MA 101 (Mathematics - I)

Supplementary Examination : Part - II

Maximum Marks : 24

Date: March 15, 2021 **Time:** 6:35 pm - 7:35 pm

Instructions:

- The answers of the questions are to be filled in the response form. You get exactly one hour time (from 6:35 pm to 7:35 pm) for doing this.
- You should submit the response form at 7:35 pm (or before). Although you get extra 3 minutes for submission only (the portal will close at 7:38 pm), it is advised not to take any risk of submitting after 7:35 pm. No request will be entertained if you fail to submit your responses through the portal due to any reason.
- The response form permits only one submission. It does not allow revision.

Type of Questions and Marking scheme:

- The first question is writing your Roll number. It is compulsory. It has no marks.
- Q.2 to Q.7 are of single correct option type questions, where exactly one option is correct. Each of these questions carries 2 marks for correct answer, -1 mark for incorrect answer, and 0 mark for no answer. No answer will be considered if you do not mark any option or if you mark option (E) given in the response form.
- Q.8 to Q.13 are of multiple correct option type questions, where one or more of the options is (are) correct. In each of these questions, you get 2 marks if you choose all the correct options and choose no wrong option. In all other cases, you get 0 mark.

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1. Write your Roll number.

2. Let $f(x,y) = 4x^5y^2 - 3x^4y^3 + 5\sin(xy^2) + 8$ and $g(x,y) = (1+x^2+y^2)\sin(1+x^2+y^2)$ for all $(x,y) \in \mathbb{R}^2$. Then
(A) both $f : \mathbb{R}^2 \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are onto
(B) $f : \mathbb{R}^2 \to \mathbb{R}$ is onto but $g : \mathbb{R}^2 \to \mathbb{R}$ is not onto
(C) $g : \mathbb{R}^2 \to \mathbb{R}$ is onto but $f : \mathbb{R}^2 \to \mathbb{R}$ is not onto
(D) neither $f : \mathbb{R}^2 \to \mathbb{R}$ nor $g : \mathbb{R}^2 \to \mathbb{R}$ is onto

Answer: (A)

Explanation: Let $\varphi(x) = f(x,1)$ and $\psi(x) = g(x,0)$ for all $x \in \mathbb{R}$. Using the IVP of continuous functions of one real variable, it is easy to see that $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are onto. Consequently f and g are onto.

3. Let S be a nonempty open set in $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$. Consider the following two statements **P** and **Q**.
P : If $D_{\mathbf{u}}f(\mathbf{x}) = 0$ for all $\mathbf{x} \in S$ and for all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then f must be a constant

function.

Q : If $f_x(x, y) = 0$ for all $(x, y) \in S$, then it is necessary that $f(x_1, y) = f(x_2, y)$ for all $(x_1, y), (x_2, y) \in S$.

Then

(A) both **P** and **Q** are true (B) **P** is true but **Q** is false

(C) **Q** is true but **P** is false (D) both **P** and **Q** are false

Answer: (D)

Explanation: Consider $S = A \cup B$, where $A = \{(x, y) \in \mathbb{R}^2 : x < 0\}$ and $B = \{(x, y) \in \mathbb{R}^2 : x > 0\}$. Define $f(x, y) = 0$ if $(x, y) \in A$ and $f(x, y) = 1$ if $(x, y) \in B$.

4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function. Let C be a closed, simple, piece-wise smooth and counter clockwise oriented curve in $\mathbb{R}^2$ which encloses domain D in $\mathbb{R}^2$. Let $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $P = (1, 1)$. Then $\iint\limits_D [(Tf') \cdot P](x, y)dxdy$ is equal to

(A) $\oint\limits_C (f, f) \cdot dR$ (B) $-\oint\limits_C (f, f) \cdot dR$

(C) $\oint\limits_C (-f, f) \cdot dR$ (D) None of $(A), (B)$ and (C) is true

Answer: (A)

Explanation: By Green's Theorem,

$$\iint\limits_D [(Tf') \cdot P](x, y)dxdy = \iint\limits_D [(f_x(x, y) - f_y(x, y)]dxdy = \oint\limits_C (f, f) \cdot dR.$$

5. Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is a never vanishing continuously differentiable function. Let $\|\nabla f\|^2 = 2f$ and $\text{div}(f\nabla f) = 5f$. Let $\mathbf{n}$ be the unit outward normal to the sphere $S = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ and $D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1\}$. If $\frac{\partial f}{\partial \mathbf{n}}$ is the directional derivative of f along $\mathbf{n}$, then

(A) $\iint\limits_S \frac{\partial f}{\partial \mathbf{n}} d\sigma = 4\pi$ (B) $\iint\limits_S \frac{\partial f}{\partial \mathbf{n}} d\sigma = 7\pi$

(C) $\iint\limits_S \frac{\partial f}{\partial \mathbf{n}} d\sigma = -7\pi$ (D) $\iint\limits_S \frac{\partial f}{\partial \mathbf{n}} d\sigma = 3 \iiint\limits_D f dV$

