Notations: Note that for all the problems $\mathbf{a}_k^T$ denotes the k-th row of A and $\tilde{\mathbf{a}}_k$ denotes the k-th column of A. The u_{ik}'s have their usual meaning.
$\mathbf{e}_i$ denotes the i th column of the identity matrix I.
Convention: For the system, $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, always we have assumed $rank(A) = m$.

Result 1: $\mathbf{x}$ is a BFS of the system $A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ if and only if $\mathbf{x}$ is an extreme point of $S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x}_{m\times n} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$.

Proof: Only If part: Let $\mathbf{x}$ be a BFS of $A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, then $\mathbf{x}$ is of the form $\begin{bmatrix} B^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix}$.
Let $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for some $\mathbf{x}_1, \mathbf{x}_2 \in S$ and $0 < \lambda < 1$.

$$\text{Then,} \quad \begin{bmatrix} B^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x}_{11} \\ \mathbf{x}_{12} \end{bmatrix} + (1-\lambda) \begin{bmatrix} \mathbf{x}_{21} \\ \mathbf{x}_{22} \end{bmatrix}. \qquad (*)$$

Since $\mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0}$ and $0 < \lambda < 1$ we get from (*), $\mathbf{x}_{12} = \mathbf{x}_{22} = \mathbf{0}$.
Also $A\mathbf{x}_1 = A\mathbf{x}_2 = \mathbf{b}$ implies $\mathbf{x}_{11} = \mathbf{x}_{21} = B^{-1}\mathbf{b}$, or $\mathbf{x} = \mathbf{x}_1 = \mathbf{x}_2$.
Hence $\mathbf{x}$ is an extreme point of S.

If part: Let $\mathbf{x}$ be an extreme point of S then to show that is a BFS of the system, $A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$.
Let $\mathbf{x} = [x_1, x_2, ..., x_n]^T$ be an extreme point of S such that $x_1 > 0, x_2 > 0, ..., x_k > 0$ and $x_{k+1} = ... = x_n = 0$.
Case 1: If the columns $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, ..., \tilde{\mathbf{a}}_k$ of A are LI then $\mathbf{x}$ is a BFS and since $rank(A) = m$, $k \leq m$.
Case 1a: If $k < m$, then since $Rank(A) = m$, we can add some $(m-k)$ linearly independent columns of A to these k columns of A, to get a basis of $\mathbb{R}^m$ and then $\mathbf{x}$ will be a basic feasible solution of $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$ corresponding to **each** such basis. The basic feasible solution $\mathbf{x}$ is then degenerate.
Case 1b: If $k = m$, then $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, ..., \tilde{\mathbf{a}}_m$ forms a basis of $\mathbb{R}^m$ and $\mathbf{x}$ is the corresponding non degenerate basic feasible solution.
Case 2: If the columns $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, ..., \tilde{\mathbf{a}}_k$ are linearly dependent, then there exists λ_i not all zeros such that $\sum_{i=1}^k \lambda_i\tilde{\mathbf{a}}_i = \mathbf{0}$.
Let $\lambda_{n\times 1} = [\lambda_1, \lambda_2, ..., \lambda_k, 0, .., 0]^T$.
We can choose $a > 0$ small enough ($a \leq min\{|\frac{x_i}{\lambda_i}| : \lambda_i \neq 0\}$) such that
$\mathbf{x}' = \mathbf{x} + a\lambda \geq \mathbf{0}$ and $\mathbf{x}'' = \mathbf{x} - a\lambda \geq \mathbf{0}$.
Note that $A\mathbf{x}' = A\mathbf{x}'' = \mathbf{b}$, hence $\mathbf{x}', \mathbf{x}'' \in S$.
Since $\mathbf{x} = \frac{1}{2}\mathbf{x}' + \frac{1}{2}\mathbf{x}''$ and $\mathbf{x}' \neq \mathbf{x}''$ it contradicts our initial assumption that $\mathbf{x}$ is an extreme point of S.
Hence $\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, ..., \tilde{\mathbf{a}}_k$ are LI $\Rightarrow$ $\mathbf{x}$ is a basic feasible solution.
Aliter: If part: If $\mathbf{x}$ is an extreme point of S then it is also a corner point of S and lies on n LI hyperplanes defining S, to show that it is a BFS of the above system.
Note that the n LI hyperplanes on which $\mathbf{x}$ lies can be taken (why?) to be $\mathbf{a}_i^T\mathbf{x} = b_i$ for $i = 1, \ldots, m$, where $\mathbf{a}_i^T$ is the i th row of A and $x_{i_1} = x_{i_2} = \ldots = x_{i_{n-m}} = 0$ for some $i_1, \ldots, i_{n-m} \in \{1, 2, \ldots, n\}$.

Let us assume WLOG that $i_1 = m+1, \dots, i_{n-m} = n$ (otherwise renumber the variables so that this is obtained), then $\mathbf{x}$ satisfies the system of equations

$D\mathbf{x} = \begin{bmatrix} B_{m\times m} & N \\ \mathbf{0} & I_{n-m} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{b} \\ \mathbf{0} \end{bmatrix}$, where $rank(D) = n$.

If we can show that $rank(B) = m$ then we are done. But since $rank(D) = n$ that is D is nonsingular, $0 \neq det(D) = det(B)$, hence B must be nonsingular or $rank(B) = m$.

Result 2: Let $S' = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \le \mathbf{b}, \mathbf{x} \ge \mathbf{0}\}$ and let $S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m} : A_{m\times n}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}, \mathbf{s} \ge \mathbf{0} \right\}$.

Then $\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix}$ is an extreme point of S if and only if $\mathbf{x}$ is an extreme point of S'.

Proof: Let $\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in S$.

If $\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{s_1} \end{bmatrix} + (1-\lambda) \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{s_2} \end{bmatrix}$,

where $0 < \lambda < 1$, $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{s_1} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{s_2} \end{bmatrix} \in S$, and $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{s_1} \end{bmatrix} \neq \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{s_2} \end{bmatrix}$

then $\mathbf{x} \in S'$, and $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ where $\mathbf{x}_1, \mathbf{x}_2 \in S'$. $\quad (*)$

If $\mathbf{x}_1 = \mathbf{x}_2$ then $A\mathbf{x}_1 = A\mathbf{x}_2$ which implies $\mathbf{s}_1 = \mathbf{s}_2$, which contradicts $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{s_1} \end{bmatrix} \neq \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{s_2} \end{bmatrix}$, hence $\mathbf{x}_1 \neq \mathbf{x}_2$.

Hence from (*) it follows that $\mathbf{x}$ is not an extreme point of S'.

Hence it follows that if $\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix}$ is not an extreme point of S then $\mathbf{x}$ is not an extreme point of S'.

Conversely if $\mathbf{x} \in S'$ is such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$,
where $0 < \lambda < 1$, $\mathbf{x}_1, \mathbf{x}_2 \in S'$, and $\mathbf{x}_1 \neq \mathbf{x}_2$,
then $A\mathbf{x} = \lambda A\mathbf{x}_1 + (1-\lambda)A\mathbf{x}_2$ (**).
Let $\mathbf{s}, \mathbf{s_1}, \mathbf{s_2} \in \mathbb{R}^m$ (uniquely determined by $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2$) be such that
$A\mathbf{x} + \mathbf{s} = \mathbf{b}$, $A\mathbf{x}_1 + \mathbf{s_1} = \mathbf{b}$ and $A\mathbf{x}_2 + \mathbf{s_2} = \mathbf{b}$,
then by (**) $\mathbf{s} = \lambda \mathbf{s_1} + (1-\lambda)\mathbf{s_2}$, that is

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{s_1} \end{bmatrix} + (1-\lambda) \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{s_2} \end{bmatrix}, \qquad (***)$$

where

$\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{s_1} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{s_2} \end{bmatrix} \in S$ and $\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{s_1} \end{bmatrix} \neq \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{s_2} \end{bmatrix}$.

Hence from (***) it follows that $\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix}$ is not an extreme point of S.

Hence if $\mathbf{x}$ is not an extreme point of S' then $\begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix}$ is not an extreme point of S.

Result 3: If $S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m} : A_{m\times n}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \ge \mathbf{0}, \mathbf{s} \ge \mathbf{0} \right\} \neq \phi$, then it has at least one basic feasible solution.

Proof: Follows from **Result 1** and **Result 2** and the fact that S' has (refer to previous notes) at least one extreme point.

Result 4: If $S_0 = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \neq \phi$ then it has atleast one basic feasible solution.

Proof: From **Result 1** it follows that the basic feasible solutions are the extreme points of S_0. Also since S_0 can be re written as $S_0 = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, -A_{m\times n}\mathbf{x} \leq -\mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, which is of the form S' and nonempty feasible regions of the form S' has atleast one extreme point (refer to previous notes), the result follows.

Remark: Note that **Result 3** follows immediately from **Result 4** but the reason why I have given **Result 3** before **Result 4** is to emphasize the relationship between the extreme points of S' and the basic feasible solutions of S.

