Post Midsem

$$AP = QR \qquad R = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} \times \;\; \times \;\; \times \;\cdots\; \times & \times \\ \hline 0 & 0 \end{array}\right]$$

$$AP = \left[\begin{array}{c|c} Q_1 R_1 & Q_1 \hat{R}_1 \\ \hline & \end{array}\right]$$

Q_1 is full rank, R_1 is full rank $\therefore Q_1 R_1$ is full rank

as $\begin{bmatrix} Q_1 R \\ 0 \end{bmatrix}$ has rank r.

26/9/24

$$QQ^T = I_m$$

$$[Q_1 \;\; Q_2] \begin{bmatrix} Q_1^T \\ Q_2^T \end{bmatrix} = I \qquad Q_1 Q_1^T + Q_2 Q_2^T = I$$

$$Q_2 Q_2^T = I - Q_1 Q_1^T$$

$$b = b_1 + b_2 \qquad c = Q^T b = \begin{bmatrix} Q_1^T b \\ Q_2^T b \end{bmatrix} \begin{matrix} c_1 \\ c_2 \end{matrix}$$

$$= Q_1 Q_1^T b + \underbrace{Q_2 Q_2^T b}_{r}$$

$$\|r\|_2 = \|Q_2 Q_2^T b\|_2 = \|Q_2^T b\|_2 = \|c_2\|_2$$

SVD

$$A = U \Sigma V^*$$

$$A^* = V \Sigma^* U^*$$

$\Sigma^* \neq \Sigma$ (cuz Σ is not a square matrix here)

(Values of Σ are real only)

$\operatorname{span}\{a_1, a_2, \dots, a_r\} \supseteq \operatorname{span}\{u_1, u_2, \dots, u_r\}$

bcuz each $u_i = \sum c_k a_k$ (using $Av_i = \sigma_i u_i$)

and $\operatorname{span}\{a_1, a_2, \dots, a_r\} \subseteq \operatorname{span}\{u_1, u_2, \dots, u_r\}$

bcuz $a_i = \sum \sigma_i \bar{v}_{ik} u_i$ (using $u_i^* A = \sigma_i v_i^*$)

c) $R(A) = \operatorname{span}\{u_1, \dots, u_r\} \Rightarrow N(A^T)^\perp = \operatorname{span}\{u_1, \dots, u_r\}$

$$\Downarrow$$

$N(A^*) = R(A)^\perp = \operatorname{span}\{u_{r+1}, u_{r+2}, \dots, u_n\}$

T.S.T $R(A) = \text{span}\{u_1, \ldots, u_r\}$

Pf:-

$Av_i = \sigma_i u_i$

$\text{span}\{u_1, \ldots, u_r\} \subset \text{span}\{a_1, \ldots, a_r\}$

as we $u_1, \ldots, u_r$ are orthonormal

$(\because tr(AB) = tr(BA))$

$\|A\|_F^2 = tr(A^* A) = tr(V\Sigma^T U^* U \Sigma V^*) = tr(V\Sigma^T \Sigma V^*) = tr(\underbrace{V^* V}_{I} \Sigma^T\Sigma)$

$\|A\|_2 = \|U\Sigma V^*\|_2 = \|\Sigma V^*\|_2 = \|(\Sigma V^*)^*\|_2 = \|V\Sigma^*\|_2 = \|\Sigma^*\|_2 \overset{?}{=} \sigma_1$

$\dfrac{\sigma_1^2 x_1^2 + \sigma_2^2 x_2^2 + \cdots}{x_1^2 + x_2^2 + \cdots}$

$\text{I} \quad \dfrac{\sigma_1^2 \left(x_1^2 + \frac{\sigma_2^2}{\sigma_1^2} x_2^2 + \cdots\right)}{x_1^2 + x_2^2 + \cdots}$

$\text{II} \geq \text{I}$

$\text{II} = \text{I}$ when $x_2 = x_3 = \cdots = 0$

If A is square or tall, and $\text{rank}(A) < n$

$\text{Minsing}(A) = 0$ bcuz $N(A) \neq \phi$ — Tall or square :- Mam told as 2 cases — broad :-

FA where $F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ — fliplr $\quad F = \begin{bmatrix} 0 & & 1 \\ & 1 & \\ 1 & & 0 \end{bmatrix}$

$A^{-1} = (U\Sigma V^*)^{-1}$

$= (V^*)^{-1}(\Sigma)(U)^{-1}$

$= (VF)(F\Sigma^{-1}F)(UF)^*$ — ?

