MA 321 (Optimization)

Quiz-2

Time: 6:15 pm - 7:30 pm | October 31, 2023 | **Total marks:** 15

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1. Consider the optimal simplex tableau of the LPP (P) given below.

	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{b}$
$\mathbf{z_j - c_j}$	0	2	1	0	1	
x_1	1	2	1	0	1	8
s_1	0	3	-1	1	1	6

(a) Give an optimal BFS of the Dual of (P). If possible give a direction of Dual of (P).
Soln: $\mathbf{y}^T = [0,1] = c_B^T B^{-1}$ is the optimal extreme point of the Dual. Since $z_1 - c_1 = 0 = \mathbf{y}^T\tilde{\mathbf{a}}_1 - c_1 = s_1'$, $z_2 - c_2 = 2 = \mathbf{y}^T\tilde{\mathbf{a}}_2 - c_2 = s_2'$ and $z_3 - c_3 = 1 = \mathbf{y}^T\tilde{\mathbf{a}}_3 - c_3 = s_3'$ where s_i' are variables added to the Dual constraints, the corresponding BFS is $[0,1,0,2,1]^T$.
The dual is of the form Min $\mathbf{b}^T\mathbf{y}$
subject to $A^T\mathbf{y} \geq \mathbf{c}$ or $\mathbf{y}^T A \geq \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$.
Since the above table is optimal, with $z_j - c_j$ values ≥ 0, $\mathbf{y}^T A \geq \mathbf{c}$, $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$.
Since $(B^{-1})_{.1}A \geq \mathbf{0}$ so $(\mathbf{y}^T + \alpha(B^{-1})_{.1})A \geq \mathbf{c}$, and $(\mathbf{y}^T + \alpha(B^{-1})_{.1}) \geq \mathbf{0}$ for all $\alpha \geq 0$, hence $(B^{-1})_{.1} = [0,1]$ written as a column is a direction of the dual of (P).

[1+1]

(b) If the constraint $x_1 - x_2 + 2x_3 \leq 3$ is added to (P) then find the optimal solution of the **new** (P) by using the Dual simplex algorithm.
Soln:
The updated table after adding the constraint is given by:

	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{s}_3$	$B^{-1}\mathbf{b}$
$\mathbf{z_j - c_j}$	0	2	1	0	1	0	
x_1	1	2	1	0	1	0	8
s_1	0	3	-1	1	1	0	6
s_3	0	-3	1	0	-1	1	-5

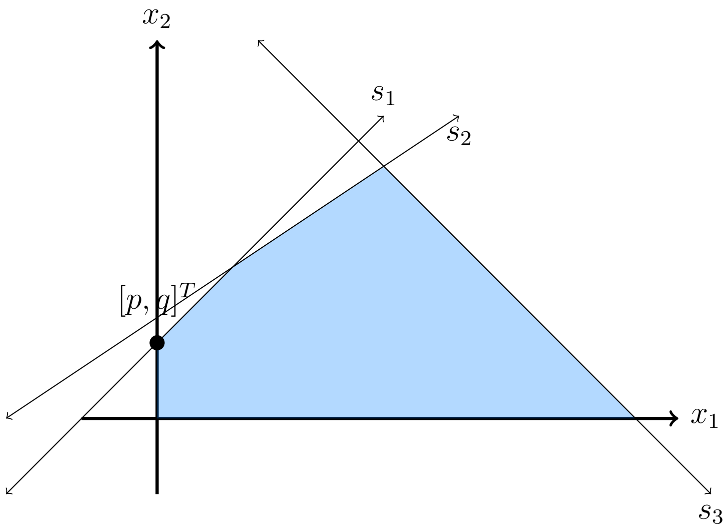
According to the Dual Simplex Algorithm, s_3 is the leaving variable and x_2 is the entering variable since $\min\{|\frac{c_j - z_j}{u_{rj}}| : u_{rj} < 0\} = \frac{2}{3}$ for x_2.
The next table is given by:

	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{s}_3$	$B^{-1}\mathbf{b}$
$\mathbf{z_j - c_j}$	0	0	$\frac{4}{3}$	0	$\frac{2}{3}$	$\frac{2}{3}$	
x_1							$\frac{14}{3}$
s_1							1
x_2	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$

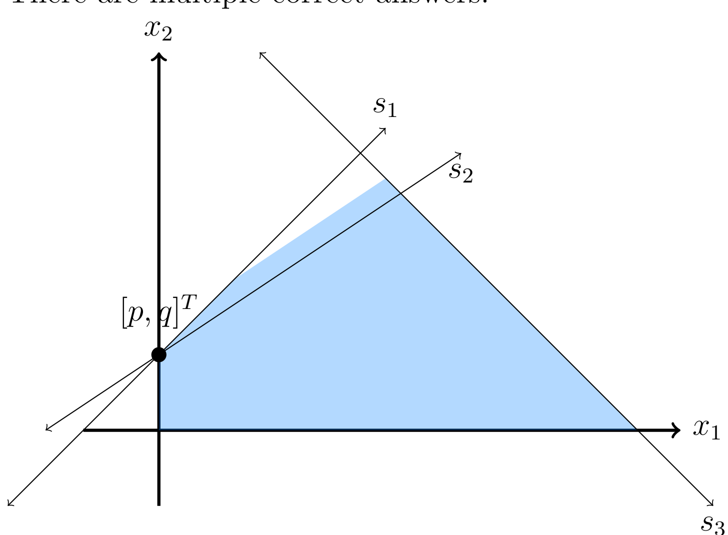
The above table is optimal and the optimal solution is given by: $[x_1, x_2, x_3]^T = [\frac{14}{3}, \frac{5}{3}, 0]^T$.

