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Deriving the Black-Scholes-Merton Formula

To obtain the Black-Scholes-Merton price of a European call, we assume a constant volatility σ, constant interest rate r, and take the derivative security payoff to be $V(T) = (S(T) - K)^+$. Then:

$$V(t) = \widetilde{E}\left[e^{-r(T-t)}\left(S(T)-K\right)^+ | \mathcal{F}(t)\right],\ 0 \le t \le T.$$

Because geometric Brownian motion is a Markov process, this expression depends on the stock price $S(t)$ and of course on the time t at which the conditional expectation is computed, but not on the stock price prior to time t. In other words, there is a function $c(t,x)$ such that:

$$c(t, S(t)) = \widetilde{E}\left[e^{-r(T-t)}\left(S(T)-K\right)^+ | \mathcal{F}(t)\right].$$

With constant σ and r, we have:

$$S(t) = S(0)\exp\left\{\sigma \widetilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\},$$

and we may thus write:

$$S(T) = S(t)\exp\left\{\sigma\left(\widetilde{W}(T) - \widetilde{W}(t)\right) + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\},$$

where Y is the standard normal random variable:

$$Y = -\left(\frac{\widetilde{W}(T) - \widetilde{W}(t)}{\sqrt{T-t}}\right),$$

and τ is the "time-to-expiration" $\tau = T - t$.

We see that $S(T)$ is the product of the $\mathcal{F}(t)$-measurable random variable $S(t)$ and the random variable $\exp\left\{-\sigma\sqrt{\tau}Y + \left(r - \frac{1}{2}\sigma^2\right)\tau\right\}$, which is independent of $\mathcal{F}(t)$.

Therefore:

$$\begin{aligned} c(t,x) &= \widetilde{E}\left[e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}Y + \left(r-\frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right] \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}y + \left(r-\frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+ e^{-\frac{y^2}{2}}\,dy. \end{aligned}$$

The integrand:

$$\left(x\exp\left\{-\sigma\sqrt{\tau}y + \left(r-\frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+,$$

is positive if and only if:

$$y < d_-(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right)\tau\right].$$

Therefore:

$$\begin{aligned}
c(t,x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}y + \left(r-\frac{1}{2}\sigma^2\right)\tau\right\} - K\right) e^{-\frac{y^2}{2}}\,dy,\\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} x\exp\left\{-\frac{y^2}{2} - \sigma\sqrt{\tau}y - \frac{\sigma^2\tau}{2}\right\} dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} e^{-r\tau} K e^{-\frac{y^2}{2}}\,dy,\\
&= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)} \exp\left\{-\frac{1}{2}(y+\sigma\sqrt{\tau})^2\right\} dy - e^{-r\tau} K N(d_-(\tau,x)),\\
&= \frac{x}{\sqrt{2\pi}} \int_{-\infty}^{d_-(\tau,x)+\sigma\sqrt{\tau}} \exp\left\{-\frac{z^2}{2}\right\} dz - e^{-r\tau} K N(d_-(\tau,x))\\
&= xN(d_+(\tau,x)) - e^{-r\tau} K N(d_-(\tau,x)),
\end{aligned}$$

where: $d_+(\tau,x) = d_-(\tau,x) + \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right)\tau\right]$. For future reference the following notation is introduced:

$$BSM(\tau,x;K,r,\sigma) = \widetilde{E}\left[e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}Y + \left(r-\frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right],$$

where Y is the standard normal variable under $\widetilde{\mathbb{P}}$. We have just shown that

$$BSM(\tau,x;K,r,\sigma) = xN(d_+(\tau,x)) - e^{-r\tau} K N(d_-(\tau,x)).$$

Martingale Representation Theorem (One-Dimension)

Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \le t \le T$, be the filtration generated by this Brownian motion. Let $M(t), 0 \le t \le T$ be a martingale with respect to this filtration (i.e., for every t, $M(t)$ is $\mathcal{F}(t)$-measurable and for $0 \le s \le t \le T$, $E[M(t)|\mathcal{F}(s)] = M(s)$). Then there is an adapted process $\Gamma(u), 0 \le u \le T$, such that:

$$M(t) = M(0) + \int_0^t \Gamma(u)dW(u),\ \ 0 \le t \le T.$$

Corollary

Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \le t \le T$, be the filtration generated by this Brownian motion. Let $\Theta(t), 0 \le t \le T$ be an adapted process. Define:

$$\begin{aligned}
Z(t) &= \exp\left\{-\int_0^t \Theta(u)dW(u) - \frac{1}{2}\int_0^t \Theta^2(u)du\right\},\\
\widetilde{W}(t) &= W(t) + \int_0^t \Theta(u)du,
\end{aligned}$$

and assume that $E\int_0^T \Theta^2(u)Z^2(u)du < \infty$. Set $Z = Z(T)$. Then $EZ = 1$ and under the probability measure $\widetilde{\mathbb{P}}$, the process $\widetilde{W}(t), 0 \le t \le T$, is a Brownian motion.

