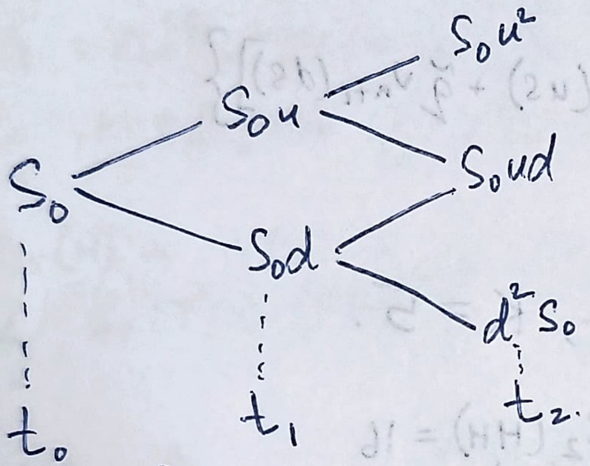
Straddles $\rightarrow f(T) = |S(T) - K|$

American Derivatives

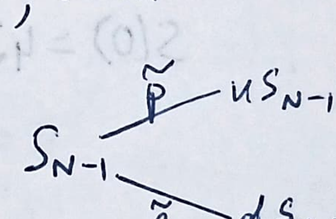
N-period binomial model with up factor u and down factor d, interest rate r.



No arbitrage condition:–

$0 < d < 1 + r < u$.

For a (european) derivative giving payoff $= g(S_N)$, then the price v_n at time $n \le N$ is given by



$$v_N(S_N) = g(S_N).$$

$$v_{N-1}(S_{N-1}) = \frac{\tilde{E}_{N-1}(g(S_N))}{1+r}$$

$$= \frac{1}{1+r}\left[\tilde{p} \cdot g(uS_{N-1}) + \tilde{q} \cdot g(dS_{N-1})\right].$$

$$v_n(S) = \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(Su) + \tilde{q}\, v_{n+1}(Sd)\right].$$

where $\tilde{p} = \dfrac{1+r-d}{u-d}$, $\tilde{q} = 1 - \tilde{p}$.

Hedging $\Delta_n = \dfrac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)S_n}$

Suppose American derivative with payoff

$$g(S_n) \quad \forall \quad n \leq N.$$

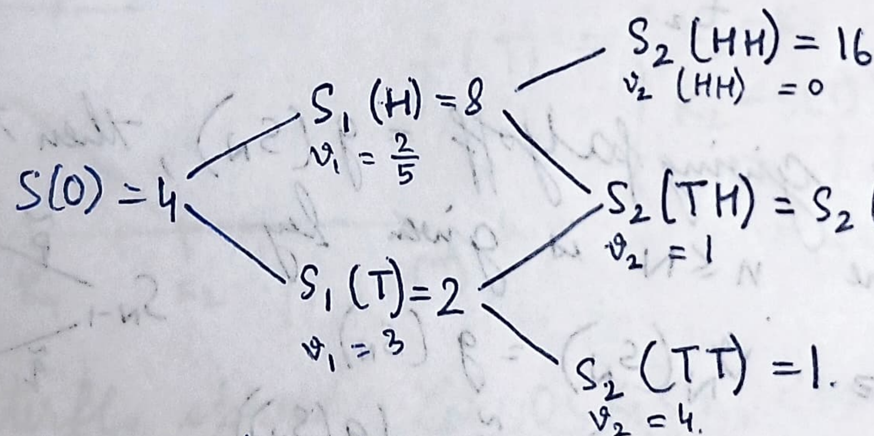
The price is given by v_n.

$$v_N(S_N) = \max\{g(S_N), 0\}.$$

$$v_{N-1}(S_N) = \max\left\{g(S_{N-1}), \frac{1}{(1+r)}\left[\tilde{p}\, v_N(uS_{N-1}) + \tilde{q}\, v_N(dS_{N-1})\right]\right\}$$

$$v_n(S) = \max\left\{g(S), \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(uS) + \tilde{q}\, v_{n+1}(dS)\right]\right\}.$$

<u>Example</u> – $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $K = 5$.



$$\tilde{p} = \frac{1+r-d}{u-d} = \frac{\frac{5}{4}-\frac{1}{2}}{2-\frac{1}{2}}$$

Consider an American put with $N = 2$. and $g(s) = K - S$.

