Enumerative Combinatorics

$$\frac{a_n}{n+1} = a_n - n + 1$$

$$= n\,(a_{n-1} - n + 2) - n + 1$$

$$= n\,((n-1)(a_{n-2} - n + 3)) - n + 1)$$

$$(a_n = n! + n)$$

·) Multiplication of 2 exp. GF

$A(x)\cdot B(x) = C(x)$

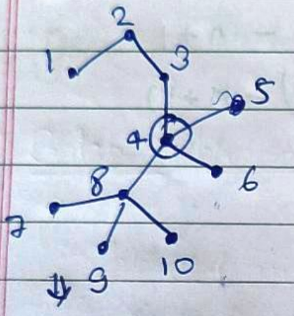
$$n! \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} = \boxed{\sum_k \binom{n}{k} a_k b_{n-k}}$$

$$\frac{a_n}{n!} \quad \frac{b_n}{n!} \qquad \frac{c_n}{n!} = \sum_{k=0}^{n} \frac{a_k}{k!} \frac{b_{n-k}}{n-k!}$$

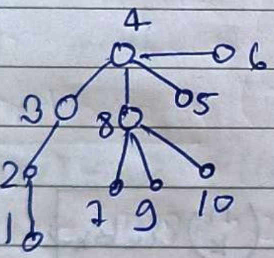
$$c_n = \sum \binom{n}{k} a_k\, b_{n-k}$$

Rooted Tree

Date- 25/10/22



degree = 1 (1, 5, 6, 10, 9, 7)
degree = 2 (2)



Height
~~depth~~ (tree) = Longest distance b/w root node and last leaf

~~here, depth =~~

depth (node) = b/w root and node

Ordered tree

→ put the children of each node in specific order.

K-ary tree → no. of children are b/w 0 and k

if k = 2 → Binary tree
full Binary tree → 0 or 2 children.

$$\frac{1(2^{\log_2 n}-1)}{n-1}$$

$$\frac{1\cdot(2^{\log_2 n}-1)}{n-1} = (n-1)$$

Lemma :- in full Binary tree with 'n' leaves

$\Rightarrow$ $(n-1)$ internal nodes

$\Rightarrow$ #nodes $= 2n-1$

$\Rightarrow$ #edges $= 2n-2$

Def :> A Dyck Word (deek)

is a string of n 1's and n 2's,

such that $\longrightarrow$ (length $= 2n$)

1's $\geq$ # 2's for any prefix

e.g. $\hookrightarrow$ "11212 11222"

for $n=2$

$\hookrightarrow$ total possibility $\binom{4}{2} = 6$

1122	1212	1221	2112	2121	2211
✓	✓	x	x	x	x

If W_n = Set of all Dyck words, with n 1's and n 2's,

claim $C_n = |W_n|$ (cathlon no.)

$$C_n = \frac{1}{(n+1)}\binom{2n}{n} = \frac{2n!}{(n!)^2\,(n+1)!}$$

w_0	w_1	w_2	w_3		
ϕ	12	1122	111222	121212	112122
		1212	112212	121122	

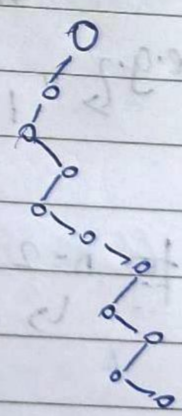
* Balanced paranthesis (Bijection with C_n)

↳ replace 1 with left and 2 with right
= n pair of balanced paranthesis

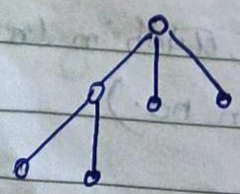
* Order trees (Bijection with C_n) (through Dyck word)

$w = \overbrace{112122}\,1212$

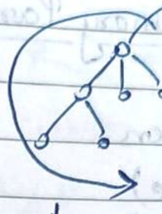
Hint :→ 1 extra node.



- Draw a root
- Read from L-R
- if 1, add a new node at rightmost side (and move to it)
- if 2 go to the parent of the current child)



- Traverse in ACW way
- Add '1' as go down
- Add '2' as go up



start from root, end should ~~be~~ here

1 1 2 1 2 2 1 2 1 2

→ $W_n \longleftrightarrow$ ordered tree with n edges.

↓

Counts C_n

→ for $n \geq 0$ any Dyck word can be written as

$u = 1x2y$ (uniquely)

$u \in W_n \ (n \geq 0)$ $\quad x \in W_i$ and $y \in W_{n-1-i}$

$i = 0, 1, 2, \dots n-1$

$$C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1} \quad (n > 0)$$

$C_0 = 1$

$$\boxed{C_n = \sum_{i=0}^{n-1} C_i C_{n-i-1}}$$

Enumeration Combinatorics

Date
Page

Complete Binary Parameterisation

↳ n letters

(a(bc))(de) ✓ (abc)(de) ✗

n letters

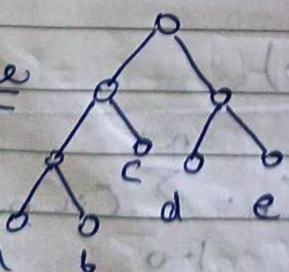
$n=1$	$n=2$	$n=3$	$n=4$	
a	ab	a(bc)	((ab)c)d	a(b(cd))
1	1	(ab)c	(a(bc))d	(ab)(cd)
		2	a((bc)d)	
			5	

b_n = # complete binary parem' of 'n' letters

$b_1 = 1$

$$b_n = \sum_{i=1}^{n-1} b_i b_{n-i} \qquad \boxed{b_n = C_{n-1}}$$

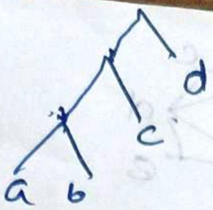
ordered full Binary tree



$\Rightarrow$ ((ab) c)(de)

↕

B) Dyck words



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Balanced parenthes

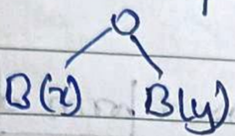
↓

Triangulation of (n+2) gon ↔ Ordered trees → $\boxed{\text{Dyck words}}$

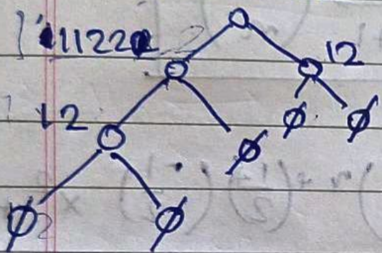
Complete binary parenthesization ↔ Ordered full binary tree

$B(\phi) = 0$

$B(x2y) =$



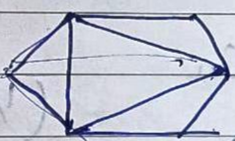
111222 12



※ Triangulating regular polygons.

