Indian Institute of echnology Guwahati
Statistical Inference and Multivariate Analysis (MA 324)
Problem Set 04

1. Suppose that $X_1,\, X_2,\, X_3,\, X_4 \overset{i.i.d.}{\sim} N(0,\, \sigma^2)$, where $\sigma > 0$ is unknown parameter. Consider the following estimators:

$$T_1 = X_1^2 - X_2 + X_4, \quad T_2 = \frac{1}{3}\left(X_1^2 + X_2^2 + X_4^2\right), \quad T_3 = \frac{1}{4}\sum_{i=1}^{4} X_i^2, \quad \text{and} \quad T_4 = \frac{1}{3}\sum_{i=1}^{4}\left(X_i - \overline{X}\right)^2.$$

   (a) Is T_i UE for σ^2 for $i = 1,\, 2,\, 3,\, 4$?

   (b) Among estimators T_1, T_2, T_3 and T_4 for σ^2, which one has the smallest MSE?

2. Suppose that $X_1,\, X_2 \overset{i.i.d.}{\sim} N(0,\, \sigma^2)$, where $\sigma > 0$ is unknown parameter. Consider the following estimators:

$$T_5 = \frac{1}{2}\left|X_1 - X_2\right|.$$

   Is T_5 UE for σ? If not, propose an UE for σ based on T_5. Calculate the MSE of T_5 as an estimator of σ.

3. Let $X_1,\, X_2,\, \ldots,\, X_n$ be a RS with a common PDF

$$f(x) = \frac{1}{\sigma}\exp\left[-\frac{x-\mu}{\sigma}\right] I_{(\mu,\,\infty)}(x).$$

   Denote $U = \sum_{i=1}^{n}\left(X_i - X_{(1)}\right)$ and let $V = cU$ be an estimator of σ, where $c > 0$ is a constant.

   (a) Find the MSE of V. Then, minimize the MSE with respect to c. Call this latter estimator W, which has smallest MSE.

   (b) How do estimators W and $\frac{U}{n-1}$ compare relative to their respective bias and the MSE?

4. Let $X_1,\, X_2,\, \ldots,\, X_n \overset{i.i.d.}{\sim} Bernoulli(p)$, where $p \in (0,\, 1)$ is unknown parameter. Show that there is no UE for the parametric functions (a) $\tau(p) = \dfrac{1}{p(1-p)}$, (b) $\tau(p) = \dfrac{1}{p(1-p)^2}$.

5. Let $X_1,\, X_2,\, \ldots,\, X_n \overset{i.i.d.}{\sim} Bernoulli(p)$, where $p \in (0,\, 1)$ and $n \geq 3$. Derive UMVUE of (a) $\tau(p) = p^2(1-p)$, (b) $\tau(p) = \left(p + qe^3\right)^2$, where $q = 1 - p$.

6. Let $X_1,\, X_2,\, \ldots,\, X_n \overset{i.i.d.}{\sim} N(\mu,\, \sigma^2)$, where $\mu \in \mathbb{R}$ is unknown, but $\sigma > 0$ is known. Derive UMVUE of (a) $\tau(\mu) = P_\mu\left(|X_1| \leq a\right)$, where a is a known positive real number. (b) $\tau(\mu) = \mu^3$.

7. Let $X_1,\, X_2,\, \ldots,\, X_n \overset{i.i.d.}{\sim} U(-\theta,\, \theta)$, where $\theta \in \mathbb{R}$ is an unknown parameter. Derive the UMVUE of $\tau(\theta) = \theta^k$, where k is a known fixed positive real number.

8. Let $X_1, X_2, \ldots, X_n$ be a RS from a common PDF

$$f(x,\,\theta) = \frac{2}{\theta} x \exp\left[-\frac{x^2}{\theta}\right] I_{(0,\,\infty)}(x),$$

where $\theta > 0$ is a unknown parameter. Find the UMVUE of (a) $\tau(\theta) = \theta$, (b) $\tau(\theta) = \theta^2$, (c) $\tau(\theta) = \theta^{-1}$. Find CRLBs and check if these UMVUEs attain CRLBs.

9. Let $X_1, X_2, \ldots, X_n$ be a RS from a common PDF

$$f(x,\,\theta) = \frac{2}{\theta} x \exp\left[-\frac{x^2}{\theta}\right] I_{(0,\,\infty)}(x),$$

where $\theta > 0$ is a unknown parameter. Show that MLEs and UMVUEs of (a) $\tau(\theta) = \theta$, (b) $\tau(\theta) = \theta^2$, (c) $\tau(\theta) = \theta^{-1}$ are all consistent.

10. Suppose $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu,\,\sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown parameters. Denote, for $n \geq 1$,

$$T_n = \frac{2X_1 + 4X_2 + 6X_3 + \ldots + 2nX_n}{n(n+1)}.$$

   (a) Evaluate $E\left(T_n\right)$ and $Var\left(T_n\right)$ for all $n \geq 1$.
   (b) Show that $\{T_n : n \geq 1\}$ is consistent for μ.
   (c) Is $\max\{0,\, T_n\}$ consistent for μ?