$$v_2(s) = \max\{5 - s, 0\}$$

$$v_n(s) = \max\left\{5 - s, \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)\right]\right\}, \quad n = 0, 1.$$

$$v_2(16) = 0, \quad v_2(4) = 1, \quad v_2(1) = 4.$$

$$v_1(8) = \max\left\{\frac{1}{5/4}\left[\tfrac{1}{2}\cdot 0 + \tfrac{1}{2}\cdot 1\right], 0\right\} = \frac{4}{5}\cdot\frac{1}{2} = \frac{2}{5}.$$

$$v_1(2) = \max\left\{\frac{1}{5/4}\left[\tfrac{1}{2}\cdot 1 + \tfrac{1}{2}\cdot 4\right], 3\right\} = 3, \quad v_0(4) = \max\left\{\frac{1}{5/4}\left[\tfrac{1}{2}\cdot\tfrac{2}{5} + \tfrac{1}{2}\cdot 3\right], 1\right\}$$

$$= 1.36$$

$$\begin{tikzcd}[column sep=small]
 & & v_2(16)=0 \\
 & v_1(8)=0.4 \arrow[ur] \arrow[dr] & \\
v_0(4)=1.36 \arrow[ur] \arrow[dr] & & v_2(4)=1 \\
 & v_1(2)=3 \arrow[ur] \arrow[dr] & \\
 & & v_2(1)=4
\end{tikzcd}$$

$3 = -3\Delta_0 + 1.7$
$3\Delta_0 = -1.3$
$\Delta_0 = -1.3/3$

$3 = 2\Delta_0(T) + \frac{5}{4}(1.36 - 2\Delta_0)$ $\quad 0.34$
$3 = -\frac{1}{2}\Delta_0(T) + 1.7$
$\frac{1}{2}\Delta_0(T) = -1.3$
$\Delta_0 = -2.6$

Hedging strategy –

$$0.4 = \Delta_0^{(H)} S_1(H) + (1+r)(X_0 - \Delta_0^{(H)} S_0)$$
$$= 8\Delta_0(H) + \frac{5}{4}(1.36 - 4\Delta_0^{(H)})$$

$\Rightarrow \Delta_0(H) = -0.43$
Similarly, $\Delta_0(T) = -0.43$ } Always comes same.

$$\Delta_n \text{ is always } = \frac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)S_n}$$

$$Y_n = \tilde{v}_n = \frac{1}{(1+r)^n} v_n$$

$$\begin{tikzcd}[column sep=small]
 & & Y_2(HH)=0 \\
 & Y_1(H)=0.32 \arrow[ur] \arrow[dr] & \\
Y_0(4)=1.36 \arrow[ur] \arrow[dr] & & Y_2(HT)=Y_2(TH)=0.64 \\
 & Y_1(T)=2.4 \arrow[ur] \arrow[dr] & \\
 & & Y_2(TT)=2.56
\end{tikzcd}$$

$\tilde{p} = \tilde{q} = \frac{1}{2}$

$r = \frac{1}{4}$

If martingale, (Y_n)
$$\tilde{E}_n[Y_{n+1}] = Y_n$$
$$\Rightarrow Y_1(T) = \frac{1}{2}(Y_2(TH) + Y_2(TT)) = \frac{1}{2}(2.56 + 0.64) = 1.6 \neq 2.4.$$

Hence, not martingale but super martingale.

Theorem – Consider a N-period binomial model with

$$0<d<1+r<u, \quad \tilde{p} = \frac{1+r-d}{u-d}, \quad \tilde{q} = 1-\tilde{p},$$

Let payoff function $g(S)$ be given and define

$v_N(S), v_{N-1}(S), \ldots, v_0(S)$ by

$$v_N(S) = \max\{g(S), 0\}.$$

$$v_n(S) = \max\left\{g(S), \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(uS) + \tilde{q}\, v_{n+1}(dS)\right]\right\}.$$

Define $\Delta_n = \dfrac{v_{n+1}(uS_n) - v_{n+1}(dS_n)}{(u-d)\,S_n}$,

$$C_n = v_n(S_n) - \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(uS_n) + \tilde{q}\, v_{n+1}(dS_n)\right].$$

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n)$$

then,

$$X_n = V_n \;\&\; X_n \geq g(S_n) \quad \forall n.$$

Stopping Time

A random variable $\tau : \Omega_N \longrightarrow \{0, 1, 2, \ldots, N, \infty\}$ satisfies if $\tau(\omega_1\omega_2 \ldots \omega_n \omega_{n+1} \ldots \omega_N) = n$, then

$$\tau(\omega_1\omega_2 \ldots \omega_n \omega'_{n+1} \ldots \omega'_N) = n \;\forall\; \omega'_{n+1} \ldots \omega'_N.$$

Our example :- (Stopping time when $g(S) > 0$).

$$\tau : \{HH, HT, TH, TT\} \longrightarrow \{1, 2, \infty\}.$$

$\tau(HH) = \infty$ $\qquad$ $\tau(TH) = 1$

$\tau(HT) = 2$ $\qquad$ $\tau(TT) = 1.$ $\quad$ stopping time ✓

Consider $\sigma(HH)=0$ $\quad \sigma(TH)=1$

$\sigma(HT)=0$ $\quad \sigma(TT)=2$ $\quad$ ∴ σ is not a stopping time.

