- **Result 1:** $\mathbf{x}$ is a BFS of $A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O} \Leftrightarrow \mathbf{x}$ is an extreme point of $S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x}_{m\times n} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}\}$.

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  Let $S' = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{O}\}$
  and $S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m} : A_{m\times n}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}, \mathbf{s} \geq \mathbf{O} \right\}$.

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  **Result 3:** If $S \neq \phi$, then it has at least one BFS.

- **Result 4:** If $S_0 = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}\} \neq \phi$, then it has atleast one BFS.

- **Result 4:** If $S_0 = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}\} \neq \phi$, then it has atleast one BFS.
- **Result 5:** $\mathbf{d}_0$ is an extreme direction of $S' \Leftrightarrow \begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}'_0 \end{bmatrix}$ is an extreme direction of $S$ where $\mathbf{d}'_0 \geq \mathbf{O}$ is such that $A\mathbf{d}_0 + \mathbf{d}'_0 = \mathbf{O}$.

- **Result 4:** If $S_0 = \{\mathbf{x} \in \mathbb{R}^n : A_{m \times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}\} \neq \phi$, then it has atleast one BFS.
- **Result 5:** $\mathbf{d}_0$ is an extreme direction of $S' \Leftrightarrow \begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}_0' \end{bmatrix}$ is an extreme direction of $S$ where $\mathbf{d}_0' \geq \mathbf{O}$ is such that $A\mathbf{d}_0 + \mathbf{d}_0' = \mathbf{O}$.
- In simplex method in any iteration we move from one extreme point to an adjacent extreme point if the current BFS is nondegenerate.
- It remains at the same extreme point if the current BFS is degenerate.

## Sensitivity Analysis:

- Consider the problem (**P**),
  Min $\mathbf{c}^T\mathbf{x}$
  subject to
  $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{O}$.
  Let $\mathbf{x}_0$ be an optimal solution of this problem. WLOG let $B = [\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_m]$ be a basis matrix corresponding to $\mathbf{x}_0$, hence a set of basic variables of $\mathbf{x}_0$ are $x_1, \ldots, x_m$.

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- If the new $c_j' - z_j' \geq 0$ for all $j = 1, \ldots, n$, then $\mathbf{x}_0$ will again be optimal for the new problem.

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- **Changing the cost vector c:**
- If the new $c'_j - z'_j \geq 0$ for all $j = 1, \ldots, n$, then $\mathbf{x}_0$ will again be optimal for the new problem.
- If not, then simplex algorithm can be used to get an optimal solution for the new problem or to conclude that the new problem has no optimal solution.

## Changing the vector b:

- If the vector **b** is changed to **b**′, and if the basic solution $\mathbf{x}_0' = [(B^{-1}\mathbf{b}')^T, \mathbf{O}_{1\times(n-m)}]^T$ corresponding to $\mathbf{x}_0 = [(B^{-1}\mathbf{b})^T, \mathbf{O}_{1\times(n-m)}]^T$, is such that $B^{-1}\mathbf{b}' \geq \mathbf{O}$, then $\mathbf{x}_0'$ is optimal for the new problem.

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- Since $\mathbf{y}^T$ satisfies $\mathbf{y}^T\tilde{\mathbf{a}}_j = z_j = c_j$ for $j = 1, \ldots, m$, it lies on $m$ LI hyperplanes defining $Fea(D)$, so $\mathbf{y}$ is an extreme point of $Fea(D)$.

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- The **dual simplex algorithm** can be used to either get an **optimal solution** of the new problem or to conclude that the new problem does not have a **feasible solution**.

- **Example :** Consider the LPP given by
  Max $-x_1 + 2x_2$
  subject to
  $-x_1 + x_2 \leq 1,$
  $x_1 + x_2 \leq 7,$
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  If we convert the above problem to a problem with equality constraints by adding (slack) variables, then it becomes
- Max $-x_1 + 2x_2$
  subject to
  $-x_1 + x_2 + s_1 = 1,$
  $x_1 + x_2 + s_2 = 7,$
  $x_1 + 3x_2 + s_3 = 15,$
  $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$

- The optimal BFS $[3, 4, 0, 0, 0]^T$ is degenerate and corresponds to three different basis matrix, $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_1]$, $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_2]$ and $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_3]$.
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  The tables corresponding to these three bases are given by
-

| $c_j - z_j$ | 0 | 0 | 0 | $\frac{5}{2}$ | $-\frac{3}{2}$ | |
|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}_1$ | $B^{-1}\tilde{\mathbf{a}}_2$ | $B^{-1}\mathbf{e}_1$ | $B^{-1}\mathbf{e}_2$ | $B^{-1}\mathbf{e}_3$ | $B^{-1}\mathbf{b}$ |
| $\tilde{\mathbf{a}}_1$ | | | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ | 3 |
| $\tilde{\mathbf{a}}_2$ | | | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 4 |
| $\mathbf{s}_1$ | | | 1 | 2 | -1 | 0 |

