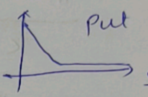
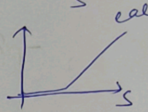
Computational finance

$V(\delta S, t)$. $\quad 0 \le S < \infty$



$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - \delta) S \frac{\partial V}{\partial S} - rV = 0$$



$V(0, t) =$

$V(S_{max}, t) =$

$$V(S, T) = \begin{cases} (K - S)^+ \\ (S - K)^+ \end{cases}$$

1) Finite Domain → equ. more complicated.

2) Infinite domain → eq. simple.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad -\infty < S \le \infty$$

$S_{min} < S < S_{max}$

$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \quad \text{FTCS}$$

$$\frac{u_m^n - u_m^{n-1}}{h} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2k} \quad \text{BTCS}$$

$$\frac{u_m^{n+1} - u_m^{n-1}}{h} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{2k} \quad \text{CTCS}.$$

$S \in [a, b] \quad N = \frac{b-a}{h} \quad h = \frac{b-a}{N} = \Delta S$

$t \in [0, T] \quad M, \quad k = \frac{T}{M} = \Delta t$

$V(S, t)$

$V_m^n \simeq V(S_m, t_n)$

FTCS $\quad \frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2} \qquad \lambda = \frac{k}{h^2}$.

$$\Rightarrow U_m^{n+1} = \lambda U_{m+1}^n + (1 - 2\lambda) U_m^n + \lambda U_{m-1}^n$$

$$0 < \lambda < 1/2$$

BTCS $\dfrac{U_m^{n+1} - U_m^n}{k} = \dfrac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2}$

$$-\lambda U_{m+1}^{n+1} + (1+2\lambda) U_m^{n+1} + \lambda U_{m-1}^{n+1} = U_m^n$$

$AU = F.$

$U = A^{-1}F.$

Crank Nicolson $\frac{1}{2}(FTCS + BTCS)$

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1} + 2U_{m+1}^n - 2U_m^n + U_{m-1}^n}{2h^2}$$

$\neq U_{m+1}^{n+1}$

$$\boxed{\frac{\lambda}{2} U_{m+1}^n - \lambda U_m^n + \frac{\lambda}{2} U_{m-1}^n = (1+\lambda) U_m^{n+1} - \lambda U_{m+1}^{n+1} + \frac{\lambda}{2} U_{m-1}^{n+1}}$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

$x^2 y'' + x y' + y = 0.$

$e^{y/x} = t.$

$V(S,T) = V_T(S)$

$y = x \log t$

$\dfrac{dy}{dx} = \log t + x \dfrac{1}{t} \dfrac{dt}{dx}$

For BSM. Transformation - ①

$\dfrac{d^2y}{dx^2} = \dfrac{1}{t}\dfrac{dt}{dx} + \dfrac{1}{t}\dfrac{dt}{dx} + x\left(-\dfrac{1}{t^2}\right).$

$$\boxed{\begin{array}{l} y = \ln S \rightarrow -\infty < y < \infty \\ \tau = T - t \\ V(S,t) = e^{-r(T-t)} v(y,\tau) \end{array}}$$

$\therefore V_t, V_S, V_{SS}$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

$$V_t = re^{-r(T-t)} v(y,\tau) - e^{-r(T-t)} v_\tau(y,t)$$

$$V_S = e^{-r(T-t)} v_y(y,\tau) \frac{1}{S}$$

$$V_{SS} = e^{-r(T-t)} v_y(y,\tau)\left(\frac{-1}{S^2}\right) + e^{-r(T-t)} v_{yy}(y,\tau)\left(\frac{1}{S^2}\right)$$

$$S^2 V_{SS} = e^{-r(T-t)}\left(V_{yy}(y,\tau) - V_y(y,\tau)\right)$$

$$S V_S = e^{-r(T-t)} V_y(y,\tau)$$

$$V_t = e^{-r(T-t)}(rv - v_\tau)$$

$$e^{-r(T-t)}(\cancel{rv} - v_\tau) + \frac{\sigma^2}{2}\left(e^{-r(T-t)}(v_{yy} - v_y)\right) + (r-\delta)\left(e^{-r(T-t)} v_y\right) - \cancel{r e^{-r(T-t)} v} = 0$$

$$-v_\tau + \frac{\sigma^2}{2}(v_{yy} - v_y) + (r-\delta) v_y = 0$$

$$\frac{\sigma^2}{2} v_{yy} + v_y\left(r - \delta - \frac{\sigma^2}{2}\right) - v_\tau = 0 \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

$$v(y,0) = V_T(e^y)$$

<u>Transformation 2</u>

$$\boxed{\begin{aligned} x &= y + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau \\ \tilde{\tau} &= \frac{1}{2}\sigma^2\tau \\ v(y,\tau) &= u(x,\tilde{\tau}) \end{aligned}}$$

$$v_y = u_x(x,\tau)$$

$$v_{yy} = u_{xx}(x,\tau)$$

$$v_\tau = \frac{\sigma^2}{2} u_{\tilde{\tau}} + \left(r - \delta - \frac{\sigma^2}{2}\right) u_x$$

$$\frac{\sigma^2}{2} u_{xx} + \cancel{u_x\left(r - \delta - \frac{\sigma^2}{2}\right)} - \frac{\sigma^2}{2} u_{\tilde{\tau}} - \cancel{\left(r - \delta - \frac{\sigma^2}{2}\right) u_x} = 0$$

$$\boxed{u_{xx} = u_{\tilde{\tau}}} \qquad -\infty < x < \infty$$

Restrict

$$u(x,0) = V_T(e^x) \qquad x_{min} < x < x_{max}$$

~~$S(t) = S(0) \exp ($~~

$$V(s,t) = e^{-r(T-t)}\, v(y,t) = e^{-r(T-t)}\, u(x, \tilde{\tau})$$

Remark: Both the transformation can be combined $k = 1$.

$$y = \ln s.$$

$$\tau = (T-t)$$

$$\boxed{\begin{aligned} x &= \ln S + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau \\ \tilde{\tau} &= \frac{1}{2}\sigma^2 (T-t) \\ V(S,t) &= e^{-r(T-t)}\, u(x, \tilde{\tau}) \end{aligned}}$$

Note: Here we have assumed $k = 1$ (strike price)

Remarks

① Since the value of the option is given at final time $t = T$ in order to make the initial time use $\tau = T - t$

② $y = \ln s$, $s = e^{y}$ is used to convert the variable coeff. to constant coefficient

③ $V = e^{-r(T-t)}\, v(y,\tau)$ to eliminate the integrating factor.

$$\frac{dv}{d\tau} - rV = f.$$

$$\text{I.f.} = e^{-r\tau}$$

If $r(t)$ is function of t,

then I.f. $e^{-\int_0^{\tau} r(T-s)\, ds}$.

Suppose the $r(t)$, $\delta(t)$, $\sigma(t)$ are function of t then the transformation will be

$$x = \ln S + \int_t^T \left(r(s) - \delta(s) - \frac{\sigma^2(s)}{2}\right) ds.$$

$$\tilde{\tau} = \frac{1}{2}\int_t^T \sigma^2(s)\, ds. \qquad V(S,t) = e^{-\int_t^T r(s)\, ds}\, u(x, \tilde{\tau})$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$$

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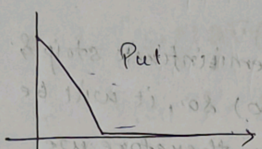
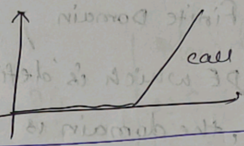
$V(S,T) = (S-K)^+$





$V_p(S,t) = 0, \quad S \to \infty$ $\qquad$ $V_c(S,t) = 0, \quad S = 0$

$V_p(S,t) = \quad S \to 0$ $\qquad$ $V_c(S,t) = \quad S \to \infty$

$C - P = x - ke^{-r\tau}$

$\cancel{V_p + V_c} = e^{-r(T-t)}$

$V_c(S,t) = S - ke^{-r(T-t)} \quad S \to \infty$

$V_p(S,t) = ke^{-r(T-t)} - S \quad S \approx 0$

$S = ke^x, \quad t = T - \frac{2\tau}{\sigma^2}$

$q = \frac{2r}{\sigma^2}, \quad q_\delta = \frac{2(r-\delta)}{\sigma^2}$

$V(S,t) = V\left(ke^x, T - \frac{2\tau}{\sigma^2}\right) = v(x,\tau)$

$v(x,\tau) = k \exp\left\{-\frac{1}{2}(q_\delta - 1)x - \frac{1}{4}\left((q_\delta - 1)^2 + q\right)\tau\right\} y(x,\tau)$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial \tau}$$

$V(S,T) = (S-K)^+ = \max(S-K, 0)$

$V(S,T) = V_T(S)$

~~$y(x,0)$~~ $y(x,0) = \begin{cases} \_\_\_ & \text{Call} \\ \_\_\_ & \text{put} \end{cases}$

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \hookrightarrow \text{get Black Scholes}$$

$V(S, t)$

$x \in (-\infty, \infty)$.

