HAND WRITTEN NOTES

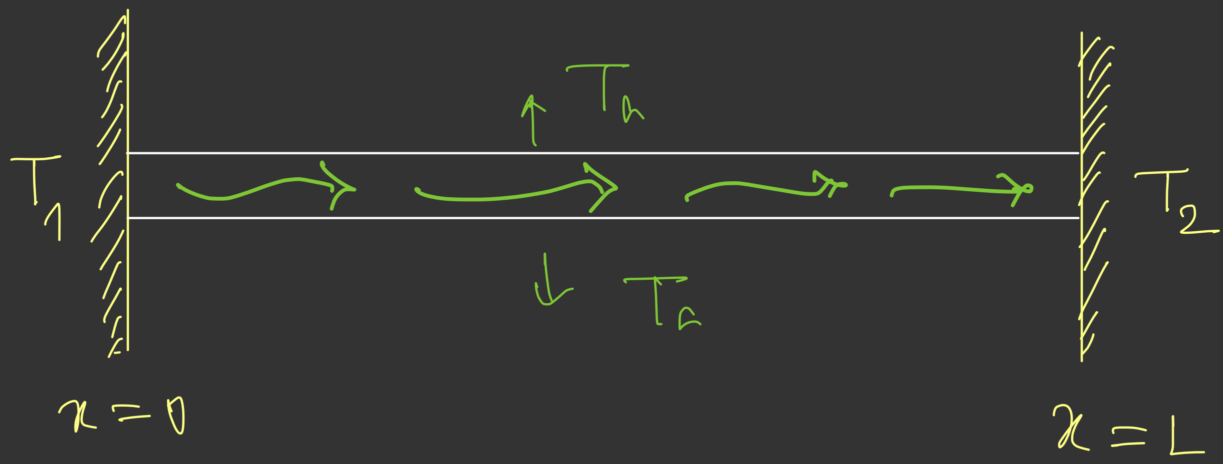
MA322

SCIENTIFIC COMPUTING

- Boundary Value Problems (BVP)
  - Finite Difference Method
  - Shooting Method
- Partial Differential Equations (PDEs)
  - Classifications
  - Numerical methods for solving parabolic type eqns
    - FTCS
    - BTCS
    - CN
    - ADI

Boundary value problem (BVP)

A heat transfer problem in 1D



Steady state problem

$$(\text{BVP}) - \begin{cases} \dfrac{d^2T}{dx^2} + h'(T_a - T) = 0 \quad \text{--- (1)} \\ \\ T(x=0) = T_1 \quad \& \quad T(x=L) = T_2 \quad \text{--- (2)} \end{cases}$$

h': a heat transfer coefficient (m^{-2})

Method 1 : Shooting Method

1. Shooting method solves a BVP by converting it into an IVP.

2. It shoots from that initial cond^{ns} to match the other boundary cond^{n}

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0$$

Define $\frac{dT}{dx} = z$

$$\frac{d^2T}{dx^2} = \frac{dz}{dx}$$

$$\Rightarrow -h'(T_a - T) = \frac{dz}{dx}$$

$$\Longrightarrow \boxed{\begin{aligned} \frac{dT}{dx} &= z \\ \frac{dz}{dx} &= -h'(T_a - T) \end{aligned}}$$

unknowns (dependent variables) are T & z.

assume

$$\boxed{\begin{aligned} T(x=0) &= T_1 \\ z(x=0) &= z_1^{(1)} \end{aligned}} \text{ — (ICs)}$$

$\frac{dT}{dx}$

IVP

Solve this resultant IVP.

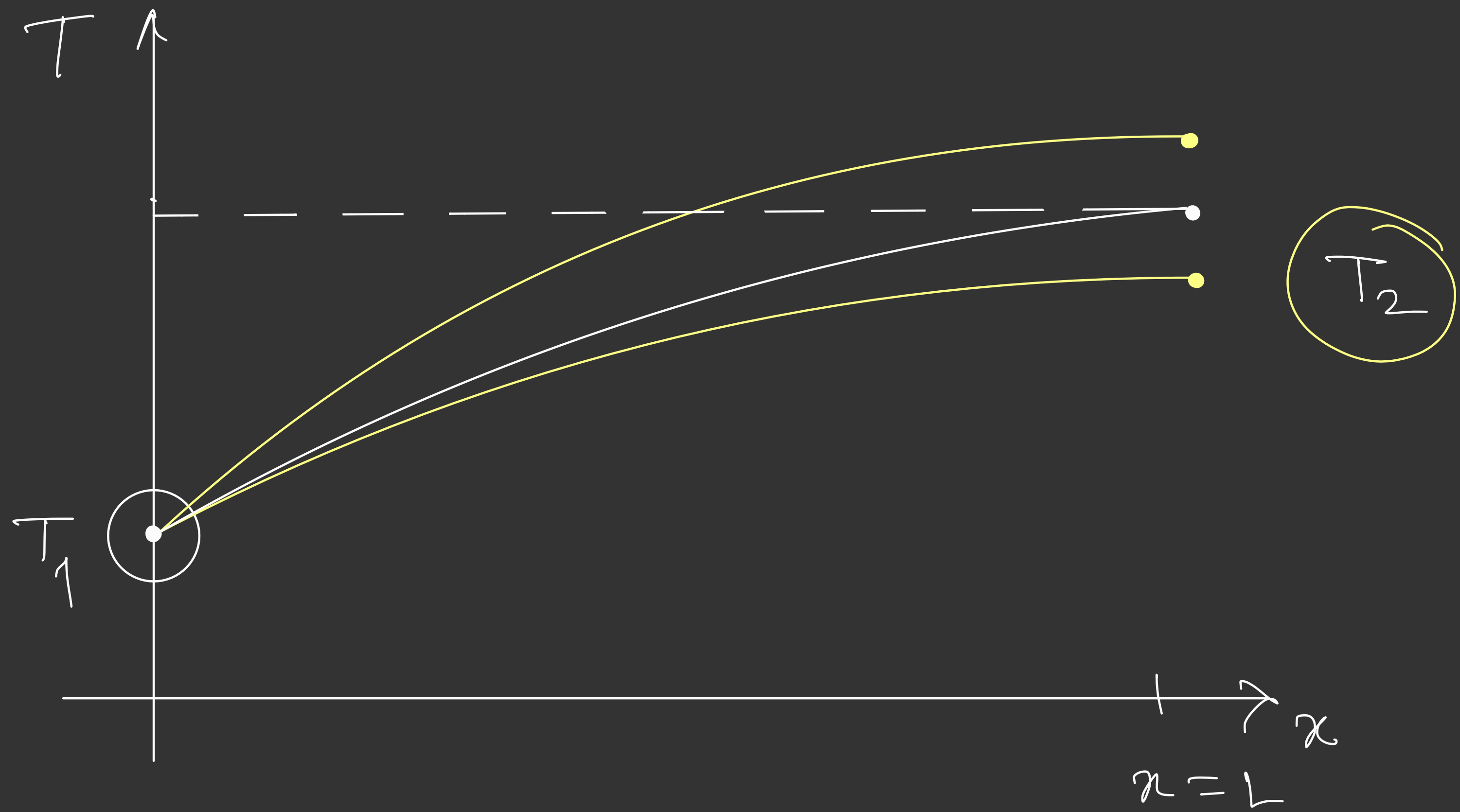
Let us denote the soln. $T^{(1)}$

Compute $T^{(1)}(x = L)$ & check if this equals to T_2

If $T^{(1)}(x = L) \neq T_2$, then take a new assumption

$$z(x=0) = z_1^{(2)}$$

Check if $T^{(2)}(x = L) = T_2$



T
T_1
T_2
x
x = L

Shooting Method (Linear eq^n)

(BVP) — $\begin{cases} \dfrac{d^2y}{dx^2} + p(x)y = 0 \quad \text{in} \quad a \leq x \leq b \\ y(a) = y_a, \quad y(b) = y_b. \end{cases}$

(IVP1) — $\left\{ \begin{array}{l} \left.\begin{array}{l} \dfrac{dy}{dx} = z \\ \dfrac{dz}{dx} = -p(x)y \end{array}\right\} \text{— (eqs)} \\ \left.\begin{array}{l} y(a) = y_a \\ \text{assume } \cancel{z(a) = z_a^{(1)}} \end{array}\right\} \text{— (IC1)} \end{array} \right.$

solve the IVP

$z(a) = z_a$

Obtain a sol^n by solving (IVP1) and compute

$$y(b) = y_b^{(1)}$$

$$(IVP2) - \begin{cases} \dfrac{dy}{dx} = z \\ \dfrac{dz}{dx} = -p(x)y \end{cases} \text{ — (ODE)}$$

$$\begin{cases} y(a) = y_a \\ \text{assume } z(a) = z_a^{(2)} \end{cases} \text{ — (IC2)}$$

