By implicit differentiation with respect to t we get,

$$\frac{d\pi(q^*)}{dt} = [R'(q^*) - C'(q^*)]\frac{dq^*}{dt} - t\frac{dq^*}{dt} - q^* \text{(using the chain rule)}$$

$$= [R'(q^*) - C'(q^*) - t]\frac{dq^*}{dt} - q^*$$

$= -q^*$, since $R'(q^*) - C'(q^*) - t = 0$

Hence, the proof.

The maximized profit falls as tax rate rises.

Example: A tree is planted at time $t = 0$, $P(t)$ is its current market value at time t. It's a differentiable function. r is the rate of interest. $P''(t) < 0$. When should the tree be cut down to maximize the present discounted value with continuous compound discounting?

Let the present value be given by $f(t)$; we know, $f(t) = P(t)e^{-rt}$

- The necessary condition for extreme points is, $f'(t^*) = 0$
- Or, $\frac{d}{dt} P(t^*)e^{-rt^*} = 0$
- Or, $P'(t^*)e^{-rt^*} + (-r)P(t^*)e^{-rt^*} = 0$
- Multiplying both sides by e^{rt^*}, we get,
- $P'(t^*) = r.P(t^*)$
- The tree will be cut down at the exact point where the instantaneous increase in the value of the tree ($P'(t^*)$) equals the return that can be obtained by putting the value of tree at the rate of interest ($r.P(t^*)$). Marginal gain = marginal loss of the instantaneous change in t.

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- The sufficient condition for maximum present value is,

$$\frac{d}{dt}\left[P'(t^*)e^{-rt^*} - r.P(t^*)e^{-rt^*}\right] < 0$$

The LHS = $\frac{d}{dt}[P'(t^*) - r.P(t^*)]e^{-rt^*}$

$= e^{-rt^*}(P''(t^*) - r.P'(t^*))$ - $re^{-rt^*}[P'(t^*) - r.P(t^*)]$

$= e^{-rt^*}(P''(t^*) - r.P'(t^*))$ (since $P'(t^*) - r.P(t^*) = 0$)

Given that $P''(t^*) < 0$ and assuming $P(t^*) > 0$ (since $P'(t^*) = r.P(t^*)$, it guarantees $P'(t^*) > 0$),

$$e^{-rt^*}(P''(t^*) - r.P'(t^*)) < 0$$

Hence, t^* is indeed a local maximum point.

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- Example/Application: The value of a parcel of land bought for speculation is increasing according to the formula, $F = 100e^{t^{1/4}}$. If the rate of interest is 10%, how long should the parcel be held to maximize the present value?
- The present discounted value of $100e^{t^{1/4}}$ is $100e^{-1/10t}e^{t^{1/4}}$
- Let, $P(t) = 100e^{t^{1/4}-1/10t}$
- The first order necessary condition is given by, $\frac{d}{dt}(100e^{t^{1/4}-1/10t}) = 0$

Or, $100e^{t^{1/4}-1/10t}[\frac{d}{dt}(t^{1/4} - 1/10t)] = 0$

Or, $¼(t^{-3/4}) - 1/10 = 0$

Or, $t^{-3/4} = 2/5$

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Or, $t^{3/4} = 5/2$

Or, $t = \left(\frac{5}{2}\right)^{4/3} = \sqrt[3]{39.06} = 3.39 = t^*$

Thus $t^* = 3.39$ is a stationary point.

- $\frac{d}{dt}(100e^{t^{1/4}-1/10t}) = 100e^{t^{1/4}-1/10t}[¼(t^{-3/4}) - 1/10] > 0$, for $t < t^*$
- And, $\frac{d}{dt}(100e^{t^{1/4}-1/10t}) < 0$, for $t > t^*$
- Thus, $t^* = 3.39$ is a global maximum.
- The land parcel should be optimally sold after 3 years 4 months (approx.).

Convex and concave functions and inflection points

Here we shall have a discussion on the meaning of the second derivative.

Recall: $f'(x) \geq 0$ on $(a, b) \leftrightarrow f(x)$ is increasing in (a, b)

$f'(x) \leq 0$ on $(a, b) \leftrightarrow f(x)$ is decreasing in (a, b)

In the same vein, as the second derivative $f''(x)$ is the derivative of $f'(x)$,

$f''(x) \geq 0$ on $(a, b) \leftrightarrow f'(x)$ is increasing in (a, b)

$f''(x) \leq 0$ on $(a, b) \leftrightarrow f'(x)$ is decreasing in (a, b)

$f'(x)$ increasing, $f(x)$ convex

$f'(x)$ decreasing, $f(x)$ concave

If f is continuous in the interval I, and twice differentiable in I' (interior of I), then, we define:

- f is convex on $I \leftrightarrow f''(x) \geq 0$, for all x in I'
- f is concave on $I \leftrightarrow f''(x) \leq 0$, for all x in I'

Example: is the function $f(x) = px^2 + qx + r$ a convex or a concave function?

From $f(x) = px^2 + qx + r$ we get,

$$f'(x) = 2px + q$$

Or, $f''(x) = 2px$

Thus, if $p > 0$, $f(x)$ is convex; if $p < 0$, $f(x)$ is concave.

If $p = 0$, it is linear function, in which case it is both convex and cancave.

- Just as an increasing function can be convex, a decreasing function can also be convex. In the latter case, the negative slope goes on rising, i.e., it become less negative.

