Physics II: Electromagnetism

PH 102

Lecture-3

March-June2022

Review :

What we have learnt so far:

- Line integrals of vector fields along a given path. The results are in general path dependent.

- Gradient fields are a class of vector fields, whose line integrals are path independent. The result depends only on the end points. **(Fundamental theorem of Gradients).**

- Such Gradient fields are called conservative vector fields that satisfy the property :

  $\vec{\nabla} \times \vec{F} = 0$ and $\oint \vec{F} \cdot .d\vec{r} = 0.$

- Surface integrals of vector fields can be calculated from the idea that elementary surfaces can be treated as vectors . In general $d\vec{a} = d\vec{x} \times d\vec{y} = da\ \hat{n}.$

- In general, line integrals are one-dimensional integrals while the surface integrals are two-dimensional ones.

Volume Integrals

Consider a closed surface in space enclosing a volume. Then,

$$\int_{\mathcal{V}} \vec{v} d\tau \text{ and } \int_{\mathcal{V}} \mathrm{T} d\tau$$

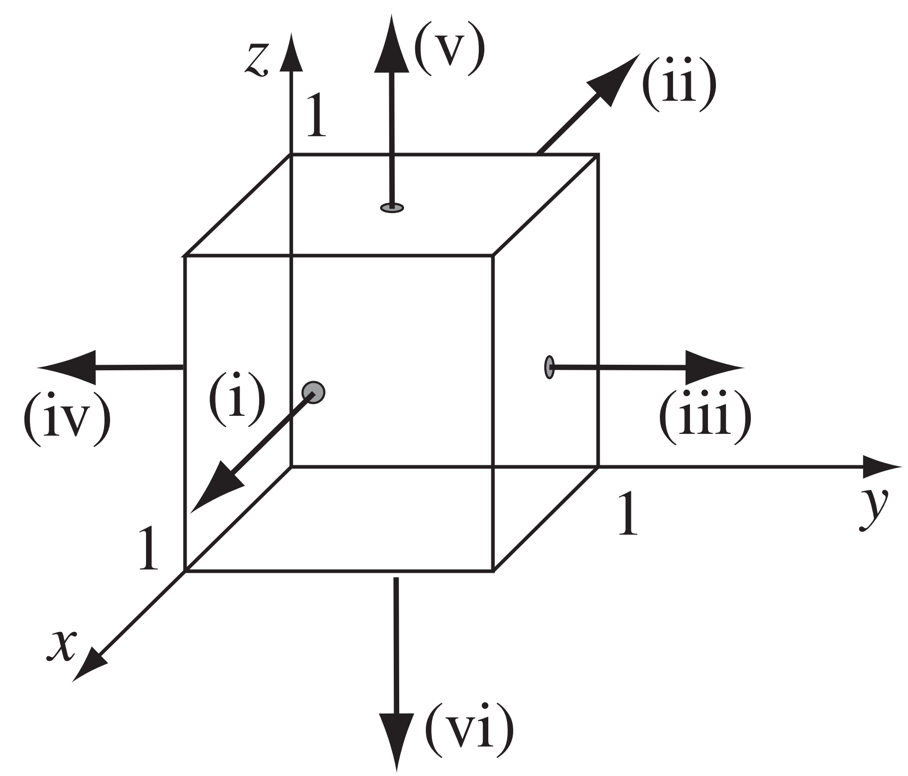
are examples of volume integrals. Here $d\tau$ represents elementary volume.

- Here the volume element is obtained from $d\tau = d\overrightarrow{a} \cdot d\hat{x}_3 = dx_1 dx_2 dx_3 \ \hat{n} \cdot \hat{x}_3$
- For the elementary surface area in the xy-plane, the corresponding unit vector $\hat{n}$ of area is in the same direction as $\hat{z}$.
- In the case of surface area, if $d\hat{x}_1$ and $d\hat{x}_2$ are in the same direction then the elementary area vanishes i.e. when $d\hat{x}_1 \parallel d\hat{x}_2$. and $d\overrightarrow{a}$ is maximum if $d\hat{x}_1 \perp d\hat{x}_2$. In the cartesian coordinate system, $\hat{x} \perp \hat{y}$ and hence $d\overrightarrow{a} = d\overrightarrow{x} \times d\overrightarrow{y} = dxdy\,\hat{z}$.
- For the case of elementary volumes, if the unit vector of the elementary area and the unit vector of height are orthogonal to each other, then the elementary volume vanishes. In cartesian coordinate system, $d\overrightarrow{a} \parallel \hat{z}$ and hence $d\tau = d\overrightarrow{a} \cdot dz\hat{z} = da\,dz$ is maximum.

Volume Integral of a scalar field

Let's consider a simple scalar field $\phi(x, y, z) = 2(x + y)$. Calculate the volume integral of this scalar field over a cube with a unit length at the origin, as shown in the figure.

Here, the elementary volume is given by $d\tau = dx\, dy\, dz$.



We need to calculate $\int_v \phi(x, y, z)\ d\tau = \iiint 2(x + y)\, dx\, dy\, dz$

The integration limits are simple and are $x = [0,1]$, $y = [0,1]$ and $z = [0,1]$

$$\int_{\mathcal{V}} 2(x+y)\, d\tau = 2\int_0^1\int_0^1\int_0^1 (x+y)\, dx\, dy\, dz,$$

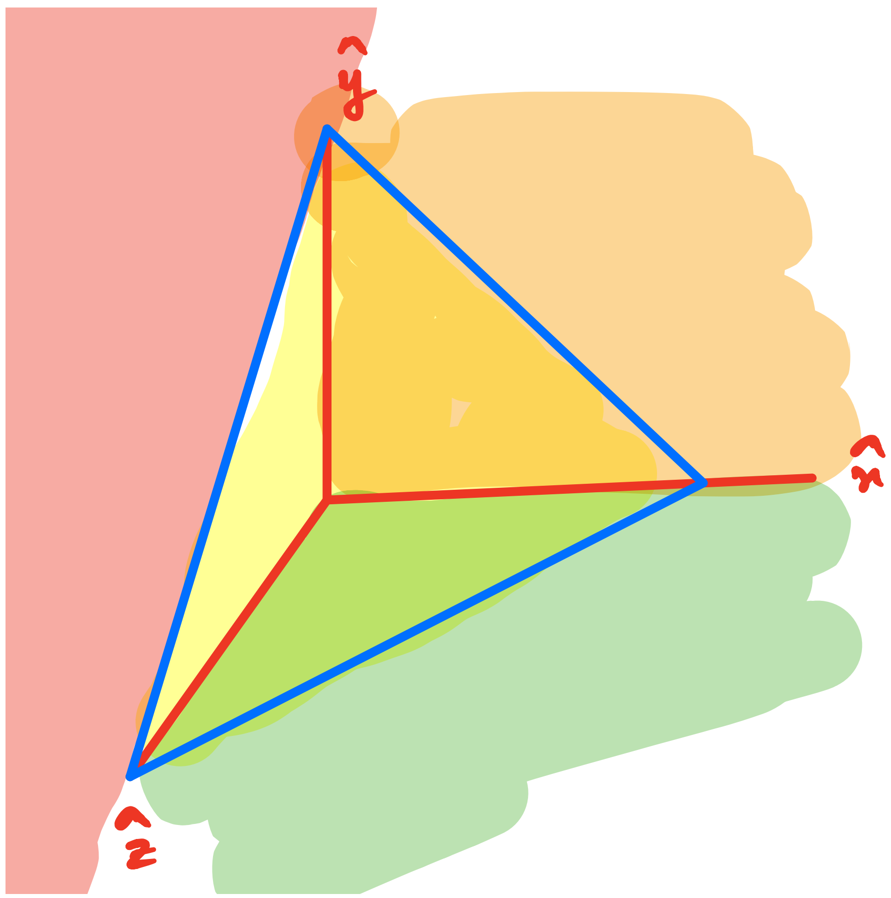
$$\int_0^1 (x+y)\, dx = \tfrac{1}{2} + y, \quad \int_0^1 (\tfrac{1}{2} + y)\, dy = 1, \quad \int_0^1 1\, dz = 1.$$

