Sequences

- Any function whose domain is the entire set of positive integers is called an **infinite sequence**.
- Example, $f(n) = 1/2n$ ($n = 1, 2, 3, \ldots$).
- The terms of the sequence are: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$

If s is an infinite sequence its terms are denoted by, $s_1, s_2, s_3, \ldots s_n, \ldots$

We use the notation $\{s_n\}_{n=1}^{\infty}$ or, $\{s_n\}$.

A sequence $\{s_n\}$ is said to **converge** to a number s if s_n is arbitrarily close to s for all n sufficiently large. $\lim_{n \to \infty} s_n = s$. A sequence which does not converge to any real, finite number is said to **diverge**.

Series

- **A finite geometric series with quotient** k is given by

$S_n = a + ak + ak^2 + \cdots + ak^{n-2} + ak^{n-1}$

Example: A man keeps 100 rupees in the savings deposit of a bank fetching him rate of interest of 10% per year.

In the first year his balance is 100.

In the second year his balance = 100+100(10%) = 100(1+0.1)

In the third year, his balance = 100(1+0.1)+100(1+0.1)(0.1) = 100(1+0.1)(1+0.1) = 100 $(1+0.1)^2$

In the 20-th year, this is, 100 $(1+0.1)^{20-1}$

- His balance over the years is, 100+100(1+0.1)+100 $(1+0.1)^2$+...+100 $(1+0.1)^{19}$
- This is same as the expression above, with a = 100, k = 1.1, n = 20
- $S_n = a + ak + ak^2 + \cdots + ak^{n-2} + ak^{n-1}$
- Also, $kS_n = ak + ak^2 + \cdots + ak^{n-1} + ak^n$
- So, $kS_n - S_n = ak^n - a$
- Or, $S_n = \frac{a(k^n - 1)}{k-1}$, for $k \neq 1$

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- For infinite geometric series summation,

$a + ak + ak^2 + \cdots ak^{n-1} + \cdots to\ \infty = \frac{a}{1-k}$, provided $|k| < 1$

Or, $\sum_{n=1}^{\infty} ak^{n-1} = \frac{a}{1-k}$

This is the case when the series **converges**.

If, $|k| \geq 1$, then the series **diverges**. A divergent series has no finite sum.

Continuity and differentiability: Implications

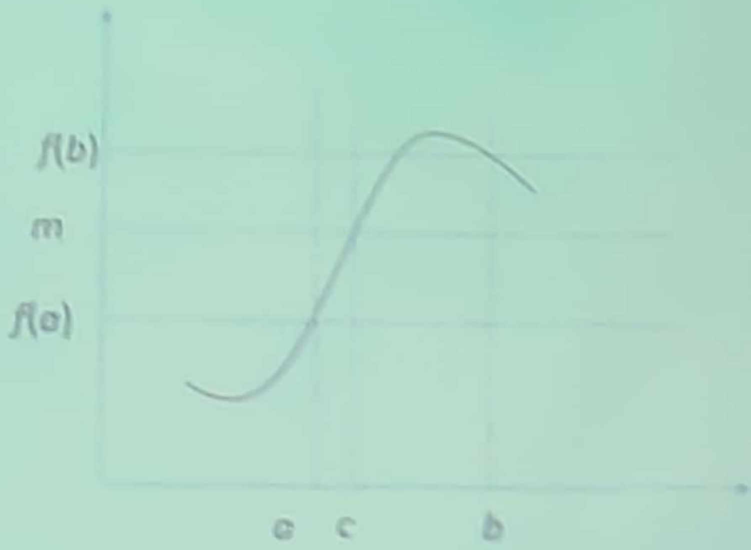
- Here are some of the implications of the properties of continuity and differentiability which have a bearing on the optimization techniques.
- **The intermediate-value theorem:**

Let f be a function that is continuous at all x in the closed interval

$[a, b]$, and assume that $f(a) \neq f(b)$. As x varies between a and b, so $f(x)$ takes on every value between $f(a)$ and $f(b)$.

The implication of it is that the function f must intersect the line $y = m$ at at least one point (c, m), where $f(c) = m$, as shown in the diagram.

A useful consequence of the Intermediate-value theorem is: let $f(a)$ and $f(b)$ have different signs, then there is at least one $c \in (a, b)$ such that $f(c) = 0$.