Increasing, convex

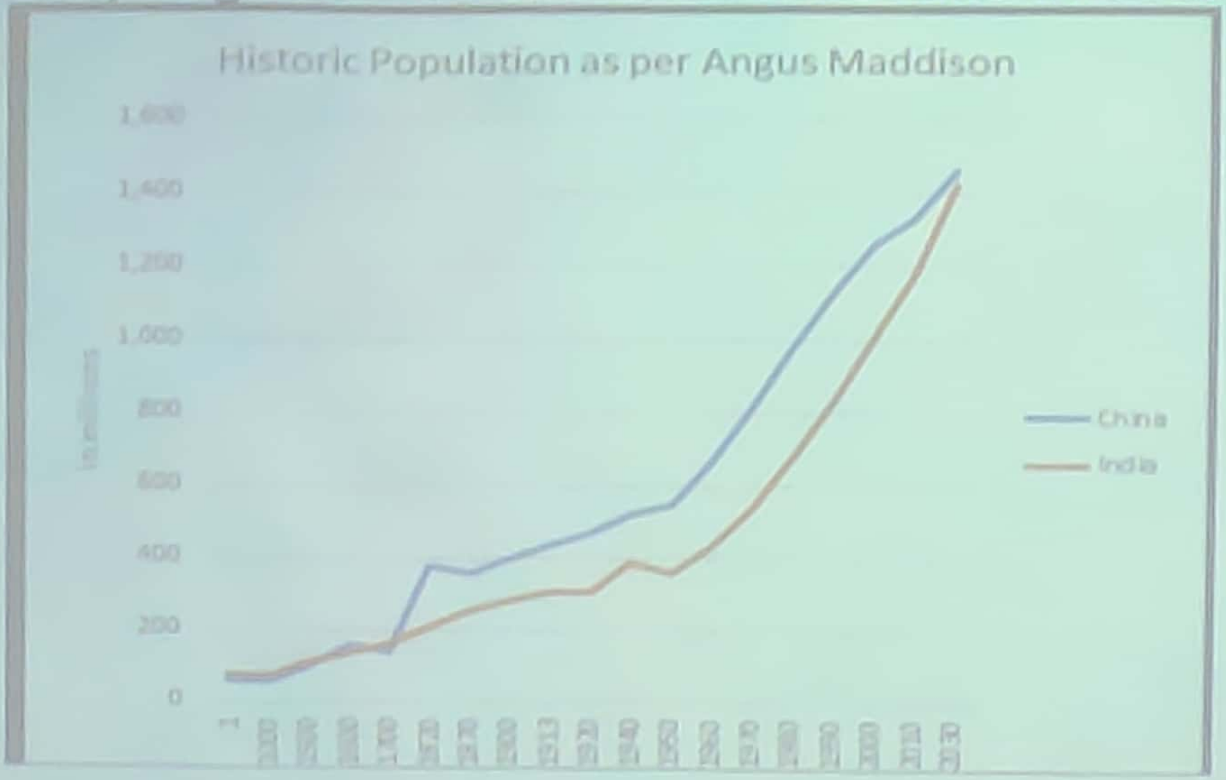
Decreasing, convex

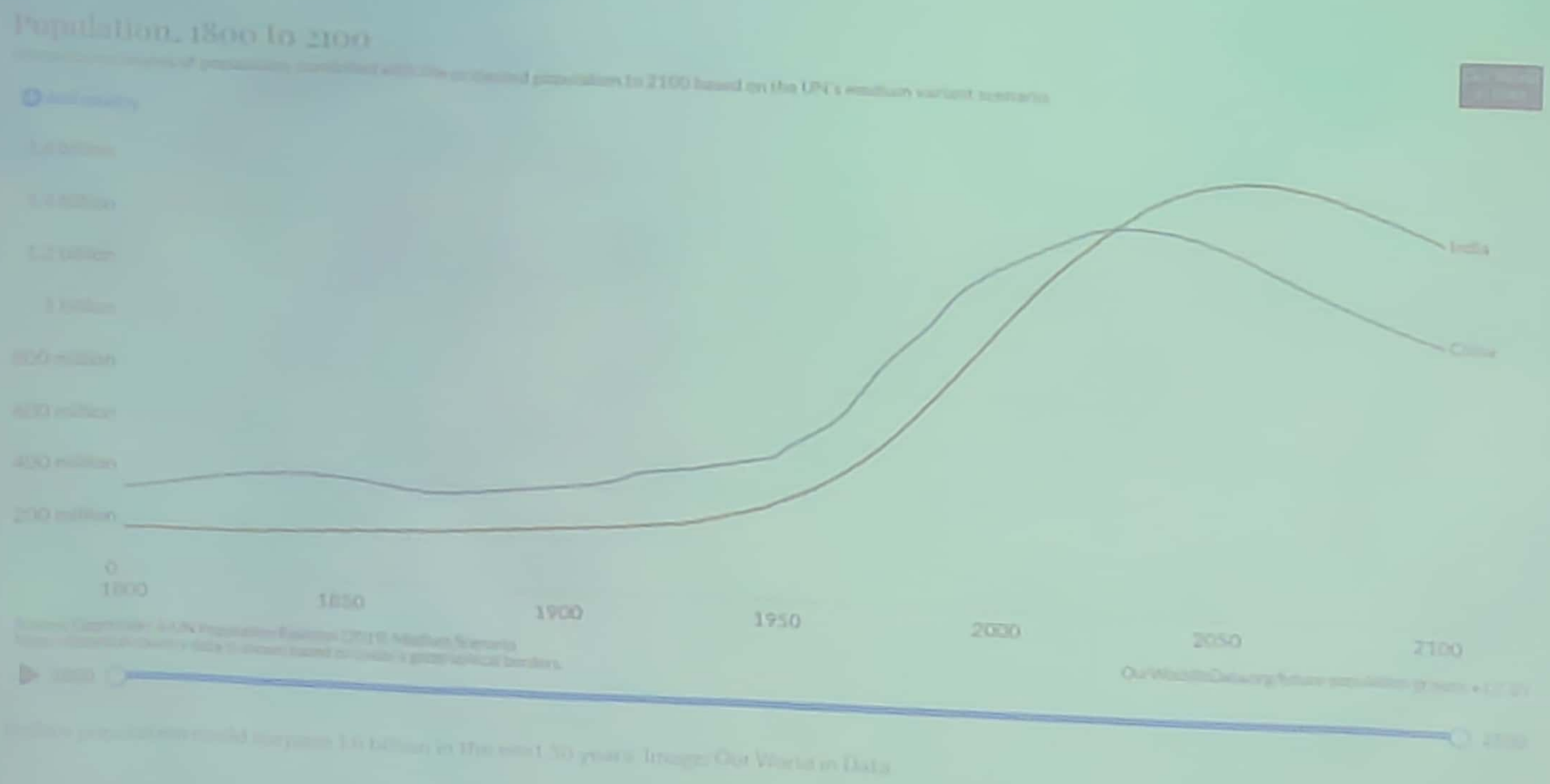
- Similarly, for concave: increasing and decreasing.

