22/8/24

$Ax=b \Rightarrow x = A\backslash b \longrightarrow$ this doesn't work for our equatⁿ.

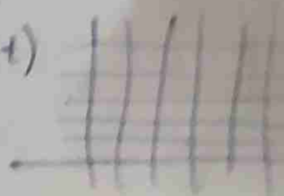
1) Jacobi
2) Gauss- Seidel
3) SoR

Fully discretized schemes:-

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

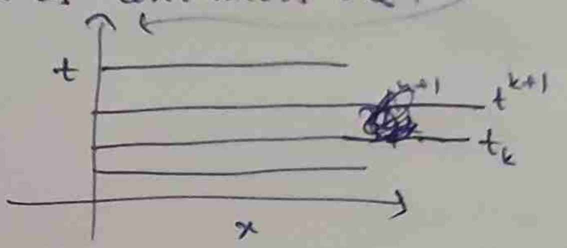
Backward Euler ← $O(\Delta t)$
Forward "

CN — $O(\Delta t^2)$



Semi-discretized:- (Method of Lines) — MoL

In order to solve parabolic IBVP numerically, one can use MoL, which is known as semi-discrete scheme. In this method, we discretize either the time domain and ~~the~~ preserve the spacial domain as continuous one.



Consider $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t) \qquad u(a,t) = u(b,t) = 0$$
$$u(x,0) = \phi(x), \quad x \in [a,b]$$

$$\frac{U^{k+1}(x) - U^k(x)}{\Delta t} = \frac{\partial^2 U^k}{\partial x^2} + f(x,t_k) \longrightarrow \text{explicit}$$

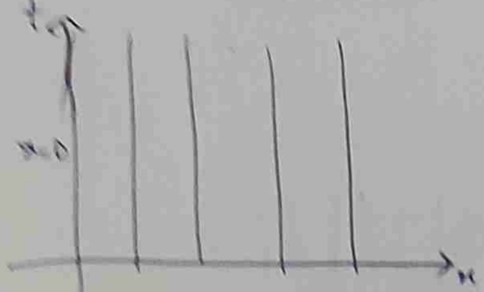
$U^k(x) \simeq U(x,t_k)$

(2) — $$\frac{U^{k+1}(x) - U^k(x)}{\Delta t} = \frac{\partial^2 U^{k+1}}{\partial x^2} + f(x,t_{k+1}) \rightarrow \text{implicit}$$

with bdry condⁿ, $U^k(a) = U^k(b) = 0$ (Using $u(a,t) = u(b,t) = 0$)

(3) — $$U^{k+1} - (\Delta t)\frac{\partial^2 U^{k+1}}{\partial x} = U^k(x) + \Delta t\, f(x,t^k)$$

Method of vertical lines:-



this can also be made higher order.

$$\frac{du_i(t)}{dt} = \frac{U_{i+1}(t) - 2U_i(t) + U_{i-1}(t)}{\Delta x^2} + f_i(t)$$

$U_0(t) = U_{N+1}(t) = 0$

$U_i(0) = \phi(x_i)$

The above eqⁿs are 1st order ODE eqⁿs. We obtain system of 1st order ODEs.

One can use explicit or implicit Euler — then it will FTCS or BTCS

In order to obtain higher order accuracy, one can use RK or multi step scheme like Adam Bashford or Adam Moulton.

Finite Element Methods — 4th editⁿ — Seydel

Added\
23/8/24

Step 1:- Weak formulatⁿ

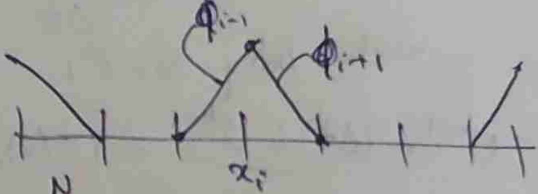
Step 2:- Domain Discretizatⁿ — finite dimension problem

3:- Numerical Integratⁿ / sys of linear algebra

Step 2:- Divide into elements — In 1D,

In 2D, Grid or Δles.

