$$\approx b(x(t)) + \sigma_b(x(t))\,[W(u) - W(t)] \qquad \forall\, u \in [t, t+h]$$

~~Since we~~ $\int_0^t dx(t) = \int_0^t a(x(u))\,du + \int_0^t b(x(u))\,dW_u$

$$\int_t^{t+h} b(x(u))\,dW_u = \int_t^{t+h} (b(x(t)) + \sigma_b(x(t))\,dW_u$$

$$= b(x(t))\,[W(t+h) - W(t)] + \sigma_b(x(t)) \underbrace{\int_t^{t+h} [W(u) - W(t)]\,dW_u}$$

$$\underbrace{\int_t^{t+h} W(u)\,dW_u}_{Y(t+h) - Y(t)} - W(t)\,[W(t+h) - W(t)]$$

where

$$\begin{cases} Y(t) = \int_0^t W(t)\,dW_t & dY(t) = W(t)\,dW_t \\ Y(0) = 0 \end{cases}$$

from Ito's formula, $Y(t) = \frac{1}{2} W^2(t) - \frac{1}{2} t$

$$\int_t^{t+h} b(x(u))\,dW_u = b(x(t))\,[W(t+h) - W(t)] + \Big[\tfrac{1}{2} W^2(t+h) - \tfrac{1}{2}(t+h) - \tfrac{1}{2} W^2(t)$$

$$+ \tfrac{1}{2} t - W(t)\,W(t+h) + W^2(t)\Big]\,\sigma_b(x(t))$$

$$= b(x(t))\,[W(t+h) - W(t)] + \Big[\tfrac{1}{2}\big[W^2(t+h) + W^2(t)\big] - \tfrac{1}{2} h$$

$$- W(t)\,W(t+h)\Big]\, b'(x(t))\, b(x(t))$$

$$= b(x(t))\,[W(t+h) - W(t)] + \frac{1}{2}\Big[[W(t+h) - W(t)]^2 - \frac{1}{2} h\Big]\,\sigma_b(x(t))$$

$$= b(x(t))\,[W(t+h) - W(t)] + \tfrac{1}{2} b'(\,)\,b(\,\cdot\,)\,\big[\{W(t+h) - W(t)\}^2 - h\big]$$

$$X(t+h) = a(X(t))\,h + b(X(t))\,\underbrace{[W(t+h) - W(t)]}_{\sqrt{h}\, z_i} + \frac{1}{2} b'(\,)\, b(\,) \Big[\underbrace{\{W(t+h) - W(t)\}^2}_{h z_i^2} - h\Big]$$

for each z_i in $z_i \in N(0,1)$ we get a random walk

$E\left[\|X(T) - X(t_n)\|\right] \le Ch^p$ — strong

$\| E(X(T)) - E(X_n) \|$ — weak

Strong & Weak Convergences :

Since the sol^n of SDE indeterministic Geomtric BM we can define the convergence in strong & weak sense. We can calculate the error.

$$\{\hat{X}(0), \hat{X}(h), \hat{X}(2h), \dots \hat{X}(nh)\}$$

T , $n = \lfloor T/h \rfloor$

$$E\left[\|\hat{X}(nh) - X(T)\|\right], \quad E\left[\|\hat{X}(nh) - X(T)\|^2\right]$$

$$E\left[\sup_{0\le t\le T} \|\hat{X}(\lfloor t/h \rfloor h) - X(T)\|\right]$$

Rather we can consider the weak error criteria for any polynomial

$f \in \mathcal{C}_p^{2\beta+2}$

$$\| E(f(\hat{X}(nh))) - E(f(X(T))) \|$$

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→ strong convergence

$$E\left[\| \hat{X}_n - X(T) \|\right] \leq C h^{p} \quad \text{——} \quad ①$$

weak

$$\| E\left[f(\hat{X}_n) - f(X(T))\right] \| \leq C h^{\beta} \quad \text{——} \quad ②$$

$f_i \in \underbrace{C_P^{2\beta+2}}$

which consicts of functions $\mathbb{R}^d \to \mathbb{R}$ whose derivatives of order ~~$2\beta+2$~~ $0, 1, 2, \cdots 2\beta+2$ are polynomially bounded

A function $g()$ is said to be polynomially bounded if

$$|g(x)| \leq k\left(1 + \|x\|^q\right) \quad \forall x \in \mathbb{R}^d, \quad k, q - \text{constants}$$

In ② C may depend on polynomial $f(\cdot)$

* Lower order strong convergence schemes maybe of higher order weak convergence.

→ Second-order schemes:

$$dX(t) = a(X(t))\,dt + b(X(t))\,dW_t$$

$$\int_0^T dX(t) \Rightarrow X(t) = X(0) + \int_0^T a + \int_0^T b\,dW_t$$

$$\mathcal{L}^0 = a\frac{d}{dx} + \frac{1}{2} b^2 \frac{d^2}{dx^2}$$

$$\mathcal{L}^1 = b \cdot \frac{d}{dx}$$

$$\int_t^{t+h} \Rightarrow X(t+h) = X(t) + \int_t^{t+h} \underbrace{a(X(u))}\,du + \int_t^{t+h} b(X(u))\,dW_u$$

if we take $X(t)$ instead of $X(u)$ we got earlier schemes.

$u \in [t, t+h]$

$$a(X(u)) = a(X(t)) + \int_t^u \mathcal{L}^0(a(X(s)))\,ds + \int_t^u \mathcal{L}^1(a(X(s)))\,dW_s \quad \text{———} (*)$$

$$= a(X(t)) + \mathcal{L}^0(a(X(t)))\,\cancel{(u-t)}\int_t^u ds + \mathcal{L}^1(a(X(t)))\,\cancel{[W(u)-W(t)]}\int_t^u dW_s$$

$$b(X(u)) = b(X(t)) + \int_t^u \mathcal{L}^0(b(X(s)))\,ds + \int_t^u \mathcal{L}^1(b(\,))\,dW_s \quad \text{———} (**)$$

$$X(t+h) = X(t) + \int_t^{t+h} a(X(u))\,du + \int_t^{t+h} b(X(u))\,dW(u)$$

$$= X(t) + \int_t^{t+h} a(X(t))\,du + \int_t^{t+h}\int_t^u \mathcal{L}^0(a(X(s)))\,ds\,du + \int_t^{t+h}\int_t^u \mathcal{L}^1(\,)\,dW_s\,du$$

$$+ \int_t^{t+h} b(X(t))\,dW_u + \int_t^{t+h}\int_t^u \mathcal{L}^0(b(X(s)))\,ds\,dW_u + \int_t^{t+h}\int_t^u \mathcal{L}^1(b(\,))\,dW_s\,dW_u$$

$$\int_t^{t+h} a(X(u))\,du = a(X(t))h + \mathcal{L}^0(a(X(t)))\underbrace{\int_t^{t+h}\int_t^u ds\,du}_{I_{(0,0)}} + \mathcal{L}^1(a(\,))\underbrace{\int_t^{t+h}\int_t^u dW(s)\,du}_{I_{(1,0)}}$$

$$\int_t^{t+h} b(X(u))\,dW(u) = b(X(t))[W(t+h) - W(t)] + \mathcal{L}^0(b(\,))\int_t^{t+h}\int_t^u ds\,dW(u)$$

$$+ \mathcal{L}^1(b(\,))\underbrace{\int_t^{t+h}\int_t^u dW(s)\,dW(u)}_{I_{(1,1)}}$$

