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Books – Sidel, Glasserman

$\boxed{\text{Sidel Sexn 4.2}}$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0 \longrightarrow (1)$$

1. Finite Domain $\longrightarrow$ Eqn more complicated
2. Infinite domain $\longrightarrow$ Eqn simple $\longrightarrow$ Domain $-\infty < S < \infty$

$S_{min} < S < S_{max}$

Eqn (1) converted to $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

Methods to solve (1):–

(1) FTCS – Explicit Euler + Central diff

~~BTCS~~

(2) BTCS – Implicit Euler + CS

(3) $\frac{1}{2}$(FTCS + BTCS)

$S \in (a, b)$, N, $h = \frac{b-a}{N} = \Delta S$ $\quad t \in [0, T]$, M, $k = \frac{T}{M}$

$V_m^n \approx V(S_m, t)$

(1)
$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}$$

$$U_m^{n+1} = \lambda U_{m+1}^n + (1-2\lambda) U_m^n + \lambda U_m^n \quad \left(0 < \lambda < \tfrac{1}{2}\right)$$

(2)
$$-\lambda U_{m+1}^{n+1} + (1+2\lambda) U_m^{n+1} + \lambda U_{m-1}^{n+1} = U_m^n$$

$$AU = F$$
$$U = A \backslash F$$

Crank-Nicolson Scheme

Transformation (1) for Black-Scholes:–

$y = \ln S$

$\tau = T - t$

$$V(S,t) = \exp(-r(T-t))\, v(y, \tau)$$
$$= \exp(-r\tau)\, v(y; \tau)$$

$y = \ln S \quad \tau = T - t \quad V(S,t) = e^{-r(T-t)} v(y, \tau)$

$$\frac{\partial V}{\partial t} + (r-\delta) S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0, \quad V(S,T) = (S-K)^+$$

$= V(e^y, T-\tau)$

$$\frac{\partial V}{\partial t} = r e^{-r(T-t)} \cdot v + e^{-r(T-t)} \frac{\partial v}{\partial \tau} \cdot \frac{\partial \tau}{\partial t}$$

$$= r e^{-r(T-t)} v - e^{-r(T-t)} \cdot v_\tau$$

$$\frac{\partial V}{\partial S} = e^{-r(T-t)} \cdot \frac{\partial v}{\partial y} \cdot \frac{1}{S}$$

$$\frac{\partial^2 V}{\partial S^2} = e^{-r(T-t)} \left(\frac{1}{S} \cdot \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right) \frac{\partial y}{\partial S} - \frac{1}{S^2} \cdot \frac{\partial v}{\partial y} \right)$$

$$= \frac{e^{-r(T-t)}}{S^2} (v_{yy} - v_y)$$

$$\cancel{rv} - v_\tau + (r-\delta) v_y + \frac{1}{2}\sigma^2 (v_{yy} - v_y) - \cancel{rv} = 0$$

$$-v_\tau + \left(r - \delta - \frac{\sigma^2}{2}\right) v_y + \frac{\sigma^2}{2} v_{yy} = 0$$

$$v(y, 0) = (e^y - K)^+$$

$x = y + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau$

$\tilde{\tau} = \frac{1}{2}\sigma^2 \tau$

$v(y, \tau) = u(x, \tilde{\tau})$

$\boxed{\cancel{v\left(x - \left(r-\delta-\frac{\sigma^2}{2}\right)\tau, 0\right) =}}$

$v_{yy} =$

$$v_\tau = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \tau} + \frac{\partial u}{\partial \tilde{\tau}} \cdot \frac{\partial \tilde{\tau}}{\partial \tau}$$

$$= u_x \cdot \left(r - \delta - \frac{\sigma^2}{2}\right) + \cancel{\frac{\partial u}{\partial \tilde{\tau}}} \frac{1}{2}\sigma^2 \cdot u_{\tilde{\tau}}$$

$$v_y = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial y} + \cancel{\frac{\partial u}{\partial \tilde{\tau}} \cdot \frac{\partial \tilde{\tau}}{\partial y}} = 0$$

$= u_x$

$$e^{\frac{\ln S}{2}\left(\frac{2(r-\delta)}{\sigma^2} + \frac{1}{2}\right)} = S^{\left(\frac{(r-\delta)}{\sigma^2} + \frac{1}{2}\right)}$$

