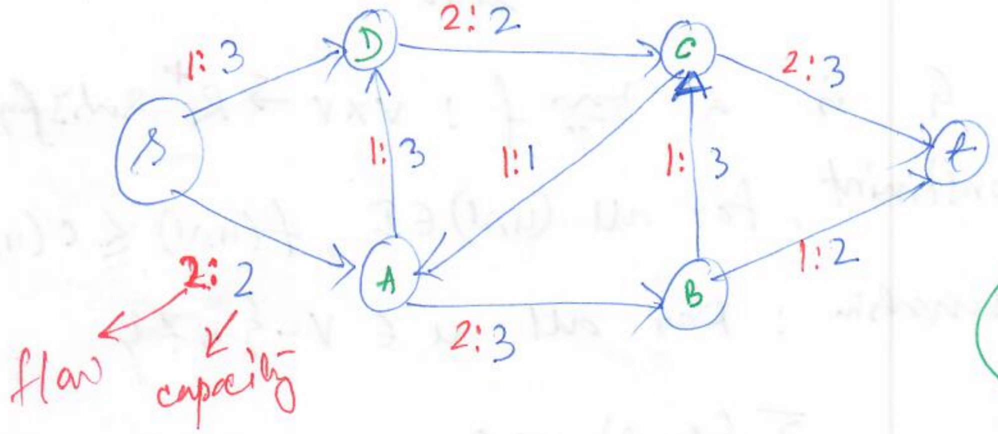
Maximum Flow

15/03/2023

P-1

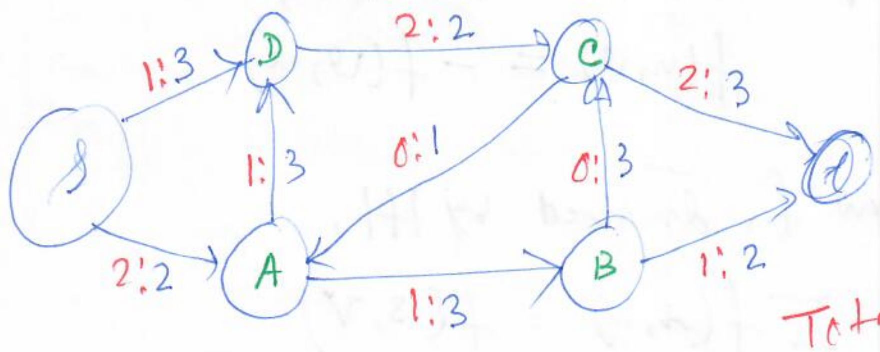
Flow network: $G(V,E) \rightarrow$ directed graph

- Two distinguished vertices source s and sink t
- Each edge $(u,v) \in E$, non-negative capacity $c(u,v)$
- If no edge $[(u,v) \notin E]$ then capacity is $c(u,v) = 0$

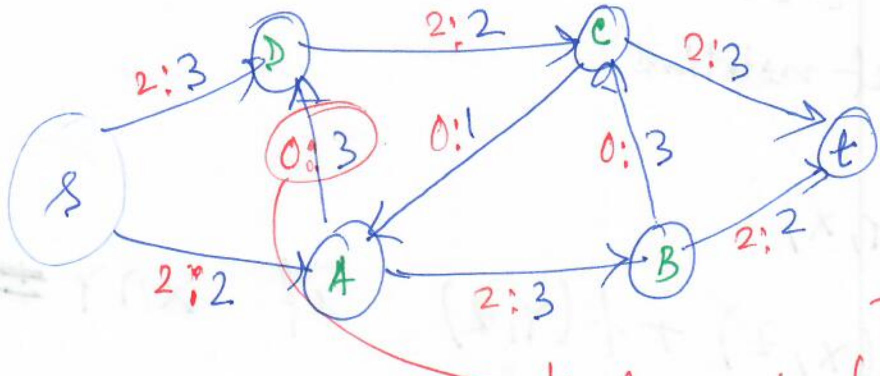


If it is not source or sink the flow amount in is equal to out flow amount → conservation law.

A, B, C makes a cycle, so modification



Total flow = 3.