↓

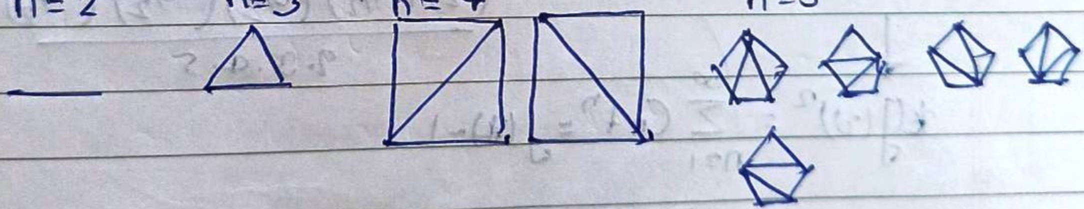
Non-crossing Diagonals



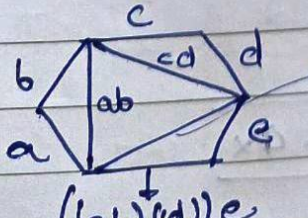
(n-gon)

↳ <u>(n-3) diagonals</u>

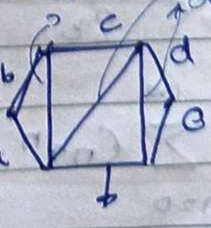
n=2 n=3 n=4 n=5



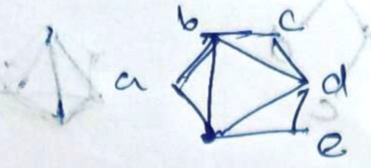
we'll label n-1 sides :]



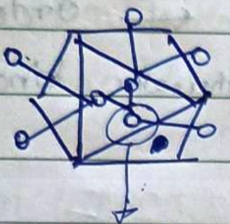
((ab)(cd))e



((ab)c)(de)



~~Composition~~ Triangulating with ordered full Binary tree :-)



Root

Closed form of C_n

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$

$$(1+x)^{1/2} = 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2 + \frac{(1/2)(-1/2)(-3/2)}{2\cdot 3} \qquad \frac{(1/2)(-1/2)(-3/2)(-5/2)(-7/2)\cdots}{2\cdot 3\cdot 4\cdot 5}$$

$$f(x) = \sum_{n=0}^{\infty} x^n C_n$$

$$(f(x))^2 = \sum_{n=0}^{\infty} x^n \left(\sum_{i=0}^{n-1} C_i C_{n-1-i}\right)$$

$$x(f(x))^2 = \sum_{n=1}^{\infty} C_n x^n = f(x) - 1$$

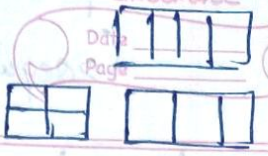
$$f(x) = \left(\frac{1 \pm \sqrt{1-4x}}{2x}\right)$$

i.e.

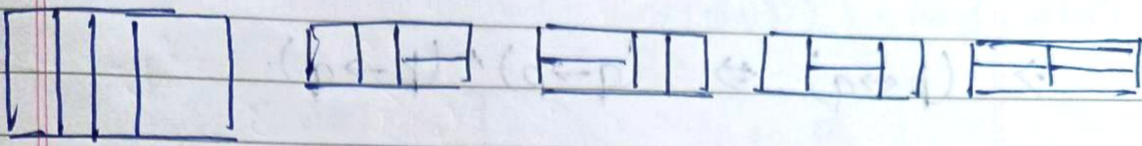
$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 \pm \sqrt{1-4x}}{2x}$$

tile $2\times n$ rectangle with 2×1 domino,

$t_n = \#$ ways.



$n=4$



$\cancel{f(n) = 2f(n-1) +}$

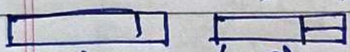
$$t_n = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \dots$$

tiling of 2n rectangle $\longleftrightarrow$ no. of ways of writing n as no of 1 and 2's. (as a composition)

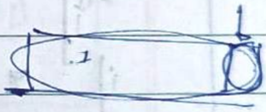
$t_n = \sum \#$ composition of k 2's and $(n-2k)$ 1's $= \sum \binom{n-k}{k}$

$\boxed{t_n = t_{n-1} + t_{n-2}}$ → Recurrence

→ fibonacci number



(n-1) (n-2)



$\cancel{A(x)} = t_{n+2} = t_{n+1} + t_n$

$= A(x) - \cancel{2x^2}\, 1x = (A(x) - \cancel{x})x + x^2 A(x)$

$= A(x)[1 - x - x^2] = \cancel{2x^2} + x =$

$\Rightarrow A(x) = \dfrac{\cancel{x(2x+1)}\; x^2 + x}{1 - x - x^2}$

$= \dfrac{x(2x+1)}{1-(x+x^2)} = [(x^2+x)^2 + \dots$

$= x(2x+1)\,\cancel{[(x+x^2) + (x+x^2)^2 + \dots]}$

$= (2x^2+x)[\dots] =$ n-2, n-1

$\vdash$ ↳ sequent

$\models$ ↳ semantic entailment

Q) $(p \wedge q) \vee r,\ r \rightarrow s \vdash p \vee s$

:> $(p \leftrightarrow q) \Leftrightarrow (q \rightarrow p) \wedge (p \rightarrow q)$

\# Exhaustive proof → can be proved ~~testing~~ by examining a relatively small no. of examples.

Soundness → propositional → Completeness

↳ if $\vdash$ then $\models$

if $\not\models$ then $\nvdash$

↳ if $\models$ then $\vdash$

if $\Gamma \models \phi \mapsto \psi$ then $\Gamma, \phi \models \psi$

→ proposition logic is consistent i.e. $\nvdash \bot$

formula ϕ, valuation v, semantics → $\boxed{[[\phi]]_v = \{T, F\}}$ ↳ truth table

$\neg(\phi \wedge \psi) \vdash \neg\phi \vee \neg\psi$

1\. P

$\neg(\neg\phi \vee \neg\psi)$

ϕ

ψ

$\perp$

$\neg\psi$

$(\neg\phi \vee \neg\psi)$

$\perp$

$\neg\phi$

$\cancel{\perp}$

$\cancel{\neg\psi}\ (\neg\phi \vee \neg\psi)$

$\cancel{\neg\phi} \cdot \perp$

$\neg\phi \vee \neg\psi$

$\neg(\phi \vee \psi) \vdash (\neg\phi \wedge \neg\psi)$

1\. P

$\neg(\neg\phi \wedge \neg\psi)$

ϕ

$\perp$

$\neg\phi$

ψ

$\perp$

$\neg\psi$

$\neg\phi \wedge \neg\psi$

$\perp$

$(\neg\phi \wedge \neg\psi)$

$\neg\forall x\phi \vdash \exists x\neg\phi$

$\neg\exists x\phi \vdash \forall x\neg\phi$

$\boxed{Q \to \mathbb{N}}$ → Bijection property

$\boxed{\text{denumerable} \to \text{countably infinite}}$

$f: A \to B \quad g: \to B \to C \quad g \circ f = g(f(a))$

a) $g \circ f \to$ injective $\Rightarrow f$ is injective $\quad g \circ f(a) = g \circ f(b) \Rightarrow a = b$

