MA 322: Scientific Computing

Department of Mathematics
Indian Institute of Technology Guwahati

February 08, 2023

Chapter 4: Numerical Integrations or Quadratures

Simpson's 3/8-rule

- Approximate
$$I = \int_a^b f(x)\mathrm{d}x.$$
- Using cubic polynomial approximation,
$$I_3(f) = \frac{3h}{8}\left[f(a) + 3f\left(\frac{a+2b}{3}\right) + 3f\left(\frac{2a+b}{3}\right) + f(b)\right], \qquad h = \frac{b-a}{3}.$$
- Error in Simpson's 3/8-rule,
$$E_2(f) = -\frac{(b-a)^5}{6480} f^{(4)}(\eta), \qquad \eta \in [a, b].$$

Composite Simpson's 3/8-rule

- Approximate, $I = \int_a^b f(x)\mathrm{d}x.$
- Using cubic polynomial approximation,

$$I_n(f) = \frac{3h}{8}\left[f(x_0) + 3\sum_{\substack{j=1 \\ 3\nmid j}}^{n-1} f(x_j) + 2\sum_{j=1}^{n/3-1} f(x_{3j}) + f(x_n)\right], \qquad h = \frac{x_{3j} - x_{3j-3}}{3}.$$

- Error in Simpson's 3/8-rule,

$$E_n(f) = -\frac{(b-a)h^4}{80} f^{(4)}(\eta), \qquad \eta \in [a, b].$$

- Asymptotic error formula, $\tilde{E}_n(f) = -\frac{1}{80}\left[f^{(3)}(b) - f^{(3)}(a)\right].$

Comparison of different quadrature formulae

$$I = \int_0^2 f(x)\mathrm{d}x.$$

$f(x)$	x^2	x^4	$1/(1+x)$	$\sqrt{1+x^2}$	$\sin x$	e^x
Exact value	2.667	6.400	1.099	2.958	1.416	6.389
Trapezoidal	4.000	16.000	1.333	3.326	0.909	8.389
Simpson's 1/3-rule	2.667	6.667	1.111	2.964	1.425	6.421
Simpson's 3/8-rule						

Newton-Cotes integration

Motivation

- Trapezoidal rule integrates any linear function exactly.
- Simpson's rules integrate any cubic function exactly.

- Can we derive a generalized numerical integration rule that will approximate any polynomial of degree n exactly?

Principle and type of Newton-Cotes formula

- Polynomial of degree n has $n+1$ free parameters.
- We look for the appropriate integral as a linear combination of $f(x_j),\ j = 0, 1, 2, \cdots n$.
- Such a general integration is called Newton-Cotes formula.
- There are two types of Newton-Cotes formula: open and closed.

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