Total flow = 4

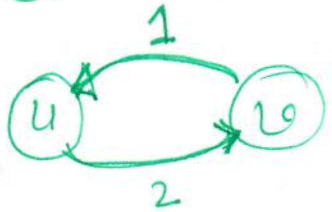
decrease flow on a particular edge leads to increase overall flow.

Given a flow network G, find a flow with maximum value on G.

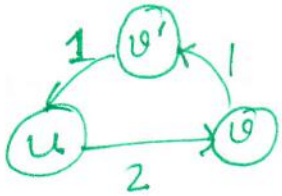
Assumptions:

No cycles with length 1 or 2.

- 
no self-loop edges allowed

- 

$\Rightarrow$ transform into



A flow on G is a ~~fun~~ $f : V \times V \to R^{+}$ satisfying

- capacity constraint, for all $(u,v) \in E$, $f(u,v) \leq c(u,v)$
- Flow conservation: For all $u \in V - \{s, t\}$

$$\sum_{v \in V} f(u,v) = 0.$$

- Skew symmetry: For all $u, v \in V$,

$$f(u,v) = -f(v,u)$$

The value of a flow f, denoted by $|f|$,

$$|f| = \sum_{v \in V} f(s,v) = f(s,V)$$

Simple notations. (set notations)

$f(X,X) = 0$

$f(X,Y) = -f(Y,X)$

$f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$ if $X \cap Y = \phi$

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p-2

Theorem: $|f| = f(V, t)$

proof: $|f| = f(s, V)$

$= f(V, V) - f(V - \{s\}, V)$

$\left[\because f(\underbrace{\{s\} \cup V - \{s\}}_{V}, V) = f(s, V) + f(V - \{s\}, V) \right.$

$\Rightarrow f(V, V) = f(s, V) + f(V - \{s\}, V)$

$\left. \Rightarrow 0 = f(s, V) + f(V - \{s\}, V) \right]$

$|f| = -f(V - \{s\}, V)$

$= f(V, V - \{s\})$

$= f(V, t) + f(V, V - \{s, t\})$

$= f(V, t) - f(V - \{s, t\}, V)$

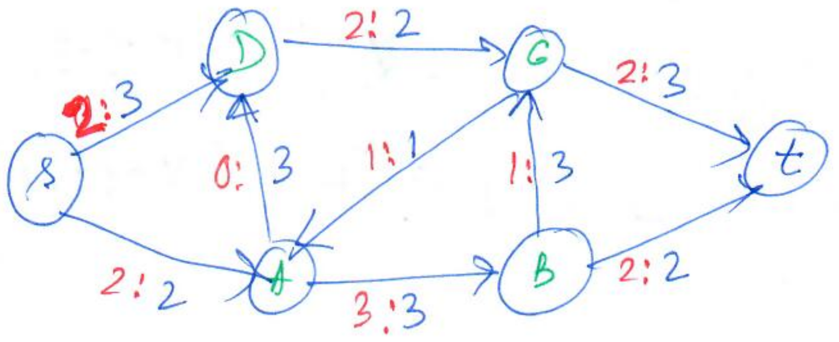
$= f(V, t) - \sum_{u \in V - \{s, t\}} f(u, V)$

$= f(V, t) - \sum_{u \in V - \{s, t\}} 0$

Cuts:

A cut (S,T) of a flow network $G(V,E)$ is a partition of V such that $s \in S$, and $t \in T$

If f is a flow on G, then the flow across the cut is $f(S,T)$.



let $S = \{s, C\}$
$T = \{A, B, D, t\}$

$$f(S,T) = \underset{s,A}{2} + \underset{s,D}{2} + (\underset{C,D}{-2} + \underset{C,A}{1} \underset{C,B}{-1} + \underset{C,t}{2})$$

$$= 4$$

capacity of cut : $c(S,T) = \underset{s,D}{3} + \underset{s,A}{2} + \underset{C,A}{1} + \underset{C,t}{3}$

$$= 9$$

- value of any flow is bounded by the capacity of any cut. [Max flow - min cut theorem]
  i.e. $f(S,T) \le c(S,T)$

Another characterization of flow value

Lemma: For any flow f and any cut (S,T) we have $|f| = f(S,T)$.