↳ let $a \neq b$

$f(a) = f(b)$

$g \circ f(a) = g \circ f(b)$

$\Rightarrow a = b$ ✗ through contradiction

b) $g \circ f \to$ surjective $\Rightarrow g$ is surjective

$g \circ f \ (A \to C)$

$\forall c \in C \ \exists \ a \in A$

$g(f(a)) = x \ \forall x \in C$

$g(b) = x \ \forall x \in C$

i.e. $\forall x \in C \ \exists \ b \in \underline{B}$

"Union of countably many countable sets is countable" → "Cartesian product of finitely many countable sets is countable"

↳ e.g. $S_i \to$ countable set → $\{S_1, S_2, S_3 \ldots$

& $\{S_n\}, n \in \mathbb{N}$

$S_{mn} \to m^{th}$ element of S_n

can be written as

$S_{11} \quad S_{12} \quad S_{13} \ldots$

$S_{21} \quad S_{22} \quad S_{23} \ldots$

$S_{31} \quad S_{32} \quad S_{33} \ldots$

→ a mapping can be done from $\mathbb{N}$ onto diagonal lines

now $S_{11} = 1/1 \quad S_{12} = 1/2 \ldots$ similar way $S_1 \cup S_2 \ldots S_n = \mathbb{Q}^+$

thus $\underline{\mathbb{Q}^+ \text{ is countable}}$

⇓

$\mathbb{Q}$ is countable

set $A \to n$ elements

↳ symmetric relations $\to 2^n \cdot 2^{\frac{n(n-1)}{2}} = \left(2^{\frac{n(n+1)}{2}}\right)$

↳ antisymmetric relation $\to \left(2^n \cdot 3^{\frac{n(n-1)}{2}}\right)$

↳ asymmetric relation $\to \left(3^{\frac{n(n-1)}{2}}\right)$

↳ irreflexive $\longrightarrow 2^{n(n-1)}$

• $\left.\begin{array}{l} f: A \to B \\ g: B \to C \end{array}\right\}$ bijective $\Longrightarrow$ $g\circ f$ is also bijective
(P.T. onto & one-one)

• $f\circ g$ is injective $\to$ $g(f(x))$ is injective $\Rightarrow g$ is injective

~~Let~~ $a \neq b$

Let $a \neq b$ $\quad g(a) = g(b)$

$= g(f(x)) = g(f(y))$

$= f\circ g(x) = f\circ g(y)$

$\Rightarrow x = y$

$= f(x) = f(y)$

$= \boxed{a = b}$

~~$g\circ f$~~ $f\circ g$ is surjective

$g(f(x))$ is surjective

⇓

$f(x)$ is surjective

$f: A \to B \quad g: B \to C$ T.P.

$\forall b \in B \quad \exists a \in A$

$\forall c \in C \ \exists \ a \in A \quad g(f(a)) = c$

$R_1 \to$ congruent modulo 3

$\hookrightarrow \{(a,b) \mid a \equiv b \bmod 3$

Every Infinite uncountable ~~subset~~ has a countable subset:

$S - \{s_1, s_2, \ldots, s_n\}$ ✓ s_{n+1} can be extracted

Set S countable

$\hookrightarrow f: N \to S$ (bijective function) ✓

$A \longrightarrow B$

$(n) \quad (m)$

R → $\boxed{\text{uncountable}}$ → $(0,1)$ is uncountable

no. of functions $\longrightarrow (m^n)$

one-one $\longrightarrow ({}^mP_n)$

onto $\longrightarrow (\ )$

2^n → $\{\phi, \{a\}, \{b\}, \{c\}, \ldots \{S\}\}$

no. of relations in $A \times A$ set (n^2 elements)

(m elements $\longrightarrow 2^m$ subsets)

(n^2 elements $\to 2^{n^2}$ subsets or relation)

no. of reflexive relations $\to \boxed{2^{n(n-1)}}$ ✓

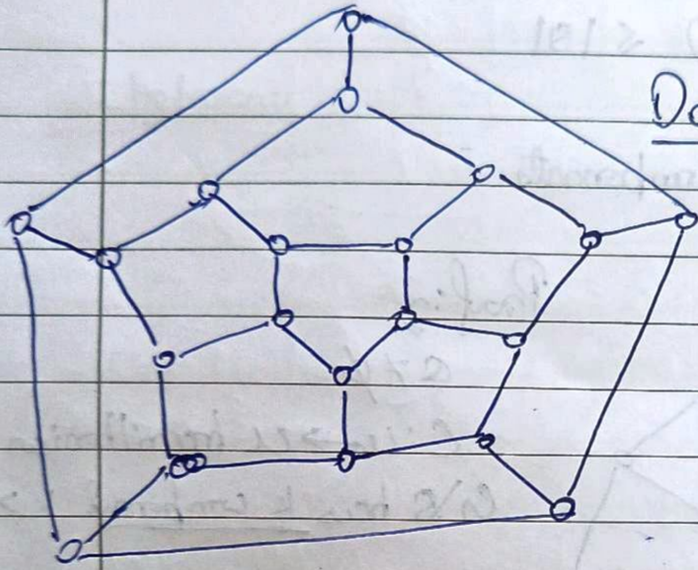
Transitive only when $R^n \subseteq R \ \forall n \in \mathbb{N}$

Irreflexive $\to \boxed{\forall a \in A \ ; (a,a) \notin R}$

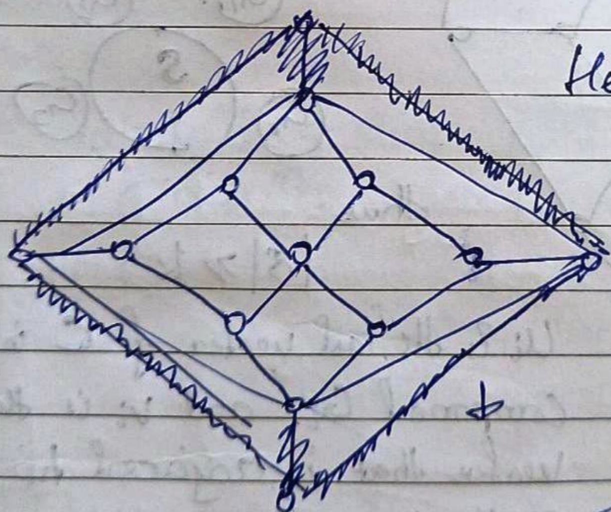
Asymmetric $\to$ ~~no~~ $\boxed{(b,a) \notin R \ \forall (a,b) \in R}$

_/_/_

Date - 07/11/22

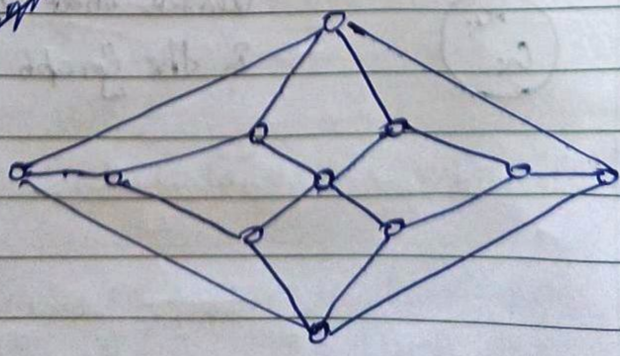


Dodecahedron
(Hamaltonian)