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| $\tilde{\mathbf{a}}_1$ | | | 0 | $\frac{3}{2}$ | $-\frac{1}{2}$ | 3 |
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| $\mathbf{s}_1$ | | | 1 | 2 | -1 | 0 |

- Note that not all the $c_j - z_j$ values in the above table are nonpositive, but the above BFS is still optimal.
- So the optimality condition, $c_j - z_j \geq 0$ for all $j$, is a **sufficient condition** but **not** a **necessary condition** for the corresponding BFS to be optimal.

- 

| $c_j - z_j$ | 0 | 0 | $-\frac{5}{4}$ | 0 | $-\frac{1}{4}$ | |
|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}_1$ | $B^{-1}\tilde{\mathbf{a}}_2$ | $B^{-1}\mathbf{e}_1$ | $B^{-1}\mathbf{e}_2$ | $B^{-1}\mathbf{e}_3$ | $B^{-1}\mathbf{b}$ |
| $\tilde{\mathbf{a}}_1$ | | | $-\frac{3}{4}$ | 0 | $\frac{1}{4}$ | 3 |
| $\tilde{\mathbf{a}}_2$ | | | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ | 4 |
| $\mathbf{s}_2$ | | | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | 0 |

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| $c_j - z_j$ | 0 | 0 | $-\frac{5}{4}$ | 0 | $-\frac{1}{4}$ | |
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| $\tilde{\mathbf{a}}_1$ | | | $-\frac{3}{4}$ | 0 | $\frac{1}{4}$ | 3 |
| $\tilde{\mathbf{a}}_2$ | | | $\frac{1}{4}$ | 0 | $\frac{1}{4}$ | 4 |
| $\mathbf{s}_2$ | | | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | 0 |

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|---|---|---|---|---|---|---|
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| $\tilde{\mathbf{a}}_1$ | | | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 3 |
| $\tilde{\mathbf{a}}_2$ | | | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 4 |
| $\mathbf{s}_3$ | | | -1 | -2 | 1 | 0 |

- The dual of the above problem is given by
  Min $y_1 + 7y_2 + 15y_3$
  subject to
  $-y_1 + y_2 + y_3 \geq -1,$
  $y_1 + y_2 + 3y_3 \geq 2,$
  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

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- The optimal solutions of the Dual obtained from the optimal tables are given by:
  $\mathbf{y}^T = \mathbf{c}_B^T B^{-1} = \mathbf{c}_B^T B^{-1}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = [z_{s_1}, z_{s_2}, z_{s_3}]$
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- $[\frac{5}{4}, 0, \frac{1}{4}]^T$ and $[\frac{3}{2}, \frac{1}{2}, 0]^T$, are both optimal solutions of the Dual as well as extreme points of $Fea(D)$.

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  where $z_{s_i}$ is the $z_j$ value corresponding to the slack variable $s_i$.
- $[\frac{5}{4}, 0, \frac{1}{4}]^T$ and $[\frac{3}{2}, \frac{1}{2}, 0]^T$, are both optimal solutions of the Dual as well as extreme points of $Fea(D)$.
- The dual has infinitely many optimal solutions.

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  subject to
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  $y_1 + y_2 + 3y_3 - s'_2 = 2,$
  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, s'_1 \geq 0, s'_2 \geq 0.$

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  subject to
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  $y_1 + y_2 + 3y_3 - s'_2 = 2,$
  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, s'_1 \geq 0, s'_2 \geq 0.$
- The BFS corresponding to the extreme point $[\frac{5}{4}, 0, \frac{1}{4}]^T$ of the Dual will have basic variables as $y_1, y_3$.

- If we convert the dual problem into a problem with equality constraints by adding (surplus) variables then we get
- Min $y_1 + 7y_2 + 15y_3$
  subject to
  $-y_1 + y_2 + y_3 - s_1' = 1,$
  $y_1 + y_2 + 3y_3 - s_2' = 2,$
  $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, s_1' \geq 0, s_2' \geq 0.$
- The BFS corresponding to the extreme point $[\frac{5}{4}, 0, \frac{1}{4}]^T$ of the Dual will have basic variables as $y_1, y_3$.
- The BFS corresponding to the extreme point $[\frac{3}{2}, \frac{1}{2}, 0]^T$ have basic variables as $y_1, y_2$.

