Statistical Inference and Multivariate Analysis (MA324)

Lecture Slides
Lecture 19

Neyman-Pearson Lemma

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Simple Null Vs. Simple Alternative

Theorem (Neyman-Pearson Lemma)

Let $\boldsymbol{\theta}_0 \neq \boldsymbol{\theta}_1$ be two fixed numbers in Θ. The MP level α test for $H_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ against $H_1 : \boldsymbol{\theta} = \boldsymbol{\theta}_1$ is given by

$$\psi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } L(\boldsymbol{\theta}_1) > kL(\boldsymbol{\theta}_0) \\ \gamma & \text{if } L(\boldsymbol{\theta}_1) = kL(\boldsymbol{\theta}_0) \\ 0 & \text{if } L(\boldsymbol{\theta}_1) < kL(\boldsymbol{\theta}_0), \end{cases}$$

where $k \geq 0$ and $\gamma \in [0,\, 1]$ such that $\beta(\boldsymbol{\theta}_0) = E_{\boldsymbol{\theta}_0}\left(\psi(\boldsymbol{X})\right) = \alpha$. Here, $L(\cdot)$ is the likelihood function.

Example 1: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where σ is known. Let $\mu_0 < \mu_1$ be two fixed real numbers. We are interested to test $H_0 : \mu = \mu_0$ against $H_1 : \mu = \mu_1$. Here, the likelihood function is

$$\begin{aligned} L(\mu) &= \frac{1}{\sigma^n (2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2\right] \\ &= \frac{1}{\sigma^n (2\pi)^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{i=1}^{n} x_i^2 - 2n\mu\bar{x} + n\mu^2\right\}\right]. \end{aligned}$$

Therefore,

$$\frac{L(\mu_1)}{L(\mu_0)} = \exp\left[\frac{1}{2\sigma^2}\left\{2n\bar{x}(\mu_1 - \mu_0) + n\left(\mu_0^2 - \mu_1^2\right)\right\}\right].$$

What is the MP level α test?

Remark: Note the way of solving the problem. We try to simplify $\frac{L(\mu_1)}{L(\mu_0)} > k$ so that we can write an equivalent condition on a statistic whose distribution under H_0 is known or can be found. If this **statistic is a continuous random variable**, we will have a **non-randomized test**. Otherwise we may need to consider $\gamma \in (0,\, 1)$ making the test a randomized one.