Herschel Graph
(non-Hamaltonia)

Bipartite graph with odd vertices

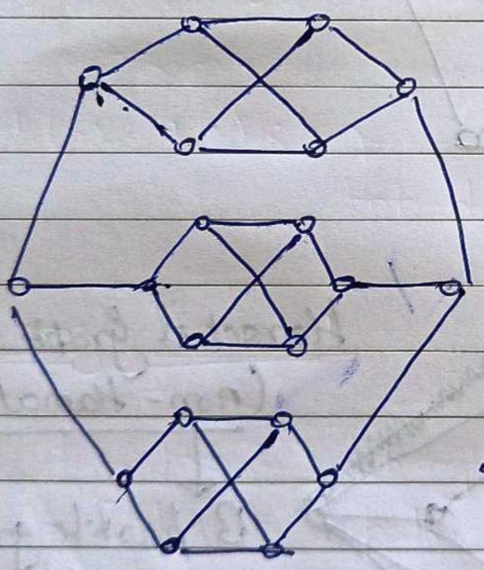


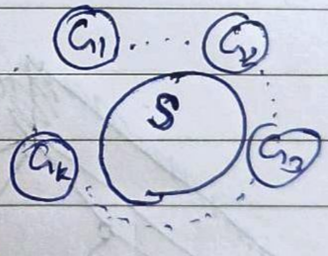
Use to prove Non-Hamaltonian graph.

If G is hamiltonian, then $\forall\ S \subseteq V_G$
$c(G \setminus S) \leq |S|$

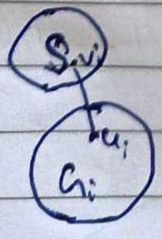
\# Connected Components in $G \setminus S$

Proof:
$S \neq \emptyset$
$C: u \to u$ hamiltonian cycle.
$G \setminus S$ has k component $k > 1$.





thus
$|S| \geq k$



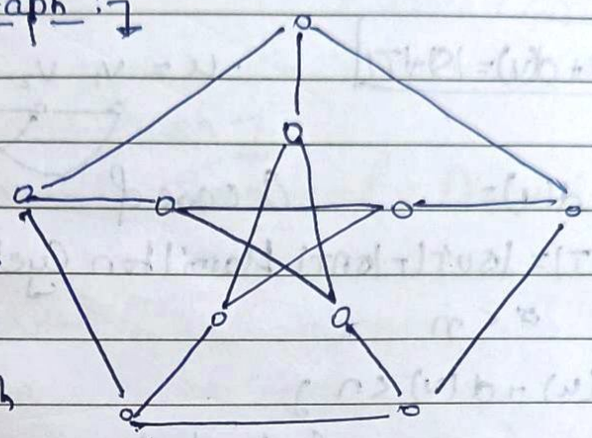
u_i is the last vertex of C in the component G_i and v_i is the vertex that is adjacent to u_i in the Graph G.

__/__/__

Maximal Path

Counter example for the Converse

peterson graph :-



Non-Hamiltonian Graph with the previous condition satisfied.

Dirac → Thm:- G is a simple graph with n vertices and $\delta \geq n/2$, then G is hamiltonian. ($n \geq 3$)

δ → min. degree

$G' \rightarrow$ maximal non-Hamiltonian satisfying → (not Complete)

$G' + uv \rightarrow$ hamiltonian

u and $v \ni uv \notin G'$

$\Rightarrow$ Each Hamilton Cycle in $G' + uv$ contains the edge uv.

$u = v_1, v_2, \ldots, v_n = v$ hamiltonian path

$S = \{v_i / uv_{i+1} \in E\}$ $T = \{v_i u \in E\}$

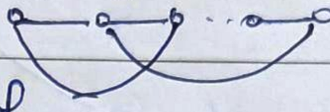
$v_n \notin S \cup T$

$|S \cup T| < n$ and $|S \cap T| = \cancel{\phi}\ 0$?

if $|S \cap T| \neq 0$

$\boxed{d(u) + d(v) = |S + T|}$

$u = v_1 \quad v_2 \quad \cdots \quad v_n = v$



$d(u) + d(v) =$ a case of

$|S| + |T| = |S \cup T| + |S \cap T|$ Hamilton Cycle (but that is not possible)

$< n$

$d(u) + d(v) < n$?

Contradiction as $(d(u) \geq n/2)$

<u>Strongly Connected</u>

A digraph G is strongly connected if $\exists$ a u to v path and v to u path. (vertices u and v)

<u>Balanced</u>

If the indegree of ~~vertex~~ v is same as out degree of v; $\forall\, v \in V_G$.

Directed Graph, Balanced, Strongly Connected

$\Downarrow$

~~Path~~ Eulerian Graph

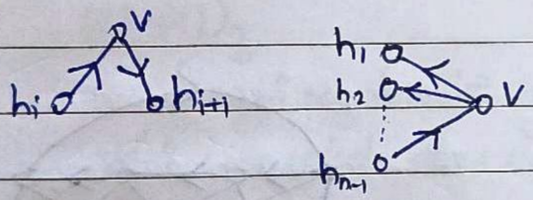
7 8 9 10 11*

→ D
px
→ N
/ D Y
→ D

Thm:→ ~~Grag~~
Graph G has closed Eulerian ~~train~~ trail $\Longleftrightarrow$ balanced and strongly ~~dish~~ connected.

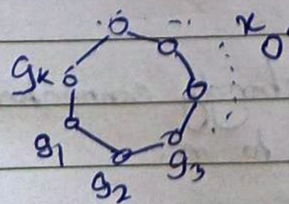
Complete Direct Graph T
* Tournament ~~Tornament~~ → Directed Complete Graph.

Thm:→ All tournaments have a hamilton path. (use MI on number of vertices)
→ T is tournament (n vertices)
$T' = T \setminus V$ (removing some vertex)
$h_1, h_2, \dots, h_{n-1}$ — hamilton path in T'



Tournament

Thm:— Tournament - Strongly Connected $\Longleftrightarrow$ Hamilton Cycle
$C = y_1, y_2 \dots y_k$ — maximal cycle length cycle in T
assume C is not Hamiltonian.

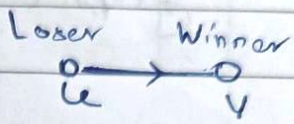


→ as it is not Hamiltonian (an edge exits to outside).

Date - 09/11/22

Tournament :

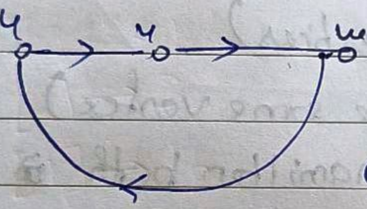
Round-Robin-Tournament



*(always have Hamilton Path)

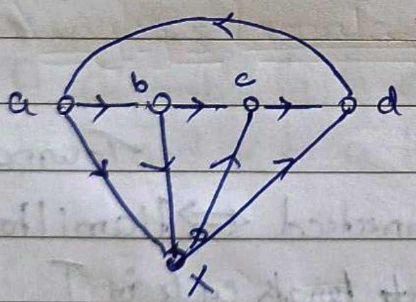
* Thm :→ Tournament is hamiltonian ⟺ Strongly Connected.