Obtain a soln. by solving (IVP2) and compute

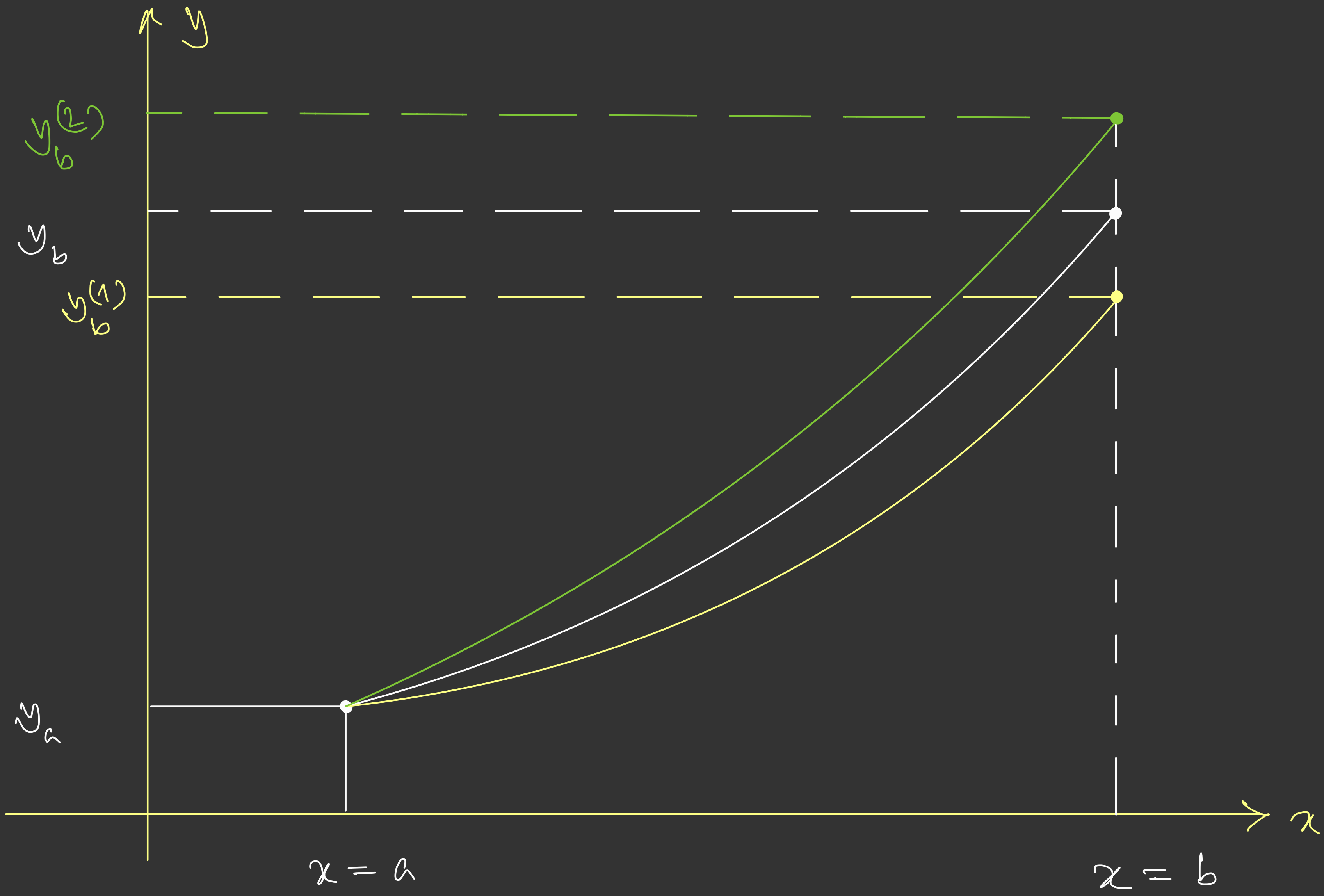
$$y(b) = y_b^{(2)}$$

Exploit the linearity of the problem.

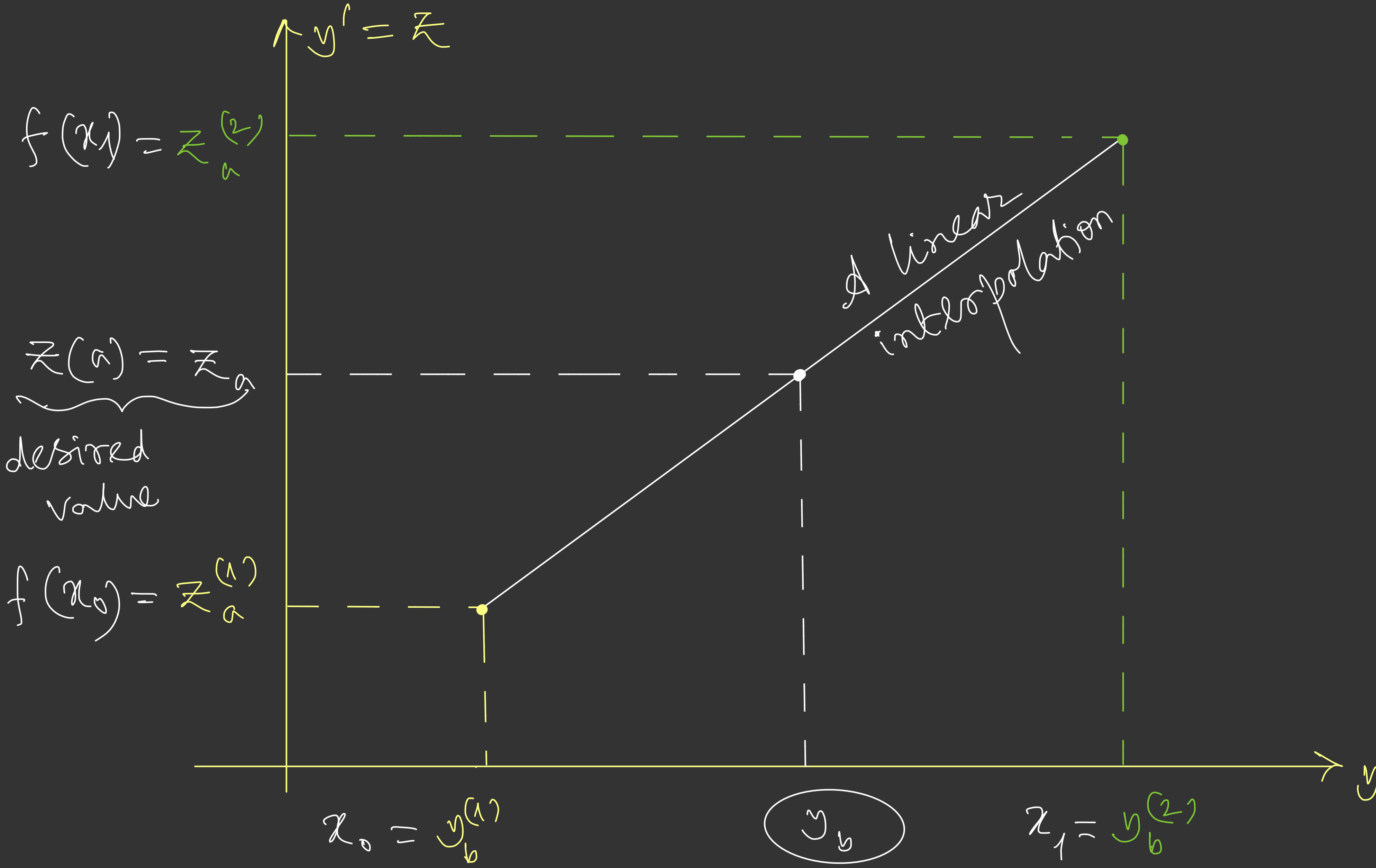
Since, the original ODE is linear

$$\left.\begin{array}{ll} z_a^{(1)}, & y_b^{(1)} \\ \& \quad z_a^{(2)}, & y_b^{(2)} \end{array}\right\} \text{ are linearly related}$$

$\boxed{\text{Use Linear inter-polation}}$



$$f_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$



$$\underline{z(x=a)} = z_a^{(1)} + \frac{z_a^{(2)} - z_a^{(1)}}{y_b^{(2)} - y_b^{(1)}} \left(y_b - y_b^{(1)}\right)$$

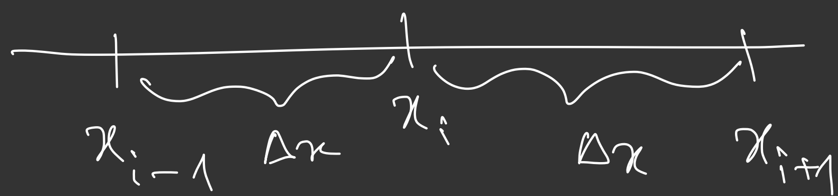
Method 2 : Finite difference

$$\frac{d^2T}{dx^2} + h'(T_a - T) = 0 \qquad \text{——— (1)}$$

$$\left.\frac{d^2T}{dx^2}\right|_{x_i} = \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} \rightarrow \text{central difference}$$

$$x_{i+1} = x_i + \Delta x$$

$$x_{i-1} = x_i - \Delta x$$



at x_i

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2} + h'(T_a - T_i) = 0$$

$$\Rightarrow \quad -T_{i-1} + (2 + h'\Delta x^2)\, T_i - T_{i+1} = h'\Delta x^2 T_a \qquad \text{--- (2)}$$

(2) is valid at all $i = 1, 2, \ldots, n$

Converted ODE (1) to a system of linear algebraic eqns (2) — tridiagonal system.

Solve using TDMA to get the desired soln.

Finite difference approximation of BVP.

Eigen value problems (SL problem)

$$\frac{d^2y}{dx^2} + \lambda y = 0$$

$\uparrow$ unknown and is called an eigenvalue.

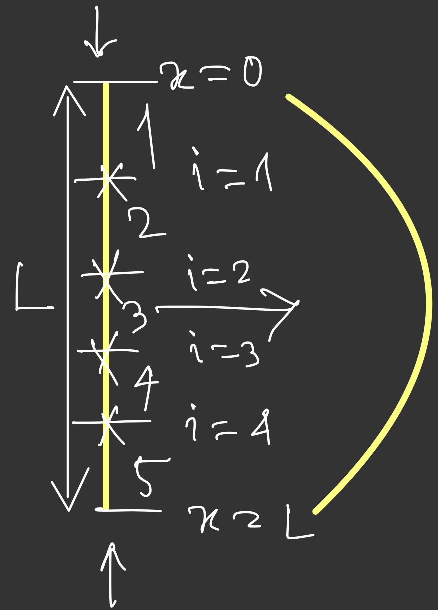
Euler buckling problem

Slender column subjected to axial load P.

$$\left.\begin{aligned} \frac{d^2y}{dx^2} &= \frac{M}{EI} \\ M &= -Py \end{aligned}\right\} \quad \frac{d^2y}{dx^2} = \frac{-Py}{EI}$$

$$p^2 = \frac{P}{EI}$$

$$\boxed{\begin{aligned} &\Rightarrow \frac{d^2y}{dx^2} + p^2 y = 0 \quad (*) \\ &y(0) = 0, \quad y(L) = 0 \end{aligned}}$$



$$p = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots$$

$$P = \frac{\pi^2 E I}{L^2} \longrightarrow \text{Euler's formula.}$$

Polynomial method

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + p^2 y_i = 0$$

$$\Rightarrow \underline{y_{i-1} - (2 - h^2 p^2) y_i + y_{i+1} = 0} \quad \text{—— } (**)$$

$$\underbrace{\begin{bmatrix} 2-h^2p^2 & -1 & 0 & 0 \\ -1 & 2-h^2p^2 & -1 & 0 \\ 0 & -1 & 2-h^2p^2 & -1 \\ 0 & 0 & -1 & 2-h^2p^2 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\det A = 0 \implies$ non-trivial sol^n.