$Y_{0\wedge\tau} = Y_0 = 1.36$

$Y_{1\wedge\tau} = Y_1$

$Y_{2\wedge\tau}(HH) = Y_2(HH) = 0$ $\qquad Y_{2\wedge\tau}(TH) = Y_1(T) = 2.4$

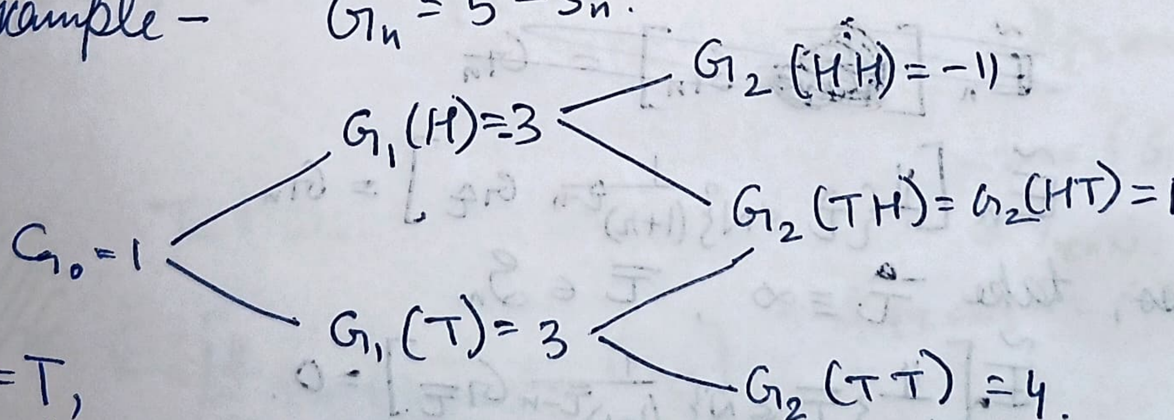
$Y_{2\wedge\tau}(HT) = Y_2(HT) = 0.64$ $\qquad Y_{2\wedge\tau}(TT) = Y_1(T) = 2.4.$

$S_n = \{\tau\text{-stopping time} : \tau : \Omega_N \to \{n, n+1, \dots, N, \infty\}\}$

Consider an american derivative with intrinsic value at n is G_n.

Define :- $V_n = \max\limits_{\tau \in S_n} \tilde{\mathbb{E}}_n\left[\mathbb{1}\{\tau \le N\} \frac{1}{(1+r)^{\tau-n}} G_\tau\right]$, $n=0,1,\dots,N.$

Our example - $\quad G_n = 5 - S_n.$



If $\omega_1 = T$,
feasible τ's -

$\tau_2(TT)=2,\ \tau_2(TH)=\infty$ $\qquad \tau_2(TT)=\infty,\ \tau_2(TH)=2$

$\tau_2(TT)=2,\ \tau_2(TH)=2$ $\qquad \tau_2(TT)=1=\tau_2(TH).$

$\tau_2(TT)=\infty,\ \tau_2(TH)=\infty$

For $\tau(TH) = 1 = \tau(TT)$

$$v_1^{\tau}(T) = \tilde{\mathbb{E}}_1\left[\mathbb{I}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-1}}\right]$$

$$= \tilde{\mathbb{E}}_1\left[G_1(T)\right]$$

$$= 3.$$

General American Derivatives

Theorem - If $v_n = \max_{\tau \in S_n} \tilde{\mathbb{E}}_n\left[\mathbb{I}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau\right]$

then, ① $v_n \ge \max\{G_n, 0\} \ \forall n$

② $\frac{1}{(1+r)^n} v_n$ is a supermartingale

③ If Y_n is another process satisfying ① and ②, then $v_n \le Y_n \ \forall n$.

Proof :- ① Take $\hat{\tau} \equiv n$, $\hat{\tau} \in S_n$.

~~$\tilde{\mathbb{E}}_n[\ G_n] = G_n$~~

$$\tilde{\mathbb{E}}_n\left[\mathbb{I}_{\{\hat{\tau} \le N\}} \frac{1}{(1+r)^{\hat{\tau}-n}} G_{\hat{\tau}}\right] = G_n$$

Also, take $\bar{\tau} \equiv \infty$, $\bar{\tau} \in S_n$.

$$\tilde{\mathbb{E}}_n\left[\mathbb{I}_{\{\bar{\tau} \le N\}} \frac{1}{(1+r)^{\bar{\tau}-n}} G_{\bar{\tau}}\right] = 0.$$

$\therefore v_n \ge G_n$ and $v_n \ge 0$

$\Rightarrow v_n \ge \max\{G_n, 0\}$.