Now. let $\widetilde{M}(t), 0 \leq t \leq T$, be a martingale under $\widetilde{\mathbb{P}}$. Then there is an adapted process $\widetilde{\Gamma}(u), 0 \leq u \leq T$ such that:

$$\widetilde{M}(t) = \widetilde{M}(0) + \int_0^t \widetilde{\Gamma}(u) d\widetilde{W}(u), 0 \leq t \leq T.$$

Definition

A probability measure $\widetilde{\mathbb{P}}$ is said to be *risk-neutral* if:

(A) $\widetilde{\mathbb{P}}$ and $\mathbb{P}$ are equivalent (i.e., for every $A \in \mathcal{F}$, $\mathbb{P}(A) = 0$ if and only if $\widetilde{\mathbb{P}}(A) = 0$), and

(B) Under $\widetilde{\mathbb{P}}$, the discounted stock price $D(t)S_i(t)$ is a martingale for every $i = 1, 2, \ldots, n$.

Lemma

Let $\widetilde{\mathbb{P}}$ be a risk-neutral measure, and let $X(t)$ be the value of a portfolio. Under $\widetilde{\mathbb{P}}$, the discounted portfolio value $D(t)X(t)$ is a martingale.

Definition

An arbitrage is a portfolio value process $X(t)$ satisfying $X(0) = 0$ and also satisfying for some time $T > 0$, $\mathbb{P}\{X(T) \geq 0\} = 1$, $\mathbb{P}\{X(T) > 0\} > 0$.

First Fundamental Theorem of Asset Pricing

If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Definition

A market model is *complete* of every derivative security can be hedged.

Second Fundamental Theorem of Asset Pricing

Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.

Dividend Paying Stock

Discounted stock prices are martingales under the risk-neutral measure provided the stock pays no dividend. The key feature of a risk-neutral measure is that it causes discounted portfolio values to be martingales, and that ensures the absence of arbitrage. In order for the discounted value of a portfolio that invests in a dividend-paying stock to be a martingale, the discounted value of the stock *with the dividends reinvested* must be a martingale, but the discounted stock price itself is not a martingale.

Continuously Paying Dividend

Consider a stock, modeled as a generalized geometric Brownian motion, that pays dividends continuously over time at the rate $A(t)$ per unit time. Here $A(t), 0 \leq t \leq T$, is a non-negative adapted process. Dividends paid by a stock reduces its value, and so we shall stake as our model of the stock price:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t) - A(t)S(t)dt.$$

If the stock were to withhold dividends, its mean rate of return would be $\alpha(t)$. Equivalently, if an agent holding the stock were to reinvest the dividends, the mean rate of return on his investment would be $\alpha(t)$. The mean rate of return $\alpha(t)$, the volatility $\sigma(t)$, and the interest rate $R(t)$ are all assumed to be adapted processes.

An agent who holds the stock receives both the capital gain or loss due to stock price movements and the continuously paying dividend. Thus, if $\Delta(t)$ is the number of shares held at time t, then the portfolio value

$X(t)$ satisfies:

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + \Delta(t)A(t)S(t)dt + R(t)[X(t) - \Delta(t)S(t)]dt \\ &= R(t)X(t)dt + (\alpha(t) - R(t))\Delta(t)S(t)dt + \sigma(t)\Delta(t)S(t)dW(t) \\ &= R(t)X(t) + \Delta(t)S(t)\sigma(t)\left[\Theta(t)dt + dW(t)\right], \end{aligned}$$

where $\Theta(t) = \dfrac{\alpha(t) - R(t))}{\sigma(t)}$ is the usual market price of risk. We define:

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u)du,$$

and use Girsanov's Theorem to change to a measure $\widetilde{\mathbb{P}}$ under which $\widetilde{W}$ is a Brownian motion. Thus we have:

$$dX(t) = R(t)X(t)dt + \Delta(t)S(t)\sigma(t)d\widetilde{W}(t).$$

The discounted portfolio value satisfies:

$$d\left[D(t)X(t)\right] = \Delta(t)D(t)S(t)\sigma(t)d\widetilde{W}(t).$$

In particular, under the risk-neutral measure $\widetilde{\mathbb{P}}$, the discounted portfolio process is a martingale. Here we denote by $D(t) = e^{-\int_0^t R(u)du}$, the usual discount process.
If we now wish to hedge a short position in a derivative security paying $V(T)$ at time T, where $V(T)$ is an $\mathcal{F}(T)$-measurable random variable, we will need to choose the initial capital $X(0)$ and the portfolio process $\Delta(t), 0 \leq t \leq T$, so that $X(T) = V(T)$. Because $D(t)X(t)$ is a martingale under $\widetilde{\mathbb{P}}$, we must have:

$$D(t)X(t) = \widetilde{E}\left[D(T)V(T)|\mathcal{F}(t)\right], \; 0 \leq t \leq T.$$

