Convergence of QR Algorithm

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Examples:

1. For
$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$
, $A(\operatorname{span}\{e_1, e_3\}) \subseteq \operatorname{span}\{e_1, e_3\}$.

2. For
$$A = \begin{bmatrix} 4 & 3 & -5 \\ 0 & -3 & 3 \\ 0 & -2 & 3 \end{bmatrix}$$
, $A(\operatorname{span}\{e_1\}) \subseteq \operatorname{span}\{e_1\}$ and $A(\operatorname{span}\{e_1 + e_2 + e_3, e_1 - e_2\}) \subseteq \operatorname{span}\{e_1 + e_2 + e_3, e_1 - e_2\}$.

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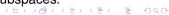
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Facts:

- 1. The trivial subspaces \mathbb{F}^n and $\{0\}$ are always invariant with respect to every $A \in \mathbb{F}^{n \times n}$.
- 2. $V \subseteq \mathbb{F}^n$ is a one dimensional subspace of \mathbb{F}^n invariant with respect to $A \in \mathbb{F}^{n \times n}$ if and only if $V = \operatorname{span}\{v\}$ for some eigenvector v of A.
- 3. Eigenvectors of $A \in \mathbb{F}^{n \times n}$ span invariant subspaces.



Theorem: Let $A \in \mathbb{F}^{n \times n}$. The first k columns of an invertible $S \in \mathbb{F}^{n \times n}$ span a subspace of \mathbb{F}^n invariant with respect to A if and only if

$$S^{-1}AS = \begin{bmatrix} A_{11} & A_{12} \\ & A_{22} \end{bmatrix}$$

where $A_{11} \in \mathbb{F}^{k \times k}$, $A_{12} \in \mathbb{F}^{k \times n - k}$ and $A_{22} \in \mathbb{F}^{n - k \times n - k}$.

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Corollary: Let $A \in \mathbb{F}^{n \times n}$ and $S = [s_1 \cdots s_n] \in \mathbb{F}^{n \times n}$ be a invertible matrix. Then the first k columns of S span subspaces of \mathbb{F}^n that are invariant with respect to A for each $k = 1, \ldots, n - 1$, if and only if $S^{-1}AS$ is an upper triangular matrix.

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Schur's Theorem: Given $A \in \mathbb{C}^{n \times n}$, there exists an orthonormal basis $\{q_1, \ldots, q_n\}$ of \mathbb{C}^n such that

$$A(\operatorname{span}\{q_1,\ldots,q_k\})\subseteq\operatorname{span}\{q_1,\ldots,q_k\}$$

for each k = 1, ..., n - 1.