$\downarrow$

a polynomial eq^n.

Power method

1. It computes the largest eigenvalue of a system of the form $\underline{Ax = \lambda x}$

2. Reconstruct another eigenvalue problem from the original one so that the largest eigenvalue is eliminated. Compute the largest eigenvalue of the resultant eigenvalue problem.

———————— X ————————

Conic Section

General Cartesian form of a conic section is

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

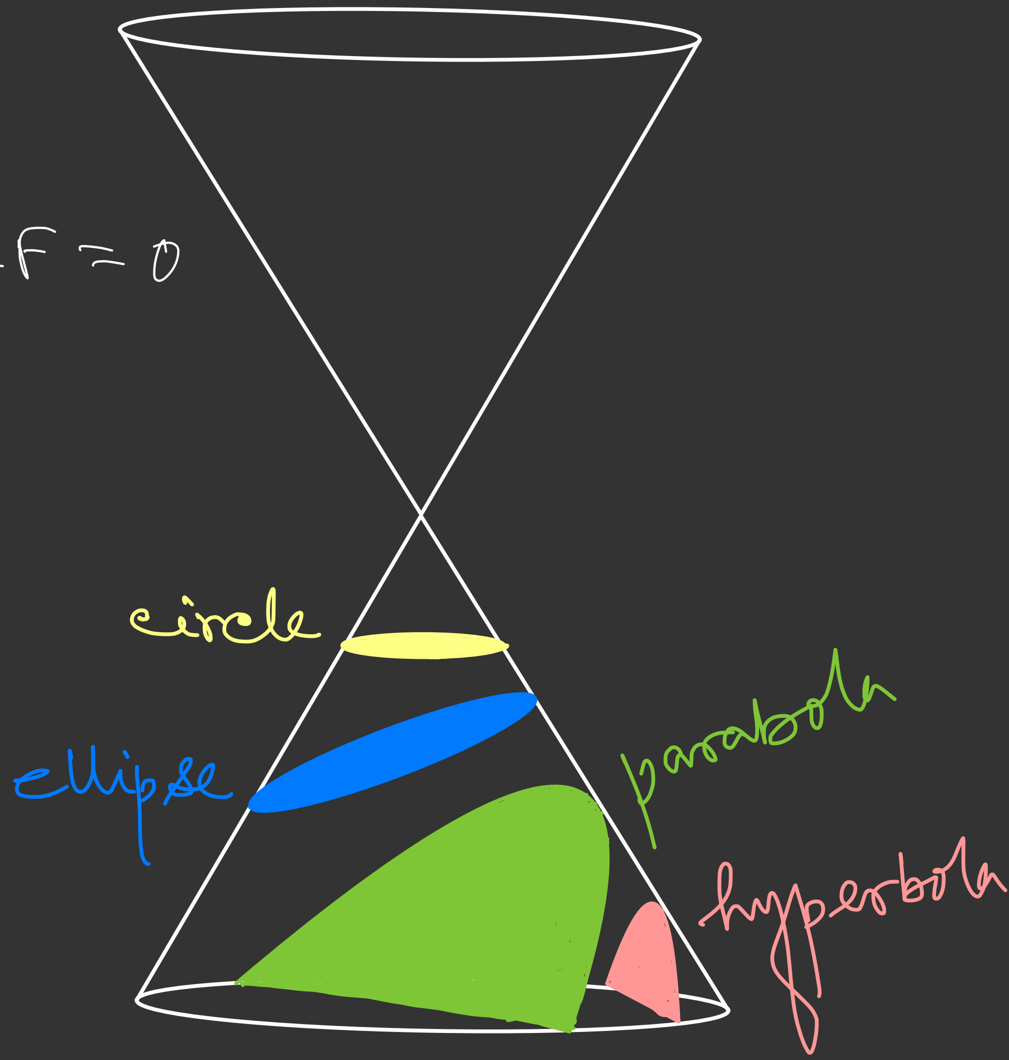
Classification

(i) $B^2 - 4AC = 0 \longrightarrow$ parabola

(ii) $B^2 - 4AC > 0 \longrightarrow$ hyperbola

(iii) $B^2 - 4AC < 0 \longrightarrow$ ellipse

If $A = C$ & $B = 0$ (circle)



Not shown is the other half of the hyperbola.

Partial differential eqns of 2^{nd} order

Consider $u = u(x, y; t)$ is a twice continuously differentiable fn. of $(2+1)$ independent variables.

A generic form of 2^{nd} order linear partial differential eqn (PDE) in two independent variables $(x_1 \;\&\; x_2)$ can be written as

$$\underbrace{\boxed{A u_{x_1 x_1} + B u_{x_1 x_2} + C u_{x_2 x_2}}}_{\text{principal part}} + D u_{x_1} + E u_{x_2} + F u + G = 0 \qquad (1)$$

where A, B, C, D, E, F, G are $\underbrace{\text{fns}}_{\text{real valued}}$ of $x_1 \,\&\, x_2$ in general.

$$u_{x_1 x_1} = \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_1}\right) = \frac{\partial^2 u}{\partial x_1^2}.$$

Classification of PDEs (2nd order linear in two independent variables)

Compute the discreminent

$$B^2 - 4AC$$

> 0 : hyperbolic

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad \text{—— (a)}$$

(Wave eqn.)

$= 0$: parabolic

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} \quad \text{—— (b)}$$

(heat eqn. or diffusion eqn.)

< 0 : elliptic

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{—— (c)}$$

Laplace eqn.

(steady state heat eqn. or steady state vibrations of a membrane)

$$A = \frac{1}{c^2}, \quad B = 0, \quad C = -1$$

$$B^2 - 4AC = 0^2 - 4 \cdot \frac{1}{c^2}(-1) = 4/c^2 > 0$$

$c \rightarrow$ wave speed.

$$A = \alpha, \quad B = 0, \quad C = 0$$

$$B^2 - 4AC = 0^2 - 4 \cdot \alpha \cdot 0 = 0$$

$$A = C = 1, \quad B = 0$$

$$B^2 - 4AC = 0^2 - 4 \cdot 1 \cdot 1 = -4 < 0$$

Elliptic Eqn. (Laplace eqn.)

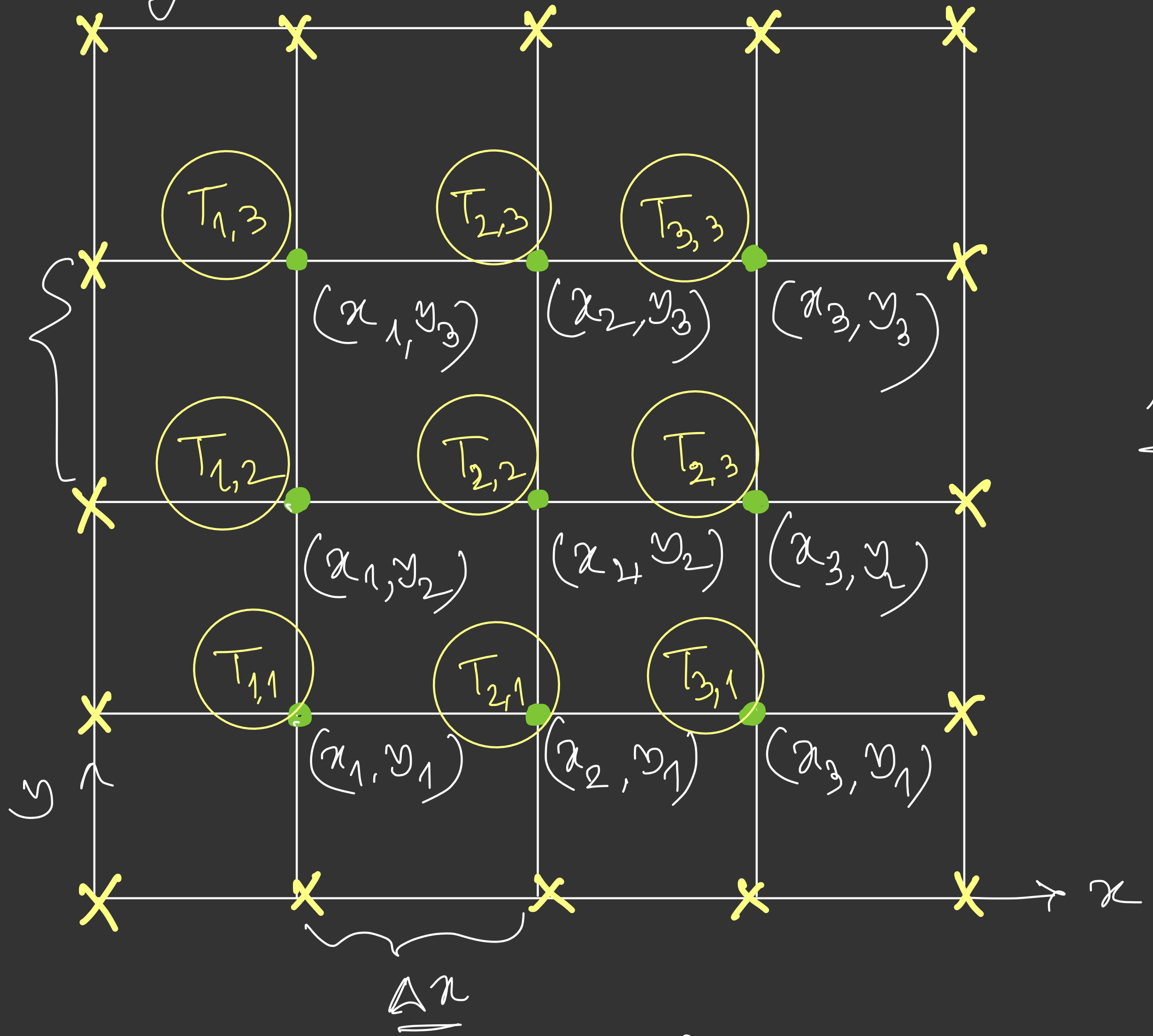
All Laplace eqns are elliptic, but all elliptic eqns are not Laplace.

