~~Now~~ 4/3/24

$$\int_a^b f(x)\,dx$$

Consider $k = x$

$$f(x) = P_3(x) + f[x_0, x_1, x_2, x_3, x]\,\psi_3(x)$$

$$x_0 = x_1 = a \qquad x_2 = x_3 = b$$

$$\psi_3(x) = (x-x_0)^2(x-x_1)^2 = (x-a)^2(x-b)^2$$

$$E(x) = \frac{1}{4!} f^{(4)}(\eta) \int_a^b (x-a)^2(x-b)^2\,dx$$

$$= \frac{f^{(4)}(\eta)}{720}(b-a)^5$$

$$I(f) = \int_a^b P_3(x)\,dx = \cancel{\int_a^b f(x_0) + f[x_0, x_0](x-x_0) + f[x_0, x_0, x_1](x-x_0)^2 + f[x_0, x_0, x_1, x_1]}$$

$$P_3(x) = f(a) + f'(a)(x-a) + f[a,a,b](x-a)^2 + f[a,a,b,b](x-a)^2(x-b)$$

$$I(f) = f(a)(b-a) + f[a,a]\frac{(b-a)^2}{2} + f[a,a,b]\frac{(b-a)^3}{3} + f[a,a,b,b]\frac{(b-a)^4}{12}$$

$$\int (x-a)^2(x-b)$$

$$= \frac{(b-a)^3}{3} - \frac{(x-a)^3}{3}$$

$I(f) =$ Corrected trapezoidal

$$= \frac{(b-a)}{2}\left[f(a)+f(b)\right] + \frac{(b-a)^2}{12}\left[f'(a) - f'(b)\right]$$

So far, the interpolate integrand we use Newton's divided differences. Now we consider the Lagrange interpolating polynomial $p_n(x)$ $n \geq 1$

and $h = \frac{b-a}{h}$, $x_j = a + jh$ $j = 0, \dots n$

Newton - Cotes /

$$I(f) = \int_a^b f(x)\,dx = I_n(f) = \int_a^b p_n(x)\,dx$$

$$= \int_a^b \sum_{j=0}^{n} l_{j,n}(x)\, f(x_j)\,dx$$

$$= \sum_{j=0}^{n} w_{j,n}(x)\, f(x_j)$$

$$w_{j,n}(x) = \int_a^b l_{j,n}(x)\,dx$$

$$I(f) = \frac{1}{2}(b-a)\left(f(b)+f(a)\right)$$

→ Take $n = 1$,

$$l_0 = \frac{(x-x_1)}{x_0-x_1}$$

$$l_1 = \left(\frac{x-x_0}{x_1-x_0}\right)$$

$$\int_a^b \frac{x-x_1}{\cancel{x_0-x_1}\; a-b}$$

$$\frac{(b-a)}{2}$$

$$\frac{(x-x_0)^2}{2\,(b-a)}(b-a)$$

consider the case $n=3$, therefore

$$w_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx$$

$x = x_0 + \mu h$

$$= \int_a^b \frac{(\mu-1)(\mu-2)(\mu-3)}{-6h^3}\,h^3\,(h\,d\mu)$$

$$= \frac{-1}{6}\int_0^3 (\mu-1)(\mu-2)(\mu-3)\,h\,d\mu$$

$$= \frac{3}{8}h$$

$$w_1 = \int_a^b l_1(x)\,dx = \int_a^b \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\,dx$$

$x = x_0 + \mu h$

$$\Rightarrow \quad w_1 = \int_0^3 \frac{(\mu)(\mu-2)(\mu-3)\,\cancel{h^3}\,h\,d\mu}{2\cancel{h^3}}$$

$$= \frac{9}{8}$$

$$\int_0^3 \mu(\mu^2-5\mu+6$$

$$\int_0^2 \mu^3-5\mu^2+6\mu$$

$$\left[\frac{\mu^4}{4} - \frac{5\mu^3}{3} + \frac{6\mu^2}{2}\right]h$$

$$\frac{81}{4} - \frac{45}{3} + 27$$

$$\frac{4-40+12}{3}$$

$$I_n(f) = \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]$$

<u>Error</u>:

(i) for even n, Assume $f(x)$ is $(n+2)$ times cont diff on $[a,b]$

$$f \in C^{n+2}[a,b]$$

$$E(f) = I(f) - I_n(f) = c_n h^{n+3} f^{(n+2)}(\eta)$$

$$c_n = \frac{1}{(n+2)!}\int_0^n \mu^2(\mu-1)\cdots(\mu-n)\,d\mu$$

(ii) for odd n, and $f(x)$ is $(n+1)$ times cont diff

$$f \in C^{n+1}[a,b]$$

$$E(f) = c_n h^{n+2} f^{(n+1)}(\eta)$$

$$c_n = \frac{1}{(n+1)!}\int_0^n \mu(\mu-1)\cdots(\mu-n)\,d\mu$$

These are "Newton cotes formulae"

Composite formulae

→ Always it is advisable to divide the interval $[a,b]$ into $a = x_0 < x_1 \cdots < x_{n-1} < x_n = b$ equidistant. $[x_i, x_{i+1}]$ and apply all the quadrature formulas.

$$I(f) = \int_a^b f(x)\,dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

one can replace $f(x)$ by polynomials of various degrees.

$$\int_{x_{i-1}}^{x_i} f(x)\,dx = \int_{x_{i-1}}^{x_i} P_{i,k}(x)\,dx$$

when $k = 0$, constant interpolant

$$I(f) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

Error : $\sum \frac{f'(\eta_i)}{2!} h^2$ } Error will be less in comparision to the original quadratures.

If I have a kth degree polynomial, can I make the formulae exact upto $2k+1$ degree

↓

Gaussian quadratures

consider the numerical quadrature of

$$\int_a^b f(x)\,dx = \int_a^b w(x)\,g(x)\,dx$$

→ To remove bad singularities.

$I(g) \approx A_0 g(x_0) + A_1 g(x_1) + \cdots A_k g(x_k)$

→ These weights are independent of the function $g(x)$

A_i's are ~~weights which are~~

So far, we have chosen nodal points $x_0, \ldots x_k$ are equispaced. and we replaced by a polynomial of degree k.

All rules derived →

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$$I(f) = \int_a^b g(x)\, w(x)\, dx$$

where w is a non-negative integrable function

$g(x) = \dfrac{f(x)}{w(x)}$ is smooth

$w(x) = (x-a)^{\alpha}$

If $f(x)$ behaves like $(x-\alpha)^{a}$ for $\alpha > -1$ (or) $a \to -\infty, \infty$ etc

we can express this as weight $w(x)$

$x_0\ x_1 \cdots x_k$ in (a,b)

$\rightarrow g(x) = p_k(x) + g[x_0\ x_1 \cdots x_k, x]\, \psi_k(x)$ } Analytic substitution

where $\psi_k(x) = (x-x_0)(x-x_1) \cdots (x-x_k)$

$$\Rightarrow I(g) = I(p_k) + \int_a^b g[x_0 \cdots x_k, x]\, \psi_k(x)\, dx$$

$I(p_k)$ is clearly of form $A_0 g(x_0) + A_1 g(x_1) \cdots + A_k g(x_k)$

$$p_k(x) = \sum_{i=0}^{k} g(x_i)\, l_i(x)$$

If $p_k(x) = g(x_0)\, l_0(x) + g(x_1)\, l_1(x) \cdots + g(x_k)\, l_k(x)$

$$l_i(x) = \prod_{j \neq i} \frac{(x-x_j)}{(x_i - x_j)}$$

$$I(p_k) = \int_a^b \sum_{i=0}^{k} g(x_i)\, l_i(x)\, w(x)\, dx$$

$$= \sum_{i=0}^{k} g(x_i) \int_a^b l_i(x)\, w(x)\, dx$$

$\rightarrow$ ~~$A_i = \int_a^b l_i\, w(x)\, dx$~~

$\rightarrow$ In general, the error in this quadrature will be

$g^{(k+1)}(\eta)$ (or) $g^{k+2}(\eta)$

$g \to$ degree $\leq K$

$$E(g) = I(g) - I(p_k) = \int_a^b g[x_0 \cdots x_k, x]\, w(x)\, \psi_k$$

Suppose $\int_a^b \psi_k(x)\, w(x)\, dx = 0$ then

$$\int_a^b g[x_0 \cdots x_k, x]\, w(x)\, \psi_k$$

$$= g[x_0 \ldots x_{k+1}] + g[x_0 \cdots x_{k+1}, x](x - x_{k+1})$$

$$E(g) = \int_a^b g[x_0 \cdots x_{k+1}, x]\, \psi_{k+2}(x)\, w(x)\, dx$$

$\to$ we can do this many times so, for certain $x_0\ x_1 \cdots x_{k+1+i}$, we have

$$\int_a^b \psi_k(x)\,(x - x_{k+1})(x - x_{k+2}) \cdots (x - x_{k+1+i})\, w(x)\, dx = 0 \quad i = 0 \ldots m-1$$

$$\to E(g) = \int_a^b g[x_0 \cdots x_{k+m+1}, x]\, \psi_{k+m+1}(x)\, w(x)\, dx$$

For several choices of $w(x)$ we can find $\psi_{k+1}(x)$ s.t

$$\int_a^b p_{k+1}(x)\, \underbrace{q(x)}_{\text{degree} \leq k}\, w(x)\, dx = 0 \quad \text{— (1)}$$

orthogonality

split ψ_{k+1} into polynomial of degree $k+1$ — ψ_{k+1} and $q(x)$

① tells us that polynomials are orthogonal

→ we can express $p_{k+1}(x)$ as

$$p_{k+1}(x) = \alpha_{k+1}(x-\xi_0) \cdots (x-\xi_k)$$

$\xi_0, \xi_1 \cdots \xi_k$ are distinct $(k+1)$ points/roots in $[a,b]$ } Here p_{k+1} vanishes

→ $x_j = \xi_j$ $j = 0, \cdots k$ and x_{j+k} are arbitrary roots in $[a,b]$

$j = 1$ to $k+1$ $m = k$

$$\varphi_{k+1}(x)\, q(x) = \psi_{2k+1}(x)$$

$$q(x) = \underbrace{(x-x_{k+1})(x-x_{k+2}) \cdots (x-x_{k+1+i})}_{\alpha_{k+1}}$$

$i = 0, \cdots m-1$

→ $$E(g) = \int_a^b g[x_0 \cdots x_{k+m+1}, x]\, \varphi_{2k+1}(x)\, w(x)\, dx$$

In order to obtain error

$$x_{k+j} = \xi_{j-1} \qquad j = 1 \cdots k+1$$

$$\psi_{2k+1}(x) = (x-x_0) \cdots (x-x_{2k+1})$$

$$= (x-\xi_0)(x-\cdot) \cdots (x-\xi_k)(x-\xi_0) \cdots (x-\xi_k)$$

$$= \left[\frac{P_{k+1}(x)}{\alpha_{k+1}}\right]^2$$

$$E(g) = g[x_0 \cdots x_{2k+1}, \eta] \int_a^b \left(\frac{1}{\alpha_{k+1}} p_{k+1}(x)\right)^2 w(x)\,dx$$

$$= \frac{1}{(2n+2)!} g^{(2k+2)}(\eta) \frac{\delta_{k+1}}{\alpha_{k+1}^2}$$

→ we have to choose the nodal pts of $p_{k+1}(x)$ of degree $k+1$ which is orthogonal to itself wrt $w(x)$

→ Gaussian quadrature is exact upto all polynomials of degree $2k+1$

→ Assume $p_{k+1}(x)$ or legendre polynomial

$w(x) = 1$

$p_1(x) = x \qquad \xi_0 = 0$

$p_2(x) = \frac{3}{2}\left(x^2 - \frac{1}{3}\right) \qquad \xi_0 = \frac{1}{\sqrt{3}} \quad \xi_1 = \frac{-1}{\sqrt{3}}$

$p_3(x) = \frac{5}{2}\left(x^3 - \frac{3x}{5}\right)$

$\xi_0 = -\sqrt{\frac{3}{5}}, \quad \xi_1 = 0, \quad \xi_2 = \sqrt{\frac{3}{5}}$

<u>Legendre</u>

$$p_{n+1} = x\, p_n(x) - \frac{n^2}{4n^2-1} p_{n-1}(x)$$

The integral $\int_a^b f(x)\,dx$ } → Bring it to -1 to 1

$x = a + b + \frac{(b-a)}{2} t$ $\qquad$ t is -1 to 1

Take $a = 1$

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$k=1$

$$\left.\begin{aligned} x_0 &= \xi_0 = -1/\sqrt{3} \\ x_1 &= \xi_1 = 1/\sqrt{3} \end{aligned}\right\} \text{ zeros of legendre polynomial}$$

$$\int_{-1}^{1} g(x)\,dx = A_0\, g\left(-1/\sqrt{3}\right) + A_1\, g\left(1/\sqrt{3}\right)$$

$$E = c\, g''''(\eta)$$

$$A_1 = \int_{-1}^{1} \frac{x - (-1/\sqrt{3})}{\left(1/\sqrt{3} - (-1/\sqrt{3})\right)}\,dx = 1$$

$$\int_{-1}^{1} g(x)\,dx = g\left(-1/\sqrt{3}\right) + g\left(1/\sqrt{3}\right)$$

$$E = \frac{1}{135}\, g^{(iv)}(\eta)$$

$$A_0 = 1$$

For $k=2$, $\rightarrow$

$\rightarrow$ we can obtain gaussian Quadratures from orthogonal polynomials

Ex: from Hermite Interpolant.