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$$x = y + \left(r - \delta - \frac{\sigma^2}{2}\right)\tau$$

$$\tilde{\tau} = \frac{1}{2}\sigma^2\tau$$

$$v(y,\tau) = u(x,\tilde{\tau})$$

$$\cancel{\frac{\partial v}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} + \left(r-\delta-\frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x}}$$

$$\frac{dv}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} + \left(r-\delta-\frac{\sigma^2}{2}\right)\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial \tau} = \frac{\partial u}{\partial \tilde{\tau}}\cdot\frac{\partial \tilde{\tau}}{\partial \tau} + \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \tau}$$

$$= \frac{\partial u}{\partial \tilde{\tau}}\cdot\frac{1}{2}\sigma^2 + \left(r-\delta-\frac{\sigma^2}{2}\right)\cdot\frac{\partial u}{\partial x}$$

$$\cancel{\frac{\partial^2 v}{\partial x^2} = \frac{\partial u}{\partial x}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial \tilde{\tau}}}$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial y} + \frac{\partial u}{\partial \tilde{\tau}}\cdot\cancel{\frac{\partial \tilde{\tau}}{\partial y}} = 0$$

$$= \frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$$

$$V(S,t) = e^{-r(T-t)}v(y,\tau) = e^{-r(\tilde{\tau}-t)}u(x,\tilde{\tau})$$

For a direct transformation,

$$x = \ln\left(\frac{S}{K}\right) + \left(r-\delta-\frac{\sigma^2}{2}\right)(T-t)$$

$$\tilde{\tau} = \frac{1}{2}\sigma^2(T-t)$$

i) $V(S,T)$ = payoff – inorder to make this

ii) $y = \ln S$ is used to make the variable coeffecients constant coeffecien

iii) $V = e^{-r(T-t)}v(y,\tau)$ has been chosen

$$\frac{dV}{\partial \tau} - rV = f$$

Suppose ~~...~~ r, δ, σ are f^ns of t, then the transformat^n will be

$$\alpha = \ln\left(\frac{S}{K}\right) + \int_t^T \left(r(s) - \delta(s) - \frac{\sigma^2}{2}(s)\right) \cdot ds$$

no transformat^n like these if f^ns of S

$$\tilde{\tau} = \frac{1}{2} \int_t^T \sigma^2(s) \cdot ds$$

$$V(S,t) = e^{-\int_t^T r(s) \cdot ds} \, u(S,t)$$

$V_p(S,t) = 0 \, , \, S \to \infty$

$V_c(S,t) = 0 \, , \, S = 0$

$V_p(S,t) = \quad , S \to 0$

$V_c(S,t) = \quad , S \to \infty$

$$V_c = V_p + Ke^{-r(T-t)}$$

$V_c(S,t) = S - Ke^{-r(T-t)} \, , S \to \infty$

$V_p(S,t) = Ke^{-r(T-t)} - S, \, S \simeq 0$

$S = Ke^{\alpha} \, , \, t = T - \frac{2\tau}{\sigma^2} \, , \, q_\tau = \frac{2\tau}{\sigma^2} \, , \, q_\delta = \frac{2(r-\delta)}{\sigma^2}$

$V(S,t) = V\left(Ke^{\alpha}, T - \frac{2\tau}{\sigma^2}\right) = v(\alpha, \tau)$ $\quad \frac{\partial^2 y}{\partial \alpha^2} = \frac{\partial y}{\partial \tau}$

$$v(x,\tau) = K\exp\left\{-\frac{1}{2}(q_\delta - 1)\alpha - \left(\frac{1}{4}(q_\delta - 1)^2 + q_\tau\right)\tau\right\} y(\alpha, \tau)$$

$V(S,T) = \max(S - K, 0)$

where $y(\alpha, 0) = \begin{cases} \\ \end{cases}$

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Transform - II :- Finite Domain

$$\boxed{\xi = \frac{S}{S+P}, \quad P>0 \qquad \tau = T-t \\ V(S,t) = (S+P)\,\bar{V}(\xi,\tau)}$$

$\boxed{P=K}$

$$S = \frac{P\xi}{1-\xi}, \quad S+P = \frac{P}{1-\xi}$$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2(S)S^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-\delta)S\frac{\partial V}{\partial S} - rV = 0$$

