f convex

1. set of all optimal sols is convex set.

<u>proof</u>:

$$S = \{\bar{x} \in \Omega : f(\bar{x}) = \min_{x \in \Omega} f(x)\}$$

$$= \{\bar{x} \in \Omega : f(\bar{x}) \leq \min_{x \in \Omega} f(x)\}$$

$$S' = \{x \in \Omega : f(x) \leq \alpha\}$$

is a convex set

f: convex function

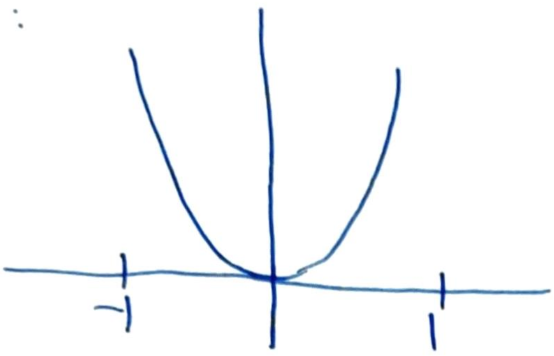
$f : \Omega \to \mathbb{R}$

convex set

$\therefore S$ is a convex set.

only true when minimising convex f

ex:



$f(x) = x^2 \quad 0 \leq x \leq 1$

$\max f(x) \quad x \in \Omega = [-1, 1]$

optimal solution $= \{-1, 1\}$

Not convex.

2. x^* is local minimum $\Rightarrow$ also global minimum

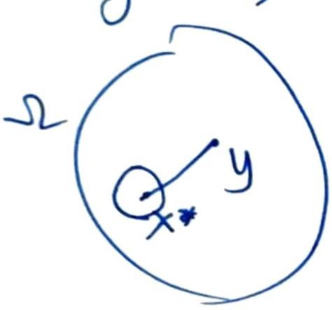
<u>Proof</u>: (contradiction)

Suppose x^* is local but not global minimum,

$\Rightarrow \exists y \in \Omega$, s.t. $f(y) < f(x^*)$.

$$f(\lambda x^* + (1-\lambda) y) \leq \lambda f(x^*) + (1-\lambda) f(y).$$

$0 \leq \lambda \leq 1$.



$$< \lambda f(x^*) + (1-\lambda) f(x^*) \quad 0 \leq \lambda < 1$$

$$< f(x^*)$$

x^* is best in it's neighbbourhood, but any neighbourhood ~~but~~ must contain this line.

So, every local minimum must be global minimum.

ex: Only for mininimizing convex function.

$\max f(x) = x^2, \quad -1 \leq x \leq 2$

-1 is local maximum, but not global maximum.

Remark :

Minimizing or maximizing ~~pot~~ linear functions, attains extremum in atleast one extreme point.

Not true for convex function (minimizing) $f(x)=x^2$ but, when maximized convex function, it attains at extreme point of Ω.

Theorem-6 : f - convex function - on closed & bounded convex set (so it has atleast one extreme point), Then $\exists$ a extreme point of Ω, where f takes maximum value.

FJ conditions & KKT conditions

Non-linear problem

$$\min f(x)$$

subject to

$$g_i(x) \leq 0 \quad i=1,2,\dots,m \quad x \in \mathbb{R}^n$$

$$f, g_i : \mathbb{R}^n \to \mathbb{R}$$

$$S = Fea(P) = \{ x : g_i(x) \leq 0 \}$$

$\Rightarrow$ A set D of feasible directions at $x^* \in Fea(P)$ is,

$$D_{x^*} = \{ d \in \mathbb{R}^n : \exists\, c>0 \text{ s.t } g_i(x^*+td) \leq 0 \ \forall i=1,2,\dots,m, \ \forall\, 0 \leq t \leq c \}$$

$\Rightarrow$ $I = \{ i \in \{1,2,\dots,m\} : g_i(x^*) = 0 \}$

let, $I^* = \{1,2,\dots,m\} \setminus I$

g_i is continuous at x^* $\leftarrow$ $g_i(x^*) < 0$, $\forall i \in I^*$ $\rightarrow$ along any direction, d allows change, if g_i continuous

$g_i(x^*) = 0$, $\forall i \in I$

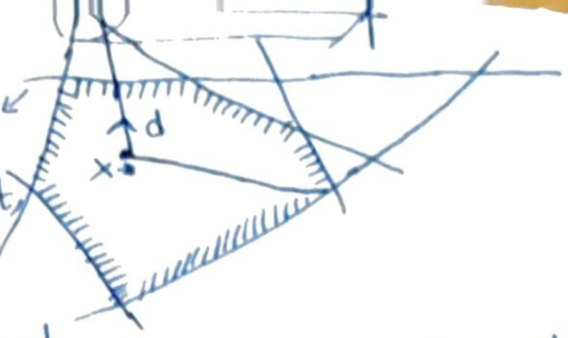
g_i is continuously differentiable at x^*

$f_i(t) = g_i(x^*+td)$

at $t=0$ $f_i(t)$ continuous, so for small t, $f_i(t)<0$

for each $i \in I^*$, $\exists c_i > 0$ s.t $g_i(x^*+td) < 0 \ \forall 0 \leq t \leq c_i, \forall d$

If $c = \min\limits_{i \in I^*} \{c_i\}$, then $g_i(x^*+td) < 0 \ \forall\, 0 \leq t \leq c, \forall d$.



for this constraint less than upper constraint

, c_i is dependent on the constraints & directions.

