MA 322: Scientific Computing

Department of Mathematics
Indian Institute of Technology Guwahati

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Chapter 5: Numerical Differentiations and Initial Value Problems for ODEs

Initial Value Problems (IVPs)

Definition (IVP)

Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ is continuous. An initial value problem (IVP) is defined as

$$y' = f(x, y) \qquad y(x_0) = Y_0,$$

where (x_0, Y_0) is a point in D.

Definition (Solution of an IVP)

We say that a function $Y(x)$ is a solution on $[a, b]$ of the IVP if $\forall x \in [a, b]$,

1. $(x, Y(x)) \in D$.
2. $Y(x_0) = Y_0$.
3. $Y'(x)$ exists and $Y'(x) = f(x, Y(x))$.

Initial Value Problems (IVPs)

Example

Consider a general first-order linear DE

$$y' = a_0(x)y + g(x) \quad a \le x \le b$$

in which the coefficients $a_0(x)$ and $g(x)$ are assumed to be continuous on $[a, b]$. The domain D for this problem is

$$D = \{(x, y) : a \le x \le b, -\infty < y < \infty\}.$$

The exact solution of this equation can be easily obtained (See MA102 for a revision). A particular case of the above DE is when $a_0(x) = \lambda$. Then the solution is

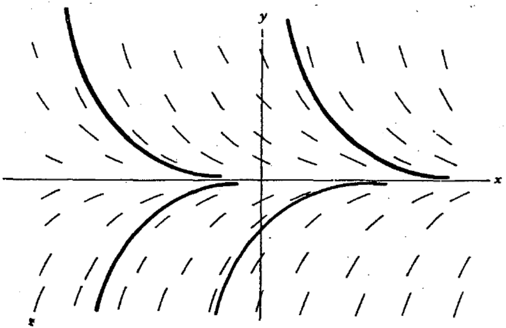
$$Y(x) = Y_0 e^{\lambda x} + \int_0^x e^{\lambda(x-t)} g(t) \mathrm{d}t \qquad 0 \le x < \infty.$$

Direction fields

If $Y(x)$ is a solution that passes through (x_0, Y_0), then the slope of $Y(x)$ at (x_0, Y_0) is $Y'(x_0) = f(x_0, Y_0)$. Within the domain D of $f(x, y)$, pick a representative set of points (x, y) and then draw a short line segment with slope $f(x, y)$ through each (x, y).

Example

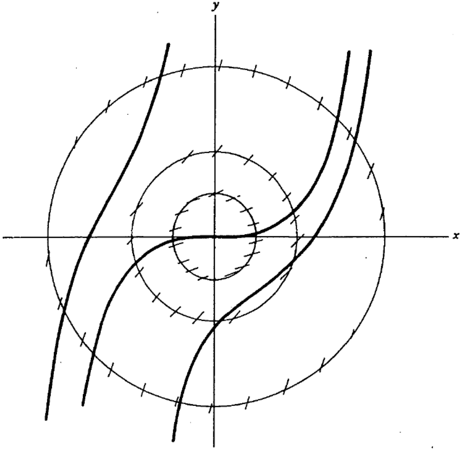
Consider the equation $y' = -y$. The direction field is shown in the below figure.



Direction fields

Example

Consider the equation $y' = x^2 + y^2$. The direction field is shown in the below figure.



Existence and Uniqueness

Theorem (Existence and uniqueness)

Let $f(x, y)$ be a continuous function of x and y, for all (x, y) in D, and let (x_0, Y_0) be an interior point of D. Assume $f(x, y)$ satisfies the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2| \qquad \forall (x, y_1), (x, y_2) \in D$$

for some $K \geq 0$. Then for a suitably chosen interval $I = [x_0 - \alpha, x_0 + \alpha]$, there is a unique solution $Y(x)$ on I of the IVP

$$y' = f(x, y) \qquad y(x_0) = Y_0.$$

Existence and Uniqueness

Example

Consider $y' = 1 + \sin(xy)$ with

$$D = \{(x, y) : 0 \leq x \leq 1,\ -\infty < y < \infty\}.$$

To compute the Lipschitz constant K, use

$$K = \max_{(x,y)\in D} \left| \frac{\partial f(x,y)}{\partial y} \right|.$$

We can show that $K = 1$. Thus for (x_0, Y_0) with $0 < x_0 < 1$, there is a solution $Y(x)$ to the associated initial value problem on some interval $[x_0 - \alpha, x_0 + \alpha] \subset [0, 1]$.

Existence and Uniqueness

Example

Discuss the existence of unique solution of the initial value problem,

$$y' = \frac{2x}{a^2} y^2 \qquad y(0) = 1, \;\; a > 0.$$

Stability of the solution

The stability of the solution $Y(x)$ is examined when the initial value problem is changed by a small amount. We consider the perturbed problem

$$y' = f(x, y) + \delta(x)$$

$$y(x_0) = Y_0 + \epsilon$$

with the same hypothesis for $f(x, y)$ as in the previous theorem (Existence and uniqueness). Further, we assume that $\delta(x)$ is continuous for all x such that $(x, y) \in D$ for some y. The problem above can be shown to have a unique solution, denoted by $Y(x; \delta, \epsilon)$.

Stability of the solution

Theorem (Stability)

Assume the same hypotheses as in "Existence and uniqueness theorem". Then the problem

$$y' = f(x, y) + \delta(x) \qquad\qquad y(x_0) = Y_0 + \epsilon$$

will have a unique solution $Y(x; \delta, \epsilon)$ *on the interval* $[x_0 - \alpha, x_0 + \alpha]$, *some* $\alpha > 0$, *uniformly for all perturbations* ϵ *and* $\delta(x)$ *that satisfy*

$$|\epsilon| \leq \epsilon_0 \qquad\qquad \|\delta\|_\infty \leq \epsilon_0$$

for ϵ_0 *sufficiently small. In addition, if* $Y(x)$ *is the solution of the unperturbed problem, then*

$$\max_{|x - x_0| \leq \alpha} |Y(x) - Y(x; \delta, \epsilon)| \leq k[|\epsilon| + \alpha\|\delta\|_\infty]$$

with $k = 1/(1 - \alpha K)$, *where* K *is the Lipschitz constant.*