Corollary 2: $\mathbf{d}_0$ is an extreme direction of $S' = \{\mathbf{x} \in \mathbb{R}^n : A_{m\times n}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$
if and only if $\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}'_{\mathbf{0}} \end{bmatrix}$ is an extreme direction of $S = \left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \end{bmatrix} \in \mathbb{R}^{n+m} : A_{m\times n}\mathbf{x} + \mathbf{s} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{s} \geq \mathbf{0} \right\}$
where $\mathbf{d}'_0 \geq \mathbf{0}$ is such that $A\mathbf{d}_0 + \mathbf{d}'_0 = \mathbf{0}$.
Proof: The proof is just a repetition of the proof of **Result 2**, but I am repeating it for the sake of completeness.
Let $D' = \{\mathbf{d} \in \mathbb{R}^n : A_{m\times n}\mathbf{d} \leq \mathbf{0}, \mathbf{d} \geq \mathbf{0}, \sum_{i=1}^n d_i = 1\}$

$$= \left\{ \mathbf{d} \in \mathbb{R}^n : \begin{bmatrix} A \\ \mathbf{1}_{1\times n} \\ -\mathbf{1}_{1\times n} \end{bmatrix} \mathbf{d}_{n\times 1} \leq \begin{bmatrix} \mathbf{0}_{m\times 1} \\ 1 \\ -1 \end{bmatrix}, \mathbf{d} \geq \mathbf{0} \right\}$$

and $D'' = \left\{ \begin{bmatrix} \mathbf{d}_{n\times 1} \\ \mathbf{d}' \end{bmatrix} \in \mathbb{R}^{m+n} : A_{m\times n}\mathbf{d} + \mathbf{d}' = \mathbf{0}, \mathbf{d} \geq \mathbf{0}, \mathbf{d}' \geq \mathbf{0}, \sum_{i=1}^n (\mathbf{d})_i = 1 \right\}$

$$= \left\{ \begin{bmatrix} \mathbf{d}_{n\times 1} \\ \mathbf{d}' \end{bmatrix} \in \mathbb{R}^{m+n} : \begin{bmatrix} A \\ \mathbf{1}_{1\times n} \\ -\mathbf{1}_{1\times n} \end{bmatrix} \mathbf{d}_{n\times 1} + \begin{bmatrix} \mathbf{d}' \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{m\times 1} \\ 1 \\ -1 \end{bmatrix}, \mathbf{d} \geq \mathbf{0}, \mathbf{d}' \geq \mathbf{0} \right\}.$$

Then D' and D'' give the distinct directions of S' and S respectively.
Only if part: Let $\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}'_0 \end{bmatrix} \in D''$ be such that $\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d}'_0 \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}'_{\mathbf{1}} \end{bmatrix} + (1-\lambda) \begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d}'_{\mathbf{2}} \end{bmatrix}$,
where $0 < \lambda < 1$, $\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}'_{\mathbf{1}} \end{bmatrix}, \begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d}'_{\mathbf{2}} \end{bmatrix} \in D''$, and $\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}'_{\mathbf{1}} \end{bmatrix} \neq \begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d}'_{\mathbf{2}} \end{bmatrix}$
then $\mathbf{d}_0 \in D'$, and $\mathbf{d}_0 = \lambda\mathbf{d}_1 + (1-\lambda)\mathbf{d}_2$ where $\mathbf{d}_1, \mathbf{d}_2 \in D'$. $(*)$

If $\mathbf{d}_1 = \mathbf{d}_2$ then $A\mathbf{d}_1 = A\mathbf{d}_2$ which implies $\mathbf{d}'_1 = \mathbf{d}'_2$, which contradicts $\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}'_{\mathbf{1}} \end{bmatrix} \neq \begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d}'_{\mathbf{2}} \end{bmatrix}$,
hence $\mathbf{d}_1 \neq \mathbf{d}_2$.
Hence from (*) it follows that $\mathbf{d}_0$ is not an extreme point of D'.

If part: Let $\mathbf{d}_0 = \lambda\mathbf{d}_1 + (1-\lambda)\mathbf{d}_2$ for $0 < \lambda < 1$ and $\mathbf{d}_1, \mathbf{d}_2 \in D'$, where $\mathbf{d}_1 \neq \mathbf{d}_2$, then $A\mathbf{d}_0 = \lambda A\mathbf{d}_1 + (1-\lambda)A\mathbf{d}_2$ (**).
Let $\mathbf{d}'_{\mathbf{0}}, \mathbf{d}'_{\mathbf{1}}, \mathbf{d}'_{\mathbf{2}} \in \mathbb{R}^m$ be such that
$A\mathbf{d}_0 + \mathbf{d}'_{\mathbf{0}} = \mathbf{0}$, $A\mathbf{d}_1 + \mathbf{d}'_{\mathbf{1}} = \mathbf{0}$ and $A\mathbf{d}_2 + \mathbf{d}'_{\mathbf{2}} = \mathbf{0}$,
then by (**) $\mathbf{d}'_{\mathbf{0}} = \lambda\mathbf{d}'_{\mathbf{1}} + (1-\lambda)\mathbf{d}'_{\mathbf{2}}$, that is

$$\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d_0'} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d_1'} \end{bmatrix} + (1-\lambda) \begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d_2'} \end{bmatrix}, \qquad (***)$$

where $\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d_0'} \end{bmatrix}, \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d_1'} \end{bmatrix}, \begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d_2'} \end{bmatrix} \in D''$ and $\begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d_1'} \end{bmatrix} \neq \begin{bmatrix} \mathbf{d}_2 \\ \mathbf{d_2'} \end{bmatrix}$.

Hence from (***) it follows that $\begin{bmatrix} \mathbf{d}_0 \\ \mathbf{d_0'} \end{bmatrix}$ is not an extreme point of D''.

Remark: In simplex method in any iteration either it moves from one extreme point (if the current basic feasible solution is nondegenerate) to an adjacent extreme point in the next iteration (along the straight line joining the two) or it may remain at the same extreme point (in that case the current basic feasible solution is degenerate).

To see the above, without loss of generality let $\mathbf{x} = [x_1, x_2, .., x_m, 0, .., 0]^T$ be the basic feasible solution obtained in some iteration of the simplex algorithm, with basic variables $x_1, x_2, .., x_m$.

If $\mathbf{x}$ is not an optimal solution, then the simplex algorithm in the next iteration will give a new basic feasible solution by making exactly one of $x_1, x_2, .., x_m$ as nonbasic which is called the leaving variable and one nonbasic variable out of $x_{m+1}, x_{m+2}, .., x_n$ as basic, which is called the entering variable.

Let x_r $(r \leq m)$ be the leaving variable and x_s ($m < s \leq n$) be the entering variable.

So that the new basic feasible solution is $\mathbf{x}' = [x_1', x_2', .., x_{r-1}', 0, x_{r+1}', .., x_m', 0, .., x_s', .., 0]^T$, where $x_i' = x_i - u_{is}x_s'$, for $i = 1, 2, .., r-1, r+1, .., m$ and $x_s' = \frac{x_r}{u_{rs}}$,

the u_{ij}'s are as defined earlier.

Let $\mathbf{x}$ be nondegenerate then $x_r > 0$.

Consider $0 < \theta < min_i\{\frac{x_i}{u_{is}} : u_{is} > 0\} = \frac{x_r}{u_{rs}}$, and define $\mathbf{x}_\theta$ as

$\mathbf{x}_\theta = [x_1 - u_{1s}\theta, x_2 - u_{2s}\theta, .., x_{r-1} - u_{r-1,s}\theta, x_r - u_{rs}\theta, x_{r+1} - u_{r+1,s}\theta, .., x_m - u_{ms}\theta, 0, .., \theta, .., 0]^T$

$$= \mathbf{x} + \theta\mathbf{d}, \text{ where } \mathbf{d} = \begin{bmatrix} -u_{1s} \\ \vdots \\ -u_{ms} \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} -B^{-1}\tilde{\mathbf{a}}_s \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}.$$

Note that this $\mathbf{d}$ is orthogonal to the normals of the $(n-1)$ hyperplanes common to $\mathbf{x}$ and $\mathbf{x}'$ (but this $\mathbf{d}$ is not a direction).

Also note that $\mathbf{x}_\theta$ can also be written as,

$\mathbf{x}_\theta = \mathbf{x} + \lambda\mathbf{d}' = \lambda\mathbf{x}' + (1-\lambda)\mathbf{x}$, where $\lambda = \frac{u_{rs}}{x_r}\theta$ $(0 \leq \lambda \leq 1)$ and $\mathbf{d}' = \frac{x_r}{u_{rs}}\mathbf{d} = \mathbf{x}' - \mathbf{x}$.

Hence $\mathbf{x}_\theta$ gives points on the straight line segment joining $\mathbf{x}$ and $\mathbf{x}'$.

If $\mathbf{x}$ is degenerate and suppose if $min_i\{\frac{x_i}{u_{is}} : u_{is} > 0\} = 0$, then $\mathbf{x} = \mathbf{x}'$.

However if $\mathbf{x}$ is degenerate but $min_i\{\frac{x_i}{u_{is}} : u_{is} > 0\} > 0$ (which will happen if the entries in the s th column, that is the u_{is}'s, corresponding to the zero valued basic variables are ≤ 0) then again $\mathbf{x}'$ will give an adjacent extreme point and $\mathbf{x}_\theta$ will again be points on the line segment joining $\mathbf{x}$ and $\mathbf{x}'$.

Sensitivity Analysis:

Consider the problem (**LPI**),
Min $\mathbf{c}^T\mathbf{x}$
subject to
$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$.
Let $\mathbf{x}_0$ be an optimal solution of this problem. WLOG let $B = [\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_m]$ be a basis matrix corresponding to $\mathbf{x}_0$, hence a set of basic variables of $\mathbf{x}_0$ are $x_1, \ldots, x_m$.

1. **Changing the cost vector c:**

   It is clear that if the cost vector $\mathbf{c}$ is changed to say $\mathbf{c}'$ then only the $c_j - z_j$ values in the final optimal table changes.
   If the new $c'_j - z'_j$ values again satisfy the optimality condition ($c'_j - z'_j \geq 0$ for all j), then $\mathbf{x}_0$ will again be optimal for the new problem.
   If not, then the simplex algorithm can be used to get an optimal solution for the new problem or to conclude that the new problem has no optimal solution (which will happen if $c'_j - z'_j$ becomes < 0 for some non positive column in the simplex table).