$\Rightarrow$ $\forall i \in I$, g_i is continuously differentiable at x^*.
If d is such that $\nabla g_i(x^*) d < 0$, $\forall i \in I$, then for each $i \in I$, $\exists\, a_i > 0$, such that $g_i(x^* + td) < 0$, $\forall\, 0 \leq t \leq a_i$.
for fixed $i \in I$.

$$g_i(x^* + td) = g_i(x^*) + \underbrace{t\left(\nabla g(x^*) d\right) + o(t)}_{<0 \text{ (for small } t)}$$

$\Rightarrow$, $g_i(x^* + td) < g_i(x^*) = 0$

$\Rightarrow$ If $a = \min_{i \in I} \{a_i\} > 0$, $g_i(x^* + td) < 0$ $\forall\, 0 \leq t \leq a$, $\forall i \in I$

$\Rightarrow$ If for $\forall i \in I^*$, g_i is continuous at x^* & $\forall i \in I$, g_i is continuously differentiable at x^*, then
$G_0 \subseteq D_{x^*}$ $\quad G_0 = \{d \in \mathbb{R}^n : \nabla g_i(x^*) d < 0, \forall i \in I\}$

$g_i(x^* + td) \leq 0 \longrightarrow \forall i \in I$, $\forall\, 0 \leq t \leq a$

$g_i(x^* + td) \leq 0 \longrightarrow \forall i \in I^*$, $\forall\, 0 \leq t \leq c$.

$\searrow$ $\forall\, 0 \leq t \leq \min(a, c)$.

(necessary condition)
$\Rightarrow$ If x^* is a local minimum, $\nabla f(x^*) d \geq 0$ $\forall d \in D_{x^*}$
then $F_0 \cap D_{x^*} = \phi$
$(F_0 = \{d \in \mathbb{R}^n : \nabla f(x^*) d < 0\})$.
Then $F_0 \cap G_0 = \phi$

Theorem 7: (FJ necessary conditions) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function.

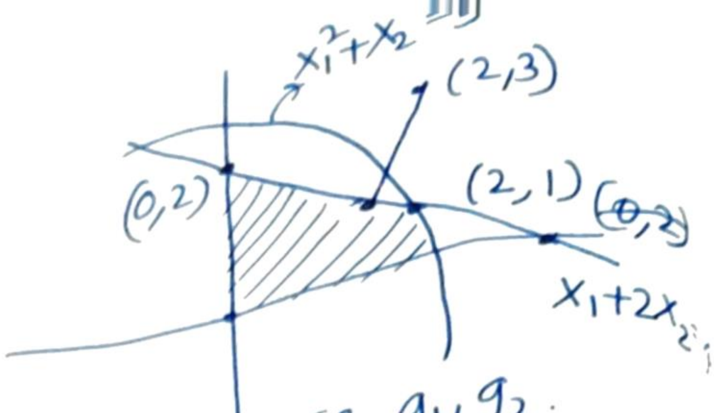
$S = Fea(P) = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, \dots, m\}$ &
$x^* \in S$.

For all $i \in I^*$, g_i is continuous at x^*.
$\forall i \in I$, g_i is continuously differentiable at x^*.

$\Rightarrow$ If x^* is a local minimum of f over S, then there exists non-negative constants, u_0, u_i, $i \in I$ not all zeros s.t

$$u_0 \nabla f(x^*) + \sum_{i \in I} u_i \nabla g_i(x^*) = 0.$$

Ex 1:



$$f(x_1,x_2)=(x_1-3)^2+(x_2-2)^2$$
$$x_1^2+x_2^2 \leq 5$$
$$x_1+2x_2 \leq 4$$
$$-x_1 \leq 0$$
$$-x_2 \leq 0$$

at $x^* = [2,1]^T$ binding constraints are g_1, g_2.

$\nabla g_1(x^*) = [4,2]$ $\quad\quad \nabla g_1(x) = [2x_1, 2x_2]$

$\nabla g_2(x^*) = [1,2]$

$\nabla f(x) = [2(x_1-3), 2(x_2-2)]$

$\nabla f(x^*) = [-2,-2]$

$u_0 = 1,\ u_1 = \frac{1}{3},\ u_2 = \frac{2}{3}$.

$\Rightarrow \nabla f(x^*) + \sum_{i=1}^{2} u_i \nabla g_i(x^*) = 0$.

$\Rightarrow [-2,-2] + \left(\frac{1}{3}\right)[4,2] + \left(\frac{2}{3}\right)[1,2] = \bar{0}$

$\Rightarrow$ $+ [2,2] = 0$.

$[2,1]$ satisfy FJ condition.

$\Rightarrow$ At $x^* = [0,2]$, binding constraints are g_2, g_3.

$\nabla g_2(x^*) = [1,2]$

$\nabla g_3(x^*) = [-1,0]$

$\nabla f(x^*) = [-6,0]$

$u_0 = 1\ u_1 = -6\ u_2 = 0$ $\times$ (all non-negative).

$\Rightarrow$ So, $x^* = [0,2]$ is not a local minimum.

$\nabla f(x^*)\, d = [-6,0]\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ $(d_1>0)$ $\searrow$ $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$= -6d_1 < 0$ (so, not ≥ 0).

$\star$ If g_i is continuously differentiable at x^* for all $i = 1,2,\ldots,m$, then

$\rightarrow$ has no solution

$F_0 \cap G_0 = \phi \iff \nabla g_1(x^*)d < 0 \ldots\ \nabla g_k(x^*)d < 0\ \nabla f(x^*)d$

WLOG $I = \{1,2,\ldots,k\}$ $\quad \{I = \{i \in \{1,2,\ldots,m\};\ g_i(x^*) = 0\}$

$$A = \begin{bmatrix} \nabla g_1(x^*) \\ \vdots \\ \nabla g_k(x^*) \\ \nabla f(x^*) \end{bmatrix}$$

$\iff Ad < 0$ has no solution

Gordon's Theorem :

Exactly one of the following two systems has a solution:

$u \neq 0, u \geq 0, u^T A = 0$ — (1)

$Ay > 0$ — (1).

$Ad < 0$ has a solution $\Leftrightarrow$ $Ad > 0$ has a solution.

