For up-and-in call option

- $S(0) < B$
- pay-off at maturity is $= (S(T)-K)^+ \mathbb{1}_{\{M_S \geq B\}}$.

For down-and-in call option

- $S(0) > B$
- pay-off at maturity is $= (S(T)-K)^+ \mathbb{1}_{\{m_S \leq B\}}$.

- <u>up-and-out call:-</u>

our underlying asset is geometric Brownian motion

$$dS(t) = r S(t) dt + \sigma S(t) d\widetilde{W}(t)$$

where $\widetilde{W}(t)$, $0 \leq t \leq T$ is a Brownian motion under the risk-neutral measure $\widetilde{\mathbb{P}}$. Consider a European call, expiring at time T, with strike price K and up-and-out barrier B. we assume that $K<B$, otherwise the option must be knock out (in order to be in the money and hence could only pay off zero). The solution to the stochastic differential equation for the asset price is

$$S(t) = S(0) e^{\sigma \widetilde{W}(t) + (r - \frac{1}{2}\sigma^2)t} = S(0) e^{\sigma \widehat{W}(t)}$$

where $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$, and $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$

we define $\widehat{M}(T) = \max\limits_{0 \leq t \leq T} \widehat{W}(t)$, so.

$$\max_{0 \leq t \leq T} S(t) = S(0) e^{\sigma \widehat{M}(T)}$$

The option knocks out if and only if $S(0) e^{\sigma \widehat{M}(T)} > B$

The pay off of the option is $\overset{V(T)}{=} (S(T)-K)^+ \mathbb{1}_{\{S(0) e^{\sigma \widehat{M}(T)} \leq B\}}$.

$$V(T) = \left(S(0)e^{\sigma \hat{W}(T)} - K\right)^+ \mathbb{1}_{\{S(0)e^{\sigma \hat{M}(T)} \le B\}}.$$

$$= \left(S(0)e^{\sigma \hat{W}(T)} - K\right) \mathbb{1}_{\{S(0)e^{\sigma \hat{W}(T)} \ge K\}} \mathbb{1}_{\{S(0)e^{\sigma \hat{M}(T)} \le B\}}.$$

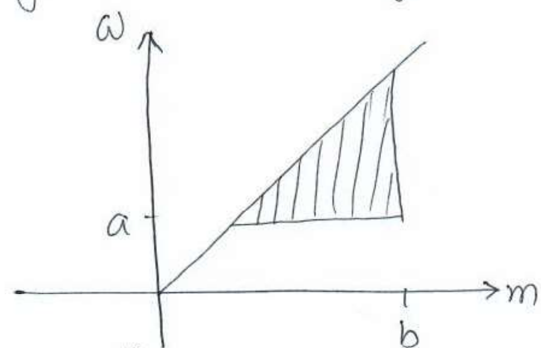
$$= \left(S(0)e^{\sigma \hat{W}(T)} - K\right) \mathbb{1}_{\{S(0)e^{\sigma \cdot \hat{W}(T)} \ge K,\ S(0)e^{\sigma \hat{M}(T)} \le B\}}.$$

$$= \left(S(0)e^{\sigma \hat{W}(T)} - K\right) \mathbb{1}_{\{\hat{W}(T) \ge a,\ \hat{M}(T) \le b\}}.$$

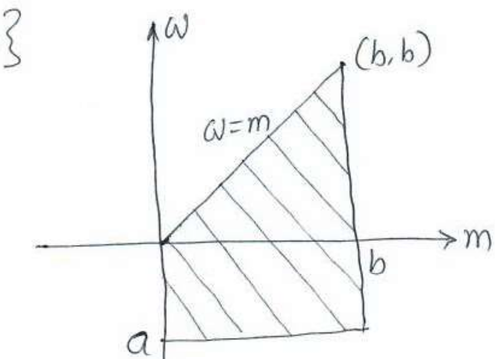
where $a = \frac{1}{\sigma} \log \frac{K}{S(0)}$ and $b = \frac{1}{\sigma} \log B/S(0)$.

The risk-neutral price at time zero of the up-and-out call is $V(0) = \widetilde{\mathbb{E}}\left[e^{-rT} V(T)\right]$.

If $a \ge 0$, we must integrate over the region $\{(m,\omega) : a \le \omega \le m \le b\}$



If $a < 0$, we integrate over the region $\{(m,\omega) : a \le \omega \le m,\ 0 \le m \le b\}$



In both cases the region can be described as
$\{(m,\omega) ;\ a \le \omega \le b,\ \omega^+ \le m \le b\}$.

Note that here $S(0) \le B \Rightarrow b \ge 0$.