Numerical Differentiation

$$f'(x_i) = \frac{f_{i+1} - f_i}{h} \quad \Big\} \quad O(h) \rightarrow$$

$$= \frac{f_i - f_{i-1}}{h}$$

$$= \frac{f_{i+1} - f_{i-1}}{2h} \qquad O(h^2)$$

$$f_{i+1} = f(x_{i+1}) = f(x_i) + h f'(x_i) + \frac{h^2}{2!} f''(\xi_n)$$

$$\frac{f_{i+1} - f_i}{h} = \underbrace{\frac{h}{2} f''(\xi_n)}_{\text{first dominant term.}} = O(h)$$

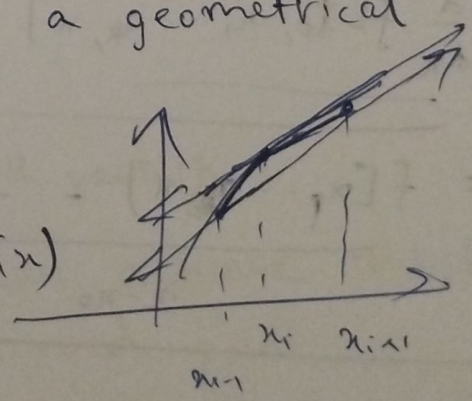
$$f_{i-1} = f(x_{i-1}) = f(x_i) - h f'(x_i) + \frac{h^2}{2!} f''(x_i) - \frac{h^3}{3!} f'''(x_i) \quad \cdots$$

$$f_{i+1} = f_i + h f_i' + \frac{h^2}{2!} f_i'' + \cdots$$

$$\frac{f_{i+1} - f_{i-1}}{2h} = \frac{2h f_i'}{2h} + \frac{\frac{2h^3}{3!} f_i'''(\eta_i)}{2h}$$

$$- f_i' = \frac{h^2}{3} f_i'''(\eta_i) \quad \Big\} \; O(h^2)$$

Central Difference is the average of forward and backwar-d difference, we can also give a geometrical interpretation



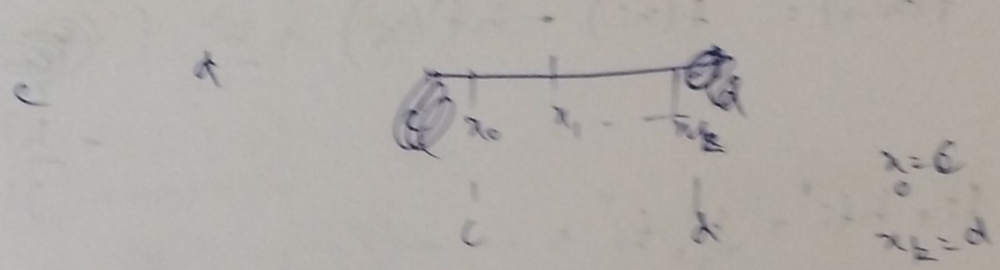
$$f(x) = P_k(x) + f[x_0 \cdots x_k, x] \, \psi_k(x)$$

$$f'(x) = P_k'(x) + E'(x)$$

Suppose we are given tabulated values (interpolating values) and we want to determine the derivative of the data,

Ex: Given displacement at different times, we can calculate velocity (first derivative) (or) acceleration (second derivative)

Let $f(x)$ be cont diff. $f \in C^1$ on $[c,d]$ which contains (a,b)



→ Since x_i's are distinct we can approximate a polynomial of degree k.

$$\psi_k(x) = \prod_{j=0}^{k} (x - x_j)$$

$$f[x_0, x] = f'(x) \quad \Big| \quad \frac{d}{dx} f[x_0, x_1] = f[x_0, x_1, x]$$

$$\frac{d}{dx} f[x_0, x_1 \cdots x_k, x] = f[x_0, \cdots x_k, x, x]$$

$$= \frac{f[x_0 \cdots x_k, x] - f[x_0 \cdots x_k, x]}{x - x_0}$$

$$f'(x) = P_k'(x) + f[x_0 \cdots x_k, x, x]\, \psi_k(x) + f[x_0 \cdots x_k, x]\, \psi_k'(x)$$

let D denote the differential operator.

$D = f'(a)$ at point a for some $a \in [c, d]$

$D(f) = P_k'(x)$

Error $= D(f) - P_k'(x) = E(f) = f[x_0 - x_k, x, x]\, \psi_k(x)$

$+ f[x_0 - x_k, x]\, \psi_k'(x)$

→ Since the error is containing two terms, by choosing $a = x_i$ for some i, then first term is zero ($\psi_k(x_i) = 0$)

~~and~~ whereas if we choose point a such that $\psi_k'(a) = 0$ then the second term disappears.

→ Suppose we pick up $\dfrac{\psi_k(x)}{x - x_i}$ $\qquad \psi_k'(a) = q(a)$

<u>Case (i)</u> $a = x_i$ for some 'i',

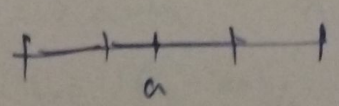
then $E(f) = \dfrac{1}{(k+1)!} f^{k+1}(\eta) \displaystyle\prod_{\substack{j=0 \\ j \neq i}}^{k} (x - x_j)$ $\quad$ (if $f \in C^{k+1}[c, d]$)

<u>Case (ii)</u> : where $\psi_k'(a) = 0$

$\psi_k'(a) = q(a) = 0 = \dfrac{\psi_k(x)}{x - x_i}$

~~of~~ Then in error, second term will vanish

Suppose k is odd, then we can achieve this by placing x_j's symmetrically around point a.



$x_{k-j} - a = a - x_j$ for $j = 0 \ldots (k-1)/2$

$$(x-x_j)(x-x_{k-j}) = (x-a+a-x_j)(x-a+a-x_{k-j})$$

$$= (x-a)^2 + \cancel{(x-a)(a-x_j)} - (a-x_j)^2$$

$$\psi_k(x) = \prod_{j=0}^{k} (x-x_j) = \prod_{j=0}^{(k-1)/2} (x-a)^2 - (a-x_j)^2$$

Derivative of $\psi_k(x)$ is

$$\psi_k'(x) = \cancel{\left(\frac{k-1}{2}\right)\left[(x-a)^2 - (a-x_j)^2\right]^{\frac{k-3}{2}} (2(x-a)) - 2(a-x_j)}$$

$$0 \quad \text{at } 'a'.$$

$$E(f) = \frac{1}{(k+2)!} f^{(k+2)}(\eta) \prod_{j=0}^{j-1/2} \left[(x-a)^2 - (a-x_j)^2\right]$$

If $k=0$ then constant interpolant,

$$D(p_k) = 0$$

<u>If $k=1$,</u> $\quad p_k(x) = f(x_0) + f[x_0, x_1](x-x_0)$

$$D(p_k) =$$

$$P(f) = D(p_k) = f[x_0, x_1, x](x-x_0) + f[x_0, x_1]$$

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$k=1,$

$$P_k = P_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$$

$$D(P_k) = f[x_0, x_1]$$

$$f'(a) = f[a, a+h] = \frac{f(a+h) - f(a)}{a}$$

$$=$$

$$E(f) = -\frac{h^2}{2} f''(x)$$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(\eta)$$

$$\frac{f(a+h) - f(a)}{h} = f'(a)$$

$\rightarrow$

$$x_1 = a$$

$\rightarrow$

$$D(P_k) = f[a-h, a]$$

$$f'(a) = f[a-h, a] = \frac{f(a) - f(a-h)}{a}$$

$$E(f) = -\frac{h^2}{2} f''(\eta)$$

$$\cancel{\frac{f(a) - f(a-h)}{a} = f'(a-h)}$$

$\rightarrow$

$$a = \frac{x_0 + x_1}{2}$$

$$x_0 = a-h \qquad x_1 = a+h$$

$$h = (x_1 - x_0)/2$$

$$f'(a) = \frac{f(a+h) - f(a-h)}{2h}$$

$$= -\frac{h^2}{6} f'''(\eta)$$

$$f(a+h) - f(a-h) = \frac{2h^3}{3!} f'''(\xi)$$

If $f \in C^3(I)$

$O(h^2) \rightarrow$ first Dominant.

(I can merge)

$$f'''(a) = \alpha f(a+h) + \beta f(a-h) + \gamma f(a)$$ } central

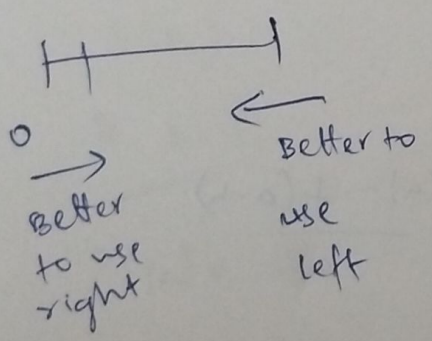
use Taylor's for each.

→ we can try forward direction

$$\alpha f(a) + \beta (a+h) + \gamma f(a+2h)$$

(or) even

$$\alpha f(a) + \beta (a-h) + \gamma f(a-2h)$$



Finite difference methods to solve differential equations

Difficulties

For some random eqn like

$$y'' + (x+1)y' - 5x^2 y = e^x / \sec x / \log x$$ etc

we try to write some power series like $y = \sum a_i x^i$

→ we ~~are~~ need domain of independent variable $x \in (a,b)$

and prescribing boundary conditions

$$\left.\begin{array}{l} y(a) = __ \\ y'(a) = __ \end{array}\right\} \text{"All from Physics"}$$

etc
etc

→ Point is stuff gets complicated. (Even linear case)

$$\begin{cases} y'(t) = f(t, y) & t \in (a,b) \\ y(a) = \alpha \end{cases}$$

→ IVP → at initial point

well posedness

i) Existence

ii) Uniqueness

iii) stability / continuous dependence

If data - changes

$z' = f(t, z) + \delta(t)$ v/s $y'(t) = f(t, y)$ $t \in (a,b)$ → Domain

$z(a) = \alpha \pm \delta$ $y(a) = \alpha$ → I.V

If $|y(t) - z(t)| \leq K(\delta)$ then stable

"ill-posed" if any of the three is violated

Lipschitz - continuity

$|f(t, y_1) - f(t, y_2)| \le L |y_1 - y_2|$

$y_1, y_2 \in D$

weaker sense of continuity of $\frac{\partial f}{\partial y}$

f - const

$\frac{\partial f}{\partial y}$ - cont

Ex:

$y' = t + y \qquad t \in (0,1)$

$y(0) = 1/2$

$e^{-y}(y' - y) = e^{-y} t$

$d(e^{-y} y) = e^{-y} t \, dy \qquad y(0) = 1/2$

$e^{-y}(y)]_0^1 = -e^{-y} t]_0^1$

$\cancel{\frac{y}{e} = -t(e)}$

$\cancel{e^{-y} y = -e^{-y} t + c}$

$\cancel{e^{-1/2}(1/2) = c}$

$\cancel{e^{-y}(y - 2t) = (e^{-1/2}(1/2))}$

$\cancel{e^{-y}}$

$\cancel{\frac{dy}{dt} = t + y}$

$\cancel{dy - y\,dt = t\,dt}$

$\cancel{e^{-t}(dy - y\,dt) = e^{-t} t\,dt}$

$\cancel{d(e^{-t} y) = e^{-y} t\,dt}$

$\cancel{e^{-y} t = \frac{1}{2} t^2 e^{-y} + c}$

$y(0) = 1/2$

$d(e^{-t} y) = e^{-t} t \, dt$

$y = \frac{3}{2} e^{t} - 1 - t$

$\cancel{\frac{3}{2} e^{t} - 1 - t}$

Step I: Discretize domain → Divide into finite no of domains

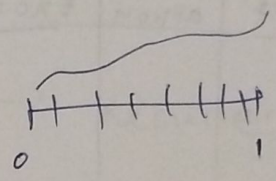
'N' → $h = \frac{b-a}{N}$

$t_0 = a$

$t_1 = a+h$

⋮

$t_n = a+nh$



If n is huge

→ looks continuous

Step II: Replace derivative by finite difference

i) $y' = \frac{y_{n+1} - y_n}{h}$ ii) $y' = \frac{y_n - y_{n-1}}{h}$ iii) $y' = \frac{y_{n+1} - y_{n-1}}{2h}$

Forward diff

$t_0 = a,\ t_1 = a+h \ldots\ t_n = a+nh$

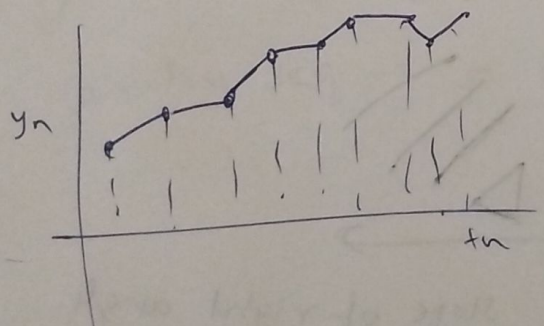
$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)$$

$$y_{n+1} = y_n + h f(t_n, y_n)$$

$n = 0, 1, \ldots N-1$

Also $y_0 = \alpha$

→ Difference equation



"This is Euler's method".