$F_0 \cap G_0 = \phi \iff Ad < 0$ has no solution.

$\Rightarrow$ Then $u \geq 0$, $u^T A = 0$, $u \neq 0$ has solution (by Gordon's Theorem).

$$\iff [u_0 \; u_1 \ldots u_k] \begin{bmatrix} \nabla f(x^*) \\ \nabla g_1(x^*) \\ \vdots \\ \nabla g_k(x^*) \end{bmatrix} = 0, \; u \geq 0, \; u \neq 0 \text{ has solution.}$$

$$\iff u_0 \nabla f(x^*) + \sum_{i=1}^{k} \nabla g_i(x^*) u_i = \underline{0} \qquad u_0, u_1, \ldots, u_k \geq 0$$ not all zero's.

$\Rightarrow$ If g_i is continuously differentiable at x^* $\forall$ $i = 1, 2, \ldots, m$, then.

$$u_0 \nabla f(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*) = 0 \;\;\&$$

$$u_i g_i(x^*) = 0, \; \forall i = 1, 2, \ldots, m$$ (non binding, Then $u_i = 0$)

has non-trivial solution (FJ necessary conditions). ($u_{k+1} = \cdots = u_m = 0$)

(u_i's dual variables)

$x^* \in S \Rightarrow$ primal feasibility condition.

* with non-negativities of u_i's is dual feasible condition.

$\Rightarrow$ x^* is said to satisfy KKT condition if $\exists$ non-negative u_i, $i \in I$. such that

$$\nabla f(x^*) + \sum_{i \in I} u_i g_i(x^*) = 0.$$

(only non-negati need not be al not zero).

(Interior minima satisfies directly) KKT point.

Theorem 8: (KKT Conditions):

If in addition to previous theorem, $\forall i\ \nabla g_i(x^*)$ are LI and x^* is a local minimum (FJ are necessary) then KKT point

~~FJ → KKT~~
KKT → FJ

ex:

$$\left.\begin{array}{l}\nabla g_1(x^*) = \begin{bmatrix}4\\2\end{bmatrix}\\ \nabla g_2(x^*) = \begin{bmatrix}1\\2\end{bmatrix}\end{array}\right\} \text{LI.}$$

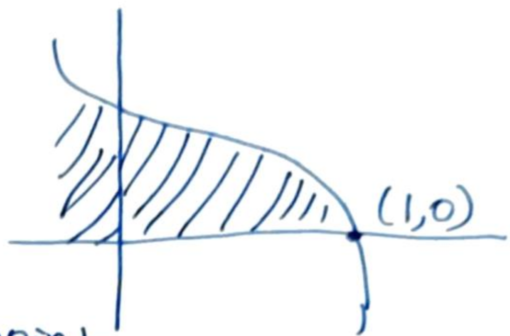
$u_0 = 1 \quad u_1 = \frac{1}{3},\ u_2 = \frac{2}{3} \qquad \nabla f(x^*) + \sum u_i \nabla g_i(x^*) = 0.$

FJ ⇒ KKT

ex: $\min f(x_1, x_2) = -x_1$

Sub. $\quad x_2 - (1-x_1)^3 \leq 0$

$\qquad -x_2 \leq 0.$



$x^* = [1,0]^T$ is a local minimum (also global minimum).

⇒ both are binding at this point

$\nabla f(x^*) = [-1, 0]$

$$\left.\begin{array}{l}\nabla g_1(x^*) = [0,1]\\ \nabla g_2(x^*) = [0,-1]\end{array}\right\} \text{not LI}$$

$u_0 = 0,\ u_1 = 1,\ u_2 = 1$. Then $u_0 \nabla f(x^*) + \sum_{i=1}^{2} u_i \nabla g_i(x^*) =$

⇒ u_0 must be zero

FJ point
not a KKT point

FJ is a necessary conditions, but
⇒ KKT is a not a necessary condition

(If LI and local minimum ⇒ KKT)

FJ $\not\Rightarrow$ KKT (may or maynot)
KKT ⇒ FJ

FJ & LI ⇒ KKT $\qquad \nabla f(x^*) + \sum_{i \in I} u_i \nabla g_i(x^*) = 0$

⇒ If x^* is local minimum, Then $-\nabla f(x^*)$ is cone generated by $\nabla g_i(x^*)$.

Gordons Theorem

Exactly one of the 2 has solution

1. $Ax > 0$
2. ~~$y^T A \geq 0$~~ $y^T A = 0, y \neq 0, y \geq 0.$

$$\Updownarrow$$

$$\begin{bmatrix} A^T \\ 1 \dots 1 \end{bmatrix} y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, y \geq 0$$

Farka's lemma: Exactly one of the following has solution

1. $Ax = b, x \geq 0$
2. $y^T A \geq 0, y^T b < 0$

from farka's lemma ~~if~~ exactly one of the following has a solution.

1. $y^T \begin{bmatrix} \cancel{A^T} \\ 1 \dots 1 \end{bmatrix} \geq 0 \quad , \quad y^T \begin{bmatrix} 0 \\ 1 \end{bmatrix} < 0$

$\Leftrightarrow \quad [y_1^T \; a] \begin{bmatrix} A^T \\ 1 \dots 1 \end{bmatrix} \geq 0 \quad [y_1^T \; a] \begin{bmatrix} 0 \\ 1 \end{bmatrix} < 0$

$\Leftrightarrow \quad y_1^T A^T \geq -a[1 \dots 1] \qquad a < 0$

$\Leftrightarrow \quad A y_1 \geq -a \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{m \times 1} \qquad a < 0$

$\Leftrightarrow \quad A y_1 > 0$ has a solution.