$I_{(0,0)} =$

$I_{(1,0)} =$

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$f(\,) < 1$

$\rightarrow$ $Ax = b$ $\quad \|A\| < 1$

$x = A^{-1}b$

$\{x^{(n)}\} \rightarrow \tilde{x}$
$n \rightarrow \infty$

$x^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$

$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$

$\Rightarrow x_1^{(k+1)} = \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}]$

Gauss-Seidal $\rightarrow$ SOR

Jacobi Iterative method

$x_2^{k+1} = \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}]$

Successive Over Relaxation

$x_3^{(k+1)} = \frac{1}{a_{33}} [b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}]$

M - pre-conditioner

$Ax = b$

$Mx = (M-A)x + b$ —— (1)

$\uparrow$
Non-singular

$x = (I - M^{-1}A)x = M^{-1}b$

$$\boxed{x^{(k+1)} = \underbrace{(I - M^{-1}A)}_{B}\, x^{(k)} + M^{-1}b}$$

$A = D - L - U$

D → diagonal matrix, L → lower Δ^{le}, U → upper Δ^{le}

$\boxed{\rho(B) < 1}$ $\qquad \rho(B) = \max|\lambda_i|$

$\rho(B^*B)$

$\Updownarrow$

$\|B\|_2 < 1$

$\|B\|_\infty < 1$

In Jacobi: $M = D$

In Gauss-sidal:

Consider $M = D$ (diagonal matrix) then $M - A = L + U$

$(1) \Rightarrow Dx = (L+U)x + b$

Jacobi $\quad Dx^{(k+1)} = (L+U)x^{(k)} + b$

$$x^{(k+1)} = \underbrace{D^{-1}(L+U)}_{B_J}x^{(k)} + D^{-1}b \qquad \text{iterative matrix } B$$

$\rho(B) < 1$

Consider $M = D - L \Rightarrow M - A = U$

$$(D-L)x^{(k+1)} = Ux^{(k)} + b$$

$$x^{(k+1)} = \underbrace{(D-L)^{-1}U}_{B_{GS}} \cdot x^{(k)} + (D-L)^{-1}b$$

Inorder to speed up the convergence of Gauss-sidal, we can introduce a parameter ω_R

SOR $\leftarrow M = \frac{1}{\omega_R}D - L \Rightarrow M - A = \left(\frac{1}{\omega_R} - 1\right)D + U$

$$\left(\frac{1}{\omega_R}D - L\right)x^{(k+1)} = \left(\left(\frac{1}{\omega_R} - 1\right)D + U\right)x^{(k)} + b$$

$$x^{(k+1)} = \underbrace{\left(\frac{1}{\omega_R}D - L\right)^{-1}\left(\left(\frac{1}{\omega_R} - 1\right)D + U\right)}_{B_{SOR}} x^{(k)} + b$$

The stopping criteria

$$\|x - \tilde{x}\| < TOL$$

$$r^{(n)} = b - Ax^{(n)}$$

$$\|r^{k}\| \to 0 \quad n \to \infty$$

$$\|r^{(n)}\| < TOL = 10^{-8}$$

Cauchy seq $\|x^{(n)} - x^{(n-1)}\| \to 0$

Optimal value of $\omega_{opt} = \dfrac{2}{\sqrt{1 + \sqrt{1 - \rho(B_J)^2}}}$

In case of American options, we have imposed the side conditions $y \geq g$ at each iteration

Correction vector : $x^{(k)} - x^{(k-1)}$

$$r_i^{(k)} = b - \sum_{j=1}^{i} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}$$

$$x_i^{(k)} = x_i^{(k-1)} + \omega_R \frac{r_i^{(k)}}{a_{ii}}$$

$$Ax = \hat{b} = b - Ag$$

PSOR (Projected SOR) $\quad x_i^{(k)} = \max\left\{0, x_i^{(k-1)} + \omega_R \frac{r_i^{(k)}}{a_{ii}}\right\}$

$$y_i = -r_i^{(k)} + a_{ii}\left(x_i^{(k)} - x_i^{(k+1)}\right)$$

Since the Crayer's problem has a unique minimum which ensures the convergence of PSOR

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$\rightarrow$ FEM

$$\begin{cases} -u'' = f, & x \in (0,1) = \Omega \\ u(0) = 0 = u(1) \end{cases}$$

$u \in C_0^\infty(\Omega)$

$$-\int u'' v \, dx = \int f v \, dx$$

$$= -\cancel{(uv)}\Big|_0^1 + \int_0^1 u'v' \, dx = \int f v \, dx$$

$u'v'$ - weak sense

$u \in L^p(\Omega)$ Lebesgue $\qquad L^p(\Omega) = \{u \in \Omega : \int_\Omega |u(x)|^p dx < \infty\}$

find $u \in H^1(\Omega)$

$$\left. \int u'v' \, dx = \int f v \, dx \quad \forall v \in H(\Omega) \right\} \text{weak formulation}$$

$u \in L^2$, $u' \in L^2$, $u \in H^1(\Omega)$

$$H^1(\Omega) = \{u \in L^2(\Omega) ; u' \in L^2(\Omega)\}$$

$\rightarrow u \in C^k(\Omega)$, $\quad v \in C_0^\infty(\Omega)$ $\qquad$ ($D^\alpha u(x)$ - weak derivative of u)

$$\int_\Omega D^\alpha u(x) \, v(x) \, dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha v(x) \, dx \quad \text{--- (1)} \qquad |\alpha| \leq k, \ \forall v \in C_0^\infty(\Omega)$$

$u \in L^1(\Omega)$

$$\int_\Omega w_\alpha(x) v(x) \, dx = (-1)^{|\alpha|} \int_\Omega u(x) D^\alpha v(x) \, dx \quad \forall v \in C_0^\infty(\Omega) \quad \text{--- (2)}$$

In (2) u may or may not be ~~co~~ differentiable

But from (1) & (2) we get $w_\alpha(x) = D^\alpha u(x)$ (weak derivative always exist)

Comparing the RHS of ① & ② $w_\alpha = D^\alpha u$. Since $u \in L^1(\ )$, the derivative is weak derivative.