2. **Changing the vector b:**

   If the vector $\mathbf{b}$ is changed to $\mathbf{b}'$, then the feasible region changes. If the basic solution $\mathbf{x}'_0 = [(B^{-1}\mathbf{b}')^T, \mathbf{0}_{1\times(n-m)}]^T$ corresponding to $\mathbf{x}_0 = [(B^{-1}\mathbf{b})^T, \mathbf{0}_{1\times(n-m)}]^T$, is feasible for the new problem (that is $B^{-1}\mathbf{b}' \geq \mathbf{0}$), then $\mathbf{x}'_0$ is optimal for the new problem, since all the other entries in the table (other than the RHS entries) for the basic feasible solution $\mathbf{x}'_0$ remains same (as in the table for $\mathbf{x}_0$), hence the $c_j - z_j$ values satisfy the optimality criterion (they will all be ≥ 0).
   If however $B^{-1}\mathbf{b}' \not\geq \mathbf{0}$, then the basic solution $\mathbf{x}'_0$ is no longer feasible for the changed problem, but since the feasible region of the dual has not changed, we still have a feasible solution of the dual which is given by $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$ (since the $c_j - z_j \geq 0$, for all $j = 1, \ldots, n$, $c_j \geq z_j = \mathbf{y}^T\tilde{\mathbf{a}}_j$ for all $j = 1, 2, \ldots, n$).
   Also we have seen that this $\mathbf{y}^T$, since it satisfies the conditions $\mathbf{y}^T\tilde{\mathbf{a}}_j = z_j = c_j$ for $j = 1, \ldots, m$, (the dual constraints corresponding to the basic variables of $\mathbf{x}_0$), it lies on m LI hyperplanes defining $Fea(D)$, so this $\mathbf{y}$ is an extreme point of $Fea(D)$.
   So the **dual simplex algorithm** can be used to either get an optimal solution of the changed problem or to conclude that the new problem does not have a feasible solution. Note that the new problem if it has a feasible solution also has an optimal solution, since its dual has a feasible solution.

   Example : Consider the LPP given by
   Max $-x_1 + 2x_2$
   subject to
   $-x_1 + x_2 \leq 1$,
   $x_1 + x_2 \leq 7$,

$x_1 + 3x_2 \leq 15$,
$x_1, x_2 \geq 0$.

Check that the optimal solution for the above problem is given by $[3, 4]^T$.
If we convert the above problem to a problem with equality constraints by adding (slack) variables, then it becomes
Max $-x_1 + 2x_2$
subject to
$-x_1 + x_2 + s_1 = 1$,
$x_1 + x_2 + s_2 = 7$,
$x_1 + 3x_2 + s_3 = 15$,
$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$.

Note that the optimal basic feasible solution $[3, 4, 0, 0, 0]^T$ is degenerate and corresponds to three different basis matrix, $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_1]$, $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_2]$ and $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_3]$.
The tables corresponding to these three bases are given by

$c_j - z_j$	0	0	0	$\frac{5}{2}$	$-\frac{3}{2}$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$			0	$\frac{3}{2}$	$-\frac{1}{2}$	3
$\tilde{\mathbf{a}}_2$			0	$-\frac{1}{2}$	$\frac{1}{2}$	4
$\mathbf{s}_1$			1	2	-1	0

Note that all the $c_j - z_j$ values in the above table are not nonpositive, but the basic feasible solution is still optimal.
So the optimality condition, $c_j - z_j \geq 0$ for all j, is a **sufficient condition** but **not** a **necessary condition** for the corresponding basic feasible solution to be optimal.
However if the basic feasible solution is **nondegenerate** then check that the optimality condition is necessary as well as sufficient for it to be an optimal solution.

$c_j - z_j$	0	0	$-\frac{5}{4}$	0	$-\frac{1}{4}$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$			$-\frac{3}{4}$	0	$\frac{1}{4}$	3
$\tilde{\mathbf{a}}_2$			$\frac{1}{4}$	0	$\frac{1}{4}$	4
$\mathbf{s}_2$			$\frac{1}{2}$	1	$-\frac{1}{2}$	0

$c_j - z_j$	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	0	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$			$-\frac{1}{2}$	$\frac{1}{2}$	0	3
$\tilde{\mathbf{a}}_2$			$\frac{1}{2}$	$\frac{1}{2}$	0	4
$\mathbf{s}_3$			-1	-2	1	0

The dual of the above problem is given by
Min $y_1 + 7y_2 + 15y_3$
subject to
$-y_1 + y_2 + y_3 \geq -1$,

$y_1 + y_2 + 3y_3 \geq 2,$
$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$

We can read the optimal solutions of the dual from the optimal tables of the primal problem since
$\mathbf{y}^T = \mathbf{c}_B^T B^{-1} = \mathbf{c}_B^T B^{-1} I = \mathbf{c}_B^T B^{-1}[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = [z_{s_1}, z_{s_2}, z_{s_3}]$
where z_{s_i} is the z_j value corresponding to the slack variable s_i.
Two optimal solutions of the dual are given by $[\frac{5}{4}, 0, \frac{1}{4}]^T$ and $[\frac{3}{2}, \frac{1}{2}, 0]^T$, that is the dual has infinitely many optimal solutions. We also know that the above two optimal solutions are extreme points of the feasible region of the dual.

If we convert the dual problem into a problem with equality constraints by adding (surplus) variables then we get

Min $y_1 + 7y_2 + 15y_3$
subject to
$-y_1 + y_2 + y_3 - s'_1 = 1,$
$y_1 + y_2 + 3y_3 - s'_2 = 2,$
$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0, s'_1 \geq 0, s'_2 \geq 0.$

The BFS of the above problem corresponding to the extreme point $[\frac{5}{4}, 0, \frac{1}{4}]^T$ of the dual problem (with greater than equal to, constraints) will obviously have basic variables as y_1, y_3. This also follows from the fact that since the corresponding primal solution has x_1, x_2, s_2 as basic variables, so by complementary slackness property, the corresponding variables in the dual which is s'_1, s'_2, y_2 should necessarily take the value 0 hence are nonbasic variables in the corresponding basic solution of the dual.
The BFS of the above problem corresponding to the extreme point $[\frac{3}{2}, \frac{1}{2}, 0]^T$ of the dual problem (with greater than equal to, constraints) will obviously have basic variables as y_1, y_2. Similarly this also follows from the fact that since the corresponding primal solution has x_1, x_2, s_3 as basic variables, so by complementary slackness property, the corresponding variables in the dual which is s'_1, s'_2, y_3 should necessarily take the value 0 hence are nonbasic variables in the corresponding basic solution of the dual.
The table corresponding to these BFSs will be given by (check this)

$c_j - z_j$	0	0	0	3	4	
	$B^{-1}\tilde{\mathbf{a}}'_1$	$B^{-1}\tilde{\mathbf{a}}'_2$	$B^{-1}\tilde{\mathbf{a}}'_3$	$B^{-1}(-\mathbf{e}_1)$	$B^{-1}(-\mathbf{e}_2)$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}'_1$					$-\frac{1}{2}$	$\frac{3}{2}$
$\tilde{\mathbf{a}}'_2$					$-\frac{1}{2}$	$\frac{1}{2}$

3.

$c_j - z_j$	0	0	0	3	4	
	$B^{-1}\tilde{\mathbf{a}}'_1$	$B^{-1}\tilde{\mathbf{a}}'_2$	$B^{-1}\tilde{\mathbf{a}}'_3$	$B^{-1}(-\mathbf{e}_1)$	$B^{-1}(-\mathbf{e}_2)$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}'_1$					$-\frac{1}{4}$	$\frac{5}{4}$
$\tilde{\mathbf{a}}'_3$					$-\frac{1}{4}$	$\frac{1}{4}$

Note that here $\tilde{\mathbf{a}}'_i$ gives the columns corresponding to the variables $y_i, i = 1, 2, 3$ in the dual constraints, when the constraints are written in the greater than equal to form. $-\mathbf{e}_i$ are the columns corresponding to the **surplus** variables s'_i.

Now suppose if the RHS of the primal problem corresponding to the third constraint is changed from 15 to 14, then which bases among the three mentioned above, will correspond to the new optimal solution?
So the new problem is given as following:

Max $-x_1 + 2x_2$
subject to
$-x_1 + x_2 + s_1 = 1,$
$x_1 + x_2 + s_2 = 7,$
$x_1 + 3x_2 + s_3 = 14,$
$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0.$

Note that the BFS corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_1]$, should have $s_2 = s_3 = 0$ hence should lie at the intersection of the two lines
$x_1 + x_2 = 7$, and $x_1 + 3x_2 = 14$
and is given by $x_1 = \frac{7}{2}, x_2 = \frac{7}{2}$. Hence $s_1 = 1$. The corresponding BFS is $[\frac{7}{2}, \frac{7}{2}, 1, 0, 0]^T$, which is a nondegenerate basic feasible solution.
But since all the $c_j - z_j$ values are not ≤ 0, (refer to the corresponding table) so by entering s_2 into the basis one can get a BFS with better value of the objective function. Hence this BFS is not optimal.
Note that the BFS corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_2]$, should have $s_1 = s_3 = 0$ hence should lie at the intersection of the two lines
$-x_1 + x_2 = 1$, and $x_1 + 3x_2 = 14$
and is given by $x_1 = \frac{11}{4}, x_2 = \frac{15}{4}$. Hence $s_2 = \frac{1}{2}$. The corresponding BFS is $[\frac{11}{4}, \frac{15}{4}, 0, \frac{1}{2}, 0]^T$, which is again a nondegenerate BFS.
Also since $c_j - z_j \leq 0$ for all $j = 1, \ldots, n$, so this basic feasible solution is the optimal solution for the new problem.
Note that the solution corresponding to the basis $[\tilde{\mathbf{a}}_1, \tilde{\mathbf{a}}_2, \mathbf{e}_3]$, should have $s_1 = s_2 = 0$ hence should lie at the intersection of the two lines
$-x_1 + x_2 = 1$, and $x_1 + x_2 = 7$
and is given by $x_1 = 3, x_2 = 4$. Hence $s_3 = -1$. The corresponding basic solution is $[3, 4, 0, 0, -1]^T$, which is not feasible for the new problem.