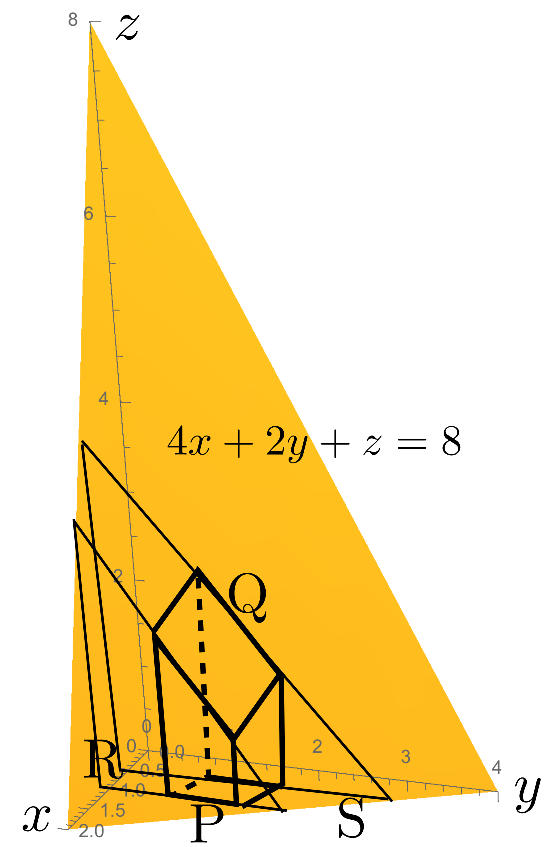
$$\therefore \int_v \phi(x, y, x)\ d\tau = 2$$

Volume Integrals of a scalar field (example)

Let $\phi = 45x^2y$ and let $\mathcal{V}$ denotes a closed region bounded by the planes $4x + 2y + z = 8$, $x = 0$, $y = 0$, $z = 0$. Evaluate the integral $\int_{\mathcal{V}} \phi d\tau$.

Solution Keep x and y constant and integrate from $z = 0$ (base of the column PQ in figure) to $z = 8 - 4x - 2y$ (top of the column PQ)



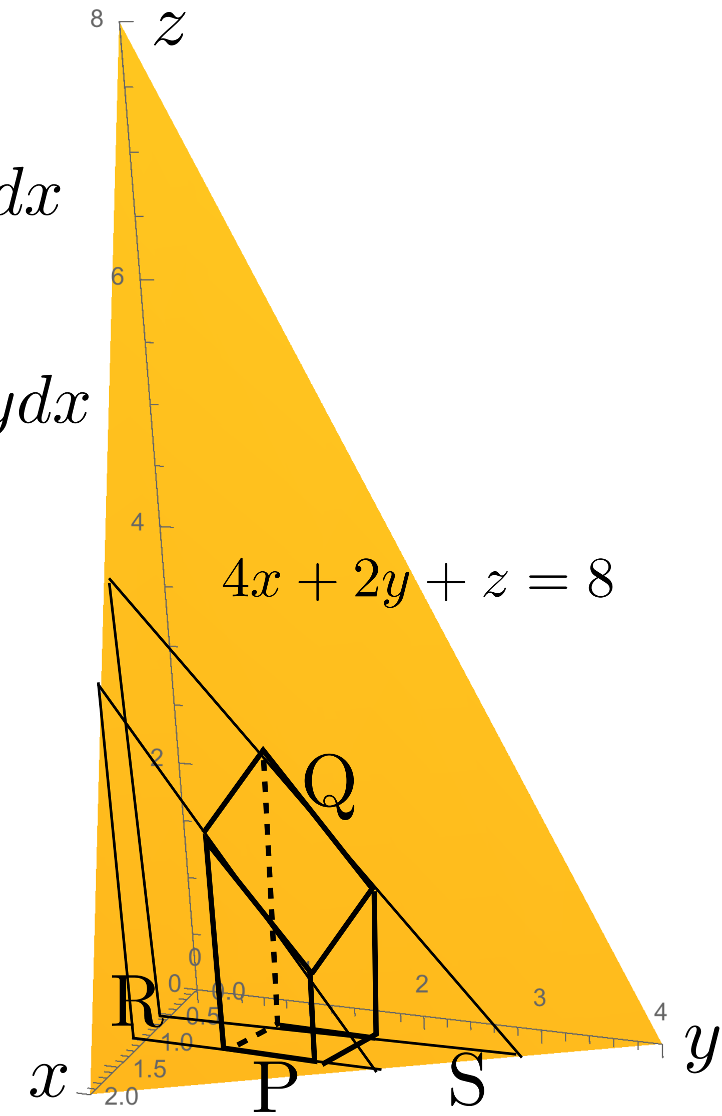


Next keep x constant and integrate w.r.t y. This amounts to addition of columns having bases in the xy plane ($z = 0$) located anywhere from R (where $y = 0$) to S (where $4x + 2y = 8$ or $y = 4 - 2x$), and the integrations from $y = 0$ to $y = 4 - 2x$.

Volume Integrals

Finally add all slabs parallel to yz plane, which amounts to integration from $x = 0$ to $x = 2$.

$$\begin{aligned}
\int_{\mathcal{V}} \phi d\tau &= \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45x^2 y dz dy dx \\
&= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2 y(8-4x-2y) dy dx \\
&= 45 \int_{x=0}^{2} \frac{8}{3} x^2 (2-x)^3 dx \\
&= 120 \int_{x=0}^{2} x^2 (2-x)^3 dx \\
&= 120 \times \frac{16}{15} = 128
\end{aligned}$$



Here, $x_{min} = y_{min} = z_{min} = 0$

For $4x = 8 - 2y - z$, x is maximum when y and z are minimum. Therefore, $x_{max} = 2$.

For a given x, then y has maximum When z is minimum in 2y=8-4x-z. Therefore, $y_{max} = 4 - 2x$.

For the given x and y, then z has a maximum of z = 8-4x-2y.

Physically the result can be interpreted as the mass of the region $\mathcal{V}$ in which the density varies according to the formula $\phi = 45x^2 y$.

