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Black-Scholes-Merton Equation

We derive the Black-Scholes-Merton partial differential equation for the price of an option on an asset modeled as a geometric Brownian motion. The idea behind this derivation is to determine the initial capital required to perfectly hedge a short position in the option.

Consider an agent, who at time $t$ has a portfolio valued at $X(t)$. This portfolio invests in a money market account paying a constant rate of interest $r$ and in a stock modeled by the geometric Brownian motion:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dW(t).$$

*Recall that: Let $\alpha$ and $\sigma > 0$ be constants, and define the geometric Brownian motion:*

$$S(t) = S(0)\exp\left\{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right\}.$$

*This is the asset-price model used in the Black-Scholes-Merton option-pricing formula.*

Suppose that at each time $t$, the investor holds $\Delta(t)$ shares of stock. The position $\Delta(t)$ can be random but must be adapted to the filtration associated with the Brownian motion $W(t), t \geq 0$. The remainder of the portfolio value, $X(t) - \Delta(t)S(t)$, is invested in the money market account.

The differential $dX(t)$ for the investor's portfolio value at each time $t$ is due to two factors, the capital gain $\Delta(t)dS(t)$ on the stock position and the interest earnings $r\left(X(t) - \Delta(t)S(t)\right)$ on the cash position. In other words:

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + r(X(t) - \Delta(t)S(t))dt, \\ &= \Delta(t)(\alpha S(t)dt + \sigma S(t)dW(t)) + r(X(t) - \Delta(t)S(t))dt, \\ &= rX(t) + \Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t). \end{aligned}$$

We shall often consider the discounted stock price, $e^{-rt}S(t)$ and the discounted portfolio value of an agent, $e^{-rt}X(t)$.

*Recall the Ito-Doeblin formula for an Ito process:*

*Let $X(t), t \geq 0$, be an Ito process and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$, $f_x(t,x)$ and $f_{xx}(t,x)$ are defined and continuous. Then:*

$$df(t, X(t)) = f_t(t, X(t))dt + f_x(t, X(t))dX(t) + \frac{1}{2}f_{xx}(t, X(t))dX(t)dX(t).$$

According to the Ito Doeblin formula with $f(t,x) = e^{-rt}x$, the differential of the discounted stock price is:

$$\begin{aligned}
d(e^{-rt}S(t)) &= df(t,S(t)), \\
&= f_t(t,S(t))dt + f_x(t,S(t))dS(t) + \frac{1}{2}f_{xx}(t,S(t))dS(t)dS(t), \\
&= -re^{-rt}S(t)dt + e^{-rt}dS(t), \\
&= (\alpha - r)e^{-rt}S(t)dt + \sigma e^{-rt}S(t)dW(t),
\end{aligned}$$

and the differential of the discounted portfolio value is:

$$\begin{aligned}
d(e^{-rt}X(t)) &= df(t,X(t)), \\
&= f_t(t,X(t))dt + f_x(t,X(t))dX(t) + \frac{1}{2}f_{xx}(t,X(t))dX(t)dX(t), \\
&= -re^{-rt}X(t)dt + e^{-rt}dX(t), \\
&= \Delta(t)(\alpha - r)e^{-rt}S(t)dt + \Delta(t)\sigma e^{-rt}S(t)dW(t), \\
&= \Delta(t)d(e^{-rt}S(t)).
\end{aligned}$$

Discounting the stock prices reduces the mean rate of return from $\alpha$ (the term multiplying $S(t)dt$) to $(\alpha - r)$ (the term multiplying $e^{-rt}S(t)dt$). The change in the discounted portfolio value is solely due to the change in the discounted stock prices.

Evolution of Option Value

Consider a European call option that pays $(S(T) - K)^+$ at time $T$. Let $c(t,x)$ denote the value of the call at time $t$, if the stock price at that time is $S(t) = x$. There is nothing random about the function $c(t,x)$. However, the value of the option is random. In fact, it is the stochastic process $c(t,S(t))$ obtained by replacing the dummy variable $x$ by the random stock prices $S(t)$ in this function. At the initial time, we do not know the future stock prices $S(t)$ and hence do not know the future option values $c(t,S(t)$. Our goal is to determine the function $c(t,x)$ so we at least have a formula for the future option values in terms of the future stock prices. We begin by computing the differential of $c(t,S(t)$. According to the Ito-Doeblin formula, it is given by:

$$\begin{aligned}
dc(t,S(t)) &= c_t(t,S(t))dt + c_x(t,S(t))dS(t) + \frac{1}{2}c_{xx}(t,S(t))dS(t)dS(t), \\
&= c_t(t,S(t))dt + c_x(t,S(t))(\alpha S(t)dt + \sigma S(t)dW(t)) + \frac{1}{2}c_{xx}(t,S(t))\sigma^2 S^2(t)dt, \\
&= \left[c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt + \sigma S(t)c_x(t,S(t))dW(t).
\end{aligned}$$