Laplace → $\nabla^2 u = 0$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Poisson → $\nabla^2 u = f$

Solve Laplace eqn. with the focus on steady state heat conduction.

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (1)$$



$\boxed{\Delta x = \Delta y}$

$\Delta x \neq \Delta y$

T is given at all four boundaries.

$$T = T(x, y)$$

at $\left.\dfrac{\partial^2 T}{\partial x^2}\right|_{(x_i, y_i)} = \dfrac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$ —— (2a)

$\dfrac{d^2 y}{dx^2} = \dfrac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$

$x_{i+1} - x_i = h$

$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{(x_i, y_i)} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} \quad \text{—— (2b)}$$

Finite difference (FD) approximation of eqn. (1) at (x_i, y_i)

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\cancel{\Delta x^2}} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\cancel{\Delta y^2}} = \cancel{0}\ f_i$$

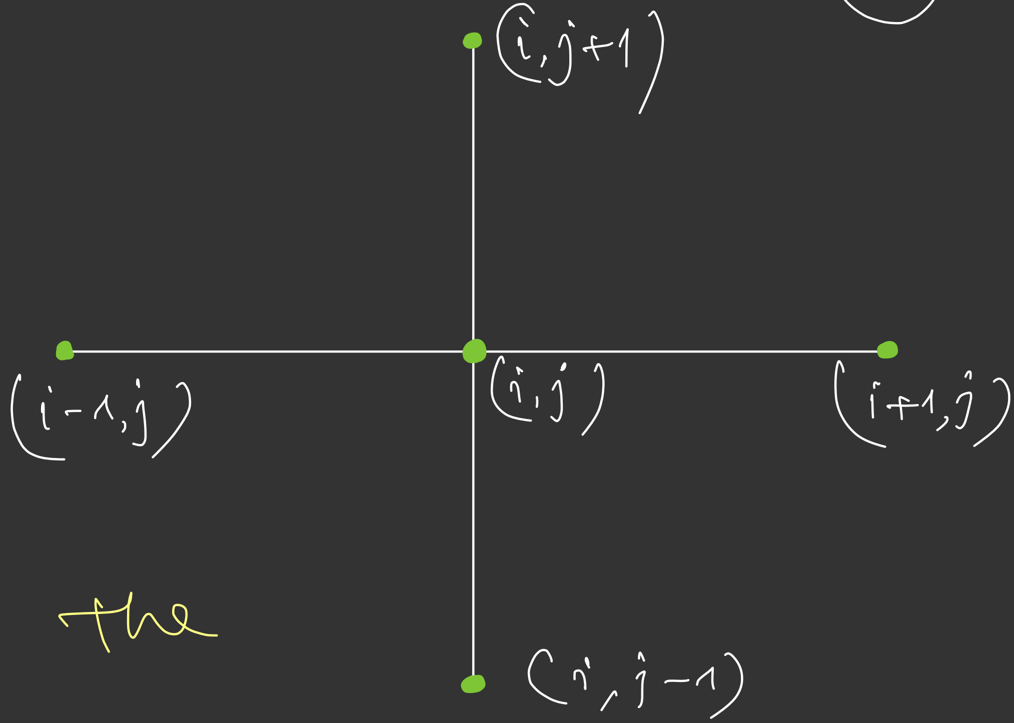
We assumed that $\Delta x = \Delta y$.

$$\left(T_{i+1,j} - 2T_{i,j} + T_{i-1,j}\right) + \left(T_{i,j+1} - 2T_{i,j} + T_{i,j-1}\right) = \cancel{0}\ \Delta x^2 f_i$$

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = \cancel{0}\ \Delta x^2 f_i \qquad (3)$$

Laplacian difference eqn.

5 point stencil formula



Eqn. holds at all the interior points.

The resultant system of linear algebraic eqns can be written in the form of a banded matrix eqn.

Solve this system using a suitable linear algebraic system solver.

If $\Delta x \neq \Delta y$

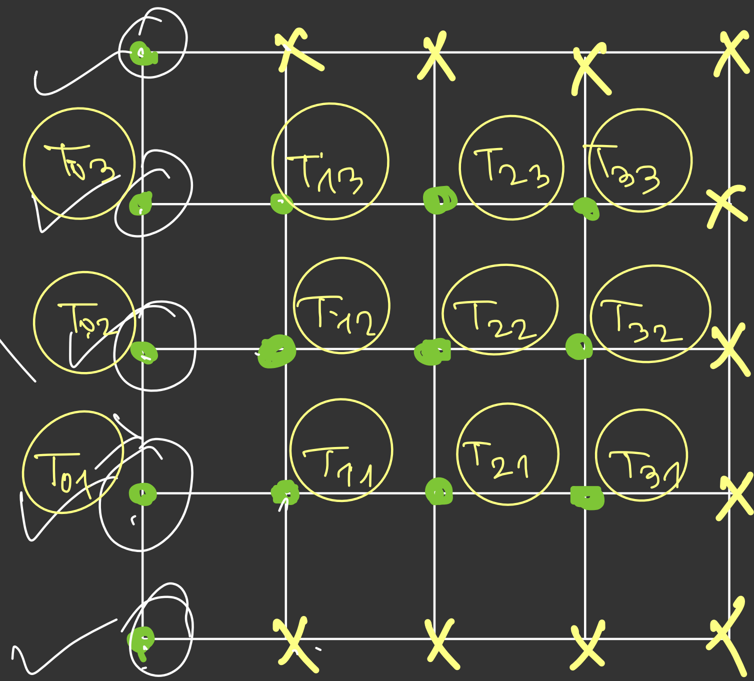
$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\underbrace{\Delta x^2}_{h}} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\underbrace{\Delta y^2}_{k}} = 0$$

$$\frac{1}{h^2}\left(T_{i+1,j} + T_{i-1,j}\right) + \frac{1}{k^2}\left(T_{i,j+1} + T_{i,j-1}\right) - 2\left(\frac{1}{h^2} + \frac{1}{k^2}\right)T_{i,j} = 0 \qquad (*)$$

Solve the system (*).

Derivative of the temp. is given @ the boundary (Neumann BC)

$$\boxed{\frac{\partial T}{\partial x} = 0}$$ at some boundary



$$\left.\frac{\partial T}{\partial x}\right|_{(i,j)} = \frac{T_{i+1,j} - T_{i,j}}{\Delta x} = 0$$

$$\Rightarrow T_{i+1,j} = T_{i,j}$$

Parabolic Eqn.

Recap

1. Classifications of PDEs (2nd order linear in two independent variables)

2. Numerical solns of elliptic PDEs (e.g., Laplace & Poisson eqns).

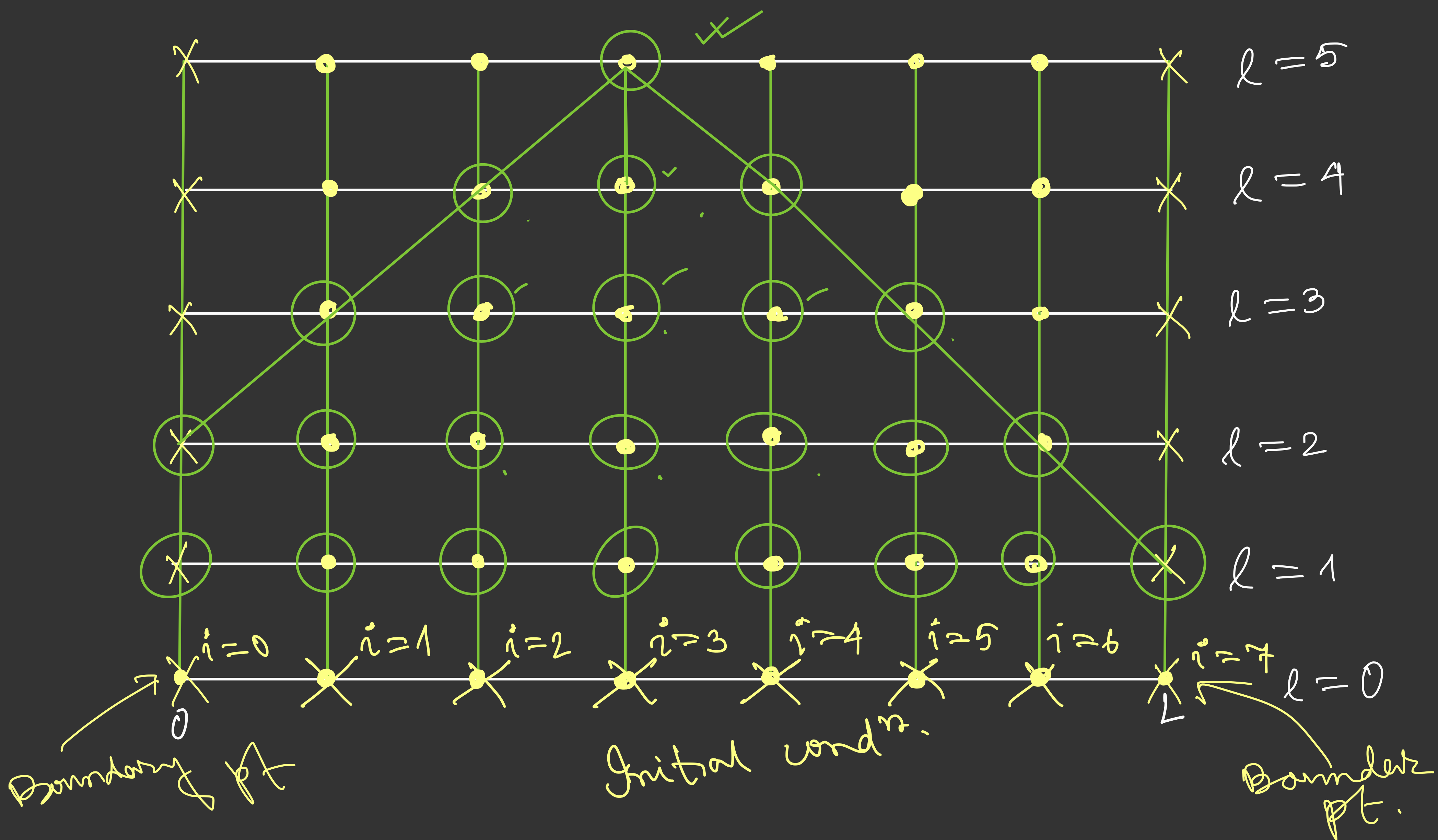
In this lecture, we will focus on numerical soln. of (1+1) heat eqn.:

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \longleftarrow (1)$$

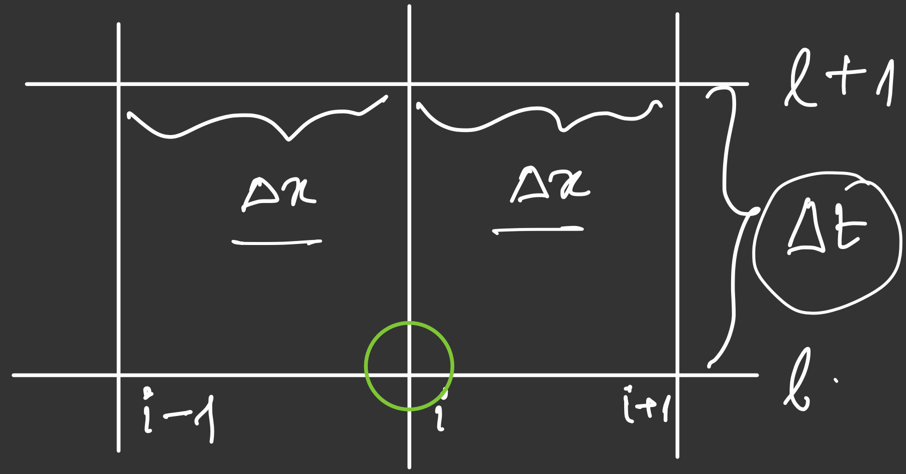
$$0 \leq x \leq L, \quad t > 0$$

Three different methods.

(1) An explicit method (Forward time central space or FTCS)



$$\left.\frac{\partial T}{\partial t}\right|_{(i,\ell)} = \frac{T_i^{\ell+1} - T_i^{\ell}}{\Delta t}$$



$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{(i,\ell)} = \frac{T_{i+1}^{\ell} - 2T_i^{\ell} + T_{i-1}^{\ell}}{\Delta x^2}$$

Differential eqn. (1) can be approximated as

$$\frac{T_i^{\ell+1} - T_i^{\ell}}{\Delta t} = K \, \frac{T_{i+1}^{\ell} - 2T_i^{\ell} + T_{i-1}^{\ell}}{\Delta x^2}$$

$$T_i^{\ell+1} = T_i^{\ell} + K\frac{\Delta t}{\Delta x^2}\left(T_{i+1}^{\ell} - 2T_i^{\ell} + T_{i-1}^{\ell}\right) \qquad (2)$$

Consistency

$$\left.\frac{\partial T}{\partial t}\right|_{(i,l)} = K \left.\frac{\partial^2 T}{\partial x^2}\right|_{(i,l)}$$

$$\frac{T_i^{l+1} - T_i^{l}}{\Delta t} = K \, \frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2}$$

$$T_i^{l+1} = T(x_i, t_{l+1}) \qquad t_{l+1} = t_l + \Delta t$$

$$= T(x_i, \underline{t_l + \Delta t})$$

$$= \underbrace{T(x_i, t_l)}_{T_i^l} + \Delta t \left.\frac{\partial T}{\partial t}\right|_{(x_i, t_l)} + \frac{\Delta t^2}{2!} \left.\frac{\partial^2 T}{\partial t^2}\right|_{(x_i, t_l)} + \cdots$$

$$\Rightarrow \frac{T_i^{l+1} - T_i^{l}}{\Delta t} = \left.\frac{\partial T}{\partial t}\right|_{(x_i, t_l)} + \frac{\Delta t}{2!} \left.\frac{\partial^2 T}{\partial t^2}\right|_{(x_i, t_l)} + \cdots \quad (2)$$

$$
\begin{aligned}
T_{i\pm1}^{l} &= T\left(x_{i\pm1},\, t_l\right) \\
&= T\left(x_i \pm \Delta x,\, t_l\right) \\
&= \underline{T(x_i, t_l)} \pm \Delta x \left.\frac{\partial T}{\partial x}\right|_{(x_i, t_l)} \\
&\quad + \frac{(\pm \Delta x)^2}{2!} \left.\frac{\partial^2 T}{\partial x^2}\right|_{(x_i, t_l)} \\
&\quad + \frac{(\pm \Delta x)^3}{3!} \left.\frac{\partial^3 T}{\partial t^3}\right|_{(x_i, t_l)} + \dots
\end{aligned}
$$

$$
\begin{aligned}
T_{i+1}^{l} + T_{i-1}^{l} &= 2\underbrace{T(x_i, t_l)}_{T_i^{l}} + \cancel{2}\frac{\Delta x^2}{\cancel{2!}} \left.\frac{\partial^2 T}{\partial x^2}\right|_{(x_i, t_l)} \\
&\quad + O\left(\Delta x^4\right)
\end{aligned}
$$

$$\Rightarrow \quad \frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2} = \left.\frac{\partial^2 T}{\partial x^2}\right|_{(x_i, t_l)} + O(\Delta x^2) \quad \text{— } (3)$$

$$(2) - k \times (3)$$

$$\frac{T_i^{l+1} - T_i^{l}}{\Delta t} - k\,\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2}$$

$$= \left.\frac{\partial T}{\partial t}\right|_{(x_i, t_l)} - k \left.\frac{\partial^2 T}{\partial x^2}\right|_{(x_i, t_l)} + O(\Delta t, \Delta x^2)$$

$$= \left[\frac{\partial T}{\partial t} - k\,\frac{\partial^2 T}{\partial x^2}\right]_{(x_i, t_l)}$$

$+ \underline{O(\Delta t, \Delta x^2)}$

FTCS method is $O(\Delta t, \Delta x^2)$ consistent

Convergence

A method is called convergent if the finite difference soln. approaches the true soln. as $\Delta t \to 0$, $\Delta x \to 0$.

Stability

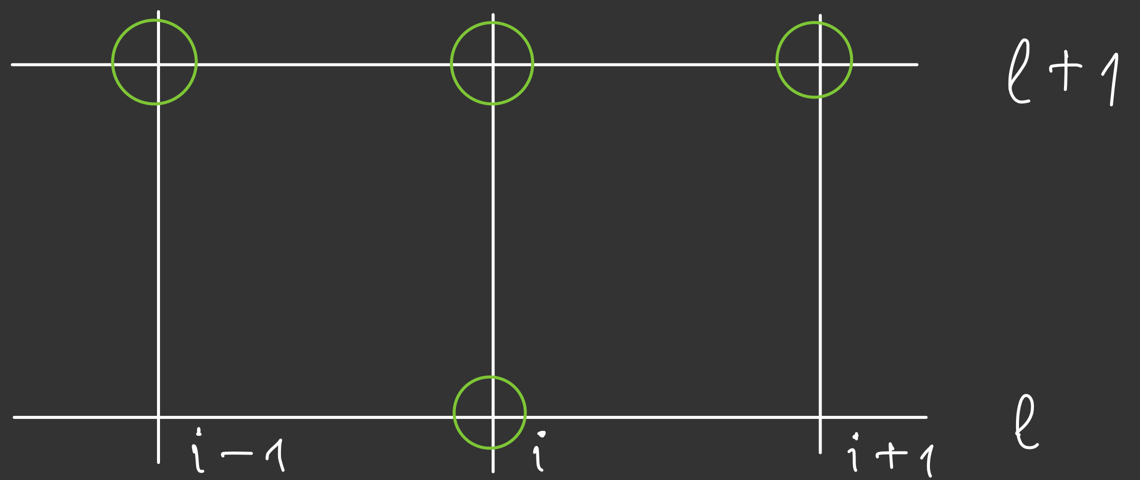
The errors at any stage do not amplify, rather they are attenuated as the computation progress.

FTCS is stable if

$$\boxed{\Delta t \leq \frac{1}{2} \frac{\Delta x^2}{K}}$$

Von - Neumann stability to check this stability criteria.

(2) An implicit method
(Backward time centered space or BTCS)



$$\left.\frac{\partial T}{\partial t}\right|_{(x_i, t_l)} = \frac{T_i^{l+1} - T_i^{l}}{\Delta x}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{(x_i, t_l)} = \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2}$$

$$\left.\frac{\partial T}{\partial t}\right|_{(x_i, t_l)} = K \left.\frac{\partial^2 T}{\partial x^2}\right|_{(x_i, t_l)}$$

$$\frac{T_i^{l+1} - T_i^{l}}{\Delta t} = K \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2}$$

$$T_i^{l+1} = T_i^{l} + \underbrace{K \frac{\Delta t}{\Delta x^2}}_{\lambda} \left(T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1} \right)$$

$$\Rightarrow \quad -\lambda T_{i-1}^{l+1} + (1+2\lambda) T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^{l} \quad \text{— } (4)$$

Apply to all the interior points

Resultant will be a system of linear algebraic eqns. $\longrightarrow$ tridiagonal system.