Answer: (A)

Explanation: Note that $\text{div}(f\nabla f) = f\nabla \cdot (\nabla f) + \|\nabla f\|^2$. Hence Since, f is never vanishing function, it follows that $\nabla \cdot (\nabla f) = 3$. By Stoke's Theorem,

$$\iint\limits_S \frac{\partial f}{\partial \mathbf{n}} d\sigma = \iint\limits_S \nabla f \cdot \mathbf{n} d\sigma = \iiint\limits_D \nabla \cdot (\nabla f) dV = 4\pi.$$

6. For $(x, y, z) \in \mathbb{R}^2$, let $\hat{n}(x, y, z) = \left(\frac{1}{x}, \frac{1}{1+y}, \frac{1}{1+|z|^2}\right)$. If $\hat{n}$ is the outward normal to the surface S in $\mathbb{R}^3$, then which of the following statements is (are) true ?

(A) S is orientable if origin is not lying on the surface S

(B) S is orientable if origin and $(0, -1, 0)$ are not lying on the surface S

(C) S is orientable if S does not intersect the curve $\{(0, -1, t) : \ t \in \mathbb{R}\}$

(D) The set in $\mathbb{R}^3$ on which possibly S could not be orientable is set of content zero in $\mathbb{R}^3$

Answer: (D)
Explanation: The set of continuity of $\hat{n}$ is $D = \{(r,s,t) \in \mathbb{R}^3 : , r \neq 0, s \neq -1\}$.

7. Let D be the domain in $\mathbb{R}^2$ bounded by the curves $y = 1-x^2$ and $y = 2(1-x^2)$. Let $\ln r$ denote natural logarithm of r. Then
(A) $\iint\limits_D \frac{x^2}{y} dydx$ exists and equals to $\frac{2}{3}\ln 2$. (B) $\iint\limits_D \frac{x^2}{y} dydx$ exists and equals to $\frac{4}{3}\ln 2$.
(C) $\iint\limits_D \frac{x^2}{y} dydx$ exists and equals to $\frac{1}{3}\ln 2$. (D) $\iint\limits_D \frac{x^2}{y} dydx$ does not exist

Answer: (A)
Explanation: Introduce change of variables $x = v$ and $y = u(1-v^2)$. Then the D will be transferred to $[1,2] \times [-1,1]$. Here $|J(u,v)| = 1 - v^2$.

$$\iint\limits_D \frac{x^2}{y} dydx = \int_{-1}^{1}\int_{1}^{2} \frac{v^2}{u} dudv = \frac{2}{3}\ln 2.$$

8. For $m,\ n,\ k,\ \ell \in \mathbb{N}$ with $k,\ \ell$ even, let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be defined by $f(x,y) = \frac{x^m y^n}{x^k + y^\ell}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Then $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists (in $\mathbb{R}$) if
(A) $m = 3,\ n = 5,\ k = \ell = 8$ (B) $m = 5,\ n = 2,\ k = 8,\ \ell = 6$
(C) $m = 5,\ n = 1,\ k = 8,\ \ell = 2$ (D) $m = 2,\ n = 4,\ k = 8,\ \ell = 6$

Answer: (C)

Explanation: In (C), $|f(x,y)| = \left|\frac{x^4 y}{x^8+y^2}\right||x| \leq |x| \leq \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and hence $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$. In (A), (B), (D), first take the sequence $\big((\frac{1}{n}, 0)\big)$. Then take the sequence $\big((\frac{1}{n^8}, \frac{1}{n^8})\big)$ in (A), the sequence $\big((\frac{1}{n^6}, \frac{1}{n^8})\big)$ in (B) and the sequence $\big((\frac{1}{n^6}, \frac{1}{n^8})\big)$ in (D).

9. Let $S = \{(x,y,z) \in \mathbb{R}^3 : x+2y+3z = 6\}$ and let $f(x,y,z) = 3(x^2+y^2+z^2)$ for all $(x,y,z) \in S$. Then
(A) there exists $\mathbf{x}_0 \in S$ such that $f(\mathbf{x}_0) = 4$
(B) there exists $\mathbf{x}_0 \in S$ such that $f(\mathbf{x}_0) = 7$
(C) there exists $\mathbf{x}_0 \in S$ such that $f(\mathbf{x}_0) = 9$
(D) there exists $\mathbf{x}_0 \in S$ such that $f(\mathbf{x}_0) = 19$

Answer: (C), (D)

Explanation: Using Lagrange multiplier method, we can see that the minimum value of $f(x,y,z)$ subject to the constraint $x+2y+3z = 6$ is $\frac{54}{7}$. Also, f is not bounded above on S. Since $f : S \to \mathbb{R}$ is continuous, it follows that the range of f is $\big[\frac{54}{7}, \infty\big)$.