Increasing, concave

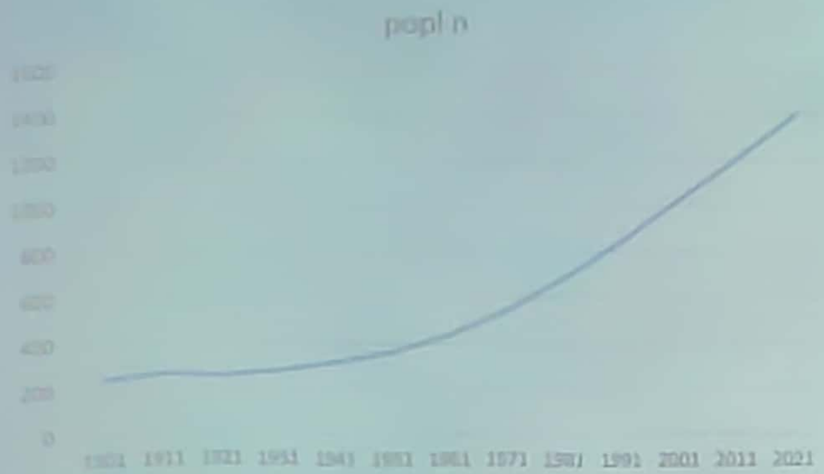
Decreasing, concave

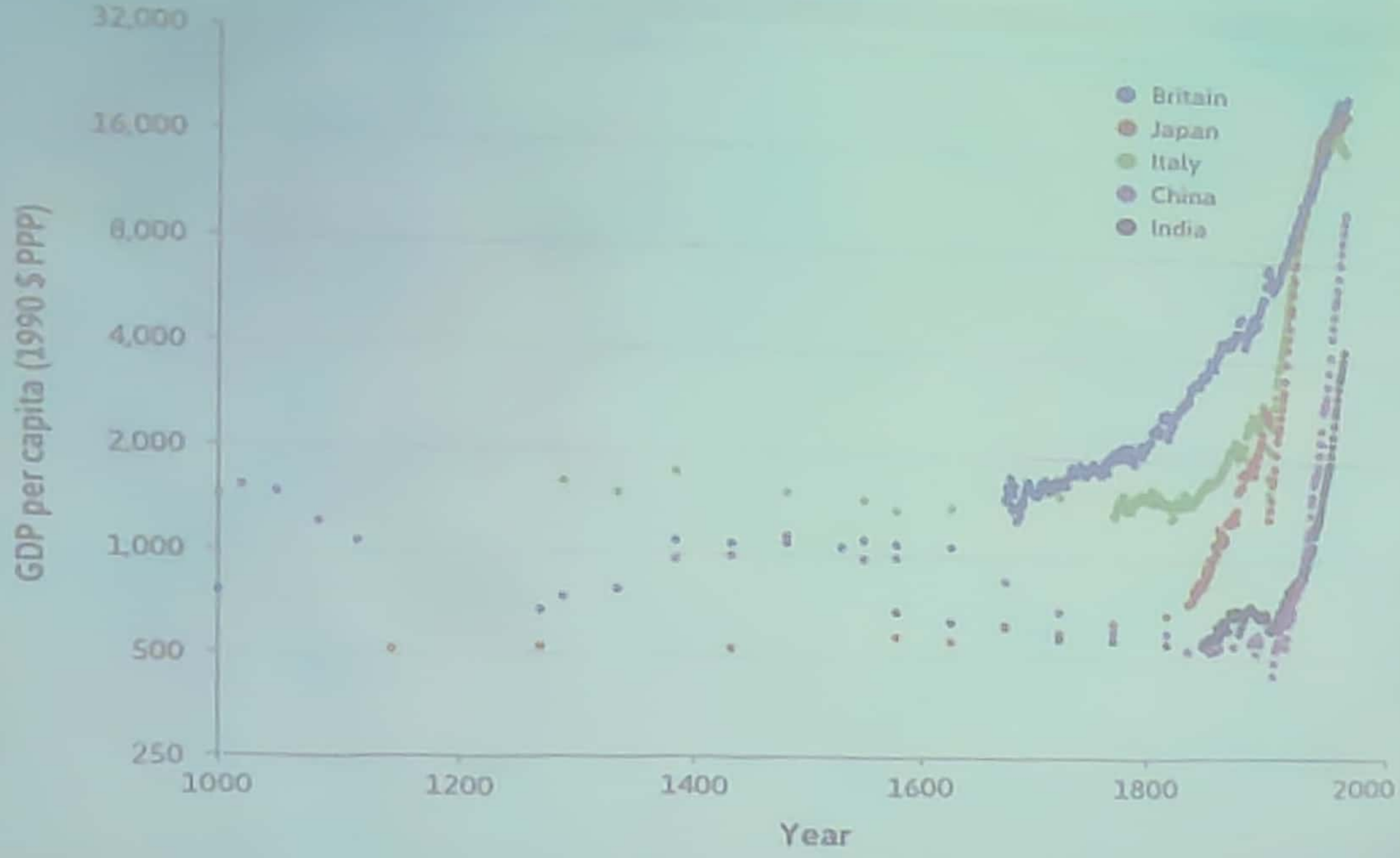
- Real life examples: The population of a country can take a shape, for some period, as given below. It is a convex and increasing function.




Population, 1800 to 2100
...projected population to 2100 based on the UN's medium variant scenario
200 million
400 million
600 million
800 million
0
1800
1850
1900
1950
2000
2050
2100
India
China
Image: Our World in Data