② Let n be given and suppose τ^* attains the maximum in the definition of V_{n+1}.

$$V_{n+1} = \tilde{\mathbb{E}}_{n+1}\left[\mathbb{I}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n-1}} G_{\tau^*}\right].$$

$$\tau^* \in S_{n+1} \Rightarrow \tau^* \in S_n.$$

$$\therefore V_n \geq \tilde{\mathbb{E}}_n\left[\mathbb{I}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*}\right].$$

$$= \tilde{\mathbb{E}}_n\left[\tilde{\mathbb{E}}_{n+1}\left[\mathbb{I}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*}\right]\right].$$

$$\cancel{\tilde{\mathbb{E}}_n[V_{n+1}]} = \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)} V_{n+1}\right].$$

$$\therefore \boxed{\frac{1}{(1+r)^n} V_n \geq \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{n+1}} V_{n+1}\right]}$$

③ Suppose Y_n is another process satisfying ① and ②,

$$Y_k \geq \max\{G_k, 0\}. \ \forall k.$$

$$\mathbb{I}_{\{\tau \le N\}} G_\tau \le \mathbb{I}_{\{\tau \le N\}} \max\{G_\tau, 0\} \le \mathbb{I}_{\{\tau \le N\}} \max\{G_{\tau\wedge N}, 0\} + \mathbb{I}_{\{\tau = \infty\}} \max\{G_{\tau \wedge N}, 0\}$$

$$= \max\{G_{\tau\wedge N}, 0\}.$$

$$\le Y_{\tau \wedge N}.$$

$$\tau \in S_n.$$

$$\therefore \mathbb{I}_{\{\tau \le N\}} G_\tau \le Y_{\tau\wedge N} \ \forall \tau.$$

$$\tilde{\mathbb{E}}_n\left[\mathbb{I}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau}} G_\tau\right].$$

$$\le \tilde{\mathbb{E}}_n\left[\frac{1}{(1+r)^{\tau\wedge N}} Y_{\tau\wedge N}\right]$$

$$\le \frac{1}{(1+r)^{\tau\wedge n}} Y_{\tau \wedge n} \quad (\text{from } ②).$$

$$= \frac{1}{(1+r)^n} Y_n \quad (\text{Because } \tau \in S_n \Rightarrow \tau \geq n).$$

$$\Rightarrow \tilde{\mathbb{E}}\left[\mathbb{1}_{\{\tau \leq N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right] \leq Y_n \quad \forall\, \tau.$$

$$\therefore \boxed{V_n \leq Y_n}$$

* 1 Lec Miss

Stopping Time

Definition - ~~A~~ random variable τ taking values in $[0,\infty]$ is called a stopping time if
$\{\tau \le t\} \in \mathcal{F}(t)\ \forall t$.

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$$

Define $\tau_m = \min\{t \ge 0 : S(t) = m\}$.

If $S(t)$ never reaches level m, then we interpret $\tau_m = \infty$.

Ex:- $X(t)$ is a martingale (sub/super) then $X(t \wedge \tau_m)$ is also a martingale (sub/super).

$$X(t \wedge \tau_m) = \begin{cases} X(t) & , t \le \tau_m \\ X(\tau_m) & , t > \tau_m. \end{cases}$$

Perpetual put -

The underlying assest price is given by
$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$$
where $\tilde{W}(t)$ is a BM under risk-neutral $\tilde{\mathbb{P}}$.
$$S(t) = S(0)\, e^{\sigma \tilde{W}(t) + (r - \frac{\sigma^2}{2})t}$$

The american perpetual put pay (intrinsic value) (NO T).

$K - S(t)$ if it is exercised at time t.

$\underline{\text{Def}^n}$ - Let $\mathcal{S}$ be the set of all stopping time

$$v_*(x) = \max_{\tau \in \mathcal{S}} \mathbb{E}\left[(K - S(\tau))\, e^{-r\tau}\right], \quad S(0) = x.$$

If $\tau = \infty$, we interpret $e^{-r\tau}$ as 0.

<u>Lemma</u> - Let $\tilde{W}(t)$ be a BM under $\tilde{\mathbb{P}}$. Let μ be a real number and let m be a positive number.

