

Francis's QR \equiv QR

Let A be properly upper Hessenberg.

Shifted QR

Set $A_0 = A$

for $j = 0, 1, \dots$

(i) Compute $p_j(A_j)$.

(ii) Find reflectors $Q_j^{(1)}, \dots, Q_j^{(n-1)}$ such that
 $Q_j^{(n-1)} \dots Q_j^{(1)} p_j(A_j) = R_j$ is upper triangular

(iii) Compute

$$A_{j+1} := Q_j^{(n-1)} \dots Q_j^{(1)} A_j Q_j^{(1)} \dots Q_j^{(n-1)}.$$

Francis's Shifted QR

Set $A_0 = A$

for $j = 0, 1, \dots$

(i) Compute $p_j(A_j)e_1$.

(ii) Find reflector $Q_j^{(1)}$

such that $Q_j^{(1)} p_j(A_j)e_1 = \alpha_j e_1$

(iii) Compute $Q_j^{(1)} A_j Q_j^{(1)}$.

(iv) Find reflectors $\hat{Q}_j^{(2)}, \dots, \hat{Q}_j^{(p)}$ such that
 $A_{j+1} := \hat{Q}_j^{(p)} \dots \hat{Q}_j^{(2)} Q_j^{(1)} A_j Q_j^{(1)} \hat{Q}_j^{(2)} \dots \hat{Q}_j^{(p)}$
is upper Hessenberg.

Shifted QR finds $Q_j := Q_j^{(1)} \dots Q_j^{(n-1)}$ such that $p_j(A_j) = Q_j R_j$ is a QR decomposition of $p_j(A_j)$ and sets $A_{j+1} = Q_j^* A_j Q_j$.

But Francis's Shifted QR **also** finds a QR decomposition

$p_j(A_j) = \tilde{Q}_j \tilde{R}_j$ and sets $A_{j+1} = \tilde{Q}_j^* A_j \tilde{Q}_j$ where

$$\tilde{Q}_j := Q_j^{(1)} \hat{Q}_j^{(2)} \dots \hat{Q}_j^{(p)} \text{ and } \tilde{R}_j := \alpha_j K(A_{j+1}, e_1) [K(A_j, e_1)]^{-1}.$$

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- ▶ diagonal if A is normal;

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The iterates A_j are symmetric tridiagonal if A is Hermitian or real symmetric and upper Hessenberg otherwise.

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→ Finding v requires forming T !

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Strategy to find eigenvectors:

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- ▶ If A is Hermitian, the columns of Q are the eigenvectors.
- ▶ If A is not Hermitian, solve upper triangular systems $Tv = \lambda v$ for v and compute Qv .
- ▶ If A is real, arrange all computations to stay in real arithmetic till the very end.

Implicit QR: One of the top 10 algorithms of the 20th century



John Francis



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