India’s population 1901-2021, changing curvature





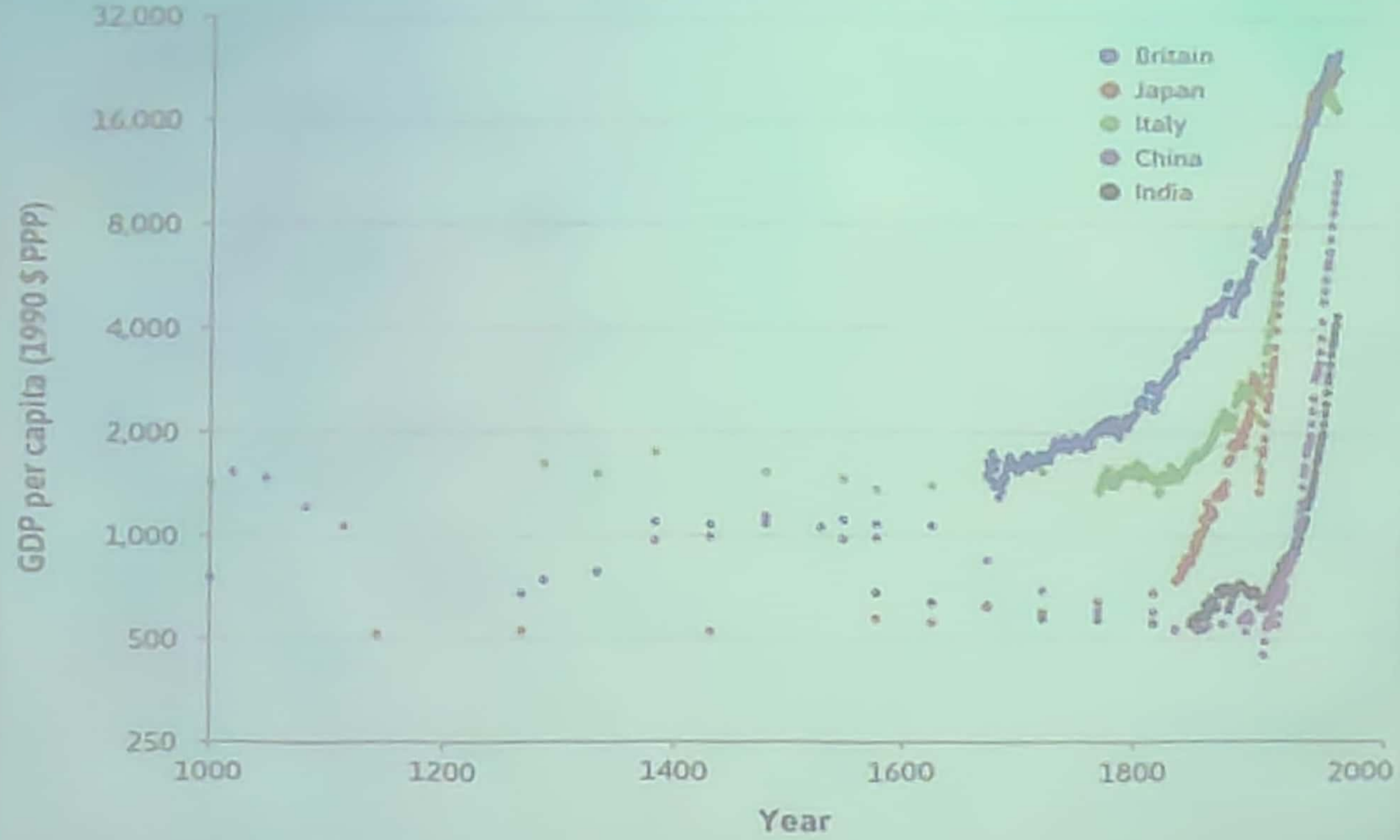
32,000
16,000
8,000
4,000
2,000
1,000
500
250
GDP per capita (1990 $ PPP)
1000
1200
1400
1600
1800
2000
Year
Britain
Japan
Italy
China
India

- Production of output, as a function of an input, is sometimes assumed to take a concave shape. It reaches a maximum and then become a decreasing function.


Population, 1800 to 2100
1.6 billion
1.4 billion
1.2 billion
1 billion
800 million
600 million
400 million
200 million
0
1800
1850
1900
1950
2000
2050
2100
India
China
India's population would surpass 1.6 billion in the next 30 years. Image: Our World in Data

India's population 1901-2021, changing curvature

India’s population 1901-2021, changing curvature



32,000
16,000
8,000
4,000
2,000
1,000
500
250
GDP per capita (1990 $ PPP)
1000
1200
1400
1600
1800
2000
Year
Britain
Japan
Italy
China
India

- Production of output, as a function of an input, is sometimes assumed to take a concave shape. It reaches a maximum and then becomes a decreasing function.

- $Y = AL^{\alpha}$, the value of α is unspecified, other than that it is positive. $A > 0$.
- We get from the above, $\frac{dY}{dL} = A\alpha L^{\alpha-1}$
- And, $\frac{d^2Y}{dL^2} = A\alpha(\alpha - 1)L^{\alpha-2}$
- Since $\alpha > 0$, marginal productivity of labour is positive.
- If $\alpha > 1$, then the second derivative is also positive, the production function is a convex function.
- If $\alpha < 1$, the second derivative is negative, the production function is a concave function.
- If $\alpha = 1$, the production function is a simple linear function of labour.

Inflection points

At some point in the domain of x, the nature of the function can change from convex to concave (and vice versa). Such points are called inflection points.

Suppose f is a twice-differentiable function, c is called an **inflection point** if there is an interval (a, b) such that $c \in (a, b)$, and either of the following conditions holds:

1. $f''(x) \geq 0, if\ a < x < c\ and\ f''(x) \leq 0\ if\ c < x < b$

Or,

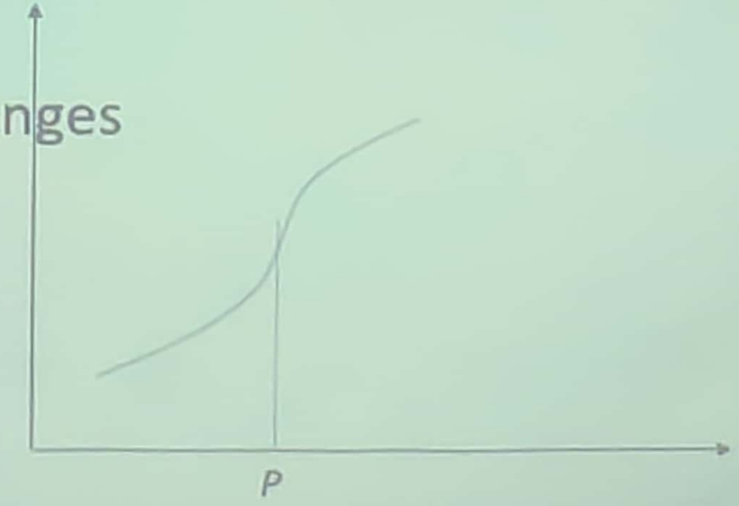
2. $f''(x) \leq 0, if\ a < x < c\ and\ f''(x) \geq 0\ if\ c < x < b$

Test for inflection points

Let f be a function with a continuous second derivative in an interval I, and c is an interior point of I.