Important theorems in vector calculus

Fundamental theorem of gradients :

$$\boxed{\int_{\mathbf{a}}^{\mathbf{b}} (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).}$$

Fundamental theorem for Curls
Also known as Stokes' theorem.

$$\boxed{\int_{\mathcal{S}} (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_{\mathcal{P}} \mathbf{v} \cdot d\mathbf{l}.}$$

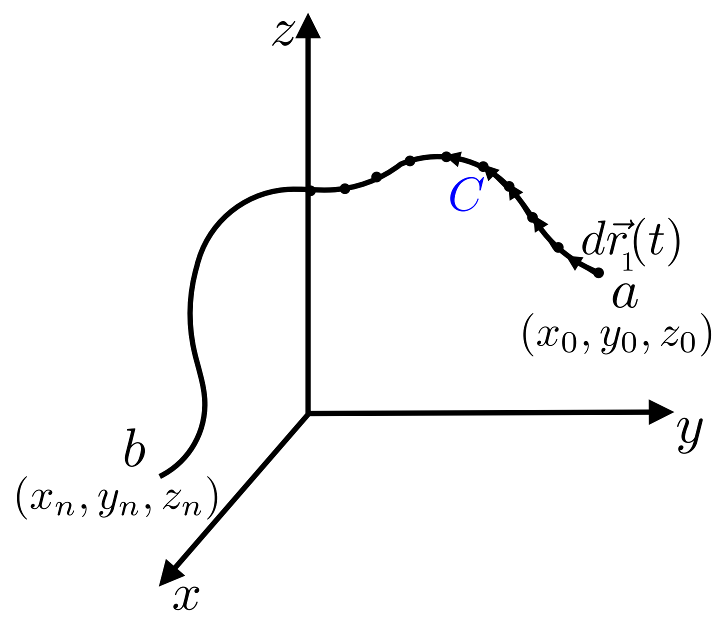
Fundamental theorem for divergences.
Also known as Gauss's theorem
or divergence theorem.

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{v})\, d\tau = \oint_{\mathcal{S}} \mathbf{v} \cdot d\mathbf{a}.}$$

These theorems are very important and have a wide range of applications in Electrodynamics. We will discuss them one by one in detail.

Fundamental theorem for gradients

- Scalar field $\phi(x, y, z)$.
- Start at a and move infinitesimal distance $d\vec{r}_1$ along path C to reach $\vec{r}_a + d\vec{r}_1$, $\vec{r}_a$ is position vector at a



Recall from first lecture that the scalar function ϕ will change by $d\phi = (\vec{\nabla}\phi).d\vec{r}_1$

- Now, move a little further, by small displacement $d\vec{r}_2$.
- The change in ϕ will be $\vec{\nabla}\phi.d\vec{r}_2$.
- In this manner, proceeding by infinitesimal steps we reach point b. At each step computing gradient of ϕ (at that point) and dot it into the displacement.
- This gives the total change in ϕ as

Fundamental theorem of calculus

$$\int_a^b \frac{df(x)}{dx}dx = f(b) - f(a)$$

$$\int_a^b (\vec{\nabla}\phi).d\vec{r} = \phi(b) - \phi(a)$$

Fundamental theorem for gradients

Like the ordinary fundamental theorem of calculus, it says that the integral (here line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (a and b)

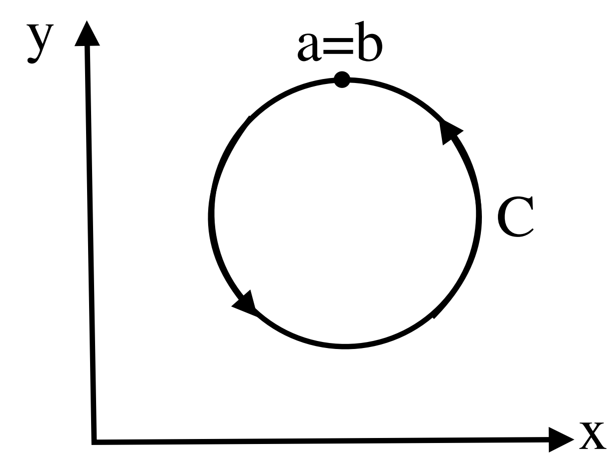
Some important corollaries

⋆ Gradients have the special property that their line integrals are path independent: $\int_a^b \vec{\nabla}\phi.d\vec{r}$ is independent of the path from a to b.

⋆ $\oint \vec{\nabla}\phi.d\vec{r} = 0$, since the beginning and end points are identical and hence $\phi(b) - \phi(a) = 0$.

Assume path independence and consider the closed path C in Figure 1. Since the starting point a and end point b are same, we get $\oint \vec{\nabla}\phi.d\vec{r} = \phi(b) - \phi(a) = 0$.

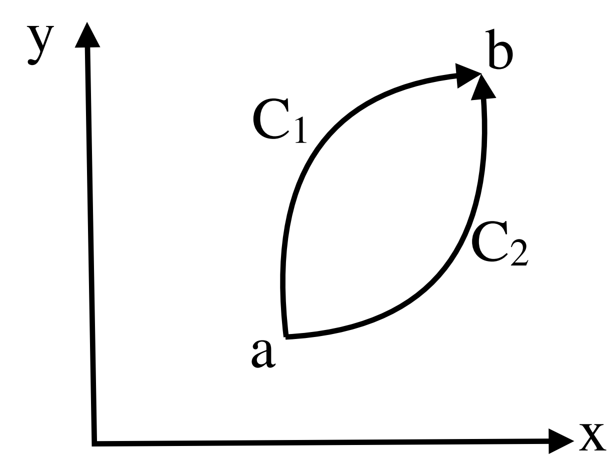
path independence $\Longrightarrow$ line integral around closed path=0



Assume $\oint \vec{\nabla}\phi.d\vec{r} = 0$ for any closed curve. If C_1 and C_2 are both paths between a and b, then $C_1 - C_2$ is a closed path. So by hypothesis

$$\oint_{C_1-C_2} \vec{\nabla}\phi.d\vec{r} = \int_{C_1} \vec{\nabla}\phi.d\vec{r} - \int_{C_2} \vec{\nabla}\phi.d\vec{r} = 0$$

$$\Longrightarrow \int_{C_1} \vec{\nabla}\phi.d\vec{r} = \int_{C_2} \vec{\nabla}\phi.d\vec{r}$$

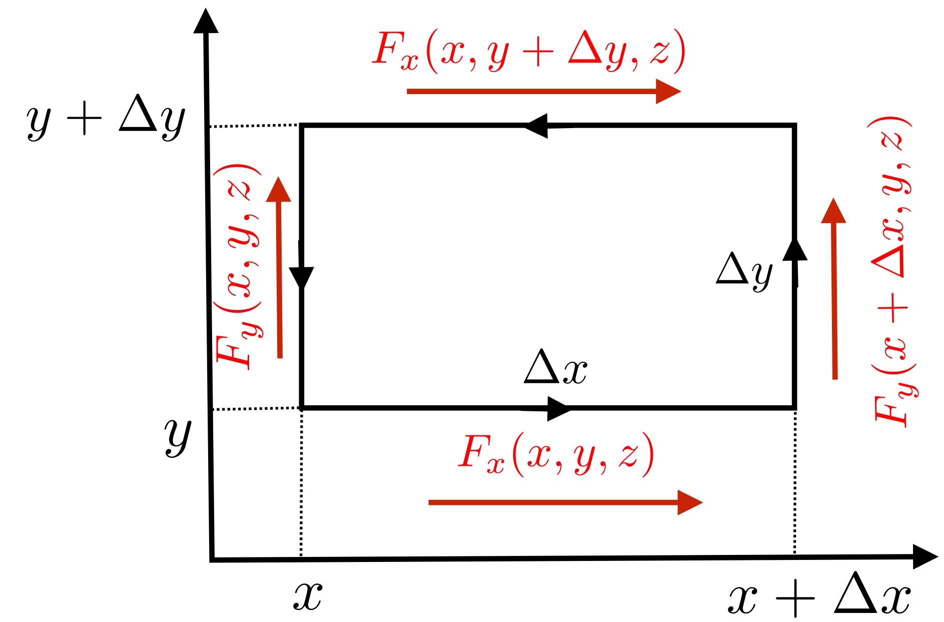


line integral around closed path=0 $\Longrightarrow$ path independence

Fundamental theorem for Curl (Stokes' Theorem)