Theorem 8 (KKT conditions): If in addition to the conditions assumed for f and g_i's, as in the previous theorem, if $G_0 \neq \emptyset$ and x^* is local minimum (FJ point), then $\nabla f(x^*) + \Sigma_{i \in I} u_i \nabla g_i(x^*) = 0.$

$G_0 = \{d : \nabla g_i(x^*) d < 0 \quad \forall i \in I = \{ i \in \{1 \dots m\} : g_i(x^*) = 0 \}\}$

$I = \{1, 2, \dots, k\}$, $\begin{bmatrix} \nabla g_1(x^*) \\ \nabla g_2(x^*) \\ \vdots \\ \nabla g_k(x^*) \end{bmatrix} d < \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$ has solution

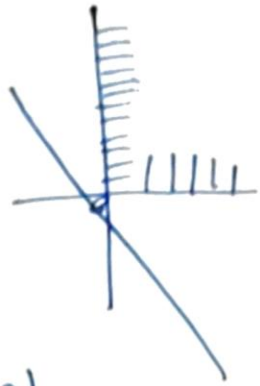
By Gordon's, $u^T \begin{bmatrix} \nabla g_1(x^*) \\ \vdots \\ \nabla g_k(x^*) \end{bmatrix} = 0, \quad u \geq 0, \; u \neq 0$ doesnot have a solution

$\Rightarrow \sum_{i=1}^{k} u_i \nabla g_i(x^*) = 0, \; u_i \geq 0, \; u \neq 0$ doesnot have solution.

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\{\nabla g_i(x^*)\}_{i \in I}$ is LD
but $G_0 \neq \emptyset$

$\min\ -x_1$
$-x_2 - x_1 \leq 0$
$-x_2 \leq 0$
$-x_1 \leq 0$



$\Rightarrow$ $u_1 \nabla g_1(x^*) + u_2 \nabla g_2(x^*) + u_3 \nabla g_3(x^*) = 0$
doesnot have non-negative, non-trivial Solution

$$u_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} + u_2 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_3 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow G_0 \neq \emptyset$$
(from Gordon's)

$\Rightarrow$ If $\nabla g_i(x^*)$, is LI, then $G_0 \neq \emptyset$ but not converse.

Theorem (KKT sufficient conditions) :

Let $f: \mathbb{R}^n \to \mathbb{R}$ be convex and continuously differentiable.

minimizing f subject to
$g_i(x) \leq 0 \quad i = 1, 2, \dots, m \qquad g_i : \mathbb{R}^n \to \mathbb{R}$.
$S = \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1, 2, \dots, m\}$
$x^* \in S$.

$\Rightarrow$ $\forall i \in I^*$ assume g_i is continuous function and $\forall i \in I$ assume g_i is continuously differentiable at x^*.

$f : \Omega \to \mathbb{R}$ is convex
$\downarrow$
convex set

$\{x \in \Omega : f(x) \leq c\}$ is a convex set
$\{x \in \mathbb{R}^n : g_i(x) \leq 0\}$ is convex set
$\Rightarrow$ So, S is also convex set.

$\Rightarrow$ let all g_i's be convex functions, so that
$S = \{x \in \mathbb{R}^n : g_i(x) \leq 0;\ i = 1, 2, \dots, m\}$ is convex.
Then x^* is global minimum, $\exists$ non-negative constants such that

$$\nabla f(x^*) + \Sigma u_i \nabla g_i(x^*) = 0.$$

ex:

$f(x_1,x_2) = (x_1-3)^2+(x_2-2)^2$

$\nabla^2 f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ positive semi-definite

$g_1(x) = x_1^2 + x_2^2 - 5 \le 0$

$\nabla^2 g_1 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

$g_2(x) = x_1 + 2x_2 \le 4$ (also convex) $\nabla^2 g_2(x) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$g_3(x) = -x_1 \le 0$

$g_4(x) = -x_2 \le 0$

$x^* = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is KKT point, so, it's global minimum.

Proof:

If $D=\phi$, Then trivially x^* is global minimum otherwise let d be a feasible direction at x^*.

$g_i(x^*+td) \le 0 \quad \forall i = 1,2,\dots,m \quad t$ sufficiently small.

Note that for $i \in I$, $g_i(x^*) = 0$.

$g_i(x^*+td) \ge g_i(x^*) + t\nabla g_i(x^*) d. \quad \forall t>0$ small

$f(y) \ge f(x) + \nabla f(x)(y-x)$

$g_i(x^*+td) \le 0 \Rightarrow \nabla g_i(x^*) d < 0$ (as $g_i(x^*)=0$).

$\forall i \in I$

x^* is KKT point, so,

$\Rightarrow \nabla f(x^*) + \sum_{i\in I} u_i \nabla g_i(x^*) = 0$ non-negative, ~~no~~ solution

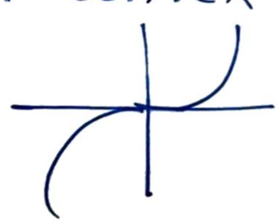
$\Rightarrow \nabla f(x^*) d + \sum_{i \in I} u_i (\nabla g_i(x^*) d) = 0$

$\Rightarrow \nabla f(x^*) d \ge 0$

which is sufficient condition, for convex function, to be global minimum.

ex: $f(x) = -x^2$ for $x \le 0$

$= x^2$ for $x \ge 0$.

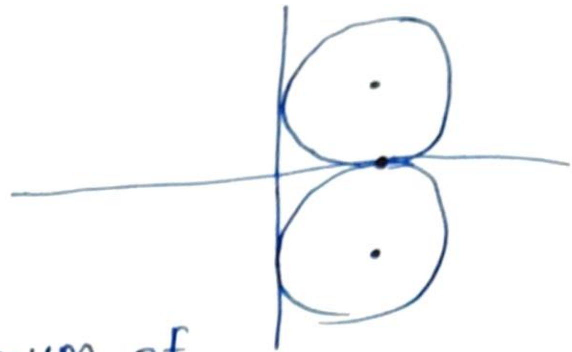


$x^*=0$, $\nabla f(x^*) = 0$, KKT, but not ~~a~~ local minimum (f not convex) ex: $f(x_1, x_2) = x_1$

subject to

$(x_1-1)^2+(x_2-1)^2 \leq 1$

$(x_1-1)^2+(x_2+1)^2 \leq 1$



$x^* = [1,0]^T$ is global minimum of f but not KKT point.