Will the new dual now have a unique optimal solution or will it again have infinitely many optimal solutions?

Dual Simplex Algorithm :
Consider the following LP problem :
Min $\mathbf{c}^T\mathbf{x}$
subject to $A_{m\times n}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
Let $\mathbf{x}_0$ be a basic solution of the above problem corresponding to a basis matrix B, such that $\mathbf{x}_0$ is not a feasible solution. That is $B^{-1}\mathbf{b} \ngeq \mathbf{0}$.

If the c_j's are such that (in this case, if $c_j \geq 0$ for all $j = 1, \ldots, n$) all the $c_j - z_j$ values in the simplex table corresponding to $\mathbf{x}_0$ are nonnegative, then $\mathbf{y}$ given by $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$ (since $z_j = c_B^T B^{-1} \tilde{\mathbf{a}}_j \leq c_j \quad \forall j = 1, \ldots, n$) is feasible for the Dual.
In this case we use the **Dual Simplex Method** to get an optimal solution of the primal or to conclude that the primal does not have a feasible solution, as the case may be.
Take x_r (make $\tilde{\mathbf{a}}_r$ leave the basis) to be the leaving basic variable if $(B^{-1}\mathbf{b})_r = min\{(B^{-1}\mathbf{b})_j : (B^{-1}\mathbf{b})_j < 0\}$.

Case 1: $u_{rj} \geq 0$ for all $j = 1, 2, ..., n$.

Then it can be shown that the primal does not have a feasible solution.
Note that the r th row of the simplex table gives an equation, which is an equation in the system of equations given by,
$B^{-1}A\mathbf{x} = B^{-1}[B : N]\mathbf{x} = B^{-1}\mathbf{b}$,
which is equivalent to the original system of equations $A\mathbf{x} = [B : N]\mathbf{x} = \mathbf{b}$.
So if the system $A\mathbf{x} = \mathbf{b}$ has a nonnegative solution $\mathbf{x}^*$, then $\mathbf{x}^*(\geq \mathbf{0})$ should also be a solution of the system $B^{-1}A\mathbf{x} = B^{-1}\mathbf{b}$.
The equation given by the r th row of the simplex table in this case, is of the form $((B^{-1})_{r.}A)\mathbf{x} = (B^{-1}\mathbf{b})_r$, where $(B^{-1})_{r.}$ denotes the r th row of B^{-1}.
But the above equation cannot be satisfied by $\mathbf{x}^*(\geq \mathbf{0})$, since the RHS entry of the equation is < 0, but all the coefficients in the LHS are nonnegative.
Alternatively note that the r th row of the simplex table (without the RHS entry) is given by $(B^{-1})_{r.}[B : N] = (B^{-1})_{r.}A$, where $(B^{-1})_{r.}$ is the r th row of B^{-1}.
If all the entries of this row is nonnegative then
$\mathbf{c}^T - \mathbf{z}^T + \alpha(B^{-1})_{r.}A \geq \mathbf{0}$, for all $\alpha \geq 0$, $\qquad (*)$
where $\mathbf{z}^T = [z_1, \ldots, z_n] = \mathbf{y}^T A$, $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$.
Hence from $(*)$ we get
$\mathbf{c}^T \geq \mathbf{z}^T + \alpha(-B^{-1})_{r.}A = (\mathbf{y}^T + \alpha(-B^{-1})_{r.})A, \qquad$ for all $\alpha \geq 0$, $\quad$ (**)
that is, $(-B^{-1})_{.r}$ is a direction for $Fea(D)$.
Let us denote this direction $(-B^{-1})_{r.}$ of $Fea(D)$ as $\mathbf{d}_0^T$.
Since $(B^{-1}\mathbf{b})_r < 0$, and $(B^{-1}\mathbf{b})_r = -(-B^{-1})_{r.}\mathbf{b}$
so $(-B^{-1})_{r.}\mathbf{b} > 0$, that is $\mathbf{d}_0^T\mathbf{b} > 0$. $\qquad (**)$
Since the dual is a maximization problem it follows from $(**)$ that the dual has does not have an optimal solution. Hence the primal problem does not have a feasible solution.

Case 2 : $u_{rj} < 0$ for atleast one $j = 1, 2, ..., n$.

So now we need an entering variable (since x_r is leaving the basis) to get a new basic solution of (P) such that the table again corresponds to a feasible solution of the Dual. Once we decide on this entering variable the table is updated by performing the necessary elementary row operations to make the column corresponding to the entering variable (in the main simplex table leaving out the $c_j - z_j$ values) as the r th column

of the identity matrix I and the corresponding $c_j - z_j$ value equal to 0 (if j is the entering variable).
Since $(\mathbf{c}^T - \mathbf{z}^T + \alpha(B^{-1})_{r.}A)_j = c_j - z_j + \alpha u_{rj} \geq 0$ for all $\alpha > 0$ if $u_{rj} \geq 0$,
if we have to enter a new variable in the basis such that the new table again corresponds to a feasible solution of the Dual that is $c_j - z_j \geq 0$ for all $j = 1, \ldots, n$ (then $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$ gives a feasible solution of the Dual) then we have to be careful of the entries in the r th row which are negative, that is the j's for which $u_{rj} < 0$.

Let $\tilde{\mathbf{a}}_s$ enter the basis, that is x_s is the **entering variable** (new basic variable replacing x_r) if $\frac{|c_s - z_s|}{|u_{rs}|} = min_j\{\frac{|c_j - z_j|}{|u_{rj}|} : u_{rj} < 0\}$. $(*)$
So the **pivot element** will be u_{rs}.
Next the table is updated by performing the necessary elementary row operations to get the new column corresponding to $\tilde{\mathbf{a}}_s$ in the table as the r th column of the identity matrix and $c_s - z_s = 0$.
Note that the new $c_j - z_j' = c_j - z_j - \frac{c_s - z_s}{u_{rs}} u_{rj}$.
(i) If $u_{rj} \geq 0$, then since $c_j - z_j \geq 0$ for all $j = 1, \ldots, n$ and $u_{rs} < 0$
$c_j - z_j' = c_j - z_j - \frac{c_s - z_s}{u_{rs}} u_{rj} \geq 0$.
(ii) If $u_{rj} < 0$, then it follows from $(*)$ that
$c_j - z_j' = c_j - z_j - \frac{c_s - z_s}{u_{rs}} u_{rj} \geq 0$.
Let $\mathbf{x}'$ be the new basic solution of the primal and $\mathbf{y}'$ be the corresponding feasible solution of the dual, then $\mathbf{b}^T\mathbf{y}' = \mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + (c_s - z_s)\frac{x_r}{u_{rs}} \geq \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$.
Since the dual is a maximization problem so $\mathbf{y}'$ is a better solution with respect to cost (or the objective function) than $\mathbf{y}$.
If the new **basic solution** $\mathbf{x}'$ is nonnegative, then $\mathbf{x}'$ is an optimal BFS for the primal. Otherwise repeat the procedure on this new table and continue till you get a BFS and hence an optimal solution of the primal or conclude that (P) has no feasible solution (If **Case 1** situation arises in any of the subsequent tables).
So once we have obtained a BFS say $\mathbf{x}$ of the primal, using duality theory we can conclude that $\mathbf{x}$ is optimal for the primal.
In **simplex method** we start with a BFS of the **primal** (that is an extreme point of Fea(P)) and the process terminates either when a **feasible** solution of the **dual** is obtained (that is when the optimality conditions are satisfied), which is an **optimal solution** and an **extreme point** of Fea(D) or with the observation that the dual has **empty feasible region**.
In **dual simplex method** we start with an **extreme point** of Fea(D) and the process terminates either when a BFS of the **primal** which is an **optimal solution** of the primal is obtained, or with the observation that the primal has **empty feasible region**.

In case (P) is **Max** $\mathbf{c}^T\mathbf{x}$
subject to $A_{m\times n}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.

Then Dual of (P) is given by
Min $\mathbf{b}^T\mathbf{x}$

subject to $A^T_{n\times m}\mathbf{y} \geq \mathbf{c}$.

Case 1': $u_{rj} \geq 0$ for all $j = 1, 2, ..., n$.
$\mathbf{c}^T - \mathbf{z}^T - \alpha(B^{-1})_{r.}A \leq \mathbf{0}$, for all $\alpha \geq 0$, $\qquad (*)$
where $\mathbf{z}^T = [z_1, \ldots, z_n] = \mathbf{y}^T A$, $\mathbf{y}^T = \mathbf{c}_B^T B^{-1}$.
Hence from $(*)$ we get
$\mathbf{c}^T \leq \mathbf{z}^T + \alpha(B^{-1})_{r.}A = (\mathbf{y}^T + \alpha(B^{-1})_{r.})A$, $\qquad$ for all $\alpha \geq 0$, $\quad$ (**)
that is, $(B^{-1})_{r.}$ is a direction for $Fea(D)$.
If we denote $(B^{-1})_{r.}$ as $\mathbf{d}_0^T$,
then $(B^{-1}\mathbf{b})_r = \mathbf{d}_0^T\mathbf{b} < 0$.
Since the **Dual** is a minimization problem it follows that the Dual **does not** have an optimal solution. Hence the primal problem (P) **does not** have a feasible solution.