Ex: $\Omega = (-1,1)$, $u(x) = |x|$

$$\int_{-1}^{1} |x| v\,dx = \int_{-1}^{0} (-x) v\,dx + \int_{0}^{1} (x) v\,dx$$

→ For $L^p(\cdot)$ functions (bounded, measurable funcns)

$k > 0$, $p \in [1, \infty)$

w_p^k → k upto k-th derivative, p → L^p

$$w_p^k(x) = \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega), |\alpha| \leq k \}$$

is known as "Sobolev space"

$$\|u\|_{w_p^k(\Omega)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^p(\Omega)} \right)^{1/p}, \quad 1 \leq p < \infty$$

$$\left(w_p^k(\Omega), \|\cdot\| \right)$$

When $p=2$, the norm will be defined as inner product.

$$H^k(\Omega) = W_2^k(\Omega) = \{ u \in L^2, D^\alpha u \in L^2 \}$$

↙ Hilbert space

$$(u,v) = \sum_{|\alpha| \leq k} (D^\alpha u, D^\alpha v)$$

$p=2$, $k=1$

$$H^1(\Omega) = \{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_j} \in L^2(\Omega) \} \quad j = 1, 2, \cdots, n$$

Here u_x denotes the weak derivative of u which belong to L^2

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2(P)} + \sum_{j=1}^{n} \left| \frac{\partial u}{\partial x_j} \right|^2 \right)^{1/2}$$

$H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0, \partial\Omega\}$ $\qquad \partial\Omega$ - boundary

$\rightarrow \begin{cases} -u'' = f \\ u(0) = 0, u(1) = 1 \end{cases} \Rightarrow H_E^1(\Omega) = \{u \in H^1(\Omega), u(0) = 0, u(1) = 1\}$

~~$-u'(1)=5$~~ $\quad -u'(1) = 5;\ u'(1) = 2,\ u(0) + u'(0) = 1,\ u(1) + u'(1) = -5$

$-\int u'' v\,dx = \int f v\,dx$

$-(vu')'_0 + \int v'u'\,dx = \int f v\,dx$

$\Rightarrow \int u'v'\,dx = \int f v\,dx + u'(1)$

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$\begin{cases} -u'' = f \\ -u'(0) = \alpha \\ u'(1) = \beta \end{cases}$ $\qquad$ find $u \in H_E^1(\Omega)$ s.t.

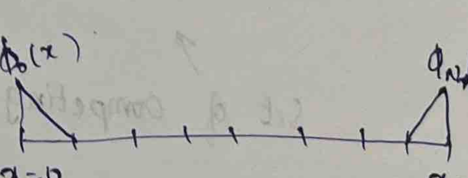
$-\int u'' v\,dx = \int f v\,dx, \ \forall\ v \in H_E^1(\Omega)$

$H_E^1(\Omega) = \{u \in H^1(\Omega) : -u'(0) = \alpha, u'(1) = \beta\}$

$\int u'v'\,dx - [u'(x)v(x)]_0^1 = \int f v\,dx$

$\int u'v'\,dx = \int f v\,dx + [u'(x)v(x)]_0^1$

$AU = F$



$u_n(x) = \sum_{i=1}^{N} U_i \phi_i(x)$

$\uparrow$

Dirichlet BVP

Neumann Problem $\rightarrow$ $u_n(x) = \sum_{i=0}^{N+1} U_i \phi_i(x)$

$\int_{x_0}^{x_1} \phi_0'(x)\phi_1$ $\qquad \int_{x_N}^{x_{N+1}} \phi_{N+1}\phi_N$

$\uparrow$ first term $\qquad \uparrow$ last

$$\begin{cases} -u'' = f \\ u(0) - u'(0) = \alpha \\ u(1) + u'(1) = \beta \end{cases}$$ Robin

$H^1_R(\Omega) := \{ u \in \Omega : u(0) - u'(0) = \alpha,\ u(1) + u'(1) = \beta \}$

find $u \in H^1_R(\Omega)$ s.t

$$\int u'v'\,dx = \int fv\,dx + \underbrace{\cdots}_{BCs} \quad \forall v \in H^1_R(\Omega)$$

→ Obstacle problem:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad \text{American}$$

$$① \begin{cases} \text{find } u,\ \text{s.t } u''(u-g) = 0,\ -u'' \geq 0,\ u - g \geq 0 \\ u \in C^1(\Omega) \qquad u(-1) = u(1) = 0 \end{cases}$$

Consider the obstacle problem inorder to solve by Finite element, we will define the following space

$$K = \{ v \in C^0[-1,1] : v(-1) = v(1) = 0,\ v(x) \geq g(x),\ \forall x \in \bar{\Omega} : v \text{ is piecewise } C^1 \}$$

↑ Set of competing functions

$\bar{\Omega}$: including boundary pts

find $u \in K$, $\forall v \in K$

The solution u of obstacle problem ① implies $u \in K$ and for any $v \in K$, $(v-g) \geq 0 \Rightarrow -u''(v-g) \geq 0$

∴ The finite element formulation tells

$$\int_{-1}^{1} -u''(v-g)\,dx \geq 0, \quad \forall v \in K \qquad ②$$

from ① $\Rightarrow$ $\int_{-1}^{1} -u''(u-g)\,dx = 0$ ——— ③

subtracting ② & ③ :-

$$\int_{-1}^{1} -u''(v-u)\,dx \geq 0 \quad \text{——— ④}, \quad \forall v \in K$$

Now the inequality ④ doesn't contain g explicitly whereas it is inside the space K.

$$\int_{-1}^{1} -d(u')(v-u)\,dx \geq 0$$

$$\underbrace{\left[-u'(v-u)\right]_{-1}^{1}}_{0} + \int_{-1}^{1} u'(v-u)'\,dx \geq 0$$

Since u and v belongs to K, $u(-1) = v(-1)$ and $u(1) = v(1)$

Weak formulation

$$⑤ \begin{cases} \text{find } u \in K, \text{ s.t} \\ \int u'(v-u)'\,dx \geq 0, \ \forall v \in K \end{cases}$$

~~Since~~ the inequality ⑤ is known as variational inequality

If we consider any approximation ~~so~~ $w \in K$, we have:

$$\int_{-1}^{1} w'(v-w)'\,dx \geq 0 \ \forall v \in K$$

$$\sum_{i=1}^{N} \int_{-1}^{1} \phi_i'(x)\left(\phi_j'(x) - \phi_i'(x)\right)dx \geq 0 \ \forall j$$

$$w_h(x) = \sum_{i=0}^{N+1} W_i \phi_i(x)$$

⑤ is a minimization problem.

When $v = u$, the integral (in ⑤) vanishes. Therefore ⑤ is a minimization problem.