$W(x) = \sum c_i \phi_i(x)$ — basis or triaⁿ fⁿ



Discretize the domain Ω, $\Omega = \bigcup_{k=1}^{N} \Omega_i$

$\Omega_i = [x_i, x_{i+1}]$

For linear ϕ_i – ~~it is~~ we get tridiagonal matrices

quadr – we get $\begin{bmatrix} \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}$ matrix

To determine c_i's, we have to form "n" no of eq^{n}s

To calculate c_i's, we can use residual f^n (R)

R: $Lw - f$

Test f^{n}s – $\psi_1, \ldots, \psi_N$

s.t $\int_\Omega R\psi_j(x).dx = 0, \; j = 1, \ldots, N$

For $-u'' = f$

$\therefore Lw = \int_\Omega \left(\sum_{i=1}^{N} c_i \phi_i''(x) - f\right)\psi_j(x).dx = 0, \; j = 1, \ldots, N$

$$\sum_{i=1}^{N} c_i \int_\Omega \phi_i''(x)\psi_j(x).dx = -\int_\Omega f.\psi_j(x).dx$$

$$\int Lw\, \psi_j(x) = \int_\Omega f.\psi_j(x).dx \qquad \text{As} \left(-\sum c_i \phi_i''(x) = Lw\right)$$

$$(Lw, \psi_j(x)) = (f, \psi_j)$$

inner product

i) $\phi_j(x) = \psi_j(x) \Rightarrow$ Buknov – Galerkin

ii) Collocat^n: $\psi_j(x) = \delta(x - x_j)$

~~$Lw(x)$~~

$\int f\psi_j(x).dx = f(x_j)$ (From properties of Dirac delta f^n)

$\int Lw\,\psi_j(x).dx = Lw(x_j)$

iii) Least square

$$\psi_j(x) = \frac{\partial R}{\partial c_j} = \frac{\partial\left(\sum c_i \phi_i(x) - f\right)}{\partial c_j}$$

$\int$

26/6/24

Using FEM to

Step I:- $-p(x)u''(x) + q(x)u(x) = r(x)$, $u(0) = 0 = u(1)$ $\Big\} \longrightarrow ①$

$v \in C_0^\infty(\Omega)$ cont ∞ diff, vanishes on boundary

$v \in H_0^1(\Omega)$ tell later, vanishes on bdry

$\int_0^1 ① \times v \Rightarrow$

$$-\int_0^1 p(x)u''v\,dx + \int_0^1 q(x)uv\,dx = \int_0^1 r(x)v(x)\,dx$$

$\int_0^1 u\,dv = (uv)_0^1 - \int_0^1 v\,dx$ By parts

$$\left(-p(x)u'v\right)_0^1 + \int_0^1 p(x)u'v'\,dx + \int q(x)v\,dx = \int rv\,dx$$

$$② \longleftarrow \begin{cases} \text{Find } u \in H_0^1(\Omega) =: V \\ \ni \int_0^1 p(x)u'v'\,dx + \int q(x)uv\,dx = \int rv\,dx, \ \forall v \in V \end{cases}$$

$① \equiv ②$

In ①, As $r(x)$ is cont, $p \in C^{(1)}$, $u \in C^{(2)}$ but in ②, this is not necessary

$u \in L^p(\Omega) \Leftrightarrow \int |u|^p(x))^{1/p} < \infty$

L-p space

Step 2:-

$V_h \subset V$ $\quad V_h = \text{span}\{\phi_1, \dots, \phi_{N-1}\}$ where ϕ_i's are piecewise fn's.

Find $u_h \in V_h \ni$

$$③ \longleftarrow \int_0^1 p(x)u_h'v_h'\,dx + \int_0^1 q(x)u_h v_h\,dx = \int_0^1 r(x)v_h\,dx \quad \forall v_h \in V_h$$

$$④ \longleftarrow u_h(x) = \sum_{i=1}^{N-1} u_i(x)\phi_i(x)$$

where u_i's are unknown to be determined.

In order to determine $N-1$ U_i's, we use

$$\sum_{i=1}^{N-1}\left(\int_{x_i}^{x_{i+1}} p(x)\,\phi_i'(x)\,\phi_j'(x) + q(x)\,\phi_i(x)\,\phi_j(x)\right)U_i = \int_0^1 r(x)\,\phi_j(x)\,dx$$

$j = 1 : N-1$

$\left(v_h = \sum_i^N V_i \phi_i\right.$ — the V_i's get cancelled$\Big)$

(Using different quadratures & different forms of ϕ_i, we get different schemes.

Assume ϕ_i's are piecewise linear polynomials.