We also assume that $S(0)>0$ so that a, b are finite.

$$V(0)=\int_{\omega=a}^{\omega=b}\int_{m=\omega^+}^{m=b} e^{-rT}\left(S(0)e^{\sigma\omega}-K\right)\frac{2(2m-\omega)}{T\sqrt{2\pi T}}\, e^{\alpha\omega-\frac{1}{2}\alpha^2T-\frac{1}{2T}(2m-\omega)^2}\, dm\, d\omega$$

$$=-\int_a^b e^{-rT}\left(S(0)e^{\sigma\omega}-K\right)\frac{1}{\sqrt{2\pi T}}\, e^{\alpha\omega-\frac{1}{2}\alpha^2T-\frac{1}{2T}(2m-\omega)^2}\Big|_{m=\omega^+}^{m=b} d\omega$$

$$=\frac{1}{\sqrt{2\pi T}}\int_a^b\left(S(0)e^{\sigma\omega}-K\right)e^{-rT+\alpha\omega-\frac{1}{2}\alpha^2T-\frac{1}{2T}\omega^2}\, d\omega$$

$$-\frac{1}{\sqrt{2\pi T}}\int_a^b\left(S(0)e^{\sigma\omega}-K\right)e^{-rT+\alpha\omega-\frac{1}{2}\alpha^2T-\frac{1}{2T}(2b-\omega)^2}\, d\omega.$$

$$=S(0)I_1-KI_2-S(0)I_3+KI_4,$$

where

$$I_1=\frac{1}{\sqrt{2\pi T}}\int_a^b e^{\sigma\omega-rT+\alpha\omega-\frac{1}{2}\alpha^2T-\frac{1}{2T}\omega^2}\, d\omega.$$

$$I_2=\frac{1}{\sqrt{2\pi T}}\int_a^b e^{-rT+\alpha\omega-\frac{1}{2}\alpha^2T-\frac{1}{2T}\omega^2}\, d\omega$$

$$I_3=\frac{1}{\sqrt{2\pi T}}\int_a^b e^{\sigma\omega-rT+\alpha\omega-\frac{1}{2}\alpha^2T-\frac{1}{2T}(2b-\omega)^2}\, d\omega$$

$$=\frac{1}{\sqrt{2\pi T}}\int_a^b e^{\sigma\omega-rT+\alpha\omega-\frac{1}{2}\alpha^2T-\frac{2b^2}{T}+\frac{2}{T}b\omega-\frac{1}{2T}\omega^2}\, d\omega.$$

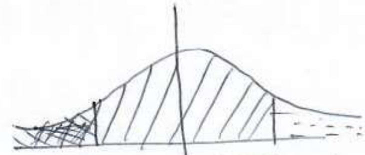
$$I_4=\frac{1}{\sqrt{2\pi T}}\int_a^b e^{-rT+\alpha\omega-\frac{1}{2}\alpha^2T-\frac{2}{T}b^2+\frac{2}{T}b\omega-\frac{1}{2T}\omega^2}\, d\omega.$$

Each of these integrals is of the form

$$J=\frac{1}{\sqrt{2\pi T}}\int_a^b e^{\beta+\gamma\omega-\frac{1}{2T}\omega^2}\, d\omega=\frac{1}{\sqrt{2\pi T}}\int_a^b e^{-\frac{1}{2T}(\omega-\gamma T)^2+\frac{1}{2}\gamma^2T+\beta}\, d\omega$$

let $y = \frac{w-\nu T}{\sqrt{T}}$, then

$$I = e^{\frac{1}{2}\nu^2 T+\beta}\frac{1}{\sqrt{2\pi}}\int_{\frac{a-\nu T}{\sqrt{T}}}^{\frac{b-\nu T}{\sqrt{T}}} e^{-\frac{1}{2}y^2}\,dy$$



$N(-z) + N(z) = 1.$

$$= e^{\frac{1}{2}\nu^2 T+\beta}\left[N\left(\frac{b-\nu T}{\sqrt{T}}\right) - N\left(\frac{a-\nu T}{\sqrt{T}}\right)\right]$$

$$= e^{\frac{1}{2}\nu^2 T+\beta}\left[N\left(\frac{-a+\nu T}{\sqrt{T}}\right) - N\left(\frac{-b+\nu T}{\sqrt{T}}\right)\right]$$

[since $N(z) = 1 - N(-z)$]

$$= e^{\frac{1}{2}\nu^2 T+\beta}\left[N\left(\frac{\frac{1}{\sigma}\log\frac{S(0)}{K}+\nu T}{\sqrt{T}}\right) - N\left(\frac{\frac{1}{\sigma}\log\frac{S(0)}{B}+\nu T}{\sqrt{T}}\right)\right]$$

$$= e^{\frac{1}{2}\nu^2 T+\beta}\left[N\left(\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{K}+\nu\sigma T\right]\right) - N\left(\frac{1}{\sigma\sqrt{T}}\left[\log\frac{S(0)}{B}+\nu\sigma T\right]\right)\right]$$

set $\delta_{\pm}(\tau, s) = \frac{1}{\sigma\sqrt{\tau}}\left[\log s + (r \pm \frac{1}{2}\sigma^2)\tau\right]$