Transform - II Finite Domain

Black Scholes PDE which is defined in semiinfinite strip & transformed k, the domain is $x \in (-\infty, \infty)$ so, it will be difficult to deal with unbounded domain, therefore we transform Black Scholes PDE in domain $[0,1]$ with appropriate transformation.

(1)
$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{\sigma^2(S)}{2} S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0 & 0 \le S < \infty \quad t \le T \\ V(S, T) = V_T(S) = (S-K)^+ \\ \qquad = (K-S)^+ \end{cases}$$

Consider following transformation

(2)
$$\begin{cases} \xi = \dfrac{S}{S+P} & P \neq 0 \quad P > 0 \\ \tau = T - t \\ V(S, t) = (S+P)\, \bar{V}(\xi, \tau). \end{cases}$$

$(S+P)\xi = S$

$P\xi = S(1-\xi)$

$$\boxed{S = \frac{P\xi}{1-\xi}}$$

$$S+P = \frac{P\xi - P\xi + P}{1-\xi} = \frac{P}{1-\xi}$$

$$\frac{\partial \xi}{\partial S} = \frac{S+P-S}{(S+P)^2} = \frac{P}{(S+P)^2} = \frac{P(1-\xi)^2}{P^2} = \frac{(1-\xi)^2}{P}$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left[(S+P)\,\bar{V}(\varepsilon_e, \tau)\right] = -(S+P)\frac{\partial \bar{V}}{\partial \tau}$$

$$= -\left(\frac{P}{1-\varepsilon_e}\right)\frac{\partial \bar{V}}{\partial \tau}$$

$$\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left[(S+P)\,\bar{V}(\varepsilon_e, \tau)\right]$$

$$= \bar{V}(\varepsilon_e, \tau) + (S+P)\frac{\partial \bar{V}}{\partial \varepsilon_e}\cdot\frac{\partial \varepsilon_e}{\partial S}$$

$$= \bar{V}(\varepsilon_e, \tau) + \cancel{(S+P)}\cdot\frac{\partial \bar{V}}{\partial \varepsilon_e}\cdot\frac{P}{(S+P)^{\cancel{2}}}$$

$$\boxed{\frac{\partial V}{\partial S} = \bar{V}(\varepsilon_e, \tau) + \frac{\partial \bar{V}}{\partial \varepsilon_e}\cdot(1-\varepsilon_e)}$$

New domain.

$0 \le S < \infty$

$\varepsilon_e = \dfrac{S}{S+P}$

$0 \le \varepsilon_e \le 1.$

$t \le T$

$\tau \geq 0$

$$\frac{\partial^2 V}{\partial S^2} = \cancel{\frac{\partial \bar{V}}{\partial \varepsilon_e}\cdot\frac{\partial \varepsilon_e}{\partial S}} + \frac{\partial^2 \bar{V}}{\partial \varepsilon_e^2}\cdot\frac{\partial \varepsilon_e}{\partial S}(1-\varepsilon_e) + \frac{\partial \bar{V}}{\partial \varepsilon_e}\cdot\cancel{\left(-\frac{\partial \varepsilon_e}{\partial S}\right)}$$

$$= \cancel{\frac{\partial \bar{V}}{\partial \varepsilon_e}\times\left(\frac{(1-\varepsilon_e)^2}{P}\right)} + \frac{\partial^2 \bar{V}}{\partial \varepsilon_e^2}\cdot\frac{(1-\varepsilon_e)^3}{P} +$$

$$\boxed{\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 \bar{V}}{\partial \varepsilon_e^2}\left(\frac{(1-\varepsilon_e)^3}{P}\right)}.$$

$$-\left(\frac{P}{1-\varepsilon_e}\right)\frac{\partial \bar{V}}{\partial \tau} + \frac{\sigma^2\left(P\varepsilon_e/(1-\varepsilon_e)\right)}{2}\,\frac{\cancel{P}^2\varepsilon_e^2}{(1-\varepsilon_e)^{\cancel{2}}}\,\frac{\partial^2 V}{\partial \varepsilon_e^2}\times\left(\frac{(1-\varepsilon_e)^{\cancel{3}}}{\cancel{P}}\right)$$

$$+(r-\delta)\left(\frac{P\varepsilon_e}{1-\varepsilon_e}\right)\left(\bar{V} + \frac{\partial \bar{V}}{\partial \varepsilon_e}(1-\varepsilon_e)\right)$$

$$-r\left(\frac{P}{1-\varepsilon_e}\right)\bar{V} = 0$$

$$-\left(\frac{P}{1-\varepsilon_e}\right)\frac{\partial \bar{V}}{\partial \tau} + \frac{\sigma^2(\varepsilon_e)}{2}P\varepsilon_e^2(1-\varepsilon_e)\frac{\partial^2 \bar{V}}{\partial \varepsilon_e^2} + (r-\delta)P\varepsilon_e\frac{\partial \bar{V}}{\partial \varepsilon_e} + \frac{P\bar{V}}{(1-\varepsilon_e)}\left((r-\delta)\varepsilon_e - r\right) = 0$$

$$\left.\begin{aligned}\frac{\partial \bar{V}}{\partial \tau} &= \frac{\sigma^2(\varepsilon_e)}{2}\,\varepsilon_e^2(1-\varepsilon_e)^2\frac{\partial^2 \bar{V}}{\partial \varepsilon_e^2} + (r-\delta)\,\varepsilon_e(1-\varepsilon_e)\frac{\partial \bar{V}}{\partial \varepsilon_e} \\ &\quad - [r(1-\varepsilon_e)+\delta\varepsilon_e]\bar{V}\end{aligned}\right\} \rightarrow (4)$$

$$0 \le \varepsilon_e \le 1$$

$$V(S,t) = (S+P)\,\bar{V}(\varepsilon_e,\tau)$$

$$t = T \qquad \tau = 0$$

$$V(S,T) = (S+P)\,\bar{V}(\varepsilon_e,0)$$

$$V_T(S) = \frac{P}{(1-\varepsilon_e)}\,\bar{V}(\varepsilon_e,0)$$

$$\bar{V}(\varepsilon_e,0) = \frac{(1-\varepsilon_e)\,V_T(S)}{P}$$

$$\boxed{\bar{V}(\varepsilon_e,0) = \frac{(1-\varepsilon_e)}{P}\,V_T\left(\frac{P\varepsilon_e}{1-\varepsilon_e}\right)} \quad \text{Initial}$$

$\varepsilon_e = 0$ $\qquad\qquad$ $\varepsilon_e = 1$

$$\frac{\partial \bar{V}}{\partial \tau} = -r\bar{V} \qquad\qquad \frac{\partial \bar{V}}{\partial \tau} = -\delta\bar{V}$$

In order to solve eq. (4) we require the boundary conditions at $\varepsilon_e = 0$ & $\varepsilon_e = 1$. When $\varepsilon_e = 0$ & $\varepsilon_e = 1$ given in (4) degenerates to ODE:

When $\varepsilon_e = 0$

$$\frac{\partial \bar{V}}{\partial \tau} = -r\bar{V}$$

$$\frac{\partial \bar{V}(0,\tau)}{\partial \tau} = -r\bar{V}(0,\tau)$$

Boundary

$$\boxed{\bar{V}(0,\tau) = \bar{V}(0,0)\,e^{-r\tau}}$$

When $\varepsilon_e = 1$

$$\frac{\partial \bar{V}}{\partial \tau} = -\delta\bar{V}$$

$$\boxed{\bar{V}(1,\tau) = \bar{V}(1,0)\,e^{-\delta\tau}}$$

$$\bar{V}(\varepsilon_e, 0) = \frac{(1-\varepsilon_e)}{p} V_T\left(\frac{p\,\varepsilon_e}{1-\varepsilon_e}\right)$$

Call

$$V(S,T) = (S-K)^+ = \max\{S-K, 0\} = V_T(S)$$

Put

$$V(S,T) = (K-S)^+ = \max\{K-S, 0\} = V_T(S) \qquad (1)$$

consider call.

$$\bar{V}(\varepsilon_e, 0) = \cancel{\left(\frac{1-\varepsilon_e}{p}\right) \max\left\{\frac{p\varepsilon_e - K + K\varepsilon_e}{(1-\varepsilon_e)}\right.}$$

$$\left(\frac{1-\varepsilon_e}{p}\right)\max\left\{\frac{p\,\varepsilon_e}{1-\varepsilon_e} - K, 0\right\} = \max\left\{\varepsilon_e - \frac{(1-\varepsilon_e)K}{p}, 0\right\}.$$

consider put

$$\bar{V}(\varepsilon_e, 0) = \frac{(1-\varepsilon_e)}{p}\max\left\{K - \frac{p\,\varepsilon_e}{1-\varepsilon_e}, 0\right\} = \max\left\{\frac{K(1-\varepsilon_e)}{p} - \varepsilon_e, 0\right\}$$

$p > 0$, $p = K$.

call

$$\bar{V}(\varepsilon_e, 0) = \max\{\varepsilon_e - (1-\varepsilon_e), 0\}$$

$$= \max\{2\varepsilon_e - 1, 0\}.$$

Put.

$$\bar{V}(\varepsilon_e, 0) = \max\{1 - 2\varepsilon_e, 0\}$$

Remark:

(1) we assume the volatility σ is function of S (i.e. asset & price) and r & δ (interest rate & dividend) are constant, the same transformation will be applicable if $\sigma(S,t)$, $r(S,t)$, $\delta(S,t)$

(2) when $\varepsilon_e = 1$, we obtain the asymptotic expression of black schole's formula. (i.e. solution of black scholes PDE).