Let us find the $\boxed{\text{circulation}}$ of a vector field $\vec{F}(x, y, z)$ around a closed curve C.

$$\boxed{\oint_C \vec{F}.d\vec{r}}$$



The fields in the x- direction at bottom and top are

$$F_x(x, y, z) \text{ and } F_x(x, y+\Delta y, z) = F_x + \frac{\partial F_x}{\partial y}\Delta y.$$

The fields in the y- direction at left and right are

$$F_y(x, y, z) \text{ and } F_y(x+\Delta x, y, z) = F_y + \frac{\partial F_y}{\partial x}\Delta x.$$

Summing around from bottom in anti-clockwise manner

$$\Delta C = \sum \vec{F}.\Delta\vec{r} = F_x(x, y, z)\Delta x + F_y(x+\Delta x, y, z)\Delta y - F_x(x, y+\Delta y, z)\Delta x - F_y(x, y, z)\Delta y$$

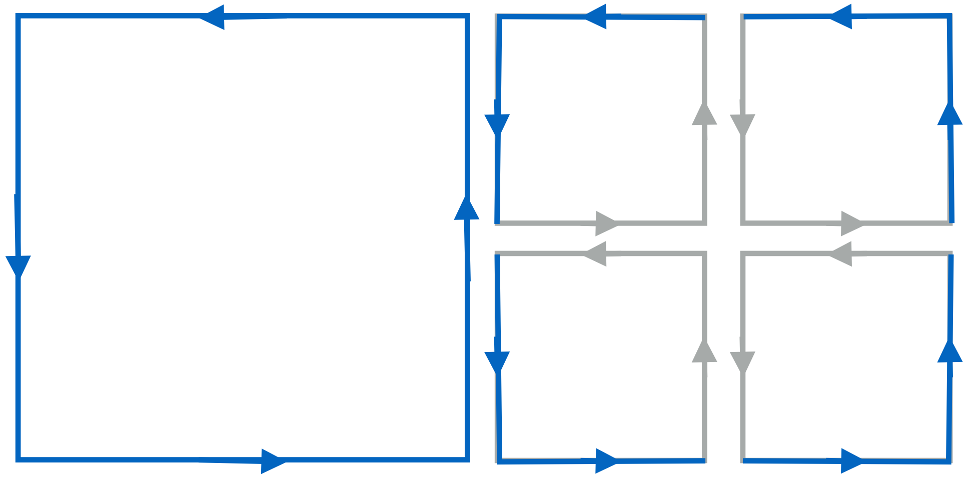
$$\begin{aligned} \sum \vec{F}.\Delta\vec{r} &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\Delta x\Delta y = (\vec{\nabla}\times\vec{F}).\Delta x\Delta y\hat{z} \\ &= (\vec{\nabla}\times\vec{F}).\Delta\vec{a} \end{aligned}$$

This implies that curl can be defined as circulation per unit area...

Fundamental theorem for Curl (Stokes' Theorem)

Now, if we add these little elementary loops together, the internal line sections cancel out because the $d\vec{r}$'s are in opposite directions, except on the bounding line.

This gives the larger bounding contour.



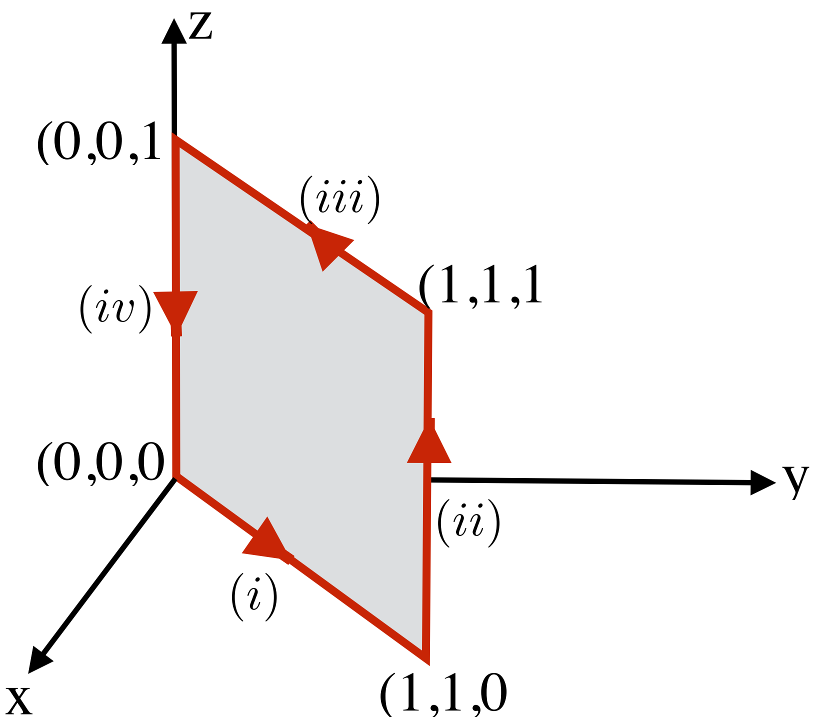
Stokes' Theorem: $$\boxed{\oint_C \vec{F}.d\vec{r} = \int_S (\vec{\nabla} \times \vec{F}).d\vec{a}}$$

Corollaries:

- $\int (\vec{\nabla} \times \vec{F}).d\vec{a}$ depends only on the boundary line, not on the particular surface used.

Example

Verify Stokes' theorem when $\mathcal{S}$ is the rectangle with vertices at $(0,0,0)$, $(1,1,0)$, $(0,0,1)$, and $(1,1,1)$, and $\vec{F} = yz\hat{x} + xz\hat{y} + xy\hat{z}$.



Direct Method:

Line integral $\oint_C \vec{F}.d\vec{r} = \oint_C yzdx + xzdy + xydz$
over path $(i) + (ii) + (iii) + (iv)$:

$\int_{(i)} \vec{F}.d\vec{r} = 0$, since $z = dz = 0$ on (i).

$\int_{(ii)} \vec{F}.d\vec{r} = \int_0^1 1.1dz = 1$, since $x = 1, y = 1$, $dx = 0 = dy$.

$\int_{(iii)} \vec{F}.d\vec{r} = \int_{(iii)} ydx + xdy = \int_1^0 xdx + xdx = -1$, since $y = x$, $z = 1, dz = 0$.