Set ~~the~~ $X(t) = \mu t + \tilde{W}(t)$.

$$\tau_m = \min\{t \geq 0 : X(t) = m\}.$$

If $X(t)$ never reaches m, then $\tau_m = \infty$.

$$\tilde{\mathbb{E}}\left[e^{-\lambda \tau_m}\right] = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})} \quad \forall \lambda > 0.$$

$$e^{-\lambda \tau_m} = 0, \text{ if } \tau_m = \infty.$$

Set a level L

If $S(0) = x < L$,

$$v_L(x) = (K - x).$$

If $S(0) = x > L$,

$$\tau_L = \min\{t \geq 0 : S(t) = L\}.$$

$$v_L(x) = \tilde{\mathbb{E}}\left[(K - S(\tau_L))\, e^{-r\tau_L}\right].$$

$$= (K - L)\, \tilde{\mathbb{E}}\left[e^{-r\tau_L}\right].$$

$$S(t) = x e^{\{\sigma \tilde{W}(t) + (r - \frac{1}{2}\sigma^2) t\}}.$$

$$S(\tau_L) = L.$$

$$\Leftrightarrow x e^{\sigma \tilde{W}(\tau_L) + (r - \frac{1}{2}\sigma^2)\tau_L} = L.$$

$$\Leftrightarrow -\sigma \tilde{W}(\tau_L) - (r - \frac{1}{2}\sigma^2)\tau_L = \ln(x/L).$$

$\Leftrightarrow -\tilde{W}(\tau_L) - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)\tau_L = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right).$

Here, $\mu = -\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$, $\lambda = r$, $m = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right)$.

$$\mu^2 + 2\lambda = \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 + 2r$$

$$= \frac{1}{\sigma^2}\left(r + \frac{1}{2}\sigma^2\right)^2.$$

$$\Rightarrow -\mu + \sqrt{\mu^2 + 2\lambda} = \cancel{\frac{1}{\sigma}\left(r + \frac{1}{2}\sigma^2\right)} \cdot \frac{2r}{\sigma}.$$

$$\tilde{\mathbb{E}}\left[e^{-r\tau_L}\right] = e^{-\frac{1}{\sigma}\ln\left(\frac{x}{L}\right)\cdot\frac{2r}{\sigma}} = \left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}.$$

$$\therefore v_L(x) = \begin{cases} (K-L)\left(\frac{x}{L}\right)^{-\frac{2r}{\sigma^2}}, & x > L \\ (K-x), & x \le L \end{cases}$$

Using $\frac{\partial v_L(x)}{\partial L} = 0$ (for maximum),

$$g(L) := (K-L)\,L^{2r/\sigma^2}.$$

$$g'(L) = -L^{2r/\sigma^2} + \frac{2r}{\sigma^2}(K-L)\,L^{\frac{2r}{\sigma^2}-1} = 0.$$

$$\Rightarrow \cancel{L^{2r/\sigma^2}} = \frac{2r}{\sigma^2}\,\frac{(K-L)\,\cancel{L^{\frac{2r}{\sigma^2}}}}{L}$$

$$\Rightarrow L = \frac{2r}{\sigma^2}(K-L)$$

$$\Rightarrow L^* = \frac{\frac{2r}{\sigma^2}K}{1 + \frac{2r}{\sigma^2}} = \frac{2rK}{\sigma^2 + 2r}.$$

$$\therefore v_{L^*}(x) = \begin{cases} (K-x), & \text{if } x \le L^* \\ (K-L^*)\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}, & x > L^* \end{cases}$$

$$v'_{L^*}(x) = \begin{cases} -1, & x \le L^* \\ -(K-L^*)\frac{2r}{x\sigma^2}\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}, & x \geq L^* \end{cases}$$

$v'_{L^*}(x)$ is cts at $x = L^*$.

$$v''_{L^*}(x) = \begin{cases} 0, & x < L^* \\ (K-L^*)\frac{2r(2r+\sigma^2)}{\sigma^4 x^2}\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}, & x > L^* \end{cases}$$

For $x > L^*$,

$$r v_{L^*}(x) - r x v'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L^*}(x)$$

$$= \left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}}(K-L^*)\left[r + \frac{2r^2}{\sigma^2} - \frac{r(2r+\sigma^2)}{\sigma^2}\right] = 0$$

If $0 \le x < L^*$,

$$r v_{L^*}(x) - r x v'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_{L^*}(x)$$

$$= r(K-x) + rx = rK.$$

$v_{L^*}(x)$ satisfies the linear complementarity conditions:

$$v(x) \geq (K-x)^+ \quad \forall x \geq 0 \quad \text{—— (1)}$$

$$r v(x) - r x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0 \;\forall x \quad \text{—— (2)}$$

and $\forall x \geq 0$, equality holds in ~~either~~ exactly one of (1) or (2) —— (3)

Theorem:- $\tau_{L^*} = \min\{t \geq 0 : S(t) = L^*\}$.

Then $e^{-rt} v_{L^*}(S(t))$ is a supermartingale and

$e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale.