proof:

$$f(S,T) = f(S,V) - f(S,S) \quad \left[\because S \cup T = V \text{ and } S \cap T = \phi\right]$$

$$= f(S,V)$$

$$= f(s,V) + \underbrace{f(S-\{s\},V)}_{\text{does not contain } t}$$

$$= |f| + \sum_{u \in S-\{s\}} f(u,V)$$

$$= |f| + \sum 0$$

$$= |f|.$$

Residual network: $G(V,E)$

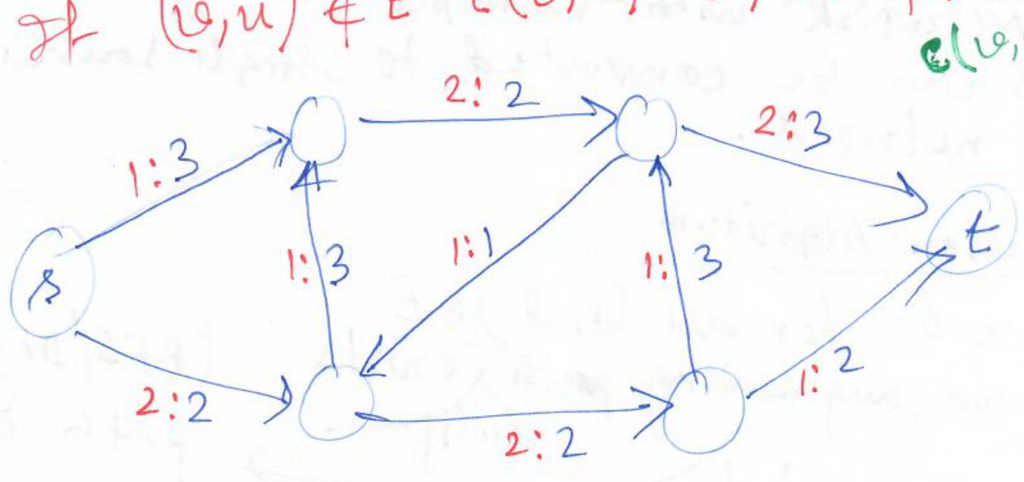
$G_f(V,E_f)$: strictly +ve residual ~~capacity~~ capacities

$$c_f(u,v) = c(u,v) - f(u,v) \;\underline{\underline{> 0}}.$$

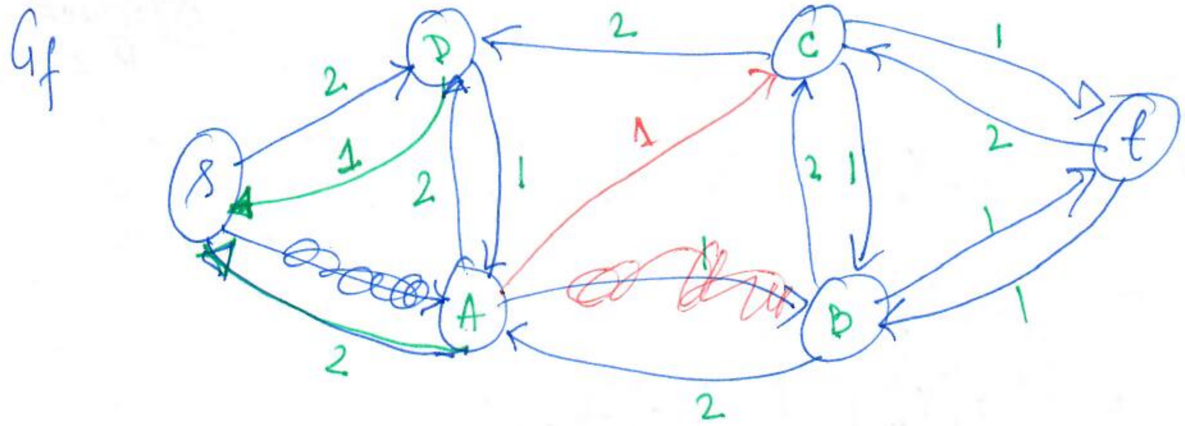
edges in E_f admit more flow

If $(v,u) \notin E$ $c(v,u) = 0$, but ~~$f(v,u) = -f(u,v)$~~

$c(v,u) = -f(v,u)$



$|f| = 3.$



Augmenting path in G_f:

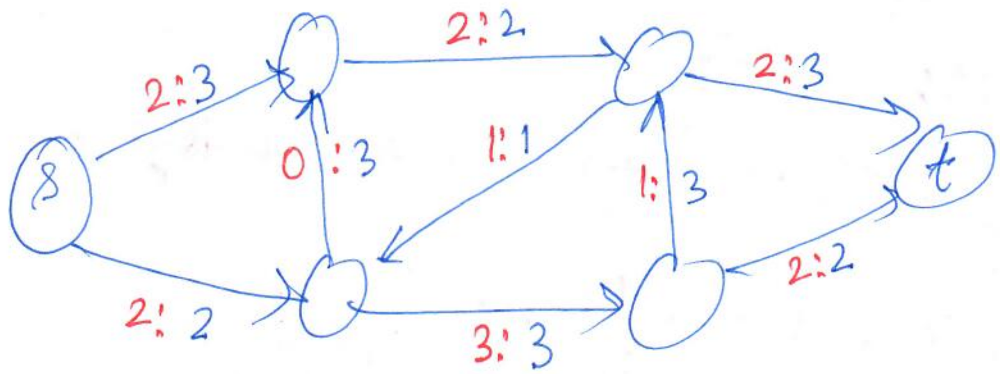
A path from s to t in G_f is call an augmenting path.

s — D — A — B — t (BFS/DFS).

C

Increment by 1 unit

Go to original network.



Note: network with multiple sources and multiple sinks can be converted to single source and single sink network.

Ford-Fulkerson Algorithm

1) $f(u,v) \leftarrow 0$ for all $(u,v) \in E$
2) while an augmenting path exists in G_f (BFS/DFS)
3) do augment f by $c_f(P) = \min_{(u,v)\in P} c_f(u,v)$

path from s to t

G: 1) $s \xrightarrow{1:2} a \xrightarrow{1:4} t$

2) $s \xrightarrow{0:2} a \xrightarrow{0:4} t$

G_f:

$$\begin{tikzcd} s \arrow[r, bend left, "1"] & a \arrow[l, bend left, "1"] \arrow[r, bend left, "3"] & t \arrow[l, bend left, "1"] \end{tikzcd}$$

Stand for a to s capacity 1
i.e. s to a we can reduce by 1

P_f : s — a — t

$c_f(p) = \min\{1, 3\} = 1$.

G: $s \xrightarrow{2:2} a \xrightarrow{2:4} t$

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Max-flow min cut theorem

The following are equivalent

1. $|f| = c(S,T)$ for some cut (S,T)
2. f is a maximum flow
3. f admits no augmenting path

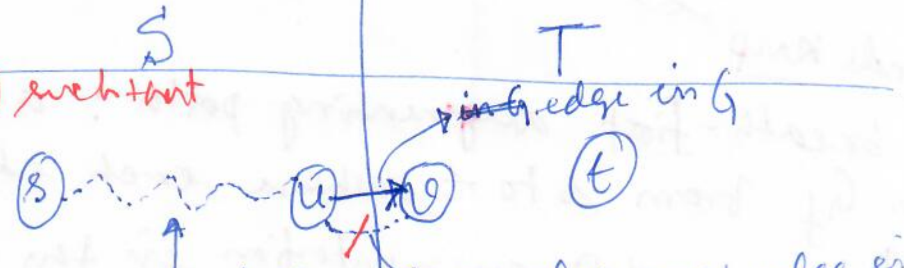
proof: $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

$1 \Rightarrow 2$ Since $|f| \le c(S,T)$ for any cut (S,T), (edge constraint)
the assumption that $|f| = c(S,T)$ implies
f is a maximum flow (because f can't be increased)

$2 \Rightarrow 3$ If there ~~were~~ is an augmenting path, then
flow value could be increased, contradicting
the maximality of $|f|$.

$3 \Rightarrow 1$ Suppose f admits no augmenting path.

