MA423 (Post Midsem).

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Ex :- $\|r\| = \|c_2\|$

$$b = b_1 + b_2$$
$$= Q_1 Q_1^T b + (I - Q_1 Q_1^T) b$$
$$(I - Q_1 Q_1^T) = Q_2 Q_2^T.$$

$$r = Q_2 Q_2^T b$$

$$\Rightarrow \|r\| = \|Q_2 Q_2^T b\| = \|Q_2^T b\| = \|c_2\|$$

(Q_2 is isometry).

$*\ (\Sigma^*)_{m\times n} \neq \Sigma_{n\times m}$

$$\|A\|_2 = \|U \Sigma V^*\|_2 = \|\Sigma V^*\|_2 = \|V \Sigma^*\|_2 = \|\Sigma^*\|_2 = \sigma_1.$$

(U is unitary)

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$$A_{n\times m}\, x = b_{n\times 1}, \quad n >> m.$$

Case I :- $S := \{x_0 \in \mathbb{R}^m : \|b - A x_0\|_2 = \min_{x \in \mathbb{R}^m} \|b - Ax\|_2\}$.

$r = \text{rank } A = m$. Let, $b = b_1 \oplus b_2$, $b_1 \in R(A)$, $b_2 \in R(A)^\perp = N(A^T)$.

$x_0 \in S \Rightarrow A x_0 = b_1$

$$\Rightarrow U_r \Sigma_r V_r^* x_0 = U_r U_r^* b$$

$$\Rightarrow U_r^* U_r \Sigma_r V_r^* x_0 = U_r^* U_r U_r^* b.$$

$$\Rightarrow \Sigma_r V_r^* x_0 = U_r^* b.$$

$$\Rightarrow V_r^* x_0 = \Sigma_r^{-1} U_r^* b.$$

$$\Rightarrow V_r V_r^* x_0 = V_r \Sigma_r^{-1} U_r^* b.$$

$\because r = m$, $\Rightarrow x_0 = V_r \Sigma_r^{-1} U_r^* b \quad (\because V_m V_m^* = I_m)$.

$$\boxed{x_0 = A^+ b} \qquad \therefore S = \{A^+ b\}.$$

Case II :- $r < m$,

Let $\hat{x}_0 \in \mathbb{R}^m$, s.t. $\boxed{\hat{x}_0 \in S}$?

$$\|\hat{x}_0\|_2 = \min\{\|x_0\|_2 : x \in S\}.$$

$$\hat{x}_0 \in \mathbb{R}^m = R(A^*) \oplus R(A^*)^{\perp} = N(A),$$
$$= R(A^*) \oplus N(A).$$

$$\Rightarrow \hat{x}_0 = \underset{\in R(A^*)}{\hat{x}_{01}} + \underset{\in N(A)}{\hat{x}_{02}}.$$

$$\Rightarrow \|\hat{x}_0\|_2^2 = \|\hat{x}_{01}\|_2^2 + \|\hat{x}_{02}\|_2^2.$$

$$\Rightarrow \|\hat{x}_0\|_2^2 \geq \|\hat{x}_{01}\|_2^2.$$

$$\hat{x}_{01} = V_r V_r^* \hat{x}_0.$$

$\hat{x}_0 \in S \Rightarrow \hat{x}_{01} = A^+ b.$ (from Case I).

$$A\hat{x}_0 = b_1$$
$$\Rightarrow A\hat{x}_{01} + A\hat{x}_{02} = b_1$$
$$\Rightarrow A\hat{x}_{01} = b_1 \Rightarrow \boxed{\hat{x}_{01} \in S}.$$

$$\Rightarrow \|\hat{x}_{01}\|_2^2 \geq \|\hat{x}_0\|_2^2 = \|\hat{x}_{01}\|_2^2 + \|\hat{x}_{02}\|_2^2$$
$$\Rightarrow \|\hat{x}_{02}\|_2 = 0$$
$$\therefore \hat{x}_0 = \hat{x}_{01} = A^+ b.$$

$$A = U \Sigma V^*, \quad A_k = U \Sigma_k V^*.$$

$$\Rightarrow A - A_k = U(\Sigma - \Sigma_k) V^*$$

$$\Sigma - \Sigma_k = \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & \sigma_{k+1} & \\ & & & & \ddots \\ & & & & & \sigma_n \end{bmatrix} \Rightarrow \boxed{\|A - A_k\|_2 = \sigma_{k+1}}$$

$$\therefore \sigma_{k+1} = \|A - A_k\|_2 \geq \min\{\|A - B\|_2 : B \in F^{n \times m}, \text{rank}(B) \leq k\}.$$

Let, $B \in F^{n \times m}$, rank $(B) \leq k$.