Example: the equation $x^6 + 3x^2 - 2x - 1 = 0$ has at least one solution between 0 and 1, because suppose $f(x) = x^6 + 3x^2 - 2x - 1$, then $f(0) = -1$, $f(1) = 1$.

The extreme-value theorem

Extreme points:

If $f(x)$ is defined over the domain D then

$c \in D$ is a **maximum point** for f if and only if $f(x) \leq f(c)$ for all $x \in D$

$d \in D$ is a **minimum point** for f if and only if $f(x) \geq f(d)$ for all $x \in D$

$f(c)$ and $f(d)$ are called **maximum value** and **minimum value** respectively.

Extreme-value theorem: If a function f is continuous in a closed, bounded interval $[a, b]$, then f attains both a maximum value and a minimum value in $[a, b]$.

- The theorem is intuitively appealing.
- If the conditions are not satisfied then extreme-values may not lie in the interval.
- If the function asymptotically approaches infinity for some value of x in the interval, then it is not continuous, hence there is not maximum in the interval. For example, $y = 1/x$, is not continuous at $x = 0$. It has no maximum of minimum in the interval [-1, 1].
- If the function is continuous but defined in an open interval, say $y = x$, defined in (0, 1). This has no maximum or minimum in the interval.

- Let f be defined in an interval I and let c be an interior point of I (not an end point I). If c is a maximum or a minimum point of f and if $f'(c)$ exists then, $f'(c) = 0$.
- Since c is a maximum point, for $h>0$ sufficiently small, $f(c+h) - f(c) \leq 0$.

Or, $\frac{f(c+h)-f(c)}{h} \leq 0$

LHS is the Newton quotient. As $h^+ \to 0$, this approaches $f'(c)$.

So, $f'(c) \leq 0$

On the other hand, if we take h be to negative then $\frac{f(c+h)-f(c)}{h} \geq 0$.

As $h^- \to 0$, $f'(c) \geq 0$.

Hence, $f'(c) = 0$.

- Such a point where $f'(c) = 0$, is called a **stationary point**.

The mean-value theorem

If f is continuous in the closed bounded interval $[a, b]$ and differentiable in the open interval (a, b), then there exists at least one interior point z in (a, b) such that, $f'(z) = \frac{f(b)-f(a)}{b-a}$

The implication is, for a continuous and differentiable function defined in an interval, at some point in the interval the slope of the tangent to the graph equals the slope of the line connecting the endpoints on the graph.

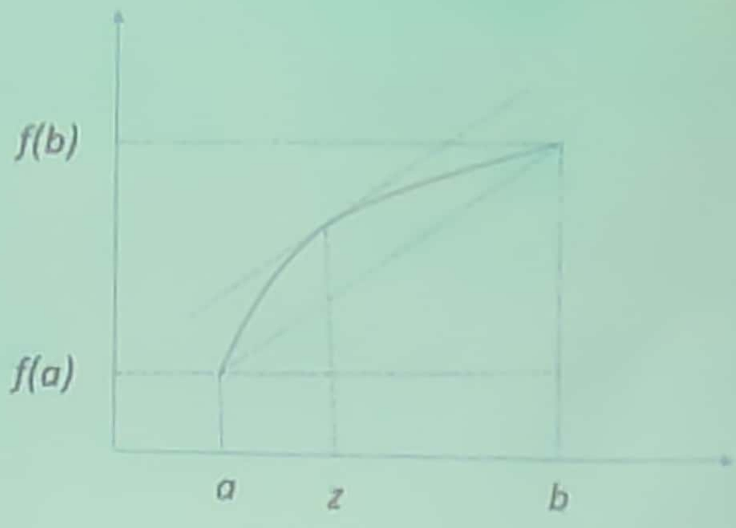
Few definitions:

1. If $f(x_1) \leq f(x_2)$ whenever $x_1 \leq x_2$, then f is **increasing.**

2. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is **strictly increasing.**

3. If $f(x_1) \geq f(x_2)$ whenever $x_2 \geq x_1$, then f is **decreasing.**

4. If $f(x_1) > f(x_2)$ whenever $x_2 > x_1$, then f is **strictly decreasing.**



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Using the mean-value theorem one can show the following.