The value $X(t)$ of this portfolio at each time t is the value (price) of the derivative security at that time, which we denote by $V(t)$. Making this replacement in the formula we obtain the risk-neutral pricing formula:

$$D(t)V(t) = \widetilde{E}[D(T)V(T)|\mathcal{F}(t)], \; 0 \leq t \leq T.$$

This is the same risk-neutral pricing formula as in the case with no dividends. The difference between the dividend and the no-dividend cases is in the evolution of the underlying stock under the risk-neutral measure:

$$dS(t) = [R(t) - A(t)]S(t)dt + \sigma(t)S(t)d\widetilde{W}(t).$$

Under the risk-neutral measure, the stock does not have mean rate of return $R(t)$, and consequently the discounted stock price is not a martingale. Indeed:

$$S(t) = S(0)\exp\left\{\int_0^t \sigma(u)d\widetilde{W}(u) + \int_0^t \left[R(u) - A(u) - \frac{1}{2}\sigma^2(u)\right]du\right\}.$$

The process:

$$e^{\int_0^t A(u)du} D(t)S(t) = S(0)\exp\left\{\int_0^t \sigma(u)d\widetilde{W}(u) - \frac{1}{2}\int_0^t \sigma^2(u)du\right\}.$$

is a martingale. This is the interest rate discounted value at time t of an account that initially purchases one share of the stock and continuously reinvests the dividends in the stock.

Continuously Paying Dividend with Constant Coefficients

In the event that the volatility σ, the interest rate r and the dividend rate a are constant, the stock price at time t, is given by:

$$S(t) = S(0)\exp\left\{\sigma\widetilde{W}(t) + \left(r - a - \frac{1}{2}\sigma^2\right)t\right\},$$

and we have:

$$S(T) = S(t)\exp\left\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + \left(r - a - \frac{1}{2}\sigma^2\right)(T-t)\right\},\ 0 \le t \le T.$$

According to the risk-neutral pricing formula, the price at time t of a European call expiring at time T with strike K is:

$$V(t) = \widetilde{E}[e^{-r(T-t)}(S(T)-K)^+|\mathcal{F}(t)], 0 \le t \le T.$$

To evaluate this, we first compute:

$$\begin{aligned}
c(t,x) &= \widetilde{E}\left[e^{-r(T-t)}\left(x\exp\left\{\sigma(\widetilde{W}(T) - \widetilde{W}(t)) + \left(r - a - \frac{1}{2}\sigma^2\right)(T-t)\right\} - K\right)^+\right] \\
&= \widetilde{E}\left[e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}Y + \left(r - a - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)^+\right]
\end{aligned}$$

where $\tau = T - t$ and $Y = -\left(\dfrac{\widetilde{W}(T) - \widetilde{W}(t)}{\sqrt{T-t}}\right)$, is a standard normal variable under $\widetilde{\mathbb{P}}$. We define:

$$d_{\pm}(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\frac{x}{K} + \left(r - a \pm \frac{1}{2}\sigma^2\right)\tau\right].$$

The call expires in the money if and only if $Y < d_-(\tau, x)$. Therefore:

$$\begin{aligned}
c(t,x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}\left(x\exp\left\{-\sigma\sqrt{\tau}y + \left(r - a - \frac{1}{2}\sigma^2\right)\tau\right\} - K\right)e^{-\frac{y^2}{2}}\,dy, \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} x\exp\left\{-\sigma\sqrt{\tau}y - \left(a + \frac{\sigma^2}{2}\right)\tau - \frac{y^2}{2}\right\}dy - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} e^{-r\tau}Ke^{-\frac{y^2}{2}}\,dy, \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_-(\tau,x)} xe^{-a\tau}\exp\left\{-\frac{1}{2}(y + \sigma\sqrt{\tau})^2\right\}dy - e^{-r\tau}KN(d_-(\tau,x)), \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_+(\tau,x)} xe^{-a\tau}\exp\left\{-\frac{z^2}{2}\right\}dz - e^{-r\tau}KN(d_-(\tau,x)), \\
&= xe^{-a\tau}N(d_+(\tau,x)) - e^{-r\tau}KN(d_-(\tau,x)).
\end{aligned}$$

Note that here we have made the change of variable $z = y + \sigma\sqrt{\tau}$.