1. If c is an inflection point for f, then $f''(c) = 0$.
2. If $f''(c) = 0$ and f'' changes sign at c, then c is an inflection point for f.

In the diagram the sign of $f''(c)$ changes from positive to negative. P is the inflection point.



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- Example: for the function $f(x) = x^4$, show that at $x = 0$, although $f''(0) = 0$, 0 is not an inflection point.
- From $f(x) = x^4$,
- $f'(x) = 4x^3, f''(x) = 12x^2$
- At $x = 0$, $f''(0) = 0$
- Does the sign of $f''(x)$ change at $x = 0$?
- For $x < 0, f''(x) = 12x^2 > 0$. It is also positive for $x > 0$. There is no change in sign. Hence $x = 0$ is not an inflection point.

- Suppose $f''(x) \geq 0$ in an interval. If there is an interior point in the interval, c which is a stationary point ($f'(c) = 0$), then $f'(c) \leq 0$, to the left of c (*since* $f'(c)$ is rising). Thus, $f(x)$ must be falling to the left of c. Similarly, it is rising to the right of c (weakly).
- In other words, c is a local minimum.
- Formally:

$f''(x) \leq 0$ for all $x \in I$, and $f'(c) = 0 \rightarrow x = c$ is a maximum point for f in I.

Applications

Example: Suppose the cost function of a firm is, $C(x) = px^2 + qx + r$, where p, q, r are positive constants, x is the level of output. Prove that the average cost of the firm has a minimum at $x = \sqrt{r/p}$ $(x > 0)$.

The average cost function is given by, $AC(x) = (px^2 + qx + r)/x = px + q + r/x$

To find the stationary point: $\frac{d}{dx}AC(x) = 0$

Or, $p - r/x^2 = 0$

Or, $(px^2 - r)/x^2 = 0$

Or, $x = \sqrt{r/p}$, since x cannot be negative

- $\frac{d^2}{dx^2}AC(x) = 2r/x^3 > 0$ (since r is positive), implying the average cost is a convex function.
- Hence the stationary point $x = \sqrt{r/p}$ is a minimum point.

- We can apply this test to the profit maximization exercise.
- Earlier we *assumed* that at a positive $q = q^*$ (say), the profit is maximized.
- Suppose the producer is operating in a perfect competition market, so that the price is given, p.

- We know, profit, $\pi(q) = R(q) - C(q)$
- We assume the profit is maximized at an interior point of I.
- First, we identify the stationary point(s) by the necessary condition,

$\pi'(q) = 0$

- This translates to, $\frac{d}{dq}\left(R(q) - C(q)\right) = 0$
- Or, $\frac{d}{dq}\left(pq - C(q)\right) = 0$
- Or, $p - C'(q) = 0$
- Or, $p = C'(q)$

- Let us suppose, $p = C'(q)$ is satisfied at $q = q^*, i.e.,$

$$p = C'(q^*)$$

Now, we take the second derivative of profit function, $R(q) - C(q)$

$$\frac{d^2}{dx^2}\big(R(q) - C(q)\big) = \frac{d}{dq}(p - C'(q))$$

$$= -C''(q)$$

For the stationary point q^* to be the maximum point, we need,

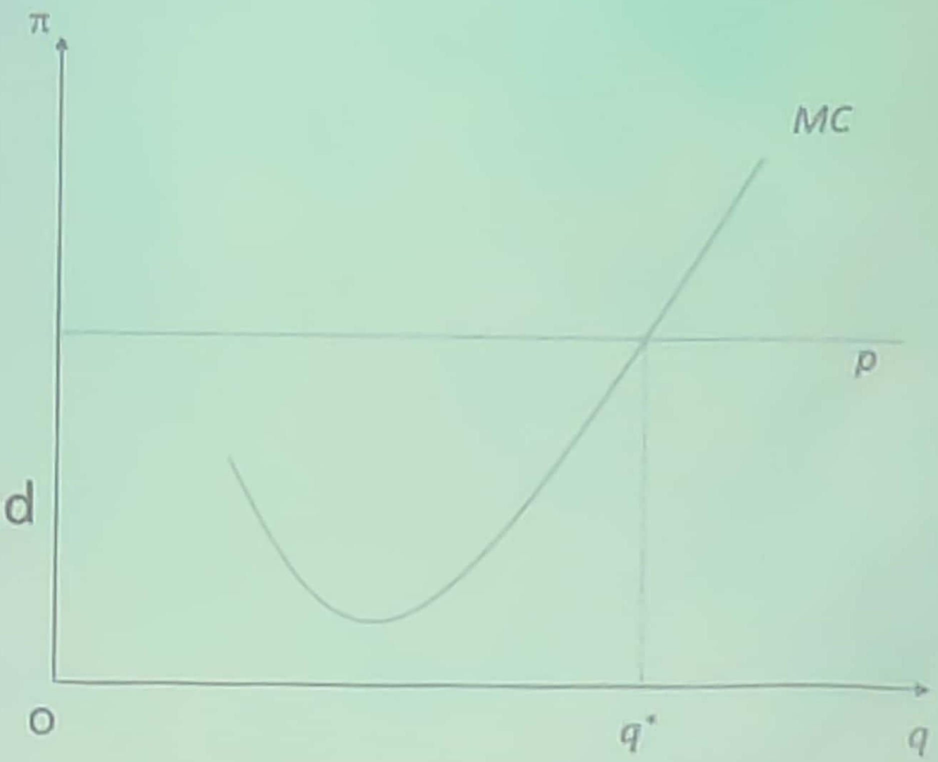
$$-C''(q) < 0$$

Or, $C''(q) > 0$

In other words, the maximum profit is obtained at q^* if the marginal cost function is increasing at q^*.

At q^*, the necessary condition is satisfied.

The graph of the marginal cost function has a positive slope at q^*, which ensures that at q^* there is maximization of profit, and not is minimization, or an inflection point.