Let f be a function continuous in the interval I and differentiable in the interior of I,

a. If $f'(x) > 0$ for all x in the interior of I, then f is **strictly increasing** in I.

b. If $f'(x) < 0$ for all x in the interior of I, then f is **strictly decreasing** in I.

Similarly for increasing and decreasing functions.

In a similar vein, if $f'(x) = 0$ for x in the interior of I, then f is **constant** in I.

Taylor's formula

- We have seen the n-th order Taylor polynomial,

$$f(x) \approx f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + .. + \frac{f^{(n)}(0)}{n!}(x)^n$$

This is not however very useful because it's only an approximation, there is an error term involved = the difference between $f(x)$ and $f(0) + \frac{f'(0)}{1!}(x) + \frac{f''(0)}{2!}(x)^2 + .. + \frac{f^{(n)}(0)}{n!}(x)^n$.

Below is the Taylor's formula which takes care of this.

$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + .. + \frac{1}{n!}f^{(n)}(0)x^n + \frac{1}{(n+1)!}f^{(n+1)}(c)x^{n+1}$, for some c between 0 and x.

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- For example for $n = 3$, the Taylor formula is,

$$f(x) = f(0) + \frac{1}{1!}f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f'''(c)x^3$$

$$= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \frac{1}{6}f'''(c)x^3$$

L'Hôpital's rule for 0/0 form

- Suppose f and g are differentiable in an interval (α, β) around a except possibly at a, and suppose that $f(x)$ and $g(x)$ both tend to 0 as x tends to a. If $g'(x) \neq 0$ for all $x \neq a$ in (α, β), and if $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ (L is finite, $L = \infty$, or $-\infty$), then
- $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = L$
- This is true also if $f(x)$ and $g(x)$ both tend to ∞, or $-\infty$

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Find: $\lim\limits_{x \to \infty} \frac{1-3x^2}{5x^2+x-1}$

As x goes to infinity, the numerator goes to minus infinity, the denominator goes to plus infinity.

We use the L'Hôpital's rule, $\lim\limits_{x \to \infty} \frac{1-3x^2}{5x^2+x-1}$

$= \lim\limits_{x \to \infty} \frac{-6x}{10x+1} = \frac{-\infty}{\infty}$

Use the L'Hôpital's rule once more, $\lim\limits_{x \to \infty} \frac{-6x}{10x+1}$

$= \lim\limits_{x \to \infty} \frac{-6}{10}$

$= -3/5$

Exponential functions

- Exponential functions have the **general form**, $y = a^x$.
- They are often used in economics and finance to deal with economic growth, compound interest rate, discounting of future profits, etc.
- An exponential function with base a is denoted by,

$$f(x) = a^x$$

- As x changes by 1 unit, the value of the function changes by the factor a.
- What is the derivative of this function?

- The Newton quotient is given by, $\frac{f(x+h)-f(x)}{f(x)}$

$= \frac{a^{x+h}-a^{x}}{h} = a^{x}.\frac{a^{h}-1}{h}$

- As h goes to zero, $f'(x) = a^{x}.\frac{a^{h}-1}{h}$
- At $x = 0$, $f'(0) = \frac{a^{h}-1}{h}$
- Or, $f'(x) = a^{x}.f'(0)$
- Derivative of the function for all value of x exists **if $f'(0)$ exists.**
- $f'(0) = \frac{f'(x)}{f(x)}$, the proportional change is invariant with respect to x, is an important quantity.

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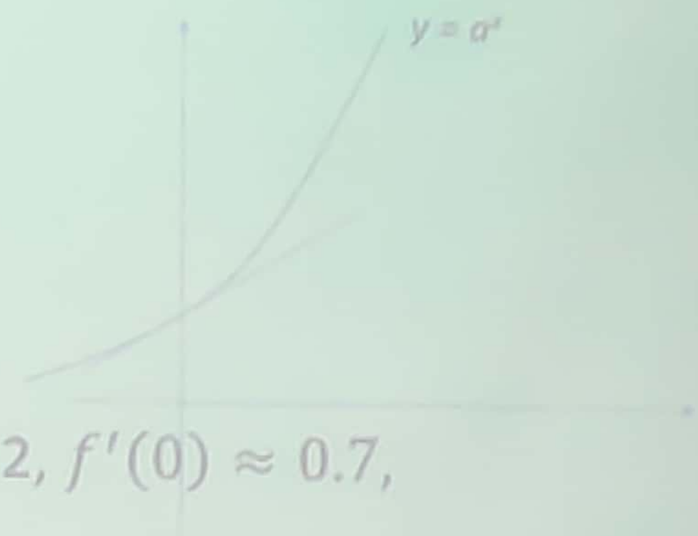
- $f'(0)$ is the slope of the function $y = a^x$ at $x = 0$.