[3]

2. Consider the LPP (P) of the form,
   Minimize $\mathbf{c}^T\mathbf{x}$, subject to $A_{3\times 2}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$,
   with the shaded feasible region given below and the hyperplanes corresponding to each of the added variables $\mathbf{s}_i$, marked accordingly:



   (a) If both (P) and the Dual of (P) have unique optimal solutions and $[p,q]^T$ is optimal for (P) then if possible by changing **exactly** one entry of $\mathbf{b}$ get $\mathbf{b}'$ such that $[p,q]^T$ is optimal for the **new** (P) and the Dual of **new** (P) has multiple optimal solutions (only give the picture of the new feasible region of (P) if it is possible). **Justify**.
   Soln:
   There are multiple correct answers.



   Since the $c_j - z_j$ values do not change due to this change in $\mathbf{b}$ and $[p,q]^T$ is still feasible for the new problem, hence $[p,q]^T$ is still optimal for the new (P).
   Since $[p,q]^T$ was the unique non degenerate optimal solution of the old (P) so all the $c_j - z_j$ values corresponding to the non basic variables are strictly positive in the optimal table.
   In the new (P) shift the line corresponding to s_2 such that $[p,q]^T$ becomes a degenerate BFS with one entry of $B^{-1}b$ equal to 0..
   Then the corresponding BFS of the Dual $[\mathbf{y_0}^T, \mathbf{s}_0'^T]^T$ will be non degenerate (since $c_j -$

$z_j > 0$ corresponding to the non basic variables) and one $c_j - z_j$ value corresponding to a nonbasic variable will be equal to 0 in that table of the BFS of the Dual. By entering that nonbasic variable at a positive value (which is possible since BFS $[\mathbf{y}_0{}^T, \mathbf{s}_0'{}^T]^T$ is non degenerate) to get new feasible solution $\mathbf{y}'$ will give alternate optimal solutions of the Dual,

comes from the formula

$\mathbf{b}^T\mathbf{y}_{\prime} = \mathbf{b}^T\mathbf{y}_0 + y_s'(c_s - z_s) = \mathbf{b}^T\mathbf{y}_0.$

(b) If possible give a BFS of the Dual of (P) which has a column with all entries ≤ 0 in its corresponding simplex table and also give the non basic variable (of the Dual) corresponding to that column. **Justify**.

Soln: There are multiple solutions for this problem:

For example the BFS which corresponds to the intersection point of $x_2 = 0$ and $s_3 = 0$. If we move along the edge which corresponds to nonbasic x_2 or the edge corresponding to nonbasic s_3 one can check that eventually in both these cases the hyperplane $x_1 = 0$ obstructs indefinite movement, hence in both these two columns the entry corresponding to the basic variable x_1 is positive.

Hence the row corresponding to basic variable x_1 in the corresponding table is non negative which gives an **extreme** direction of the Dual. Since x_1, s_1, s_2 are positive for this BFS, by complementary slackness the corresponding BFS of the dual has non basic variables as s_1', y_1 and y_2 the rest are basic variables.

The non positive column in the Dual is corresponding to the nonbasic variable s_1' which corresponds to the basic variable x_1 of the Primal.

Similarly if we consider the BFS which corresponds to the intersection point of $x_1 = 0$ and $x_2 = 0$ that is $[0, 0]^T$ then we get that the row corresponding to basic variable s_3 in the corresponding table is non negative which gives an **extreme** direction of the Dual. Since s_1, s_2, s_3 are positive for this BFS, the corresponding BFS of the dual has non basic variables as y_1, y_2 and y_3 the rest are basic variables.

The non positive column in the Dual is corresponding to the nonbasic variable y_3 which corresponds to the basic s_3 of the Primal.

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[3+3]

3. Does there exist a balanced transportation problem with 4 supply stations and 4 destinations with a BFS such that all the non basic columns in the corresponding simplex table has the same number of non zero entries? If yes, then give an example and **justify**. If no, then **justify**.

Soln: Yes for example the one given below:

					a_i
	10	10	10	10	40
	10				10
	10				10
	10				10
d_j	40	10	10	10	

The above gives a BFS for the TP with basic cells $\{(1,1),(1,2),(1,3),(1,4),(2,1),(3,1),(4,1)\}$. Since for any nonbasic cell (p,q), $B \cup \{(p,q)\}$ has a θ-loop with exactly 4 cells, so any non basic column for this BFS has exactly 3 nonzero entries, two +1 and one -1. [4]

No credit will be given for answers without justification