$0 \le S \le \infty$

$0 \le \xi \le 1$

$$V(S,T) = V_T(S) = (S-K)^+$$
$$= (K-S)^+$$

$$\cancel{\frac{\partial V}{\partial S} = \frac{\partial \bar V}{\partial \xi} \times \frac{\partial \xi}{\partial S} = \frac{(1-\xi)^2}{P} \times \frac{\partial \bar V}{\partial \xi}} \qquad = \frac{P}{(P+S)^2}$$

$$\cancel{\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left[(S+P)\bar V(\xi,\tau)\right] = (S+P)\frac{\partial \bar V}{\partial \tau} = \frac{P}{1-\xi}\cdot\frac{\partial \bar V}{\partial \tau}}$$

$$\cancel{\frac{\partial^2 V}{\partial S^2} = \frac{\partial \bar V}{\partial \xi} \times P \times \frac{-2}{(P+S)^3}} \qquad = \bar V + (S+P)\cdot\frac{\partial \bar V}{\partial t}$$

$$= \bar V + (S+P)\cdot\frac{\partial \bar V}{\partial \xi}\cdot\frac{\partial \xi}{\partial S}$$

$$\frac{\partial v(x(t), y(t))}{\partial t} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial t} \qquad = \bar V + (1-\xi)\frac{\partial \bar V}{\partial \xi}$$

$$\frac{\partial V}{\partial t} = \frac{\partial}{\partial t}\left((S+P)\,\bar V(\xi,\tau)\right) = -(S+P)\frac{\partial \bar V}{\partial \tau} = \frac{-P}{1-\xi}\cdot\frac{\partial \bar V}{\partial \tau}$$

~~PDE~~

Black Schole PDE
→ heat conduxn
(Seydel)

$$\frac{\partial V}{\partial S} = \frac{\partial}{\partial S}\left[(S+P)\bar V\right] = \bar V + (S+P)\cdot\frac{\partial \bar V(\xi,\tau)}{\partial S}$$

$$= \bar V + (S+P)\left(\frac{\partial \bar V}{\partial \xi}\cdot\frac{\partial \xi}{\partial S}\right)$$

$$= \bar V + \cancel{(S+P)}\,(1-\xi)\cdot\frac{\partial \bar V}{\partial \xi}$$

~~Seydel~~

Transf (notes)

heat conduxn → eqs (pdf what, app)
→ ODE

Green f^n (?)
→ heat conduxn eqs

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial \xi}\left[\bar V + (1-\xi)\cdot\frac{\partial \bar V}{\partial \xi}\right]\cdot\frac{\partial \xi}{\partial S} = \frac{(1-\xi)^3}{P}\cdot\frac{\partial^2 \bar V}{\partial \xi^2}$$

$0 \le \xi \le 1$

$$-\frac{P}{1-\xi}\cdot\frac{\partial \bar{V}}{\partial \tau} + \frac{\sigma^2(\xi)\cdot\left(\frac{P\xi}{1-\xi}\right)^2}{2}\cdot\frac{(1-\xi)^3}{P}\cdot\frac{\partial^2 \bar{V}}{\partial \xi^2} + (r-\delta)\left(\frac{P\xi}{1-\xi}\right)\cdot\left(\bar{V}+(1-\xi)\cdot\frac{\partial \bar{V}}{\partial \xi}\right) = \frac{rP}{(1-\xi)}\cdot\bar{V} \longrightarrow (4)$$

(Write the final eq^n)

$$V(S,t) = (S+P)\,\bar{V}(\xi,\tau)$$

$$V(S,T) = \left(\frac{P}{1-\xi}\right)\bar{V}(\xi,0) = \left(\frac{P}{1-\xi}\right)\cdot\bar{V}(\xi,0)$$

$$\bar{V}(\xi,0) = V_T\left(\frac{P\xi}{1-\xi}\right)\left(\frac{1-\xi}{P}\right)$$

For solving (4), We need bdary cond^{n}s at $\xi=0$ & $\xi=1$ (that is discussed in next class)

