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First order necessary conditions for a local minimum

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- So $[6, 9]^T$ and $[0, 0]^T$ satisfies the **first order necessary conditions** for a **local minimum.**
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- **Exercise:** Will $[0, 0]^T$ be a **local minimum point** of f given in **Example 1**, if $\Omega = \mathbb{R}^2$?

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- Check that f has a **global minimum** at $x_1 = \frac{1}{2}, x_2 = 0$

Second order necessary conditions for a point to be a local minimum

- **Theorem 3:** Let $f : \Omega \to \mathbb{R}$ be a **twice continuously differentiable** function (that is all the second order partial derivatives of f (given by $\frac{\partial^2 f}{\partial x_j \partial x_i}$) exists and are continuous as functions from Ω to $\mathbb{R}$).

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- For example the **0** matrix has all n eigenvalues equal to 0, the identity matrix I_n has all n eigenvalues equal to 1, and for an **upper triangular matrix**, the diagonal entries are its eigenvalues.

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- **Remark:** A nonsingular (nonzero determinant) **positive semidefinite matrix** is **positive definite.**

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- **Theorem 1 :** If f is a **convex function** on Ω (a convex set), then the set $S = \{\mathbf{x} \in \Omega : f(\mathbf{x}) \leq c\}$ is a **convex set** (for all real c).

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- **Theorem 6:** Let f be a **convex function** defined on a **closed** and **bounded convex set** Ω (so it has atleast one extreme point), then there exists an extreme point of Ω, where f takes its maximum value.