$\bar{V}(1,0)(S+P) \simeq V(S,T)$

$V(S,t)$? $S \to \infty$ $\quad \varepsilon_e = 1$.

$$V(S,t) = (S+P)\,\bar{V}(\varepsilon_e, \tau)$$

$$= (S+P)\,\bar{V}(1,\tau) = (S+P)\,\bar{V}(1,0)\,e^{-\delta\tau} \simeq V(S,T)\,e^{-\delta(T-t)}$$

Obtain the black scholes formula from 1D heat conduction eqn.

$$(1)\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, & -\infty < x < \infty \\ & \tau \geq 0 \\ u(x,0) = u_0(x). \\ u(x,\tau) = V(S,t) \end{cases}$$

Cauchy Pbm

Pure IVP.

No Boundary condition

consider : $\underline{x \in (0, l)}$

$u(0,\tau) = \phi(\tau)$.

$u(l,\tau) = \varphi(\tau)$

$\boxed{u(x,\tau) = X(x)T(\tau)}$

Since the domain for x is $(-\infty, \infty)$, we don't have Boundary condition
$\therefore$ Method of separation of variable is not applicable to pbm. (1)
We will try to look for a special solution of this form

$$\boxed{u(x,\tau) = \frac{1}{\sqrt{\tau}} U(\eta) \qquad \eta = \frac{x - \xi}{\sqrt{\tau}}}$$

$\tau^{-1/2} U(\eta)$

ξ = parameter

$$\frac{\partial u}{\partial x} = \cancel{\frac{1}{\sqrt{\tau}} U_\eta} \quad \frac{1}{\sqrt{\tau}} \frac{\partial U(\eta)}{\partial \eta} \cdot \frac{1}{\sqrt{\tau}} = \frac{1}{\tau} \frac{\partial U(\eta)}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\tau} \frac{\partial^2 U(\eta)}{\partial \eta^2} \cdot \frac{1}{\sqrt{\tau}} = \frac{\partial^2 U}{\partial \eta^2} \tau^{-3/2}$$

$$\frac{\partial u}{\partial \tau} = \frac{-1}{2} \tau^{-3/2} U(\eta) + \frac{\partial U}{\partial \eta} \frac{1}{\sqrt{\tau}} (x - \xi) \cdot \left(-\frac{1}{2} \tau^{-3/2}\right)$$

$$= -\frac{\tau^{-3/2}}{2} \left(U + \eta \frac{\partial U}{\partial \eta}\right) = -\frac{\tau^{-3/2}}{2} \frac{\partial(\eta U)}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial \hat{t}}$$

$$\cancel{\hat{t}^{-3/2}} \frac{\partial^2 U}{\partial \eta^2} = -\frac{\hat{t}^{-3/2}}{2} \frac{\partial(\eta U)}{\partial \eta}$$

$$\Rightarrow \frac{\partial^2 U}{\partial \eta^2} + \frac{1}{2} \frac{\partial(\eta U)}{\partial \eta} = 0$$

$$\Rightarrow \frac{\partial}{\partial \eta}\left[\frac{\partial U}{\partial \eta} + \frac{1}{2}\eta U\right] = 0$$

$$\therefore \Rightarrow \boxed{\frac{\partial U}{\partial \eta} + \frac{1}{2}\eta U = C}$$

$$\frac{\partial U}{\partial \eta} = C - \frac{1}{2}\eta U$$

for simplicity $C = 0$

$$\cancel{\frac{\partial U}{C - \eta U} = \partial \eta}$$

$$\frac{\partial U}{U} = -\frac{1}{2}\eta \, \partial \eta$$

$$\log U = -\frac{\eta^2}{4} + c \Rightarrow U = C_1 e^{-\eta^2/4}$$

$$\boxed{U(\eta) = C_1 e^{-\frac{(x-\xi_e)^2}{4\hat{t}}}}$$

We require that $u(x, \hat{t}) = \frac{1}{\sqrt{\hat{t}}} U(\eta) = C \hat{t}^{-1/2} e^{-(x-\xi_e)^2/4\hat{t}}$

We require that $\int_{-\infty}^{\infty} C \hat{t}^{-1/2} e^{-(x-\xi_e)^2/4\hat{t}} \, d\xi_e = 1.$

$$C = \frac{1}{\int_{-\infty}^{\infty} \hat{t}^{-1/2} e^{-(x-\xi_e)^2/4\hat{t}} \, d\xi_e}$$

$$\boxed{\frac{x - \xi_e}{\sqrt{\hat{t}}} = \eta}$$

$$= \frac{1}{\sqrt{2} \int_{-\infty}^{\infty} e^{-\eta^2/2} \, d\eta} = \frac{1}{2\sqrt{\pi}}$$

$$u(x,\hat{t}) = \frac{1}{2\sqrt{\pi \hat{t}}} e^{-(x-\xi)^2/4\hat{t}}$$

↳ fundamental soln. of Green's fn.

$$u(x,0) = u_0(x)$$

$$\frac{\partial u}{\partial \hat{t}} = \frac{\partial^2 u}{\partial x^2}.$$

For any function $G(\xi, x, \hat{t})$ where ξ is a parameter s.t.

$$\frac{\partial G(\xi, x, \hat{t})}{\partial \hat{t}} = \frac{\partial^2 G(\xi, x, \hat{t})}{\partial x^2}$$

$$\int_{-\infty}^{\infty} u_0(\xi) \frac{\partial}{\partial \hat{t}} G(\xi, x, \hat{t})\, d\xi = \int_{-\infty}^{\infty} u_0(\xi) \frac{\partial^2 G}{\partial x^2} (\xi, x, \hat{t})\, d\xi$$

$$\frac{\partial}{\partial \hat{t}} \int_{-\infty}^{\infty} u_0(\xi) G(\xi, x, \hat{t})\, d\xi = \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} u_0(\xi) G(\xi, x, t)\, d\xi.$$

$$u(x,\hat{t}) = \int_{-\infty}^{\infty} u_0(\xi) \frac{1}{2\sqrt{\pi \hat{t}}} e^{-(x-\xi)^2/4\hat{t}}\, d\xi.$$

$$\lim_{\hat{t}\to 0} u(x,\hat{t}) \overset{?}{=} u_0(x)$$

$$\lim_{\hat{t}\to 0} \frac{1}{2\sqrt{\pi \hat{t}}} e^{-(x-\xi)^2/4\hat{t}} = \begin{cases} 0 & x \neq \xi \\ \infty & x = \xi \end{cases} = \delta(x-\xi)$$

Further we have $\displaystyle\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi \hat{t}}} e^{-(x-\xi)^2/4\hat{t}}\, d\xi = 1.$

$$\lim_{\hat{t}\to 0} \frac{1}{2\sqrt{\pi \hat{t}}} e^{-(x-\xi)^2/4\hat{t}} = \delta(x-\xi).$$

$$u_0(x) = \lim_{\hat{\tau} \to 0} \int_{-\infty}^{\infty} u_0(\xi) \frac{1}{2\sqrt{\pi \hat{\tau}}} e^{-(x-\xi)^2/4\hat{\tau}} \, d\xi .$$

$$u(x, \hat{\tau}) = \frac{1}{2\sqrt{\pi \hat{\tau}}} e^{-(x-\xi)^2/4\hat{\tau}}$$

eq.

$u(x, \hat{\tau})$ along with $u_0(x)$ satisfies heat conduction eq condition with initial conditions. we have to retrieve. Black Scholes formula from eq.

$$V(S,t) = e^{-r(T-t)} u(x, \hat{\tau})$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} u_0(\xi) \frac{1}{2\sqrt{\pi\hat{\tau}}} e^{-(x-\xi)^2/4\hat{\tau}} \, d\xi .$$

$$V(S,T) = V_T(S)$$

$$x = \ln S + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau \qquad x - \xi = \left(\ln S + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau - \xi\right)$$

$$\hat{\tau} = \frac{1}{2}\sigma^2 \tau \qquad 4\hat{\tau} = 2\sigma^2\tau$$

$$V(S,t) = e^{-r(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} u_0(\xi) e^{-\frac{\left(\ln S + \left((r-\delta) - \frac{\sigma^2}{2}\right)(T-t) - \xi\right)^2}{2\sigma^2(T-t)}} d\xi$$

$$V(S,t) = e^{-r(T-t)} \frac{1}{\sigma\sqrt{2\pi(T-t)}} \int_{0}^{\infty} V_T(\tilde{S}) \times \exp\left(-\ln\tilde{S} - \left(\ln S + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)\right)^2 \Big/ 2\sigma^2(T-t)\right) \times \frac{d\tilde{S}}{\tilde{S}}$$

3

Black Scholes formula can be written.

$$V(S,t) = e^{-r(T-t)} \int_0^{\infty} V_T(\tilde{S})\, G(\tilde{S}, T, S, t)\, d\tilde{S}.$$

$$G(\ ) = \frac{1}{\tilde{S}} \cdot \frac{1}{\sigma\sqrt{2\pi(T-t)}} \exp\left(\frac{-\ln\tilde{S} - \left(\ln S + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2(T-t)}\right)$$

G is Green's function for Black Scholes PDE.