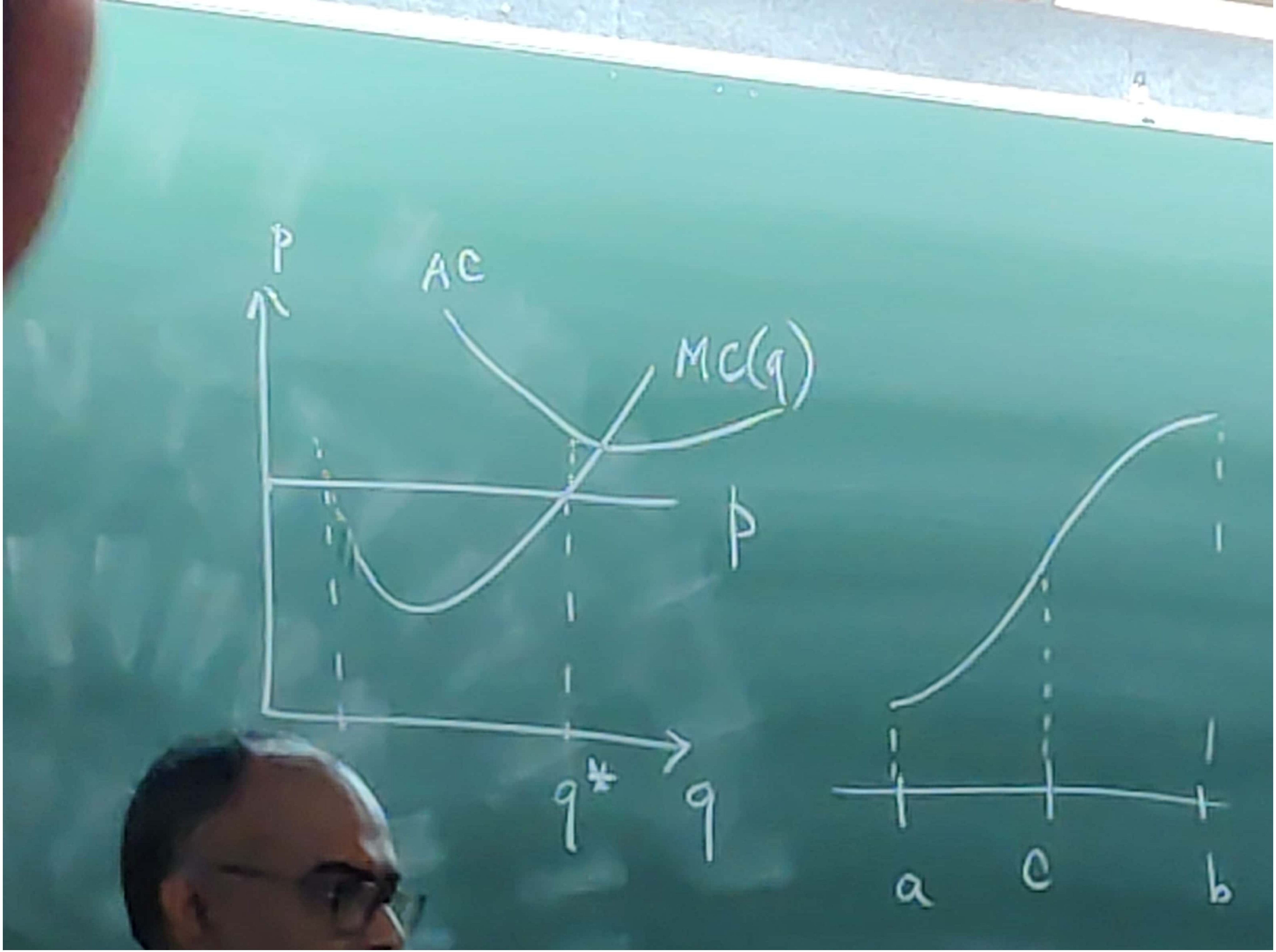


The profit function, $\pi(q) = aq^2 + bq + c$, should reflect the following assumptions. What parametric restrictions are required?

a. If nothing is produced profit is negative due to fixed costs.

b. The profit function is strictly concave.

c. The maximum profit occurs at a positive output level.


p
AC
MC(q)
p
q*
q
a
c
b

$$\pi(q) = aq^2 + bq + c$$

a. At $q = 0, \pi(q) = c$. Thus the value of c must be less than zero.

b. From $\pi(q) = aq^2 + bq + c$ we get,

$$\frac{d\pi}{dq} = 2aq + b$$

Taking the second derivative, $\frac{d^2\pi}{dq^2} = 2a$

Since the function is strictly concave, $\frac{d^2\pi}{dq^2} < 0$

Thus, $a < 0$

c. The profit maximizing output can be identified from the first order codition $\frac{d\pi}{dq} = 2aq + b = 0$

Or, $q = -\frac{b}{2a} = q^*$, say

Now, $q^* > 0$ implies $-\frac{b}{2a} > 0$

From above we know $a < 0$.

Hence the required restriction on b is, $b > 0$

In sum, the conditions are, $a < 0, b > 0, c < 0$.

A wine dealer has a certain quantity of wine, which he can sell at present and get *K* rupees. Alternatively, he can keep the wine in stock, as time passes the value of the stock rises by the formula, $V(t) = Ke^{\sqrt{t}}$, where *t* is the time at which the stock is sold. Find the optimal time at which the present value of the stock is maximized.

Let us denote the present value of $V(t) = Ke^{\sqrt{t}}$ by $P(t)$

Thus, $P(t) = V(t)e^{-rt}$, where *r* is the rate of interest

Or, $P(t) = Ke^{\sqrt{t}}e^{-rt} = Ke^{\sqrt{t}-rt}$

The dealer will like to maximize $P(t)$ by choosing the optimal time.

- By the first order condition, $\frac{dP(t)}{dt} = Ke^{\sqrt{t}-rt}(\frac{1}{2}t^{-\frac{1}{2}} - r) = 0$

Or, $\frac{1}{2}t^{-\frac{1}{2}} = r$, implying $t = \frac{1}{4r^2}$

It can be verified that the rate of increase of the value of the stock,

$\frac{1}{V(t)}\frac{dV(t)}{dt} = \frac{1}{2}t^{-\frac{1}{2}}$

Thus, the first order necessary condition implies that rate the increase of the value of the wine stock must be equal to the rate of interest, at the optimal time.