$x_2<0$, first inequality is not satisfied

$x_2>0$, second inequality is not satisfied.

$x_2=0$, so, $(x_1-1)^2 \leq 0$, so, $x_1=1$.

$\Rightarrow$ $\nabla f[x^*] = [1, 0]$

$\nabla g_1[x^*] = [2(x_1-1), 2(x_2-1)] = [0, -2]$ > binding constraint

$\nabla g_2[x^*] = [2(x_1-1), 2(x_2+1)] = [0, 2]$

$$1\begin{bmatrix}1\\0\end{bmatrix} + u_1\begin{bmatrix}0\\-2\end{bmatrix} + u_2\begin{bmatrix}0\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

not possible, not KKT.

But FJ conditions are satisfied, and also global minimum.

$\Rightarrow$ KKT is sufficient under convex conditions, but not necessary.

If global minimum $\Rightarrow$ Conditions satisfied.

not satisfied — need not be — not global minimum.

<u>LPP</u>

min $f(x) = c^Tx$

s.t $Ax=b$

$x \geq 0$

$g_i(x) \leftarrow a_i^Tx \leq b_i$

$g_i'(x) \leftarrow -a_i^Tx \leq -b_i$

$-x_i \leq 0$

binding

$$\nabla f(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*) = 0$$

$u_i g_i(x) = 0 \quad \forall i=1,2,..,m$

$\Downarrow$

$$c^T + \sum_{i=1}^{m}(u_i a_i^T - u_i' a_i^T) + \sum_{i=1}^{m}(-v_i e_i^T) = 0$$

$v_i x_i = 0 \quad \forall i=1,2,..,m$

$$\Leftrightarrow c^T + \sum_{i=1}^{m}(u_i - u_i')a_i^T - \sum_{i=1}^{n} v_i e_i^T = 0$$

$v_i x_i = 0 \quad \forall i=1,2,..,n$

$$\Leftrightarrow c^T - y^TA - v^TI = 0 \quad -(1)$$

$v_i x_i = 0 \quad \forall i=1,2,..,n$ $\left(\begin{matrix} u_i \geq 0 \\ u_i' \geq 0 \\ v_i \geq 0 \end{matrix}\right)$

y_i is unrestricted and $v_i \geq 0$.

$\Leftrightarrow$ $\cancel{y^T A - c^T \geq 0}$

$c^T - y^T A \geq 0$

$v_i x_i = 0 \quad i = 1, 2, .., n$

$v_i \geq 0$ & y_i is unrestricted in sign.

$\Leftrightarrow$ $y^T A \leq c^T$

$v_i x_i = 0$

$v_i \geq 0$

$\min c^T x$	$\text{Max } b^T y$
s.t $Ax = b$	s.t $A^T y \leq c$
$x \geq 0$	

$x^* \in Fea(P)$ is a KKT point $\Leftrightarrow$ $\exists$ a feasible solution y of the dual. s.t x^* satisfies complementary slackness ~~satisfies~~ with y.

$v_i x_i$ is like $((c^T)_i - (y^T A)_i) x_i = 0$. from ①.

$\Leftrightarrow$ x^* is an optimal solution of Ⓟ.

Conditions are equivalent to dual feasibility & complementary slackness conditions of LPP.

$\Rightarrow$ $x^* \in Fea(P)$ is optimal $\Leftrightarrow$ x^* is KKT point of (P).

$\Rightarrow$ Every point of Feasible region is FJ point.

Theorem 10: (FJ necessary conditions):

Let $f: \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable function. Consider P of minimizing f.

$g_i(x) \leq 0 \quad i = 1, 2, .., m \quad$ where $g_i : \mathbb{R}^n \to \mathbb{R} \ \forall i$

$h_j(x) = 0 \quad j = 1, 2, .. J \quad h_j : \mathbb{R}^n \to \mathbb{R} \ \forall j$.

$S = Fea(P)$ & $x^* \in S$.

$\forall i \in I^*$, g_i's are continuous at x^*. and $\forall i \in I$, g_i's are continuously differentiable at x^*, $\forall j$, h_j's are continuously differentiable at x^*.

$\Rightarrow$ If x^* is local minimum of f, Then $\exists$ $u_0, u_i, i \in I$ non-negative, v_j unrestricted, not all zero. s.t

$$u_0 \nabla f(x^*) + \sum_{i \in I} u_i \nabla g_i(x^*) + \sum_{j \in J} v_j \nabla h_j(x^*) = 0$$

(Not all points are FJ necessary points) minimize x_1+x_2

Sub to

$$x_2-x_1=0$$

$\nabla f(x)=[1,1]$

$\nabla h_1(x)=[-1,1]$

$$u_0[1,1]+v_1[-1,1]=[0,0]$$

(no point is FJ-necessary point)

ex: min $(x_1-1)^2+x_2$

Sub.

$$x_2-x_1=1$$
$$x_1+x_2\leq 2$$

$x^* = \left[\frac{1}{2},\frac{3}{2}\right]^T$

$\Rightarrow$ $\nabla h_1(x)=[-1,1]$

$\nabla f(x^*)=[2(x_1-1),1] = [-1,1]$

$\nabla g(x^*)=[1,1]$

$\Rightarrow$ $u_0\begin{bmatrix}-1\\1\end{bmatrix}+u_1\begin{bmatrix}1\\1\end{bmatrix}+v_1\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$u_0,u_1\geq 0.$

$\left.\begin{matrix}u_0=1\\ v_1-1\\ u_1=0\end{matrix}\right\}$ ~~So, It is local minimum~~ $\left[\frac{1}{2},\frac{3}{2}\right]$ is FJ point.

