MA 322: Scientific Computing

Department of Mathematics
Indian Institute of Technology Guwahati

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CHAPTER 2: ROOT FINDINGS

Newton's Method: Convergence Analysis

Theorem

Assume $f(x) \in C^2,\ \forall x \in N_\delta(\alpha)$*, and assume* $f(\alpha) = 0,\ f'(\alpha) \neq 0$*. Then if* x_0 *is chosen sufficiently close to* α*, the iterates* x_n

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0$$

will converge to α*. Moreover,*

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^2} = -\frac{f''(\alpha)}{2f'(\alpha)}$$

providing that the iterates have an order of convergence $p = 2$*.*

Newton-Fourier Method

Assume $f(x) \in C^2([a,b])$ such that $\alpha \in (a,b)$. Further assume $f(a) < 0,\ f(b) > 0$, and that

$$f'(x) > 0 \ \ f''(x) > 0 \text{ for } a \leq x \leq b.$$

Then $f(x)$ is strictly increasing on $[a,b]$., and there is a unique root α in $[a,b]$. Also, $f(x) < 0$ for $a \leq x < \alpha$, and $f(x) > 0$ for $\alpha < x \leq b$. Let $x_0 = b$ and define the Newton iterates x_n as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n \geq 0.$$

Next define a new sequence of iterates by

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}, \qquad n \geq 0$$

with $z_0 = a$. With the use of $\{z_n\}$, we obtain excellent upper and lower bounds for α. This method is called the *Newton-Fourier method*.

Newton's Method: Convergence Analysis

Theorem

Assume $f(x) \in C^2([a,b])$, $f(a) < 0$, $f(b) > 0$, and $f'(x) > 0$ $f''(x) > 0$ for $a \leq x \leq b$. Then the iterates x_n are strictly decreasing to α, and the iterates

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(x_n)}, \qquad n \geq 0$$

with $z_0 = a$, are strictly increasing to α. Moreover,

$$\lim_{n\to\infty} \frac{x_{n+1} - z_{n+1}}{(x_n - z_n)^2} = \frac{f''(\alpha)}{2f'(\alpha)}$$

showing the distance between x_n and z_n decreases quadratically with n.

Newton's Method: Algorithm

1. Set df as the derivative function $f'(x)$, itmax as the maximum number of iterations to be computed, and ier as an error flag to the user.
2. itnum $:= 1$
3. denom $:= \mathrm{df}(x_0)$
4. If denom $=0$, then ier$:= 2$ and exit
5. $x_1 := x_0 - f(x_0)/\mathrm{denom}$
6. If $|x_1 - x_0| \leq \epsilon$, then set ier $:= 0$, root $:= x_1$, and exit.
7. If itnum $=$ itmax, set ier $:= 1$ and exit
8. Otherwise, itnum $:=$ itnum $+ 1$, $x_0 := x_1$ and go to step 3.