~~$tr(U\Sigma V^*)$~~

$tr(V\Sigma^T \cancel{V^* V} \Sigma V^*)$

~~$tr(V\Sigma^T\Sigma V^*)$~~

$tr(V\Sigma^T\Sigma V^* V\Sigma^T\Sigma V^*)$

27/9/24

1) $A = U \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \ddots \\ & & & & 0 \end{bmatrix} V^*$ $\quad \epsilon > 0$

$B = U \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & \epsilon/2 & \\ & & & & \ddots \\ & & & & & \epsilon/2 \end{bmatrix} V^*$ $\quad \text{rank } B = n$

$\|A - B\|_2 = \left\| U \left(\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \ddots \\ & & & & 0 \end{bmatrix} - \begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_r & \\ & & & \epsilon/2 \\ & & & & \ddots \\ & & & & & \epsilon/2 \end{bmatrix} \right) V^* \right\|_2$

$= \left\| \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \epsilon/2 \\ & & & & \ddots \\ & & & & & \epsilon/2 \end{bmatrix} V^* \right\|_2 = \left\| \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \epsilon/2 \\ & & & & \ddots \\ & & & & & \epsilon/2 \end{bmatrix} \right\|_2$ = max of diagonal entries $= \epsilon/2$

Hly we can do for Frobenius norm with ϵ/n

MPP inverse :-

If $\underbrace{P^2 = P}_{\text{projexn}}$, $\underbrace{P^T P = I}_{\text{orthogonal}}$ then $P = I$

d) Let $P = (AA^+)^*$, then $P^* = P$ & $P^2 = P$

30/9/24

$A_{n\times m} x = b_{n\times 1}$, $n \geq m$ $\quad S = \{x_0 \in \mathbb{R}^m : \|b - Ax_0\|_2 = \min_{x \in \mathbb{R}^m} \|b - Ax\|_2\}$

Case 1:-

rank $A = m$:- Let $b = b_1 \oplus b_2$ where $b_1 \in R(A)$, $b_2 \in R(A)^{\perp} = N(A^T)$

$x_0 \in S \Rightarrow Ax_0 = b_1$

$U_r \Sigma_r V_r^* x_0 = U_r U_r^* b$

$\therefore R(A) = \text{span}\{u_1, \dots, u_r\}$
If u take orthonormal basis of $R(A)$, represented by B, $BB^* b$ is projexn of b on $R(A)$

$U_r^* U_r \Sigma_r V_r^* x_0 = U_r^* U_r U_r^* b$

$\Sigma_r V_r^* x_0 = U_r^* b$

$V_r^* x_0 = \Sigma_r^{-1} U_r^* b$

$V_r V_r^* x_0 = V_r \Sigma_r^{-1} U_r^* b$

If $r = m$, then

$V_r V_r^* = V_m V_m^* = I_m$

$\therefore x_0 = V_r \Sigma_r^{-1} U_r^* b$

From g) of Moore Penrose Properties,

$x_0 = A^+ b$

$\therefore S = \{A^+ b\}$ when $r = m$

Case 2:- Let $r = \text{rank } A < m$

Let $\hat{x}_0 \in \mathbb{R}^m \ni$

$\|\hat{x}_0\|_2 = \min\{\|x_0\|_2 : x_0 \in S\}$

$\hat{x}_0 \in \mathbb{R}^m = R(A^*) \oplus \underbrace{R(A^*)^{\perp}}_{N(A)}$

$= R(A^*) \oplus N(A)$

$\therefore \hat{x}_0 = \hat{x}_{01} + \hat{x}_{02}$ where $\hat{x}_{01} \in R(A^*)$ & $\hat{x}_{02} \in N(A)$

$\|\hat{x}_0\|_2^2 = \|\hat{x}_{01}\|_2^2 + \|\hat{x}_{02}\|_2^2$

Here also same

$\hat{x}_{01} = V_r V_r^* \hat{x}_0$

$\hat{x}_0 \in S \Rightarrow V_r V_r^* \hat{x}_0 = A^+ b$

$\Rightarrow \hat{x}_{01} = A^+ b$

$A\hat{x}_0 = b_1 \Rightarrow A(\hat{x}_{01} + \hat{x}_{02}) = b_1 \Rightarrow A\hat{x}_{01} = b_1$