Case 2' : $u_{rj} < 0$ for atleast one $j = 1, 2, ..., n$.
x_s is the entering variable if $\frac{c_s - z_s}{u_{rs}} = min_j\{\frac{c_j - z_j}{u_{rj}} : u_{rj} < 0\}$.
So the pivot element is u_{rs}.
The table is updated by performing the necessary elementary row operations. The new column corresponding to $\tilde{\mathbf{a}}_s$ in the table is the r th column of I_m and $c_s - z_s = 0$.

If $\mathbf{x}'$ is the new basic solution of the primal (P) and $\mathbf{y}'$ be the corresponding feasible solution of the Dual, then $\mathbf{b}^T\mathbf{y}' = \mathbf{c}^T\mathbf{x}' = \mathbf{c}^T\mathbf{x} + (c_s - z_s)\frac{x_r}{u_{rs}} \leq \mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$.
Since the Dual is a minimization problem so $\mathbf{y}'$ is a better solution with respect to cost (or the objective function) than $\mathbf{y}$.
If the new basic solution $\mathbf{x}'$ is non negative, then $\mathbf{x}'$ is an optimal BFS of (P).
If not repeat the procedure till you get a BFS and hence an optimal solution of (P) or conclude that (P) has no feasible solution.

Example 1: Max $-3x_1 + 2x_2$
subject to
$-x_1 + x_2 \leq 1$,
$x_1 + x_2 \leq 7$,
$x_1 + 3x_2 \leq 15$,
$x_1, x_2 \geq 0$.
By adding variables the problem is changed to

Max $-3x_1 + 2x_2$
subject to
$-x_1 + x_2 + s_1 = 1$,
$x_1 + x_2 + s_2 = 7$,
$x_1 + 3x_2 + s_3 = 15$,
$x_1 \geq 0, x_2 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$.

The table corresponding to the basic solution with $x_1 = -1$ and $x_2 = 0$ is given by

4.

$c_j - z_j$		-1	-3			
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_1$		-1	-1			-1
$\mathbf{s}_2$		2	1			8
$\mathbf{s}_3$		4	1			16

According to the Dual Simplex method, x_1 is the leaving variable, since $(B^{-1}\mathbf{b})_1 < 0$ and $(B^{-1}\mathbf{b})_2 > 0$, $(B^{-1}\mathbf{b})_3 > 0$.

Also since in the row corresponding to the leaving variable, only two entries are negative, u_{12}, and u_{13} and $\frac{c_2 - z_2}{u_{12}} = 1 < \frac{c_3 - z_3}{u_{13}} = 3$, hence the entering variable is x_2 and the **pivot** is u_{12}, where u_{ij}'s have their usual meaning.

After doing the necessary elementary row operations we get the following table :

5.

$c_j - z_j$	-1	0	-2			
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{e}_3$	$B^{-1}\mathbf{b}$
$\tilde{\mathbf{a}}_2$	-1	1	1			1
$\mathbf{s}_2$	2	0	-1			6
$\mathbf{s}_3$	4	0	-3			12

Hence an optimal solution of the primal is given by, $x_2 = 1$ and $x_1 = 0$.
The optimal solution of the Dual is given by, $y_1 = 2, y_2 = 0, y_3 = 0$,
where the Dual is given by:
Min $y_1 + 7y_2 + 15y_3$
subject to
$-y_1 + y_2 + y_3 \geq -3$,
$y_1 + y_2 + 3y_3 \geq 2$,
$y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$.

Example 2: Consider the linear programming problem **LP(1)** given below:

Minimize $x_1 + x_2$
subject to
$2x_1 + x_2 \geq 4$
$x_1 - x_2 \leq 1$
$x_1 \geq 0,\ x_2 \geq 0$.
The above problem is same as
Maximize $-x_1 - x_2$
subject to
$-2x_1 - x_2 + s_1 = -4$
$x_1 - x_2 + s_2 = 1$
$x_1 \geq 0,\ x_2 \geq 0$.

The dual (D) of **LP(1)** is given by
Minimize $-4y_1 + y_2$
subject to
$-2y_1 + y_2 \geq -1$
$-y_1 - y_2 \geq -1$
$y_1 \geq 0,\ y_2 \geq 0.$

The initial table corresponding to the basic variables s_1 and s_2 of **LP(1)** (the objective function is min $x_1 + x_2$) is given below.

	$c_1 - z_1 = +1$	$c_2 - z_2 = +1$	$c_3 - z_3 = 0$	$c_4 - z_4 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{b}$
s_1	-2	-1	1	0	-4
s_2	1	-1	0	1	1

Since the $c_j - z_j$ values are nonnegative for all j we are given a feasible solution of the dual. The corresponding feasible solution of the dual is given by $y_1 = 0, y_2 = 0$, which are obtained from the $c_j - z_j$ values corresponding to s_1 and s_2.
But the initial basic solution of the primal given by $s_1 = -4$ and $s_2 = 1$ and $x_1 = x_2 = 0$ is not feasible for the primal.
By following the dual simplex method, s_1 will be the leaving variable and x_1 will be the entering variable for the next table which is given by,

	$c_1 - z_1 = 0$	$c_2 - z_2 = \frac{1}{2}$	$c_3 - z_3 = \frac{1}{2}$	$c_4 - z_4 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{b}$
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
s_2	0	$-\frac{3}{2}$	$\frac{1}{2}$	1	-1

Now s_2 will be the leaving variable and x_2 will be the entering variable.

	$c_1 - z_1 = 0$	$c_2 - z_2 = 0$	$c_3 - z_3 = \frac{4}{6}$	$c_4 - z_4 = \frac{1}{3}$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{e}_1$	$B^{-1}\mathbf{e}_2$	$B^{-1}\mathbf{b}$
x_1	1	0	$-\frac{1}{3}$	0	$\frac{5}{3}$
x_2	0	1	$-\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$

The above table is the optimal table.

6. **Introduction of a new variable:**
Suppose a new variable x_{n+1} is added to the LPP given above. So addition of a new variable means that a column is added to the matrix A call it $\tilde{\mathbf{a}}_{n+1}$ and a component c_{n+1} is added to the cost vector $\mathbf{c}$, which gives the cost associated with the new variable x_{n+1}.
Note that since $\mathbf{x}_0$ is an optimal solution for **LPI**
$\mathbf{x}'_0 = [\mathbf{x}_0^T, 0]^T$ is a feasible solution for the problem,
Min $[\mathbf{c}, c_{n+1}]^T \mathbf{x}_{(n+1)\times 1}$
subject to
$[A : \tilde{\mathbf{a}}_{n+1}]\mathbf{x}_{(n+1)\times 1} = \mathbf{b}, \quad \mathbf{x}_{(n+1)\times 1} \geq \mathbf{0}.$

In order to check whether $\mathbf{x}_0'$ is optimal for the new problem, add a new column to the optimal table and calculate $c_{n+1} - z_{n+1}$.
If $c_{n+1} - z_{n+1} > 0$, then $\mathbf{x}_0'$ is optimal for the new problem.
If not, then enter $\tilde{\mathbf{a}}_{n+1}$ in the basis and use the simplex algorithm to get an optimal solution or conclude that the problem does not have an optimal solution.
Note that adding a variable to the problem **LPI** results in adding a constraint to the dual, which might result in the feasible region of the dual to become an empty set, in that case the problem **LPI** will not have an optimal solution.

7. **Introduction of a new constraint:**
Addition of a new constraint makes the feasible region of **LPI** smaller.
Let us assume WLOG that the new constraint added is of the form
$\mathbf{a}_{m+1}{}^T\mathbf{x} \leq b_{m+1}$, where $\mathbf{a}_{m+1}{}^T$ is a row vector.
Case 1: $\mathbf{x}_0$ satisfies $\mathbf{a}_{m+1}{}^T\mathbf{x} \leq b_{m+1}$.
Then it can be shown that $\mathbf{x}_0$ is optimal for the changed problem also.
Let $S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and
$S' = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, a_{m+1}{}^T\mathbf{x} \leq b_{m+1}, \mathbf{x} \geq \mathbf{0}\}$, then $S' \subseteq S$ and $\mathbf{x}_0 \in S'$.
For any $\mathbf{x} \in S'$, since $\mathbf{x} \in S$
$\mathbf{c}^T\mathbf{x} \geq min_{\mathbf{x} \in S}\mathbf{c}^T\mathbf{x} = \mathbf{c}^T\mathbf{x}_0$.

Case 2: $\mathbf{x}_0$ does not satisfy the new constraint, that is $\mathbf{a}_{m+1}{}^T\mathbf{x}_0 > b_{m+1}$,
or $\quad \mathbf{a}^T_{B,m+1}B^{-1}\mathbf{b} > b_{m+1}, \quad (**)$
where $\mathbf{a}^T_{B,m+1}$ and $\mathbf{a}^T_{N,m+1}$ are the components of $\mathbf{a}^T_{m+1}$ corresponding to the basic variables and nonbasic variables of $\mathbf{x}_0$, respectively. Note that the new constraint can be written as $\mathbf{a}_{m+1}{}^T\mathbf{x} + s_{m+1} = b_{m+1}$, where $s_{m+1} \geq 0$.
If B denotes the basis matrix corresponding to $\mathbf{x}_0$, then

$$\begin{bmatrix} B & \mathbf{0} \\ \mathbf{a}^T_{B,m+1} & 1 \end{bmatrix}$$

is the new basis matrix after the new constraint is added, the new basic variable added being s_{m+1}.
The inverse of this basis matrix is given by

$$\begin{bmatrix} B^{-1} & \mathbf{0} \\ -\mathbf{a}^T_{B,m+1}B^{-1} & 1 \end{bmatrix}.$$

Hence the new row added to the (main) simplex table will be of the form

$$[-\mathbf{a}^T_{B,m+1}B^{-1}, 1]\begin{bmatrix} B & N & \mathbf{0} \\ \mathbf{a}^T_{B,m+1} & \mathbf{a}^T_{N,m+1} & 1 \end{bmatrix} = [\mathbf{0}, -\mathbf{a}^T_{B,m+1}B^{-1}N + \mathbf{a}^T_{N,m+1}, 1].$$

The new RHS becomes

$$\begin{bmatrix} B^{-1} & \mathbf{0} \\ -\mathbf{a}^T_{B,m+1}B^{-1} & 1 \end{bmatrix}\begin{bmatrix} \mathbf{b} \\ b_{m+1} \end{bmatrix} = \begin{bmatrix} B^{-1}\mathbf{b} \\ b_{m+1} - \mathbf{a}^T_{B,m+1}B^{-1}\mathbf{b} \end{bmatrix}.$$

Hence if $\mathbf{x}_0$ does not satisfy the newly added constraint then from $(**)$,
the $(m+1)$ th entry of the RHS will be strictly less than zero. Note that since the cost associated with s_{m+1} is equal to zero, all the $c_j - z_j$ will remain same as in the optimal table corresponding to $\mathbf{x}_0$, hence all these values will be nonnegative. Hence

the dual simplex method can be used to get a basic feasible solution, hence an optimal solution of the primal or conclude that the primal is infeasible, if the addition of the new constraint makes the feasible region of **LPI** empty.