- The table corresponding to these basic feasible solutions will be given by (check this)

| $c_j - z_j$ | 0 | 0 | 0 | 3 | 4 | |
|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}'_1$ | $B^{-1}\tilde{\mathbf{a}}'_2$ | $B^{-1}\tilde{\mathbf{a}}'_3$ | $B^{-1}(-\mathbf{e}_1)$ | $B^{-1}(-\mathbf{e}_2)$ | $B^{-1}$ |
| $\tilde{\mathbf{a}}'_1$ | | | | | $-\frac{1}{2}$ | $\frac{3}{2}$ |
| $\tilde{\mathbf{a}}'_2$ | | | | | $-\frac{1}{2}$ | $\frac{1}{2}$ |

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| $\tilde{\mathbf{a}}'_1$ | | | | | $-\frac{1}{2}$ | $\frac{3}{2}$ |
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-

| $c_j - z_j$ | 0 | 0 | 0 | 3 | 4 | |
|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}'_1$ | $B^{-1}\tilde{\mathbf{a}}'_2$ | $B^{-1}\tilde{\mathbf{a}}'_3$ | $B^{-1}(-\mathbf{e}_1)$ | $B^{-1}(-\mathbf{e}_2)$ | $B^{-1}$ |
| $\tilde{\mathbf{a}}'_1$ | | | | | $-\frac{1}{4}$ | $\frac{5}{4}$ |
| $\tilde{\mathbf{a}}'_3$ | | | | | $-\frac{1}{4}$ | $\frac{1}{4}$ |

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| | $B^{-1}\tilde{\mathbf{a}}'_1$ | $B^{-1}\tilde{\mathbf{a}}'_2$ | $B^{-1}\tilde{\mathbf{a}}'_3$ | $B^{-1}(-\mathbf{e}_1)$ | $B^{-1}(-\mathbf{e}_2)$ | $B^{-1}$ |
| $\tilde{\mathbf{a}}'_1$ | | | | | $-\frac{1}{2}$ | $\frac{3}{2}$ |
| $\tilde{\mathbf{a}}'_2$ | | | | | $-\frac{1}{2}$ | $\frac{1}{2}$ |

-

| $c_j - z_j$ | 0 | 0 | 0 | 3 | 4 | |
|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}'_1$ | $B^{-1}\tilde{\mathbf{a}}'_2$ | $B^{-1}\tilde{\mathbf{a}}'_3$ | $B^{-1}(-\mathbf{e}_1)$ | $B^{-1}(-\mathbf{e}_2)$ | $B^{-1}$ |
| $\tilde{\mathbf{a}}'_1$ | | | | | $-\frac{1}{4}$ | $\frac{5}{4}$ |
| $\tilde{\mathbf{a}}'_3$ | | | | | $-\frac{1}{4}$ | $\frac{1}{4}$ |

Here $\tilde{\mathbf{a}}'_i$ gives the columns corresponding to the variables $y_i, i = 1, 2, 3$ in the dual constraints, when the constraints are written in the greater than equal to form.

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- Note that the BFS corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_1]$, should have $s_2 = s_3 = 0$ hence should lie at the intersection of the two lines
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- The corresponding BFS is $[\frac{7}{2}, \frac{7}{2}, 1, 0, 0]^T$, which is a nondegenerate BFS.

- Since all the $c_j - z_j$ values are not $\leq 0$, so by entering $s_2$ into the basis one can get a BFS with better value of the objective function.
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- The corresponding BFS is $[\frac{11}{4}, \frac{15}{4}, 0, \frac{1}{2}, 0]^T$, a nondegenerate BFS.
- Since $c_j - z_j \leq 0$ for all $j = 1, \ldots, n$, so this BFS is optimal solution for the new problem.

- The basic solution corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_3]$, should have $s_1 = s_2 = 0$ hence should lie at the intersection of the two lines
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- The corresponding basic solution is $[3, 4, 0, 0, -1]^T$ which is not feasible for the new problem.
- Will the new dual now have a unique optimal solution or will it again have infinitely many optimal solutions?

## Dual Simplex Algorithm :

- Consider the following LP problem (P) :
  Min $\mathbf{c}^T\mathbf{x}$
  subject to $A_{m\times n}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{O}$.

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- In this case the **Dual Simplex Method** is used to get an optimal solution of (P) or to conclude that (P) **does not** have a feasible solution.