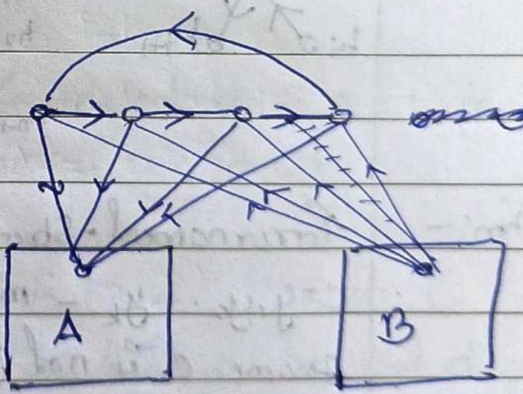
⇐ Tournament that is strongly connected can't be a transitive tournament



cycle of length 3

$C = \langle v_1, v_2, \ldots, v_k, v_1 \rangle$





Tournament of n vertices that is strongly connected will have cycles of all lengths 3 to n.

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walk ⇒ path

Trees :-
minimally connected graph
(one-edge removed → disconnected)

Thm:- G is a simple connected graph of n-vertices. then FAE
~~to~~ ⇒ G is minimally connected
⇒ G doesn't contain any cycle

P.T. (1) ⇒ (2) :-
G is simply connected, containing a cycle C
then removing an edge ab from C, the graph remains connected, let $x, y \in V$,

(2) ⇒ (1) :-
through negation :- even if we remove an edge, it is still connected. thus another path P ⇒ a cycle is present

Corr:→ Connected graph H is a tree iff for each pair of vertices (x,y) ∃ exactly one path joining x and y.
P Δ Q → union of cycles

lemma:→ T be a tree on n vertices ($n \geq 2$)
then T has at least 2 vertices of degree 1.
→ through maximal path*

Martin - Aigner

THE Book

Notes / ____

Thm:- no. of edges is N-1, in all trees of N vertices and all connected graphs of vertices n and n-1 edges is a tree.

* edges in a forest with k connected componenets = $\underline{\underline{n-k}}$

* tree with vertex set [n]

$[1] \to 1$

$[2] \to 1$

$[3] \to 3$

$[4] \to 16$

$\boxed{n^{n-2}}$ *

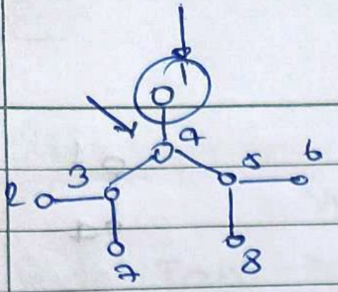
* Cayley's formula → +ve integer n

tree with vertex set [n]

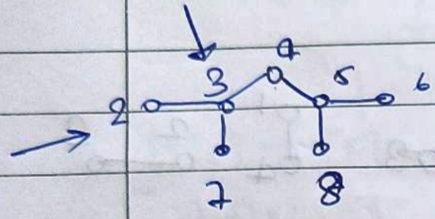
is $A_n = n^{n-2}$

$n^{n-2} \Rightarrow$ # seq. of length $(n-2)$ from $[n]$

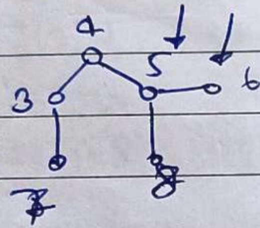
sequence of length 6



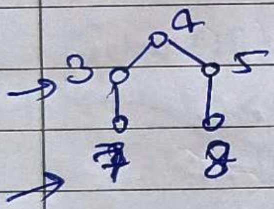
(4, . . (neighbour of the least no. with degree = 1 .)



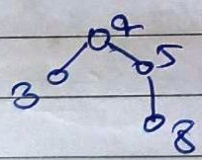
(4, 3, . . .



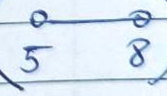
(4, 3, 5, . . .



(4, 3, 5, 7, . .



$\Rightarrow$ (4, 3, 5, 7, 4, 5) → final sequence

left with ()

Given Sequence

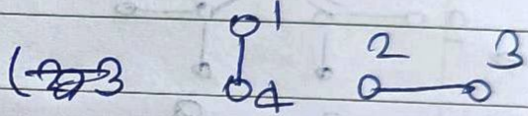
$t = (\cancel{4}, 3, 5, 3, 4, 5)$

not app. ⇒ $(\cancel{1}, 2, 7, 8)$

(order wise)

$t = (\cancel{4}, \cancel{3}, 5, 3, 4, 5)$

$N = (\cancel{2}, 7, 8)$



$t = (\cancel{4}, \cancel{3}, \cancel{5}, 3, 4, \cancel{5})$

$N = (\cancel{2}\cancel{3}, 7, 8)$

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Caylay's thm

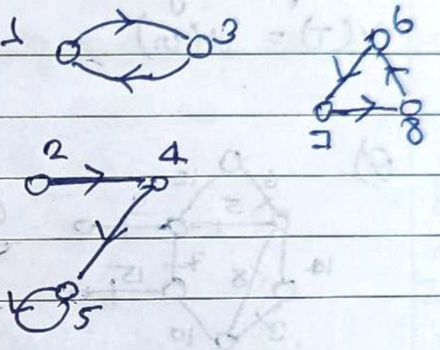
→ # trees with vertices from $[n] = n^{n-2}$

Andre Joyal → $n^n = n^2 A_n$ → Doubly rooted tree (head-tail)

function from $[n] \to [n]$

$f: [8] \to [8]$

$f(1)=3$ $f(2)=4$
$f(3)=1$ $f(4)=5$
$f(5)=5$ $f(6)=7$
$f(7)=8$ $f(8)=6$



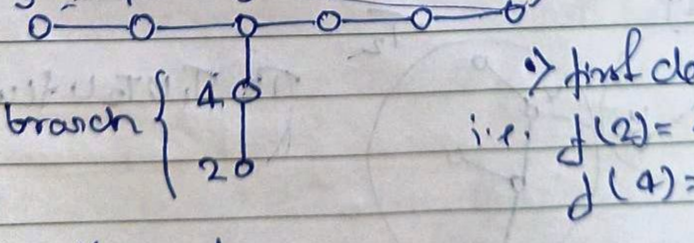
Cycles: (13) (5) (678)

head ← $c_1 < c_2 <$ → tail

$c = \{1\ 3\ 5\ 6\ 7\ 8\}$

$d_i = f(c_i)$

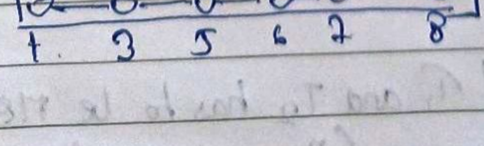
$d = \{3\ 1\ 5\ 7\ 8\ 6\}$ → spine of the graph