$\therefore \hat{x}_{01} \in S \Rightarrow \|\hat{x}_{01}\|_2 \geq \|\hat{x}_0\|_2$

$\|\hat{x}_{01}\|_2^2 \geq \|\hat{x}_0\|_2^2 = \|\hat{x}_{01}\|_2^2 + \|\hat{x}_{02}\|_2^2$

$\therefore \hat{x}_{02} = 0$

Another pf in book

Eckart - Young Thm :-

$A = U \Sigma V^*$

$A_k = U \Sigma_k V^*$

$\therefore A - A_k = U(\Sigma - \Sigma_k) V^* = U \begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & \ddots & & \\ & & & \sigma_{k+1} & \\ & & & & \ddots \\ & & & & & \sigma_r \end{bmatrix} V^*$

$\therefore \|A - A_k\|_2 = \sigma_{k+1}$

$\therefore \sigma_{k+1} = \|A - A_k\|_2 \geq \min\{\|A-B\|_2 : B \in \mathbb{F}^{n\times m}\ \text{rank } B \leq k\}$

Let $B \in \mathbb{F}^{n\times m}$, rank $B \leq k$ $\therefore$ dim null $B \geq m-k$ (from rank-Nullity thm)

null $B \subseteq \mathbb{F}^m$

Let $S = \text{span}\{v_1, \ldots, v_{k+1}\} \subseteq \mathbb{F}^m$. dim $S = k+1$

Thm :- $\dim(S_1 + S_2) = \dim S_1 + \dim S_2 - \dim S_1 \cap S_2$ (MA102)

$\dim(\text{null } B \cap S) = \dim \text{null } B + \dim S - \dim(\underbrace{\text{null } B + S}_{\subseteq \mathbb{F}^m})$

$\geq m - k + k + 1 - m$

≥ 1

$\therefore \exists\ x_0 \in S \cap \text{null } B$

$(A-B)x_0 = Ax_0$

Ignore

$A = U\Sigma V^* \in \mathbb{F}^{n\times m}$, rank $A = r$, $A_k = U\Sigma_k V^*$ where $\Sigma_k = \text{diag}(\sigma_1, \ldots, \sigma_k, 0, 0, \ldots)$

$\|A - A_k\|_2 = \min\{\|A-B\|_2 : B \in \mathbb{F}^{n\times m}, \text{rank } B \leq k\} = \sigma_{k+1}$

rank $A_k = k$ and $\|A - A_k\|_2 = \sigma_{k+1}$ $\therefore \sigma_{k+1} \geq \min\{\|A-B\|_2 : B \in \mathbb{F}^{n\times m}, \text{rank } B \leq k\}$

Let $B \in \mathbb{F}^{n\times m}$, rank $B \leq k$, then dim null $B \geq m-k$.

Let $S = \text{span}\{v_1, \ldots, v_{k+1}\}$. Then $\dim(\text{null } B \cap S) \geq 1$.

~~σ~~
1/10/24

$\exists x_0 \in \text{null } B \cap S \ni x_0 \in \mathbb{R}^m,\ x \neq 0$

$\therefore (A-B)x_0 = Ax_0$ (as $x_0 \in \text{null}(B)$). Now $x_0 \in S \Rightarrow x_0 = \sum_{j=1}^{k+1} \alpha_j v_j$ & $\|x_0\|_2^2 = \sum_{j=1}^{k+1} |\alpha_j|^2$

$$\|(A-B)x_0\|_2^2 = \|Ax_0\|_2^2 = \|A(\sum_{j=1}^{k+1}\alpha_j v_j)\|_2^2 = \|\sum_{j=1}^{k+1}\alpha_j Av_j\|_2^2$$