10. For $\alpha \in \mathbb{R}$, let $f : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x,y,z) = \begin{cases} \frac{xyz}{(x^2+y^2+z^2)^\alpha} & \text{if } (x,y,z) \neq (0,0,0), \\ 0 & \text{if } (x,y,z) = (0,0,0). \end{cases}$
Then
(A) f is differentiable at $(0,0,0)$ if $0 < \alpha \leq \frac{1}{2}$
(B) f is differentiable at $(0,0,0)$ if $\frac{1}{2} < \alpha < 1$

(C) f is differentiable at $(0,0,0)$ if $\alpha \geq 1$

(D) f is differentiable at $(0,0,0)$ only if $0 < \alpha < 1$

Answer: (A), (B)

Explanation: If $\alpha \leq 0$, then clearly f is differentiable at $(0,0,0)$. Again, we have $f_x(0,0,0) = f_y(0,0,0) = f_z(0,0,0) = 0$. If $0 < \alpha < 1$, then $\lim\limits_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{h})-f(\mathbf{0})-0|}{\|\mathbf{h}\|} = 0$ and if $\alpha \geq 1$, then $\lim\limits_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{h})-f(\mathbf{0})-0|}{\|\mathbf{h}\|} \neq 0$.

11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2 e^{-x^4-y^2}$ for all $(x,y) \in \mathbb{R}^2$. Then

(A) f has at least six critical points

(B) f has at least one saddle point

(C) f has at least one local maximum

(D) f attains absolute maximum on $\mathbb{R}^2$

Answer: (A), (C), (D)

Explanation: Solving the system of equations $f_x(x,y) = 2x(1-2x^4)e^{-x^4-y^2} = 0$ and $f_y(x,y) = -2x^2ye^{-x^4-y^2} = 0$, we see that $\big(\pm\frac{1}{\sqrt[4]{2}}, 0\big)$ and $(0,y)$ for $y \in \mathbb{R}$, are all the critical points of f. Since $f(x,y) \geq 0 = f(0,y)$ for all $x, y \in \mathbb{R}$, f has a local minimum at each of the points $(0,y)$ for $y \in \mathbb{R}$. Also, f has local maximum at each of the points $\big(\pm\frac{1}{\sqrt[4]{2}}, 0\big)$ and since $\lim\limits_{\|(x,y)\|\to\infty} f(x,y) = 0$, it follows that f attains absolute maximum at each of the points $\big(\pm\frac{1}{\sqrt[4]{2}}, 0\big)$.

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function. Let $\alpha = \int\limits_0^x \int\limits_0^y \int\limits_0^z f(t,y)dtdzdy$ and $\beta = \int_0^x \int_0^y (y-t)f(t,y)dtdy$. Then which of the following statements is (are) true ?

(A) $\alpha = \beta$ if f is continuous on $\mathbb{R}^2$

(B) $\alpha = \beta$ if f is continuous on every bounded set $D \subset \mathbb{R}^2$ except possibly on a set $E \subset D$ of content zero

(C) $\alpha = \beta$ if f is Riemann integrable on every bounded set D in $\mathbb{R}^2$

(D) $\alpha \neq \beta$ even if f is Riemann integrable on every bounded set D in $\mathbb{R}^2$

Answer: (A), (B), (D)

Explanation: By Fubini's Theorem

$$\beta = \int_0^x \int_0^y (y-t)f(t,y)dtdy = \int_0^x \int_0^y \int_t^y f(t,y)dzdtdy = \int\limits_0^x \int\limits_0^y \int\limits_0^z f(t,y)dtdzdy = \alpha$$

Note that Fubini's theorem can be applied in case f is discontinuous except on a set of content zero in bounded set.

13. Let $f : D = [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}^c \cap [0,1] \text{ and } y \in \mathbb{Q} \cap [0,1]; \\ 1 - \frac{1}{q}, & \text{if } x = \frac{p}{q} \text{ in lowest term and } y \in \mathbb{Q} \cap [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Then which of the following statement is (are) true ?

(A) f is integrable and $\iint\limits_D f(x,y)dxdy = 1$.

(B) Repeated integral $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dx \right) dy$ exists

(C) Repeated integral $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dx \right) dy$ does not exist

(D) Repeated integral $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dy \right) dx$ does not exist

Answer: (C), (D)

Explanation: Let $P_n = Q_n \times R_n = \{\frac{i}{n} : \ i = 0, 1, \ldots, n\} \times \{\frac{j}{n} : \ j = 0, 1, \ldots, n\}$. Note that $M_{ij} = 1$ and $m_{ij} = 0$ for all i, j. Hence f is not Reimann integrable. For $y \in \mathbb{Q}$, we get $m_i = 1$ and $m_i \geq 1 - \frac{1}{n}$. This implies $\int_0^1 f(x,y)dx = 1$. When $y \in \mathbb{Q}^c$, we get $\int_0^1 f(x,y)dx = 0$. Hence $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dx \right) dy$ does not exist. Similarly $\int\limits_0^1 \left(\int\limits_0^1 f(x,y)dy \right) dx$ does not exist.

——— **END** ———