For I_1, $\beta = -rT - \frac{1}{2}\alpha^2 T$, $\nu = \sigma + \alpha$, so $\frac{1}{2}\nu^2 T + \beta = \frac{1}{2}T(\alpha+\sigma)^2 - rT - \frac{1}{2}\alpha^2 T$

$$= \cancel{\tfrac{1}{2}\alpha^2 T} + \tfrac{1}{2}\sigma^2 T + \sigma\alpha T - rT - \cancel{\tfrac{1}{2}\alpha^2 T} = (\tfrac{1}{2}\sigma^2 - r)T + T\sigma\alpha$$

$$= (\tfrac{1}{2}\sigma^2 - r)T + T(r - \tfrac{1}{2}\sigma^2) = 0.$$

and $\nu\sigma = (\sigma+\alpha)\sigma = \sigma^2 + \alpha\sigma = \sigma^2 + (r - \frac{1}{2}\sigma^2)$

$$= r + \tfrac{1}{2}\sigma^2$$

Therefore $I_1 = N\left(\delta_+\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_+\left(T, \frac{S(0)}{B}\right)\right)$

$$I_2 = e^{-rT}\left[N\left(\delta_-\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_-\left(T, \frac{S(0)}{B}\right)\right)\right]$$

$$I_3 = \left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}-1}\left[N\left(\delta_+\left(T, \frac{B^2}{KS(0)}\right)\right) - N\left(\delta_+\left(T, \frac{B}{S(0)}\right)\right)\right]$$

Finally for I_4, we have $\beta = -rT - \frac{1}{2}\alpha^2 T - \frac{2b^2}{T}$ and

$$\nu = \alpha + \frac{2b}{T}, \text{ so}$$

$$\tfrac{1}{2}\nu^2 T + \beta = -rT + \log\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}$$

$$\nu\sigma T = (r - \tfrac{1}{2}\sigma^2)T + \log\left(B/S(0)\right)^2$$

Therefore

$$I_4 = e^{-rT}\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(T, B^2/KS(0)\right)\right) - N\left(\delta_-\left(T, \frac{B}{S(0)}\right)\right)\right]$$

Putting all this together, under the assumption $0 < S(0) \leq B$, we have the up-and-out call price formulla.

$$\begin{aligned}
V(0) = {} & S(0)\left[N\left(\delta_+\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_+\left(T, \frac{S(0)}{B}\right)\right)\right] \\
& - e^{-rT}K\left[N\left(\delta_-\left(T, \frac{S(0)}{K}\right)\right) - N\left(\delta_-\left(T, \frac{S(0)}{B}\right)\right)\right] \\
& - B\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}}\left[N\left(\delta_+\left(T, \frac{B^2}{KS(0)}\right)\right) - N\left(\delta_+\left(T, \frac{B}{S(0)}\right)\right)\right] \\
& + e^{-rT}K\left(\frac{S(0)}{B}\right)^{-\frac{2r}{\sigma^2}+1}\left[N\left(\delta_-\left(T, \frac{B^2}{KS(0)}\right)\right) - N\left(\delta_-\left(T, \frac{B}{S(0)}\right)\right)\right].
\end{aligned}$$

Theorem:- Let $V(t,x)$ denote the price at time t of the up-and-out call under the assumption that the call has not knocked out prior to time t and $S(t) = x$. Then $v(t,x)$ satisfies the Black-Scholes-Merton partial differential equation

$$v_t(t,x) + rx\, v_x(t,x) + \tfrac{1}{2}\sigma^2 x^2\, v_{xx}(t,x) = r v(t,x)$$

in the rectangle $\{(t,x) : 0 \le t < T,\ 0 \le x \le B\}$ and satisfies the boundary conditions

$$v(t,0) = 0 \ ; \quad 0 \le t \le T$$

$$v(t,B) = 0 \ ; \quad 0 \le t < T$$

$$v(T,x) = (x-K)^+, \quad 0 \le x \le B.$$

Let us begin with an initial asset price $S(0) \in (0,B)$ and

$$V(T) = \left(S(0)\, e^{\sigma \widehat{W}(T)} - K\right) \mathbb{1}_{\{\widehat{W}(T) \ge k,\ \widehat{M}(T) \le b\}}$$

where $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$ & $\widehat{M}(T) = \max\limits_{0 \le t \le T} \widehat{W}(t)$