Solve using TDMA to obtain the desired soln.

BTCS is $O(\Delta t, \Delta x^2)$ consistent.

BTCS is convergent.

BTCS is unconditionally stable.

Crank–Nicolson Method

$$\frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial x^2} \quad \text{——— (1)}$$

(1) An explicit or forward time centered space (FTCS) method

$$T_i^{l+1} = T_i^l + \lambda\left(T_{i+1}^l - 2T_i^l + T_{i-1}^l\right) \quad \text{——— (2a)}$$

where $\lambda = K\frac{\Delta t}{\Delta x^2}$

Stability: Conditionally stable : stability condn. $\lambda \le 1/2$

Consistency: 1^{st} order in time, 2^{nd} order in space, i.e., $O(\Delta t, \Delta x^2)$

Convergence: Stability + consistency $\Rightarrow$ convergence.

② An implicit or backward time centered space (BTCS) method

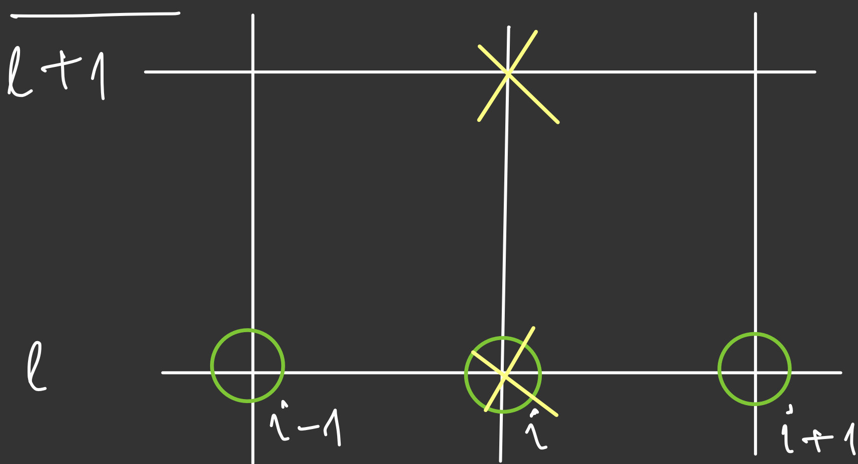
$$-\lambda T_{i-1}^{l+1} + (1+2\lambda) T_i^{l+1} - \lambda T_{i+1}^{l+1} = T_i^{l} \quad \text{——} \quad (2b)$$

Consistency Same as FTCS

Stability Unconditionally stable

Convergence Consistency + stability $\Rightarrow$ convergence

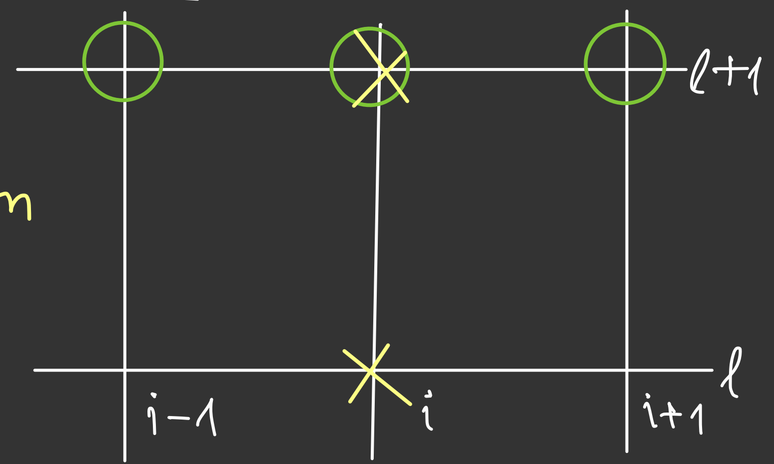
FTCS



X time discretization

O space discretization

BTCS



(3) Another implicit scheme (Crank-Nicolson)

$$-\lambda T_{i-1}^{l+1} + 2(1+\lambda)\, T_i^{l+1} - \lambda T_{i+1}^{l+1} = \lambda T_{i-1}^{l} + 2(1-\lambda)\, T_i^{l} + \lambda T_{i+1}^{l} \qquad (2c)$$

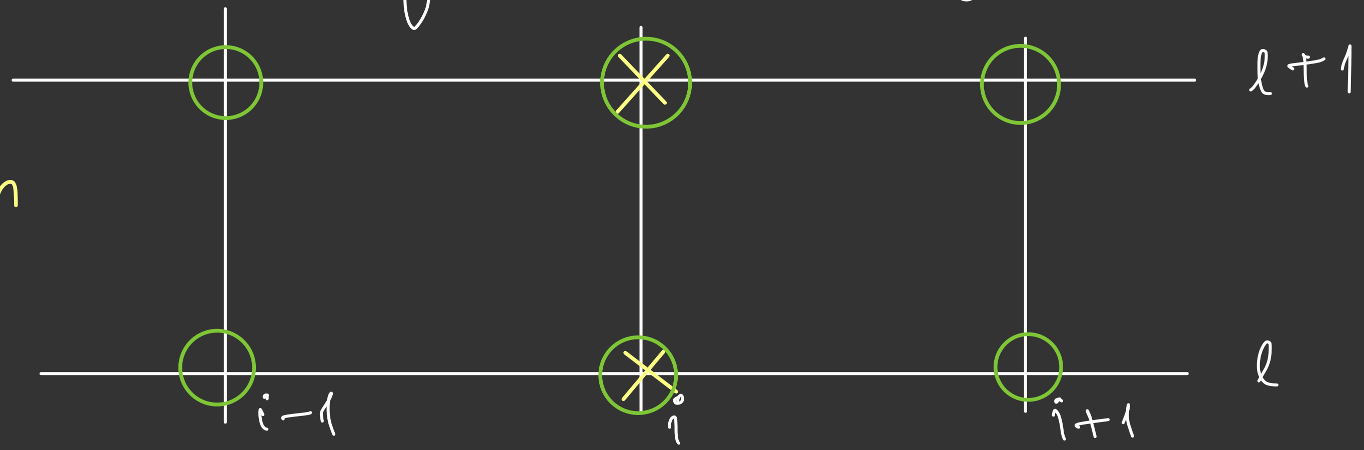
Consistency $\quad O(\Delta t^2, \Delta x^2)$

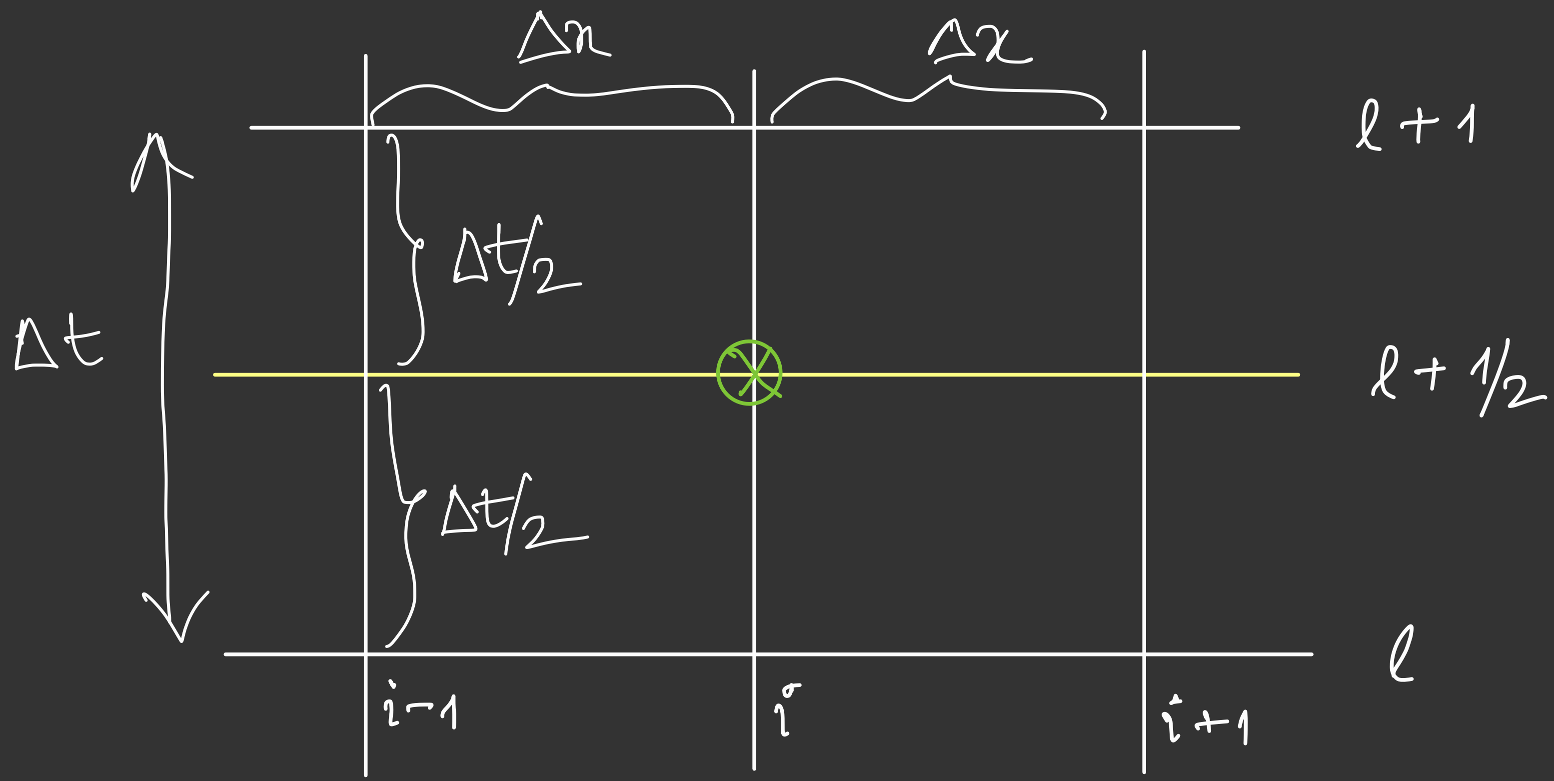
Stability $\quad$ Unconditionally stable