Proof:- $d\left(e^{-rt} v_{L^*}(S(t))\right)$

$$= -re^{-rt} v_{L^*}(S(t))\,dt + e^{-rt}\,d\,v_{L^*}(S(t))$$

$$= -re^{-rt} v_{L^*}(S(t))\,dt + e^{-rt}\left[v_{L^*}'(S(t))\,dS(t) + \frac{1}{2}v_{L^*}''(S(t))\,dS(t)\,dS(t)\right]$$

$$\cancel{= -re^{-rt} v_{L^*}(S(t))\,dt +}$$

$$= e^{-rt}\left[-r\,v_{L^*}(S(t)) + r\,S(t)\,v_{L^*}'(S(t)) + \frac{1}{2}\sigma^2 S^2(t)\,v_{L^*}''(S(t))\right]dt + e^{-rt}\,v_{L^*}'(S(t))\,\sigma\,S(t)\,d\tilde{W}(t).$$

$$= e^{-rt}\,rK\,\mathbb{1}_{\{S(t)<L^*\}}\,dt + e^{-rt}\sigma S(t)\,d\tilde{W}(t)\,v_{L^*}'(S(t)).$$

$\Rightarrow e^{-rt} v_{L^*}(S(t))$ is a supermartingale.

$$\int_0^{u\wedge\tau_{L^*}} d\left(e^{-rt} v_{L^*}(S(t))\right) = -\int_0^{u\wedge\tau_{L^*}} e^{-rt}\,rK\,\mathbb{1}_{\{S(t)<L^*\}}\,dt + \int_0^{u\wedge\tau_{L^*}} e^{-rt}\sigma\,S(t)\,v_{L^*}'(S(t))\,d\tilde{W}(t)$$

$$\Rightarrow e^{-r(u\wedge\tau_{L^*})} v_{L^*}(S(u\wedge\tau_{L^*})) - v_{L^*}(S(0)) = \int_0^{u\wedge\tau_{L^*}} e^{-rt}\sigma\,S(t)\,v_{L^*}'(S(t))\,d\tilde{W}(t).$$

$\therefore e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale.

Corollary –

$$v_{L^*}(x) = \max_{\tau \in S} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right]; \quad S(0) = x.$$

Proof :– $e^{-rt} v_{L^*}(S(t))$ is a supermartingale:

$$v_{L^*}(x) = v_{L^*}(S(0)) \geq \tilde{\mathbb{E}}\left[e^{-rt} v_{L^*}(S(t))\right].$$

Let $\tau \in S$, then $e^{-r(t\wedge\tau)} v_{L^*}(S(t\wedge\tau))$ is also supermartingale

$$v_{L^*}(x) \geq \tilde{\mathbb{E}}\left[e^{-r(t\wedge\tau)} v_{L^*}(S(t\wedge\tau))\right].$$

$t \to \infty$, by DCT (v_{L^*} is bdd).

$$v_{L^*}(x) \geq \tilde{\mathbb{E}}\left[e^{-r\tau} v_{L^*} S(\tau)\right]$$

$$\geq \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right] \quad (\text{by } 1).$$

$$v_{L^*}(x) \geq \max_{\tau \in S} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

Now, $e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))$ is a martingale.

$$\therefore v_{L^*}(x) = \tilde{\mathbb{E}}\left[e^{-r(t\wedge\tau_{L^*})} v_{L^*}(S(t\wedge\tau_{L^*}))\right].$$

$t \to \infty$, by DCT,

$$v_{L^*}(x) = \tilde{\mathbb{E}}\left[e^{-r\tau_{L^*}} v_{L^*}(S(\tau_{L^*}))\right].$$

$$= \tilde{\mathbb{E}}\left[e^{-r\tau_{L^*}}(K - S(\tau_{L^*}))\right] \quad (\text{by } 1).$$

$$\therefore v_{L^*}(x) \leq \max_{\tau \in S} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

$$\therefore v_{L^*}(x) = \max_{\tau \in S} \tilde{\mathbb{E}}\left[e^{-r\tau}(K - S(\tau))\right].$$

Linear Complementarity conditions –

$$v(x) \geq (K-x)^+ \quad \forall\, x \quad \text{—— (a)}$$

$$r v(x) - r x v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0 \quad \forall\, x \geq 0 \quad \text{— (b)}$$

and $\forall\, x > 0$, equality holds in (a) or (b) — (c)

$$d\left(e^{-rt} v_{L^*}(S(t))\right) = -e^{-rt} rK \mathbb{I}_{\{S(t) \leq L^*\}} dt + e^{-rt} \sigma S(t)\, v'_{L^*}(S(t))\, d\tilde{W}(t).$$

Corollary – Consider an agent with initial capital $X(0) = v_{L^*}(S(0))$. Suppose agent uses the portfolio $\Delta(t) = v'_{L^*}(S(t))$ and consumed cash at the rate $C(t) = rK\mathbb{I}_{\{S(t)<L^*\}}$.

Then, $X(t) = v_{L^*}(S(t))$, $\forall\, t$ and $X(t) \geq (K-x)^+$.