Theorem 11 (KKT necessary conditions):

f is continuously differentiable. & same as before

$\Rightarrow$ let $\{\nabla g_i(x^*), \nabla h_j(x^*)$ for $i\in I, j\in\{1,..,I\}\}$ be LI. Then, if x^* is local minimum of f

$\Rightarrow \exists$ u_i non-negative, v_j unrestricted such that

$$\nabla f(x^*)+\Sigma_{i\in I}u_i\nabla g_i(x^*)+\sum_{j=1}^{I}v_j\nabla h_j(x^*)=0.$$

$\{$ local min $\Rightarrow$ $u_0\nabla f(x^*)+\sum_{i\in I}u_i\nabla g_i(x^*)+\sum_{j=1}^{I}v_j\nabla h_j(x^*)=0$

So, u_0 must be non-zero, so, we can divide by $u_0\}$.

a) min

$f(x) = -x_1x_2 + x_1^2 + 2x_2^2 - 2x_1 + e^{x_1+x_2}$ over $\mathbb{R}^2$

First order necessary conditions →

If x^* is local minimal, then

$\nabla f(x^*) = 0$ → necessary is sufficient.

$$\nabla^2 f(x^*) = \begin{bmatrix} 2+e^{x_1+x_2} & -1+e^{x_1+x_2} \\ -1+e^{x_1+x_2} & 4+e^{x_1+x_2} \end{bmatrix} \rightarrow \text{so convex } f$$

$D_{x^*} = \{$ feasible directions at $x^* \}$
$= \mathbb{R}^2$

b)

$\Rightarrow \nabla f(x^*) = [2x_1 - x_2 + e^{x_1+x_2} - 2, \ -x_1 + 4x_2 + e^{x_1+x_2}]$

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad \nabla f(\begin{bmatrix} 0 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\nabla f(x^*) d < 0$

$\Rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} < 0$

$\Rightarrow -d_1 + d_2 < 0$

$d_2 < d_1 \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

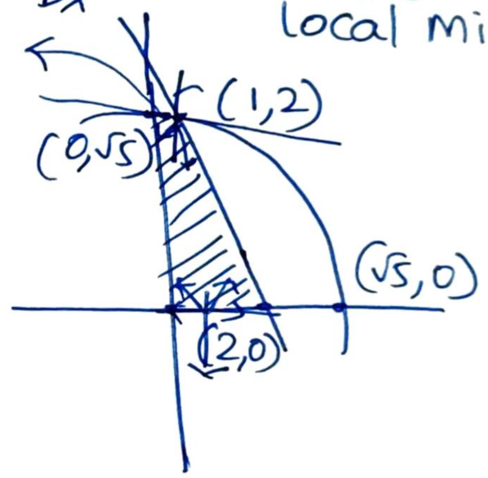
$f(x^* + td) = f(x^*) + t\left(\nabla f(x^*) d + \frac{o(t)}{t}\right)$

t sufficient small, then contradiction to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ being local minimum

q) min $(x_1-2)^2 + (x_2-3)^2$

Sub to

$x_1^2 + x_2^2 \leq 5$
$2x_1 + x_2 \leq 4$
$-x_1 \leq 0$
$-x_2 \leq 0$



1. $x^* = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is KKT point.
2. x^* is global minimum of f
3. $x^* \in \text{Fea}(P)$ s.t. $G_{0,x^*} = D_{x^*}$.

$G_{0,x^*} = \{ d \in \mathbb{R}^2 : \nabla g_i(x^*) d < 0 \quad \forall i \in I \}$

$I = \{ i \in \{1, \dots, m\} : g_i(x^*) = 0 \}$

$a_i^T x = b_i \qquad g_i(x) = a_i^T x - b_i \qquad a_i^T x \leq b_i$

outward normal $\leftarrow \nabla g_i(x) = a_i^T$

$\Rightarrow \quad G_{0,x^*} \subset D_{x^*}$ + all points on ~~by~~ defining curves.

$\Rightarrow G_{0,x^*} = D_{x^*} \to \{\text{on circle}\}$

$\nabla g_i(x^*) d < 0$

$\Rightarrow$ Does there exist an FJ point which is not KKT point.

$\Rightarrow$ only one KKT point

$$u_0 \nabla f(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*) = 0$$

$$u_i g_i(x^*) = 0 \quad \forall i = 1, 2, \ldots, m$$

only one $x_2 = 0$ line. $\leftarrow$ $u_0 \nabla f(x^*) + u_4 \nabla g_4(x^*) = 0$ $\quad u_0, u_4 \geq 0$ non-trivial

$$\Rightarrow u_0 \begin{bmatrix} 2(x_1-2) \\ 2(x_2-3) \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad u_0 \begin{bmatrix} 2(x_1-2) \\ -6 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad \underline{-6u_0 - u_4 = 0.} \to$ no non-negative non-trivial $\underline{\text{sol}}$

$\Rightarrow \quad 2u_0(x_1-2) = 0 \qquad x_1 = 2$ or $u_0 = 0$

$x^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$u_0 \begin{bmatrix} -4 \\ -6 \end{bmatrix} + u_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \to \text{no sol}$$

9) min $-x_1^2 - 4x_1x_2 - x_2^2$

sub. $x_1^2 + x_2^2 = 1$ $\quad \Rightarrow$ every point is FJ point

$\searrow$ no feasible direction.

closed & bounded continuous function

optimal solution

$$\begin{bmatrix} 2(x_1-2) \\ 2(x_2-3) \end{bmatrix} + u_1 \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} + u_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + u_3 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2(x_2-3) + u_1 2x_2 + u_2 - u_4 = 0.$ $\qquad u_1, u_2, u_3, u_4 \geq 0$