$\therefore \dim(N(B)) \geq m - k$.

$\therefore N(B) \subseteq F^m$.

Let, $S = \text{span}\{v_1, \dots, v_{K+1}\} \subseteq F^m$.

$\dim(S) = K+1$.

$\therefore \dim(N(B)+S) = \dim(N(B)) + \dim(S) - \dim(N(B) \cap S)$.

$\Rightarrow \dim(N(B) \cap S) = \dim(N(B)) + \dim(S) - \dim(N(B)+S)$.

$$\geq (m-K) + (K+1) - m$$
$$= 1.$$

$\therefore x_0 \in S \cap N(B)$.

$(A-B)x_0 = Ax_0$

$x_0 \in S \Rightarrow x_0 = \sum_{j=1}^{K+1} \alpha_j v_j$ & $\|x_0\|_2^2 = \sum_{j=1}^{K+1} \alpha_j^2$

$\alpha_i = \langle x_0, v_i \rangle$

$$\|(A-B)x_0\|_2^2 = \|Ax_0\|_2^2 = \|A(\sum_{j=1}^{K+1} \alpha_j v_j)\|_2^2 = \|\sum_{j=1}^{K+1} \alpha_j Av_j\|_2^2 = \|\sum_{j=1}^{K+1} \alpha_j \sigma_j u_j\|_2^2$$

$$\text{"} = \sum_{j=1}^{K+1} \sigma_j^2 |\alpha_j|^2$$

$$\geq \sigma_{K+1}^2 \sum_{j=1}^{K+1} |\alpha_j|^2 = \sigma_{K+1}^2 \|x_0\|^2.$$

$$\Rightarrow \frac{\|(A-B)x_0\|_2}{\|x_0\|_2} \geq \sigma_{K+1}$$

$\therefore \|A-B\|_2 \geq \sigma_{K+1}$

Schur's theorem :-

$A \in \mathbb{C}^{n \times n}$

Let, $\lambda \in \mathbb{C}$, $v \neq 0, v \in \mathbb{C}^n \ni Av = \lambda v$.

If $n=1$, $A = [a]$, ✓ (trivial).

Suppose that ✓ for sizes $< n$.

Let, $q_1 = \frac{v}{\|v\|_2} \Rightarrow \|q_1\|_2 = 1$, $Aq_1 = \lambda q_1$

Let $\{q_1, w_2, \dots, w_n\}$ be an LI set in $\mathbb{C}^n$.

Then let $S := [q_1\ w_2 \cdots w_n] = \tilde{Q}R$ be a QR decomposition of S.

$\tilde{Q} = [q_1\ q_2 \cdots q_n] \leftarrow$ orthonormal basis

$$\tilde{Q}^* A \tilde{Q} = \begin{bmatrix} q_1^* \\ \vdots \\ q_n^* \end{bmatrix} A \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$$

$$= \begin{bmatrix} q_1^* \\ \vdots \\ q_n^* \end{bmatrix} \begin{bmatrix} Aq_1 & Aq_2 & \cdots & Aq_n \end{bmatrix}$$

$$= \begin{bmatrix} q_1^* A q_1 & q_1^* A q_2 & \cdots & q_1^* A q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_n^* A q_1 & q_n^* A q_2 & \cdots & q_n^* A q_n \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \lambda q_1^* q_1 & * \\ \hline \lambda q_2^* q_1 & \\ \vdots & \tilde{A} \\ \lambda q_n^* q_1 & \end{array}\right] = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{A} \end{array}\right]$$