Convergence $\quad$ Stability + consistency $\Rightarrow$ convergent

X time discretization

O space discretization





$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \quad \text{——} \quad (1)$$

$$\left.\frac{dy}{dx}\right|_{x_i} = \frac{y_{i+1} - y_i}{\Delta x} + O(\Delta x)$$ Forward difference

$$\left.\frac{dy}{dx}\right|_{x_i} = \frac{y_i - y_{i-1}}{\Delta x} + O(\Delta x)$$ Backward difference

$$\left.\frac{dy}{dx}\right|_{x_i} = \frac{y_{i+1} - y_{i-1}}{2\Delta x} + O(\Delta x^2)$$

central difference

$$\left.\frac{\partial T}{\partial t}\right|_{(i,\, l+1/2)} = \frac{T_i^{l+1/2+1/2} - T_i^{l+1/2-1/2}}{\left(\frac{\Delta t}{2} + \frac{\Delta t}{2}\right)} + O\left(\cancel{\left(\frac{\Delta t}{2}\right)^2}\right) \quad \Delta t^2$$

$$= \frac{T_i^{l+1} - T_i^{l}}{\Delta t} + O(\Delta t^2) \qquad \text{(2a)}$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{(i,\, l+1/2)} = \frac{1}{2}\left(\underbrace{\left.\frac{\partial^2 T}{\partial x^2}\right|_{(i,l)}} + \left.\frac{\partial^2 T}{\partial x^2}\right|_{(i,\, l+1)} \right)$$

$$= \frac{1}{2}\left[\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2} + O(\Delta x^2) \right.$$

$$\left. + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2} + O(\Delta x^2) \right]$$

$$= \frac{1}{2}\left[\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x2} \right]$$

$$+ O(\Delta x^2) \quad \text{—— (2b)}$$

Using (2a) & (2b) in (1)

$$\frac{T_i^{l+1} - T_i^{l}}{\Delta t} = K\frac{1}{2}\left[\frac{T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l}}{\Delta x^2} + \frac{T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{l+1}}{\Delta x^2}\right]$$

Use $\lambda = K\dfrac{\Delta t}{\Delta x^2}$ to get

$$T_i^{l+1} - T_i^{l} = \frac{1}{2}\lambda\left[T_{i+1}^{l} - 2T_i^{l} + T_{i-1}^{l} + T_{i+1}^{l+1} - 2T_i^{l+1} + T_{i-1}^{+1}\right]$$

$$\Rightarrow -\frac{1}{2}\lambda T_{i-1}^{l+1} + (1+\lambda)T_i^{l+1} - \frac{1}{2}\lambda T_{i+1}^{l+1}$$
$$= \frac{1}{2}\lambda T_{i-1}^{l} + (1-\lambda)T_i^{l} + \frac{1}{2}\lambda T_{i+1}^{l}$$

$$\Rightarrow -\lambda T_{i-1}^{l+1} + 2(1+\lambda)T_i^{l+1} - \lambda T_{i+1}^{l+1}$$
$$= \lambda T_{i-1}^{l} + 2(1-\lambda)T_i^{l} + \lambda T_{i+1}^{l} \quad \text{---} \; (3)$$

Eqn. 3 is applied at all the interior points. $\Longrightarrow$ Resultant is a system of linear algebraic eqns.

This is a tridiagonal matrix system.

Solve it using TDMA.

Consistency $\quad O(\Delta t^2, \Delta x^2)$

Stability $\quad$ Unconditionally stable.

Convergence $\quad$ Stability + consistency

$\Longrightarrow$ stable.

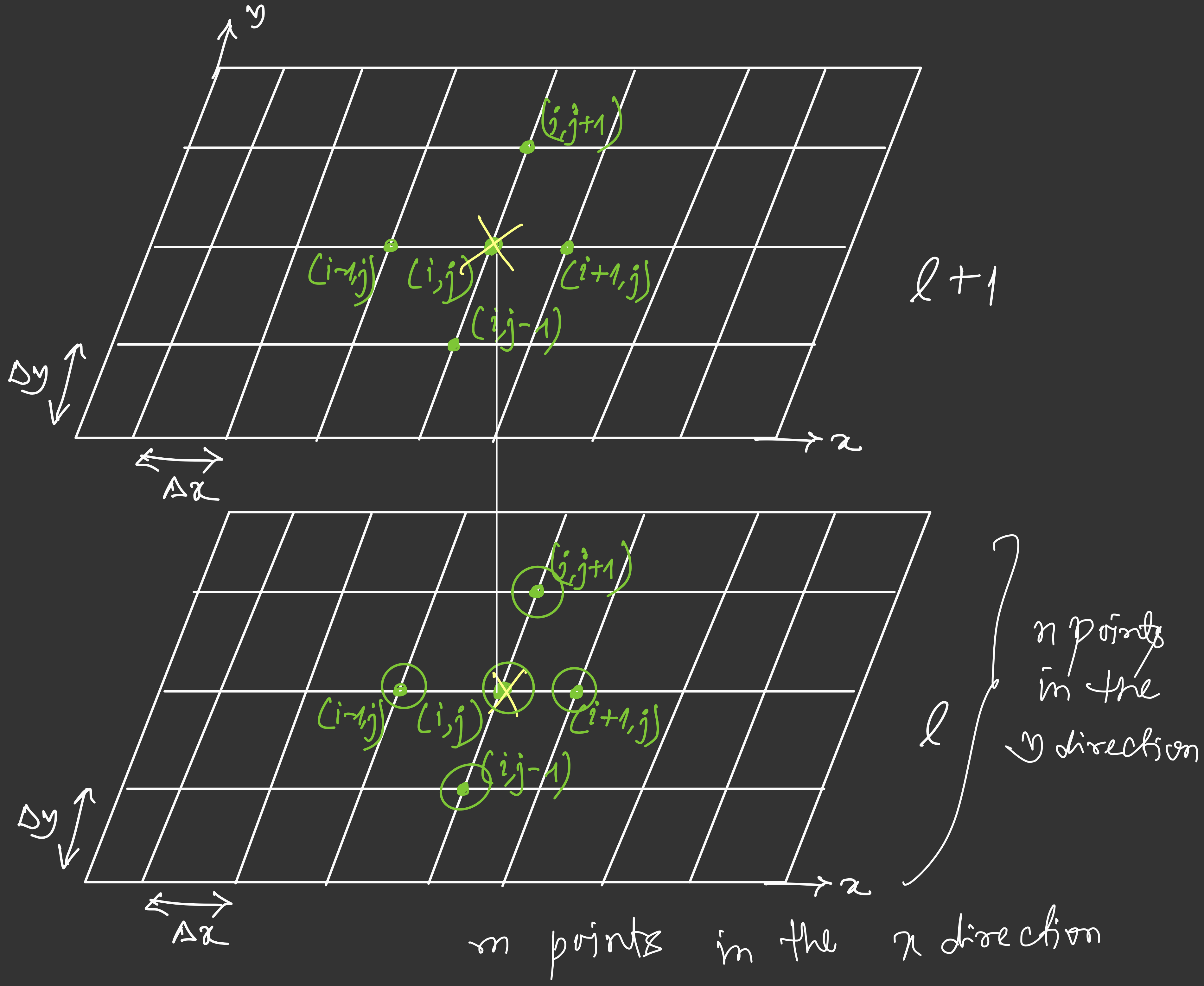
$$\frac{\partial T}{\partial t} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$$\frac{\partial T}{\partial t} = K\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \longrightarrow (1)$$

(1) Explicit method.

$$\frac{T_{i,j}^{l+1} - T_{i,j}^{l}}{\Delta t} = K\left[\frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{\Delta x^2} + \frac{T_{i,j+1}^{l} - 2T_{i,j}^{l} + T_{i,j-1}^{l}}{\Delta y^2}\right]$$

$$\Rightarrow T_{i,j}^{l+1} = T_{i,j}^{l} + K\Delta t\left[\frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{\Delta x^2} + \frac{T_{i,j+1}^{l} - 2T_{i,j}^{l} + T_{i,j-1}^{l}}{\Delta y^2}\right] \longleftarrow (2)$$


y
(i,j+1)
(i-1,j)
(i,j)
(i+1,j)
(i,j-1)
l+1
Δy
Δx
x
(i,j+1)
(i-1,j)
(i,j)
(i+1,j)
(i,j-1)
l
n points in the y direction
Δy
Δx
x
m points in the x direction