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- The table is updated by performing the necessary elementary row operations.
- The new column corresponding to $\tilde{\mathbf{a}}_s$ in the table is the $r$ th column of $I_m$ and $c_s - z_s = 0$.
- New $c_j - z'_j = c_j - z_j - \frac{c_s - z_s}{u_{rs}} u_{rj}$.

- If $\mathbf{x}'$ is the new basic solution of the primal (P) and $\mathbf{y}'$ be the corresponding feasible solution of the Dual, then $\mathbf{b}^T\mathbf{y}' = \mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + (c_s - z_s)\frac{x_r}{u_{rs}} \geq \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$.

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  $-x_1 + x_2 + s_1 = 1$,
  $x_1 + x_2 + s_2 = 7$,
  $x_1 + 3x_2 + s_3 = 15$,
  $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$
  The table corresponding to the basic solution with $x_1 = -1$ and $x_2 = 0$ is given by

- Max $-3x_1 + 2x_2$
  subject to
  $-x_1 + x_2 + s_1 = 1$,
  $x_1 + x_2 + s_2 = 7$,
  $x_1 + 3x_2 + s_3 = 15$,
  $x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$
  The table corresponding to the basic solution with $x_1 = -1$ and $x_2 = 0$ is given by
-

| $c_j - z_j$ | | -1 | -3 | | | |
|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}_1$ | $B^{-1}\tilde{\mathbf{a}}_2$ | $B^{-1}\mathbf{e}_1$ | $B^{-1}\mathbf{e}_2$ | $B^{-1}\mathbf{e}_3$ | $B^{-1}\mathbf{b}$ |
| $\tilde{\mathbf{a}}_1$ | | -1 | -1 | | | -1 |
| $\mathbf{s}_2$ | | 2 | 1 | | | 8 |
| $\mathbf{s}_3$ | | 4 | 1 | | | 16 |

- According to the Dual Simplex method, $x_1$ is the leaving variable, since $(B^{-1}\mathbf{b})_1 < 0$ and $(B^{-1}\mathbf{b})_2 > 0$, $(B^{-1}\mathbf{b})_3 > 0$.

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- Also since in the row corresponding to the leaving variable, only two entries are negative, $u_{12}$, and $u_{13}$ and $\frac{c_2 - z_2}{u_{12}} = 1 < \frac{c_3 - z_3}{u_{13}} = 3$, hence the entering variable is $x_2$ and the **pivot** is $u_{12}$, where $u_{ij}$'s have their usual meaning.

- According to the Dual Simplex method, $x_1$ is the leaving variable, since $(B^{-1}\mathbf{b})_1 < 0$ and $(B^{-1}\mathbf{b})_2 > 0, (B^{-1}\mathbf{b})_3 > 0$.
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- | $c_j - z_j$ | -1 | 0 | -2 | | | |
  |---|---|---|---|---|---|---|
  | | $B^{-1}\tilde{\mathbf{a}}_1$ | $B^{-1}\tilde{\mathbf{a}}_2$ | $B^{-1}\mathbf{e}_1$ | $B^{-1}\mathbf{e}_2$ | $B^{-1}\mathbf{e}_3$ | $B^{-1}\mathbf{b}$ |
  | $\tilde{\mathbf{a}}_2$ | -1 | 1 | 1 | | | 1 |
  | $\mathbf{s}_2$ | 2 | 0 | -1 | | | 6 |
  | $\mathbf{s}_3$ | 4 | 0 | -3 | | | 12 |

| $c_j - z_j$ | -1 | 0 | -2 | | | |
|---|---|---|---|---|---|---|
| | $B^{-1}\tilde{\mathbf{a}}_1$ | $B^{-1}\tilde{\mathbf{a}}_2$ | $B^{-1}\mathbf{e}_1$ | $B^{-1}\mathbf{e}_2$ | $B^{-1}\mathbf{e}_3$ | $B^{-1}\mathbf{b}$ |
| $\tilde{\mathbf{a}}_2$ | -1 | 1 | 1 | | | 1 |
| $\mathbf{s}_2$ | 2 | 0 | -1 | | | 6 |
| $\mathbf{s}_3$ | 4 | 0 | -3 | | | 12 |

Hence an optimal solution of the primal is given by, $x_2 = 1$ and $x_1 = 0$.

The optimal solution of the Dual is given by,

$y_1 = 2, y_2 = 0, y_3 = 0$,

where the Dual is given by:

Min $y_1 + 7y_2 + 15y_3$

subject to

$-y_1 + y_2 + y_3 \geq -3$,

$y_1 + y_2 + 3y_3 \geq 2$,

$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$.