MA 322: Scientific Computing

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CHAPTER 2: ROOT FINDINGS

Secant Method: Convergence Analysis

Theorem

Assume $f(x) \in C^2$, $\forall x \in N_\delta(\alpha)$, and assume $f(\alpha) = 0$, $f'(\alpha) \neq 0$. Then if the initial guesses x_0 and x_1 are chosen sufficiently close to α, the iterates x_n

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}, \qquad n \geq 1$$

will converge to α. The order of convergence will be $p = (1+\sqrt{5})/2$.

Fixed-point method

Theorem

Assume that $g(x) \in C([a, b])$, *that* $g([a, b]) \subset [a, b]$ *(We say,* g *sends* $[a, b]$ *onto* $[a, b]$*). Then* $x = g(x)$ *has at least one solution in* $[a, b]$.

Proof.

Apply intermediate value theorem on $f(x) = g(x) - x$. Note that $f \in C([a, b])$. □

Fixed-point method

Theorem (Contraction Mapping Theorem)

Assume that $g(x) \in C([a, b])$, *that* $g([a, b]) \subset [a, b]$. *Furthermore, assume there is a constant* $0 < \lambda < 1$, *with*

$$|g(x) - g(y)|\lambda|x - y|, \qquad \forall x, y \in [a, b].$$

Then $x = g(x)$ *has a solution* $\alpha \in [a, b]$. *Also, the iterates*

$$x_n = g(x_{n-1}) \qquad n \geq 1$$

will converge to α *for any choice of* $x_0 \in [a, b]$, *and*

$$|\alpha - x_n| \leq \frac{\lambda^n}{1 - \lambda}|x_1 - x_0|.$$

Fixed-point method

Theorem

Assume that $g(x) \in C'([a, b])$, *that* $g([a, b]) \subset [a, b]$, *and that*

$$\lambda := \max_{a \leq x \leq b} |g'(x)| < 1.$$

Then

1. $x = g(x)$ *has a unique solution* α *in* $[a, b]$.
2. *For any choice of* $x_0 \in [a, b]$, *with* $x_{n+1} = g(x_n)$, $n \geq 0$,
$$\lim_{n \to \infty} x_n = \alpha.$$
3. $$|\alpha - x_n| \leq \lambda^n |\alpha - x_0| \leq \frac{\lambda^n}{1 - \lambda} |x_1 - x_0| \quad \textit{and} \quad \lim_{n \to \infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

Fixed-point method

Theorem

Assume α is a root of $x = g(x)$, and suppose that $g(x)$ is continuously differentiable in some neighbouring interval of α with $|g'(\alpha)| < 1$. Then the results of the previous theorem are still true, provided x_0 is chosen sufficiently close to α.

Higher order one-point method

Theorem

Assume α is a root of $x = g(x)$, and that $g(x)$ is p times continuously differentiable for all x near α, for some $p \geq 2$. Furthermore, assume

$$g'(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0.$$

Then if the initial guess x_0 is chosen sufficiently close to α, the iteration

$$x_{n+1} = g(x_n) \qquad n \neq 0$$

will have order of convergence p, and

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} = (-1)^{p-1} \frac{g^{(p)}(\alpha)}{p!}.$$