By induction hypothesis, $\exists$ a unitary $\tilde{Q}_1$ and an upper-triangular $\tilde{T}$ $\ni$ $\tilde{Q}_1^* \tilde{A} Q_1 = \tilde{T}$. $\Rightarrow \tilde{A} = \tilde{Q}_1 \tilde{T} \tilde{Q}_1^*$

$$\tilde{Q}^* A^* \tilde{Q} = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{A} \end{array}\right] = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{Q}_1 \tilde{T} \tilde{Q}_1^* \end{array}\right] = \left[\begin{array}{c|c} 1 & \\ \hline 0 & \tilde{Q}_1 \end{array}\right] \underbrace{\left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{T} \end{array}\right]}_{=: T} \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q}_1^* \end{array}\right].$$

$$\therefore \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q}_1^* \end{array}\right] \tilde{Q}^* A \underbrace{\boxed{\tilde{Q} \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q}_1 \end{array}\right]}}_{=: Q} = T$$

$$\therefore \boxed{Q^* A Q = T}$$

Spectral theorem for Symmetric Matrices

$A_{n\times n} \rightarrow$ real sym.

Let, $\lambda \in \mathbb{R}$, $v \in \mathbb{R}^n \ni Av = \lambda v$.

Let, $q_1 = \frac{v}{\|v\|_2} \Rightarrow Aq_1 = \lambda_1 q_1$, $\|q_1\|_2 = 1$.

~~① Base case~~

① Bluid an orthogonal matrix $\tilde{Q} \in \mathbb{R}^{n\times n} \ni \tilde{Q} = [q_1 q_2 \cdots q_n]$.

② $\tilde{Q}^T A \tilde{Q} = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & \tilde{A} \end{array}\right]$

③ Use induction hypothesis on $\tilde{A}$ where $\tilde{A}^T = \tilde{A}$

Normal matrix: $Q^* A Q = D \Rightarrow AQ = QD$
$\Rightarrow Aq_i = \lambda_i q_i$
where $Q = [q_1 \cdots q_n]$.

$\{q_1, \cdots, q_n\} \leftarrow$ orthonormal basis

Quasi upper triangular - $A = \begin{bmatrix} A_1 & \times & \times & \times & \times \\ & A_2 & \times & \times & \times \\ & & A_3 & \times & \times \\ & 0 & & \ddots & \times \\ & & & & A_p \end{bmatrix}$

$|A - \lambda I| = \prod_{i=1}^{p} |A_i - \lambda I|$

size of A_i is at most 2×2.

$x = c_1 v_1 + \dots + c_n v_n$, $c_1 \neq 0$

$A^j x = c_1 \lambda_1^j v_1 + c_2 \lambda_2^j v_2 + \dots + c_n \lambda_n^j v_n$.

$\Rightarrow \frac{A^j x}{\lambda_1^j} = c_1 v_1 + c_2 v_2 \left(\frac{\lambda_2}{\lambda_1}\right)^j + \dots + c_n v_n \left(\frac{\lambda_n}{\lambda_1}\right)^j$

$\Rightarrow \frac{A^j x}{\lambda_1^j} - c_1 v_1 = c_2 v_2 \left(\frac{\lambda_2}{\lambda_1}\right)^j + \dots + c_n v_n \left(\frac{\lambda_n}{\lambda_1}\right)^j$

$\Rightarrow \left\| \frac{A^j x}{\lambda_1^j} - c_1 v_1 \right\| = \left\| \sum_{i=2}^{n} c_i v_i \left(\frac{\lambda_i}{\lambda_1}\right)^j \right\| \leq \sum_{i=2}^{n} \left|\frac{\lambda_i}{\lambda_1}\right|^j |c_i| \, \|v_i\|$

$\therefore \lim_{j\to\infty} \left\| \frac{A^j x}{\lambda_1^j} - c_1 v_1 \right\| \leq 0$.