The PDF of a log normal distribution with $m = \left(r - \delta - \frac{\sigma^2}{2}\right)$ and

$$E(\tilde{S}) = S\exp((r-\delta)(T-t))$$

$$G(\ ) = \cancel{\frac{1}{\sqrt{2\pi}\, b\tilde{S}}} \left[e^{-\left(\ln(\tilde{S}/a) + \frac{b^2}{2}\right)^2 / 2b^2}\right] \cdot \frac{1}{\sqrt{2\pi}\, b\tilde{S}}$$

$a = S e^{(r-\delta)(T-t)}$ Call $\to$ $N(d_1)S_t - N(d_2)Ke^{-rt}$

$b = \sigma\sqrt{T-t}$

In order to obtain Black & Scholes formula we have to prove following identity.

$$\int_c^{\infty} G(\tilde{S}, T, S, t)\, d\tilde{S} = N\left(\frac{\ln(a/c) - b^2/2}{b}\right)$$

$$\int_c^{\infty} \tilde{S}\, G(\tilde{S}, T, S, t)\, d\tilde{S} = aN\left(\frac{\ln(a/c) + b^2/2}{b}\right).$$

$$N(\varphi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\varphi} e^{-\xi^2/2}\, d\xi$$

Let $\eta(\tilde{s}) = \dfrac{\ln(\tilde{s}/a) + b^2/2}{b}$

$\tilde{s} = a e^{b\eta - b^2/2}$

$d\tilde{s} = a e^{-b^2/2} b \cdot e^{b\eta} \cdot d\eta$

$= b\tilde{s}\, d\eta$

$G = e^{-\frac{\eta^2}{2}} \cdot \dfrac{1}{\sqrt{2\pi}\, ba\tilde{s}}$

B-S model.

1. FDM
2. FEM
3. Spline Method
4. NNS

European 1D & 2D

1. Generalised one
2. Non linear
3. Jump difference.

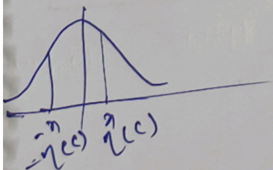
American

free bvp

Exotic Asian

$$\int_C^\infty G\, d\tilde{s} = \int_C^\infty \frac{e^{-\eta^2/2}}{\sqrt{2\pi}\, b\tilde{s}}\, ds = \int_C^\infty \frac{e^{-\eta^2/2}}{\sqrt{2\pi}}\, d\eta$$

$$= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln(\frac{c}{a}) + \frac{b^2}{2}}{b}}^{\infty} e^{-\eta^2/2}\, d\eta$$



$$\int_{\frac{\ln(\frac{c}{a}) + \frac{b^2}{2}}{b}}^{\infty} \frac{e^{-\eta^2/2}}{\sqrt{2\pi}}\, d\eta$$

$$\int_{-\infty}^{(\ln(\frac{a}{c}) - \frac{b^2}{2})/b} \frac{e^{-\eta^2/2}}{\sqrt{2\pi}} = N\left(\frac{\ln(\frac{a}{c}) - \frac{b^2}{2}}{b}\right)$$

$$\int_c^{\infty} \tilde{S}\, G\, d\tilde{S} = \int_{\eta(c)}^{\infty} \cancel{\tilde{S}}\, \frac{e^{-\eta^2/2}}{\sqrt{2\pi}\, b\tilde{S}}\, b\tilde{S}\, d\eta$$

$$\int_{\eta(c)}^{\infty} \cancel{\tilde{S}}\, \frac{e^{-\eta^2/2}}{\sqrt{2\pi}\, \cancel{b}\cancel{\tilde{S}}} \cdot \tilde{S}\, \cancel{b}\, d\eta \qquad \tilde{S} = a e^{b\eta - \frac{b^2}{2}}$$

$$\int_{\eta(c)}^{\infty} a\, \frac{e^{-\eta^2/2 - \frac{b^2}{2} + b\eta}}{\sqrt{2\pi}}\, d\eta$$

$$= \int_{\eta(c)}^{\infty} a\, \frac{e^{-\frac{(\eta - b)^2}{2}}}{\sqrt{2\pi}}\, d\eta$$

$$= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)}^{\infty} e^{-\frac{(\eta-b)^2}{2}}\, d\eta$$

$$= \frac{a}{\sqrt{2\pi}} \int_{\eta(c)-b}^{\infty} e^{-\eta^2/2}\, d\eta$$

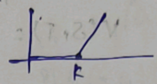
$$= a\, N\left(b - \eta(c)\right)$$

$$= a\, N\left(b + \frac{\ln(a/c) - b^2/2}{b}\right)$$

$$= a\, N\left(\frac{\ln(a/c) + b^2/2}{b}\right)$$

$$V(S,t) = \int e^{-r(T-t)} \int_0^\infty \max(\tilde{S}-K,0)\, G\, d\tilde{S}$$

$V_T(S) = \max(S-K,0)$



$$= e^{-r(T-t)} \left[\int_0^K 0\, G\, d\tilde{S} + \int_K^\infty \tilde{S}\, G\, d\tilde{S} - \int_K^\infty K\, G\, d\tilde{S} \right]$$

$$= \left[a N\left(\frac{\ln(a/K) + b^2/2}{b} \right) - K N\left(\frac{\ln(a/K) - b^2/2}{b} \right) \right] e^{-r(T-t)}$$

$a = S e^{(r-\delta)(T-t)}$, $\quad \ln\left(\frac{a}{K}\right) = \ln\left(\frac{S}{K}\right) + (r-\delta)(T-t)$

$b = \sigma\sqrt{T-t}$

$\frac{b^2}{2} = \frac{\sigma^2(T-t)}{2}$

$$d_1 = \frac{\ln\left(\frac{a}{K}\right) + \frac{b^2}{2}}{b} = \frac{\ln\left(\frac{S}{K}\right) + \left(\frac{\sigma^2}{2} + (r-\delta)\right)(T-t)}{\sigma\sqrt{T-t}}$$

~~conside~~

$$d_2 = \frac{\ln\left(\frac{a}{K}\right) - \frac{b^2}{2}}{b} = \frac{\ln\left(\frac{S}{K}\right) + \left((r-\delta) - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$V(S,t) = \boxed{S e^{-\delta(T-t)} N(d_1) - K e^{-r(T-t)} N(d_2)}$$

Consider the payoff for a forward contract where.

$$V(S,T) = S - k.$$

$$V(S,t) = e^{-r(T-t)} \int_0^k (\tilde{S} - k)\, G(\tilde{S}, \tilde{\tau}, S, t)\, d\tilde{S}.$$

$$= e^{-r(T-t)} \left(S\, e^{(r-\delta)(T-t)} - k\right)$$

$$= S e^{-\delta(T-t)} - k e^{-r(T-t)}.$$

Since for a forward contract the buyer does not need to pay any premium at initial time. ∴

∴ $t = 0$

$$V(S,0) = S e^{-\delta T} - k e^{-rT} = 0.$$

$$k = e^{(r-\delta)T} S_0$$

$\delta \to S$ $\quad \gamma \to SS$ $\quad \theta \to t$ $\quad \to \sigma$ $\quad \rho$

<u>Greeks</u> i.e. delta, gamma, theta, vega, rho $\to r$

$\Theta = V_\theta$ Derivative of option w.r.t. parameter, S, t, r

Calculate greeks, corresponding errors and correponding plot

$Ax=b$

1) Jacobi
2) Gauss-Seidel
3) SOR (Relaxation wala)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (x,t) \in (a,b) \times [0,T]$$

$$u(a,t) = u(b,t) = 0$$

$$u(x,0) = \phi(x) \qquad x \in [a,b]$$

In FTCS, BTCS, CN we discretize domain in both ~~and~~ x & t.

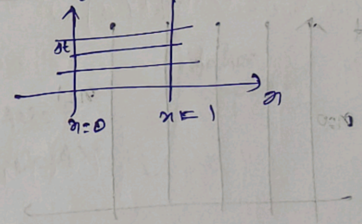
<u>Semi-discretization</u> → yield more accuracy than above one.

discretize either x or t ⇒ get system of ODEs
⇓
get IVP/BVPs

<u>Semi-discrete schemes</u> (Method of lines) (MOL)

In order to solve parabolic IBVP numerically, one can use Method of lines which is known as Semi discrete scheme. In this method we discretize either time domain and preserve x domain continuous (or) vice versa. This method is semi discretize

Time discretization ⇒ horizontal



Consider this model.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

$$u(a,t) = u(b,t) = 0$$

$$u(x,0) = \phi(x) \qquad x \in [a,b]$$

Let discretize t and use Euler. (Method of horizontal lines)

$$\frac{U^{k+1}(x) - U^k(x)}{\Delta t} = \frac{\partial^2 U^k(x)}{\partial x^2} + f(x,t^k).$$

$$U^{k+1}(x) = \Delta t\left(\frac{\partial^2 U^k(x)}{\partial x^2} + f(x,t^k)\right) + U^k(x)$$

This is explicit euler for

For Stability ~~region~~ condition we use Implicit euler.