- We know, $f'(0) = \frac{a^h - 1}{h}$ is a function of a.
- As a rises, $f'(0)$ rises.
- One can geometrically calculate that at $a = 2$, $f'(0) \approx 0.7$,

And at $a = 3$, $f'(0) \approx 1.1$

- Invoking the intermediate-value theorem, at some value of a between 2 and 3, $f'(0) = 1$.
- This particular value of a, is an irrational number, and has been given the name, e. $e = 2.71828...$

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- Since $f'(x) = a^x . f'(0)$ and for $a = e$, $f'(0) = 1$, therefore, $f'(x)/f(x) = f'(0) = 1$, or $f'(x) = f(x) = e^x$
- This is called the **natural exponential function**.
- For the natural exponential function, the slope at $x = 0$ is 1.

Logarithmic functions

- Suppose $f(t) = a^t$, we want to find the time it takes for the value of the function to double, or treble, etc.
- Example: I have 1000 rupees in the bank which pays me 5% rate of interest annually, how long will it take for the money to become 10,000 rupees?
- We need to solve, $1000\,(1+0.05)^T = 10{,}000 \Rightarrow (1.05)^T = 10$
- In general the solution of such problems involves solving an equation of the form, $a^x = b$
- Let us take the natural exponential base, e.
- Thus, $e^x = b$

- Solve for x: (i) $7e^{-2x} = 16$, (ii) $B\theta e^{-ax} = p$

(i) $7e^{-2x} = 16$

Or, $e^{-2x} = 16/7$

Or, $\ln e^{-2x} = \ln 16/7$

Or, $-2x = \ln 16/7$

Or, $x = -1/2(\ln 16/7) = \frac{1}{2} \ln \frac{7}{16}$

(ii) $B\theta e^{-ax} = p$

Or, $e^{-ax} = \frac{p}{B\theta}$

Or, $\ln e^{-ax} = \ln \frac{p}{B\theta}$

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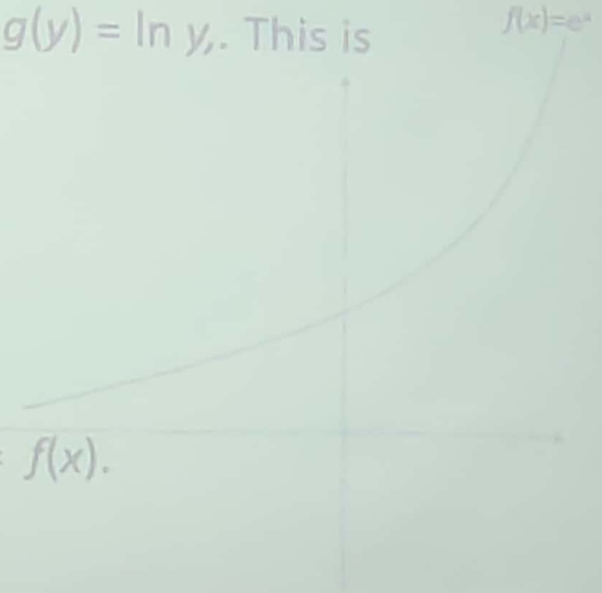
Or, $-\alpha x \ln e = \ln \frac{p}{B\theta}$