$\phi_i = \hat{a}_i + \hat{b}_i x$ for $\Omega_i = [x_i, x_{i+1}]$

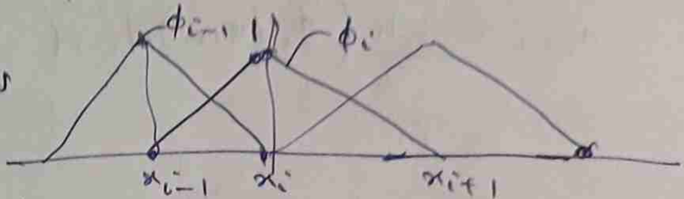
$= 0$ o.w

$\phi_i(x_j) = \begin{cases} 1 & i=j \\ 0 & \text{o.w} \end{cases}$

$\phi_i \phi_j = 0$ if $|j-i| > 1$

o.w

if $x \in [x_{i-1}, x_{i+1}]$



Again we have to solve a tridiagonal matrix (called stiffness matrix)

$a_{ji} = \int p(x)\,\phi_i'\,\phi_j' + \int q(x)\,\phi_i\,\phi_j\,dx$

$r_j = \int r(x)\,\phi_j(x)\,dx$

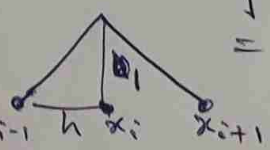
~~$= \int p(x)\,\hat{b}_i\,\hat{b}_j + \int q(x)$~~

For $j = i$,

it is p only not q

$$a_{ji} = p(x)\cdot\left(\frac{-1}{h}\right)\left(\frac{-1}{h}\right)\times h + p\left(\frac{1}{h}\right)\left(\frac{1}{h}\right)h$$

($\phi_i = \hat{a}_i + \hat{b}_i x$ is the general sol^n & it is apparently (told by sir) same as



$= \phi_i$)

For $j = i-1$,

$$a_{i\,i-1} = \int r\,\phi_i'(x)\,\phi_{i-1}'(x)\,dx = p\left(\frac{1}{h}\right)\left(\frac{-1}{h}\right)h = \frac{-p}{h}$$

29/8/24

$$a_{ji} = \sum_{j=1}^{N-1}\int_{x_j}^{x_{j+1}} q(x)\,\phi_i(x)\,\phi_j(x)\,dx \qquad i = 1, \dots, N-1$$

Assume q as const (will show var later)

$$a_{ii} = \int_{x_{i-1}}^{x_i} q\,\phi_i(x)\,\phi_i(x)\,dx + \int_{x_i}^{x_{i+1}} q\,\phi_i(x)\,\phi_i(x)\,dx$$

$$a_{ii} = \frac{q h}{2}\left[\cancel{\phi_i(x_{i-1})} + \phi_i(x_i)\right] + \frac{q h}{2}\left(\phi_i(x_i) + \cancel{\phi_i(x_{i+1})}\right]$$

$= q.h$ (if q were a f^n, it woud be $q(x_i).h$)

from trapezoid rule

$j = i-1$

$$a_{i-1} = \int_{x_{i-1}}^{x_i} q\,\phi_{i-1}\phi_i\,dx$$

$$= q\,\frac{h}{2}\left(\phi_{i-1}(x_i)\,\phi_i(x_i) + \phi_{i-1}(x_{i-1})\,\phi_i(x_{i-1})\right]$$

$= 0$

$j = i+1$

$$a_{i+1} = \frac{q h}{2}\left(\phi_i(x_{i+1})\phi_{i-1}(\quad) + \ldots\right)$$

$= 0$

$$A = \begin{bmatrix} qh & & & \\ & qh & & \\ & & \ddots & \\ & & & qh \end{bmatrix}$$

$$A \leq a_{ij} = \begin{cases} -p/h + 0, & j = i-1 \\ 2p/h + q^h, & j = i \\ -p/h + 0, & j = i+1 \end{cases}$$

$$RHS = \int_0^1 -r(x)\,\phi_j(x).dx$$

$$= -\sum_{j=1}^{N-1} \int_{x_{j-1}}^{x_j} r(x)\phi_j(x).dx$$

$$= \int_{x_{j-1}}^{x_j} r(x)\phi_j(x) + \int_{x_j}^{x_{j+1}} r(x)\,\phi_j(x).dx$$

$$= \frac{h}{2}\left[r(x_j)\phi_j(x_j) + r(x_j)\,\phi_j(x_j)\right]$$

$AU = F$

$U = A \backslash F$

Using Simpson's rule

$$a_{ji} = \underbrace{\int_0^1 p(x)\,\phi_i'(x)\,\phi_j'(x)\,dx}_{I_1} + \underbrace{\int_0^1 q(x)\,\phi_i(x)\,\phi_j(x)\,dx}_{I_2}$$