→ Consider the American

$$\begin{cases} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g) = 0 & y \in \mathcal{C}^1, \text{ w.r.t } x \\ \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0 \quad , \quad y - g \geq 0 \\ y(x,0) = g(x,0) \\ y(x_{min}, \tau) = g(x_{min}, \tau) \\ y(x_{max}, \tau) = g(x_{max}, \tau) \end{cases}$$

space of admisible funcs:

$$K = \left\{ v \in \mathcal{C}^0 ; \ \frac{\partial v}{\partial x} \text{ is piecewise } \mathcal{C}^0 : \ v(x,\tau) \geq g(x,\tau) \ \forall x, \tau. \ v(x,0) = g(x,0), \ v(x_{min},\tau) = g(x_{min},\tau), \ v(x_{max},\tau) = g(\ ) \right\}$$

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$$v \geq g ; \quad \frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0 \qquad v \in K \quad y \in K$$

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-g)\,dx \geq 0 \quad \text{—— (1)}$$

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g)\,dx = 0 \quad \text{—— (2)}$$

Subtract (1) & (2)

$$u''(v-g)$$

$$\Rightarrow \int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-y)\,dx \geq 0$$

$$\Rightarrow \int_{x_{min}}^{x_{max}} \left[\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right] dx + \underbrace{\left[\frac{\partial y}{\partial x}(v-y)\right]_{x_{min}}^{x_{max}}}_{0} \geq 0$$

$$\int_{x_{min}}^{x_{max}} \left[\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right] dx \geq 0 \quad \text{—— (3)}$$

If $v = y$, ③ will be a minimization problem and we can denote it by

find $y \in K$

$$I(y; v) = \int_{x_{min}}^{x_{max}} (\quad) \, dx \geq 0 \quad \forall v \in K$$

$$\min_{v \in K} I(y; v) = I(y; y) = 0$$

For American option, we require the sol^n $(\hat{y} \in K)$, $\hat{y} \in \mathcal{C}^2$ and we expect the sol^n $\hat{y}$ such that we can have

$$\inf_{v \in K} I(\hat{y}; v) = 0$$

$\text{span}\{\phi_i\} = K$

$$\left.\begin{aligned} \hat{y} &= \sum_i w_i \phi_i(x) = \sum_i w_i(\tau) \phi_i(x) \\ v &= \sum v_i \phi_i(x) = \sum_i v_i(\tau) \phi_i(x) \end{aligned}\right\} ④$$

using ④ in ③ we get

$$\int_{x_{min}}^{x_{max}} \left[\sum_i \frac{\partial w_i}{\partial \tau} \phi_i * \sum_j (v_j - w_j) \phi_j(x) + \sum_i w_i(\tau) \phi_i'(x) \left(\sum_j (v_j(\tau) - w_j(\tau)) \phi_j'(x) \right) \right] dx \geq 0$$

We can rewrite the matrix vector notation

$$-\sum_i \sum_j \frac{\partial w_i}{\partial \tau}(v_j - w_j) \underbrace{\int \phi_i(x) \phi_j(x) \, dx}_{B} + \sum_i \sum_j w_i (v_j - w_j) \underbrace{\int \phi_i' \phi_j' \, dx}_{A} \geq 0$$

$$\left(\frac{\partial w}{\partial \tau}\right)^T B (v - w) + w^T A (v - w) \geq 0 \Rightarrow (v - w)^T \left[B \frac{dw}{d\tau} + A w \right] \geq 0 \quad — ⑤$$

$$B = \int \phi_i \phi_j \, dx, \quad A = \int \phi_i' \phi_j' \, dx$$

Inequality ⑤ can be discretized by $\theta \in [0, 1]$

$$\left(v^{(n+1)} - w^{(n+1)}\right)^T \left[B \cdot \frac{1}{\delta \tau} \left(w^{(n+1)} - w^{(n)} \right) + \theta A w^{(n+1)} + (1 - \theta) A w^{(n)} \right] \geq 0$$

$\theta = 0 \rightarrow$ Explicit

$\theta = \frac{1}{2} \rightarrow$ Crank-Nicolson $\quad \theta = 1 \rightarrow$ Implicit Euler

We can rewrite

$$\left(v^{(n+1)} - w^{(n+1)}\right)^T \left[(B + \Delta\tau\theta A) w^{(n+1)} - (B - \Delta\tau(1-\theta)A) w^{(n)} \right] \geq 0$$

If we denote $r = (B - \Delta\tau(1-\theta)A) w^{(n)}$

$C = B + \Delta\tau\theta A$

$$\Rightarrow \left(v^{(n+1)} - w^{(n+1)}\right)^T \left(C \cdot w^{(n+1)} - r\right) \geq 0$$

For the side condition :- $\hat{y}(x,\tau) \geq g(x,\tau)$

$$\sum w_i(\tau)\phi_i(x) \geq g(x,\tau) \quad \text{—— } (*)$$

$x = x_j$

$(*) \Rightarrow w_j(\tau) \geq g(x_j,\tau)$

$w^{(n)} \geq g^{(n)}$

$v \geq g$

$\theta = 1/2$

$w^{(0)}$

for $n = 1:N$

$r = (B - \Delta\tau(1-\theta)A) w^{(n)} \quad \forall v \geq g$

$\Rightarrow (v - w)^T (Cw - r) \geq 0, \; w \geq g$

Inorder to incorporate side conditions, one has to use PSOR

07/10/24 - absent

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Exotic options

$\rightarrow V(S,t) \rightarrow \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0$

$V(S_1, S_2, t) \rightarrow$ system of SDEs

$dX_t = \cancel{dt}\ a(X(t))dt + b\,X(t)\,dW(t) \qquad dX = \sigma X dt + \mu X dW \rightarrow$ for SDE

(1)

$$\begin{cases} dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dW^{(1)}(t) \\ dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dW^{(2)}(t) \\ E\left(dW^{(1)}, dW^{(2)}\right) = \rho dt \end{cases}$$

$$\Rightarrow \frac{dV}{dt} + \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij}\sigma_i\sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-\delta_i) S_i \frac{\partial V}{\partial S_i} - rV = 0$$

$\Omega = (S^1_{min}, S^1_{max}) \times (S^2_{min}, S^2_{max})$

$\Omega \times (0, T]$

$$\mathrm{Cov}\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right) = \mathbb{E}\left(\sigma_1 dW^{(1)}, \sigma_2 dW^{(2)}\right) = \rho\sigma_1\sigma_2 dt$$

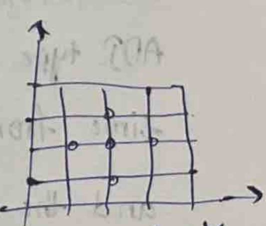
for $n=2$:-

0 (cross derivative)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left(\rho_{12}\sigma_1\sigma_2 S_1 S_2 \left(\frac{\partial^2 V}{\partial S_1^2} + \frac{\partial^2 V}{\partial S_2^2} + \cancel{\frac{\partial^2 V}{\partial S_1 \partial S_2}}\right) + \cdots \quad \text{—— (1)}$$

2D Parabolic PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{—— (2)} \qquad (x,y) \in (0,1)\times(0,t) \quad t \in (0,T]$$