The second derivative of the present value, $\frac{d^2P(t)}{dt^2}$ is,

$Ke^{\sqrt{t}-rt}(-\frac{1}{4}t^{-\frac{3}{2}})$

[after simplifying, making use of the first order condition]

- This is negative
- Hence the present value is indeed maximized at the optimal time, $\frac{1}{4r^2}$, identified by the first order condition.

A firm in a perfectly competitive market faces the price 5. It has a cost function given by,

$$C(q) = 3q^2 - 595q + 40000, \text{ if } q > 0;$$
$$= 0, \text{ if } q = 0.$$

What is the output level of the firm if it wants to maximize its profit? What is the magnitude of maximum profit?

- The price is 5, hence the total revenue is, $TR = 5q$

The profit, $\pi(q) = 5q - (3q^2 - 595q + 40000)$

The first order necessary condition of profit maximization is,

$$\frac{d\pi(q)}{dq} = 0$$

Applying this, we get, $5 - 6q + 595 = 0$

- Or, $6q = 600$, implying, $q = 100$
- Checking for the second order condition: $\frac{d^2\pi}{dq^2} = -6$, thus the profit is indeed maximized at $q = 100$.
- The maximized profit, $\pi(100) = 600.100 - 3.100^2 - 40000$

$= 60000 - 70000 = -10000$

- The maximized profit is negative. If the firm instead of producing anything at all shuts down, its profit is 0.
- Thus the optimal output and profit is 0.

Suppose the production function of firms is $f(L) = AL^{\alpha}$ ($1 > \alpha > 0$, $A > 0$), the price of the good produced = p, wage rate of labour = w. What is the employment level of a profit maximizing firm (let's call it L^*). Comment on $\frac{\delta L^*}{\delta w}, \frac{\delta L^*}{\delta A}, \frac{\delta L^*}{\delta p}$.

Suppose the government announces a policy to give wage subsidy of $(1-\beta)w$ per unit of labour employed ($1 > \beta > 0$). What is the effect on employment level?

Let the profit of the firm be, π.

- $\pi(L) = pf(L) - wL$

- The firm will decide the employment by satisfying the necessary and sufficient conditions of profit maximization.
- The necessary condition, $\frac{d\pi}{dL} = 0$

Or, $\frac{d}{dL}(pAL^{\alpha} - wL) = 0$

Or, $p\alpha AL^{\alpha-1} - w = 0$

Or, $L = (\frac{w}{p\alpha A})^{\frac{1}{\alpha-1}} = \frac{1}{(\frac{w}{p\alpha A})^{\frac{1}{1-\alpha}}}$

This is termed as $L^* = L^*(w, A, p)$

The second order condition is satisfied, because $\frac{d}{dL}(p\alpha AL^{\alpha-1} - w)$

$= p\alpha(\alpha - 1)AL^{\alpha-2} < 0$, since $\alpha < 1$

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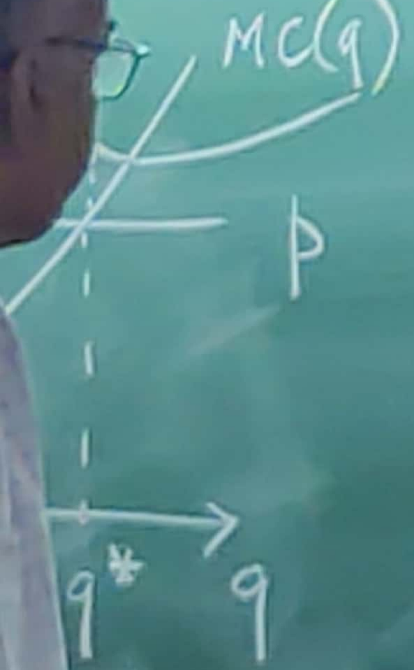
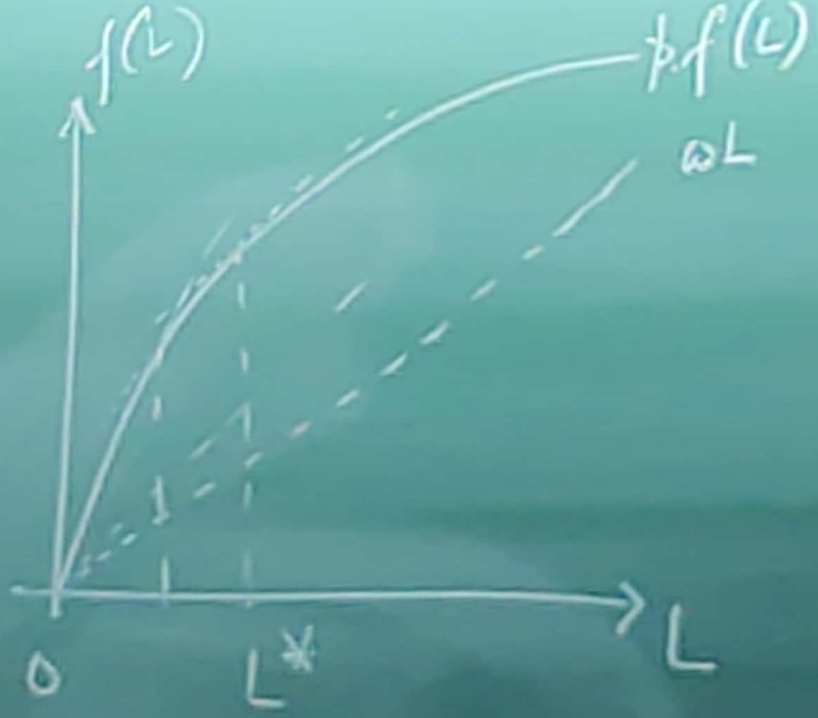
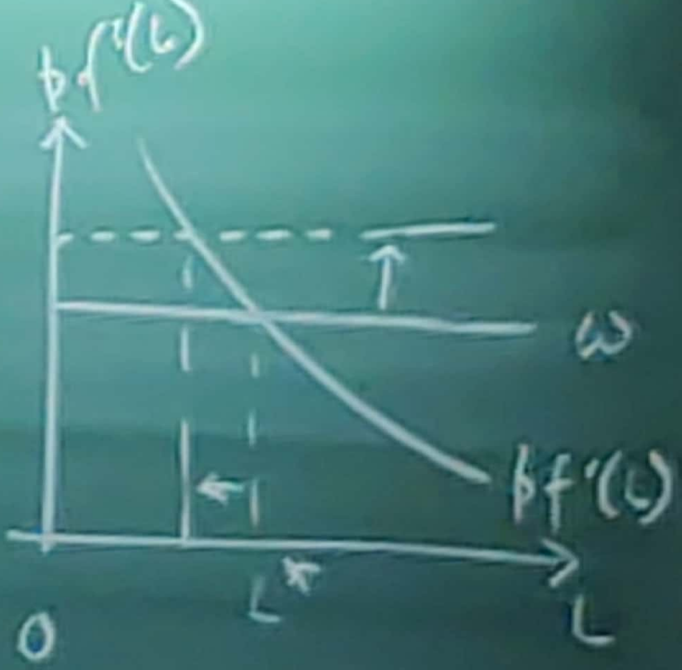