$$= \|\sum_{j=1}^{k+1}\alpha_j\sigma_j u_j\|_2^2$$

$$\cancel{= \sum_{j=1}^{k+1} |\sigma_j|^2} \Rightarrow$$

$$= \sum_{j=1}^{k+1} \sigma_j^2 |\alpha_j|^2$$

$$\geq \sigma_{k+1}^2 \sum_{j=1}^{k+1} |\alpha_j|^2$$

$$= (\sigma_{k+1})^2 \|x_0\|_2^2$$

$\therefore \|(A-B)x_0\|_2^2 \geq \sigma_{k+1}^2 \|x_0\|_2^2 \Rightarrow$

$$\frac{\|(A-B)x_0\|_2}{\|x_0\|_2} \geq \sigma_{k+1}$$

$$\therefore \|A-B\|_2 = \max_{x\neq 0} \frac{\|(A-B)x\|_2}{\|x\|_2} \geq \frac{\|(A-B)x_0\|_2}{\|x_0\|_2} = \sigma_{k+1}$$

$$\therefore \min\{\|A-B\|_2 : B \in \mathbb{F},\ \text{rank } B \leq k\} \geq \sigma_{k+1}$$

Corollary:-

$$\frac{1}{\kappa_2(A)} = \min\left\{\frac{\|\Delta A\|_2}{\|A\|_2} : A+\Delta A \text{ is singular}\right\}$$

Use ~~$\Delta A =$~~ $A = U\begin{pmatrix}\sigma_1 & & \\ & \ddots & \\ & & \sigma_n\end{pmatrix}V^*$

$$A_{n-1} = U\begin{pmatrix}\sigma_1 & & & \\ & \ddots & & \\ & & \sigma_{n-1} & \\ & & & 0\end{pmatrix}V^*$$

$$\Delta A = -A + A_{n-1} = \sigma_n u_n v_n^*$$

4/10/24

Stability - Sensitivity of LSP:-

c) If $fl(VA) = V(A+\delta A)$ for an invertible V, then
$$\frac{\|\delta A\|_2}{\|A\|_2} \le n \frac{\|\delta A\|_1}{\|A\|_1} \le n^2 \beta \kappa_2(V) \quad \text{where } |\beta| \le nu + O(u^2)$$

For unitary $V = Q$ (or even an isometry)
$$fl(QA) = Q(A+E) \quad \frac{\|E\|_2}{\|A\|_2} \le n^3 u + O(u^2)$$
$$(\text{As } \kappa_2(V) = 1)$$

Also $fl(AQ) = fl(((AQ)^T)^T) = fl((Q^T A^T)^T)$ does not bring in any error

Q^T is also unitary

So in finding R using householder reflectors,
$$fl(R) = fl[Q_p \cdots Q_2 Q_1 A] = Q_p \cdots Q_1 (A+E) \quad \text{where } \frac{\|E\|_2}{\|A\|_2} \approx O(u)$$
$$A+E = (Q_1 \cdots Q_p)\, fl(R)$$

Even we formed Q by $Q_1 \cdots Q_k e_k \xrightarrow{Q}$ backward stable process

Perturbing only b

v) $\kappa_2(A) = \dfrac{\max mag(A)}{\min mag(A)}$ not $\|A^{-1}\|_2 \|A\|_2$

Remember

8/10/24

Schur Thm:-

→ Diagonalizability of an $n \times n$ matrix A is equivalent to the existence of a basis of C^n consisting of eigen vectors of A.

Pf:- $A = S D S^{-1}$

$AS = SD$

$$A[s_1\ s_2 \cdots s_n] = [s_1 \cdots s_n]\begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix} = \begin{bmatrix} s_{11}d_1 & s_{12}d_2 & \cdots \\ s_{21}d_1 & \cdots & \\ & & \end{bmatrix}$$

$As_i = d_i s_i$

Spectral Thm for Normal matrices :-

1) $T^* T = Q^* A^* \underbrace{Q Q^*}_{I} A Q = Q^* A A^* Q = Q^* A Q Q^* A^* Q = T T^*$

2) $\cancel{(T^*) = T T^* T^{-1}}$

$\cancel{T^* T = T T^*}$ $\cancel{T^* T = T T^*}$

$\cancel{T = (T^*)^{-1} T T^*}$ $\cancel{\bar{t}_{11} \; \bar{t}_{21} \; \dots \; \bar{t}_{n1}}$