Put $\xi=0$ in (4) (Verified)

$$\frac{\partial \bar{V}}{\partial \tau}(0,\tau) = -r\bar{V}(0,\tau)$$

$$\boxed{\bar{V}(0,\tau) = ce^{-r\tau} = \bar{V}(0,0)\,e^{-r\tau}}$$

$$\frac{\partial \bar{V}}{\partial \tau} = -\delta\,\bar{V}(1,\tau)$$

Put $\xi=1$ in (4) (Verified)

$$\boxed{\bar{V}(1,\tau) = ce^{-\delta\tau}}$$

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$$\bar{V}(\xi,0) = \left(\frac{1-\xi}{P}\right)V_T\left(\frac{P\xi}{1-\xi}\right) = \left(\frac{1-\xi}{P}\right)\max\left\{\frac{P\xi}{1-\xi}-K,0\right\}$$

For Call,

$$V(S,T) = (S-K)^+$$

$$= \max\left\{\xi - \frac{K(1-\xi)}{P},0\right\}$$

Now, take $P=K$

Ally For Put,

$$\bar{V}(\xi,0) = \max\left\{\frac{K}{P}(1-\xi)-\xi,0\right\}$$

Remark:-
~~Assuming~~

We assume σ as a f^n of S as r, δ are const. The same transformat^n will be applicable even if $\sigma(S,t)$, $r(S,t)$, $\delta(S,t)$

When $\xi = 1$, $S \to \infty$:
$$V(S,t) = (S+P)\,\bar{V}(\xi,\tau)$$
$$= (S+P)\,\bar{V}(1,\tau)$$
$$= (S+P)\,\bar{V}(1,0)\,e^{-\delta\tau}$$
$$= V(S,T)\,e^{-\delta\tau}$$

Obtaining Black-scholes formula from the 1-D heat conduxn eq^n

$$\frac{\partial u}{\partial \bar{\tau}} = \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \; \bar{\tau} \geq 0$$

$$u(x,0) = u_0(x)$$
$$\cancel{u(x,\bar{\tau}) = V(S,t)}$$
$$u(0,\bar{\tau}) = \phi(\bar{\tau})$$
$$u(l,\bar{\tau}) = \psi(\bar{\tau})$$

Converting $u(x,\bar{\tau})$ to $U(\eta)$ is called "similiarity solution"

Can't use variable-seperat^n method becz $-\infty < x < \infty$ is infinite. Rather, we try to look for a special sol^n of the form

$$u(x,\bar{\tau}) = \frac{1}{\sqrt{\bar{\tau}}}\,U(\eta) \qquad \boxed{\eta = \frac{x-\xi}{\sqrt{\bar{\tau}}}} \quad \xi\text{-parameter}$$

$$\frac{\partial u}{\partial \bar{\tau}} = \frac{1}{\sqrt{\bar{\tau}}}\cdot\left(\frac{\partial U}{\partial \eta}\cdot(x-\xi)\left(\frac{-1}{2}\right)\bar{\tau}^{-3/2}\right) + U(\eta)\cdot\left(\frac{-1}{2}\right)\bar{\tau}^{-3/2}$$

$$= -\frac{\bar{\tau}^{-3/2}}{2}\left(U + \eta\cdot\frac{\partial U}{\partial \eta}\right) = -\frac{\bar{\tau}^{-3/2}}{2}\,\frac{\partial}{\partial \eta}(\eta U)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{\bar{\tau}}}\cdot\frac{\partial U}{\partial \eta}\cdot\frac{1}{\sqrt{\bar{\tau}}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \eta}\left(\frac{\partial U}{\bar{\tau}\,\partial \eta}\right)\cdot\frac{\partial \eta}{\partial x}$$

$$= \frac{1}{\bar{\tau}}\,\frac{\partial^2 U}{\partial \eta^2}\cdot\frac{1}{\bar{\tau}^{1/2}}$$

$$V(S,T) = (S(T)-K)^+$$

$$-\frac{\tau^{-3/2}}{2}\frac{\partial}{\partial\eta}(\eta U) = \tau^{-3/2}\frac{\partial^2 U}{\partial\eta^2}$$

$$\cancel{\frac{\partial^2 U}{\partial\eta^2} + \frac{1}{2}}$$

$\Downarrow$ U is only a f^n of η so PDE $\Rightarrow$ ODE

$$\boxed{\frac{d^2U}{d\eta^2} + \frac{1}{2}\frac{d}{d\eta}(\eta U) = 0}$$

$$\frac{d}{d\eta}\left(\frac{dU}{d\eta} + \frac{1}{2}\eta U\right) = 0$$

$$\cancel{\frac{dU}{c - \frac{1}{2}\eta U}} = d\eta$$

$$\frac{dU}{d\eta} + \frac{1}{2}\eta U = c \quad \left(\because U = o\left(\frac{1}{\eta}\right)\right)$$