(i) ↓ First order $O(h)$

Input: a, b, N, α

$h = \frac{b-a}{N}$

$t = a:h:b$

for

$y(i+1) = y(i) + f(t(i), y(i))$

end

plot ✓✓

Take error ——

Table - 1

N

t	approx	Exact	Error

$N = 10 \rightarrow$ maximum error entry

$E^N = \max(\text{Error})$

$$\text{order} = \ln 2 \left(\frac{E^N}{E^{2N}} \right)$$

in matlab

N	E^N	order
20		
40		$\ln 2 \left(\frac{E_{40}}{E_{20}} \right)$
80		

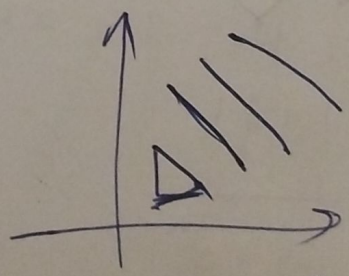
0.942
0.89
etc

Plots:

1) y v/s t
Both approx, exact

2) Error v/s t

3) loglog (N, E^N)



slope of right angle triangle ??

$O(h)$

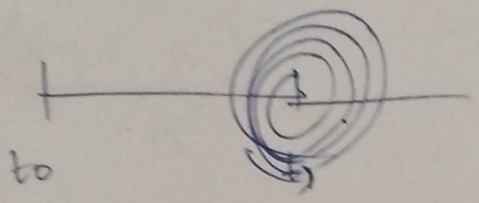
Exercise: Replace by backward formula.

$y_n = y_{n-1} + hf(t_n, y_n)$ → implicit

v/s

$y_{n+1} = y_n + hf(t_{n+1}, y_n)$ → Recurrence

How?



$$\left| y_n^{(k+1)} - y_n^{(k)} \right| < Tol$$

we have

$$y_{n+1}^{(k+1)} = y_n + hf(t_{n+1}, y_{n+1}^{(k)})$$

start with some initial iterate.

This is implicit Euler's

we have an inner loop now,

Cauchy seq → so it should converge if well posed.

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Deriving the same scheme

"using Taylor's expansion".

$$\frac{dy}{dx} = f(t, y)$$

$$y(0) = \alpha$$

$$y(t_{i+1}) = y(t_i + h)$$

$$= y(t_i) + h y'(t_i) + \underbrace{\frac{h^2}{2!} y''(\xi_i)}_{\text{stopped here at degree 2}}$$

$$= y(t_i) + h f(t_i, y(t_i)) + \underbrace{\cancel{\frac{h^2}{2!} y''(\xi_i)}}_{\text{Truncation error}}$$

$O(h)$

$$y_{i+1} = y_i + h(f(t_i, y_i))$$

---

we talk about.

1) Consistency (Truncation error)

$$\text{Truncation error} = \frac{LHS - RHS}{h}$$

If scheme is given, then we use Taylor's and $y(t_{i+1}) = y_i + hf(h, y_i)$ get the Truncation error

$$y_{i+1} \rightarrow \left(y_i + hy_i' + \frac{h^2}{2!} y_i'' + \dots\right)$$

$$\underbrace{\cancel{\left(\frac{y_{i+1} - y_i}{h}\right)}}_{LHS} - \underbrace{\left(y_i + hf(h, y_i)\right)}_{RHS}$$

$$= \frac{1}{h}\left(\frac{h^2}{2!} y_i''(\xi_i)\right)$$

$$= \frac{h}{2!} y_i''(\xi_i) \rightarrow o(h)$$

As step size $\rightarrow 0$
Truncation error $\rightarrow 0$ } then scheme is "consistent".

2) <u>Stability</u>: Related to Roundoff error (after a while)

computer computes

$$\tilde{y}_{i+1} = \tilde{y}_i + hf(t_i, \tilde{y}_i)$$ } Roundoff error

3) <u>Convergence</u>:

computed ~~by~~ 'computer'

$$|y(t_i) - y_i| \leq ch^p \quad \text{— (1)}$$

"as $h \rightarrow 0$"

$\rightarrow$ If the difference $\rightarrow 0$ following (1), then point wise convergence occurs with order $= p$

If any scheme if it is consistent + stable then it is convergent.

<u>Lemma 1:</u> For all $x \geq -1$, and any positive m, we have

$$0 \leq (1+x)^m \leq \exp(mx)$$

$$(1+x) \geq 0 \quad \Rightarrow \quad (1+x)^m \geq 0$$

$$\exp(mx) = \cancel{1 + mx + \frac{m^2x^2}{2!} + \frac{m^3x^3}{3!}}$$

$$e^x = 1 + x + \frac{x^2}{2!} e^{\xi} \qquad \xi \in (0, x)$$

since $x \geq -1$, $1+x \geq 0$

and $1+x \leq e^x$

$$\Rightarrow \boxed{0 \leq (1+x)^m \leq e^{mx}}$$

<u>Lemma 2:</u> Let s, t be two +ve real no's and $\{a_i\}$ with $a_0 \geq -t/s$ and $a_{i+1} \leq (1+s)a_i + t \quad i = 0 \ldots k$

~~consider~~

then

$$a_{i+1} \leq \exp((i+1)s)\left[a_0 + \frac{t}{s}\right] - t/s$$

$$\cancel{(a_0 + t/s) \to \geq 0 \Rightarrow}$$

$$a_{i+1} \leq (1+s)a_i + t \qquad i = 0 \ldots k$$

$$\leq (1+s)\left[(1+s)a_{i-1} + t\right] + t$$

$$\le (1+s)\, t + (1+s)^2 a_{i-1}$$

$$\cancel{(1+s)^3 t + (1+s)^2 t^2}$$

$$(1+s)^3 a_{i-2} + (1+s)^2 t$$

$$\le (1+s)^{i+1} a_0 + \left[(1+s) + (1+s)^2 + \cdots + (1+s)^i\right] t - \frac{t}{s}$$

$$\frac{(1+s)^{i+1} - 1}{s}$$

$$\le (1+s)^{i+1} \left[a_0 + \frac{t}{s} \right] - t/s$$

$$\le \exp((1+i)s) \left[a_0 + t/s \right] - t/s$$

<u>Theorem:</u> Convergence of Explicit Euler's Scheme.

f is <u>continuous</u>, $y' = f$ and satisfies (~~Lipschitz condition~~) with constant L on domain $D = \{(t,y) : a \le t \le b, \ -\infty < y < \infty\}$ and $\exists\, M$ such that $|y''(t)| \le M$, $t \in [a,b]$

$$\boxed{y'' = f_t + f_y y'}$$

(Partial derivative exists)

Let $y(t)$ be unique soln of IVP

$$\begin{cases} y'(t) = f(t,y) \\ y(a) = \infty \end{cases}$$

$w_0, w_1, \ldots w_N$

$$w_{n+1} = w_n + h f(t_n, w_n) \qquad n = 0, \ldots N-1$$

Then

$$\text{Error} = |y(t_i) - w_i|$$

$$\leq \frac{hM}{2L} \exp\{L(t_i - a) - 1\}$$

$$\leq Ch$$

$$y(t_{i+1}) = y(t_i) + h y'(t_i) + \frac{h^2}{2!} y''(\xi_i)$$

$$w_{i+1} = w_i + h f(t_i, w_i)$$

$$|w_{i+1} - y(t_{i+1})| = (y(t_i) - w_i) + h(\underbrace{y'(t_i)}_{f(t_i, y_i)} - f(t_i, w_i)) + \frac{h^2}{2!} y''(\xi_i)$$

$$|w_{i+1} - y_{i+1}|$$

$$\leq |y(t_i) - w_i| + h|f(t_i, y_i) - f(t_i, w_i)| + \frac{h^2}{2!} |y''(\xi_i)|$$

$$|a_{i+1}| \leq \underbrace{(1 + hL)}_{(1+s)} \underbrace{|y(t_i) - w_i|}_{a_i} + \underbrace{\frac{h^2}{2!} M}_{t}$$

using previous lemma,

$$a_{i+1} \leq \left(e^{(i+1)hL} \right) [$$

$$|y(t_{i+1}) - w_{i+1}| \le |y_i - w_i| [1 + hL] + \frac{h^2}{2!} M$$

$s = hL \qquad t = \frac{h^2}{2!} M$ choose and apply prev lemma.

$t_0 = a$

$t_i = a + ih \qquad h = \frac{b-a}{n}$

$$\text{Error} \le \frac{hM}{2L} \left[\exp(L(t_i - a)) - 1 \right]$$

---

we assumed there is no roundoff error in prev theorem, this is not true practically.

Therefore if we consider roundoff error then the scheme becomes

$$w_0 = \alpha + \delta_0$$

$$w_{i+1} = w_i + \underbrace{h f(t_i, w_i)}_{\downarrow} + \delta_{i+1}$$

may be non-linear & accumulate, we get this δ_{i+1} for $i = 0, \dots N-1$

<u>Theorem</u>: Assume $|\delta_i| \le \delta$ → unit round off

<u>Error</u>: $$|y(t_i) - w_i| \le \frac{1}{L} \left(\frac{hM}{2} + \frac{\delta}{h} \right) \left[\exp\{L(t_i - a)\} - 1 \right] + |\delta_0| \exp\{L(t_i - a)\}$$

$\forall i = 0 \dots N$

no more linear

Due to initial error

Proof: similar way

Homework

$$\lim_{h \to 0} \left(\frac{hM}{2} + \frac{\delta}{h} \right) = \infty$$

Let $E(h) = \frac{hM}{2} + \frac{\delta}{h}$

$E'(h) = \frac{M}{2} - \frac{\delta}{h^2} = 0$

$h^2 = \frac{2\delta}{M}$

$h < \sqrt{\frac{2\delta}{M}}$

If $h < \sqrt{\frac{2\delta}{M}}$

$E' < 0$ and E is dec

$E' > 0$ and E is inc

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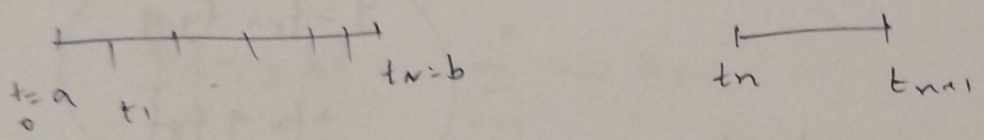
$$\begin{cases} y'(t) = f(t, y) & t \in (a, b) \\ y(a) = \alpha \end{cases}$$



$$\int_{t_n}^{t_{n+1}} y'(s)\,ds = \underbrace{\int_{t_n}^{t_{n+1}} f(s, y(s))\,ds}_{\text{constant interpolant}}$$

$\Rightarrow$

$$= f(t_n, y(t_n))\,h$$

$\Rightarrow$ $\boxed{w_{n+1} = w_n + h f(t_n, w_n)}$ $\longrightarrow$ Explicit Euler's

If we replace upper limit,

then

$\boxed{w_{n+1} = w_n + h f(t_{n+1}, w_{n+1})}$ $\longrightarrow$ Implicit Euler's

$\rightarrow$ we can use weights to generalize

$$= \int_{t_n}^{t_{n+1}} \left[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\right] ds$$

$$= \left[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})\right] (t_{n+1} - t_n)$$

$\theta = 1/2 \rightarrow$ Trapezoidal.

$\theta = 0 \rightarrow$ Implicit $\qquad \theta = 1 \rightarrow$ Explicit.

we know $f = y'$

Truncation Error $= \frac{1}{h}(\text{LHS} - \text{RHS})$

for $\theta = 1/2$

$\rightarrow O(h^2)$

$y'' = f_t + f_y y'$

$= f_t + f_y f$

and constant is bound of y'' terms

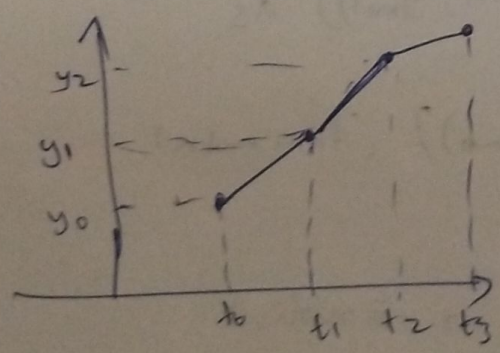
Starting point of Runge - Kutta

→ The underlying idea of Runge-Kutta method is to take a weighted average of slope of tangent in order to increase order of Truncation error.

$$\begin{cases} y(t_{n+1}) = y(t_n) + h\,\Phi(t_n, y(t_n), h) \\ y_0 = \alpha \end{cases}$$

Φ is continuous on all its arguments

→ called increment.



$O(h^p)$ $\quad$ $p \rightarrow$ goes upto 6

Optimal $h = 4$

y_{n+1} at t_{n+1} depends only at t_n

Hence single step.