$\therefore \lim = 0$.

$$q_1 = \frac{Aq_0}{s_1} = \frac{Ax}{s_0 s_1} = \lambda_1 \left(\frac{\sum_{k=1}^{n} \lambda_k c_k v_k}{\lambda_1} \Big/ s_0 s_1 \right) = \frac{\lambda_1}{s_0 s_1} \left[c_1 v_1 + \sum_{k=2}^{n} \frac{\lambda_k}{\lambda_1} c_k v_k \right].$$

Let, $\tilde{q}_j = \frac{A^j x}{\lambda_1^j} \Rightarrow q_1 = \frac{\lambda_1}{s_0 s_1} \tilde{q}_1$

Similarly, $q_2 = \frac{\lambda_1^2}{s_0 s_1 s_2} \tilde{q}_2$

$\vdots$

$$q_j = \frac{\lambda_1^j}{\prod_{k=0}^{j} s_k} \tilde{q}_j$$

Let, $\mu_j = \frac{\prod_{k=0}^{j} s_k}{\lambda_1^j} \Rightarrow q_j = \frac{\tilde{q}_j}{\mu_j}$ — (*)

Let $i_j = \min\{1, \dots, n : |Aq_{j-1}(i)| = \|Aq_{j-1}\|_\infty\}$

$\Rightarrow i_j = \min\{1, \dots, n : q_j(i) = 1\}$.

(*) $\Rightarrow \frac{\tilde{q}_j(i_j)}{\mu_j} = q_j(i_j) = 1$

$$\Rightarrow \mu_j = \tilde{q}_j(i_j) = \left[\frac{A^j x}{\lambda_1^j}\right](i_j) = \Big[c_1 v_1 + \underbrace{\sum_{k=2}^{n} \left(\frac{\lambda_k}{\lambda_1}\right)^j c_k v_k}_{\xrightarrow{j \to \infty} 0}\Big](i_j)$$

$\therefore \lim_{j\to\infty} \mu_j = \lim_{j\to\infty} c_1 v_1(i_j)$

$$\therefore \lim_{j\to\infty} q_j = \frac{\lim_{j\to\infty} \tilde{q}_j}{\lim_{j\to\infty} \mu_j} = \lim_{j\to\infty} \frac{\cancel{c_1} v_1}{\cancel{c_1} v_1(i_j)} = \lim_{j\to\infty} \frac{v_1}{v_1(i_j)}$$

Also, $\lim_{j\to\infty} \left\| \frac{v_1}{v_1(i_j)} \right\|_\infty = \lim_{j\to\infty} \|q_j\|_\infty = 1$

$$\lim_{j\to\infty} Aq_j = A \lim_{j\to\infty} q_j = A \lim_{j\to\infty} \frac{v_1}{v_1(i_j)} = \lim_{j\to\infty} \frac{Av_1}{v_1(i_j)} = \lambda_1 \lim_{j\to\infty} \frac{v_1}{v_1(i_j)}$$

$$= \lambda_1 \lim_{j\to\infty} q_j$$

$$\Rightarrow \lim_{j\to\infty} Aq_j = \lambda_1 \lim_{j\to\infty} q_j.$$

$\therefore$ For large enough j,

$$Aq_j \approx \lambda_1 q_j$$

$\Rightarrow \|Aq_j\|_\infty \approx |\lambda_1|$ — ①, $\quad Aq_j(i_j) \approx \lambda_1 q_j(i_j) = \lambda_1$, — ②

$|Aq_j(i)| \approx |\lambda_1 q_j(i)| \leq |\lambda_1| \approx \|Aq_j\|_\infty \; \forall i$ — ③

① & ② $\Rightarrow |Aq_j(i_j)| \approx \|Aq_j\|_\infty$

& ③ $\Rightarrow |Aq_j(i)| \leq |\lambda_1| \approx \|Aq_j\|_\infty \;\; \forall i \neq i_j$.

Also, by def of i_j, if $i_j \neq 1$,

$$|Aq_j(i)| \approx |\lambda_1||q_j(i)| < |\lambda_1| \quad \forall i = 1, \dots, i_j - 1.$$

$\therefore i_{j+1} = i_j$ for large enough j

Suppose $\lim\limits_{j\to\infty} i_j = i_0$.

$$\Rightarrow \lim_{j\to\infty} q_j = \lim_{j\to\infty} \frac{v_1}{v_1(i_j)} = \frac{v_1}{v_1(i_0)}$$

$$A v_i = \lambda_i v_i$$

$$(A - \rho I) v_i = (\lambda_i - \rho) v_i$$

$$\Rightarrow v_i = (\lambda_i - \rho)(A - \rho I)^{-1} v_i$$

$$\Rightarrow (A - \rho I)^{-1} v_i = \left(\frac{1}{\lambda_i - \rho}\right) v_i$$

