sum_{n=1}^{\infty} m(F_n) \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} m(F_n). \\ &= \lim_{N\to\infty} m\left(\bigcup_{n=1}^{N} F_n\right) \\ &= \lim_{N\to\infty} m(E_N) \end{aligned}$$

② $F_n = E_1 \setminus E_n$.

$\boxed{\begin{array}{l} E := \bigcap E_n. \\ \text{T.P. } m(E) = \lim (m(E_n)). \end{array}}$

$$\begin{aligned} \bigcup F_n &= \bigcup (E_1 \setminus E_n) \\ &= \bigcup (E_1 \cap E_n^c). \\ &= E_1 \cap \left(\bigcup E_n^c\right)^c \\ &= E_1 \setminus E. \end{aligned}$$

From ①,
$$\begin{aligned} m\left(\bigcup F_n\right) &= \lim (m(F_n)). \\ &= \lim (m(E_1 \setminus E_n)). \\ &= \lim (m(E_1) - m(E_n)). \end{aligned}$$

$$m(E_1 \setminus E) = \lim (m(E_1) - m(E_n))$$

$$= \cancel{m(E_1)} - m(E) = \cancel{m(E_1)} - \lim m(E_n)$$

$$\therefore m(E) = m\left(\bigcap E_n\right) = \lim m(E_n) \quad //$$

Measureable Function: (Lebesgue m'ble f^n).

Let f be an extended real valued function defined on m'ble set E. f is called m'ble f^n if $\forall \alpha \in \mathbb{R}$, the set

$$\{x : f(x) > \alpha\} \text{ is m'ble.}$$

FAE :-

1. f is m'able
2. $\{x : f(x) \geq \alpha\}$ is m'ble
3. $\{x : f(x) < \alpha\}$ is m'ble
4. $\{x : f(x) \leq \alpha\}$ is m'ble.

Proof:- 1 $\Rightarrow$ 2,

$$\{x : f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) > \alpha - \tfrac{1}{n}\} \longrightarrow \text{m'ble.}$$

2 $\Rightarrow$ 3,

$$\{x : f(x) < \alpha\} = \{x : f(x) \geq \alpha\}^c \longrightarrow \text{compliment of m'ble set.}$$

3 $\Rightarrow$ 4,

$$\{x : f(x) \leq \alpha\} = \bigcap_{n=1}^{\infty} \{x : f(x) < \alpha + \tfrac{1}{n}\} \longrightarrow \text{intersection of m'ble sets.}$$

4 $\Rightarrow$ 1,

compliment.

eg:- f is cts, $f : \mathbb{R} \to \mathbb{R}$.

$$\{x : f(x) > \alpha\} = f^{-1}(\alpha, \infty) \longrightarrow \text{open set.}$$

open is m'ble.

eg:- $f'(x) > 0$.

$$\chi_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}$$, E is m'ble $\Rightarrow \chi$ is m'ble.

$\rightarrow$ f & g be 2 real-valued m'ble functions, $c \in \mathbb{R}$.

$\{x : cf(x) > \alpha\} = \{x : f(x) > \frac{\alpha}{c}, c > 0\}$ ✓ $= \{x : f(x) < \frac{\alpha}{c}, c < 0\}$ ✓

if c is 0, cf is constant.

$E = \{x : f(x) + g(x) > \alpha\}$. $\quad \mathbb{Q} = \{r_n\}$.

$x \in E \Rightarrow f(x) > \alpha - g(x)$.

$$f(x) > r_n > \alpha - g(x)$$

$x \in \{x : f(x) > r_n\}$

$\cup \{x : g(x) > \alpha - r_n\}$

① P.T.
f is m'ble $\Rightarrow f^2$ is m'ble

② fg is m'ble.

$\phi : \mathbb{R} \xrightarrow{\text{m'ble}} \{a_1, a_2, \dots, a_n\}$.

For i, $E_i = \{x \in E, : \phi(x) = a_i\}$

$\phi(x) = \sum a_i \chi_{E_i}$ (simple f^n).

Assume $\phi(x) \geq 0 \; \forall \; x \in \mathbb{R}$

$$\int \phi(x)\,dx := \sum a_i \, m(E_i)$$

* For non-negative m'ble f^n_s,

$$\int f = \sup_{0 \leq \phi \leq f} \int \phi$$