8. **Changing entries in the coefficient matrix A:**

   (a) **Case 1:** If the column corresponding to a nonbasic variable say $\tilde{\mathbf{a}_j}$ of the optimal solution is changed to say $\tilde{\mathbf{a}}'_j$, then accordingly changes has to be made in the column corresponding to that variable in the optimal simplex table.
   $B^{-1}\tilde{\mathbf{a}}_j$ is changed to $B^{-1}\tilde{\mathbf{a}}'_j$.
   If the new $c_j - z'_j$ value satisfies the optimality condition, then the previous optimal solution will be optimal for the new problem as well.
   If not, then use the simplex method to obtain the new optimal solution or to conclude that the primal does not have an optimal solution (this will happen if the new column $B^{-1}\tilde{\mathbf{a}}'_j \leq \mathbf{0}$ and $c_j - z'_j < 0$).

   (b) **Case 2:** If the column corresponding to the j th basic variable say $\tilde{\mathbf{a}}_j$ is changed to $\tilde{\mathbf{a}}'_j$. Then treat this case as the case when a variable (say a variable x_{n+1}) is added to the problem with column $\tilde{\mathbf{a}}'_j$ and cost c_j (that is the cost associated with the j th variable) and calculate $B^{-1}\tilde{\mathbf{a}}'_j$ and the corresponding $c_{n+1} - z_{n+1}$ value.

      i. If $u_{j,(n+1)} = 0$, then it implies that $\tilde{\mathbf{a}}'_j$ can be expressed as a linear combination of the other (m-1) columns of the basis given by $\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_{j-1}, \tilde{\mathbf{a}}_{j+1}, \ldots, \tilde{\mathbf{a}}_m$, that is, the set $\{\tilde{\mathbf{a}}_1, \ldots, \tilde{\mathbf{a}}_{j-1}, \tilde{\mathbf{a}}'_j, \tilde{\mathbf{a}}_{j+1}, \ldots, \tilde{\mathbf{a}}_m\}$ is a linearly dependent set. Hence in this case we have to find an initial basic feasible solution for the changed problem.

      ii. If $u_{j,(n+1)} \neq 0$, then pivot on this element and make the variable x_{n+1} enter the basis and x_j leave the basis. Perform the necessary elementary row operations to make this (the (n+1) th) column as the j th column of the Identity matrix and the $c_{n+1} - z_{n+1}$ value equal to 0.
      Then delete the column corresponding to $\tilde{\mathbf{a}}_j$ from all subsequent calculations, and the variable x_{n+1} is now indexed as the variable x_j.
      Note that necessary calculations mentioned above might disturb and optimality as well as the feasibility of the optimal table (that is some of the $c_j - z_j$ values might become negative and also some of the RHS entries might become negative), hence again an initial basic feasible solution of the new problem has to be found.
      If however the new $c_j - z_j$ values are all nonnegative and all the RHS entries remain nonnegative, then the table is an optimal table and the corresponding basic feasible solution is optimal for the new problem.
      If the new table has all RHS entries nonnegative, but atleast one of the $c_j - z_j$ values is negative then use the simplex method to obtain the new optimal solution, or to conclude that the new problem does not have an optimal solution, as the case may be.
      If all the $c_j - z_j$ values are nonnegative but atleast one of the RHS entries

in the simplex table is negative then the dual simplex method to obtain the new optimal solution or to conclude that the new problem has no feasible solution, as the case may be.

Artificial Variable Method to find an initial basic feasible solution of linear programming problems: Big- M method:

Consider the problem **LP1**:
Min $\mathbf{c}^T\mathbf{x}$
subject to
$A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$.
We can assume WLOG that, $\mathbf{b} \geq \mathbf{0}$.
(The equations can be multiplied throughout by a (-1) if required, to get a nonnegative RHS).
Add artificial variables $w_1, \ldots, w_m$ with costs M (M very large) to the above problem such that we get an initial basic feasible solution to the following problem **LP(M)** given by,
Min $\mathbf{c}^T\mathbf{x} + [M, \ldots, M]\mathbf{w}$
subject to
$A\mathbf{x} + \mathbf{w} = [A : I] \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{w} \geq \mathbf{0}$,
where $\mathbf{w} = [w_1, \ldots, w_m]^T$ and the initial basic feasible solution is $[\mathbf{0}_{1\times n}, w_1, \ldots, w_m]^T$.

Case 1: **LP(M)** has an optimal solution.

Case 1a: $[{\mathbf{x}_*}^T, \mathbf{0}_{1\times m}]^T$ is an optimal basic feasible solution of **LP(M)** with all the artificial variables as nonbasic variables.
For any $\mathbf{x}$ feasible for **LP1**, $[\mathbf{x}^T, \mathbf{0}_{1\times m}]^T$ is feasible for **LP(M).**
Since $[{\mathbf{x}_*}^T, \mathbf{0}]^T$ is optimal for **LP(M)**
$\mathbf{c}^T\mathbf{x}_* + [M, \ldots, M]\mathbf{0}_{m\times 1} \leq \mathbf{c}^T\mathbf{x} + [M, \ldots, M]\mathbf{0}_{m\times 1}$ for all $\mathbf{x} \in$ **Fea(LP1)**,
or $\mathbf{c}^T\mathbf{x}_* \leq \mathbf{c}^T\mathbf{x}$ for all $\mathbf{x} \in$ **Fea(LP1)**.
Since $\mathbf{x}_* \in Fea(\mathbf{LP1})$, hence the above condition implies that $\mathbf{x}_*$ is optimal for **LP1**, so we got an optimal solution for the original problem.

Case 1b: $[{\mathbf{x}_*}^T, {\mathbf{w}_*}^T]^T$ is optimal for **LP(M)** and $\mathbf{w}_* \neq \mathbf{0}$.
Then, $\mathbf{c}^T\mathbf{x}_* + [M, \ldots, M]\mathbf{w}_* \leq \mathbf{c}^T\mathbf{x} + [M, \ldots, M]\mathbf{0}_{m\times 1} = \mathbf{c}^T\mathbf{x}$, for all $\mathbf{x} \in$ **Fea(LP1)**, for all M sufficiently large.
But this is a contradiction.
Hence **(LP1)** does not have a feasible solution.

Remark: Note that there exists an N such that for all $M > N$, the optimal solution for **LP(M)** does not change. In the optimal table since $c_j - z_j \geq 0$ for all $j = 1, \ldots, n$,
if $c_j - z_j$ is dependent on M (is a function of M), then $c_j - z_j$ is taken to be positive only if it is of the form $aM + c$, where $a > 0$ (since M is large). Hence even though the cost associated with the artificial variables M is taken to be variable which is large, the optimal solution will not change.

Case 2: $\mathbf{LP(M)}$ does not have an optimal solution.

Case 2a: In some iteration (or simplex table), there exists a k such that $c_k - z_k < 0$, the corresponding column $B^{-1}\tilde{\mathbf{a}}_k \leq \mathbf{0}$ and $\mathbf{w} = \mathbf{0}$ in the corresponding basic feasible solution for $\mathbf{LP(M)}$.
So there exists a direction $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix}$ of $Fea(\mathbf{LP(M)})$ such that
$\mathbf{c}^T\mathbf{d}_1 + [M, \ldots, M]\mathbf{d}_2 < 0.$ $\qquad (*)$
But since M is very large and $\mathbf{d}_2 \geq \mathbf{0}$, $(*)$ implies that $\mathbf{d}_2 = \mathbf{0}$ and $\mathbf{c}^T\mathbf{d}_1 < 0$.
Note that since $\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix}$ is a direction of $Fea(\mathbf{LP(M)})$, and $\mathbf{d}_2 = \mathbf{0}$,
$[A : I] \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \end{bmatrix} = A\mathbf{d}_1 = \mathbf{0}.$
Since $\mathbf{d}_1 \geq \mathbf{0}$, the above condition implies that $\mathbf{d}_1$ is a direction of $Fea(\mathbf{LP1})$ and since $\mathbf{c}^T\mathbf{d}_1 < 0$, hence $\mathbf{LP1}$ does not have an optimal solution but **has** a feasible solution.