$\int_{(iv)} \vec{F}.d\vec{r} = 0$, since $x = 0,\ y = 0$, on (iv).

$$\boxed{\oint_C \vec{F}.d\vec{r} = \left(\int_{(i)} + \int_{(ii)} + \int_{(iii)} + \int_{(iv)}\right) \vec{F}.d\vec{r} = 0}$$

By Stokes' Theorem:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = \hat{x}(x - x) + \hat{y}(y - y) + \hat{z}(z - z) = 0$$

$$\boxed{\therefore \int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).d\vec{a} = 0}$$

Fundamental theorem for Divergence (Gauss's Theorem)

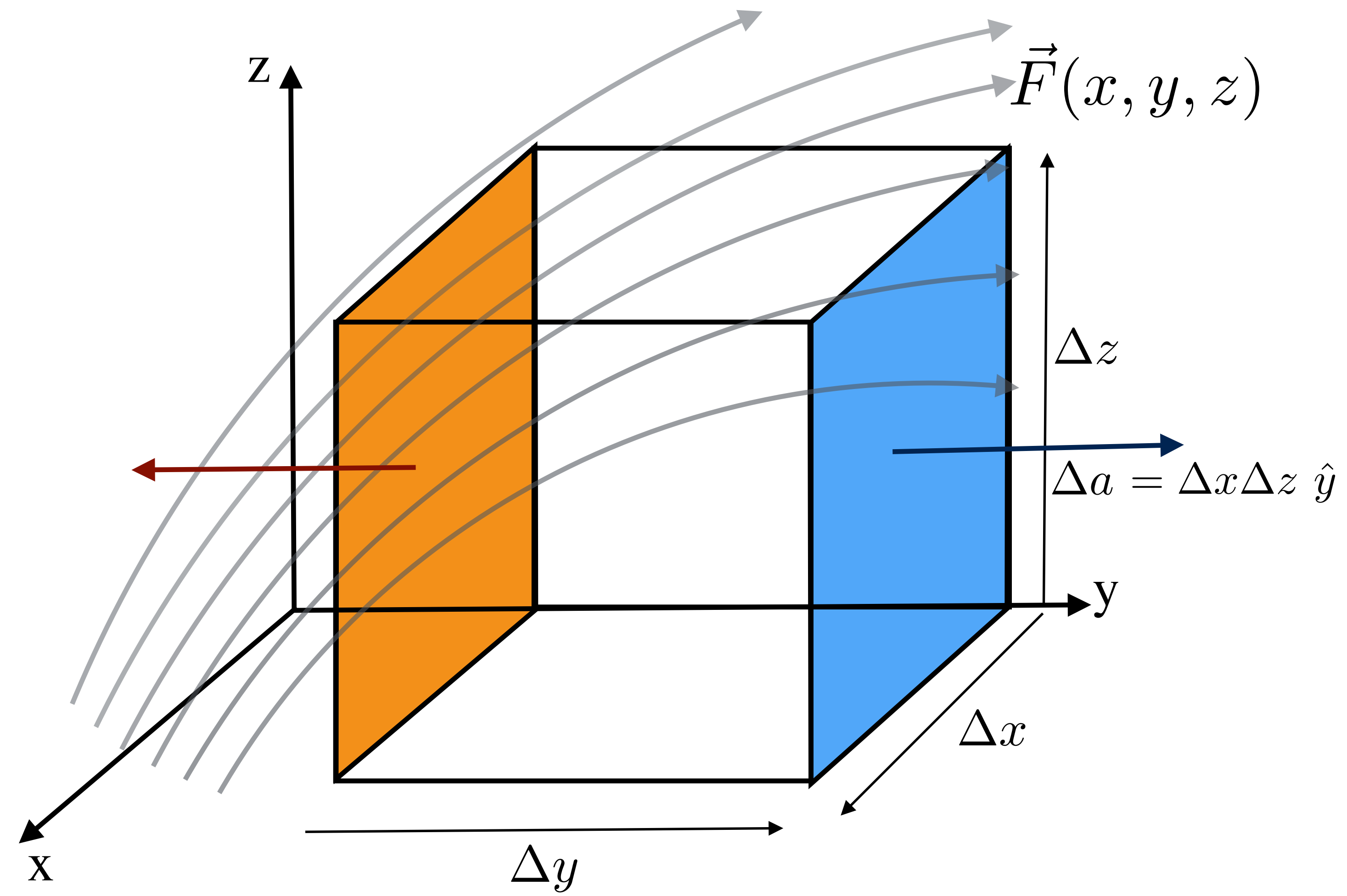
Consider a vector field $\vec{F}(x, y, z)$.

Say, flow of water...

We want to find out the flux of $\vec{F}(x, y, z)$.

To do that, take an infinitesimal volume element like a cube as shown.

..and look at the opposite faces (coloured in figure) first.

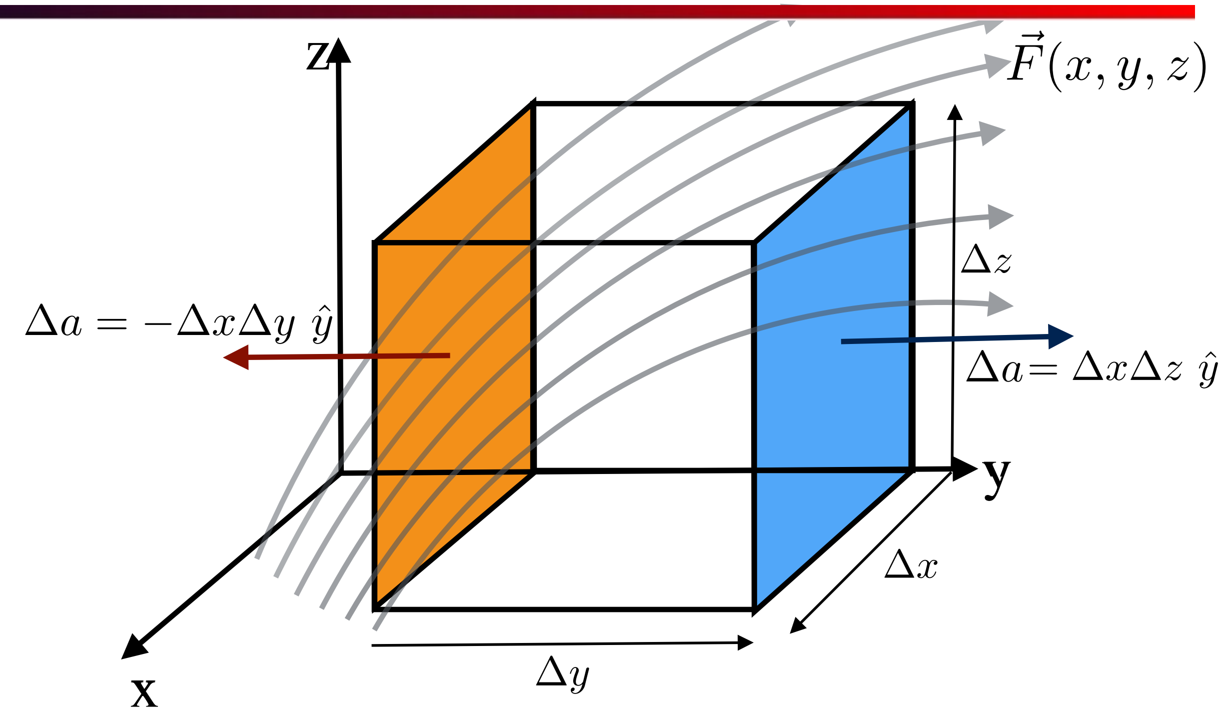


The left most face (orange) is at a fixed value of y and the incoming flux into this surface is $\vec{F}(x, y, z).\Delta\vec{a} = \vec{F}(x, y, z).\hat{n}\Delta a = (F_x\hat{x}+F_y\hat{y}+F_z\hat{z}).(-\Delta x\Delta z\hat{y}) = -F_y(x, y, z)\Delta x\Delta z$.