Let $c = 0$ ("to simplify things")

$$U e^{\frac{-\eta^2}{4}} =$$

$$\cancel{U(\eta) =}$$

$$U(\eta) = c_1 e^{\frac{-\eta^2}{4}}$$

$$= c_1 \exp\left(\frac{-(x-\xi)^2}{4\tau}\right)$$

~~We require that~~

$$u(x,\tau) = \frac{1}{\sqrt{\tau}}U(\eta) = c\,\tau^{-1/2}e^{-\frac{(x-\xi)^2}{4\tau}}$$

We require that $\int_{-\infty}^{\infty} c\,\tau^{-1/2}e^{-\frac{(x-\xi)^2}{4\tau}}\,d\xi = 1 \rightarrow$ "so that the integral is finite"

$$\cancel{c = \frac{\sqrt{\tau}\;1}{-\int_{+\infty}^{-\infty} e^{-\frac{\eta^2\tau}{4}} \cdot \frac{\sqrt{\tau}}{x-\sqrt{\tau}\eta}\cdot d\eta}}$$

$$c = \frac{1}{\sqrt{\tau}\int_{-\infty}^{\infty} e^{-\eta^2/2}\,d\eta}$$

$$= \frac{1}{2\sqrt{\pi}} \qquad \boxed{c = \frac{1}{2\sqrt{\pi}}}$$

$$\cancel{\eta = \frac{x-\xi}{\sqrt{\tau}}}$$

$$\cancel{d\eta = \frac{-\xi}{\sqrt{\tau}}\cdot d\xi}$$

$$\boxed{u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}}\exp\left(\frac{-(x-\xi)^2}{4\tau}\right)}$$

$$\cancel{= \frac{1}{\int_{-\infty}^{\infty} e^{\frac{-\eta^2\tau}{4}}\cdot\frac{1}{x-\sqrt{\tau}\eta}\cdot d\eta}}$$

$$=$$

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$$\boxed{\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}} \quad u(x,0) = u_0(x)$$

$$u = \int^{\infty} G(\ ) \cdot u_0(x)$$

$G(\xi; x, \tau)$ ξ is a parameter

Using Green's f^n, we are considering the initial condit^n.

$$\frac{\partial}{\partial \tau} G(\xi; x, \tau) = \frac{\partial^2}{\partial x^2} G(\xi; x, \tau)$$

$$u(x,0) = u_0(x)$$

$$\int_{-\infty}^{\infty} u_0(\xi) \cdot \frac{\partial}{\partial \tau} G(\cdot\cdot) \cdot d\xi = \int_{-\infty}^{\infty} u_0(\xi) \cdot \frac{\partial^2}{\partial x^2} G(\cdot\cdot\cdot) \cdot d\xi$$

Topic:- similiarity solutions

$$\frac{\partial}{\partial \tau} \int_{-\infty}^{\infty} u_0(\xi) \cdot G(\cdot\cdot) \cdot d\xi = \frac{\partial^2}{\partial x^2} \int_{\infty}^{\infty} u_0(\xi) \cdot G(\xi; x, \tau) d\xi$$

$$\boxed{\begin{aligned} u(x, \tau) &= \frac{1}{2\sqrt{\pi \tau}} e^{-(x-\xi)^2/4\tau} = u_0(x) \\ u(x_0, \tau) &= \int_{-\infty}^{\infty} u_0(\xi) \cdot \frac{1}{2\sqrt{\pi \tau}} e^{-(x-\xi)^2/4\tau} \cdot d\xi \end{aligned}} \longrightarrow \text{I}$$

$$\lim_{\tau \to 0^+} \frac{1}{2\sqrt{\pi \tau}} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) = \begin{cases} 0 & \text{if } \xi \neq x \\ \frac{1}{2\sqrt{\pi \tau}} \infty & \text{if } \xi = x \end{cases} = \delta(x-\xi)$$

dirac delta f^n

& also

$$\int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi \xi}} \exp\left(-\frac{(x-\xi)^2}{4\tau}\right) \cdot d\xi = 1$$