$U^k(x) = U(x, t^k)$

$$(2)\quad \frac{U^{k+1}(x) - U^k(x)}{\Delta t} = \frac{\partial^2 U^{k+1}(x)}{\partial x^2} + f(x, t^{k+1})$$

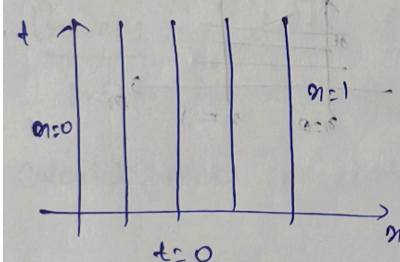
$$(3)\quad \begin{cases} U^k(a) = U^k(b) = 0 \\ U^{k+1} - \Delta t \dfrac{\partial^2 U^{k+1}(x)}{\partial x^2} = U^k(x) + f(x, t^{k+1})\,\Delta t. \end{cases}$$

$$u''(x) = f(x),$$

$$\frac{U_{m+1} - 2U_m + U_{m-1}}{\Delta x^2} = f_m.$$

Eq. Given in (3) are system of 2nd order ODEs with boundary conditions i.e. system of BVP which can be solved by 2nd order methods or any higher order methods

Discretize x first and preserve t continuous (Method of vertical lines)



We descretise special domain with vertical lines.

We write the eq. (1)

$$(4)\quad \begin{cases} \dfrac{du_i(t)}{dt} = \dfrac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\Delta x^2} + f_i(t) \\ U_0(t) = U_{N+1}(t) = 0 \\ U_i(0) = \phi(x_i) \quad i = 0, \ldots, N+1 \end{cases}$$

Equations given in (4) are system of 1st order ODEs with initial conditions i.e. we obtain system of IVP for 1st order ODEs.

Explicit Euler → FTCS

Implicit Euler → BTCS

One can use either explicit or Implicit Euler scheme the resultant scheme will be FTCS or BTCS. In order to obtain higher order scheme one can use RK of 4th order, or multisteps scheme like Adam's Bashforth or Adams Moulton

$$\frac{d\bar{y}}{dt} = \bar{f}(\bar{y}, t)$$

euler RK.
$$\begin{cases} \frac{dy_1}{dt} = f_1(y_1, y_2, t) \\ \frac{dy_2}{dt} = f_2(y_1, y_2, t) \end{cases}$$

→ Systematic apprach.

Finite Element Method (FEM) Ch. 5

$u'' = f(x)$, $x \in (0,1)$

$u(0) = 0 = u(1)$

$x_0 \quad x_i \quad x_{N+1}$

Adhoc approach

$$\frac{U_{i+1} - 2U_i + U_{i-1}}{h^2} = f_i \quad 1 \le i \le N$$

$U_0 = U_{N+1} = 0$ (B)

$AU = f$

$U = A \backslash f$

Adhoc approach

In FEM we use systematic approach.

$L_E = \|u'' - f\| \to 0$

Residue or Loss

(or) ↓

$$\int_0^1 (u'' - f)\,\varphi(x)\,dx = 0$$

$$\boxed{\int_0^1 u''\varphi(x)\,dx = \int_0^1 f\,\varphi(x)\,dx}$$ → weak formulation.

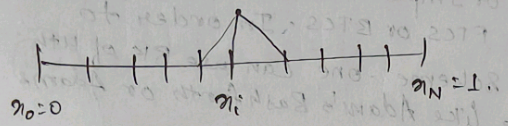
$$\begin{cases} Lu = f, \ \Omega \\ \cancel{Lu = u''} \\ Lu = -u'' \quad \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

FEM

Step 1: weak formulation

Step 2: Domain discretization finite dimensional plm

Step 3: Numerical Integration / system of linear algebraic



$u(x_i) = \sum C_i \phi_i(x) \to$ Basis function $\to$ hat function

We discretize domain $\Omega = \bigcup_{k=1}^{N} \Omega_k$, $\Omega_i = [x_i, x_{i+1}]$.

Approx $w(x) = \sum_{i=1}^{N} c_i \phi_i(x)$ — (*)

where $\phi_i(x)$ is called basis functions and constants c_i's has to be determined.

In order to determine the constants c_i's we have to form n no. of equations.

One way of calculation of c_i's by using residual function.

Residual fn. $R = Lw - f$

Idea is such that to minimise the residual in such a way that approximate soln. given in that (*) eq. in such ~~that~~ a way that error is minimum.

$$\text{Error} = \| u(x_i) - w_i \|$$

To minimise the residue we will consider N weight function (test function). $\psi_1, \dots \psi_N$ s.t. $\int_\Omega R\,\psi_j(x)\,dx = 0$, $j = 1, \dots N$.

$Lw = -\Sigma C_i \phi_i''(x)$

On Substituting w in R we obtain.

$$\int_\Omega -\left(\sum_{i=1}^{N} c_i \phi_i''(x) + f\right) \varphi_j(x)\, dx = 0$$

$$\boxed{\sum_{i=1}^{N} c_i \int_\Omega \phi_i''(x)\, \varphi_j(x)\, dx = -\int_\Omega f \varphi_j(x)\, dx} \rightarrow (**)$$

$j = 1, \ldots, N.$

$$\int Lw\, \varphi_j(x)\, dx = \int f \varphi_j(x)\, dx \quad j = 1, \ldots, N. \leftarrow (3)$$

$(Lw, \varphi_j) = (f, \varphi_j) \rightarrow$ Innerproduct.

If we choose

(i) $\phi_j(x) = \varphi_j(x)$ i.e. test and trial function is same.

But non- Galerkin.

(ii) Collocation Scheme.

$\varphi_j(x) = \delta(x - x_j)$ Dirac function

$$\delta(x - x_j) = \begin{cases} 1 & x = x_j \\ 0 & x \neq x_j \end{cases}$$

Substitute. $\varphi_j(x) = \delta(x - x_j)$ in (3).

$\int f \varphi_j(x)\, dx = \cancel{\int f(x_j)\, dx}\ f(x_j)$ ∵

$\int Lw\, \varphi_j(x)\, dx = Lw(x_j).$

Least Square method

$$\varphi_j(x) = \frac{\partial R}{\partial c_j}$$

$$\int L w\, \varphi_j(x)\, dx = \int f \varphi_j(x) dx.$$

$$\int_\Omega L w \frac{\partial R}{\partial c_j} dx = \int f \frac{\partial R}{\partial c_j} dx.$$

$$\frac{\partial R}{\partial c_j} = \frac{\partial}{\partial c_j}(Lw - f)$$

$$\int_\Omega (Lw - f) \frac{\partial R}{\partial c_j} dx = 0$$

$$\int_\Omega R \frac{\partial R}{\partial c_j} dx = 0$$

Model Pbm.

1DBVP

$$① \begin{cases} -(p(x)u'(x))' + q(x)u(x) = r(x) & x \in (0,1) \\ u(0) = 0 = u(1). \end{cases}$$

Solve using FEM.

Step 1: Weak formulation $\quad v \in H_0^1(\Omega)$

$v \in C_0^\infty(\Omega)$

Continuous many times differentiable

$① \times v \Rightarrow -(p(x)u')'v + q(x)uv = r(x)v$

Integration by parts.

$$\int_0^1 -(p(x)u')' v\, dx + \int_0^1 q(x) uv\, dx = \int_0^1 r(x) v\, dx.$$

$$\int_0^1 u\, dv = (uv)_0^1 - \int_0^1 v\, du$$

$$\left(-(p(x)u')v\right)_0^1 + \int_0^1 p(x)u'v'\,dx + \int_0^1 q(x)uv\,dx = \int_0^1 r\,v(x)\,dx.$$

$\to 0$?

$$\text{(2)}\begin{cases} \text{find } u \in H_0^1(\Omega) =: V \\ \text{s.t.} \\ \displaystyle\int_0^1 p(x)u'v'\,dx + \int_0^1 q(x)uv\,dx = \int_0^1 r(x)\,dx \quad \forall v \in V \end{cases}$$

$\because$ $r(x)$ is continuous in (1)

(*) $\therefore$ $u \in C^2(\Omega)$ & $p \in C^1(\Omega)$

In (2) we don't need (*)

$u \in L^p(\Omega)$

i.e. $\left(\int |u|^p\,dx\right)^{1/p} < \infty$

borded measurable fn.

Step (2) Discretise the domain

$\Omega =$ $x_0 = 0$ ... x_i ... $x_{N_i} = 1.$

V_h is a subspace of V which is of finite dimension.

$$V_h = \mathrm{span}\{\phi_1, \ldots, \phi_{N-1}\}.$$

ϕ_i's are peacewise basis function.

Finite element problem

$$\text{(3)}\begin{cases} \text{find } u_h \in V_h \text{ s.t.} \\ \displaystyle\int_0^1 p(x)\,u_h'\,v_h'\,dx + \int_0^1 q(x)\,u_h v_h\,dx = \int_0^1 r(x)\,v_h(x)\,dx \quad \forall v_h \in V_h. \end{cases}$$

(2) $\to$ Infinite

(3) $\to$ Finite dimensional space

$$\text{Express } u_h(x) = \sum_{i=1}^{N-1} U_i\,\phi_i(x) \longrightarrow \text{(4)}$$

U_i's are unknown to determined.

