MA 322: Scientific Computing

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Chapter 3: Interpolation

A quick review of interpolation: Motivation

- Independent variable x takes values $x_0, x_1, \ldots, x_n$ such that $\Delta x_i := x_{i+1} - x_i > 0, \; i = 1, 2, \ldots, n-1$.
- The values of the dependent variables y corresponding to x_i is denoted by $y_i = f(x_i)$ and are called entries.
- We ask the question: Can we approximate $f(x)$ for some x between x_0 and x_n? In particular, Can we approximate $f(x)$ $\forall x \in [x_0, x_n]$?
- We can approximate $f(x)$ by a polynomial of degree $\leq n$, $p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ such that $f(x_j) = p_n(x_j), \; j = 0, 1, \ldots, n$, e.g., Newton divided-difference polynomial interpolation, Lagrange's polynomial interpolation.
- Error in the approximation: $|f(x) - p_n(x)|$.

A quick review of interpolation: Vandermonde matrix

- $f(x_j) = p_n(x_j),\ j = 0, 1, \ldots, n$ gives

$$\begin{aligned}
a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n &= f(x_0), \\
a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n &= f(x_0), \\
&\vdots \\
a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n &= f(x_n).
\end{aligned}$$

A quick review of interpolation: Lagrange interpolation

- Define,

$$\phi_0(x) = (x - x_1)(x - x_2)\cdots(x - x_n), \quad \ell_0(x) = \frac{\phi_0(x)}{\phi_0(x_0)}$$

$$\phi_1(x) = (x - x_0)(x - x_2)\cdots(x - x_n), \quad \ell_1(x) = \frac{\phi_1(x)}{\phi_1(x_1)}$$

$$\phi_j(x) = (x - x_0)(x - x_1)\cdots(x - x_{j-1})(x - x_{+-1})\cdots(x - x_n), \quad \ell_j(x) = \frac{\phi_j(x)}{\phi_j(x_j)}$$

$$\phi_n(x) = (x - x_1)(x - x_2)\cdots(x - x_n), \quad \ell_n(x) = \frac{\phi_n(x)}{\phi_n(x_n)}$$

- $p_n(x) = \sum_{j=0}^{n} \ell_j(x) f(x_j)$ satisfies $f(x_j) = p_n(x_j),\ j = 0, 1, \ldots, n.$

A quick review of interpolation: Newton divided-difference interpolation

- The following degree n polynomial,

$$p_n(x) = f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \cdots \\ +(x - x_0)(x - x_1)\ldots(x - x_{n-1})f[x_0, x_1, \ldots, x_n],$$

satisfies $f(x_j) = p_n(x_j),\ j = 0, 1, \ldots, n.$

- Here,

$$f[x_0, x_1, \ldots, x_j] = \frac{f[x_0, x_1, \ldots, x_{j-1}] - f[x_1, x_2, \ldots, x_j]}{x_j - x_0}, \quad j = 2, 3, \ldots, n,$$

and

$$f[x_1, x_0] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

A quick review of interpolation: An example

Example (Direct computation)

For the data set $(-2,-27),(0,-1),(1,0)$, we have the following system of equation to compute a_0, a_1, a_2:

$$\begin{bmatrix} 1 & -2 & 4 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -27 \\ -1 \\ 0 \end{bmatrix}$$

yielding $[a_0, a_1, a_2] = [-1, 5, -4]$. Thus,

$$p_2(x) = -1 + 5x - 4x^2.$$

A quick review of interpolation: An example

Example (Lagrange basis)

For the data set $(-2,-27),(0,-1),(1,0)$, we have

$$\begin{aligned} p_2(x) &= -27\frac{(x-0)(x-1)}{(-2-0)(-2-1)} - 1\frac{(x+2)(x-1)}{(0+2)(0-1)} + 0\frac{(x+2)(x-0)}{(1+2)(1-0)} \\ &= -\frac{9}{2}x(x-1) + \frac{1}{2}(x+2)(x-1) = -1+5x-4x^2. \end{aligned}$$

Example (Newton divided-difference)

For the data set $(-2,-27),(0,-1),(1,0)$, we have

$$\begin{aligned} p_2(x) &= -27+13(x+2)-4x(x+2) \\ &= -1+5x-4x^2. \end{aligned}$$