The right most face (blue) is at a fixed value of $y + \Delta y$ and the flux going out of this surface is $\vec{F}(x, y + \Delta y, z).\Delta\vec{a} = \vec{F}(x, y + \Delta y, z).\hat{n}\Delta a = (F_x\hat{x} + F_y\hat{y} + F_z\hat{z}).(\Delta x\Delta z\hat{y}) = F_y(x, y + \Delta y, z)\Delta x\Delta z$.

Fundamental theorem for Divergence (Gauss's Theorem)

So, the outward flux from the blue face is



$$\vec{F}(x, y+\Delta y, z).\Delta\vec{a} = F_y(x, y+\Delta y, z)\Delta x\Delta z$$
$$= \left(F_y + \frac{\partial F_y}{\partial y}\Delta y\right)\Delta x\Delta z$$

Net flux out of the box through these two opposite (orange and blue) faces

$$-F_y\Delta x\Delta z + \left(F_y + \frac{\partial F_y}{\partial y}\Delta y\right)\Delta x\Delta z = \frac{\partial F_y}{\partial y}\Delta y\Delta x\Delta z = \frac{\partial F_y}{\partial y}\Delta\tau$$

Net flux going out through the sides parallel to yz plane: $(\frac{\partial F_x}{\partial x})\Delta x\Delta y\Delta z$

Net flux going out through the sides parallel to xy plane: $(\frac{\partial F_z}{\partial z})\Delta z\Delta y\Delta x$

Summing over all the faces: $\vec{F}.\hat{n}\Delta a\Big|_{\text{all surfaces}} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)\Delta x\Delta y\Delta z = \vec{\nabla}.\vec{F}\Delta\tau$

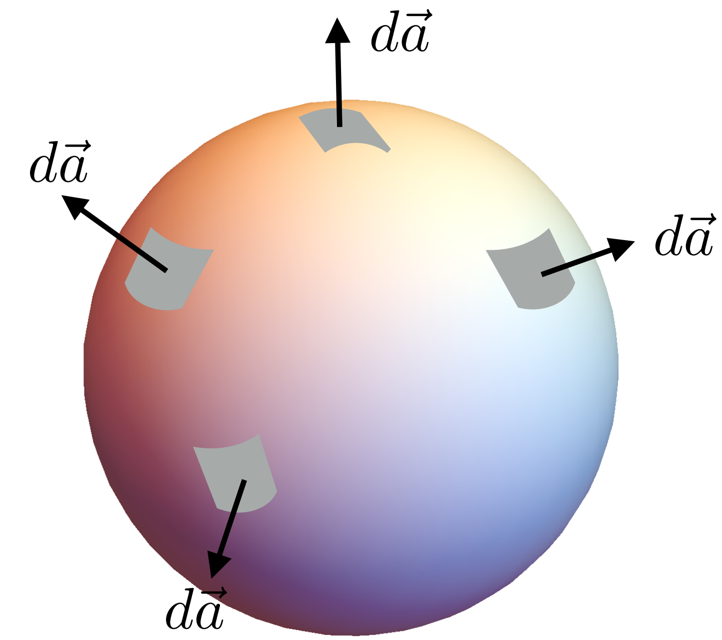
Flux over a closed surface can be written as a sum over the surfaces of elemental volumes that make the volume: $\sum\vec{F}.\hat{n}\Delta a = \sum(\vec{\nabla}.\vec{F})\Delta\tau$

In the limit $\Delta x, \Delta y, \Delta z \to 0$ $\quad \int_{\mathcal{V}}(\vec{\nabla}.\vec{F})d\tau = \oint_{\mathcal{S}}\vec{F}.\hat{n}da$ $\quad$ Gauss's Divergence Theorem

Fundamental theorem for Divergence (Gauss's Theorem)

What does it mean?

We want to find the total outward flux of the vector field $\vec{F}(\vec{r})$ across a surface $\mathcal{S}$ that bounds a volume $\mathcal{V}$: $\oint_{\mathcal{S}} \vec{F}.d\vec{a}$



$d\vec{a}$ is

- normal to the local surface element
- points everywhere out of the volume

Gauss's theorem tells us that we can calculate the flux of vector field across a surface $\mathcal{S}$, by considering the total flux generated inside the volume $\mathcal{V}$.

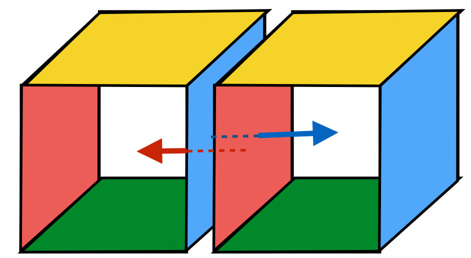
Gauss's Theorem: $$\boxed{\int_{\mathcal{V}} (\vec{\nabla}.\vec{F})d\tau = \oint_{\mathcal{S}} \vec{F}.\hat{n}da}$$

Volume integrals are easier than the surface integrals: computational efficiency!

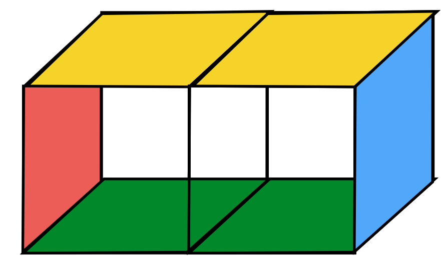
How to “see” this?

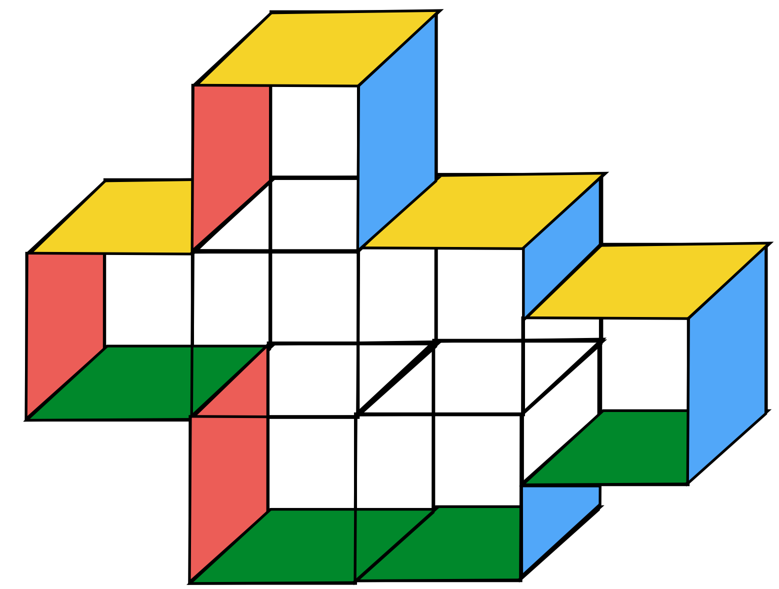
If we sum over the volume elements, this results in a sum over the surface elements!