In order to determine N-1 unknowns we need to have that many equations. i.e. System of linear algebraic equation

Using ④ in ③

$$\sum_{i=1}^{N-1}\left(\int_{x_i}^{x_{i+1}} p(x)\,\phi_i'(x)\,\phi_j'(x) + q(x)\,\phi_i(x)\,\phi_j(x)\right) U_i = \int_0^1 r(x)\,\phi_j(x)\,dx$$

$\forall j = 1, \dots N-1$

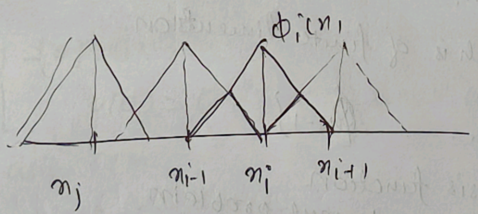
$AU = F \Leftarrow$ System of linear equations

$U = [U_1, \dots U_{N-1}]^T$

Assume the basis elements ϕ_i's are peacewise linear polynomial

i.e. $\phi_i(x) = a_i + b_i x \qquad \Omega_i = [x_i, x_{i+1}]$

$$\phi_i(x_j) = \begin{cases} 1 & i = j \\ 0 & \text{o.w.} \end{cases}$$

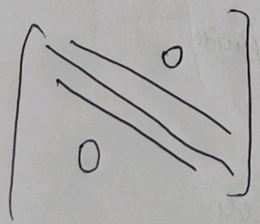


$\phi_i \phi_j \neq 0$ when
$j = i-1$
$j = i+1$
$j = i$

Matrix A is called stiffness matrix which is 3-diagonal matrix

i.e. ~~$a_{ij} \neq 0$~~ $\{ |j-i| \le 1$

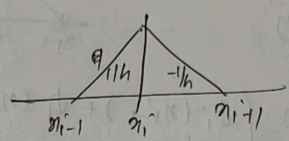
$$a_{ij} = \begin{cases} 0 & |i-j| > 1 \\ \neq 0 & |i-j| \le 1 \end{cases}$$



$a_{ij} = \int p(x)\,\phi_i'\phi_j' + \int q(x)\,\phi_i\phi_j\,dx \qquad r_j = \int r(x)\,\phi_j(x)\,dx.$

$J_1 = \int_{x_{i-1}}^{x_i} + \int_{x_i}^{x_{i+1}}$

$j=i$



Use Trapezoidal rule to evaluate Integrals

$$a_{ii} = \int_{x_{i-1}}^{x_i} p(x)\,\phi_i'\phi_i' + \int_{x_i}^{x_{i+1}} p(x)\,\phi_i'\phi_i'\,dx.$$

$$= p\left(\frac{1}{h}\right)\left(\frac{1}{h}\right)\int_{x_{i-1}}^{x_i} dx + p\left(\frac{-1}{h}\right)\left(\frac{-1}{h}\right)\int_{x_i}^{x_{i+1}} dx.$$

$$= \frac{2p}{h}$$

$j = i-1.$

$$a_{i\,i-1} = \int_{x_{i-1}}^{x_i} p\,\phi_i'(x)\,\phi_{i-1}'(x)\,dx = p\left(\frac{1}{h}\right)\left(\frac{-1}{h}\right)h = -p/h$$

$j = i+1$

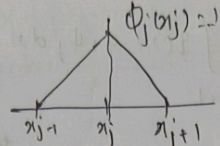
$$a_{i\,i+1} = \int_{x_i}^{x_{i+1}} p\,\phi_i'(x)\,\phi_{i+1}'(x)\,dx = p\left(\frac{-1}{h}\right)\left(\frac{1}{h}\right)h = -p/h$$

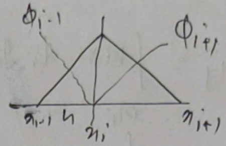
$$a_{ji} = \sum_{j=1}^{N-1}\int_{x_j}^{x} q(x)\,\phi_i(x)\,\phi_j(x)\,dx. \qquad i = 1, \dots N-1.$$

$j=i$

$$a_{ii} = \int_{x_{i-1}}^{x_i} q\,\phi_i(x)\,\phi_i(x)\,dx + \int_{x_i}^{x_{i+1}} q\,\phi_i(x)\,\phi_i\,dx.$$

$$\phi_i(x_j) = \begin{cases} 1 & j=i \\ 0, & o.w. \end{cases} \quad \delta_{ij}$$





$$a_{ii} = \frac{qh}{2}\left[\phi_i(x_{i-1}) + \phi_i(x_i)\right] + \frac{qh}{2}\left[\phi_i(x_i) + \phi_i(x_{i+1})\right].$$

$$= qh.$$

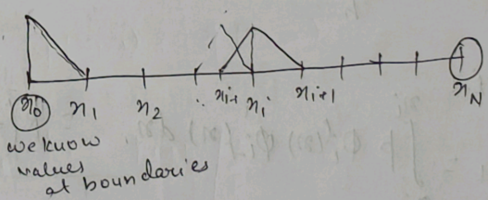
$j = i-1$

$$a_{ii-1} = \int_{x_{i-1}}^{x_i} q\,\phi_{i-1}\phi_i\,dx = \frac{qh}{2}\left[\phi_{i-1}(x_{i-1})\phi_i(x_{i-1}) + \phi_{i-1}(x_i)\phi_i(x_i)\right]$$

$$= 0$$

$j = i+1$. using trapezoidal rule.

$$a_{ii+1} = \int_{x_i}^{x_{i+1}} q\,\phi_i(x)\phi_{i+1}\,dx = 0$$



Now we get $AU=F$ matrix.

$\cancel{a_{ii} = \frac{2k}{h}} +$

$$a_{ij} = \begin{cases} -p/h & j = i-1 \\ 2p/h + qh & j = i \\ -p/h & j = i+1 \end{cases}$$

$$RHS = \int_0^1 r(x)\,\phi_j(x)\,dx.$$

$$= \sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} r(x)\,\phi_j(x)\,dx.$$

$$= \int_{x_{j-1}}^{x_j} r(x)\,\phi_j(x)\,dx + \int_{x_j}^{x_{j+1}} r(x)\,\phi_j(x)\,dx.$$

$$= \frac{h}{2}\left[\left(r(x_{j-1})\,\cancel{\phi_j(x_{j-1})}^{0}\right) + r(x_j) + \phi_j(x_j)\right]$$

$$+ \frac{h}{2}\left[r(x_j)\,\phi_j(x_j) + r(x_{j+1})\,\cancel{\phi_j(x_{j+1})}^{0}\right]$$

$$= \frac{h}{2}\,r(x_j)\times 2 = \boxed{r(x_j)\cdot h}$$

$F_j = r(x_j)\cdot h$

$AU = F \Rightarrow U = A\backslash F$

After this.

$$U_h(x) = \sum_{i=1}^{N-1} U_i\,\phi_i(x)$$

Q. $u(x_j) \approx U_i$

Exercise if $\phi_j(x)$ is quadratic.

Using Simpson's rule evaluate the entries

$$a_{ji} = \underbrace{\int_0^1 p(x)\,\phi_i'(x)\,\phi_j'(x)\,dx}_{I_1} + \underbrace{\int_0^1 q(x)\,\phi_i(x)\,\phi_j(x)\,dx}_{I_2}.$$

when $j=i$

$$I_1 = \int_{x_{i-1}}^{x_i} p(x)\,\phi_i'(x)\,\phi_i'(x)\,dx + \int_{x_i}^{x_{i+1}} q(x)\,\phi_i'(x)\,\phi_i'(x)\,dx.$$

$\phi_i' = \frac{1}{h}$

$$I_1 = \int_{x_{i-1}}^{x_i} p(x)\, \phi_i'(x) \phi_i'(x)\, dx + \int_{x_i}^{x_{i+1}} p(x)\, \phi_i'(x) \phi_i'(x)\, dx.$$

$$\frac{1}{h^2}\int_{x_{i-1}}^{x_i} p(x)\, dx. + \frac{1}{h^2}\int_{x_i}^{x_{i+1}} p(x)\, dx.$$

$$\frac{1}{h^2} \times \frac{h}{6}\left[p(x_{i-1}) + 4p\left(\frac{x_i + x_{i-1}}{2}\right) + p(x_i)\right]$$

$$+ \frac{1}{h^2} \times \frac{h}{6}\left[p(x_i) + 4p\left(\frac{x_i + x_{i+1}}{2}\right) + p(x_{i+1})\right]$$

$$\frac{1}{6h}\left[2p(x_i) + p(x_{i-1}) + p(x_{i+1}) + 4p\left(\frac{x_i + x_{i-1}}{2}\right) + 4p\left(\frac{x_i + x_{i+1}}{2}\right)\right]$$

$$= \frac{1}{6h}\left[2p(x_i) + p(x_{i-1}) + p(x_{i+1}) + 2p(x_i) + 2p(x_{i-1}) + 2p(x_i) + 2p(x_{i+1})\right]$$