I_1

$$a_{ii} = \int_{x_{i-1}}^{x_i} p(x)(\phi_i'(x))^2\,dx + \int_{x_i}^{x_{i+1}} p(x)(\phi_i'(x))^2\,dx$$

$$= \cancel{\int_{x_{i-1}}^{x_i} p(x)}\ \frac{1}{h^2}\left(\frac{h}{6}\left[p(x_{i-1}) + 4p\left(x_{i-1}+\frac{h}{2}\right) + p(x_i)\right]\right)$$

$$+ \frac{h}{6}\left[p(x_{i+1})\underbrace{\phi_i^2(x_{i+1})}_{0} + 4p\left(\frac{x_{i+1}+x_i}{2}\right)\phi_i^2\left(\frac{x_i + x_{i+1}}{2}\right) + p(x_i)\underbrace{\phi_i^2(x_i)}_{1}\right]$$

here $p\left(\frac{x_{i-1}+x_i}{2}\right) \sim \frac{p(x_i)+p(x_{i-1})}{2}$

$\frac{1}{2h}$ $\qquad \frac{1}{8} \vee \frac{1}{6}$

$\cancel{\frac{1}{6h}(p_{i-1} + p_i}$

$$a_{ij} = \begin{cases} -\frac{h}{2}(p_{i-1}+p_i) + \\ \cancel{\frac{1}{2h}(p_{i-1}+p_i)} + h(p_i \\ \qquad + \end{cases}$$

Assume ϕ_i are quadratic eq do

2/4/07

$$\int_0^1 p(x)\underbrace{\phi_i'\phi_j'}_{A}dx + \int q(x)\underbrace{\phi_i\phi_j}_{B}\, dx = \int f\phi_j\, dx \simeq F_i B$$

$$\simeq \int_{x_{j-1}}^{x_j} f\phi_j + \int_{x_j}^{x_{j+1}} f\phi_j \longrightarrow$$ It is given this way in book

$$A = \frac{1}{h}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & & \ddots & \\ & & & & \ddots \end{bmatrix} \qquad B = \frac{h}{6}\begin{bmatrix} 2 & 1 & \cdot & & \cdot \\ 1 & 4 & 1 & \cdot & \cdot \\ & & & & \cdot \\ & & & 1 & 2 \end{bmatrix}$$

Parabolic

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad (x,\tau) \in (-\infty,\infty)\times(0,T] \longrightarrow ①$$

By using the method of horizontal lines, we can convert the parabolic PDE into system of boundary value problems (2nd order ODEs) which can be solved by FEM.

Rather, one can follow discretizing both time & space by FEM. The solⁿ $u(x,\tau) = \sum_{i=1}^{N} w_i(\tau)\phi_i(x) + \phi_0(x,\tau) \longrightarrow ②$

where ϕ_0 satisfies the Boundary & initial conditⁿs.

~~② → u(x, τ) =~~

Using ② in ①, $\displaystyle\int_{x_0}^{x_m}\left[\sum_{i=1}^{m-1}\dot{w}_i\phi_i + \dot{\phi}_0\right]\phi_j\, dx$

$$= \int_{x_0}^{x_m}\left(\sum_{i=1}^{m-1} w_i\phi_i'' + \phi_0''\right)\phi_j\, dx \longrightarrow ③$$

$j = 1, \ldots, m-1$

Use integratⁿ by parts

Now, ③ can be ~~returned~~ written as

$$B\dot{w} + b = -Aw - a \longrightarrow ④$$

where $b = \begin{pmatrix} \int \dot{\phi}_0\phi_1\, dx \\ \int \dot{\phi}_0\phi_2\, dx \\ \vdots \\ \int \dot{\phi}_0\phi \end{pmatrix}$ $\qquad a = \begin{pmatrix} \int \phi_0''\phi_1\, dx \\ \int \phi_0''\phi_2\, dx \\ \vdots \\ \int \phi_0''\phi_{m-1}\, dx \end{pmatrix}$

one part will vanish on the bdry

otherwise also, we take it to RHS

This is a system of 1^{st} order ODEs which is the method of vertical lines.

At $\tau = 0$, $u(x, 0) = \alpha(x) \longrightarrow$ (5)

From (2) & (5),

$$\sum_{i=1}^{N} w_i(0)\,\phi_i(x) + \phi_0(x_i) = \alpha(x)$$

$\because \phi_i(x_j) = \cancel{\delta(x-x_i)}\ \begin{cases} 0 & \text{if } j \neq i \\ 1 & \text{if } j = i \end{cases}$

At $x = x_j$,

$$\boxed{w_j(0) \cdot 1 = \alpha(x_j) - \phi_0(x_j, 0)}$$