QR decomposition by Rotators and Reflectors

The strategy to compute a QR decomposition

The strategy to compute a QR decomposition of A is to find some 'elementary' $n \times n$ matrices $Q_1, \ldots, Q_k$ that are orthogonal if A is real and unitary if A is complex such that

$$Q_k^\star \cdots Q_1^\star A \text{ is upper triangular.}$$

Here $\star = T$ is A is real and $\star =^*$ if A is complex.

This strategy will be elaborated for $A \in \mathbb{R}^{n \times m}$, $n \geq m$ although everything extends to the complex case as well with appropriate modifications.

Rotators

A real Givens (or plane) rotator is an orthogonal matrix $Q \in \mathbb{R}^{n\times n}$, of the form

$$Qx = \begin{bmatrix} 1 &&&&&&&&&& \\ & \ddots &&&&&&&&& \\ && 1 &&&&&&&& \\ &&& c &&&& -s &&& \\ &&&& 1 &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& 1 &&&& \\ &&& s &&&& c &&& \\ &&&&&&&& 1 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{i-1} \\ x_i \\ \vdots \\ x_j \\ x_{j+1} \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ \vdots \\ x_{i-1} \\ c x_i - s x_j \\ x_{i+1} \\ \vdots \\ x_{j-1} \\ s x_i + c x_j \\ x_{j+1} \\ \vdots \\ x_n \end{bmatrix}$$

where $c = cos\,\theta, s = \sin\,\theta$. Evidently, $QQ^T = I_n = Q^T Q$.

Rotators

Assuming that $i < j$ and $Q\{i,j\} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$, if $y = Q^T x$ for $x = [x_1\; x_2\; \cdots\; x_n]^T \in \mathbb{R}^n$, then

$$\begin{bmatrix} y_i \\ y_j \end{bmatrix} = \begin{bmatrix} cx_i + sx_j \\ -sx_i + cx_j \end{bmatrix} \text{ with } y_k = x_k \text{ for } k \neq i \text{ or } j.$$

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So if $\sqrt{x_i^2 + x_j^2} \neq 0$, then for, $c = \frac{x_i}{\sqrt{x_i^2+x_j^2}}$ and $s = \frac{x_j}{\sqrt{x_i^2+x_j^2}}$, $y_i = \sqrt{x_i^2 + x_j^2}$ and $y_j = 0$. In particular if $n = 2$,

$$Q^T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \sqrt{x_1^2 + x_2^2} \\ 0 \end{bmatrix}.$$

QR decomposition by Rotators

Let $A \in \mathbb{R}^{n\times m}$, $n \geq m$. Find Givens rotators $Q_1^{(1)}, Q_2^{(1)}, \ldots Q_{n-1}^{(1)}$ such that

$$(Q_{n-1}^{(1)})^T \cdots (Q_2^{(1)})^T (Q_1^{(1)})^T A(:,1) = \begin{bmatrix} \pm\|A(:,1)\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $A(:,1) = [a_{11}\ a_{21}\ \cdots\ a_{n1}]^T$ is the first column of A.

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where $A(:,1) = [a_{11}\ a_{21}\ \cdots\ a_{n1}]^T$ is the first column of A. Then,

$$(Q_{n-1}^{(1)})^T \cdots (Q_2^{(1)})^T (Q_1^{(1)})^T A = \underbrace{\begin{bmatrix} \pm\|A(:,1)\|_2 & a_{12}^{(1)} & \cdots & a_{1m}^{(1)} \\ 0 & a_{22}^{(1)} & \cdots & a_{2m}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2}^{(1)} & \cdots & a_{nm}^{(1)} \end{bmatrix}}_{=:A_1}$$

QR decomposition by Rotators

Next find Givens rotators $Q_1^{(2)}, Q_2^{(2)}, \ldots Q_{n-2}^{(2)}$ such that

$$(Q_{n-2}^{(2)})^T \cdots (Q_2^{(2)})^T (Q_1^{(2)})^T A_1(:,2) = \begin{bmatrix} a_{12}^{(1)} \\ \pm\|A_1(2:n,2)\|_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $A_1(2:n,2) = [a_{22}^{(1)}\ a_{23}^{(1)}\ \cdots\ a_{n2}^{(1)}]^T$ is the second column of A_1 from entries 2 to n.

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$$(Q_{n-2}^{(2)})^T \cdots (Q_2^{(2)})^T (Q_1^{(2)})^T A_1 = \underbrace{\begin{bmatrix} \pm\|A(:,1)\|_2 & a_{12}^{(1)} & \cdots & a_{1m}^{(1)} \\ 0 & \pm\|A_1(2:n,2)\|_2 & \cdots & a_{2m}^{(2)} \\ 0 & 0 & \cdots & a_{3m}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nm}^{(2)} \end{bmatrix}}_{=:A_2}$$

QR decomposition by Rotators

Set

$$Q^{(k)} := Q_1^{(k)} \cdots Q_{n-k}^{(k)} \text{ for } k = 1, \ldots, p$$

where $p = m$ if $n > m$ and $p = n - 1$ otherwise. Then,

$$(Q^{(p)})^T \cdots (Q^{(1)})^T A =: R \in \mathbb{R}^{n \times m} \text{ is upper triangular.}$$

Setting $A_0 := A$, $R(i,i) = \pm \|A_{i-1}(i:n,i)\|_2$ for $i = 1, \ldots, m$.

So for the orthogonal matrix $Q := Q^{(1)} \cdots Q^{(p)}$, we have

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Exercise: Given $A \in \mathbb{F}^{n \times m}$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, and $n \geq m$, use mathematical induction to show that A has a QR decomposition.

Flop Count of finding R by Rotators

Total number of rotators used: $\Sigma_{k=1}^{p}(n-k)$.
Flop count of constructing each rotator: 5 flops and 1 square root.
Flop count of applying each rotator to a matrix with j columns: 6j flops.
So total flop count of finding R is

$$\underbrace{6\Sigma_{k=1}^{p}(n-k)(m-k)}_{\text{applying the rotators}} + \underbrace{(5+\alpha)\Sigma_{k=1}^{p}(n-k)}_{\text{creating the rotators}}.$$

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Exercise: Show that the flop count for finding R of a QR decomposition of $A \in \mathbb{R}^{n\times m}$ by rotators is
$3nm^2 - m^3 + O(nm) + O(m^2)$ if $n > m$ and $2n^3 + O(n^2)$ if $n = m$.