Multistep → Central difference etc (2-step)

General R-stage Runge Kutta

$$y_{n+1} = y_n + \Phi(t_n, y_n, h) = y_{n+1}$$

$$\Phi(t, y, h) = \sum_{r=1}^{R} w_r K_r$$

$$K_1 = h f(t, y)$$

$$K_r = h f\left(t + h c_r,\ y + \sum_{s=1}^{r-1} a_{rs} K_s\right)$$

$$r = 2 \ldots R$$

$$c_r = \sum_{s=2}^{r-1} a_{rs} \qquad r = 2 \ldots R$$

Still one-step if we consider t_n, y_n and t_{n+1}, y_{n+1}

For $R = 2$

$$y_{n+1} = y_n + w_1 K_1 + w_2 K_2 \quad - (1)$$

$$K_1 = h f(t_n, y_n), \qquad K_2 = h f(t + c_2 h,\ y_n + a_{21} K_1)$$

→ we have 4 unknowns - w_1, w_2, c_2, a_{21} (2)

In order to determine values for parameters, we use Taylor's

$$y(t_{n+1}) = y(t_n) + h y'(t_n) + \frac{1}{2!} h^2 y''(t_n) + \frac{1}{3!} h^3 y'''(t_n) + \ldots$$

$y' = f(t, y)$

$y'' = f_t + f_y \cdot y' = f_t + f_y f$

$y''' = f_{tt} + f_t f_y + f_{yt} + f_{yy} f + f_{yt} f + f_y^2$

$y'''' = 2 f_{yt} f + f_{yy} f^2 + f^2 f_y + f_y f_t + f_{tt}$

→ Taylor's for LHS gives me

$y(t_{n+1}) = y(t_n) + h y'(t_n) + \frac{h^2}{2!} y''(t_n) + \frac{h^3}{3!} y'''(t_n)$

→ Replace

→ will give (3)

For RHS

$K_2 = hf(t_n + c_2 h, y_n + a_{21} K_1)$ and $K_1 = hf(t_n, y_n)$

$= hf(t_n, y_n) + c_2 h^2 f_t(t_n, y_n)$

$+ a_{21} h f_y(t_n, y_n) \cdot h f(t_n, y_n)$

$+ \frac{1}{2} h (c_2 h)^2 f_{tt}(t_n, y_n)$

$+ \frac{1}{2} h (a_{21})^2 f_{yy}(t_n, y_n) (hf(t_n, y_n))^2$

$+ h (c_2 h)(a_{21}) f_{ty}(t_n, y_n) \, hf(t_n, y_n)$

→ $y(t_{n+1}) = y(t_n) + w_1 hf(t_n, y_n)$

$+ w_2 (hf + c_2 h^2 f_t + a_{21} h^2 f_y f$

$+ \frac{1}{2} h^3 c_2^2 f_{tt} + \frac{1}{2} h^3 a_{21}^2 f^2 f_{yy} +$

$c_2 a_{21} h^3 f_{ty}(t_n, y_n))$

From ③,

$$y(t_{n+1}) = y(t_n) + hf(t,y) + \frac{h^2}{2!}(f_t + f_y f)$$

$$+ \frac{h^3}{3!} f_{tt} + f_t f_y + f_{yt} + f_{yy} f + f_{yt} f + f_y^2$$

$$w_1 hf + w_2 hf = hf$$

$$\Rightarrow \boxed{w_1 + w_2 = 1} \quad \text{— ①}$$

$$\boxed{c_2 w_2 = \tfrac{1}{2}} \quad \text{— ②}$$

$$\boxed{a_{21} w_2 = \tfrac{1}{2}} \quad \text{— ③}$$

we get 3 eqns but 4 unknowns

→ c_2 is arbitrarily chosen

$$\boxed{\begin{aligned} a_{21} &= c_2 \\ w_2 &= \frac{1}{2c_2} \\ w_1 &= 1 - \frac{1}{2c_2} \end{aligned}}$$

c_2 - parameter

For different c_2 values, we can obtain

$$y_{n+1} = y_n + w_1 K_1 +$$

$$y_n + hf_n + \frac{h^2}{2}(f_t + f_n f_y) + \frac{c_2 h^3}{4}(f_{tt} + 2 f_n f_{ty} + f_n^2 f_{yy})$$

Truncation error $= \frac{1}{h}(LHS - RHS) = O(h^2)$

Butcher's Table

$$\begin{array}{c|c} c_2 & a_{21} \\ \hline & w_1 \quad w_2 \end{array}$$

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array}$$

improved tangent

$$\begin{array}{c|cc} 2/3 & 2/3 & \\ \hline & 1/4 & 3/4 \end{array}$$

optimal method

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

← Euler Cauchy

$R = 3, 4$

4th Stage R-K

$$\begin{array}{c|cccc} \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \end{array}$$

classical method

$$\begin{array}{c|ccc} 0 & & & \\ c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \\ \vdots & \vdots & & \\ c_s & a_{s1} & a_{s2} \cdots & a_{s,s-1} \\ \hline & w_1 & w_2 \cdots & w_{s-1} \quad w_s \end{array}$$

$$\begin{array}{c|ccc} \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & -\frac{1}{3} & 1 & \\ 1 & 1 & -1 & 1 \\ \hline & \frac{1}{8} & \frac{3}{8} \quad \frac{3}{8} & \frac{1}{8} \end{array}$$

kutta method

$$TE = \frac{1}{h}(LHS - RHS)$$

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Adams - Moulton

→ The implicit versions of the multistep schemes

To interpolate f at points $t_{n-m}, t_n \ldots t_n, t_{n+1}$

for some $m > 0$

Now this is also included

→ so we have $m+2$ points

Newton's backward difference formula which interpolates f at these $m+2$ points of degree $m+1$

$$P_{m+1}(s) = \sum_{k=0}^{m+1} (-1)^k \binom{1-s}{k} \Delta^k f_{n+1-k} \quad \text{—— (2)}$$

$$s = \left(\frac{t - t_n}{h}\right)$$

$$\int_{t_n}^{t_{n+1}} y'(s)\, ds = \int_{t_n}^{t_{n+1}} f(s, y(s))\, ds \quad \text{—— (1)}$$

using (2) in (1)

$$y_{n+1} = y_n + h\left(\gamma_0' f_{n+1} + \gamma_1' \Delta f_n + \ldots \gamma_{m+1}' \Delta^{m+1} f_{n-m}\right)$$

where $\gamma_k' = (-1)^k \int_0^1 \binom{1-s}{k} ds \qquad k = 0 \ldots m+1$

$\gamma_0' = 1$, $\gamma_1' = -1/2$, $\gamma_2' = -\frac{1}{12}$, $\gamma_3' = -\frac{1}{24}$

$\gamma_4' = \frac{-19}{720}$

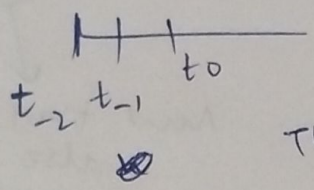
Remainder $= \gamma_{n+1}' h^{m+3} y^{(n+3)}(\xi)$

$$y_{n+1} = y_n + \frac{h}{24}\left(9 f_{n+1} + 19 f_n - 5 f_{n-1} + f_{n-2}\right)$$

$$E_{AM} = \frac{-19}{720} h^5 y^{(5)}(\xi)$$

$$O(h^4)$$

$y_0 = \alpha$



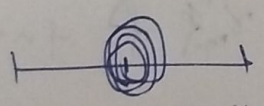
This is not self starting, we need Runge Kutta (or) something of order (4) for the first few terms

Here we can start from $n=2$ i.e. 3 points.

In Runge-Kutta, we evaluate K_1 K_2 K_3 K_4

But multistep schemes' - computational cost is less compared to Runge-Kutta of order 4. we have to evaluate K_1 K_2 K_3 K_4 at each point (the slopes)

In multistep we use only existing info.



Implicit $\Rightarrow$ another loop.

The pair of methods

Adams - Bashforth $\leftarrow$ Predictor

Adams - Moulton $\leftarrow$ Corrector

Similarly Explicit & Implicit Euler

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$\bar{y}_{n+1} = \bar{y}_n + h f(t_n, \bar{y}_n)$$

$$|y_n - \bar{y}_n| - \text{Roundoff}$$

Roundoff stability

Difference Equations

The well known examples of Difference eqns → AP, GP, Newton's, Euler's Fibonacci etc.

Some comparisions b/w Difference eqns and differential eqns.

Consider the general difference eqns

$$\Delta^N(y_n) = f(n, y_n, \Delta y_n, \ldots \Delta^{N-1} y_n) \quad \text{—— (1)}$$

In case of linear, we have eqn of type

$$a_{n,N}\, y_{n+N} + a_{n,N-1}\, y_{n+N-1} + a_{n,N-2}\, y_{n+N-2} + \ldots + a_{n,0}\, y_0 = b_n$$

$a_{nN}, a_{n,0} \neq 0$

Diff b/w highest and lowest indices = order

$y_{n+1} - y_n = 1 \quad \forall n \longrightarrow y_n = n + c$

$y_{n+1} - y_n = n \quad n \geq 0 \longrightarrow y_n = \frac{n(n+1)}{2} + c$

$y_{n+1} - (n+1) y_n = 0 \rightarrow$ Homogeneous $\rightarrow$ Factorial

$y_{n+2} - 2\cos(\theta)\, y_{n+1} + y_n = 1$

$\cancel{\frac{(n+1)(n+2)}{2} - \frac{n(n+1)}{2}}$

$$\boxed{y_{n+N} + a_{N-1} y_{n+N-1} + \cdots + a_0 y_n = 0} \quad \text{— (1)}$$

$a_0 \neq 0$

We try to take

$$\boxed{y_n = \beta^n \neq 0 \text{ for all } n,} \quad \text{— (2)}$$

$$\boxed{\beta^{n+N} + a_{N-1} \beta^{n+N-1} + \cdots + a_0 \beta^n = 0} \quad \text{— (3)}$$

$$\beta^n \underbrace{(\beta^N + a_{N-1} \beta^{N-1} + \cdots a_0)}_{\text{polynomial of degree } N} = 0$$

$$\boxed{P(\beta) = \beta^N + a_{N-1} \beta^{N-1} + \cdots + a_0 = 0} \quad \text{— (4)}$$

Roots of this eq. characteristic eqn can be real distinct real with multiplicity, complex.

$\beta_1 \cdots \beta_N$ } are solns and basis

$\beta_1^n \cdots \beta_N^n$ } are also solns } Homogeneous

$y_n = c_1 \beta_1^n + c_2 \beta_2^n + \cdots c_N \beta_N^n$

$c_1 c_2 \cdots c_N$ can be determined uniquely by given condi-tions.

Ex: $y_{n+3} - 2y_{n+2} - y_{n+1} + 2y_n = 0$

$$\beta^3 - 2\beta^2 - \beta + 2 = 0$$

$$\beta = \pm 1, 2$$

$$y_n = c_1 (1)^n + c_2 (-1)^n + c_3 (2)^n$$

conditions,

$y_0 = 0$

$y_1 = 1$

$y_2 = 1$

$y_0 = c_1 + c_2 + c_3 = 0$

$y_1 = c_1 - c_2 + 2c_3 = 1$

$y_2 = c_1 + c_2 + 4c_3 = 1$

$\Rightarrow c_3 = 1/3$

$y_3 = c_1 - c_2 + 8c_3$

$y_4 = \cdots$

$c_1 = 0$

$c_2 = -1/3$

$$y_n = (-1)^n \left(-\frac{1}{3}\right) + 2^n \frac{1}{3}$$

If characteristic polynomial has pair of conjugate complex roots then,

$\beta_1 = \alpha + i\beta$
$\beta_2 = \alpha - i\beta$

using polar

$\beta_1 = re^{i\theta}$

$\beta_2 = re^{-i\theta}$

$r = \sqrt{\alpha^2 + \beta^2}$

$\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

Since $\beta_1 \beta_2$ are L.I,

$\rightarrow c_1 \beta_1^n + c_2 \beta_2^n$

$= r^n (C_1 \cos n\theta + C_2 \sin n\theta)$

$y_{n+2} - 2y_{n+1} + 2y_n = 0$

<u>En.</u>

$$\beta^2 - 2\beta + 2 = 0$$

$$2 \pm \sqrt{4 - 4(2)} - 4$$

$$1 \pm i$$

$$\Rightarrow \sqrt{2}\, e^{i\pi/4}$$

$$\sqrt{2}\, e^{-i\pi/4}$$

$$2^{n/2}\left(C_1 \cos n\frac{\pi}{4} + C_2 \sin n\frac{\pi}{4}\right)$$

27/3/24

Suppose there is a root that is repeated (or) has multiplicity > 1.

Say multiplicity $= 2$

$$\beta_1^n, \; n\beta_1^n$$

<u>Case 3:</u> β_1 is of multiplicity 2 then

β_1^n and $n\beta_1^n$

$$p(\beta_1) = 0 \qquad p'(\beta_1) = 0$$

$\rightarrow$ <u>Proof:</u> $\beta^{n+N} + a_{N-1}\beta^{n+N-1} + \cdots + a_0\beta^n = 0$

$$p(\beta) = \beta^N + a_{n-1}\beta^{N-1} + \cdots + a_0 = 0$$

$$p'(\beta) = 0 \Rightarrow N\beta^{N-1} + (N-1)a_{n-1}\beta^{N-2} \cdots + a_1 = 0$$

$$\cancel{p(n\beta^n)} \quad y_n = n\beta_1^n$$

$$(n+N)\beta^{n+N} + a_{n-1}(n+N-1)\beta^{n+N-1} + \cdots + a_0(n)\beta^n$$

$$\Rightarrow n\beta^n(\beta^N + a_{n-1}\beta^{N-1} + \cdots + a_0)$$

$$+ \beta^{n+1}(N\beta^{N-1} + (N-1)a_{n-1}\beta^{N-2} + \cdots + a_1)$$

Ex: $y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0$

$\beta^3 - 5\beta^2 + 8\beta - 4 = 0$

$1, 2, 2$

$c_1(1)^n + c_2(2)^n + c_3(n2^n)$

Particular solution

Consider Non-homogeneous equation

$y_{n+N} + a_{N-1}y_{n+N-1} + \cdots + a_n y_n = \underbrace{b_n = b}$

Then solution can be expressed as

$y_n = y_n^h + y_n^p$

Consider case independent of n.