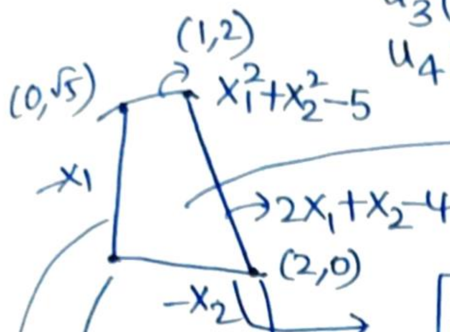
$\Rightarrow 2x_1 - 4 + 2u_1 x_1 + 2u_2 - u_3 = 0$

$u_1(x_1^2 + x_2^2 - 5) = 0$ $\qquad \begin{bmatrix} 2 \\ 3 \end{bmatrix} \to$ KKT point

$u_2(2x_1 + x_2 - 4) = 0$

$u_3(x_1) = 0$

$u_4(x_2) = 0.$



$\to$ Interior only $u_0 \begin{bmatrix} 2(x_1-2) \\ 2(x_2-3) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{matrix} FJ \\ KKT \end{matrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ outside so no FJ, no KKT

$$\begin{bmatrix} 2x_1-4 \\ -6 \end{bmatrix} + u \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad u_0(-6) + u(-1) = 0$$

no FJ

$u < 0 \to$ not KKT.

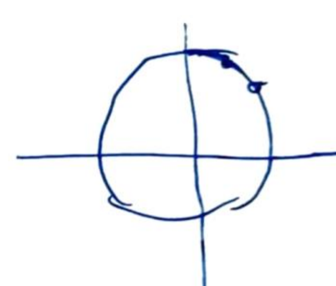
$$\begin{bmatrix} 0 \\ -6 \end{bmatrix} + u_1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + u_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad u_0(-6) + u_1(-1) = 0$$

no FJ.

$u_2 = 0 \quad u_1 < 0 \to$ not KKT

$\Rightarrow$ In all region's check for FJ & KKT points and see if there is some FJ point which is not KKT point.

$$\begin{bmatrix} -4 \\ -6 \end{bmatrix} + u_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$ not KKT no FJ.



$$\begin{bmatrix} -4 \\ 2x_2-6 \end{bmatrix} + u \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$ no KKT no FJ.

at $(0, \sqrt{5})$. $$\begin{bmatrix} -4 \\ 2\sqrt{5}-6 \end{bmatrix} + u_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 2(0) \\ 2\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\not\Rightarrow$ no FJ no KKT.

between $(0, \sqrt{5})$ & $(1,2)$

$$\Rightarrow \begin{bmatrix} 2(x_1-2) \\ 2(x_2-3) \end{bmatrix} + u \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \text{no FJ (also as } u_0 = 0) \text{ (no sol)}$$

$$\begin{array}{r} 36 \\ 16 \\ \hline 52 \end{array}$$

$\Rightarrow (2+2u)x_1 - 4 = 0$ $\qquad x_1^2 + x_2^2 = 5.$

$(2+2u)x_2 - 6 = 0.$ $\qquad \Rightarrow \dfrac{16+36}{2^2(1+u)^2} = 5$

$\Rightarrow \sqrt{\dfrac{13}{5}} = (1+u)$ $\qquad \dfrac{\cancel{52}^{13}}{\cancel{4} \times 5} = (1+u)^2$

no KKT $\leftarrow$ $x_1 = \dfrac{4}{2(1+u)} = \dfrac{2\sqrt{5}}{\sqrt{13}} \Rightarrow = 1.24 > 1$

$x_2 = \dfrac{3\sqrt{5}}{\sqrt{13}} = 1.86$ outside at $\begin{bmatrix}1\\2\end{bmatrix}$

$$\begin{bmatrix}-2\\-2\end{bmatrix}+u_1\begin{bmatrix}2\\4\end{bmatrix}+u_2\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$-2+2u_1+2u_2=0.$
$-2+4u_1+u_2=0.$

$u_2=2-4u_1$
$-2+2u_1+2(2-4u_1)=0$
$\Rightarrow -2+4-8u_1+2u_1=0$
$\Rightarrow 6u_1=2$
$u_1=1/3$

$u_2=2-4/3$
$=2/3$

KKT, so also FJ
(no other FJ)
$u_0=0$ no sol.

between $\begin{bmatrix}1\\2\end{bmatrix}$ & $\begin{bmatrix}2\\0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2x_1-4\\2x_2-6\end{bmatrix}+u_1\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ no FJ, that's not KKT

$\Rightarrow$ To check FJ, not KKT, take $u_0=0$ find FJ, then check if they are KKT or not.

Q. $g_2=-g_3$ $\forall x\in Fea(P)$

$g_2(x)\le 0$
$g_3(x)\le 0$
$\Rightarrow g_2(x)=g_3(x)=0.$
$\nabla g_2=-\nabla g_3$
$\Rightarrow$ so all points are FJ points. $(u_2=u_3)$

- If $\nabla f(x^*)=0$, x^* is KKT
- x^* is FJ & $u_0\ne 0$ has no sol, then x^* is not KKT
- KKT → FJ
- x^* is FJ, $\{\nabla g_i(x^*)\}_{i\in I}$ are LD, Then $G_{0,x^*}=\phi$.
(This is not true)

Gordon's Theorem:

$Ad<0$ —(1)
$u^TA=0,\ u\ge 0,\ u\ne 0.$ —(2)
} only one of these have solution

$\sum_{i\in I} u_i \nabla g_i(x^*)=0$ has non-negative solution
Then $\nabla g_i(x^*)$ are LD.

* If $\nabla g_i(x^*)$ are LI, then $G_{0,x^*} \neq \phi$.