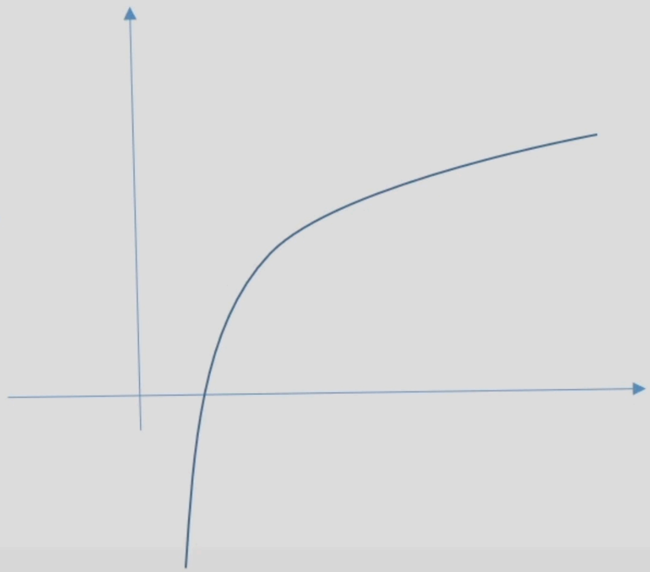
Or, $x = -\frac{1}{\alpha} \ln \frac{p}{B\theta} = \frac{1}{\alpha} \ln \frac{B\theta}{p}$

If $y = f(x) = e^x$ one can define a function, $x = g(y) = \ln y$,. This is illustrated here.

The graph of $y = e^x$ intersects the y-axis at 1. For each positive value of y, one can read the corresponding value of x, from the graph. The function $x = g(y)$ is defined for positive values of y. It's the inverse of $y = f(x)$.



- One can turn graph around and get the natural logarithmic function as a rising, concave to the x-axis function, $y = g(x) = \ln x$.
- $\ln 1/e = -1$,
- $\ln e = 1$
- $\ln 1 = 0$



- Taking derivative of $x = e^{g(x)}$ where $g(x) = \ln x$, we get,

$1 = g'(x).\ e^{g(x)}$

Or, $g'(x) = 1/\ e^{g(x)} = 1/x$,

- Thus, derivative of ln x is $1/x$.
- Since this is positive, it implies the first derivative of natural logarithmic function is always positive.
- The second derivative is negative ($-1/x^2$).
- If we have a composite function like, $y = \ln u$, where $u = f(x)$, then

$$y' = \frac{1}{u}\frac{d}{dx}u = \frac{f'(x)}{f(x)}$$

Logarithmic differentiation: An example:

Suppose, $y = x^r(bx - c)^p$, to find y'.

Taking the natural logarithm of both sides,

$\ln y = r\ln x + p\ln(bx - c)$

- Taking implicit differentiation of both sides with respect to *x*,

$$\frac{y'}{y} = r\frac{1}{x} + p\frac{b}{bx - c}$$

Or, $y' = y\left(r\frac{1}{x} + p\frac{b}{bx-c}\right) = x^r(bx-c)^p\left(r\frac{1}{x} + p\frac{b}{bx-c}\right)$

Elasticity of *y* with respect to *x* is given by

$e_x^y = x\frac{y'}{y}$.

Find the elasticity of $y = e^x$, $y = \ln x$

From $y = e^x$ we get, $\ln y = x$

Or, $\frac{y'}{y} = 1$

Or, $e_x^y = x\frac{y'}{y} = x$

From $y = \ln x$ we get,

$y' = 1/x$

Or, $e_x^y = x\frac{y'}{y} = x\left(\frac{1}{x}\right)(1/\ln x\,) = 1/\ln x$

Logarithm with other bases:

If for any fixed positive number a, $a^x = b$, we call ***x* to be the logarithm of *b* to base *a***. $x = \log_a b$.

In general, $a^{\log_a x} = x$.

Example: $\log_{10} 1000 = \log_{10} 10^3 = 3$

- From $a^{\log_a x} = x$, taking ln of both sides, we get, $\log_a x = \frac{\ln x}{\ln a}$
- Thus log of any number with base *a* is proportional to the natural log of the same number.
- The rules of ln are applicable to log as well (product, quotient, etc.).