I - satisfies the 1D heat conduct^n eq^n. Now we have to retrieve

$$V(S,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} u_0(\xi) \frac{1}{2\sqrt{\pi \tau}} e^{-(x-\xi)^2/4\tau} \cdot d\xi \qquad V(S,T) = V_T(S)$$

$$e^{\xi} = \tilde{S}$$

$$= e^{-r(T-t)} \int_{-\infty}^{\infty} V_T(e^{\xi}) \cdot d\xi$$

$$= e^{-r(T-t)} \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{\infty} V_T(\tilde{S}) \times \exp^{-} \frac{\left(-\ln \tilde{S} + \left(r - \delta - \frac{\sigma^2}{2}\right)(T-t)\right)^2}{2\sigma^2 (T-t)} \cdot \frac{d\tilde{S}}{\tilde{S}}$$

$\ln \tilde{S} -$

Green f^n for Black-Sc eq^n

$$V(S,t) = e^{-r(T-t)} \int_0^\infty u_T(\tilde{S}) \cdot G(\tilde{S}, \ldots) \cdot d\tilde{S}$$

If $m = r - \delta - \frac{\sigma^2}{2}$ & $E(\tilde{S}) = S\exp\left((r-\delta)(T-t)\right) = a$

Let $b = \sigma\sqrt{T-t}$ $\quad a = Se^{(r-\delta)(T-t)}$

$$G(\ldots) = \frac{1}{\sqrt{2\pi}\, b\tilde{S}} e^{-\left(\ln\left(\frac{\tilde{S}}{a}\right) + \frac{b^2}{2}\right)^2 / 2b^2} \rightarrow (1)$$

In order to obtain the BS formula, we have to prove the following identities.

1) $\displaystyle\int_c^\infty G(\tilde{S}, T, S, t)\, d\tilde{S} = N\left(\frac{\ln\left(\frac{a}{c}\right) - \frac{b^2}{2}}{b}\right)$

bcuz a is f^n of S

2) $\displaystyle\int_c^\infty \tilde{S} G(\tilde{S}, T, S, t) \cdot d\tilde{S} = aN\left(\frac{\ln\left(\frac{a}{c}\right) + \frac{b^2}{2}}{b}\right)$

$$N(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2/2} \cdot d\xi$$

Pf:- Use (1) & do. $\quad \eta(\tilde{S}) = \dfrac{\ln\left(\frac{\tilde{S}}{a}\right) + \frac{b^2}{2}}{b}$ $\quad \tilde{S} = a\exp\left(b\eta - \frac{b^2}{2}\right)$

$d\tilde{S} = b\tilde{S}\, d\eta$

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$V_T(S) = (S-K)^+$

$C \quad \int_{-\infty}^{\infty} (\tilde{S}-K)^+ G(\ldots$

call p

For a call

$$C(S,t) = e^{-r(T-t)} \int_{K}^{\infty} (\tilde{S}-K) \cdot G(\tilde{S} \ldots) \cdot d\tilde{S}$$

$$= e^{-r(T-t)} \, S e^{(r-\delta)(T-t)} N\left(\frac{\ln\left(\frac{S e^{(r-\delta)(T-t)}}{K}\right) + \left(\frac{\sigma^2 (T-t)}{2}\right)}{\sigma\sqrt{T-t}} \right)$$

$$- e^{-r(T-t)} K N\left(\frac{\ln\left(\frac{S e^{(r-\delta)(T-t)}}{K}\right) - \frac{\sigma^2 (T-t)}{2}}{\sigma\sqrt{T-t}} \right)$$

~~called~~ canceled out by

$\cancel{S e^{(r-t)(T-t)}}$

$S e^{(r-\delta)(T-t)}$

$S(1)_y \quad P(S,T)$

Consider the payoff of a forward contract

$$V(S,t) = e^{-r(T-t)} \int_{0}^{\infty} (\tilde{S}-K) G(\tilde{S}, \ldots, t) \cdot d\tilde{S}$$

For a forward contract, the buyer doesn't need to pay any premium

$$V(S,T) = S e^{-\delta T} - K e^{-rT} = 0$$

$$K = e^{(r-\delta)T} S_0$$

$\rightarrow$ Find ~~for~~ all vega, delta, (Greeks)