$Av = \lambda v, \quad \|v\|_2 = 1 = \|q\|_2, \quad \rho = q^* A q.$

$$\Rightarrow |\lambda - \rho| = |v^* A v - q^* A q| = |v^* A v - v^* A q + v^* A q - q^* A q|$$

$$= |v^* A (v - q) + (v - q)^* A q|.$$

$$\leq |v^* A (v - q)| + |(v - q)^* A q|.$$

$$= |v^* A (v - q)| + |q^* A^* (v - q)|$$

$$\leq \underbrace{\|v\|_2}_{1} \|A(v - q)\|_2 + \underbrace{\|q\|_2}_{1} \|A^*(v - q)\|_2.$$

$$\leq 2\|A\|_2 \|v - q\|_2.$$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = q_0$ cannot be solved by Rayleigh Quotient method

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QR algorithm

Step 1 $[M, H] = hess(A)$, $A_0 = H$

(a) Find reflectors $Q_1 \ldots Q_{n-1}$ $\ni$ $Q_{n-1} \cdots Q_1 H = R$

(b) Set $A_1 = R \underbrace{Q_1 \ldots Q_{n-1}}_{Q}$

$O(n^2)$.

$O(n)$ if $A^T = A$.

H is upper hessenberg $\Rightarrow$ $Q^* H Q$ is upper hessenberg for unitary/orthogonal Q coming from $qr(H)$.

A is singular properly hessenberg matrix.

After 1 iterate, last row is 0.

Proof: In A, col(A) is LD

col(A(:, 1:n-1)) is LI

$\Rightarrow$ rank(A) = n-1.

$A = QR$

$A_1 = RQ$.

rank(R) = n-1.

$A = QR$

$\Rightarrow \underbrace{[A_1 \ldots A_{n-1}]}_{LI} = Q \underbrace{[R_1 \ldots R_{n-1}]}_{LI}$ $\Longrightarrow$ LI

$\therefore$ rank(R) = n-1

$\Rightarrow$ One diagonal entry of R is 0.

All other diagonal entries $\neq 0$ ($\because$ rank(R) = n-1).

$\therefore R(i,i) \neq 0 \;\; \forall i = 1, 2, \ldots, n-1$

and $R(n,n) = 0$

Double shift QR

$A - \rho I = Q_\rho R_\rho$ — (1)

$\hat{A} - \tau I = Q_\tau R_\tau$ — (2)

$R_\rho Q_\rho + \rho I = \hat{A} = Q_\rho^* A Q_\rho$ — (3)

$R_\tau Q_\tau + \tau I = \tilde{A} = Q_\tau^* \hat{A} Q_\tau$ — (4)

(3) and (4), $\Rightarrow \tilde{A} = Q_\tau^* Q_\rho^* A Q_\rho Q_\tau$

$= (Q_\rho Q_\tau)^* A (Q_\rho Q_\tau)$

$= Q^* A Q$, $\quad Q = Q_\rho Q_\tau$.

$QR = Q_\rho Q_\tau R_\tau R_\rho$

$= Q_\rho(\hat{A} - \tau I) R_\rho.$

$= Q_\rho(Q_\rho^* A Q_\rho - \tau Q_\rho^* Q_\rho) R_\rho.$

$= Q_\rho Q_\rho^*(A - \tau I) Q_\rho R_\rho.$

$= (A - \tau I) Q_\rho R_\rho$

$= (A - \tau I)(A - \rho I).$

If $\tau = \bar{\rho}$, $(A - \rho I)(A - \tau I) = (A - \rho I)(A - \bar{\rho} I)$

$= A^2 - 2Re(\rho) A + |\rho|^2 I.$

which is Real.

$2Re(\rho) = a_{n-1,n-1} + a_n a_n$

$|\rho|^2 = a_{n-1,n-1} * a_{nn} = \det[A(n-1:n, n-1:n)].$

$(A - \rho I)(A - \tau I) = Q_1 R_1$

$Q_1 = QD$ for $D = \begin{bmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{bmatrix}$ if

$(A - \rho I)(A - \tau I)$ is nonsingular

$Q_1^* A Q_1 = (QD)^* A (QD) = D^* \underbrace{Q^* A Q} D.$

Francis QR

Key properties of Krylov matrix $K(A, x)$:- PMI.