t_n

$T^* T = T T^*$

$$\begin{bmatrix} \bar{t}_{11} & & & \\ \bar{t}_{12} & \bar{t}_{22} & & \\ \vdots & \vdots & \ddots & \\ \bar{t}_{1n} & \vdots & & \ddots \end{bmatrix} \begin{bmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ & \ddots & & \\ & & \ddots & \\ & & & t_{nn} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & \dots & t_{1n} \\ & \ddots & & \\ & & \ddots & \\ & & & t_{nn} \end{bmatrix} \begin{bmatrix} \bar{t}_{11} & & & \\ \vdots & \ddots & & \\ \vdots & & \ddots & \\ \bar{t}_{1n} & & & \bar{t}_{nn} \end{bmatrix}$$

$$\begin{pmatrix} \bar{t}_{11} t_{11} & \bar{t}_{11} t_{12} & \bar{t}_{11} t_{13} & \dots & \bar{t}_{11} t_{1n} \\ \bar{t}_{12} t_{11} & \bar{t}_{12} t_{12} + \bar{t}_{22} t_{22} & \dots & & \end{pmatrix} = $$

Solve all eqn's

(QR)

Pf by induxn :- Base case true ($n = 1$) size of T

$$T = \begin{bmatrix} S & x \\ 0 & c \end{bmatrix}$$

$$T^* = \begin{bmatrix} S^* & 0 \\ x^* & \bar{c} \end{bmatrix}$$

$T T^* = T^* T$

$$\begin{bmatrix} S S^* + x x^* & \bar{c} x \\ c x^* & |c|^2 \end{bmatrix} = \begin{bmatrix} S^* S & S^* x \\ x^* S & x^* x + |c|^2 \end{bmatrix}$$

$x^* x = 0 \implies x = 0$ — This makes T diagonal

3) Assume Schur Thm to be true

$\Leftarrow$ $Q^* A Q = D$ $\quad A^* A = Q D^* Q^* Q D Q^*$

$A = Q D Q^*$ $\quad = Q D^* D Q^*$

$A^* = Q D^* Q^*$ $\quad = Q D D^* Q^*$

$\Rightarrow$ $\quad = Q D Q^* Q D^* Q^*$

$A^* A = A A^*$ $\quad = A A^*$

From Schur Thm,

$\exists\, Q, T \ni Q^* A Q = T$

$A = Q T Q^*$

$A^* = Q T^* Q^*$ — I

$A^* A = Q T^* Q^* Q T Q^* = Q T^* T Q^*$

$A A^* = Q T Q^* Q T^* Q^* = Q T T^* Q^*$

$A^* A = A A^* \Rightarrow$

$Q (T^* T - T T^*) Q^* = 0$

As Q is non-singular,

$(T^* T - T T^*) Q^* = 0$ — II

$Q (T^* T - T T^*)^* = 0$

As Q is non-singular,

$T^* T - T T^* = 0$

Spectral thm for Hermitian

$\Rightarrow$ A is Hermitian $\Rightarrow$ $A^* = A$

Same as I $\rightarrow$ $Q T^* Q^* = Q T Q^*$

$T^* = T = D$

lower Δ^{lar} upper Δ^{lar}

D is real as $T^* = T \Rightarrow \bar{t_{ii}} = t_{ii}$

$\Leftarrow$ $Q^* A Q = D$

As D is real, $D^* = D$

$Q^* A^* Q = Q^* A Q \Rightarrow$ (Same as in II)

$A = A^*$

5/10/24

$A \to$ real symmetric $n \times n$. Proving spectral thm for symmetric matrices:-

Pf:- $\exists\ \lambda \in \mathbb{R}$ and $v \in \mathbb{R}^n \ni Av = \lambda v$ (All λ & v are real. Prove — ?)

(We will have to go through Schur Thm proof again.)

Let $q_1 = \frac{v}{\|v\|_2}$. Then $Aq_1 = \lambda_1 q_1$, $\|q_1\|_2 = 1$.

1) Build an orthogonal matrix $\tilde{Q} \in \mathbb{R}^{n \times n} \ni \tilde{Q} = [q_1 \cdots q_n]$

2) $\tilde{Q}^T A \tilde{Q} = \left[\begin{array}{c|ccc} \lambda_1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & \tilde{A} & \\ 0 & & & \end{array}\right]$ $\left[\because A^T = A \Rightarrow (Q^T A Q)^T = Q^T A Q\right]$

3) Use the induxn hypothesis for $\tilde{A}$ where $\boxed{\tilde{A}^T = \tilde{A}}$

(OR)

For Normal Matrix, $Q^* A Q = D \Rightarrow AQ = QD \Rightarrow Aq_i = \lambda_i q_i \quad i = 1:n$

where $Q = [q_1 \cdots q_n]$, $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$

$\{q_1, \ldots, q_n\} \leftarrow$ orthonormal basis of $\mathbb{C}^n$ of eigenvalues of A.

