MA 201: Partial Differential Equations
Lecture - 5

Integral surfaces through a given curve

- Suppose we have found two solutions

$$\phi(x, y, u) = c_1, \;\; \psi(x, y, u) = c_2 \tag{1}$$

of the auxiliary equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

- Any solution of the corresponding quasi-linear equation $a(x, y, u)\dfrac{\partial u}{\partial x} + b(x, y, u)\dfrac{\partial u}{\partial y} = c(x, y, u)$ is of the form

$$G(\phi, \psi) = 0. \tag{2}$$

- Now, we want to find the integral surface which passes through a given curve Γ whose parametric equations are

$$x(0) = x_0(s), \; y(0) = y_0(s), \; u(0) = u_0(s),$$

with s as the parameter.

- Recall that, for any point (x, y, u) on the integral surface, we have

$$\phi(x, y, u) = c_1, \;\; \psi(x, y, u) = c_2. \tag{3}$$

- In particular (i.e., also on the given curve Γ), we must have

$$\phi\{x_0(s), y_0(s), u_0(s)\} = c_1, \; \psi\{x_0(s), y_0(s), u_0(s)\} = c_2. \tag{4}$$

- We eliminate the single parameter s from the equations in (4) to obtain a relation of the type

$$f(c_1, c_2) = 0 = f(\phi, \psi) = f(x, y, u). \tag{5}$$

- **Example:** Find an integral surface of the quasi-linear PDE

$$x(y^2 + u)p - y(x^2 + u)q = (x^2 - y^2)u$$

which contains the straight line $x + y = 0, \; u = 1$.
- **Solution:** The auxiliary equations are

$$\frac{dx}{x(y^2 + u)} = \frac{dy}{-y(x^2 + u)} = \frac{du}{(x^2 - y^2)u}.$$

- Taking

$$\frac{y\ dx + x\ dy}{xy^3 + xyu - yx^3 - yxu} = \frac{du}{(x^2 - y^2)u}$$

will ultimately give rise to

$$\frac{d(xy)}{xy} = -\frac{du}{u}.$$

- Its solution is

$$\phi(x, y, u) = xyu = c_1. \tag{6}$$

- Similarly taking

$$\frac{x\ dx + y\ dy}{x^2y^2 + x^2u - y^2x^2 - y^2u} = \frac{du}{(x^2 - y^2)u},$$

we ultimately obtain

$$\psi(x, y, u) = x^2 + y^2 - 2u = c_2. \tag{7}$$

- The general solution can be written as

$$G = (\phi, \psi) = 0 \quad \Rightarrow G(xyu, x^2 + y^2 - 2u) = 0.$$

- For the initial curve $x + y = 0,\ u = 1$, we have the parametric equations

$$x_0(s) = s, y_0(s) = -s, u_0(s) = 1.$$

- Then we have, from (6) and (7), respectively,

$$x_0(s)y_0(s)u_0(s) = c_1 \ \ \& \ \ x_0(s)^2 + y_0(s)^2 - 2u_0(s) = c_2,$$

so that

$$-s^2 = c_1, \ \ \& \ \ 2s^2 - 2 = c_2.$$

Now eliminating s from them to get

$$2c_1 + c_2 + 2 = 0.$$

- The desired integral surface is

$$x^2 + y^2 + 2xyu - 2u + 2 = 0.$$

Example

$$\textbf{PDE}:\ (y+u)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x - y;\ \ \textbf{IC}:\ u(x,1) = 1 + x.$$

Solution.

- **Step 1.** (Write the parametric form of the initial curve)
  To solve the IVP, we parameterize the initial curve Γ as

$$x_0(s) = s,\ \ y_0(s) = 1,\ \ u_0(s) = 1 + s.$$

- **Step 2.** (Write the initial conditions)

$$x(0) = x_0(s) = s,\ \ y(0) = y_0(s) = 1,\ \ u(0) = u_0(s) = 1 + s.$$

- **Step 3.** (Solve the characteristic equations.)

$$\frac{dx}{dt} = y + u, \quad \frac{dy}{dt} = y, \quad \frac{du}{dt} = x - y$$

By taking the second equation to get

$$y(t) = c_1 e^t,$$

which upon using the condition $y(0) = 1$ gives $y(t) = e^t$.
Next adding the first and third equations,

$$u(t) + x(t) = c_2 e^t.$$

Upon using the initial conditions $x(0) = s,\ u(0) = 1 + s$, we get

$$u(t) + x(t) = (1 + 2s)e^t.$$

From Step 3, we again obtain

$$\begin{aligned} &\frac{dx}{dt} + x = (u + x) + y = (1 + 2s)e^t + e^t = (2 + 2s)e^t \\ &\Rightarrow x(t) = (1 + s)e^t - e^{-t}, \end{aligned}$$

after finding the integrating factor and using $x(0) = s$.
Therefore,

$$u(t) = se^t + e^{-t}.$$

Step 4. (If possible get the explicit or implicit form of the solution)

Observe that the Jacobian

$$J = \begin{vmatrix} 2+s & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0 \text{ on } \Gamma.$$

Therefore, transformation $(t, s) \to (x, y)$ is possible around Γ.
Any point (x, y, u) on the characteristic curve is given by

$$\begin{aligned} x &= x(t) = (1+s)e^t - e^{-t}, \\ y &= y(t) = e^t, \\ u &= u(t) = se^t + e^{-t}. \end{aligned}$$

Noting that s can be found as $s = \dfrac{xy + 1 - y^2}{y^2}$, hence $u(t)$ is obtained as

$$u = u(t) = se^t + e^{-t} = x - y + \frac{2}{y}.$$

Note that the solution is not global (it becomes singular on the x-axis), but it is well-defined near the initial curve.