→ first clear the branch

i.e. $f(2) = 4$
$f(4) = 5$ (going towards spine)

for the spine



$f(1)=3$ $f(3)=1$ $f(5)=5$
$f(6)=7$ $f(7)=8$ $f(8)=6$

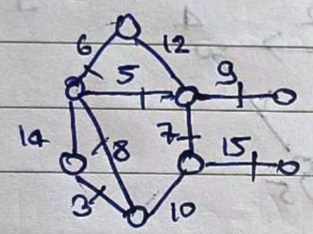
Krushkal's Algorithm

A connected graph $G = (V, E)$
weight func^{c} $W: E \rightarrow \mathbb{R}$
Qn) Spanning tree of min. weightage

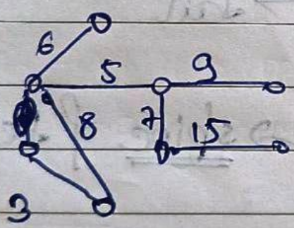
T- spanning tree
$V(T) = V(G)$ $\quad W(T) = \sum_{e \in T} w(e)$

$T = T_1 + T_2 + uw$
$T_2 = T_1' + T_2 + uw$

Q)



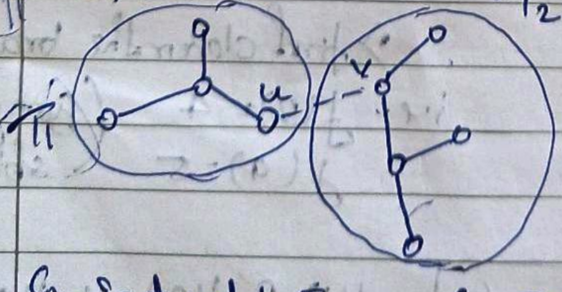
Greedy approach $\longrightarrow$
Choose the lowest weight and don't take the edge which will form a cycle



* Any ~~sb~~ subtree of minimum spanning tree is a MST

G, T-MST

$T = T_1 \cup T_2 \cup \{(u,v)\}$



$T_1 < T_1'$

G_1 - induced by T_1 $\quad G - T_2$
$V(G_1) = V(T_1)$

→ T_1 and T_2 has to be MST
(By contradiction) → weights

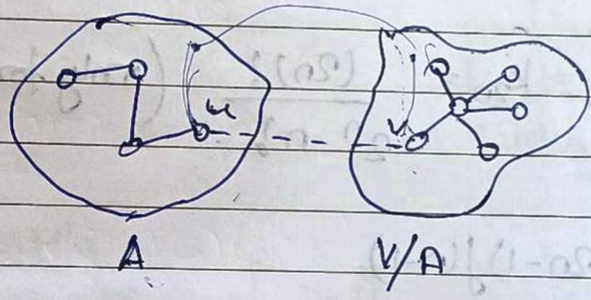
Union find

__/__/__

MST → Min. Expenditure transversal

Thm:- let T be a MST of graph $G=(V,E)$ and let $A \subseteq V$, suppose $(u,v) \in E$ is the least weight edge containing A and $V \setminus A$, then $(u,v) \in T$

→ Assume there already a ~~tree~~ ~~MSE~~ MST for A and V/A, then (u,v) has to be part of the final MST. cause if add any other edge, it will not be MST.



(M) Matching $G=(V,E)$

- an independent set of edges

$M \subseteq E$ → no single common vertex

$(u,v) \in M \Rightarrow M$ saturates u & v.

⇒ Those vertex which are not part of any edges is Matching are called unsaturated.

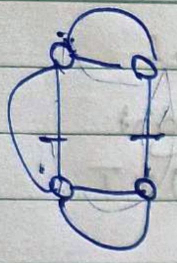
Maximal Matching
→ no extra edge can be added

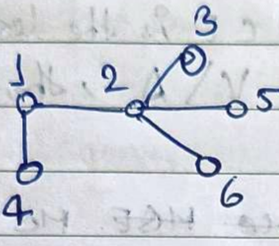
Maximum Matching
→ maximum cardinality

Perfect Matching - matching that saturates all vertices

$$\frac{n!}{(n-2)!\,2!} \quad \frac{(n-2)!}{2!\,(n-4)!} \quad \dots \quad \frac{2}{2!}$$

$$\frac{n!}{2^{n/2}\,(n/2)!}$$





{(2, 3)} → Random
{(2,5) (1,4)} → Maximal, Maximum
{(1,2)} → Maximal

K_n ⟶ # perfect matching → $n/2$ edge disjoint cycles of size 2.
Complete graph

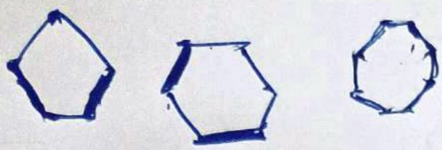
$$K_{2n} = \frac{(2n)!}{2^n\, n!} \quad \text{(only for even cases)}$$

$$f(n) = (2n-1)\,f(n-1)$$

$$\frac{2n \times (2n-1) \times (2n-2)!}{2^{n-1}\, 2.\, n.(n-1)!} = \frac{(2n-1)}{}$$

Thm:- M is a maximum matching ⟺ G has no M-augumenting Path; ~~argumenting~~

M-alternating path – edges alternates b/w M and E/M
↓
M-augumenting Path.
↳ M-alternating that unsaturate the end vertices



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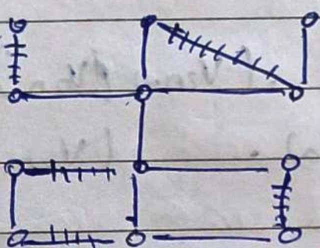
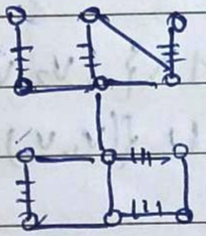
* Lemma: M and M' are 2 matchings in graph G = (V,E)
M△M', will contain either a path or a even cycle. (~~the~~ every components of M△M')

proof:→

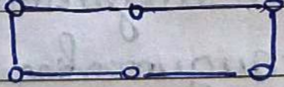


Matching (I)



Matching (II)

M△M' →



F = M△M'
$deg(v) \leq 2 \ \forall \ v \in F(V)$ → By PHP, suppose that $\exists$ v, with $deg(v) \geq 3$ then 2 of them belong to M and M'

Proof:- M is maximum matching.
suppose G has a M-augumenting path.

o—o—o. . . .o—o—o

$v_1 \quad v_2 \quad v_3 \qquad\qquad v_{2n} \quad v_{2n+1}$

$$M' = M \setminus \{(v_2,v_3), \ldots (v_{2n-1}, v_{2n})\}$$
$$\cup \{(v_1,v_2), (v_3,v_4) \ldots (v_{2n}, v_{2n+1})\}$$

now, $|M'|$ is more than $|M|$ by 1.