$h = 1/4$

$(N+1)^2 \times (N+1)^2$

$$\frac{U^{n+1}_{lm} - U^n_{lm}}{\delta t} = \left(\frac{U^{n+1}_{l+1\,m} - 2U^{n+1}_{lm} + U^{n+1}_{l-1\,m}}{h^2}\right) + \left(\frac{U^{n+1}_{l\,m+1} - 2U^{n+1}_{lm} + U^{n+1}_{l\,m-1}}{h^2}\right)$$

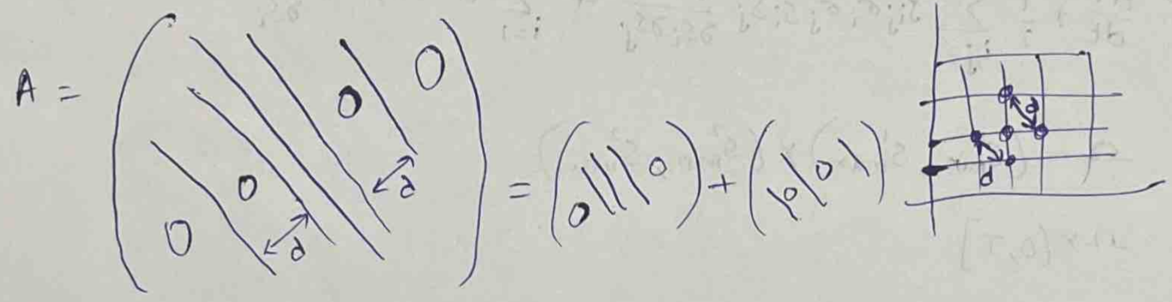
Cross derivative

$$\frac{\partial^2 V}{\partial S_1 \partial S_2} = \frac{U_{l+1\,m} - 4U_{lm} + U_{l-1\,m} + U_{l\,m+1} + U_{l\,m-1}}{h^2}$$

discretisation of $\frac{\partial V}{\partial S_i}$:-

$$\frac{U^{n+1}_{l+1\,m} - U^{n+1}_{l-1\,m}}{2h}, \quad \frac{U^{n+1}_{l\,m+1} - U^{n+1}_{l\,m-1}}{2h}$$

If we discretise the PDEs corresponding to the exotic option given in ① or the 2-D heat conduction eq^n given in ② by any implicit scheme like BTCS or the Crank-Nicolson then the bandwidth of the matrix (A in $AU=F$) will be large



$\therefore$ it takes anormous CPU time & memory to solve the system of linear algebraic eq^{n}s

Inorder to overcome these computational complexities, one can use ADI type (Alternative direction Implicit Scheme). Here when we progress time from $t^n \to t^{n+1}$ we introduce an artificial time step at $t^{n+1/2}$ and the whole scheme will be written in the following way:-

$V(x,y,0) = \phi(x,y)$

$t^n \to t^{n+1/2}$ (either x-explicit, y-implicit or the other way)

$t^{n+1/2} \to t^{n+1}$ (~~either~~ y-explicit, x-implict ; . . .)

$$\frac{U_{lm}^{n+1/2} - U_{lm}^{n}}{(\delta t/2)} = \left(\frac{U_{l+1m}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l-1m}^{n+1/2}}{h^2}\right) + \left(\frac{U_{lm+1}^{n} - 2U_{lm}^{n} + U_{lm-1}^{n}}{h^2}\right) \quad \text{—— (3)}$$

$$\frac{U_{lm}^{n+1} - U_{lm}^{n+1/2}}{(\delta t/2)} = \left(\frac{U_{l+1m}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l+1m}^{n+1/2}}{h^2}\right) + \left(\frac{U_{lm+1}^{n+1} - 2U_{lm}^{n+1} + U_{lm-1}^{n+1}}{h^2}\right) \quad \text{—— (4)}$$

Starting from the given initial conditions we can solve (3), By using the sol^n of (3) as the initial value for (4)

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$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$t_n \rightarrow t_{n+1/2} \longrightarrow t_{n+1}$$

$$\delta_x^2 U_{ij}^n = U_{i+1j}^n - 2U_{ij}^n + U_{i-1j}^n$$

$$\delta_y^2 U_{ij}^n = U_{ij+1}^n - 2U_{ij}^n + U_{ij-1}^n$$

$$\frac{U_{lm}^{n+1/2} - U_{lm}^{n}}{(\delta t/2)} = \frac{1}{h^2}\left[\delta_x^2 U_{lm}^{n+1/2} + \delta_y^2 U_{lm}^{n}\right] \quad \text{—— (5)}$$
x-explicit y-implicit

$$\frac{U_{lm}^{n+1} - U_{lm}^{n+1/2}}{(\delta t/2)} = \frac{1}{h^2}\left[\delta_x^2 U_{lm}^{n+1/2} + \delta_y^2 U_{lm}^{n+1}\right] \quad \text{—— (6)}$$
x-implicit y-explicit.

Rearranging, we get

$$\cancel{U_{lm}^{n+1/2}\left(\frac{2}{\delta t} - \frac{\delta x^2}{h^2}\right) = U}$$

$$A_1 U^{n+1/2} = B_1 U^n$$
$$A_2 U^{n+1} = B_2 U^{n+1/2}$$

A_1, A_2 — three diagonal

$$\left(1 - \frac{\lambda\,\delta x^2}{2}\right) U_{lm}^{n+1/2} = \left(1 + \frac{\lambda\,\delta y^2}{2}\right) U_{lm}^{n} \quad \text{—— (7)}$$

$$\lambda = \frac{\delta t}{h^2}$$

$$\left(1 - \frac{\lambda\,\delta y^2}{2}\right) U_{lm}^{n+1} = \left(1 + \frac{\lambda\,\delta x^2}{2}\right) U_{lm}^{n+1/2} \quad \text{—— (8)}$$

If we eliminate $U_{lm}^{n+1/2}$ from ⑦ & ⑧ we get

$$\left(1-\frac{\lambda}{2}\delta x^2\right)\left(1-\frac{\lambda}{2}\delta y^2\right)U_{lm}^{n+1} = \left(1+\frac{\lambda}{2}\delta x^2\right)\left(1+\frac{\lambda}{2}\delta y^2\right)U_{lm}^{n}$$

$$\Rightarrow \left[1-\frac{\lambda}{2}(\delta x^2+\delta y^2)+\frac{\lambda^2}{4}\delta x^2\delta y^2\right]U_{lm}^{n+1} = \left[1+\frac{\lambda}{2}(\delta x^2+\delta y^2)+\frac{\lambda^2}{4}\delta x^2\delta y^2\right]U_{lm}^{n} \quad \text{--- ⑨}$$

Truncation error :- $O(h^2+k^2)$

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Crank-Nicolson scheme:-

$$\frac{U_{lm}^{n+1}-U_{lm}^{n}}{\delta t} = \frac{\theta}{h^2}\left(\delta x^2 U_{lm}^n + \delta y^2 U_{lm}^n\right) + \frac{(1-\theta)}{h^2}\left(\delta x^2 U_{lm}^{n+1} + \delta y^2 U_{lm}^{n+1}\right)$$

$\searrow O(k^2+h^2)$

$\theta = 0, \frac{1}{2}, 1$ ~~$\delta = \delta t$~~ $\lambda = \frac{\delta t}{h^2}$

Crank- Nicolson

$$\left[1-\lambda(1-\theta)(\delta x^2+\delta y^2)\right]U_{lm}^{n+1} = \left[1+\lambda\theta(\delta x^2+\delta y^2)\right]U_{lm}^{n} \quad \text{--- ⑩}$$

Comparing ⑨ & ⑩, $\frac{\lambda^2}{4}\delta x^2\delta y^2$ term is extra in ⑨ (ADI scheme). Additional term which is present in the ADI scheme is propoetional to $\left(\frac{\delta t}{h^2}\right)^2$ which is the same order as the Crank-Nicolson scheme which is 2nd order in time & space.