Case 2b: In some iteration (or simplex table), $c_k - z_k = min\{c_j - z_j : c_j - z_j < 0\}$, $B^{-1}\tilde{\mathbf{a}}_k \leq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$ in the corresponding basic feasible solution for $\mathbf{LP(M)}$, where B is a basis matrix corresponding to this basic feasible solution.
Let $x_1, \ldots, x_p$ be the basic variables which are not artificial variables and let $w_1, \ldots, w_{m-p}$ be the artificial variables which are basic variables in the current basic feasible solution for $\mathbf{LP(M)}$.
Note that $z_k = \sum_{i=1}^{p} c_i u_{ik} + \sum_{i=p+1}^{m} M u_{ik}$, where the u_{ik}'s have their usual meaning.
Since $c_k - z_k = c_k - \sum_{i=1}^{p} c_i u_{ik} - M(\sum_{i=p+1}^{m} u_{ik}) < 0$, and $\sum_{i=p+1}^{m} u_{ik} \leq 0$,
we must have $\sum_{i=p+1}^{m} u_{ik} = 0$, which implies that the coefficient of M in the above expression is non negative (should actually be equal to 0).
Let x_j be a non basic variable which is not an artificial variable then since
$c_j - z_j = c_j - \sum_{i=1}^{p} c_i u_{ij} - M(\sum_{i=p+1}^{m} u_{ij}),$ $\qquad (*)$
and the coefficient of M in the expression for $c_k - z_k$ is nonnegative,
so $\sum_{i=p+1}^{m} u_{ij} \leq 0$ (otherwise the coefficient of M in (*) will be negative which will contradict that $c_k - z_k$ is the most negative among the $c_j - z_j$ values).
Note that for any $\mathbf{x} \in Fea(\mathbf{LP1})$, $\qquad [\mathbf{x}^T, \mathbf{0}_{1\times m}]^T \in Fea(\mathbf{LP(M)})$.
Hence $[\mathbf{x}^T, \mathbf{0}_{1\times m}]^T$ should satisfy the system of equations :
$B^{-1}[A : I] \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} = B^{-1}\mathbf{b}$ $\qquad (1)$
which is equivalent to the system
$[A : I] \begin{bmatrix} \mathbf{x} \\ \mathbf{w} \end{bmatrix} = \mathbf{b}.$ $\qquad (2)$
Hence $[\mathbf{x}^T, \mathbf{0}_{1\times m}]^T$ should satisfy the equation

$$[\mathbf{0}_{1\times p}, \mathbf{1}_{1\times(m-p)}, \sum_{i=p+1}^m u_{i,(m+1)}, \ldots, \sum_{i=p+1}^m u_{i,(m+n)}] \begin{bmatrix} x_1 \\ \ldots \\ x_p \\ w_1 \\ \ldots \\ w_{m-p} \\ x_{p+1} \\ \ldots \\ x_n \\ w_{m-p+1} \\ \ldots \\ w_m \end{bmatrix} = \sum_{i=p+1}^m (B^{-1}\mathbf{b})_i = \sum_{i=p+1}^m w_i,$$

which is obtained by adding the last $(m-p)$ equations of the system (1).
(The vector $\mathbf{1}_{1\times(m-p)}$ is a row vector with all components equal to 1.)
Hence $[\mathbf{x}^T, \mathbf{0}_{1\times m}]^T$ should satisfy
$\sum_{j=p+1}^n (\sum_{i=p+1}^m u_{i,(m-p+j)})x_j = \sum_{i=p+1}^m w_i$.

But this is a contradiction since $x_j \geq 0$ for all j,
$\sum_{i=p+1}^m u_{i,(m-p+j)} \leq 0$, for all $j = p+1, \ldots, n$ and $\sum_{i=p+1}^m w_i > 0$.
Hence **LP1** does not have a feasible solution.

Note that the above conclusion that is **LP1** is infeasible may not be true if $c_k - z_k$ is not the most negative among the $c_j - z_j$ values.

Case 2c: In some iteration (or simplex table), there exists a k such that $c_k - z_k < 0$, the corresponding column $B^{-1}\tilde{\mathbf{a}}_k \leq \mathbf{0}$ and $\mathbf{w} \neq \mathbf{0}$ in the corresponding BFS for **LP(M)**. Then combining **Case 2(a)** and **Case 2(b)** we can say **LP1** does **not** have an **optimal** solution (may be **infeasible**, **may not** be **infeasible**).

Remark 1: If at some stage of solving the artificial variables leave the basis, then the corresponding basic solution is a feasible basic solution for the original problem (without artificial variables). Then delete all columns corresponding to the artificial variables and continue.

Remark 2: If $[\mathbf{x}^T, \mathbf{0}]$ is optimal for **LP(M)**, but one or more artificial variables remain in the basis for the optimal basic feasible solution at zero value.
Then we have already seen that $\mathbf{x}$ will be optimal for **LP1**.
In order to get an optimal basic feasible solution for **LP1**, pivot appropriately on a nonzero entry in the rows corresponding to the basic artificial variables and make one of the variables x_j enter the basis in place of each of the artificial variables.
Since in each of the rows corresponding to the artificial basic variables, there will always be a nonzero entry in atleast one of the columns corresponding to the (nonbasic) original variables $x_{p+1}, \ldots, x_n$. (*)
Hence it will always be possible to pivot appropriately and remove the artificial variable from the basis and bring in one of $x_{p+1}, \ldots, x_n$ in the basis.
If (*) is not true then it would contradict that $rank(A) = m$ (check this).

Example 1 : Consider the problem **(P)**,
Minimize $x_1 - x_2$
subject to
$2x_1 + x_2 \geq 4$
$x_1 - x_2 \leq 1$
$x_1 \geq 0,\ x_2 \geq 0.$

By adding variables we get the following problem,
Minimize $x_1 - x_2$
subject to
$2x_1 + x_2 - s_1 = 4$
$x_1 - x_2 + s_2 = 1$
$x_1 \geq 0,\ x_2 \geq 0,\ s_1 \geq 0,\ s_2 \geq 0.$
If we consider the initial basic solution with basic variables s_1 and s_2, then the $c_j - z_j$ values are not ≥ 0 for all j, here we do not have a feasible solution of the dual to use Dual Simplex to solve (P).
Also we are not provided with an easy initial basic feasible solution of **(P)** (since $s_1 = -4$)to start with Simplex algorithm.
Hence the (**Big-M**) is used which provides an initial basic feasible solution and hence an optimal solution of the primal.
Consider the modified problem
Minimize $x_1 - x_2 + Mw$
subject to
$2x_1 + x_2 - s_1 + w = 4$
$x_1 - x_2 + s_2 = 1$
$x_1, x_2, s_1, s_2, w \geq 0.$
Here w is called the artificial variable and cost M associated with it is very large.
The initial table corresponding to the basic variables w and s_2 is given below.
(Note that we have used notations $B^{-1}\mathbf{s}_1, B^{-1}\mathbf{s}_2, B^{-1}\mathbf{w}$ instead of the vectors $-B^{-1}\mathbf{e}_1, B^{-1}\mathbf{e}_2, B^{-1}\mathbf{e}_1$, respectively, to remember which columns correspond to which variables and also to avoid unnecessary confusion).

	$c_1 - z_1 = 1 - 2M$	$c_2 - z_2 = -M - 1$	$c_3 - z_3 = M$	$c_4 - z_4 = 0$	$c_5 - z_5 = 0$	
	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
w	2	1	-1	0	1	4
s_2	1	-1	0	1	0	1

s_2 will be the leaving variable and x_1 will be the entering variable for the next table.

	$c_1 - z_1 = 0$	$c_2 - z_2 = -3M$	$c_3 - z_3 = M$	$c_4 - z_4 = 2M - 1$	$c_5 - z_5 = 0$	
	$B^{-1}\tilde{\mathbf{a}_1}$	$B^{-1}\tilde{\mathbf{a}_2}$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
w	0	3	-1	-2	1	2
x_1	1	-1	0	1	0	1

Now the artificial variable w will be the leaving variable and x_2 will be the entering variable.

	$c_1 - z_1 = 0$	$c_2 - z_2 = 0$	$c_3 - z_3 = 0$	$c_4 - z_4 = -1$	$c_5 - z_5 = M$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
x_2	0	1				$\frac{2}{3}$
x_1	1	0				$1 + \frac{2}{3}$

Now continue with the basic feasible solution (this is the initial basic feasible solution of the original problem, without the artificial variable) corresponding to the basic vectors x_1 and x_2 and drop the column corresponding to the artificial variable w from all future calculations. Use simplex method to get the optimal basic feasible solution.

Example 2: Consider the problem,
Minimize $x_1 + x_2$
subject to
$x_1 + 2x_2 \leq 2$
$3x_1 + 5x_2 \geq 15$
$x_1 \geq 0$, $x_2 \geq 0$.
We consider the corresponding modified problem
Minimize $x_1 + x_2 + Mw$
subject to
$x_1 + 2x_2 + s_1 = 2$
$3x_1 + 5x_2 - s_2 + w = 15$
$x_1, x_2, s_1, s_2, w \geq 0$.

Note that this problem can also be solved by dual simplex method.
But we will solve it by using the **Big-M** method.

The initial table is given by

	$c_1 - z_1 = 1 - 3M$	$c_2 - z_2 = 1 - 5M$	$c_3 - z_3 = 0$	$c_4 - z_4 = M$	$c_5 - z_5 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
s_1	1	2	1	0	0	2
w	3	5	0	-1	1	15

Here x_2 is the entering variable and s_1 the leaving variable, hence the next table is given by,

	$c_1 - z_1 = \frac{1}{2}(1 - M)$	$c_2 - z_2 = 0$	$c_3 - z_3 = \frac{1}{2}(5M - 1)$	$c_4 - z_4 = M$	$c_5 - z_5 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
x_2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	1
w	$\frac{1}{2}$	0	$-\frac{5}{2}$	-1	1	10

The next table is given by

	$c_1 - z_1 = 0$	$c_2 - z_2 = M - 1$	$c_3 - z_3 = 3M - 1$	$c_4 - z_4 = M$	$c_5 - z_5 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{w}$	$B^{-1}\mathbf{b}$
x_1	1	2	1	0	0	2
w	0	-1	-3	-1	1	9

Since the optimal table has an artificial variable taking positive value hence the conclusion is that the original problem (without the artificial variable) has no feasible solution.

