The Matrix Singular Value Decomposition

Singular Value Decomposition(SVD)

The Singular Value Decomposition (SVD) of a matrix $A \in \mathbb{R}^{n \times m}$ is a decomposition of the form

$$A = U \Sigma V^T$$

where $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ are orthogonal matrices and $\Sigma := \mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_p) \in \mathbb{R}^{n \times m}$ is a diagonal matrix with

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The numbers $\sigma_1, \sigma_2, \dots, \sigma_p$ are called the singular values of A.

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Every matrix has an SVD. For example, the SVD of

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T,$$

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Clearly if $A = U\mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_p)V^T$ is the SVD of A and $\mathrm{rank}\, A = r$, then the first r singular values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ with $\sigma_k = 0$ for $k = r+1, \ldots, p$ if $r < p$.

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If $U = [u_1 \cdots u_n]$ and $V = [v_1 \cdots v_m]$, then for $i = 1, \ldots, p$,

$$Av_i = \sigma_i u_i \text{ and } u_i^* A = \sigma_i v_i^*$$

Hence u_i and v_i are respectively left and right singular vectors of A corresponding to σ_i.

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$$\begin{aligned} R(A) &= \operatorname{span}\{u_1, \ldots, u_r\}, & N(A) &= \operatorname{span}\{v_{r+1}, \ldots, v_m\} \\ R(A^*) &= \operatorname{span}\{v_1, \ldots, v_r\} & N(A^*) &= \operatorname{span}\{u_{r+1}, \ldots, u_n\}. \end{aligned}$$

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(e) $\|A\|_F = \sqrt{\Sigma_{k=1}^{r} \sigma_k^2}$.

(Here $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$.)

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(a) If A is square and nonsingular, then $A^{-1} = (VF)(F\Sigma^{-1}F)(UF)^*$ is an SVD of A^{-1} and where F is the $n \times n$ 'flip' matrix and $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$.

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(b) If $p = \min\{m, n\}$, then assuming $\kappa_2(A) = \frac{\operatorname{maxmag} A^T}{\operatorname{minmag} A^T}$ if $n < m$,

$$\kappa_2(A) = \begin{cases} \frac{\sigma_1}{\sigma_p} & \text{if } \operatorname{rank} A = p, \\ \infty & \text{otherwise} \end{cases}$$

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(d) If $n = m$ and A is a singular matrix, then for any $\epsilon > 0$, there exists a nonsingular matrix $B \in \mathbb{F}^{n\times n}$ such that $\|A - B\|_2 < \epsilon$.