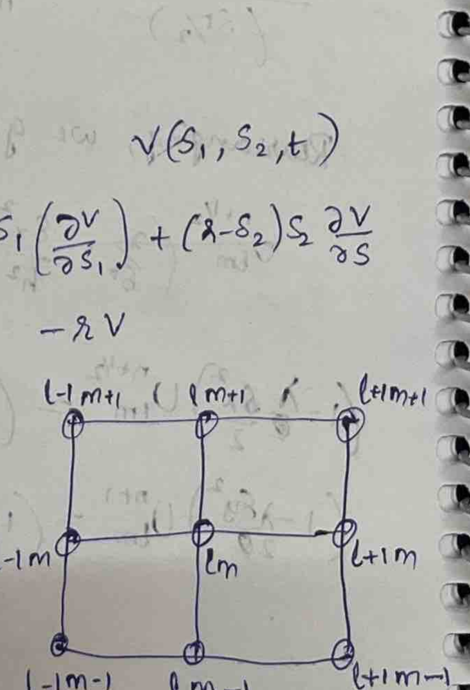
⑨ is known as the "approximate factorisation".

$V(S_1, S_2, t)$

$$\frac{\partial V}{\partial t} = \frac{\sigma_1^2}{2}\frac{\partial^2 V}{\partial S_1^2} + \frac{\sigma_2^2}{2}\frac{\partial^2 V}{\partial S_2^2} + \rho\sigma_1\sigma_2\frac{\partial^2 V}{\partial S_1\partial S_2} + (r-\delta_1)S_1\left(\frac{\partial V}{\partial S_1}\right) + (r-\delta_2)S_2\frac{\partial V}{\partial S} - rV$$

If $\rho = 0$, $\delta S_1 \delta S_2 V_{lm}^n = \delta S_1\left(\frac{V_{l+1m}-V_{l-1m}}{2h}\right)$

$$D^+D^-V_{lm}^n = D^+\left(\frac{V_{l+1m}^n - V_{lm}^n}{h}\right)$$



$$= \frac{1}{4h^2}\left(V_{l+1\,m+1} - V_{l+1\,m-1} - V_{l-1\,m+1} + V_{l-1\,m-1}\right)$$

$D^+ V_l = \frac{V_{l+1} - V_l}{h}$

$D^- V_l = \frac{V_l - V_{l-1}}{h}$

If $\frac{\partial^2 V}{\partial S_1 \partial S_2} \neq 0$ then we will have this term

If we evaluate the underlying at diff time instances $t_1, \dots t_m$. The corresponding values $S_{t_1}, \dots S_{t_m}$ then one can take either Arithmetic or Geometric mean or if we consider the continuous time $(0, T]$

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→ If we sample the underlying asset S_t at discrete time intervals t_i with ~~equidi~~ uniform time interval $\left(h = \frac{T}{n} \text{ or } t_{i+1} - t_i = h\right)$: we obtain the time series $S_{t_1}, S_{t_2}, \dots S_{t_n}$ then we can consider the average as AM:-

$$\boxed{\hat{S} = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} = \frac{h}{T}\sum_{i=1}^{n} S_{t_i}} \quad \text{— (1)}$$

(or) if the time period is continuous $0 \le t \le T$ then we can have the average will be

$$\boxed{\hat{S} = \frac{1}{T}\int_0^T S_\theta \, d\theta} \quad \text{— (2)}$$

Instead of the AM we can consider GM in both the cases then the mean will be:

$$\boxed{\left(\prod_{i=1}^{n} S_{t_i}\right)^{1/n} = \exp\left(\frac{1}{n}\log\prod_{i=1}^{n} S_{t_i}\right) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}\log S_{t_i}\right)} \text{ — discrete} \quad \text{— (3)}$$

$$\boxed{\hat{S} = \exp\left(\frac{1}{T}\int_0^T \log S_t \, dt\right)} \text{ — continuous} \quad \text{— (4)}$$

In general $\underset{\text{avg}}{A} = \hat{S} = \frac{1}{t}\int_0^t f(S,\theta)\,d\theta$ ——— simplest we can take $f(S,\theta) = S$

In the Asian option with avg $\hat{S}$, S_t, $t < T$. payoff $= (\hat{S} - K)^+$ — Avg price call (or)

$= (K - \hat{S})^+$ — Avg price put

or payoff $= (S_T - \hat{S})^+$ — Avg strike call

$= (\hat{S} - S_T)^+ \Rightarrow$ Avg strike put

Let us denote the average $A_t = \int_0^t f(S_\theta, \theta)\, d\theta$

Instead of regular SDE, we will have

$$dA = a_A(t)\,dt + b_A\,dW_t \quad ; \quad \underbrace{a_A = f(S_t, t), \; b_A = 0}_{\text{simplest}}$$

Then, $$dV_t = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + f(S,t) \frac{\partial V}{\partial A} \right) dt + \sigma S \frac{\partial V}{\partial S}\, dW_t$$

$$\Rightarrow \underbrace{\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S}}_{\text{Black-Scholes like PDE}} + \underbrace{f(S,t) \frac{\partial V}{\partial A}}_{\text{extra term due to Avg}} - rV = 0 \qquad (*)$$

~~In the~~ $V(S, A, t)$

We can consider the simplest case where $f(S,t) = S$

BTCS :-

$$\frac{V^l_{mn} - V^{l-1}_{mn}}{\delta t} + \frac{\sigma^2}{2} S^2 \left(\frac{V^l_{m+1n} - 2V^l_{mn} + V^l_{m-1n}}{2(\delta S)^2} \right) + rS \left(\frac{V^l_{m+1n} - V^l_{m-1n}}{2\delta S} \right)$$

$$+ S \frac{V^l_{mn+1} - V^l_{mn-1}}{2\delta A} - rV = 0$$

Exercise: Check if ADI-type will work for PDE (*)