Proving the contrapositive

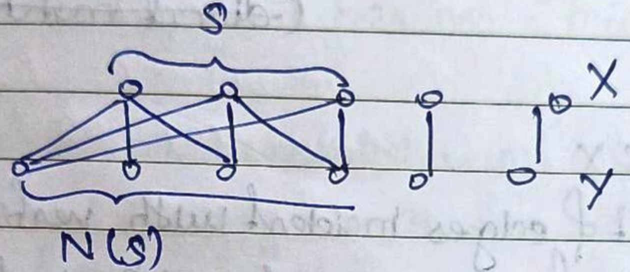
$\exists$ a matching M' larger than M,
construct a M-augumenting path
$F = M \Delta M'$
F has only paths and even cycle
$|M'| > |M|$

Perfect Matching in a BiPartite Graph

G- BiPartite Graph (X, Y)
a matching M saturates every vertex in X,
then $\forall$ subset S of X

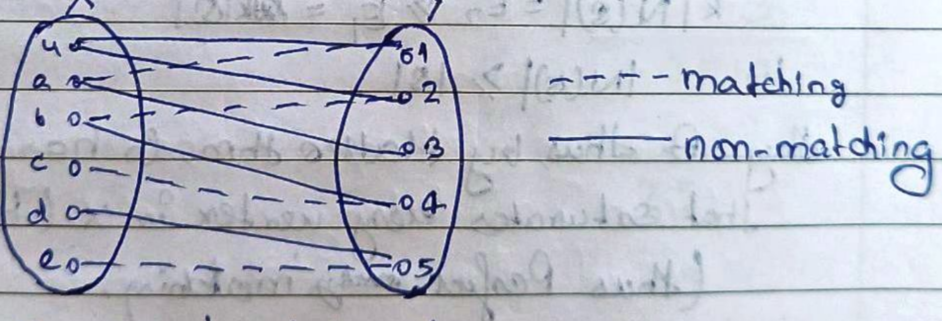
then ~~∃~~ there exits atleast $|S|$ edges that have ~~atg~~ neighbours in S

$\therefore\ \forall S \subseteq X\ ;\ |N(S)| \geq |S|$ (Hall's Condition)



(⇐) Contrapositive

if M is a matching (Maximum matching) in G which doesn't saturate X, then $\exists S \subseteq X,\ |N(S)| < |S|$.



$u \in X$ m-saturated vertex

$V_u = \{$all vertices that are reachable from u by an M-alternating path$\}$

$S = V_u \cap X$

$T = V_u \cap Y$

$|T| = |S \setminus \{u\}| = |S| - 1$

$|N(S)| = |T| = |S| - 1 < |S|$

$T \subseteq N(S)$

Corr: $k>0$, every k-regular bipartite graph has a perfect matching
→ every deg = k

no. of edges $= k|X| = k|Y| \Rightarrow |X| = |Y|$
(~~direct matching~~) (perfect)

let $S \subseteq X$
E_1 = set of edges incident with vertices in S
E_2 = set of edges incident with vertices in $N(S)$

Clearly $(E_1 \subseteq E_2)$
i.e.
$k|N(S)| = E_2 \geq E_1 = k|S|$
i.e. $|N(S)| \geq |S|$
~~they~~ ~~S~~ thus, by Hall's thm, G has matching that saturates every vertex in X (i.e. also of Y)
(thus Perfect matching.)

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Stable matching Problem

$\begin{pmatrix} x, a' \\ x', a \end{pmatrix}$

i/p :- 'n' men and 'n' women
priority ordering of each man & Woman.
o/p :- matching that has no "mistake pair"

(x,a) is called unstable w.r.t. to matching M if x and a prefer each other over their current pairs in M.

Boy			Girls	
CBEAD	1		A	35214
ABECD	2		B	52143
DCBAE	3		C	43512
ACDBE	4		D	12345
ABDEC	5		E	23415

- free - not paired
- engaged - paired during the run of algo
- married - paired at the end of algo

Nisha

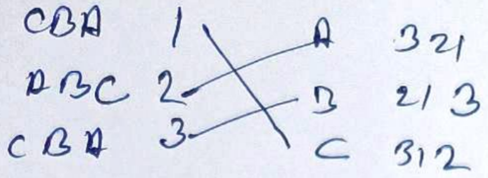
while ($\exists$ a free man m)
{
1. m proposes woman w (top list)
2. if w is free
- (m, w) get engaged
else /* (m', w) */
if $L_w(m) > L_w(m')$ then
(m, w) get engaged.

3. m deletes w from the list
}

Terminates:→ No man can get rejected by all woman.
- woman can reject only if she is engaged
- if man is rejected by the last woman on his list, then are ⇒ not possible*

- # woman = # man and no man is engaged to 2 woman

steps $O(n^2)$
- each man can propose to any woman



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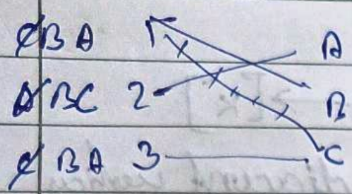


- Gives a perfect matching
→ each man is engaged with ≤1 woman.
→ algo halts when no man is free ≥1 woman.

- Gives a Stable matching

o/p (m, w) & (m', w')
Lm (w') > Lm (w) are
Lw' (m) > Lw'(m')
then it is unstable.

(m, w)
&
unstable

∴ m never proposed to w'
∴ m was rejected by w' when he proposed.

This is a male - optimized matching
↳ Suppose Not
- let in k^{th} run on while loop, let it be the first time a man m get rejected by optimal match 'w'.
- w rejects m because (m*, w)
$L_w (m^*) > L_w (m)$
- ∃ a stable matching in which (m, w) are matched.
In that say (m*, w') are matched

then (m*, w') is unstable ⇒⇐

Vertex Colouring

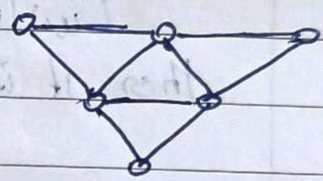
G is k colourable if $\exists\ \alpha : V_G \to [k]$

proper colouring → no 2 adjacent vertices get the same colour

chromatic no.

$\chi(G)$ – min no. of colours required for proper colouring

e.g. $\chi(G) = 3$



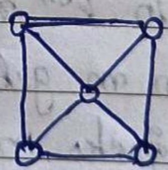
$\chi(K_n^c) = 1$

$\chi(\text{paths}) = 2$ $\chi(\text{even cycles}) = 2$

$\chi(\text{odd cycles}) = 3$

$\chi(\text{complete graph}) = n$

$\chi(W_{even}) = 3$



W_4

$\chi(W_{odd}) = 4$

$\chi(\text{Bipartite}) = 2 \Rightarrow \Leftarrow$

__/__/__

* if a graph has ~~as~~ complete subgraph (k_n)

$\downarrow$ $\chi \geq n^*$

But converse is not always true.

P.T. there are Δ-freegraph with arbitarly large element number.

e.g.