$$= \frac{1}{6h}\left[6p(x_i) + 3p(x_{i-1}) + 3p(x_{i+1})\right)$$

$$= \left[\frac{p_{i-1}}{2} + p_i + \frac{p_{i+1}}{2}\right]\frac{1}{h}.$$

$$a_{i\,i-1} = \int_{x_{i-1}}^{x_i} p(x)\, \phi_{i-1}'(x)\, \phi_i'(x)\, dx =$$

$$= \left(\frac{-1}{h}\right)\left(\frac{1}{h}\right) \int_{x_{i-1}}^{x_i} p(x)\, dx$$

$$= -\frac{1}{h^2} \times \frac{h}{6}\left[p_{i-1} + 2p_i + 2p_{i-1} + p_i\right]$$

$$= -\left(\frac{p_{i-1} + p_i}{2}\right) \times \frac{1}{h}$$

$$a_{i\,i+1} = \int_{x_i}^{x_{i+1}} p(x)\,\phi_i'(x)\,\phi_{i+1}'(x)\,dx$$

$$= \left(\frac{1}{h}\right)\left(-\frac{1}{h}\right)\int_{x_i}^{x_{i+1}} p(x)\,dx.$$

$$= -\frac{1}{h^2} \times \frac{h}{6}\left[p_i + 2p_i + 2p_{i+1} + p_{i+1}\right]$$

$$= -\frac{h}{2}\left[p_i + p_{i+1}\right]$$

when $j = i$

I_2. $$\int_{x_{i-1}}^{x_i} q(x)\left(\phi_i(x)\right)^2 dx + \int_{x_i}^{x_{i+1}} q(x)\left(\phi_i(x)\right)^2 dx.$$

$$= \frac{h}{6}\left[q_{i-1}\cancel{\left(\phi_i(x_{i-1})\right)^2}^{\,0} + 4q\left(\frac{x_{i-1}+x_i}{2}\right)\left(\phi_i\left(\frac{x_{i-1}+x_i}{2}\right)\right)^2 + q(x_i)\left(\phi_i(x_i)\right)^2\right]$$

$$+ \frac{h}{6}\left[q_i\left(\phi_i(x_i)\right)^2 + 4q\left(\frac{x_i+x_{i+1}}{2}\right)\left(\phi_i\left(\frac{x_i+x_{i+1}}{2}\right)\right)^2 + q(x_{i+1})\cancel{\left(\phi_i(x_{i+1})\right)^2}^{\,0}\right]$$

$$= \frac{h}{6}\left[\cancel{4q\left(\frac{x_{i-1}}{2}\right)}\; 4q\left(\frac{x_{i-1}+x_i}{2}\right)\left(\frac{\phi_i(x_i)+\phi_i(x_{i-1})}{2}\right)^2 + q_i\right]$$

$$\frac{h}{6}\left[q\left(\frac{x_{i-1}+x_i}{2}\right) + q_i\right] + \frac{h}{6}\left[q_i + q\left(\frac{x_i+x_{i+1}}{2}\right)\right]$$

$$\frac{h}{3}q_i + \frac{h}{6}\left[q\left(\frac{x_{i-1}+x_i}{2}\right) + q\left(\frac{x_i+x_{i+1}}{2}\right)\right]$$

$$a_{i\,i-1} = \frac{h}{6}\left[q\left(\frac{x_{i-1}+x_i}{2}\right) + q_i\right]$$

$$a_{i\,i+1} = \frac{h}{6}\left[q_i + q\left(\frac{x_i+x_{i+1}}{2}\right)\right]$$

Assume $\phi_i(x)$ is Quadratic then find A and F

$-u'' = f.$

$$\begin{cases} -(p(x)u')' + q(x)u = r(x) \\ u(0) = u(1) = 0. \end{cases}$$

$\text{Span}\{\phi_1, \ldots \phi_{N-1}\} = V_h.$

$\tilde{A}U = f.$

$$\int_0^1 p(x)\,\phi_i'\,\phi_j'\,dx + \int_0^1 q(x)\,\phi_i\,\phi_j\,dx = \int_0^1 f\phi_j\,dx$$

$$\int \phi_i'\phi_j'\,dx \quad \int \phi_i\phi_j\,dx.$$

Right hand side vector can be calculated directly or it can be calculated from $f_j B$.

$$A = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

$$B = h\begin{bmatrix} 2 & 1 & \\ 1 & 4 & 1 \\ \vdots & & \end{bmatrix}$$

$$(1)\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & (x,t) \in (-\infty, \infty) \times [0,T] \\ & (x_{min}, x_{max}) \times [0,T] \\ B.C. & \\ I.C. & u(x,t) \longrightarrow \text{Basis } \phi(x,t) \end{cases}$$

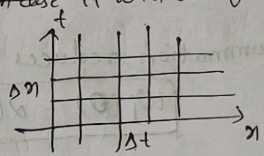
By using Method of Horizental lines we can convert PDE to system of 2nd order ODE's which can be solved by Finite Element method.

Rather one can follow discretizing both time and space by finite element.

The soln. $$\boxed{u(x,t) = \sum_{i=1}^{N} w_i(t)\phi_i(x) + \underbrace{\phi_0(x,t)}_{\text{Boundary condition}}} \rightarrow (2)$$

where ϕ_0 satisfies the boundary and initial conditions.

In earlier case of ODE the soln. of $u_h(x) = \sum U_i \phi_i(x)$.
where as in the case of parabolic PDEs. ~~in case~~ it will be of function of times also.



Using (2) in eq. (1). We obtain

$$\boxed{\int_{x_0}^{x_m} \left[\sum_{i=1}^{m-1} \mathring{w}_i \phi_i + \mathring{\phi}_0\right] \phi_j \, dx = \int_{x_0}^{x_m} \left[\sum_{i=1}^{m-1} w_i \phi_i'' + \phi_0''\right] \phi_j \, dx.} \rightarrow (3)$$

$j = 1, \ldots, m-1.$

$\circ$ denotes w.r.t. t.

$\mathring{w}_i$ denotes derivative w.r.t. t and ϕ_i'' denotes derivative w.r.t. x.

eq. (3) can be written in terms of ~~ODE~~ matrix vector form.

$$(4) \leftarrow \boxed{B\mathring{w} + b = -Aw - a}$$

$$a(t) = \begin{pmatrix} \int \phi_0'' \phi_1 \, dx \\ \int \phi_0'' \phi_2 \, dx \\ \vdots \\ \int \phi_0'' \phi_{n-1} \, dx \end{pmatrix} \qquad b(t) = \begin{pmatrix} \int \mathring{\phi}_0 \phi_1 \, dx \\ \int \mathring{\phi}_0 \phi_2 \, dx \\ \vdots \\ \int \mathring{\phi}_0 \phi_{n-1} \, dx \end{pmatrix}$$

(4) is system of first order ODEs which is method of vertical lines

$$w = [w_1(t) \ldots\ldots w_{n-1}(t)]^T$$

We have to determine w which is soln. of eq. (4)

At $t=0$ $\boxed{u(x,0) = \alpha(x)} \rightarrow (5)$

From (2) and (5)

$$\sum_{i=1}^{N} w_i(0)\, \phi_i(x) + \phi_0(x,0) = \alpha(x)$$

At $x = x_j$

Since $\phi_i(x_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

Summation reduces to 1 term.

$$\boxed{w_j(0) = \alpha(x_j) - \phi_0(x_j, 0)}$$ Initial condition.

$B = \int \phi_i \phi_j \, dx \qquad A = \int \phi_i' \phi_j' \, dx.$

$$B = \begin{bmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{bmatrix}$$

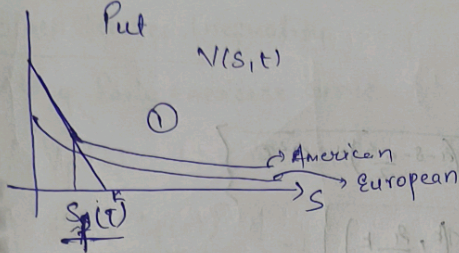
$$\phi_i(x_j) = \delta_{ij}$$

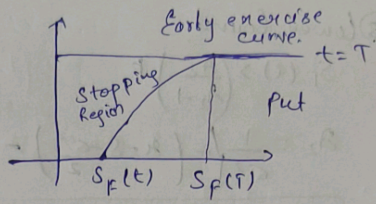
$$B\dot{w} + b = -Aw - a$$

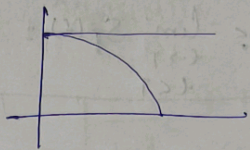
$$w_j(0) = \alpha(x_j) - \phi_0(x_j, 0)$$

Using crank Nicolson

$$\left(B + \frac{\delta t}{2} A\right) w^{n+1} = \left(B - \frac{\delta t}{2} A\right) w^n - \frac{\delta t}{2}\left(a^n + a^{n+1} + b^n + b^{n+1}\right)$$







American options

Free Boundar

Value of American option never goes below european option

$V^{Am} \geq V^{E}$

$V^{Am}_{p(s,t)} \geq (K-S)^+$ $\quad V(S,t)$.