When $f(S,t)$ is linear i.e $f(S,t) \equiv S$ then we can reduce the 2-D Black-Scholes like PDE (*) into a 1-D PDE. This is called Dimension-Reduction

Consider (*) the Black-Scholes like PDE which is defined on 2-dimension

$$V(\underbrace{S, A}_{2\text{-D}}, t) \qquad S > 0,\ A > 0,\ 0 \le t \le T$$

In the particular case when $f(S,t) \equiv S$ we can reduce the dimension 1-D

Let us consider the European Arithmetic Average Strike Call with payoff $(S_T - 3)^+$

$$(S_T-3)^+ = \left(S_T - \frac{1}{T} A_T\right)^+ = S_T\left(1 - \frac{1}{T S_T}\int_0^T S_\theta d\theta\right)^+ \quad \text{— ①}$$

Let us denote $R_t = \frac{1}{S_t}\int_0^t S_\theta d\theta$, $A_t = S_t R_t$

$$① \Rightarrow S_T\left(1 - \frac{1}{T} R_T\right)^+ = S_T \cdot \underbrace{\text{function}(R,T)}_{H(R,T)}$$

$$V(S,A,t) = S \cdot H(R,t)$$

where $H(\cdot)$ is a function of R and t and R is an independent variable.

$$dR_t = R_{t+\Delta t} - R_t$$

$$dS_t =$$

Note that R_t satisfies the following SDE ②

$$\begin{cases} R_{t+dt} = R_t + dR_t \\ dS_t = \mu S_t dt + \sigma S_t dW_t \end{cases}$$

$$dR_t = \left(1 + (\sigma^2 - \mu) R_t\right) dt - \sigma R_t dW_t \quad \text{— ②}$$

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$$A_t = S_t R_t \qquad t < T$$

$$A = S \cdot R$$

$$V(S,A,t) = S \cdot H(R,t)$$

$$V(S,A,t) = \tilde{V}(S,R,t) = S \cdot H(R,t) \qquad \text{where } R = \frac{A}{S}$$

$$V_t = S \cdot H_t$$

$$V_S = H(R,t) - S \cdot H_R \frac{A}{S^2} = H - \frac{H_R \cdot A}{S}$$

$$V_{SS} = -H_R \frac{A}{S^2} + \frac{H_R A}{S^2} - \frac{H_{RR} A^2}{S^3} = -\frac{A^2 H_{RR}}{S^3}$$

$$V_A = S \cdot H_R\left(\frac{1}{S}\right) = H_R$$

$$\boxed{\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0} \quad \text{——} \circledast$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} + S \cdot \frac{\partial V}{\partial A} - rV = 0$$

substituting V_t, V_S, V_{SS}, V_A we get $\circledast$

$$H(R,T) = \left(1 - \frac{1}{T} R_T\right)^+$$

$R_T \to \infty \quad H(R_T, T) = 0$

for $R \to \infty$ (boundary condition) we obtain from payoff

$$R_t = \frac{1}{S_t} \int_0^t S_\theta \, d\theta$$

The integral R_t is bounded. Therefore as $S \to 0$, $R \to \infty$.

This is the European call option $\Rightarrow$ We cannot exercise call option

$$\boxed{H(R,t) = 0, \; R \to \infty}$$

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$$\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0 \quad \text{——} \circledast$$

$$dR_t = \left(1 + (\sigma^2 - \mu) R_t\right) dt - \sigma R_t \, dW_t \quad ②$$

$dR_0 = dt \qquad R_T = 0$

We cannot get boundary conditions for $R = 0$

For left hand boundary condition when $R = 0$, we face some difficulties.

for example if $R \neq 0$ then from ② we can obtain $dR_0 = dt$ i.e

R_t won't remain at zero. Therefore we can't expect $R_T = 0$

$\therefore$ To obtain the boundary condition at $R=0$, we use the PDE ⊛ directly.

$$⊛ \Rightarrow \frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0 \qquad \left(\text{assumption } \frac{\partial^2 H}{\partial R^2} \approx 0\right)$$

~~provided $\frac{\partial^2 H}{\partial R^2}$ remain~~ if H is bounded then we conclude that when

$$R = 0 \Rightarrow \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R} = 0$$

suppose $R^2 \frac{\partial^2 H}{\partial R^2} = c \Rightarrow \frac{\partial^2 H}{\partial R^2} = O\left(\frac{1}{R^2}\right)$

$$\frac{\partial^2 H}{\partial R^2} = \frac{c}{R^2}$$

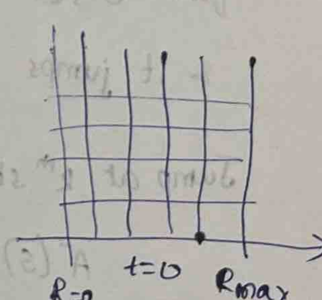
$$H = -c_1 \ln R + c_2 R + C_3 \Rightarrow \text{as } R \to 0, \ H \text{ is unbounded}$$

$\therefore R^2 \frac{\partial^2 H}{\partial R^2}$ cannot be non-zero.

$H(R,t)$

BVP

$$\begin{cases} R=0 :- \quad \dfrac{\partial H}{\partial t} + \dfrac{\partial H}{\partial R} = 0 \\ \text{from payoff we have } H(R_T, T) = \left(1 - \frac{1}{T} R_T\right)^+ \\ \dfrac{\partial H}{\partial t} + \dfrac{\sigma^2}{2} R^2 \dfrac{\partial^2 H}{\partial R^2} + (1 - rR) \dfrac{\partial H}{\partial R} = 0 \end{cases}$$



$$\frac{H_m^{n+1} - H_m^n}{\delta t} + \frac{H_{m+1}^{n+1} - H_m^{n+1}}{\delta R} = 0$$

$$\frac{\partial H}{\partial R} = \frac{H_{m+1}^{n} - H_{m-1}^{n}}{\delta R} \times$$

$$\left.\frac{\partial H}{\partial R}\right|_{R=0} = \frac{-3H_0^n + 4H_1^n - H_2^n}{2\,\delta R} + O(\delta R^2)$$

Discrete time levels $t_1, \cdots t_M$ $S_{t_1}, S_{t_2}, \cdots S_{t_M}$

If suppose we determine

$$A_{t_k} = \frac{1}{k} \sum_{i=1}^{k} S_{t_i} \quad , k = 1, \cdots, M$$

$$A_{t_k} = A_{t_{k-1}} + \frac{1}{k}\left(S_{t_k} - A_{t_{k-1}}\right) \quad \text{—— ①}$$

It is more appropriate to have this integration backwards because we have value at $t = T$.

$$A_{t_{k-1}} = A_{t_k} + \frac{1}{k-1}\left(A_{t_k} - S_{t_k}\right) \quad \text{—— ②}$$

from ①, ② we can observe that A_t is const b/w the sampling times & it jumps at t_k with $\frac{1}{(k-1)}(A_{t_k} - S_{t_k})$

Jump at k^{th} step :

- $$A^-(S) = A^+(S) + \left(\frac{1}{k-1}\right)\left(A^+(S) - S\right) \quad \text{—— ③} \quad S = S_{t_k}$$

from no arbitrage principle we can, the continuity of value of that option at t_k for any realisation of the random walk.

$$V(S, A^+, t_k) = V(S, A^-, t_k) \quad \text{—— ⊛}$$

for any fixed S and A, ③ defines the jump at t_k.