Assume $y_n^p = A$

$\Rightarrow A = \dfrac{b}{1 + a_{N-1} + \cdots + a_n}$ provided denominator is non-zero

Ex: $y_{n+2} - 2y_{n+1} + 2y_n = 1$

Homogeneous

$y_{n+2} - 2y_{n+1} + 2y_n = 0$ $\rightarrow$ sum $\neq 0$

$\beta^2 - 2\beta + 2$

$2^{n/2}\left(c_1 \cos n\frac{\pi}{4} + c_2 \sin n\frac{\pi}{4}\right) +$

$y_n^p = A$

$\Rightarrow A = \frac{1}{1} = 1$

Sum = 0 then ? linear ?

Multistep

$y' = f(t, y)$

$\frac{y_{n+1} - y_{n-1}}{2h} = f(t_n, y_n)$

$y_{n+1} = y_{n-1} + 2h f(t_n, y_n)$

$TE = O(h^2) \rightarrow \frac{\frac{1}{6} h^3 y'''(\xi_1)}{h}$

Obviously better but difficulties we see now

$$y' = -2y+1$$
$$y(0) = 1$$

Exact:

$y(t)$

$$e^{2t} y = \frac{e^{2t}}{2} + c$$

$$y = \frac{1}{2} + c e^{-2t}$$

$$\boxed{y(t) = \tfrac{1}{2}(1+e^{-2t})} \quad (1)$$

$\rightarrow$

Central diff:

$$y_{n+1} = y_{n-1} + 2h\, f(t_n, y_n)$$

$$f(t_n, y_n) = -2y_n + 1$$

$$y_{n+1} = y_{n-1} + 2h(-2y_n + 1)$$

$$\boxed{y_{n+1} + 4hy_n - y_{n-1} = 2h}$$

Particular soln $= \frac{2h}{4h} = \frac{1}{2}$

General

$\rightarrow$ $\beta^2 + 4\beta - 1 = 0$

$$\frac{-4h \pm \sqrt{16h^2+4}}{2}$$

$$-2h \pm \sqrt{1+4h^2} \quad \Big\} \text{ Distinct real}$$

$y_n =$

$$c_1 \beta_1^n + c_2 \beta_2^n + \tfrac{1}{2}$$

c_1

Approximating this,

$$\sqrt{1+4h^2} = 1 + 2h^2 +$$

$B_1 = 1-2h$ } Approximated to linear terms.

$B_2 = -(1+2h)$ $+ O(h^2)$

$$y_n = c_1(1-2h)^n + c_2(-1)(1+2h)^n + \tfrac{1}{2}$$

as $\lim_{\varepsilon \to 0} (1+\varepsilon)^{1/\varepsilon} = \exp(1)$

$\cancel{\lim_{h\to 0} (1-2h)^{1/h} = e^{-2}}$

$\cancel{\lim_{h\to 0} (1+2h)^{1/h} = e^{2}}$

$\lim_{h\to 0} (1+2h)^n = \lim_{h\to 0} (1+2h)^{\frac{2tn}{2h}}$ $\quad tn = nh$

$= \exp(2tn)$

$\lim_{h\to 0} (1-2h)^n = \exp(-2tn)$

$\lim_{r\to\infty} \left(1-\frac{2tn}{r}\right)^r = e^{-2tn}$

$$\boxed{y_n = \left[c_1 \exp(-2tn) + \tfrac{1}{2}\right] + c_2(-1)^n \exp(2tn)}$$ — (2)

Extra term

oscillatory term

and inc amplitude

If Explicit

$y_{n+1} = y_n + hf(t_n, y_n)$

$y_{n+1} - (1-2h)y_n = h$

$y_n = c_1(1-2h)^n + \tfrac{1}{2}$

$$\boxed{y_n = c_1 \exp(-2tn) + \tfrac{1}{2}}$$ — (3)

$y_0 = 1 \Rightarrow c_1 = \tfrac{1}{2}$

How did the extra term come?

→ We got two distinct roots in the central diff

Because of order of central diff } we can observe

"Extraneous"

$c_2 \neq 0 \rightarrow$ ~~extra~~ term, we see oscillations

soln of diff eqn corresponding to explicit Euler exactly replicates exact soln as $n \to \infty$.

whereas ② oscillates - obtained from central diff contains one extra term. with increasing amplitude.

If $c_2 = 0$ then (from I.C say) we dont have any problem. unfortunately due to Roundoff errors $c_2 \neq 0$.

→ we replaced first order ODE by second order diff eqn. and the extra term has nothing to do with the analytical/exact solution of ODE.

For stability of multistep schemes, one root of diff eqn corresponds to og solution.

β_1 corresponds to og solution,

If $y_n = c_1 \beta_1^n + c_2 \beta_2^n + \cdots + c_n \beta_n^{n-2}$

If $|\beta_i| < 1$, i.e. they sit in the unit disk then we say the multistep scheme is absolutely stable.

1/4/24

Adams Bash forth

$$y_n = c_1\beta_1^n + c_2\beta_2^n + \cdots + c_k\beta_k^n$$

$$y' = \lambda y \qquad \lambda < 0$$

$$y(0) = y_0$$

$$y(t) = y_0 e^{-\lambda t}$$

$$y_{n+1} = y_n - \frac{\lambda h}{24}\left(55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3}\right) = 0$$

$$\beta^4 - \beta^3 - \frac{\lambda h}{24}\left[\quad\right] = 0$$

$$\rho(\beta) + \lambda h\,\sigma(\beta) = 0$$

$$\rho(\beta) = \beta^4 - \beta^3 = \beta^3(\beta - 1)$$

$\beta = 0$ / $\beta = 1$ is root

bounded

as $h \to 0$ the term goes to 0

Then we have $\beta^3(\beta - 1) = 0$

$\beta = 1$

$\beta = 1 \to$ corresponds to exact

0 Phase

$\to$ strongly stable / Absolutely)

Milne's schema

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left(f_{n+1} + 4f_n + f_{n-1}\right)$$

$$y' = \lambda y$$

$$y' = f$$

$t = \lambda y$

$\beta^2 - 1 - \frac{h}{3}(\beta^2 + 4\beta + 1)$

$(\beta^2 - 1) - \frac{h}{3}(\lambda y_{n-1} + 4\lambda y_n + \lambda y_{n+1})$

$\rightarrow \rho(\beta) = \beta^2 - 1$

$\sigma(\beta) = -\frac{1}{3}(\beta^2 + 4\beta + 1)$

$\beta^2 - 1 - \frac{h}{3}\lambda(\beta^2 + 4\beta + 1)$

$y_{n+1} = y_n + hf(t_n, y_n)$

$\tilde{y}_{n+1} = \tilde{y}_n + hf(t_n, y_n)$

$|y_n - \tilde{y}_n|$

Conditionally?

2 roots of this quad

$y_n = c_1\beta_1^n + c_2\beta_2^n$

$y_0 = \alpha \pm \varepsilon$ Round off error

Error: $|y(t) - y_n| = |y(t_n) - \tilde{y}_n + \tilde{y}_n - y_n|$

$\leq |y(t_n) - \tilde{y}_n| + |\tilde{y}_n - y_n| \leq Ch^p$

consistency constant

we get from Taylor's

TE

Round off

Replace LHS − RHS

Taylor's for both.

---

system of first order ODEs

$y_1' = f_1(t, y_1, y_2)$

$y_2' = f_2(t, y_1, y_2)$

$\bar{y}' = \bar{f}(t, \bar{y})$

$y_1(0) = \alpha$

$y_2(0) = \beta$

$\frac{y_{n+1} - y_n}{h} = f(y_n)$

$y''(t) + p(t)\, y'(t) + q(t)\, y(t) = 0$ $\quad t \in (0,1)$

$y(0) = \alpha$

$y'(0) = \beta$

$y_1 = y$

$y_2 = v = y_1'$

$(y, v) \rightarrow$ 2 diff eqns

$$\begin{cases} y' - v = 0 \\ v' + p(t)\, v + q(t)\, y = 0 \end{cases}$$

Now system is linear

BVP — The Shooting technique

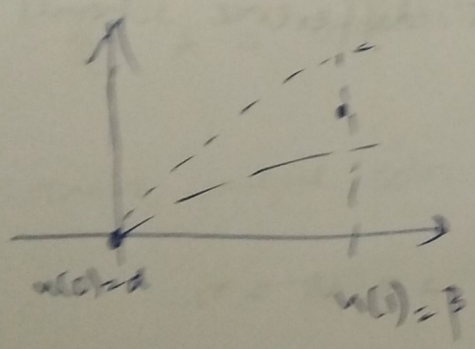
$u''(x) + p(x)\, u'(x) + q(x)\, u(x) = r(x) \qquad x \in (0,1)$

$u(0) = \alpha \qquad u(1) = \beta$

I dont have $u'(0)$

Assume $u'(0) = \varepsilon$

In IVP we had recurrence eqns.



Update $u'(0) = \varepsilon \pm \delta$

Based on over shooting or under shooting

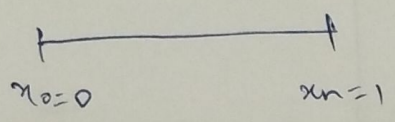
→ Converting a BVP into an IVP with some initial condition and getting an approximate value based on $|u(1) - \beta| < tol$.

Merit of shooting technique: we can solve non-linear differential eqns because all methods studied for IVPs can deal with non-linear eqns.

Demerit: we have to solve IVP for various ξ and δ.

Finite Difference Schemes

→ Discretize the domain



$0 = x_0 < x_1 \cdots < x_{n-1} < x_{n+1} = 1$

→ Replace the derivatives by finite difference ~~quotients~~ schemes

$$u'(x_i) = \begin{cases} \dfrac{u_{i+1} - u_i}{h} & \text{FD} \quad O(h) \\[2ex] \dfrac{u_i - u_{i-1}}{h} & \text{BD} \quad O(h) \\[2ex] \dfrac{u_{i+1} - u_{i-1}}{2h} & \text{CD} \quad O(h^2) \end{cases}$$

→ Ultimately we obtain 3 different difference schemes

$u''(x_i)$

$$= \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + p(x_i)\frac{v_{i+1} - v_{i-1}}{2h} + q_i v_i = r_i$$

left side $\quad$ right side

$v_0 = \alpha \qquad v_{N+1} = \beta$

~~Not exactly~~

$$= \left(\frac{1}{h^2} + \frac{p_i}{2h}\right) v_{i+1} - \left(\frac{2}{h^2} - q_i\right) v_i + \left(\frac{1}{h^2} - \frac{p_i}{2h}\right) v_{i-1} = r_i$$

If $i = 1$

$$\left(\frac{1}{h^2} + \frac{p_1}{2h}\right) v_2 - \left(\frac{2}{h^2} - q_1\right) v_1 + \left(\frac{1}{h^2} - \frac{p_1}{2h}\right) \alpha = r_1$$

$$-(\ )v_1 + (\ )v_2 = r_1 - (\ )\alpha$$

$i = 2$

$$(\ )v_1 - (\ )v_2 + (\ )v_3 = r_2$$

$i = 2$ to $n-1$

$i = n$

$$-\left(\frac{2}{h^2} - 2\eta\right)v_n + \left(\frac{1}{h^2} - \frac{\beta_n}{2h}\right)v_{n-1} = \left(r_n - \left(\frac{1}{h^2} + \frac{\beta_n}{2h}\right)\beta\right)$$

System of ~~Linear~~ algebraic equations

'N' unknowns - N eqns.

Tridiagonal sparse matrix } nonzero in main, sub, super diagonals.

$\boxed{AV = r}$

If A is invertible (or) rank $\neq 0$ (or) diagonally dominant ... etc

then unique soln.

2/4/24

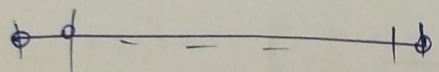
$u''(x) + p(x)u' + q(x)u = r(x)$ $(0,1)$

$\boxed{u(0) = \alpha}$ $u(1) = \beta$

$\rightarrow$ $\boxed{AV = b}$ $u_0 = \alpha$ $u_{n+1} = \beta$

Dirichlet boundary conditions:

Neumann BCs: $-u'(0) = \alpha$, $u'(1) = \beta$



Forward Backward.

$$-\left(\frac{v_1 - v_0}{h}\right) = \alpha$$

$$\frac{v_{N+1} - v_N}{h} = \beta$$

$$\left.\begin{aligned} \left(\frac{+1}{h}\right)(v_0) - \left(\frac{1}{h}\right) v_1 &= \alpha \\ \left(\frac{-1}{h}\right) v_N + \left(\frac{1}{h}\right) v_{N+1} &= \beta \end{aligned}\right\} O(h)$$

TE: LHS − RHS $O(h^2)$ at Boundaries $O(h)$

Dominant is $O(h)$

$$TE = -\frac{h^2}{24}\left(u^{(4)}(\xi_i) - 2p_i u^{(3)}(\eta_i)\right)$$

For second orders we need $u \in C^4(\Omega)$

Robin type: Mixed BCs

$u(0) - u'(0) = \alpha$

$u(1) + u'(1) = \beta$

$$u_0 - \frac{u_1 - u_0}{h} = \alpha$$

$$u_{N+1} - \frac{u_{N+1} - u_N}{h} = \beta$$

$$\left(1 + \frac{2}{h}\right) u_0 - \left(\frac{1}{h}\right) u_1 = \alpha$$

$$-\frac{1}{h} u_N + \left(1 - \frac{2}{h}\right) u_{N+1} = \beta$$

<u>Inputs:</u> a, b, N, α, β.

```
A(1,1)
A(1,2) =
b(1) =

for i = 2:N
    A(i,i)
    A(i,i-1) =
    A(i,i+1)
    b(i) =
end
A(N+1,N) =
A(N+1,N+1) =
b(N+1) =
```

$AU = b$

$U = A \backslash b$

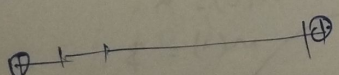
(or) Gaussian

(or) Jacobi

(or) Gauss Sidel

(or) SOR → successive relaxation

Only one computation unlike shooting technique.