Characterization of *e*

- It can be shown that, $\lim_{h \to 0} \ln(1 + h)^{1/h} = 1$
- Which implies $\lim_{h \to 0} (1 + h)^{1/h} = e$
- Or, $\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$

- Log linear relation:

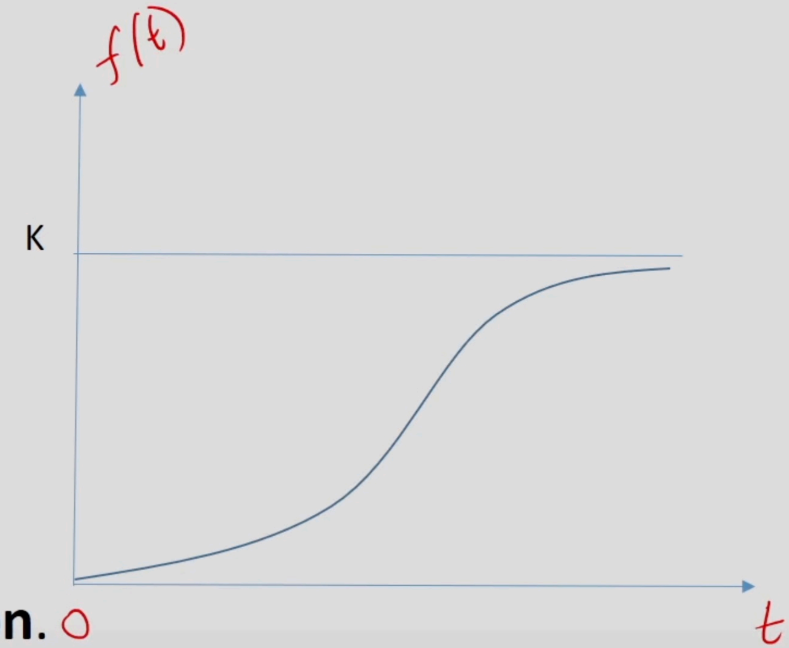
Suppose y and x are related as follows:

$y = Ax^b$

Taking the log of both sides,

$\log y = \log A + b \log x$

log of y and log of x here are linearly related. This is called a **log linear relation**.



y can be an exponential function of x: $y = Ab^x$

Taking log, $\log y = \log A + x \log b$

Here the log of y is linear of function x, with $\log b$ as the slope.

- Elasticities are closely related to logarithmic differentiation.
- Elasticity of y with respect to x = $e_x^y = \frac{xdy}{ydx} = \frac{\frac{dy}{y}}{\frac{dx}{x}} = \frac{dlny}{dlnx} = \frac{d \log_a y}{d \log_a x}$
- For a log linear relation, log y = log A + b log x, b therefore is the elasticity.
- That is, b in $y = Ax^b$ is the elasticity.

- Using Taylor's formula we can also show for some c between x and 0,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}e^c$$

Applications

- Let, there is a function, $f(t) = Ae^{rt}$, for this kind of functions, $f'(t) = rf(t)$, for all t.
  In other words, for such functions the proportional rate of change, $\frac{f'(t)}{f(t)} = r$, a fixed quantity.
- The opposite is also true, if the proportional rate of change is constant, the form of the function is as given above.
- Suppose, $f(t)$ denotes the population of a country at time t.
- Let $f'(t)/f(t)$ be called the per capita growth of population.
- Now, $f'(t)/f(t) = r$ implies a very simplistic model of population growth, where the population rises exponentially all the time.

- It might be more realistic to assume that population starts to decline above a particular limit, the carrying capacity of the environment.
- $f'(t) = rf(t)(1 - \frac{f(t)}{K})$
- Here, for $f(t)$ very small compared to *K,* the carry capacity, population rises exponentially. But as $f(t)$ approaches *K* it slows down.
- It can be shown that, $f(t) = \frac{K}{1+Ae^{-rt}}$
- As, $t \to \infty, f(t) \to K\ (A > 0)$. This is called a logistic function.

- It might be more realistic to assume that population starts to decline above a particular limit, the carrying capacity of the environment.
- $f'(t) = rf(t)(1 - \frac{f(t)}{K})$
- Here, for $f(t)$ very small compared to K, *carrying capacity*, population indeed rises exponentially. But as $f(t)$ approaches K it slows down.
- It can be shown that, $f(t) = \frac{K}{1+Ae^{-rt}}$
- As, $t \to \infty, f(t) \to K$ $(A > 0)$. This is called a **logistic** function.