Proof – $dX(t) = \Delta(t)\, dS(t) + r(X(t) - \Delta(t)S(t))dt - C(t)dt.$

$$d(e^{-rt}X(t)) = -re^{-rt}X(t)dt + e^{-rt}dX(t).$$

$$= e^{-rt}\left[\Delta(t)dS(t) - r\Delta(t)S(t)dt - C(t)dt\right].$$

$$= e^{-rt}\left[\sigma S(t)\Delta(t)d\tilde{W}(t) - C(t)dt\right].$$

$$= \sigma e^{-rt} v'_{L^*}(S(t))S(t)d\tilde{W}(t) - rK\mathbb{I}_{\{S(t)<L^*\}}dt.$$

$$= d\left(e^{-rt} v_{L^*}(S(t))\right).$$

$$\therefore \cancel{e^{-rt}} X(t) = \cancel{e^{-rt}} v_{L^*}(S(t)) \Rightarrow \boxed{X(t) = v_{L^*}(S(t))}.$$

Stopping set $\Gamma = \{x \geq 0 : v_{L^*}(x) = (K-x)^+\}$.

Continuation set $\mathcal{C} = \{x \geq 0 : v_{L^*}(x) > (K-x)^+\}$.

Let $V(t)$ be the value process of the American perpetual put ~~process~~ option, then

① $V(t) \geq (K - S(t))^+ \quad \forall\, t \geq 0$

② $e^{-rt} V(t)$ is a supermartingale

③ $\exists$ a stopping time τ^* such that

$$V(0) = \tilde{\mathbb{E}}\left[e^{-r\tau^*}(K - S(\tau^*))^+\right].$$

American put :- $[0, T]$

Underlying asset - $dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$

Defn :- Let $0 \leq t \leq T$ and $x \geq 0$ be given, $S(t) = x$.

$\mathcal{F}_u(t) = \sigma\{S(v) : t \leq v \leq u\}$.

$\mathcal{S}_{t,T} \Rightarrow$ set of all stopping times for $\mathcal{F}_u^{(t)}$ $t \leq u \leq T$, taking values in $[t, T]$ or ∞.

$$v_{t,x} = \max_{\tau \in \mathcal{S}_{t,T}} \tilde{\mathbb{E}}\left[e^{-r(\tau - t)}(K - S(\tau)) \mid S(t) = x\right].$$

$$\tau = \infty \Rightarrow e^{-r(\tau - t)}(K - S(\tau)) = 0.$$

Linear complementarity conditions —

$$v(t,x) \geq (K-x)^+ \quad \forall\, x \geq 0,\ t \in [0,T] \quad \text{— (a)}$$

$$rv(t,x) - v_t(t,x) - rx\, v_x(t,x) - \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x) \geq 0$$
$$\forall\, x \geq 0,\ t \in [0,T] \quad \text{— (b)}$$

$\forall\, t \in [0,T],\ x \geq 0$ equality holds either in (a) or (b).

$$\underset{\sim}{S} = \{(t,x) : v(t,x) = (K-x)^+\}$$

$$\mathcal{C} = \{(t,x) : v(t,x) > (K-x)^+\}.$$

$$\tau^* = \min\{u \in [t,T] : v(u, S(u)) = (K - S(u))^+\}.$$

Theorem :- $e^{-rt} v(t, S(t))$ is a supermartingale $t \leq u \leq T$. and the stopped process $e^{-r(u \wedge \tau^*)} v(u \wedge \tau^*, S(u \wedge \tau^*))$ is a martingale.

Corollary - Consider an agent with initial capital $X(0) = v(0, S(0))$. Suppose this agent uses the portfolio process: $\Delta(u) = v_x(u, S(u))$ and consume at rate $C(u) = rK\, \mathbb{I}_{\{S(u) < L(T-u)\}}$. Then $X(u) = v(u, S(u))\ \forall\, u \geq 0$ and $X(u) \geq (K - S(u))^+$.

1 Lec Miss — Thomas Björk

Continuous time Portfolio theory

S_t^i = price of 1 unit of i^{th} asset at time t

h_t^i = No. of units of i^{th} asset at t.

h_t = portfolio process $(h_t^1, h_t^2, \ldots, h_t^N)$.

V_t^h = value of portfolio h_t at t.

$$V_t^h = h_t S_t.$$

$$dV_t^h = h_t dS_t = \sum_{i=1}^{N} h_t^i dS_t^i$$

Defn:- For given portfolio h, corresponding relative portfolio or portfolio weights w is given by

$$W_t^i = \frac{h_t^i S_t^i}{V_t^h}, \quad i = 1, 2, \ldots, N$$

$$\sum_{i=1}^{N} W_t^i = 1.$$

Lemma:- A portfolio is self financing if

$$dV_t^h = \sum_{i=1}^{N} h_t^i dS_t^i$$

$$= V_t^h \sum \frac{w_t^i dS_t^i}{S_t^i}$$

Def^n:- Consider a financial market with vector price process. A contingent claim with date of maturity T also called T-claim is any RV $\mathcal{X} \in \mathcal{F}_t^S$.