MC(q)
p
q*
q


f(L)
p.f(L)
ωL
0
L*
L


pf'(L)
ω
pf'(L)
0
L*
L

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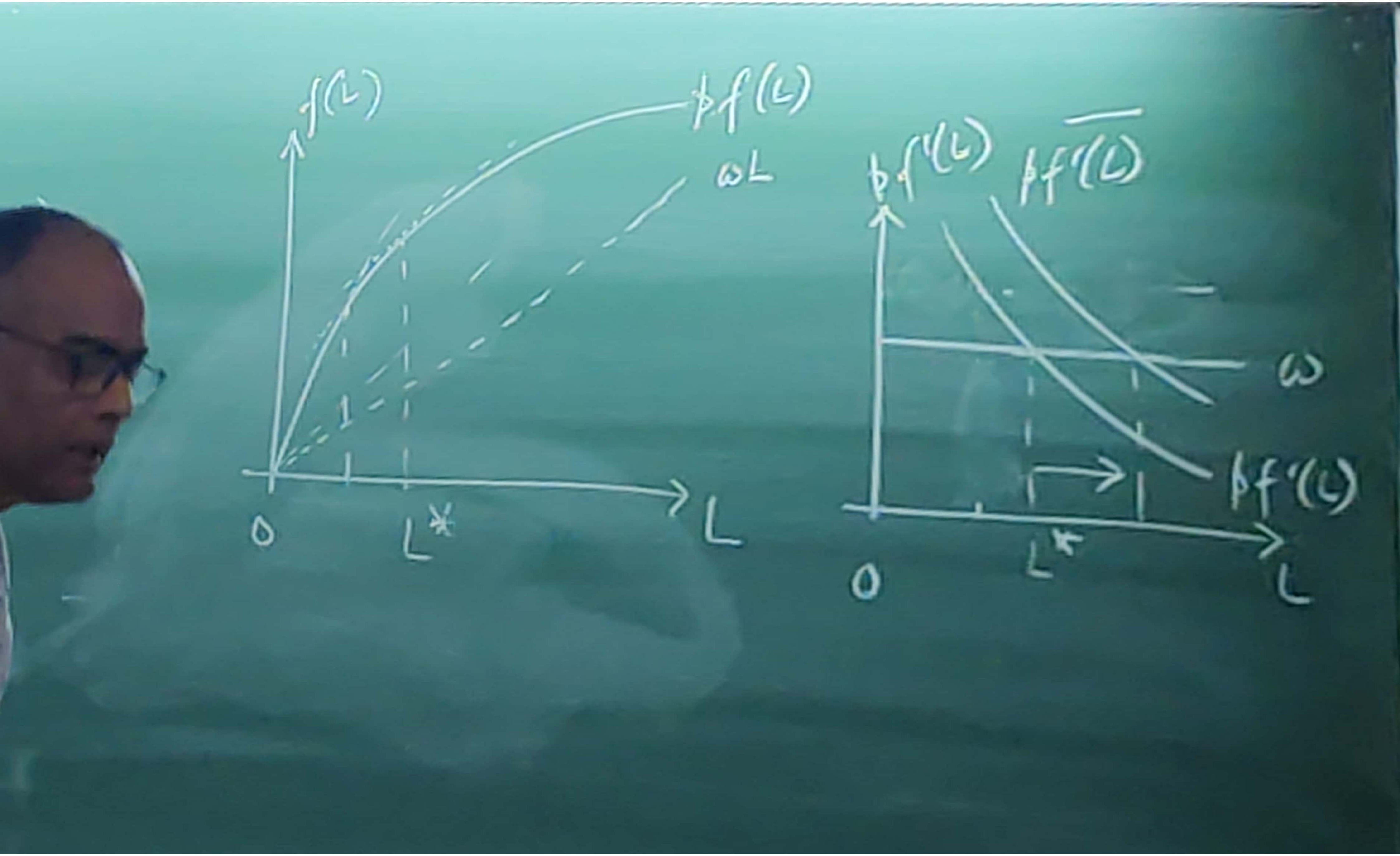
f(L)
pf(L)
ωL
1
0
L*
L
pf'(L)
pf'(L) with an overline
ω
pf'(L)
0
L*
L

- The government announced a subsidy of $(1-\beta)w$ per unit of labour employed.
- For the employer, the cost of labour is $w-(1-\beta)w = \beta w$
- The employment level is now decided by, $p\alpha AL^{\alpha-1} = \beta w$

This gives us, $L = (\frac{\beta w}{p\alpha A})^{\frac{1}{\alpha-1}} = \frac{1}{(\frac{\beta w}{p\alpha A})^{\frac{1}{1-\alpha}}}$, let us call this, L'

Since $\beta < 1$, $\beta w < w$, hence, $1/\beta w > 1/w$, thus,

$$\frac{1}{(\frac{\beta w}{p\alpha A})^{\frac{1}{1-\alpha}}} > \frac{1}{(\frac{w}{p\alpha A})^{\frac{1}{1-\alpha}}}$$

The wage subsidy boosts the employment level.

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A monopolist producer faces the market demand function: $q = 26 - p$. The cost function is, $C(q) = C + 6q + \frac{3}{2}q^2$

(i) Find the inverse demand function.

(ii) Obtain the expression of total revenue function $R(q)$ and profit function $\pi\ (q)$.

(iii) Find the profit maximizing level of output.

(iv) How does the profit maximizing output level change when (a) $C = 20$, (b) $C = 60$?