(by Gordon's Theorem, This is true).

* If $G_{0,x^*} = \phi$, then $\nabla g_i(x^*)$'s are LD. (True by Gordon's Theorem)

* If x^* is not interior point & $F_{0,x^*} = \phi$, then x^* is local minimiser.

$\Rightarrow$ not true.

$F_{0,x^*} \to \{d \text{ s.t } \nabla f(x^*) d < 0\}$

Then $\nabla f(x^*) = 0$

$\nabla f(x^*) d < 0$

$-\nabla f(x^*) d < 0$

$\Rightarrow \nabla f(x^*) = 0$

By contra-diction

$\Rightarrow$ orthogonal to n LI vectors $\Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

* If all g_i's and f are convex functions and x^* is such that $F_{0,x^*} \cap G_{0,x^*} = \phi$, then x^* is global-minimum.

$\downarrow$

Then FJ point. (By gordon's Theorem).

(KKT is sufficient for global minimum, but FJ is not sufficient)

LPP $\to$ all feasible are FJ $\Rightarrow$ but not global minima.

* If x^* is local minima with $F_{0,x^*} \neq \phi$, $G_{0,x^*} = \phi$ then it's not KKT point.

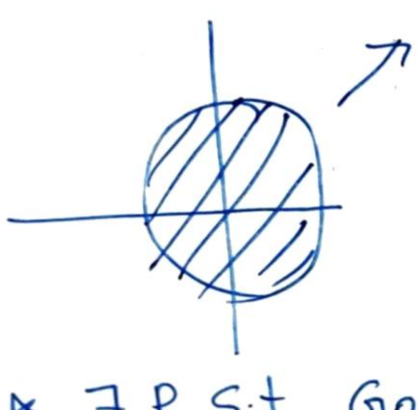
(Not true).

$\Downarrow$

$\nabla f(x^*) \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

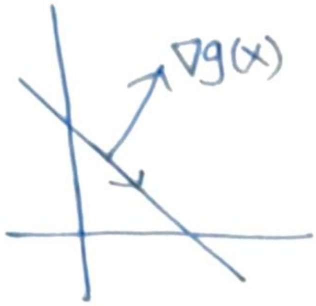
[If $F_0 = \phi = G_0$, then KKT]

* $\exists P$ such that $\forall x^* \in S$, $G_{0,x^*} = D_{x^*} \neq \phi$



* $\exists P$ s.t $G_{0,x^*} \neq D_{x^*}$ $\forall x^* \in S$.

★ $\exists P$, $fea(P) \neq \emptyset$, such that $G_{0,x^*} = \emptyset \quad \forall x^* \in S$



$a^Tx = b \in \mathbb{R} \rightarrow \begin{array}{l} a^Tx \le b \\ -a^Tx \le -b \end{array}$

$\nabla g_1(x) = a^T \qquad \nabla g_2(x) = -a^T$

binding

$G_{0,x^*} = \{ d : \nabla g_1(x)d < 0 \ \& \ \nabla g_2(x)d < 0$

$\Rightarrow a^Td < 0, \ -a^Td < 0$

$\Rightarrow \cancel{d = 0} \} = \emptyset \quad \forall x^* \in S$

★ $\exists P$, st $fea(P) \neq \emptyset$, then $D_{x^*} = \emptyset \quad \forall x^* \in S$

Q) For a LPP of form,

$\min c^Tx$

sub $Ax \le b, \ x \ge 0$

$\xrightarrow{\text{dual}}$ $\max -c^Tx$ s.t $Ax \le b$, $x \ge 0$

$\downarrow$ dual

$\min b^Ty$ s.t $A^Ty \ge -c$, $y \ge 0$

$\nabla f(x^*) + \sum_{i=1}^{m} u_i \nabla g_i(x^*) + \cancel{\sum_{i=1}^{n} v_i} = 0$

$u_i g_i(x) = 0 \ \forall i$

$a_i^T x - b_i \le 0 \quad i = 1, 2, \ldots, m$

$-x_i \le 0 \quad i = 1, 2, \ldots, n$

$c^T + \sum_{i=1}^{m} u_i a_i^T - \sum_{i=1}^{n} v_i e_i^T = 0$

$u_i \ge 0$

$v_i \ge 0.$

$c^T + u^TA = v^T \quad u \ge 0, \ v \ge 0.$

$v_i x_i = 0 \quad \forall i = 1, 2, \ldots, n$

$u_i ((Ax)_i - b_i) = 0 \quad \forall i = 1, 2, \ldots, m$

$\Rightarrow c^T + u^TA \ge 0, \ u \ge 0.$

$\Updownarrow$

$u^TA \ge -c^T, \ u \ge 0$

(dual feasibility condition)

We can solve with complementary slackness, this is optimal $\rightarrow$ also KKT point.

Not KKT, but FJ and local minimum.

q)

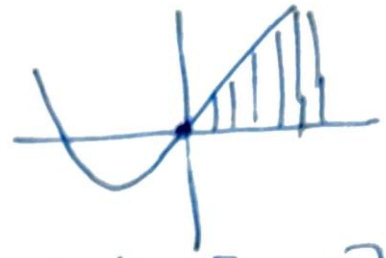
$$\min \quad 4x_1^2 - x_2^2 + 8x_1x_2$$

sub.

$$2x_1 + x_1^2 - x_2 \geq 0$$

$$\Rightarrow (x_1+1)^2 \geq (x_2+1)$$

$$\Rightarrow x_1 \geq 0, \; x_2 \geq 0.$$



$$\nabla g_1(x^*) = \begin{bmatrix} 2x_1+2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ +1 \end{bmatrix}$$

(Gordons)

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow u_0 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + u_1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u_0 = 1 \quad u_2 = 1 \Rightarrow u_1 < 0. \Rightarrow$ no sol

So, $G_0 \neq \phi$.