A contingent claim is called simple if $\mathcal{X} = \phi(S(T))$.

The function ϕ is called the contract f^n.

Eg:- European call $\rightarrow \max\{x - K, 0\} = \phi(x)$.

Notation:- $\Pi(t, \mathcal{X})$ for price process of the contingent claim $\mathcal{X}$.

$$\Pi(T, \mathcal{X}) = \mathcal{X} = \phi(S(T)) \text{ (for simple claim)}$$

Def^n:- An arbitrage ~~process~~ possibility on a financial market is an self-financing portfolio $h \ni$

$$V^h(0) = 0, \; P(V^h(T) \geq 0) = 1, \; P(V^h(T) > 0) > 0.$$

Property 1:- Suppose $\exists$ a self-financing portfolio $h \ni$

$$dV^h(t) = K(t) V^h(t)\, dt$$

where K is adapted. Then $K(t) = r(t) \; \forall t$.

$dB(t) = rB(t)dt$

$dS(t) = S(t)\mu(t, S(t))\, dt + S(t)\sigma(t, S(t))\, dW(t)$.

Take $\mathcal{X} = \phi(S(T))$

We assume that the price process is given by

$$\Pi(t, \mathcal{X}) = F(t, S(t)) :- \Pi(t).$$

$$d\Pi(t) = dF(t, S(t)) = F_t(t, S(t))\,dt + F_x(t, S(t))\,dS(t) + \frac{1}{2}F_{xx}(t, S(t))\,dS(t)\,dS(t)$$

$$= \left(F_t + \mu S F_x + \frac{1}{2}S^2\sigma^2 F_{xx}\right)^{(t,S(t))} dt + S\sigma F_x^{(t,S(t))}\,dW(t).$$

$$d\Pi(t) = \mu_\Pi(t)\Pi(t)\,dt + \sigma_\Pi(t)\Pi(t)\,dW(t),$$

where $\mu_\Pi(t) = \dfrac{F_t + \mu S F_x + \frac{1}{2}S^2\sigma^2 F_{xx}}{F}$

and $\sigma_\Pi(t) = \dfrac{S\sigma F_x}{F}$

Consider a portfolio $(\underset{\uparrow \text{ asset}}{u_S}, \underset{\uparrow \text{ claim}}{u_\Pi})$

$$dV(t) = V(t)\left[\frac{u_S\,dS(t)}{S(t)} + \frac{u_\Pi\,d\Pi(t)}{\Pi(t)}\right]$$

$$= V(t)\left[u_S\mu + u_\Pi\mu_\Pi\right]dt + V(t)\left[u_S\sigma + u_\Pi\sigma_\Pi\right]dW(t)$$

$$u_S + u_\Pi = 1.$$

Let $\begin{cases} u_S\sigma + u_\Pi\sigma_\Pi = 0 \\ u_S + u_\Pi = 1 \end{cases} \Rightarrow u_S = \dfrac{\sigma_\Pi}{\sigma_\Pi - \sigma},\ u_\Pi = \dfrac{-\sigma}{\sigma_\Pi - \sigma}$

$$dV(t) = V(t)\left[u_S\mu + u_\Pi\mu_\Pi\right]dt$$

$$u_S\mu + u_\Pi\mu_\Pi = r.$$

$$u_S = \frac{\sigma_\Pi}{\sigma_\Pi - \sigma} = \frac{\cancel{\sigma} S F_x/F}{\cancel{\sigma} S F_x/F - \cancel{\sigma}} = \frac{S F_x}{S F_x - F}$$

$$u_\Pi = \frac{-\sigma}{\sigma_\Pi - \sigma} = \frac{-F}{S F_x - F}$$

$$\frac{SF_x}{SF_x - F}\mu + \frac{-F}{SF_x - F}\left[\frac{F_t + \mu S F_x + \frac{1}{2}\sigma^2 S^2 F_{xx}}{F}\right] = r$$

$$\Rightarrow \boxed{F_t + rSF_x + \frac{1}{2}\sigma^2 S^2 F_{xx} = rF}$$

$$\Pi(T, \Phi) = \Phi(S(T))$$

$$F(T, x) = \Phi(x)$$

Meta-theorem

Let M denote the number of underlying traded assets in the model (excluding the risk-free asset) and let R denote the number of random sources. Then,

1. The model is arbitrage free iff $M \leq R$
2. The model is complete iff $M \geq R$
3. The model is complete & arbitrage free iff $M = R$.