~~If~~ instead of manually taking constants in first and last point, we can do "something else" !? and include all those points.

$$-u''(x) + f(x, u) = 0 \qquad x \in (0,1)$$

$$u(0) = \alpha \qquad u(1) = \beta$$

$$\cancel{L = \frac{d^2}{dx^2} + \frac{d}{dx} + q I}$$

$$L = \frac{d^2}{dx^2} + p\frac{d}{dx} + q I$$

$$L(\gamma_1 u_1 + \gamma_2 u_2) = \gamma_1 L(u_1) + \gamma_2 L(u_2)$$

"Linear" but if not linear

$$Lu = u''(a) + p(x)\, u' + q(x)\, u = r(x)$$

Nonlinear

Quasilinear

semilinear

Source/sink

Diffusion

convection

reaction

If $p(x) \equiv 0$

$$+ u'' + q(x) u = r(x)$$

$Av = b \rightarrow$ positive definite

If $p \neq 0$ then we saw the matrix is not symmetric

$$G = \frac{1}{2} A x^T - x^T b$$

$$\nabla G = 0$$

Optimization method

$$\cancel{x A x^T = x b}$$

$$2 A x^T = b$$

---

for $\boxed{\text{Semi Linear}}$ $-u''(x) + f(x, u) = 0 \qquad x \in (0,1)$

$$u(0) = \alpha$$

$$u(1) = \beta$$

$$\frac{-u_{i+1} + 2u_i - u_{i-1}}{h^2} + f(x_i, u_i) = 0$$

$$-\frac{1}{h^2}(u_{i-1}) + \frac{2}{h^2}(u_i) - \frac{1}{h^2} u_{i+1} = f(x_i, u_i)$$

Non linear system

To solve this system one can use ~~fixed point~~ Multidimensional Newton's method to solve eqn

$(x, u^{(m)})$ $\quad u^{(0)}$ - initial approximation

$$f(x,u) = f(x,u^{(m)}) + (u-u^{(m)})\, f_u(x,u^{(m)}) + \underbrace{h\,\Delta t}_{\text{truncating}}$$

$$\underbrace{-u_{xx} + \left[f(x,u^{(m)}) + (u-u^{(m)})\, f_u(x,u^{(m)})\right] = 0}$$

Linearized version of the same problem.

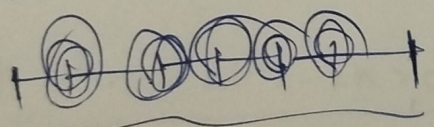
System of linear DEs

$$-u_{xx}^{(m+1)} - f_u(x,u^{(m)})\, u^{(m+1)} = -f(x,u^{(m)}) + f_u(x,u^{(m)})\, u^{(m)}$$

$u(0) = \alpha$

$u(1) = \beta$ $\qquad m \geq 0$

Here we have initial approx for all points



All at same time

```
for i = 1: N+1                    for j = 1: maxiters | ov -> old v
    ov(i) = 0.5;
                                  manually
    A(1,1)                        linearize the
    A(1,2)                        semi linear thing
    A(i,1)
    ⋮
    A(i, i-1)
    ⋮
    v = A\b
    if abs(ov - v) < tol = 10^-8 -> break
        ov = v
```

For general soln

$u'' + p(x, u, u') = 0$

manually. linearize and solve system of BVPs

$\|u(x_i) - v_i\| \le Ch^p$.

Stability Analysis

$u''(x) + k\,u'(x) = 0$

Consider second order PE subject to suitable BCs

and $|k| >> 1$

Forward / Backward / central.

Forward / Backward

1) $$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + k\,\frac{u_{i+1} - u_{i-1}}{2h} = 0$$

2) $$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + k\,\frac{v_{i+1} - v_i}{h} = 0$$

3) $$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + k\,\frac{v_i - v_{i-1}}{h} = 0$$

① $\rightarrow$ $$\frac{1}{h^2}\beta^2 - \frac{2\beta}{h^2} + \frac{1}{h^2} + \frac{k\beta^2}{2h} - \frac{k}{2h} = 0$$

$$\beta^2\left(\frac{kh+2}{2h^2}\right) + \frac{-2\beta}{h^2} + \left(\frac{2-kh}{2h^2}\right) = 0$$

$\beta_1 \quad \beta_2$

$$\frac{\frac{2}{h^2} \pm \sqrt{\frac{4}{h^4} - 4\left(\frac{kh+2}{2h^2}\right)\left(\frac{2-kh}{2h^2}\right)}}{2\left(\frac{kh+2}{2h^2}\right)}$$

$$\cancel{\frac{4 + kh^2 + 4 - 2h + 2kh}{h^2}}$$

$$\beta_1 = 1 \qquad \beta_2 = \frac{2-kh}{2+kh}$$

Actual:

$$u(x) = A + Be^{-kx}$$

$$u_x = C_1 + C_2\left(\frac{2-kh}{2+kh}\right)^n$$

$x = nh$

consider case when $k > 0$ then the exact solution goes to zero as $x \to \infty$

If $h > 0$ $\quad 2 - kh > 0$ so $h < \frac{2}{k}$ $\qquad \frac{kh-2}{kh+2}$

for k +ve and too large, $h > \frac{2}{k}$

$\rightarrow$ Then $-\frac{kh+2}{kh+2} \longrightarrow (-1)$ } then oscillation.

If $k < 0$

same restriction is applicable $h < \frac{2}{|k|}$

3/2/24

$h < \frac{2}{|k|}$

$u'' + ku' = 0$

$\downarrow$

convection coefficient

If k is large, then h should be very small

Second scheme: Forward difference

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + k\frac{u_{i+1} - u_i}{h} = 0$$

$$\left(\frac{1}{h^2} + \frac{k}{h}\right) u_{i+1} - \left(\frac{2}{h^2} + \frac{k}{h}\right) u_i + \frac{1}{h^2} u_{i-1} = 0$$

$\frac{\frac{1}{h^2}}{\frac{1}{h^2} + \frac{k}{h}}$ and 1 are the roots $= \frac{1}{1+kh}, 1$

$k > 0 \quad h > 0$

All these $\frac{1}{1+kh}$, $\frac{1-kh}{1+kh}$ etc are called Padé approximation

$\boxed{k > 0, h > 0 \rightarrow \text{No restriction}}$

$\boxed{\text{If } k < 0 \Rightarrow h < \frac{1}{|k|}}$ } Not advisable if $|k|$ is large

This is to replicate exponential

not because unit disc

Backward Difference

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + k\,\frac{v_i - u_{i-1}}{h} = 0$$

$$\frac{1}{h^2} u_{i+1} - \left(\frac{2}{h^2} - \frac{k}{h}\right) v_i + \left(\frac{1}{h^2} - \frac{k}{h}\right) u_{i-1} = 0$$

1 and $1-kh$ are the roots

$\boxed{\text{If } k < 0 \rightarrow \text{no restriction}}$

$\boxed{k > 0 \qquad h < \frac{1}{k}}$

Backward if $k < 0$

Forward if $k > 0$

These are called affine scheme

If central irrespective of $k>0$ (or) $k<0$ we see that restrictions are there in both.

→ we cant arbitrarily assign forward and backward differences while approximating the convective term as it depends on convective coefficient 'k'.

→ Think of this as flow directions and in central which is average of both forward and backward

→ we are going against flow in both

$$u_i' = \begin{cases} D^+ v_i = \dfrac{v_{i+1} - u_i}{h} & \text{if } k > 0 \\[2ex] D^- v_i = \dfrac{v_i - v_{i-1}}{h} & \text{if } k < 0 \end{cases}$$

$D^c = \frac{1}{2}(D^+ + D^-)$

$\boxed{h < \frac{2}{|k|}}$ is region of stability

$P_{p-} Q_f = R$

$u_{tt} - u_{xx} = 0 \leftarrow$ Hyperbolic

$u_t - u_{xx} = 0 \leftarrow$ parabolic

$u_{tt} = c^2 u_{xx} \leftarrow$ Hyperbolic

} Time dependent

$\Delta u = u_{xx} + u_{yy} = 0 \leftarrow$ Elliptic } Steady state

$u_t - (u_{xx} + u_{yy}) = 0$ steady state

→ Heat eqn

1) Heat eqn

Parabolic PDE — IBVP

Model problem
$$\begin{cases} u_t - u_{xx} = 0 & (x,t) \in (0,1) \times (0,T) \\ u(x,0) = \phi(x) \\ u(0,t) = f_1(t) & u(1,t) = f_2(t) \end{cases}$$

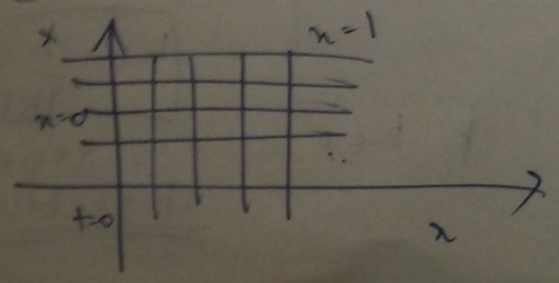
Seperation of variables

$$u(x,t) = X(x)\,T(t)$$

---

we seek numerical approximations

1) Step 1 We discretize the domain using Cartesian Grid



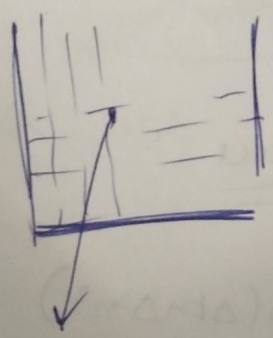
$$\left.\begin{aligned} \Delta t &= k = \frac{T}{N} \\ \Delta x &= h = \frac{1}{M} \end{aligned}\right\} \text{ number of intervals}$$

Step 2

Replace derivatives by finite difference quotients

$$u_m^n = u(x_m, t_n)$$

$$u_t = \begin{cases} \dfrac{u_m^{n+1} - u_m^n}{k} \longrightarrow D^+ \\[2ex] \dfrac{u_m^{n} - u_m^{n-1}}{k} \\[2ex] \dfrac{u_m^{n+1} - u_m^{n-1}}{2k} \end{cases}$$



x_m, t_n

$$u_m^0 = \phi(x_m)$$

$$u_0^n = f_1(t_n)$$

$$u_M^n = f_2(t_n)$$

$$u_{xx}(x_m, t_n) = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}$$

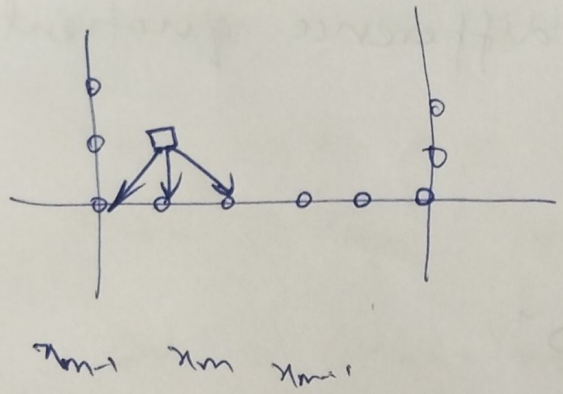
$$\frac{u_m^{n+1} - u_m^n}{k} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2} \quad \text{--- (1)}$$

$$u_m^{n+1} = \cancel{u_m^n + \frac{\lambda}{k}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)}$$

$$\boxed{u_m^{n+1} = \lambda u_{m+1}^n + (1-2\lambda) u_m^n + \lambda u_{m-1}^n}$$

$0 \le n \le N-1$

$1 \le m \le M-1$



Put $n=0$

$$u_m^1 = \lambda u_{m-1}^0 + (1-2\lambda) u_m^0 + \lambda u_{m+1}^0$$

Explicit scheme → Recurrence relation

→ we have two loops

$\boxed{\text{FTCS}}$ → "Forward Time" "Central space"

→ Euler

Truncation Error : $\boxed{\frac{1}{k}(LHS - RHS)} = O(\Delta t + \Delta x^2)$

Implicit Euler, central Diff

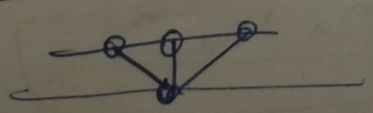
$\boxed{\text{BTCS}}$ Replace LHS by D^-

$$\frac{u_m^{n+1} - u_m^n}{k} = u_m^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1}$$

$$\boxed{-\lambda u_{m-1}^{n+1} + (1+2\lambda) u_m^{n+1} - \lambda u_{m+1}^{n+1} = u_m^n}$$

$0 \le n \le N-1$

$1 \le m \le M-1$



→ Because of implicit nature one has to solve system of linear algebraic eqns - Tridiagonal system at each time step.