$V^{Am}_{c}(S,t) \geq (S-K)^+$

Consider Fig. ① $\quad V_p^{Am}(S,t) > (K-S)^+ \quad S > S_f \cancel{(K-S)^+}(t)$

$V_p^{Am}(S,t) = K-S \quad S = S_f$

As time varies $S_f(t)$ divides domain into 2 region stopping region and holding region where Black Scholes PDE.

The early exercise curve is monotonic and smooth curve because $S_f(t)$ is not known apriori (don't know in advanced) it changes with time.

∴ American options are preferred as free boundary problems because boundary at one side vary w.r.t. time.

Properties of Early exercise curve.

Put

1) S_f is continuously diff. $0 \leq t \leq T$

2) S_f is non decreasing.

3) ~~lower bound~~ $S_f(t) > \left(\frac{\lambda_2}{\lambda_2 - 1}\right) K$

③ Lower bound

$$S_f(t) > \left(\frac{\lambda_2}{\lambda_2 - 1}\right) K$$

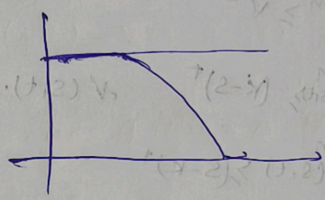
$$\lambda_2 = \frac{1}{\sigma^2}\left\{-\left(r-\delta-\frac{\sigma^2}{2}\right) - \sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}\right\}$$

④ $$\boxed{S_f(t) \le \lim_{\substack{t \to T \\ t < T}} S_f(t) = \min\left(K, \frac{r}{\delta} K\right)}$$

<u>Call:</u>

① S_f continuously differentiable (smooth) $0 \le t \le T$

② S_f non-increasing



③ upper bound

$$S_f(t) < \left(\frac{\lambda_1}{\lambda_1 - 1}\right) K$$

$$\lambda_1 = \frac{1}{\sigma^2}\left\{-\left(r-\delta-\frac{\sigma^2}{2}\right) + \sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r}\right\}$$

④ Lower bound

$$S_f(t) = \lim_{\substack{t \to T \\ t < T}} S_f(t) > \max\left(K, \frac{r}{\delta} K\right)$$

In case of Put $r \to 0$ $\quad S_f \to 0$ No stopping region: it becomes similar to European

In case of call, $\delta \to 0$ $\quad S_f \to \infty$
non-dividend paying

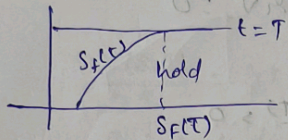
~~Black-Schol~~

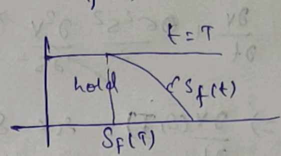
Black Scholes inequality.

Along Early exercise curve.

Call

$$\begin{cases} V_p(S_f(t), t) = k - S_f(t) \\ \dfrac{\partial V_p(S_f(t))}{\partial S_f(t)} = -1 \end{cases} \qquad V_c(S_f(t), t) = S_f(t) - k \qquad \frac{\partial V_c(S_f(t))}{\partial S_f(t)} = 1.$$





$$\mathcal{L}_{BS}(V) = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0 \quad (1)$$

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = 0$$

Consider case of American put.

$$V = k - S$$

$$\frac{\partial V}{\partial t} = 0$$

$$\frac{\partial V}{\partial S} = -1$$

$$\frac{\partial^2 V}{\partial S^2} = 0$$

$\frac{\partial V}{\partial S}(r-\delta)S$

(1) $\Rightarrow$

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V)$$

$$= (-1)(r-\delta)S - r\overset{(k-S)}{V} \neq 0$$

$\cancel{(r-\delta)}$

$$-S\cancel{r} + \delta S - rk + \cancel{rS} \neq 0$$

$$\delta S - rk$$

From the upper bound of early curve.

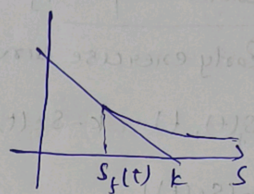
$$S < S_f(T)$$

$$\delta S < \delta S_f(t) < rk$$

$$\delta S - rk < 0 \longrightarrow \frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) \leq 0$$

$$\begin{cases} \dfrac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) < 0 & S \leq S_f(t) \Rightarrow V(S,t) = k - S. \\ \dfrac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = 0 & S > S_f(t) \Rightarrow V(S,t) \text{ satisfies BS PDE.} \end{cases}$$

$S_f(t) < \lim_{\substack{t \to T \\ t<T}} S_f(t) = \min\left(K, \frac{r}{\delta}K\right)$



WKT when $t=T$

$$V_p(S,T) = K - S \quad \boxed{S<K}$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2}\cancel{\frac{\partial^2 V}{\partial S^2}}^{\,0} + (r-\delta)S\cancel{\frac{\partial V}{\partial S}}^{\,-1} - rV = 0$$

$\Rightarrow \frac{\partial V(S,T)}{\partial t} = rK - \delta S$

$$\boxed{\frac{\partial V}{\partial t}(S,T) \le 0}$$

if $\frac{\partial V(S,T)}{\partial t} > 0$.

$\frac{V(S,T) - V(S,T-\delta t)}{\partial t} > 0$

$(K-S) > V(S,T-\delta t)$

we can have $\frac{\partial V(S,T)}{\partial t} \le 0$ otherwise. if it is > 0 then

$V \geq$ payoff.

$\because$ $rK - \delta S \le 0$ $\quad t=T$

$S<K.$

$rK \le \delta S$

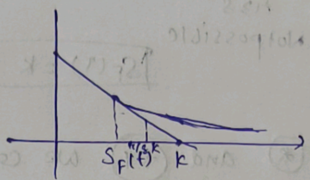
$S \ge \frac{r}{\delta}K$ It is meaningful when $r<\delta$

$\Rightarrow S_f(T) = \lim_{\substack{t\to T \\ t<T}} S_f(t)$

$\Rightarrow \boxed{S_f(T) = \frac{r}{\delta}K}$ — (*)

$S_F(T) < \frac{r}{\delta}k \rightarrow (1)$ $\quad r < \delta$

$S_F(T) \geq \frac{r}{\delta}k \rightarrow (2)$



Consider case (1) $S_F(T) < \frac{r}{\delta}k$

Choose S s.t. $S_F(T) < S < \frac{r}{\delta}k$

$$\frac{\partial V(S,T)}{\partial t} = rk - \delta S > 0$$

This contradicts $\frac{\partial V(S,T)}{\partial t} \leq 0$ $\therefore$ This case is not true.

Consider case (2) $S_F(T) > \frac{r}{\delta}k$

Choose S s.t. $\Rightarrow \frac{r}{\delta}k < S < S_F(T)$.

$rk < \delta S$

$$\int_0^t (rk)\, e^{rt} dt < \int (\delta S)\, e^{\delta t} dt$$

$rk(e^{rt}-1) < \delta S(e^{\delta t}-1) \Rightarrow$ dividend earns more than interest that means early exercise curve is not optimal.

$\therefore$ This is ruled out.

$r < \delta$

$\hookrightarrow \boxed{S_F(T) = \frac{r}{\delta}k}$

Now consider $r \geq \delta$. it is not possible that from the defn. of contact point $S_F(T)$ $\cancel{**}$

$S_F(T) > k$.

Now consider $S_F(T) < k$

$S_F(T) < S < k$, $t \simeq T$

$\frac{\partial V(S,T)}{\partial t \leq 0} = \underset{>0}{rk - \delta S} \rightarrow$ contradictions

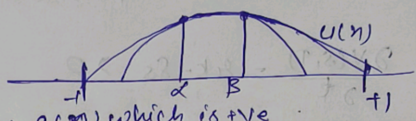
$r \geq \delta$

$\boxed{\begin{matrix} S_f(T) < k \\ S_f(T) > k \end{matrix}}$ Not possible

$\boxed{S_f(T) = k}$ (**)

Combining (*) and (**) we can get limiting value of $S_f(t)$

$$\lim_{\substack{t \to T \\ t < T}} S_f(T) = \min\left(k, \frac{r}{\delta} k\right)$$



Obstacle problem

Consider an obstacle denoted by $g(x)$ which is +ve

i.e.

$$\boxed{\begin{matrix} g(x) > 0, \ \alpha < x < \beta \\ g \in C^2 \\ g'' < 0 \\ g(-1) < 0 \quad g(1) < 0 \end{matrix}}$$

$u(x)$??

Now

we can reformulate the obstacle problem as determine.

$u \in C'[-1,1]$

$$(*) \begin{cases} \text{for } -1 < x < \alpha & : u'' = 0 & u > g \\ \text{for } \alpha < x < \beta & u = g & u'' = g'' < 0 \\ \text{for } \beta < x < 1. & u'' = 0 & u > g. \end{cases}$$

From (*) we can rewrite the obstacle problem by eliminating contact points α and β that means ~~if $u = g$~~

$$\left.\begin{matrix} \text{if} \quad u > g \quad u'' = 0 \\ \text{if} \quad u = g \quad u'' < 0 \end{matrix}\right\} \Rightarrow \begin{matrix} V > \text{payoff.} & \text{B.S. PDE.} \\ V = \text{payoff.} & \text{B.S. Inequality} \end{matrix}$$