Example 3: Consider the problem,
Minimize $-x_1 + x_2$
subject to
$x_1 - 2x_2 - x_3 = 1$
$-x_1 + 2x_2 - x_4 = 1$
$x_i \geq 0$, for all $i = 1, 2, 3, 4$.
Consider the corresponding modified problem,
Minimize $-x_1 + x_2 + Mw_1 + Mw_2$
subject to
$x_1 - 2x_2 - x_3 + w_1 = 1$
$-x_1 + 2x_2 - x_4 + w_2 = 1$
$x_i \geq 0$ and $w_j \geq 0$ for all $i = 1, 2, 3, 4$ and $j = 1, 2$.
The first table is given by

	$c_1 - z_1 = -1$	$c_2 - z_2 = 1$	$c_3 - z_3 = M$	$c_4 - z_4 = M$	$c_5 - z_5 = 0$	$c_6 - z_6 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{w}_1$	$B^{-1}\mathbf{w}_2$	$B^{-1}\mathbf{b}$
w_1	1	-2	-1	0	1	0	1
w_2	-1	2	0	-1	0	1	1

	$c_1 - z_1 = 0$	$c_2 - z_2 = -1$	$c_3 - z_3 = M - 1$	$c_4 - z_4 = M$	$c_5 - z_5 = 1$	$c_6 - z_6 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{w}_1$	$B^{-1}\mathbf{w}_2$	$B^{-1}\mathbf{b}$
x_1	1	-2	-1	0	1	0	1
w_2	0	0	-1	-1	1	1	2

Note that here the modified problem (with artificial variables) has no optimal solution and the original problem has no feasible solution (since the column corresponding to the most negative $c_j - z_j$ value, (only one negative) is non positive).

Example 4: Consider the problem,
Minimize $-x_1 - x_2$
subject to
$x_1 - x_2 - x_3 = 1$
$-x_1 + x_2 + 3x_3 - x_4 = 0$
$x_i \geq 0$, for all $i = 1, 2, 3, 4$.
Note that this problem has a feasible solution and given by $x_1 = 2, x_2 = 0, x_3 = 1$ and $x_4 = 1$.
Consider the corresponding problem,
Minimize $-x_1 - x_2 + Mw_1 + Mw_2$
subject to
$x_1 - x_2 - x_3 + w_1 = 1$
$-x_1 + x_2 + 3x_3 - x_4 + w_2 = 0$
$x_i \geq 0$ and $w_j \geq 0$ for all $i = 1, 2, 3, 4$ and $j = 1, 2$.

The simplex tables are given by

	$c_1 - z_1 = -1$	$c_2 - z_2 = -1$	$c_3 - z_3 = -2M$	$c_4 - z_4 = M$	$c_5 - z_5 = 0$	$c_6 - z_6 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{w}_1$	$B^{-1}\mathbf{w}_2$	$B^{-1}\mathbf{b}$
w_1	1	-1	-1	0	1	0	1
w_2	-1	1	3	-1	0	1	0

	$c_1 - z_1 = 0$	$c_2 - z_2 = -2$	$c_3 - z_3 = -2M - 1$	$c_4 - z_4 = M$	$c_5 - z_5 = 1$	$c_6 - z_6 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{w}_1$	$B^{-1}\mathbf{w}_2$	$B^{-1}\mathbf{b}$
x_1	1	-1	-1	0	1	0	1
w_2	0	0	2	-1	1	1	1

Note that although here also corresponding to a negative $c_j - z_j$ value $(c_2 - z_2)$, the column is non positive, but the original problem has a feasible solution ($c_2 - z_2$ is not the most negative among the $c_j - z_j$ values).

Two Phase Method :

This method is generally used when neither a feasible solution of the dual nor the primal is provided. This is also used sometimes to get an initial basic feasible solution of the primal, if the dual has a feasible solution.

Example 1 :

Minimize $x_1 - x_2$

subject to

$2x_1 + x_2 \geq 4$

$x_1 - x_2 \leq 1$

$x_1 \geq 0$, $x_2 \geq 0$.

Consider the corresponding Phase I problem.

Phase I problem when Two phase method is applied to **Example 1:** is given by:

Minimize x

subject to

$2x_1 + x_2 - s_1 + x = 4$

$x_1 - x_2 + s_2 = 1$

$x_1, x_2, x \geq 0.$

The initial table corresponding to the basic variables $\mathbf{x}$ and s_2 is given below.

	$c_1 - z_1 = -2$	$c_2 - z_2 = -1$	$c_3 - z_3 = 1$	$c_4 - z_4 = 0$	$c_5 - z_5 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{x}$	$B^{-1}\mathbf{b}$
x	2	1	-1	0	1	4
s_2	1	-1	0	1	0	1

The next table will be given by

	$c_1 - z_1 = 0$	$c_2 - z_2 = -3$	$c_3 - z_3 = 1$	$c_4 - z_4 = 2$	$c_5 - z_5 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{x}$	$B^{-1}\mathbf{b}$
x	0	3	-1	-2	1	2
x_1	1	-1	0	1	0	1

Now the artificial variable x will leave the basis.
So the next table will be

	$c_1 - z_1 = 0$	$c_2 - z_2 = 0$	$c_3 - z_3 = 0$	$c_4 - z_4 =$	$c_5 - z_5 =$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\mathbf{s}_1$	$B^{-1}\mathbf{s}_2$	$B^{-1}\mathbf{x}$	$B^{-1}\mathbf{b}$
x_2	0	1				$\frac{2}{3}$
x_1	1	0				$1 + \frac{2}{3}$

Rule 1: If the optimal solution of Phase 1 has optimal value 0 then the original problem has a feasible solution.
If not then the original problem does not have a feasible solution.

Phase II

In Phase II start with the same table only since the objective function is now changed, in the previous table the $z_j - c_j$ values will change.
We have an initial basic feasible solution of the original problem, use simplex method to obtain the optimal basic feasible solution.

Hence from **Phase I** of the Two phase method applied to **Example 1** we got an initial BFS which is given by $x_1 = \frac{2}{3}$, $x_2 = \frac{5}{3}$, $s_1 = 0$, $s_2 = 0$.
Now use the obtained BFS to get the optimal solution (if it exists) in **Phase II**.

Example 2 : Consider the problem,
Minimize $-x_1 + x_2$
subject to
$x_1 - 2x_2 - x_3 = 1$
$-x_1 + 2x_2 - x_4 = 1$
$x_i \geq 0$, for all $i = 1, 2, 3, 4$.
Consider the corresponding **Phase I** problem,
Minimize $w_1 + w_2$
subject to
$x_1 - 2x_2 - x_3 + w_1 = 1$
$-x_1 + 2x_2 - x_4 + w_2 = 1$
$x_i \geq 0$ and $w_j \geq 0$ for all $i = 1, 2, 3, 4$ and $j = 1, 2$.
The first table is given by

	$c_1 - z_1 = 0$	$c_2 - z_2 = 0$	$c_3 - z_3 = 1$	$c_4 - z_4 = 1$	$c_5 - z_5 = 0$	$c_6 - z_6 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{w}_1$	$B^{-1}\mathbf{w}_2$	$B^{-1}\mathbf{b}$
w_1	1	-2	-1	0	1	0	1
w_2	-1	2	0	-1	0	1	1

Conclusion: The primal does not have a feasible solution.

Example 3: Consider the problem,
Minimize $-x_1 - x_2$
subject to
$x_1 - x_2 - x_3 = 1$
$-x_1 + x_2 + 3x_3 - x_4 = 0$
$x_i \geq 0$, for all $i = 1, 2, 3, 4$.
Note that this problem has a feasible solution and given by $x_1 = 2, x_2 = 0, x_3 = 1$ and $x_4 = 1$.
Consider the corresponding problem,
Minimize $w_1 + w_2$
subject to
$x_1 - x_2 - x_3 + w_1 = 1$
$-x_1 + x_2 + 3x_3 - x_4 + w_2 = 0$
$x_i \geq 0$ and $w_j \geq 0$ for all $i = 1, 2, 3, 4$ and $j = 1, 2$.

The first two tables are given by:

	$c_1 - z_1 = 0$	$c_2 - z_2 = 0$	$c_3 - z_3 = -2$	$c_4 - z_4 = 1$	$c_5 - z_5 = 0$	$c_6 - z_6 = 0$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{w}_1$	$B^{-1}\mathbf{w}_2$	$B^{-1}\mathbf{b}$
w_1	1	-1	-1	0	1	0	1
w_2	-1	1	3	-1	0	1	0

	$c_1 - z_1 = -\frac{2}{3}$	$c_2 - z_2 = \frac{2}{3}$	$c_3 - z_3 = 0$	$c_4 - z_4 = \frac{1}{3}$	$c_5 - z_5 = 0$	$c_6 - z_6 = -\frac{1}{3}$	
	$B^{-1}\tilde{\mathbf{a}}_1$	$B^{-1}\tilde{\mathbf{a}}_2$	$B^{-1}\tilde{\mathbf{a}}_3$	$B^{-1}\tilde{\mathbf{a}}_4$	$B^{-1}\mathbf{w}_1$	$B^{-1}\mathbf{w}_2$	$B^{-1}\mathbf{b}$
w_1	$\frac{2}{3}$	$-\frac{2}{3}$	0	$-\frac{1}{3}$	1	$\frac{1}{3}$	1
x_3	$-\frac{1}{3}$	$\frac{1}{3}$	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	0

Check for calculation mistakes.
Continue solving and check that Phase I will give a basic feasible solution of the original problem.