<u>Consistency</u> $O(\Delta t, \Delta x^2, \Delta y^2)$

<u>Stability</u> Conditionally stable : stability cond^{n}. is

$$\Delta t \leq \frac{1}{8} \frac{\Delta x^2 + \Delta y^2}{K}$$

If $\Delta x = \Delta y$, then

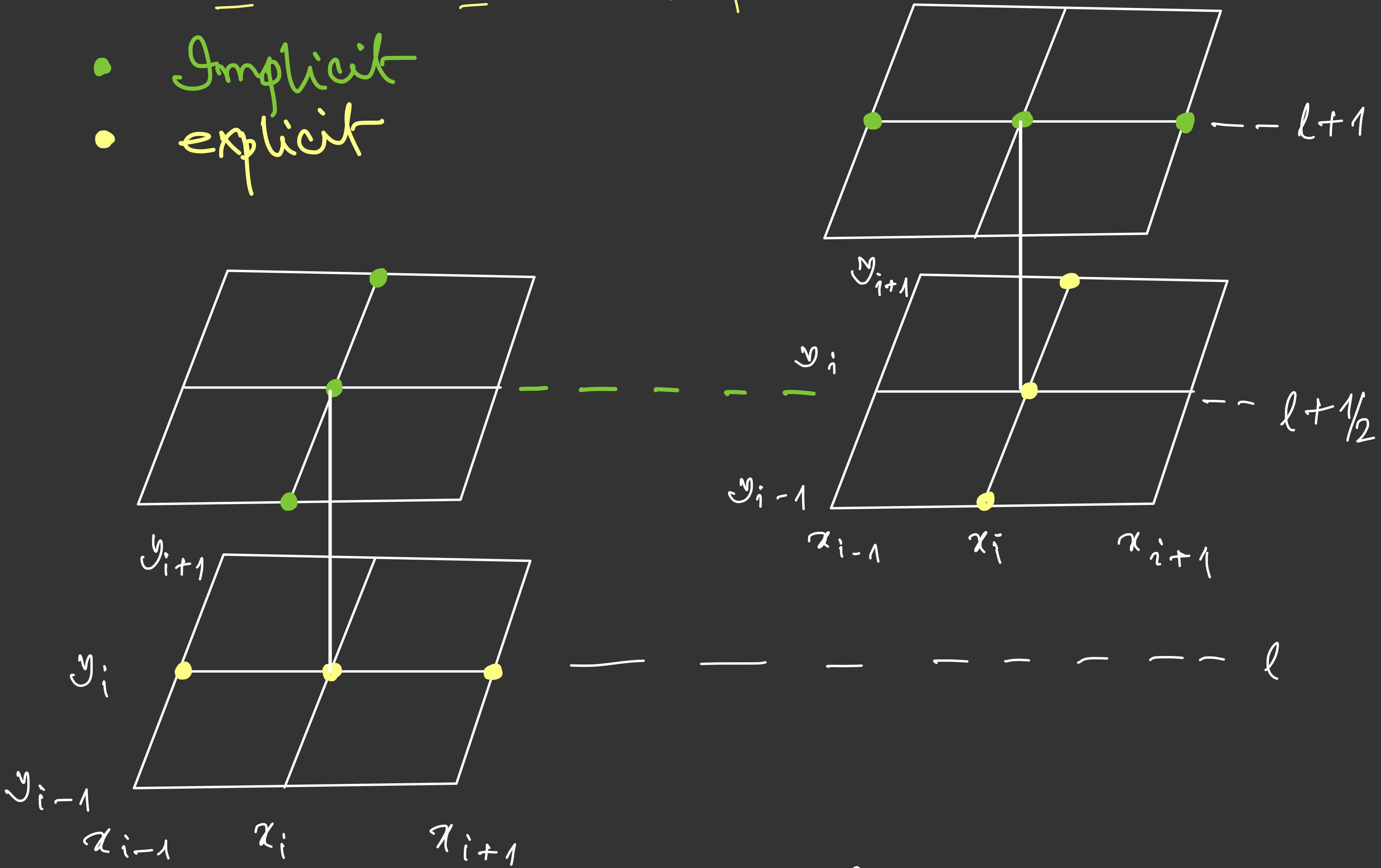
$$\Delta t \leq \frac{1}{4K} \Delta x^2$$

<u>Convergence</u> consistency + stability

$\Rightarrow$ convergence

Alternate Direction Implicit (ADI) method

- Implicit
- explicit



(First half-step) (second half-step)

$$\frac{T_{i,j}^{l+1/2} - T_{i,j}^{l}}{\Delta t/2} = k\left[\frac{T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l}}{\underline{\Delta x^2}} + \frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{\underline{\Delta y^2}}\right]$$

Let $\underline{\underline{\Delta x = \Delta y}}$

$$\Rightarrow T_{i,j}^{l+1/2} - T_{i,j}^{l} = \underbrace{\frac{\Delta t}{2} k \frac{1}{\Delta x^2}}_{\frac{\lambda}{2}}\left[T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l} + T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}\right]$$

Recall $\lambda = k\dfrac{\Delta t}{\Delta x^2}$

$$\Rightarrow \quad T_{i,j}^{l+1/2} - T_{i,j}^{l} = \frac{\lambda}{2}\Big[T_{i+1,j}^{l} - 2T_{i,j}^{l} + T_{i-1,j}^{l} + T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2} \Big]$$

$$\Rightarrow \quad -\frac{\lambda}{2} T_{i,j-1}^{l+1/2} + (1+\lambda) T_{i,j}^{l+1/2} - \frac{\lambda}{2} T_{i,j+1}^{l+1/2}$$

$$= \frac{\lambda}{2} T_{i-1,j}^{l} + (1-\lambda) T_{i,j}^{l} + \frac{\lambda}{2} T_{i+1,j}^{l}$$

$$\Rightarrow \quad -\lambda T_{i,j-1}^{l+1/2} + 2(1+\lambda) T_{i,j}^{l+1/2} - \lambda T_{i,j+1}^{l+1/2}$$

$$= \lambda T_{i-1,j}^{l} + 2(1-\lambda) T_{i,j}^{l} + \lambda T_{i+1,j}^{l} \qquad 3a$$

When 3a is written for the system then

it becomes a tridiagonal system.
(solve using TDMA)

<u>second - half step</u>

$$\frac{T_{i,j}^{l+1} - T_{i,j}^{l+1/2}}{\Delta t/2} = K\left[\frac{T_{i+1,j}^{l+1} - 2T_{i,j}^{l+1} + T_{i-1,j}^{l+1}}{\Delta x^2} + \frac{T_{i,j+1}^{l+1/2} - 2T_{i,j}^{l+1/2} + T_{i,j-1}^{l+1/2}}{\Delta y^2}\right]$$

$$\Rightarrow -\lambda T_{i-1,j}^{l+1} + 2(1+\lambda)T_{i,j}^{l+1} - \lambda T_{i+1,j}^{l+1}$$
$$= \lambda T_{i,j-1}^{l+1/2} + 2(1-\lambda)T_{i,j}^{l+1/2} + \lambda T_{i,j+1}^{l+1/2}$$

(3b)

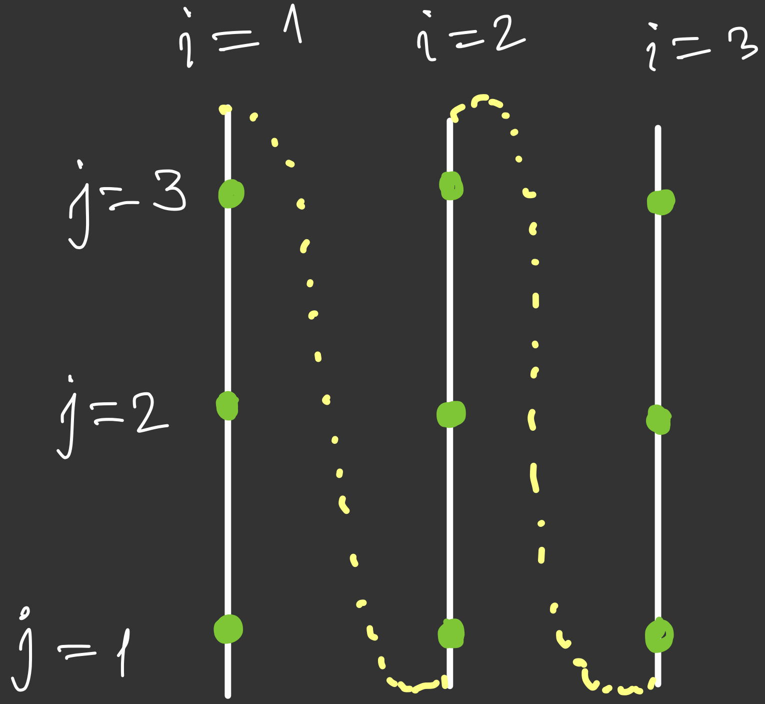
When written in the system form — a tridiagonal system

— Solve using TDMA.

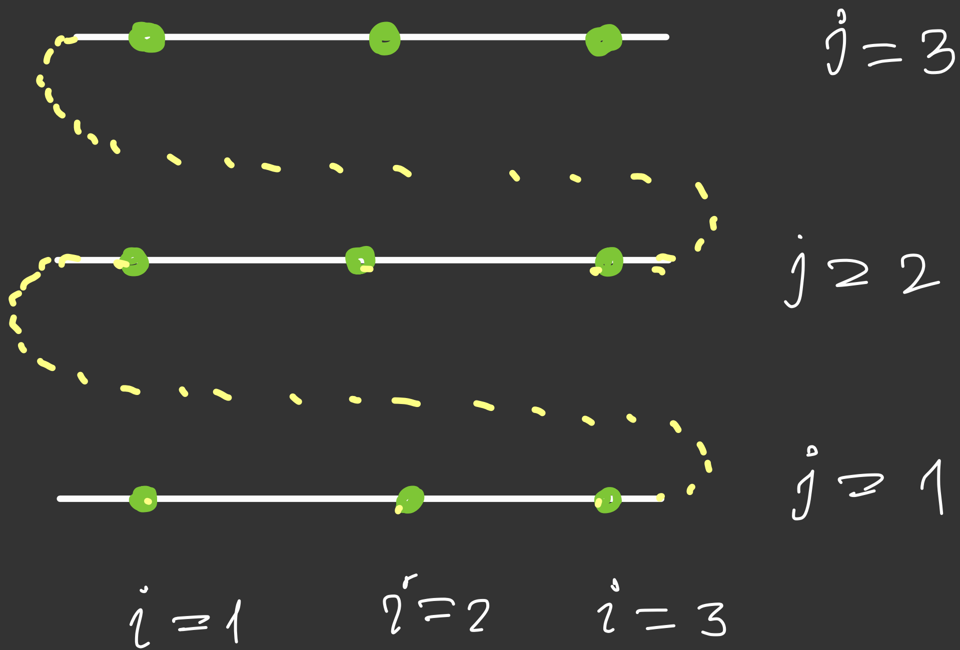
ADI Method for eqn. 1 consist of 3a & 3b

ADI method is unconditionally stable

consistency + stability $\Rightarrow$ convergence.



(First direction)



(Second direction)

X