→ Computationally expensive

TE: $O(\Delta t, \Delta x^2)$

$$\cancel{\frac{u_m^{n+1} - u_m^{n-1}}{2k} + \frac{u_{m-1}^{n} - 2u_m^{n} + u_{m+1}^{n}}{h^2}}$$

Homework: Calculate Truncation Error of FTCS BTCS

$$\cancel{u_m^{n+1} - u_m^{n-1} = \lambda u_{m-1}^{n} - 2\lambda u_m^{n} + \lambda u_{m+1}^{n}}$$

Take avg

$$\cancel{\frac{u_{m-1}^{n} - 2u_m^{n} + u_{m+1}^{n}}{h^2}}$$

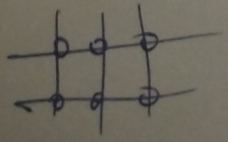
$$-\lambda u_{m-1}^{n+1} + (1+2\lambda)\, u_m^{n+1} - \lambda u_{m+1}^{n+1} = u_m^{n}$$

$$u_m^{n+1} = \lambda u_{m+1}^{n} + (1-2\lambda)\, u_m^{n} + \lambda u_{m+1}^{n}$$

$$-\frac{\lambda}{2} u_{m-1}^{n+1} + (1+\lambda)\, u_m^{n+1} - \frac{\lambda}{2} u_{m+1}^{n+1}$$

$$= \frac{\lambda}{2} u_{m-1}^{n} + (1-\lambda)\, u_m^{n} + \frac{\lambda}{2} u_{m+1}^{n}$$

Average of prev terms.



Again Implicit scheme at each time level we solve system of

Linear Algebraic eqns. $\underline{AV = b}$

Advantage: $O(\Delta t^2, \Delta x^2)$

This was Crank - Nicolson Scheme

All these schemes are single-step = 2 Level schemes

Fourth scheme $\boxed{\delta = \frac{2k}{h^2}}$

$$\frac{u_m^{n+1} - u_m^{n-1}}{2k} = \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{h^2}$$

$$u_m^{n+1} = u_m^{n-1} + \left(\delta u_{m-1}^n - 2\delta u_m^n + \delta u_{m-1}^n\right)$$

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2D Heat equation

$u_t - \Delta u = f(x, y, t)$

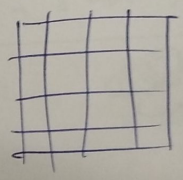
$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u_{xx} + u_{yy}$

Elliptic BVP } time independent

$\Delta u = f(x,y), \Omega$

$u(x,y) = g(x,y) \quad \partial\Omega = \Gamma$

Step 1: Discretize domain



$h = (b-a)/N$

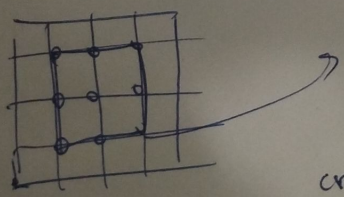
$k = (d-c)/M$

$h = k, \; M = N$

Step 2: Replacing the derivative by difference quotient

$$\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) + \left(\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{h^2}\right) = f_{ij}$$

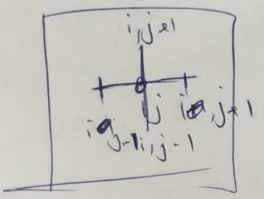
$$u_{i+1,j} - 2u_{ij} + u_{i-1,j} + u_{i,j+1} + u_{i,j+1} = h^2 f_{ij}$$

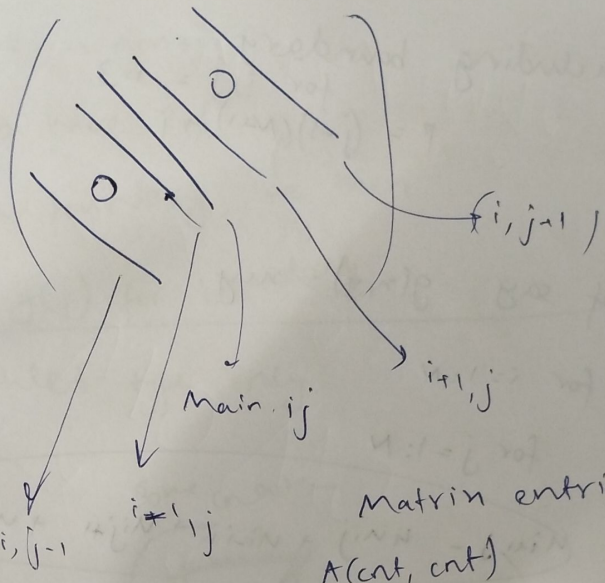


$(n-1)^2 \times (n-1)^2$ Matrix

~~for ij = 1:5~~ for $i,j = 1:5$

$cnt = (i-1)(N-1) + i$





Matrix entries

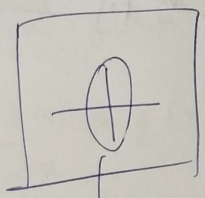
A(cnt, cnt)

A(cnt, cnt-1)

A(cnt, cnt+1)

A(cnt-N, cnt)

A(cnt+N, cnt)



interchanging

$p = (j-1)(N+1) + i$

stiffness matrix

$\boxed{AV = F}$ → A is symmetric matrix which has 5 non-zero diagonol entries (main, super, sub,

→ (Sparse matrix)

→ $A^T = A$

→ Diagonally ~~diagonal~~ dominant:

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{N} |a_{ij}|$$

→ A is diagonally dominant and ~~positive definite~~

→ A is irreducible

→ ~~False~~

$x_1 - \frac{1}{2}x_2 - x_3 + 4x_4 + x_5 = 0$

Excluding boundary

for $i, j = 1:3$

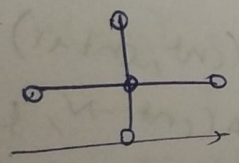
$p = (j-1)(N+1) + i$

→ If say $g(x,y) = x+y$

for $i = 1:N$

for $j = 1:N$

$$u_{i+1,j} - 4u_{ij} + u_{i-1j} + u_{ij+1} + u_{ij-1} = h^2 f_{ij} \quad \text{—— (1)}$$



$p = (j-1)(N+1) + i$

"Including the boundary" 1D – Automatic Boundary thing

$u'' = f(x)$

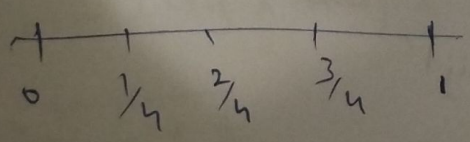
$x \in (0,1)$

$u(0) = \alpha \quad u(1) = \beta$

→ 1D → $-u'' = f(x)$

$u_{i-1} - 2u_i + u_{i+1}$

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f_i$$



3×3

Now for 5×5

$A(1,1) = 1 \quad b(1) = \alpha$

$A(N+1, N+1) = 1 \quad b(N+1) = \beta$

spy(A) → MATLAB → nonzero

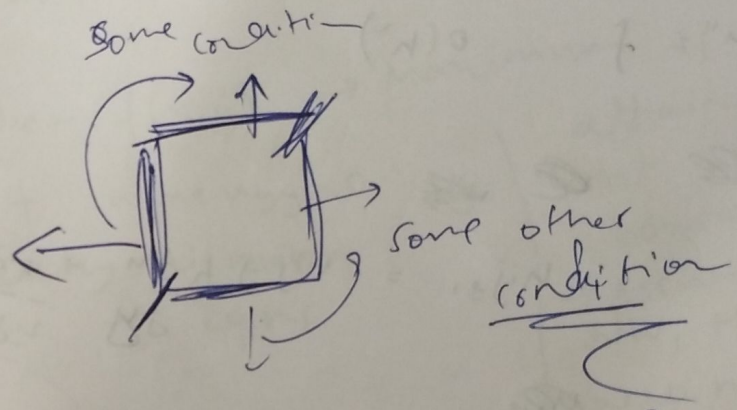
only for interior 5 have (1)

→ For including the boundary, we then write $u(x,y) = g(p)$ for boundary.

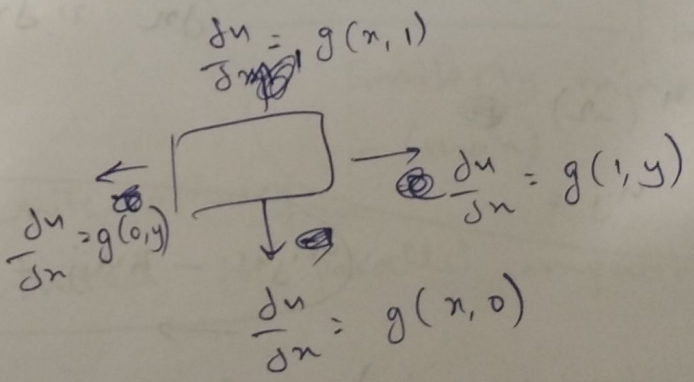
This works for Dirichlet type only.

$u(x,u) = g(x,u)$, Γ_1

$\frac{\partial u}{\partial x} = g(x,y)$



$\frac{\partial u}{\partial x} = g(x,y)$ } How to write for this boundary?



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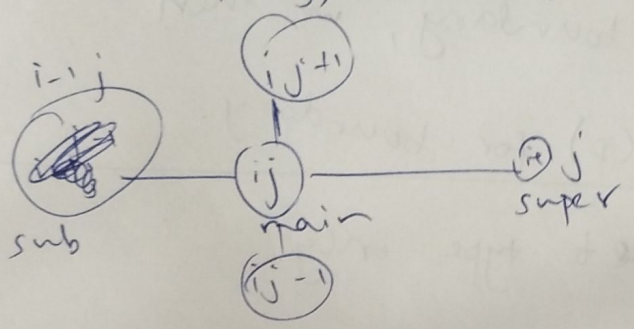
Poisson equation

$\Delta u = f(x,y) \qquad \Omega = (0,1)^2$



$TE = O(h^2)$

$= LHS - RHS$

$u'' = f \qquad O(h^2)$

$$u_{ij\pm1} = u_{ij} \pm h\frac{\partial u}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial y^2} \pm \frac{h^3}{3!}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial y^4}$$

$$u_{i\pm1 j} = u_{ij} \pm h\frac{\partial u}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial x^2} \pm \frac{h^3}{3!}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial x^4}$$

$u \in C^4(\Omega)$

$u_{ij} =$

$$u_{ij+1} - 4u_{ij} + u_{ij-1} + u_{i-1j} + u_{i+1j}$$

$$\frac{(u_{i+1j} - 2u_{ij} + u_{i-1j})}{h^2} + \frac{(u_{ij+1} - 2u_{ij} + u_{ij-1})}{h^2} = f_{ij}$$

$$u_{ij}^{(k+1)} = \frac{1}{4}\left(u_{i+1j}^{(k)} + u_{ij}^{(k)} + u_{ij+1}^{(k)} + u_{ij-1}^{(k)}\right) - \frac{h^2}{4} f_{ij}$$

Rather than making the matrix $AV = F$

→ Jacobi iteration → Poisson Solvers

$$u_{ij}^{(k+1)} = \frac{1}{4}\left(u_{i+1j}^{(k)} + u_{i-1j}^{(k+1)} + u_{ij+1}^{(k)} + u_{ij-1}^{(k+1)}\right) - \frac{h^2}{4} f_{ij}$$

these two are calculated already.

Stopping criteria:

$$\left| u_{ij}^{(k+1)} - u_{ij}^{(k)} \right| < Tol = 10^{-8}$$

~~→ maximum is attained at boundary, proof~~

~~If max is inside~~

maximum principle: $\Delta u = f(x,y)$ → maximum is attained at boundary

→ Assume in b/w, then keep moving towards boundary.

Function is harmonic if $u_{xx} + u_{yy} = 0$ by Cauchy Riemann eqns

Similarly minimum. $-(\max)$

~~Numbering~~

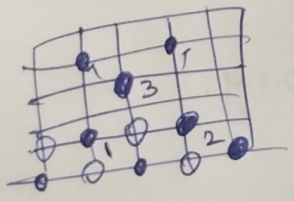
counting / ordering

Red Black ordering

(or)

Black white ordering

→ well suited for parallel computing



Discrete Maximum principle: For any mesh function v_{ij}, defined on discrete domain Ω^h, the following max principle holds true

$$Lu \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

if $L^h v_{ij} \geq 0 \quad \forall\, i,j$

$\Rightarrow$ The maximum of v_{ij} over the boundary Γ^h

$\Rightarrow$ max v_{ij} interior

$$L^h v_{ij} = \left(\frac{v_{i+1j} + v_{i-1j} + v_{ij+1} + v_{ij-1}}{h^2} - 4 \left(\quad \right) \right)$$

Proof: Assume the contrary,

take $i_0 j_0$ is interior where max exists

$$v(i_0 j_0) = \max_{\Omega^h} v_{ij}$$

$L^h v_{ij} \geq 0 \quad \forall ij$

$\Rightarrow$ from $v_{i_0 j_0}^{(k+1)} = \frac{1}{4}\left(v_{i+1j}^{(k)} + v_{i-1j}^{(k+1)} + v_{ij+1}^{(k)} + v_{ij-1}^{(k)} \right) - \frac{h^2}{4} f_{ij}$

$v_{i_0 j_0} \leq \frac{1}{4}\left(\quad \right)$ as $f_{ij} \geq 0$

But $v_{i_0 j_0} \geq$ max of all neighbouring points.