We next compute the differential of the discounted option price $e^{-rt}c(t,S(t))$. Let $f(t,x) = e^{-rt}x$. According to the Ito-Doeblin formula:

$$\begin{aligned} d(e^{-rt}c(t,S(t))) &= df(t,c(t,S(t))), \\ &= f_t(t,c(t,S(t)))dt + f_x(t,c(t,S(t)))dc(t,S(t)) + \frac{1}{2}c_{xx}(t,c(t,S(t)))dc(t,S(t))dc(t,S(t)), \\ &= -re^{-rt}c(t,S(t))dt + e^{-rt}dc(t,S(t)), \\ &= e^{-rt}\left[-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right]dt, \\ &\quad +e^{-rt}\sigma S(t)c_x(t,S(t))dW(t). \end{aligned}$$

Equating the Evolutions

A (short option) hedging portfolio starts with some initial capital $X(0)$ and invests in the stock and money market account so that the portfolio value $X(t)$ at each time $t \in [0,T]$ agrees with $c(t,S(t))$. This happens if and only if:

$$e^{-rt}X(t) = e^{-rt}c(t,S(t))$$

for all $t$. One way to ensure this equality is to make sure that:

$$d(e^{-rt}X(t)) = d(e^{-rt}c(t,S(t))) \text{ for all } t \in [0,T)$$

and $X(0) = c(0,S(0)$. Integrating from 0 to $t$ yields:

$$e^{-rt}X(t) - X(0) = e^{-rt}c(t,S(t)) - c(0,S(0)), \text{ for all } t \in [0,T).$$

If $X(0) = c(0,S(0))$, then we cancel this term and get the desired equality. We see that the above holds only if:

$$\begin{aligned} &\Delta(t)(\alpha - r)S(t)dt + \Delta(t)\sigma S(t)dW(t) \\ = &\left(-rc(t,S(t)) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t))\right)dt + \sigma S(t)c_x(t,S(t))dW(t). \end{aligned}$$

We first equate the $dW(t)$ terms, which gives:

$$\Delta(t) = c_x(t,S(t)) \text{ for all } t \in [0,T).$$

This is called *delta hedging rule*. At each time $t$ prior to expiration, the number of shares held by the hedge of the stock option position is the partial derivative with respect to the stock prices of the option value at that time. The quantity $c_x(t,S(t))$ is called the *delta* of the option. We next equate all the $dt$ terms to obtain:

$$(\alpha - r)S(t)c_x(t,S(t)) = -rc(t,S(t) + c_t(t,S(t)) + \alpha S(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)) \text{ for all } t \in [0,T).$$

Finally we have:

$$rc(t,S(t)) = c_t(t,S(t)) + rS(t)c_x(t,S(t)) + \frac{1}{2}\sigma^2 S^2(t)c_{xx}(t,S(t)) \text{ for all } t \in [0,T).$$

In conclusion, we should seek a continuous function $c(t,x)$ that is solution to the *Black-Scholes-Merton partial differential equation:*

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}^2(t,x) = rc(t,x) \text{ for all } t \in [0,T), x \geq 0$$

and that satisfies the *terminal condition*:

$$c(T,x) = (x-K)^+.$$

Solution to the Black-Scholes-Merton Equation

The solution to the Black-Scholes-Merton equation with the terminal and boundary conditions ($c(t,0) = 0$ and $\lim_{x\to\infty} \left[c(t,x) - \left(x - e^{-r(T-t)}K\right)\right] = 0$ for all $t \in [0,T]$) is:

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)), 0 \leq t < T, x > 0$$

where,

$$d_\pm(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\left[\log\left(\frac{x}{K}\right) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right],$$

and $N$ is the cumulative standard normal distribution:

$$N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}}\,dz.$$

The Greeks

The derivation of the function $c(t,x)$ with respect to various variables are called the *Greeks*,

(A) The *Delta* ($\Delta$) is:

$$c_x(t,x) = N(d_+(T-t,x)).$$

(B) The *Theta* ($\Theta$) is:

$$c_t(t,x) = -rKe^{-r(T-t)}N(d_-(T-t,x)) - \frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)).$$

(C) The *Gamma* ($\Gamma$) is:

$$c_{xx}(t,x) = N'(d_+(T-t,x))\frac{\partial}{\partial x}d_+(T-t,x) = \frac{1}{\sigma x\sqrt{T-t}}N'(d_+(T-t,x)).$$