Example

$$\textbf{PDE}:\ u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0;\ \ \textbf{IC}:\ u(x,0) = f(x).$$

- **Step 1.** (Write the parametric form of the initial curve)
  To solve the IVP, we parameterize the initial curve as

$$x_0(s) = s, \quad y_0(s) = 0, \quad u_0(s) = f(s).$$

- **Step 2.** (Write the Initial Conditions)

$$x(0) = x_0(s) = s,\ \ y(0) = y_0(s) = 0,\ \ u(0) = u_0(s) = f(s).$$

- **Step 3.** (Solve the Characteristic Equations)

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 0$$

  to have

$$u(t) = u(0) = f(s),\ \ y(t) = t + y(0) = t,\ \ x(t) = f(s)t + x(0).$$

- **Step 4.**(If possible get the explicit or implicit form of the solution)
  Note that (x, y, u) on the integral surface satisfies

$$u = u(t) = f(s), \;\; y = y(t) = t, \;\; x = x(t) = f(s)t + s.$$

For the transformation $(t, s) \to (x, y)$, check the transversality condition. Here, $J \neq 0$, along the entire initial curve. So, we can solve for s and t in terms of x and y

$$t = y, \;\; s = x - f(s)t = x - uy.$$

- Thus, the solution can also be given in implicit form as

$$u = f(s) = f(x - yu).$$

Example

Surfaces orthogonal to a given system of surfaces

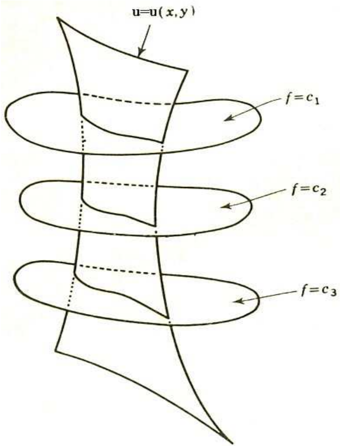
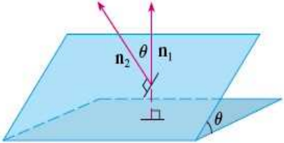


Figure : Orthogonal Surfaces

- Suppose a one-parameter family of surfaces is characterized by the equation

$$f(x, y, u) = c. \tag{8}$$

- We want to find a collection of surfaces which cut each of these given surfaces at right angles.



Definition. The angle between two surfaces at a point of intersection is the angle between their tangent planes.

- Since both the surfaces intersect orthogonally,
  at the point of intersection (x, y, u), their respective normals **are perpendicular.** Therefore, we have following PDE:

$$\nabla f \cdot \nabla F = f_x u_x + f_y u_y - f_u = 0. \tag{9}$$

- Consequently, the integral surface $u = u(x, y)$ obtained from quasi-linear pde (9) is orthogonal to the given surface $f(x, y, u) = c$.

- We know that if we want to solve a PDE of the form

$$au_x + bu_y = c,$$

we solve the equation with the help of the following auxiliary equations:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}.$$

- Therefore, it is clear that, in order to solve (9), we need to solve the following auxiliary equations:

$$\frac{dx}{f_x} = \frac{dy}{f_y} = \frac{du}{f_u}.$$

Example

Find a surface which intersects the surface $u(x+y) = (3u+1)$ orthogonally and which passes through the circle $x^2+y^2=1,\ u=1$.

- **Solution:** Here $f = u(x+y) - (3u+1)$ and hence we have

$$\frac{\partial f}{\partial x} = u,\ \frac{\partial f}{\partial y} = u, \frac{\partial f}{\partial u} = x+y-3.$$

- The integral curves are given by

$$\frac{dx}{u} = \frac{dy}{u} = \frac{du}{x+y-3}$$

$$\Rightarrow \frac{dx}{u} = \frac{dy}{u} = \frac{du}{x+y-3} = \frac{dx+dy}{2u}.$$

- Taking the first two relations, we get
$$\phi = x - y = c_1 \tag{10}$$
and taking the third and fourth relation, we get
$$\psi = (x+y)^2 - 6(x+y) - 2u^2 = c_2. \tag{11}$$

- We write the given curve in parametric form as
$$\{(x_0(s), y_0(s), u_0(s)) : s \in I \;\&\; u_0(s) = 1,\; x_0(s)^2 + y_0(s)^2 = 1.\}$$

Observe that
$$2x_0(s)y_0(s) = 1 - c_1^2, \quad \text{and}$$
$$x_0(s)^2 + y_0(s)^2 + 2x_0(s)y_0(s) - 6(x_0(s) + y_0(s)) - 2u_0^2(s) = c_2$$
which together give a relation between c_1 and c_2 as (Check the calculation)
$$36(2 - c_1^2) = (c_2 + c_1^2)^2.$$

- Therefore, the desired integral surface is obtained as
$$36(2 - \phi^2) = (\phi^2 + \psi)^2.$$