

Solving Triangular Systems

A Lower Triangular System of Equations

Consider

$$Lx = b$$

where

$L = [l_{ij}] \leftarrow n \times n$ lower triangular matrix

$b \leftarrow n \times 1$ vector.

In expanded form,

$$l_{11}x_1 = b_1$$

$$l_{21}x_1 + l_{22}x_2 = b_2$$

$$\vdots$$

$$l_{k1}x_1 + l_{k2}x_2 + \cdots + l_{kk}x_k = b_k$$

$$\vdots$$

$$l_{n1}x_1 + l_{n2}x_2 + \cdots + l_{nk}x_k + \cdots + l_{nn}x_n = b_n$$

Row Oriented Forward Substitution

$$x_1 = b_1/l_{11}; \quad x_k = \left(b_k - \sum_{j=1}^{k-1} l_{kj}x_j \right) / l_{kk}, \quad k = 2, \dots, n;$$

```
for k = 1 : n
    for j = 1 : k - 1
        b_k = b_k - l_kj*b_j;
    end
    if l_kk ≠ 0
        b_k = b_k/l_kk;
    else
        exit {'matrix is singular'}
    end
end
```

Row Oriented Forward Substitution

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for $k = 1 : n$

for $j = 1 : k - 1$

← entries of L are being accessed row wise

$$b_k = b_k - l_{kj}b_j; \quad (2 \text{ flops})$$

end

if $l_{kk} \neq 0$

$$b_k = b_k / l_{kk}; \quad (1 \text{ flop})$$

else

exit {'matrix is singular'}

end

end

Total flops:

$$1 + \sum_{k=2}^n \left\{ \left(\sum_{j=1}^{k-1} 2 \right) + 1 \right\} = \sum_{k=1}^n 2(k-1) + n = n^2.$$

Column Oriented Forward Substitution

```
for  $j = 1 : n$   
    if  $l_{jj} \neq 0$   
         $b_j = b_j / l_{jj};$       (1 flop)  
    end  
    for  $i = j + 1 : n$       ← entries of  $L$  are being accessed column wise  
         $b_i = b_i - l_{ij}b_j;$   (2 flops)  
    end  
end
```

Total flops: n^2 .

Likewise, $n \times n$ upper triangular systems may be solved in n^2 flops each by performing row and column oriented backward substitutions.

Blocked forward substitution:

The $n \times n$ lower triangular system $Lx = b$ may be rewritten as

$$\begin{bmatrix} L_{11} & \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where the blocks of L are

$$\begin{aligned} L_{11} &\rightarrow k \times k \text{ lower triangular}; L_{21} \rightarrow (n - k) \times k; \\ L_{22} &\rightarrow (n - k) \times (n - k) \text{ lower triangular} \end{aligned}$$

and the partitions x_1, x_2 of x and b_1, b_2 of b are such that

$$x_1, b_1 \rightarrow k \times 1 \text{ and } x_2, b_2 \rightarrow (n - k) \times 1.$$

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$$x_1, b_1 \rightarrow k \times 1 \text{ and } x_2, b_2 \rightarrow (n - k) \times 1.$$

Pseudocode for blocked forward substitution:

1. Solve $L_{11}x_1 = b_1$ for x_1 .
2. Compute $b_2 - L_{21}x_1$.
3. Solve $L_{22}x_2 = b_2 - L_{21}x_1$ for x_2 .