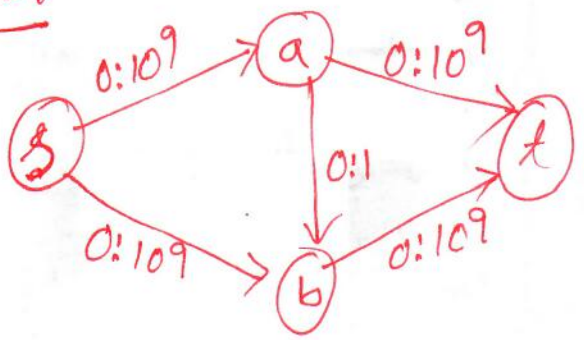
Define $S = \{u \in V :$ there exists a path in G_f from s to $u\}$
$T = V \setminus S$ $\quad [\because s \in S,\ t \in T]$
$\therefore (S,T)$ is a cut

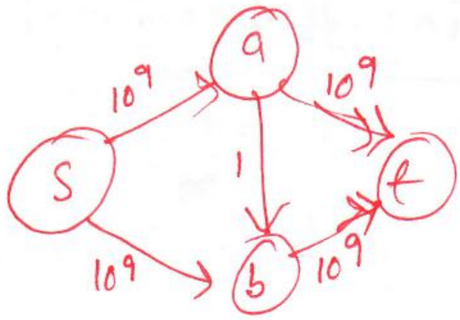


$\therefore$ Every edge (u,v) such that $u \in S$ and $v \in T$ is saturated. Summing over all $u \in S$, $v \in T$ yields $f(S,T) = c(S,T)$. Thus, the theorem.

no ~~edge~~ such edge in G_f according to the definition of u and v i.e,
$c_f(u,v) = 0 \Rightarrow f(u,v) = c(u,v)$

Run time:

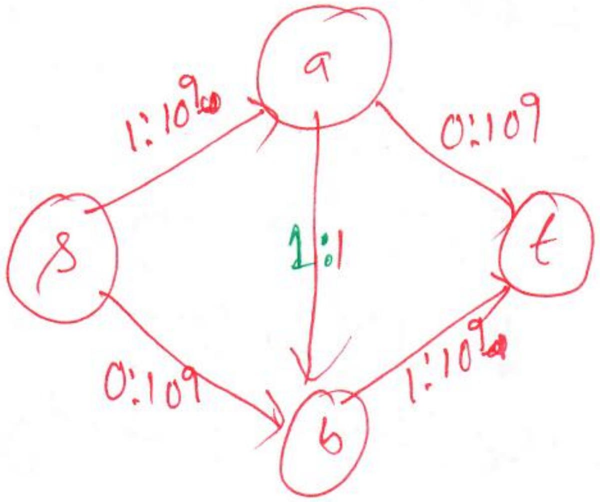




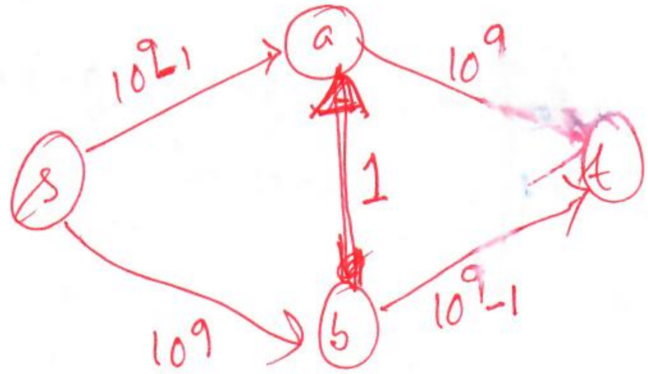
G_f

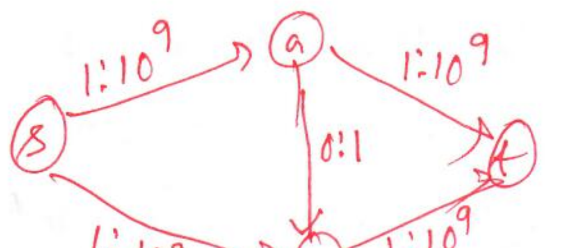
Augmenting path

$s \to a \to b \to t$



G_f





$2\times10^9 \to$ #iterations

Soln: Edmonds Karp

breath-first augmenting path: a shortest path in G_f from s to t where each edge has weight 1.

$\Rightarrow O(|V||E|)$ augmentation in the worst case

Overall complexity $O(|V||E|^2)$ time.

2011: King, Rao, Tarjan $O(VE \log_{E/(V\log V)} V)$

Orlin $O(VE)$

fast matrix multiply