The price of the option at time t is given by the risk-neutral pricing formula

$$V(t) = \widetilde{\mathbb{E}}\left[e^{-r(T-t)}\, V(T) \,\middle|\, \mathcal{F}(t)\right], \quad 0 \le t \le T$$

Note that for $s < t$

$$\widetilde{\mathbb{E}}\left[V(t) e^{-rt} \,\middle|\, \mathcal{F}(s)\right] = \widetilde{\mathbb{E}}\left[\widetilde{\mathbb{E}}\left[e^{-rT} V(T) \,\middle|\, \mathcal{F}(t)\right] \,\middle|\, \mathcal{F}(s)\right]$$

$$= \widetilde{\mathbb{E}}\left[e^{-rT} V(T) \,\middle|\, \mathcal{F}(s)\right] = e^{-rs}\, V(s)$$

Therefore $e^{-rt} V(t)$ is a martingale.

By the Markov property $V(t) = v(t, S(t))$.

~~Define~~ Let $\rho = \inf\{t \geq 0 \mid S(t) = B\}$.

Then $$e^{-r(t\wedge\rho)} V(t\wedge\rho) = \begin{cases} e^{-rt} V(t) & \text{if } 0 \leq t \leq \rho \\ e^{-r\rho} V(\rho) & \text{if } \rho < t \leq T. \end{cases}$$

we have $e^{-rt} v(t, S(t))$ is a $\widetilde{\mathbb{P}}$-martingale. The optional sampling theorem asserts that the stop process

$$e^{-r(t\wedge\rho)} v(t\wedge\rho, S(t\wedge\rho)), \quad 0 \leq t \leq T$$

is a martingale under $\widetilde{\mathbb{P}}$.

compute the differential

$$d(e^{-rt} v(t, S(t))) = e^{-rt}\Big[-r v(t, S(t))dt + v_t(t, S(t))\,dt + v_x(t, S(t))\,dS(t) + \tfrac{1}{2} v_{xx}(t, S(t))\,dS(t)\,dS(t)\Big]$$

$$= e^{-rt}\Big[-r v(t, S(t)) + v_t(t, S(t)) + r S(t) v_x(t, S(t)) + \tfrac{1}{2}\sigma^2 S^2(t) v_{xx}(t, S(t))\Big]dt$$

$$+ e^{-rt} \sigma S(t) v_x(t, S(t))\, d\widetilde{W}(t).$$

The dt term must be zero for $0 \leq t \leq \rho$. But since $(t, S(t))$ can reach any point in $\{(t,x) : 0 \leq t \leq T, 0 \leq x \leq B\}$ before the option knocks out, the B-S-M equation must hold for every $t \in [0,T)$ and $x \in [0,B]$.

Remark:- setting dt term equal to zero, we obtain

$$d(e^{-rt} v(t, S(t))) = e^{-rt} \sigma S(t) v_x(t, S(t))\, d\widetilde{W}(t)$$

The discount value of a portfolio that each time t holds

$\Delta(t)$ shares of the underlying asset is given by

$$d(e^{-rt}X(t)) = e^{-rt}\,\sigma S(t)\Delta(t)\,d\widetilde{W}(t)$$

At least theoretically, if an agent begins with a short position in the up-and-out call and with initial capital $X(0) = v(0, S(0))$, the usual delta-hedging formula

$$\Delta(t) = v_x(t, S(t))$$

will cause her portfolio value $X(t)$ to track the option value $v(t, S(t))$ up to time ρ of knock out or up to expiration T, whichever comes first.

<u>Lookback Options:-</u>

An option whose payoff is based on the maximum or minimum of the underlying asset price attains over some interval of time prior to expiration is called a lookback option. Examples

$V(T) = \max\left[\max_{t\le T} S(t) - K, 0\right]$, $V(T) = \max\left[K - \min_{t\le T} S(t), 0\right]$, $V(T) = S(T) - \min_{t\le T} S(t)$.

Here we consider floating strike lookback option. The payoff of this option is given by $V(T) = Y(T) - S(T)$, at expiration time T,

where $S(t) = S(0)e^{\sigma \widehat{W}(t)}$, $\widehat{W}(t) = \alpha t + \widetilde{W}(t)$, $\alpha = \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)$

with $\widehat{M}(t) = \max_{0\le u\le t} \widehat{W}(u)$ and $Y(t) = \max_{0\le u\le t} S(t) = S(0)e^{\sigma\widehat{M}(t)}$

$S(t) = rS(t)\,dt + \sigma S(t)\,d\widetilde{W}(t)$, $\widetilde{W}(t)$ is a Brownian motion under risk-neutral measure $\widetilde{\mathbb{P}}$.

$$S(t) = e^{(r - \frac{1}{2}\sigma^2)t + \sigma\widetilde{W}(t)}$$