Note: if two elementary surfaces touch, their $d\vec{a}$ vectors are in opposite directions!



Therefore the $d\vec{a}$ vectors cancel whenever there are two surfaces in touch





Thus the sum over surface elements gives the overall bounding surface!

$$\int_{\mathcal{V}} \vec{\nabla}.\vec{F}\, d\tau = \oint_{\text{Surface of } \mathcal{V}} \vec{F}.d\vec{a}$$

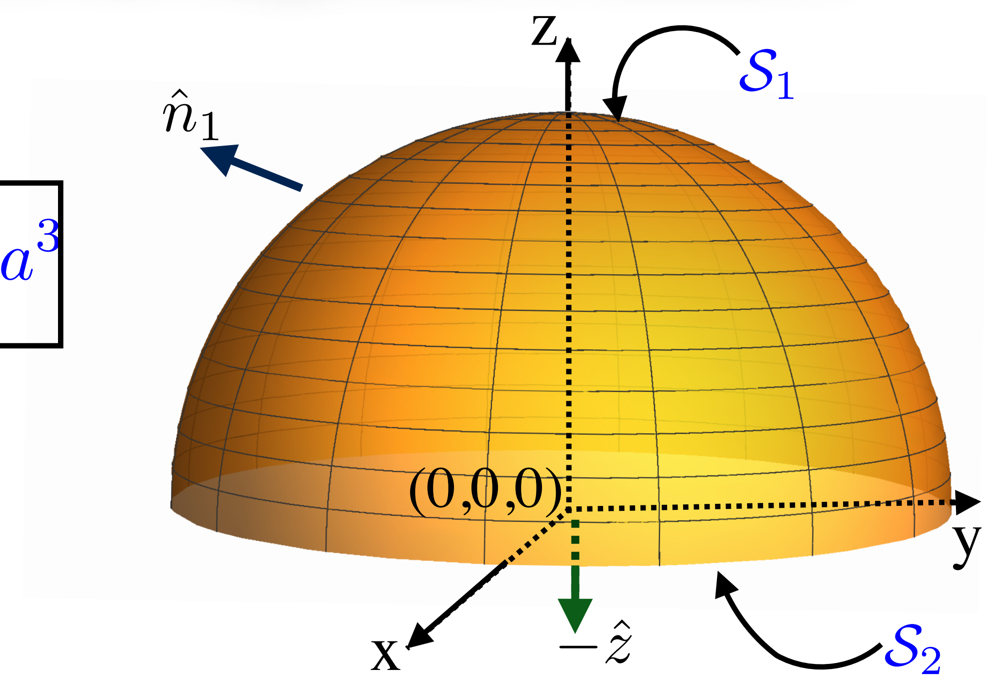
Seems reasonable! Because, the “boundary” of a line are its endpoints, and boundary of a volume is a closed surface!

Example

Verify the divergence theorem when $\vec{F} = x\hat{x} + y\hat{y} + z\hat{z}$ and $\mathcal{S}$ is the surface composed of the upper half of the sphere of radius a and centred at the origin, together with the circular disc in xy-plane centred at the origin and of radius a.

$$\vec{\nabla}.\vec{F} = \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) = 1 + 1 + 1 = 3$$

$$\boxed{\therefore \int_{\mathcal{V}} \vec{\nabla}.\vec{F}\, d\tau = 3.(\text{vol. of hemisphere}) = 3\,\frac{2}{3}\pi a^3 = 2\pi a^3}$$



To check the result, we need to calculate the surface integral of $\vec{F}$ over the closed surfaces $\mathcal{S}_1$ and $\mathcal{S}_2$.

Normal vector on $\mathcal{S}_1$ (hemisphere) : $\hat{n}_1 = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{\sqrt{x^2 + y^2 + z^2}} = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{a}$

Normal vector on $\mathcal{S}_2$ (disc at the base): $\hat{n}_2 = -\hat{z}$.

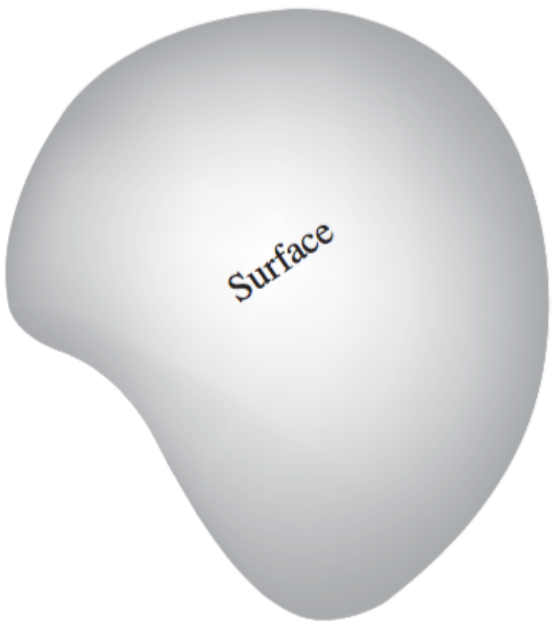
Surface integral for the flux through $\mathcal{S}_1 + \mathcal{S}_2$:

$$\oint_{\mathcal{S}_1+\mathcal{S}_2} \vec{F}.d\vec{a} = \int_{\mathcal{S}_1} \frac{x^2 + y^2 + z^2}{a} da + \int_{\mathcal{S}_2} \underbrace{(-z)da}_{= 0 \text{ because } z = 0 \text{ on } \mathcal{S}_2} = \int_{\mathcal{S}_1} a da$$

$$\boxed{\text{So, the value of the surface integral is } a(\text{area of } \mathcal{S}_1) = a(2\pi a^2) = 2\pi a^3.}$$

Summary

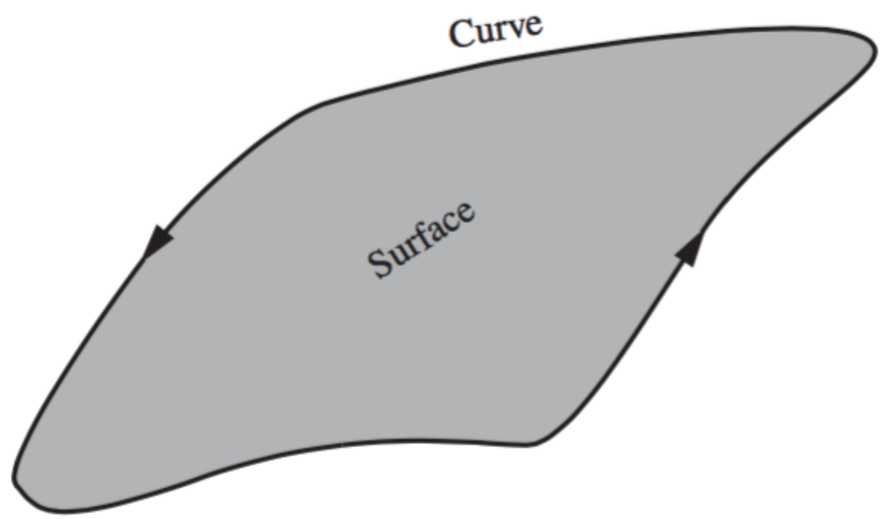
GAUSS



Surface encloses volume

$$\int_{\mathcal{S}} \vec{F}.d\vec{a} = \int_{\mathcal{V}} \vec{\nabla}.\vec{F} d\tau$$