$\Rightarrow$ All values are same

$v_{i_0 j_0} = v_{ij}$ ij are neighbours

$\Rightarrow$ If any of these four neighbouring points are on the boundary then contradiction,

Repeat the same till we reach boundary.

$\rightarrow$ similarly the minimum.

$\rightarrow$ If max/min exists in interior then function remains constant

If $L^h v_{ij} \leq 0$ then $\min_{\Gamma^h} v_{ij} \leq \min_{\Omega^h} v_{ij}$

Corollary: Uniqueness of solution

Let u_{ij}, v_{ij} be two solutions

of $L^h u_{ij} = f_{ij}$ then both are identical.

$\geq \max_{\Omega^h} v_{ij}$ interior

$$L^h v_{ij} = \left(\frac{v_{i+1j} - 2u_{ij} + u_{i-1j}}{h^2}\right) + \left(\quad\right)$$

Proof: Assume the contrary,

take i_0, j_0 is interior where max exists

$v(i_0, j_0) > \max_{\Gamma^h} v_{ij}$

$L^h v_{ij} \geq 0 \quad \forall ij$

$\Rightarrow$ From $v_{i_0 j_0}^{(k+1)} = \frac{1}{4}\left(v_{i+1j}^{(k)} + u_{i-1j}^{(k+1)} + v_{ij+1}^{(k)} + v_{ij-1}^{(k)}\right) - \frac{1}{4} f_{ij}$

$v_{i_0 j_0} \leq \frac{1}{4}\left(\quad\right)$ as $f_{ij} \geq 0$

But $v_{i_0 j_0} \geq$ max of all neighbouring points.

$\Rightarrow$ All values are same

$v_{i_0 j_0} = v_{ij}$ ij are neighbours

→ If any of these four neighbouring points are on the boundary then contradiction,

Repeat the same till we reach boundary.

→ similarly the minimum.

→ If max/min exists in interior then function remains constant

If $L^h v_{ij} \leq 0$ then $\min_{\Gamma^h} v_{ij} \leq \min_{\Omega^h} v_{ij}$

Corollary: Uniqueness of solution

Let u_{ij}, v_{ij} be two solutions of $L^h u_{ij} = f_{ij}$ then both are identical.

$$\begin{cases} Lu = f & \text{on } \Omega \\ u = g & \text{on } \Gamma \end{cases}$$

Consider $w_{ij} = u_{ij} - v_{ij}$

$$L^h w_{ij} = L^h u_{ij} - L^h v_{ij} = \begin{cases} f_{ij} - f_{ij} = 0 & \Omega^h \\ g_{ij} - g_{ij} = 0 & \Gamma^h \end{cases}$$

Apply both max and min principle

$$\underbrace{\boxed{\max_{\Gamma^h} w_{ij}}}_{=0} \geq \max_{\Omega^h} w_{ij} \geq \min_{\Omega^h} w_{ij} \geq \underbrace{\boxed{\min_{\Gamma^h} w_{ij}}}_{=0}$$

$\boxed{w_{ij} = 0}$ identically.

<u>Theorem</u>: Upper bound for discrete soln.

Discrete solution v_{ij} satisfies the following bound

$$\max_{\Omega^h} |v_{ij}| \leq \underbrace{\max_{\Gamma^h} |v_{ij}|}_{g} + \frac{1}{2} \underbrace{\max_{\Omega^h} L^h v_{ij}}_{f_{ij}}$$

<u>Proof</u>:

Consider the mesh function

$Q_{ij} = (ih)^2/2$ then max on boundary

$$\max_{\Gamma^h} |Q_{ij}| = \frac{1}{2}$$

$$L^h Q_{ij} = \frac{1}{4}\left(Q_{i+1j} + Q_{i-1j} + \ldots \right)$$

$$= 1$$

$M = \max_{\Omega^h} |L^h v_{ij}|$ and define

$w_{ij}^+ , \quad w_{ij}^-$

$$w_{ij}^{\pm} = \pm v_{ij} + M Q_{ij}$$

$$L^h w_{ij}^{\pm} = L^h(\pm v_{ij}) + L^h M Q_{ij}$$

$$= \pm L^h v_{ij} + M L^h Q_{ij}$$

$$\max L^h w_{ij}^{\pm} = \pm M + M \geq 0$$

and from max principle

$$\max_{\Omega^h} w_{ij}^{\pm} \leq \max_{\Gamma^h} w_{ij}^{\pm}$$

$$\leq \max_{\Omega^h} (\pm v_{ij} + M Q_{ij}) \leq \max_{\Gamma^h} v_{ij} + M Q_{ij}^{\pm}$$

$$\max_{\Omega^h} (\pm v_{ij}) \qquad \leq \max_{\Gamma^h} v_{ij} + M$$

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$$\max_{\Omega^h} (\pm v_{ij} + M\theta_{ij}) \le \max_{\Gamma^h} |v_{ij}| + M\theta_{ij}$$

$$\Rightarrow \max_{\Omega^h} |v_{ij}| \le \max_{\Gamma^h} |v_{ij}| + M \max_{\Gamma^h} |\theta_{ij}|$$

$$\le \max_{\Gamma^h} |v_{ij}| + M \max_{\Omega^h} L^h |v_{ij}|$$

$$\le \max_{\Gamma^h} |v_{ij}| + \tfrac{1}{2} \max_{\Omega^h} L^h |v_{ij}|$$

for arbitrary mesh function.

If $v_{ij} = u_{ij}$

$$\max_{\Omega^h} |u_{ij}| \le \max_{\Gamma^h} |g_{ij}| + \tfrac{1}{2} \max |f_{ij}| \quad \text{—— (1)}$$

$\Delta u = f$ "f, g" are data

$u = g$ perturbation quantity

say $\Delta u = f \pm \varepsilon$, $u = g \pm \delta$ } change is also bounded in RHS

→ this change typically comes from round-off error

The inequality (1) shows that any perturbation in data either in source function f or in the boundary value g, the solution remains bounded wrt change. The perturbation can be thought of as the roundoff error.

<u>Truncation Error</u> (TE):

$$e_{ij} = u(x_i, y_j) - u_{ij}$$

let error at nodal point is diff b/w exact and calculated solution,

Local truncation error at i,j

$$= L^h e_{ij} = L^h u(x_i, y_j) - L^h(u_{ij})$$

$$= u_{xx} + u_{yy} + O(h^2) - f_{ij}$$

$$\leq O(h^2)$$

If we replace v_{ij} as e_{ij} then

$$\max |v_{ij}| = \max |u(x_i, y_j) - u_{ij}|$$

$$\leq \max |e_{ij}| \rightarrow 0 \text{ on boundary}$$

$$+ \frac{h^2}{2} \max |L^h u_{ij}|$$

$$\leq c\, h^2$$

"consistency constant" which depends on higher order derivative of u. $u \in C^4$

Hence this error is "apriori error estimate."

$u(x, y, t)$

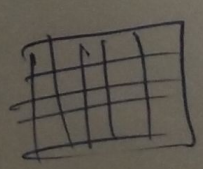
$$u_t - \Delta u = f(x, y, t)$$

$$u(x, y, 0) = \phi(x, y)$$

$$u(x, y, t) = g(t)$$

Consider 2D Parabolic PDE x, y, t

Heat conduction eqn, IBVP



$u_t \rightarrow$ can be replaced with three methods.

$$\frac{u_{lm}^{n+1} - u_{lm}^{n}}{k} - \left(\frac{u_{l+1m}^{n} - 2u_{lm}^{n} + u_{l-1m}^{n}}{h^2} + \frac{u_{lm+1}^{n} - 2u_{lm}^{n} + u_{lm-1}^{n}}{h^2} \right) = f(x_l, y_m, t_n) \quad \text{— (1)}$$

FTCS

BTCS

$$\frac{u_{lm}^{n+1} - u_{lm}^{n}}{k} - \left(\frac{u_{l+1m}^{n+1} - 2u_{lm}^{n+1} + u_{l-1m}^{n+1}}{h^2} + \frac{u_{lm+1}^{n+1} - 2u_{lm}^{n+1} + u_{lm-1}^{n+1}}{h^2} \right) = f(x_l, y_m, t_n) \quad \text{— (2)}$$

Implicit scheme, at each time we need to solve elliptic PDE

$O(\Delta t, \Delta x^2, \Delta y^2)$

$O(k, h^2)$

Average - Crank Nicolson Method

$O(k^2, h^2)$

Hyperbolic

$u_t + a u_x = 0$

wave eqn.

$x \in (0, l)$

$t \in [0, T]$

$u(x,0) = f(x)$

$u(0,t) = f(t)$

Draw characteristics

$\rightarrow \dfrac{du}{dt} + a \dfrac{dx}{dt} = 0$

$\downarrow$

$u(x,t)$

$x - at = 0$

characteristic curve

$u_{tt} = c^2 u_{xx}$

$\downarrow$

$u(x,t)$

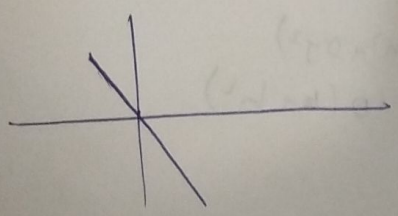
$u(x,0) = f(x)$

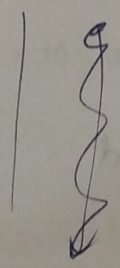
$u(0,t) = g(t)$

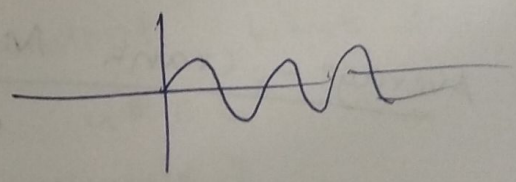
$A \sin(kx + \omega t) + B \cos(kx + \omega t)$

$x \pm ct = 0$

characteristic curves







$u(x,t) = f(x - ct) + f(x + ct)$

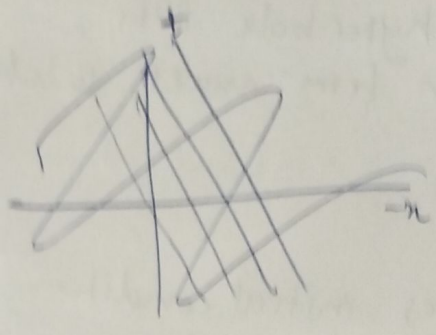
$+ \frac{1}{2} \int _ ___$

$u(x,t) = e^{-\pi t} \sin(\pi x)$

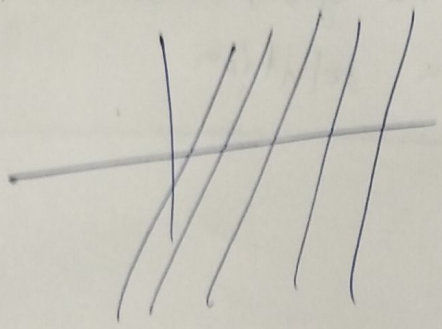
$u_t = u_{xx}$

$u(x,0) = \sin \pi x$

$u(0,t) = 0$



$x - at$



$x = at$

$x - at = 0$

$u_t + a u_x \rightarrow$ 3x3 choices

Extended real line

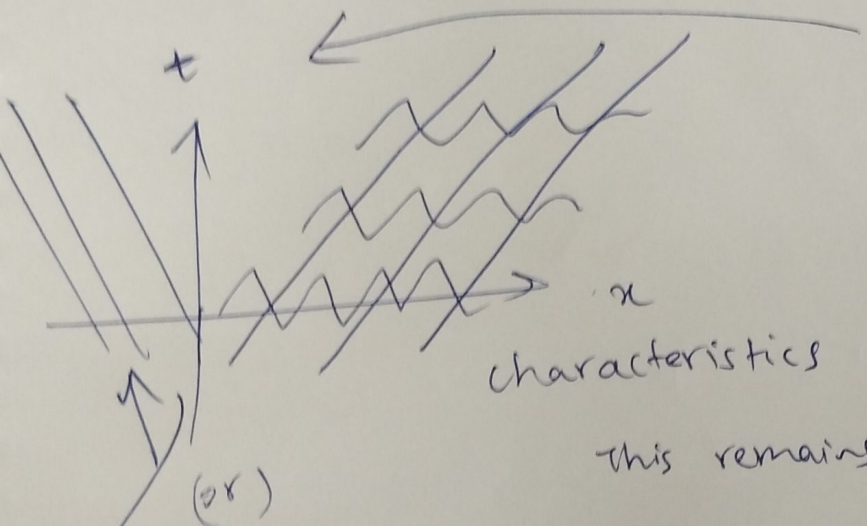
IVP

$u_t + a u_x = 0 \quad x \in \mathbb{R}$

$u(x,0) = f(x)$

soln is given

$u(x,t)$
$= f(x - at)$
for all $x \in \mathbb{R}$
$t > 0$



characteristics

This remains forever

$u(x,t) = e^{-\pi t} \sin \pi x$

this one has amplitude decreasing (we are constant)

we know solution is constant along characteristic curve $x - at = c$

a, 0 characteristic curve is given here which completely determines soln at (x,t) by projecting along characteristic back to x-axis $t = 0$

one of the major properties of soln of hyperbolic PDEs, is that of the wave propagation. seen from exact solution

$f(x-at)$

→ we have soln is $f(x-at)$, $t=0$ gives initial condition, "back propagating" we get the solution