Proof of theorem :- $A \to$ properly upper hessenberg

$Q \to$ unitary

$Qe_1 = \alpha\, p_1(A) e_1$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.

$\hat{A} = Q^* A Q$ is upper Hessenberg.

$K(\hat{A}, e_1)$ is upper triangular.

$= K(Q^* A Q, e_1)$

$= Q^* K(A, Qe_1).$

$= Q^* K(A, \alpha\, p(A) e_1)$

$= \alpha Q^* K(A, p(A) e_1).$

$K(\hat{A}, e_1) = \alpha Q^* p(A) \underbrace{K(A, e_1)}_{\text{upper triangular + non singular}}.$

$\Rightarrow p(A) = \frac{1}{\alpha} Q K(\hat{A}, e_1) (K(A, e_1))^{-1}.$

Let, $R = \frac{1}{\alpha} K(\hat{A}; e_1) [K(A, e_1)]^{-1}.$

$\therefore p(A) = QR.$

$$\boxed{Q_j = Q_{j-1}^{(1)} \hat{Q}_{j-1}^{(2)} \ldots \hat{Q}_{j-1}^{(n-1)}}$$

$p_j(A) = (A_{j-1} - \sigma_{j-1} I)(A_{j-1} - \tau_{j-1} I).$

T.P. $\boxed{Q_j e_1 = \alpha p_j(A) e_1}$

$Q^{(1)} p_j(A) e_1 = \alpha e_1$, $\alpha \neq 0 \Leftarrow p_j(A) e_1 \neq 0$

$\Rightarrow p_j(A) e_1 = \alpha Q^{(1)} e_1$

$\Rightarrow Q^{(1)} e_1 = \frac{1}{\alpha} p_j(A) e_1.$

$\hat{Q}^{(2)}, \ldots, \hat{Q}^{(n-1)}$ do not affect 1^{st} column.

Algo:- $[\hat{Q}, H] = \text{hess}(A)$

$A_0 := \hat{Q}^* A \hat{Q}$

$A_1 := Q_1^* A_0 Q_1$. ↳ these Q_i's are themselves of the form $Q_1^{(1)} Q_1^{(2)} \ldots Q_1^{(n)}$ ← applying all takes $O(\bar{n})$ flops.

$\vdots$

$D = A_j := Q_j^* A_{j-1} Q_j$

$\therefore Q^* A Q = D$

where $Q = \hat{Q} Q_1 Q_2 \ldots Q_j$.

For general matrices,

$$Q^* A Q = T \text{ (upper triangular (quasi))}.$$

$$Tv = \lambda v.$$

$$Q^* A Q v = \lambda v$$

$$A(Qv) = \lambda(Qv).$$

Invariant Subspace Theorem.

$S = [s_1 \ldots s_k \; s_{k+1} \ldots s_n] \rightarrow$ invertible

Suppose span $\{s_1, \ldots, s_k\}$ is invariant subspace wrt A.

$\Rightarrow \forall i = 1, 2, \ldots, k$

$$A s_i = \alpha_{1i} s_1 + \ldots + \alpha_{ki} s_k.$$

$\forall i = k+1, \ldots, n$

$$A s_i = \alpha_{1i} s_1 + \ldots + \alpha_{ki} s_k + \alpha_{k+1\,i} s_{k+1} + \ldots + \alpha_{ni} s_n.$$

$$AS = [As_1 \ldots As_k \; As_{k+1} \ldots As_n]$$

$$= [s_1 \ldots s_k \, s_{k+1} \ldots s_n] \left[\begin{array}{ccc|ccc} \alpha_{11} & \ldots & \alpha_{1k} & \alpha_{1,k+1} & \ldots & \alpha_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{k1} & & \alpha_{kk} & \vdots & & \vdots \\ \hline & & & \vdots & & \vdots \\ & & & \alpha_{n1} & & \alpha_{nn} \end{array}\right]$$

$$\therefore AS = S \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right]$$

$\Rightarrow S^{-1} A S = \nearrow$

Suppose $\exists S \ni S^{-1} A S = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline & A_{22} \end{array}\right]$