Interest Periods, Effective Rates

- The **interest period** is the time which elapses between successive dates when interest is added to an account (let us say, savings bank account). Usually it is one year, but it can vary. Many US bank accounts add interest daily, monthly.
- If a bank offers 10% annual rate of interest with interest payment each month, 10/12 = 0.83% of the principal accrues at the end of each month.
- The annual rate is divided by the number of interest periods to get the **periodic rate**, which is the rate of interest per period.
- Suppose, S_0 is the principal which yields r rate of interest per period (say a year). Then after t periods, it will increase to $S_t = S_0(1+r)^t$

- Suppose, the interest is paid twice a year now. After half year the principal will increase to, $S_0(1+r/2)$, hence after a year it will increase to $S_0(1+r/2)^2$.
- Thus, after t periods, it will increase to $S_0(1+r/2)^{2t}$.
- Since, $(1+r/2)^2 = 1+r+r^2/4 > 1+r$, for a lender biannual interest payment at the rate r/2 is a better deal than an annual interest payment at the rate r.
- In general, a year can be divided in n equal periods, after each period r/n interest is added to the principal. After t years, the principal will have increased to, $S_0(1+\frac{r}{n})^{nt}$.
- Example: How long will it take for 5000 rupees to double to 10000 rupees, if it is invested with annual interest rate 10% paid quarterly?

- Suppose it takes T periods to double. Then we have,
- $5000(1+\frac{0.10}{4})^T = 10000$

Or, $(1+0.025)^T = 2$

Or, T.ln(1.025) = ln2
Or, T = 0.6931/0.0247 = 28.07

28.07 quarters mean a little more than 7 years.

- **Effective interest rate**: When interest is added n times during the year at the rate of r/n per period, then the **effective yearly rate** R is defined as, $R = (1+\frac{r}{n})^n - 1$. R is rising in n.
- Example: There are two investment schemes, 5.9% with interest paid quarterly, or 6% with interest paid twice a year. Which scheme is better for the investor?

- Using the formula for effective interest rate, the R's in the two cases are as follows.
- R = $(1+\frac{0.059}{4})^4-1 \approx 0.0603$
- R = $(1+\frac{0.06}{2})^2-1 \approx 0.0609$
- Thus the second scheme is better for the investor.

- **Continuous compounding:** We have seen that if r annual interest rate is paid after each of n periods within a year, after t years the principal rises to, $S_0(1+\frac{r}{n})^{nt}$. Let Let $n = rm$, then the sum is,

$$S_0(1+1/m)^{rmt} = S_0[(1+1/m)^m]^{rt}$$

As $n \to \infty, m = \frac{n}{r} \to \infty$, thus $(1 + 1/m)^m \to e$.

Thus the sum of money goes to, $S_0 e^{rt}$ as *n* approaches infinity. This is called **continuous compounding of interest**.

$$S_t = S_0\, e^{rt}$$

As we know, for this function, $\dot{S}_t / S_t = r$

The proportional rate of change is constant for the deposit in this case, given by the rate of interest. The principal rises by a constant relative rate.

Each year, the principal gets multiplied by e^r:

because, $\frac{S_{t+1}}{S_t} = \frac{S_0 e^{r(t+1)}}{S_0 e^{rt}} = \frac{S_0 e^{rt} e^r}{S_0 e^{rt}} = e^r$

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- In case of continuous compounding the effective interest rate is defined as, $R = e^r - 1$. Since $(1 + \frac{r}{n})^n < e^r$, as long as n is finite, continuous compounding gives the best return to the investor.

Example:

- The value of a machine depreciates continuously at 6% per year. How long will it take for the value to halve?
- Let the value after t periods be, $v_t = v_0 e^{-0.06t}$, where v_0 is the initial value of the machine.
- Let T be the time at which the value exactly halves. Then,

$$e^{-0.06T} = 0.5$$

Or, $-0.06T = \ln(0.5)$

Or, T = 0.6931/0.06 ≈ 11.55. Thus, it will take about 11 and half years.

Present discounted value

- Suppose 100 rupees is available to a man today. He invests it in a business which fetches 20% return per year.
- After 5 years the money will accumulate to $100(1+0.2)^5 = 100(1.2)^5 = 100(2.49) = 249$ rupees.
- Thus any given amount of money today is equivalent to more money at a future date. This is a fundamental principle of financial economics.
- In the above example 100 rupees is the **present value** of 249 rupees five years later, at 20% rate of return per year.
- It is also called the **present discounted value (PDV)** of 249 rupees because, after all, 100 rupees is less than 249 rupees.