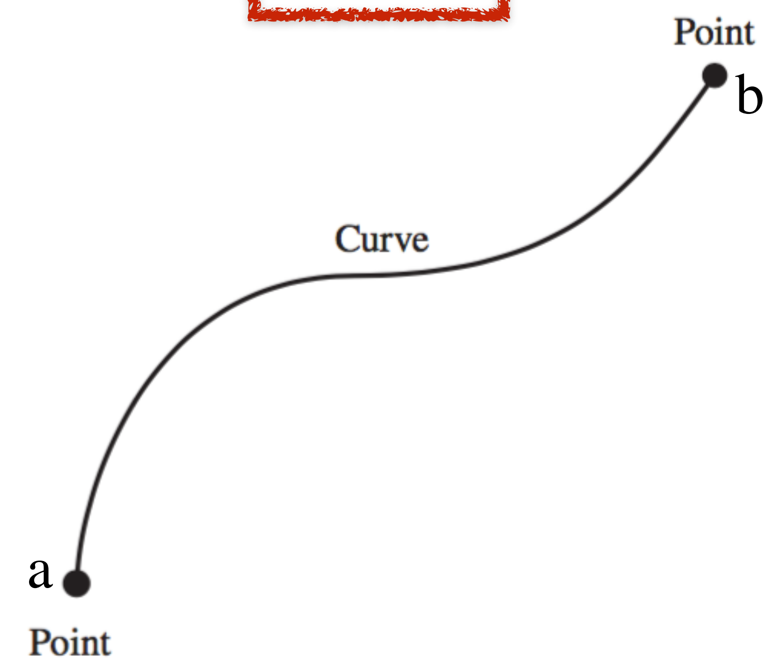
STOKES



Curve encloses surface

$$\oint_{\mathcal{C}} \vec{F}.d\vec{r} = \int_{\mathcal{S}} (\vec{\nabla} \times \vec{F}).d\vec{a}$$

GRAD



Points enclose curve

$$\int_{\mathcal{C}} \vec{\nabla}\phi.d\vec{r} = \phi(b) - \phi(a)$$

Remember:

In Cartesian Coordinates, with $\vec{\nabla} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)$

$$\vec{\nabla}.\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

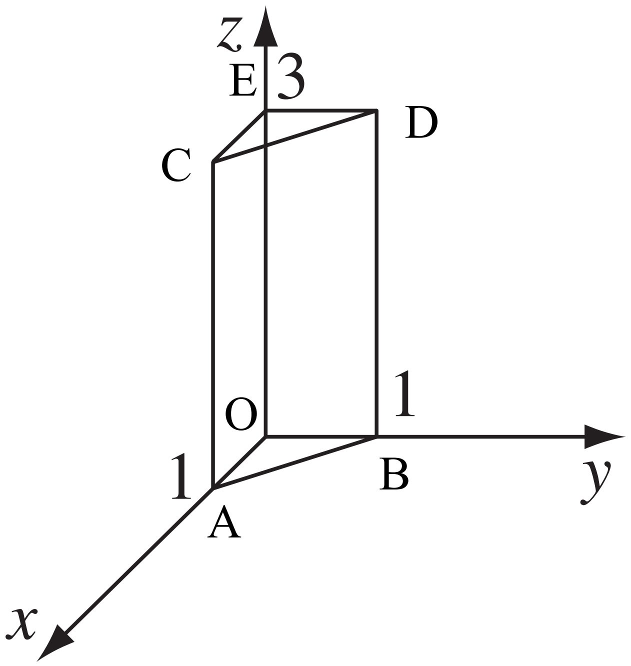
$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\vec{\nabla}\phi = \hat{x}\frac{\partial \phi}{\partial x} + \hat{y}\frac{\partial \phi}{\partial y} + \hat{z}\frac{\partial \phi}{\partial z}$$

Take home exercises

Question 1 : Calculate the volume integral of the scalar field $\phi(x, y, z) = x\, y^2\, z^3$ over the prism as shown in the figure.

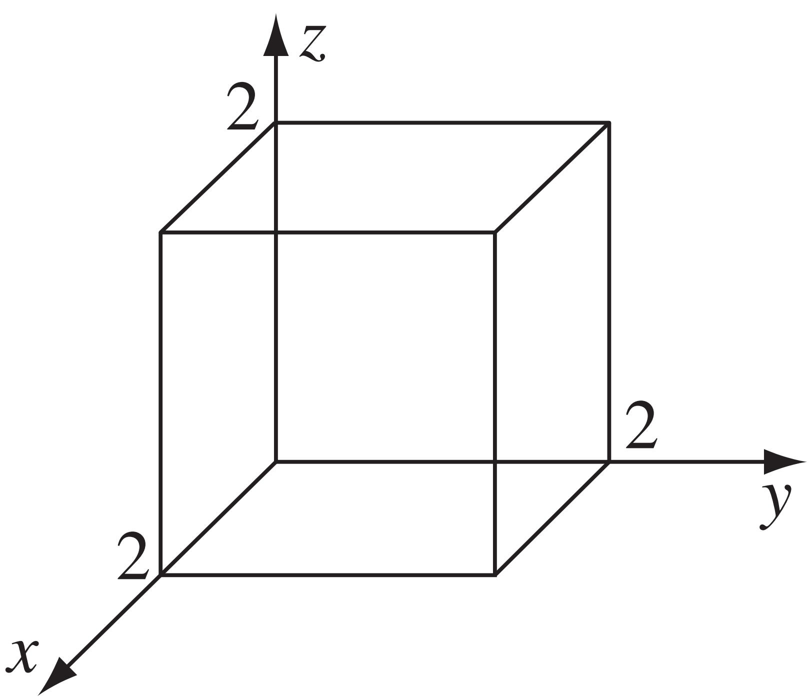
Hint : Use the constraint (x+y)=1 along AB or CD



Occasionally, we come across the volume integrals of vector functions over a certain volume. They are defined as

$$\int \mathbf{v}\, d\tau = \int (v_x\, \hat{\mathbf{x}} + v_y\, \hat{\mathbf{y}} + v_z\, \hat{\mathbf{z}}) d\tau = \hat{\mathbf{x}} \int v_x d\tau + \hat{\mathbf{y}} \int v_y d\tau + \hat{\mathbf{z}} \int v_z d\tau$$

Question 2: Given $\overrightarrow{F}(x, y, z) = y^2\, \hat{x} + (2xy + z^2)\, \hat{y} + 2yz\, \hat{z}$, calculate the volume integral of it over the cube at the origin with sides of length 2 (as shown below)



Thank You