$G - \chi(G) = k$ $\quad$ $G' \ni \chi(G) = k+1$

$\downarrow$ $\quad$ $\downarrow$

Δ free $\quad$ Δ-free

deg. of

$1 \leq \chi(G) \leq \Delta + 1$ $\quad$ Brooks

$\downarrow$ $\quad$ $\hookrightarrow$ max. deg. $\doteq (n-1)$

discrete $\quad$ (odd-cycles, k_n)

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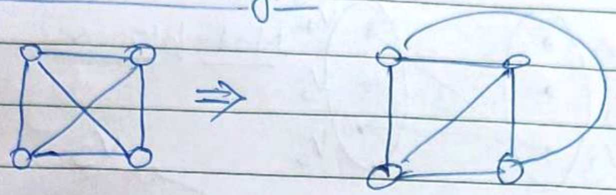
* Vertex Colouring

$\chi(\text{trees}) = 2$

Planer Embedding

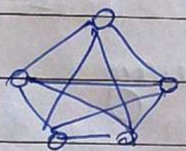
K_4



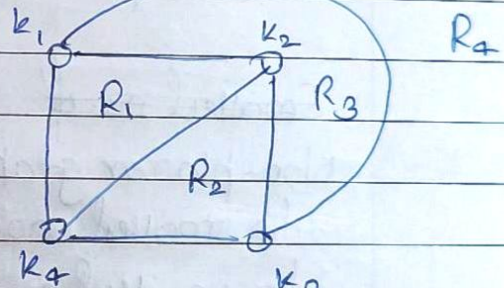
planer

A graph is called planer if $\exists$ a planer embedding of the graph.

K_5 → K_4 along with one more vertex

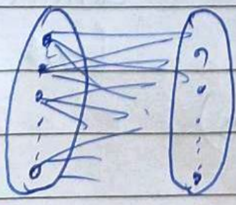


non-planer



any vertex takes in region R_1, R_2, R_3 or R_4, we will cross the boundry.

$K_{m,n}$



Complete bipartite graph

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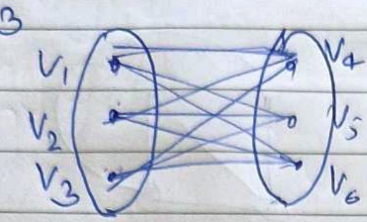
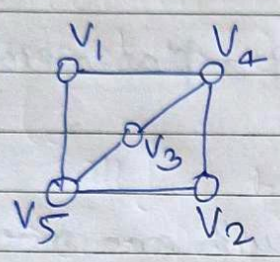
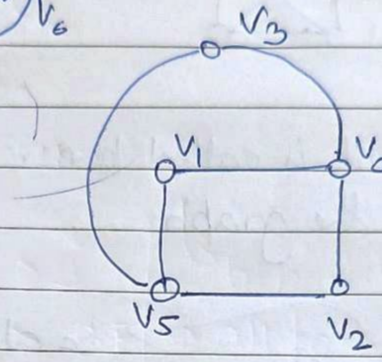
$K_{3,3}$



Not-planer





Can't place V_6 in any way

~~Smallest no. of~~

Non-planer graphs
↳ smallest no. of vertices :- K_5
↳ smallest no. of edges :- $K_{3,3}$

* Euler formula for planer graphs
↳ $V - E + F = 2$ ← only for connected (but not a sufficient condition)
↓
\# faces (regions)

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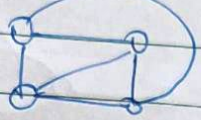
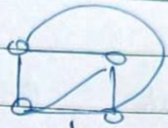
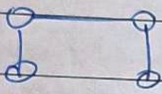
-E+F remain same



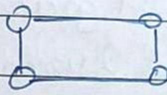
remove a cycle $\longrightarrow$



$V-E+F=2$ $\qquad$ $V-E+F=2$



$\longleftarrow$



$V-E+F=2$ $\qquad$ $V-E+F=2$

i.e. in a spanning tree of any graph

V

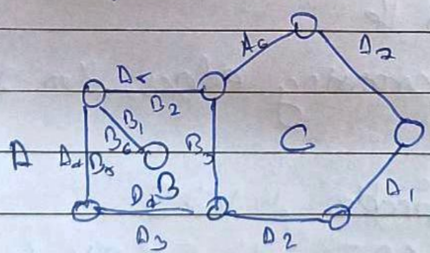
$E = V-1 \Rightarrow \boxed{V-E+F=2}$

$F=1$

* degree of a face

sum of degree of vertices

$$\boxed{\begin{array}{l}\text{Sum of degree of faces}\\ 18 = 2E\end{array}}$$



$deg(A) = 7 \qquad deg(C) = 5$

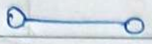
$deg(B) = 6$ (min closed walk)

__/__/__

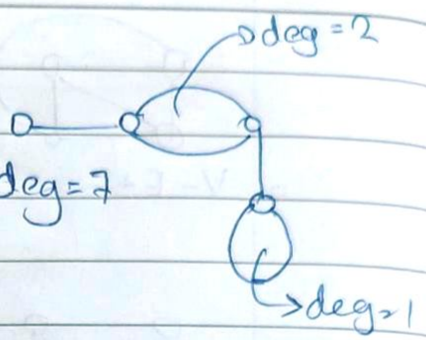
Face degree

empty graph

0



deg = 2



deg = 7

Simple connected graph with at least 3 vertices.

→ all face degree ≥ 3

→ sum of all face degree $\geq 3F$

i.e. → $\boxed{2E \geq 3F}$

→ bipartite

→ all face deg ≥ 4

→ sum of all face deg $\geq 4F$

⇒ $2E \geq 4E$

$\boxed{E \geq 2F}$

Thm:→ In a connected graph drawn in the plane without crossing edges, then:

a) $V - E + F = 2$

b) if G is simple & $V \geq 3$

  a) $3F \leq 2E$ b) $E \leq 3V - 6$

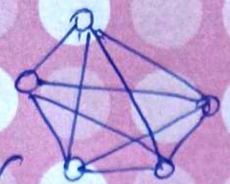
  c) $F \leq 2V - 4$

c) G is simple and bipartite

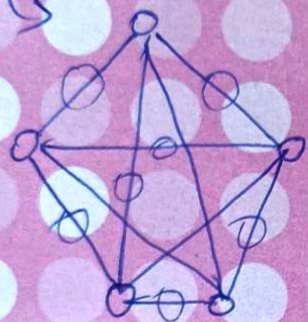
  a) $2F \leq E$ b) $E \leq 2V - 4$

  c) $F \leq V - 2$

Homiomorphism (edge equivalency)



Chickenpox



Kuratowski's theorem

G is planer iff it doesn't have a subgraph homeomorphic to K_5 or $K_{3,3}$

Thm

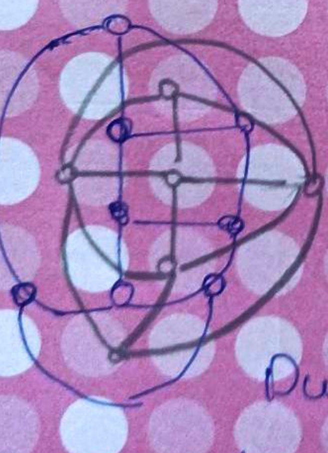
↳ A connected simple planer graph has a vertex with degree at most 5.

* proof by contradiction

$\forall$ v $deg(v) \geq 6$

$2E \geq 6V$

$E \geq 3V \Rightarrow \Leftarrow$ contradiction



Dual → * Vertex for each face (joining adjacent regions)

	G	H
V	8	6
E	12	12
F	6	8

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