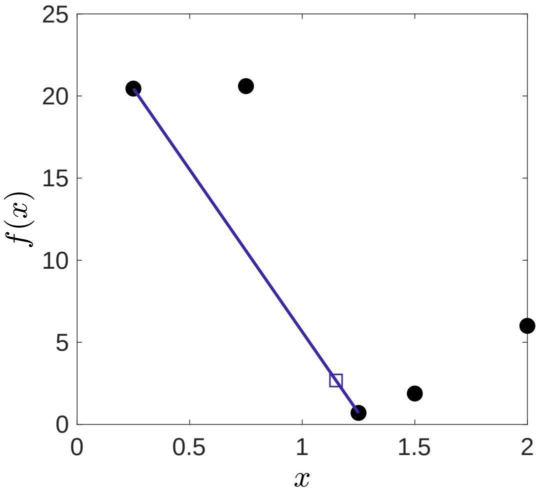
Polynomial interpolation: Lagrange's formula

Theorem

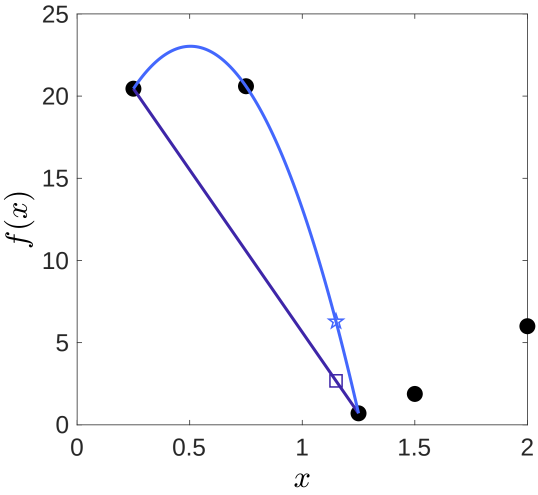
Let $x_0, x_1, \ldots, x_n$ be distinct real numbers, and let f be a given real valued function with $n+1$ continuous derivatives on the interval $I_t = \mathcal{I}\{t; x_0, \ldots, x_n\}$ (i.e., $f \in C^{(n+1)}(I_t)$), with t some given real number. Then $\exists\, \xi \in I_t$ with

$$f(t) - \sum_{i=0}^{n} f(x_i) l_i(t) = \frac{(t-x_0)(t-x_1)\cdots(t-x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

Polynomial interpolation: Graphical representation



Polynomial interpolation: Graphical representation



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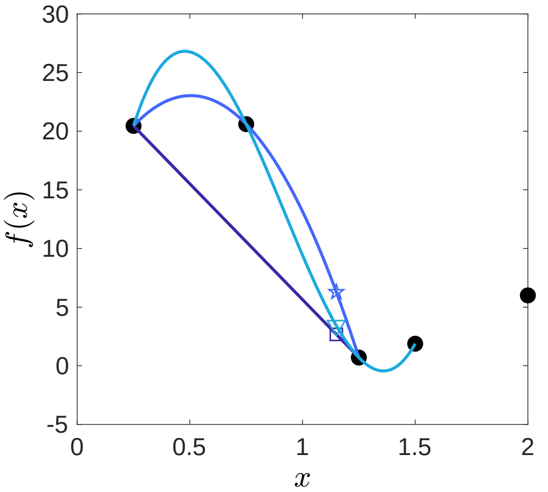
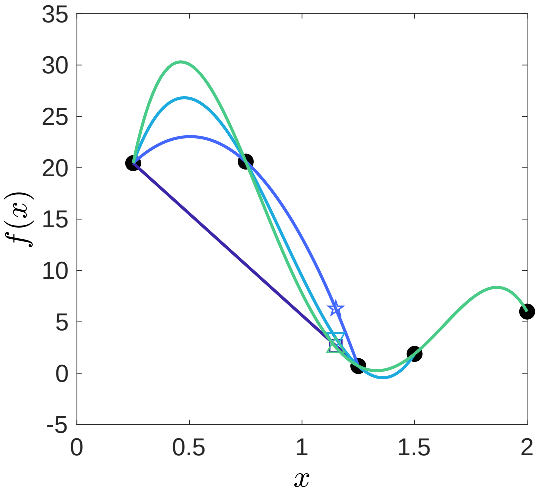


Polynomial interpolation: Graphical representation



Polynomial interpolation: Example

Example (Graphical representation)

For the given data set (0.25, 20.45), (0.75, 20.60), (1.25, 0.70), (1.5, 1.88) and (2.0, 0.60), approximate $f(1.15)$.