For the numerical calculations of this ~~com~~ jump condition, if we discretize the A axis into discrete values, A_j, $j=1,\cdots,J$. Then for each time period b/w two consecutive samples $t_{k+1} \to t_k$ the option value is independent of A because in the discretization of A_t, A_t is piecewise constant. Therefore $\frac{\partial V}{\partial A} = 0$

$\therefore$ we obtain $1,2,\cdots,J$ i.e J no. of 1D Black-Scholes PDE are integrated separately & independently from t_{k+1} to t_k which is more suitable for parallel computing.

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$\longrightarrow$ SDE :-

derivative form

$$\begin{cases} y' = f(t,y), \quad t \in [0,1] \\ y(0) = \alpha \end{cases}$$

$$y(t) = y(x) = \phi(x)$$

(i) $M(x,y)\,dx + N(x,y)\,dy = 0$

Differential form

Exact DEs

(ii) $y'(t) + P(t)y = Q(t)$

$$\begin{cases} dX(t) = a(t,X)\,dt + b(t,X)\,dW_t \\ X(0) = X_0 \end{cases}$$

$$\int_0^t dX(t) = \int_0^t a(t,X)\,dt + \int_0^t b(\ )\,dW_t$$

$$dX = \mu X\,dt + \sigma X\,dW_t$$

$$X(t) = \exp\{$$

$$\int_a^b f(x)\,dx = \sum_{i=0}^{n} f(x^*)\,\delta x$$

Ito's integral

$$\int_a^b f(x)\, dW_t = \sum_{i=0}^{n} f(t_{i-1})\, \Delta W_i \qquad \Delta W_i = W_{i+1} - W_i$$

$$= \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} f(s)\, dW_s$$

$$\boxed{X(t) = \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W(t) \right\}}$$

$$I = \int_a^b f(t)\, dW_t$$

$$dI = f\, dW_t$$

$$Y = f(t, X)$$

$$dY = f_t dt + f_x dX + \frac{1}{2} f_{xx}\, \cancel{dx}\, dx$$

$\cancel{dx}$ $dt\, dt = 0$

$dt\, dW_t = dW\, dt = 0$

$dW \cdot dW = dt$

The solⁿ of Black-Scholes diffusion eqⁿ :-

$$\boxed{X(t) = X_0 \cdot \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \right\}}$$

$X := f(t, Y) = X_0 e^Y$ where $Y = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t$

$$dY = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma\, dW_t$$

Ch-6

$$dY\, dY = \sigma^2 dt$$

Sidal Ch-3

$$dX = \cancel{\sigma}\, \mu X\, dt + \sigma X\, dW_t$$

In principle, it is difficult to obtain closed form analytical solⁿs for SDEs. Therefore one has to seek numerical approximate solⁿs for SDEs

$$\begin{cases} y'(t) = f(t,y), & t \in (0,1) \\ y(0) = \alpha \end{cases} \qquad N,\ h = 1/N$$

$$h = t_i - t_{i-1}$$

$$\int_{t_n}^{t_{n+1}} y'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds \longrightarrow \text{Euler}$$

$$\Rightarrow y_{n+1} - y_n = f(t_n, y_n)h \quad . \quad TE = O(h)$$
$$= LHS - RHS$$

Euler-Maruyama

$$\begin{cases} dX(t) = a(t,X)dt + b(t,X)\,dW_t, & t \in (0,T] \\ X(0) = X_0 \end{cases} \qquad h = T/N$$

$$\int_{t_n}^{t_{n+1}} dX(s) = \int_{t_n}^{t_{n+1}} a(s, X(s))\,ds + \int_{t_n}^{t_{n+1}} b(s, X(s))\,dW_s$$

$$\boxed{X(t_{n+1}) - X(t_n) = a(t_n, X_n)\,h + b(t_n, X_n)\,\Delta W_{n+1}}$$

$z_i \in N(0,1)$ $\qquad TE = O(\sqrt{\Delta t}) = O(\sqrt{h})$

$\Delta W_{i+1} = z_i\sqrt{\Delta t}$

Because of the geometric Brownian motion, the order of Euler-Mariyama scheme get diminished by $1/2$.

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$$\rightarrow \quad dX(t) = a(X(t))\,dt + b(X(t))\,dW_t \qquad t \in [0,T]$$

$$X(0) = X_0$$

$$X(t_{n+1}) = X(t_n) + a(X(t_n))h + b(X(t_n))\,\Delta W_h = W(t_{n+1}) - W(t_n)$$
$$= z_i\sqrt{h}$$

$$z_i \in N(0,1)$$

Since the Euler Maruyama is of $O(\sqrt{h})$ which is ~~of~~ less than the order of Euler scheme which is 1st order & the order got diminished by ½ due to approximation taken for ~~diffusn~~ diffusion term.

In order to enhance the order, we have to consider more no. of terms for $b(X(t))$

$$\int_t^{t+h} dX(t) = \int_t^{t+h} a(X(t))\,dt + \int_t^{t+h} b(X(t))\,dW_t \quad \text{———} (*)$$

$$X(t+h) = X(t) + a(X(t))h + b(X(t))\underbrace{[W(t+h) - W(t)]}_{O(\sqrt{h})} \quad \text{———} (**)$$

Instead of using (**), we use the ito's formula

$$db(X(t)) = b'(X(t))\cdot dX(t) + \frac{1}{2}b''(X(t))\cdot b^2(X(t))\,dt$$

replace $dX(t)$ from SDE

$$db(X(t)) = b'(X(t))\left[a(X(t))\,dt + b(X(t))\,dW(t)\right] + \tfrac{1}{2}b''(X(t))\,b^2(\ )\,dt$$

$$= \underbrace{\left[a(X(t))\,b'(X(t)) + \tfrac{1}{2}b''(X(t))\,b^2(X)\right]}_{\mu_b} dt + \underbrace{b(X(t))\,b'(X(t))}_{\sigma_b}\,dW_t$$

$[t, t+h]$, $t \le u \le t+h$

$$b(X(u)) \approx b(X(t)) + \mu_b(X(t))\underbrace{[u-t]}_{O(u-t)} + \sigma_b(X(t))\underbrace{[W(u) - W(t)]}_{\cancel{O(\sqrt{h})}\ O(\sqrt{u-t})},$$

~~Since $[W(u) - W(t)]$ is~~

Since we want to retain the 1st order convergence of the Euler scheme, we drop $\mu_b(X(t))[u-t]$.