Quasi-upper Δ^{lar}:-

$$A = \left[\begin{array}{c|c|c|c} A_1 & \times & \times & \times \\ \hline 0 & A_2 & \times & \times \\ \hline 0 & 0 & A_k & \times \\ \hline 0 & 0 & 0 & A_p \end{array}\right]$$

$\det(A - \lambda I)$

$= \prod_{i=1}^{k} \det(A_i - \lambda I_{\text{size of } n_i})$ $\cancel{= 0}$

(when A_i is of size 2×2 atmost

Ex:- $A = \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 1 & -1 \\ \hline 0 & 0 & 5 \end{array}\right]$

$x = c_1v_1 + c_2v_2 + \dots + c_nv_n$, $c_1 \neq 0$, $Av_i = \lambda_i v_i$, $i = 1:n$

$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$

$$A^2x = A(Ax)$$
$$= A(c_1\lambda_1 v_1 + c_2\lambda_2 v_2 + \dots)$$
$$= c_1\lambda_1 Av_1 + c_2\lambda_2 Av_2 + \dots = c_1\lambda_1^2 v_1 + c_2\lambda_2^2 v_2 + \dots,$$

Ill^y $$A^jx = A^j\left(\sum_{i=1}^n c_iv_i\right) = \sum_{i=1}^n c_iA^jv_i = \sum_{i=1}^n c_i\lambda_i^j v_i$$

$$\frac{A^jx}{\lambda_1^j} = \sum_{i=1}^n c_i\left(\frac{\lambda_i}{\lambda_1}\right)^j v_i = c_1v_1 + \sum_{i=2}^n \left(\frac{\lambda_i}{\lambda_1}\right)^j c_iv_i$$

$$\underset{j\to\infty}{Lt}\left\|\frac{A^jx}{\lambda_1^j} - c_1v_1\right\| = \underset{j\to\infty}{Lt}\left\|\sum_{i=2}^n\left(\frac{\lambda_i}{\lambda_1}\right)^j c_iv_i\right\| \leq \sum_{i=2}^n \left|\frac{\lambda_i}{\lambda_1}\right|^j |c_i|\,\|v_i\| = 0$$

Rate of convergence $= \underset{j\to\infty}{Lt}\left\|\sum_{i=2}\left(\frac{\lambda_i}{\lambda_1}\right)^{j+1}c_iv_i\right\| \Big/ \left\|\sum_{i=2}\left(\frac{\lambda_i}{\lambda_1}\right)^j c_iv_i\right\| = \dfrac{\left(\frac{\lambda_2}{\lambda_1}\right)^{j+1}\|c_2v_2+\dots\|}{\left(\frac{\lambda_2}{\lambda_1}\right)^j\|c_2v_2+\dots\|} = \left|\frac{\lambda_2}{\lambda_1}\right|$

In Power method, $x = \sum c_iv_i$ we assume $c_1 \neq 0$ (Good).

14/10/24

Power-method:

$$q_1 = \frac{Aq_0}{s_1} = \frac{A\left(\frac{x}{s_0}\right)}{s_1} = \frac{Ax}{s_0s_1} = \lambda_1\frac{\left(\sum_{k=1}^n \lambda_kc_kv_k\right)}{\lambda_1}\Big/ s_0s_1 = \frac{\lambda_1}{s_0s_1}\left[c_1v_1 + \sum_{k=2}^n \frac{\lambda_kc_kv_k}{\lambda_1}\right] \qquad \frac{Ax}{\lambda_1}$$

Let $\tilde{q}_j = \dfrac{A^jx}{\lambda_1^j}$, $j = 1, \dots, n$

$\therefore q_1 = \dfrac{\lambda_1}{s_0s_1}\tilde{q}_1$, $q_2 = \dfrac{\lambda_1^2}{s_0s_1s_2}\tilde{q}_2$, $q_j = \left[\dfrac{\lambda_1^j}{\prod_{k=0}^j s_k}\right]\tilde{q}_j$, $j = 0, 1, \dots$

Let $\mu_j = \left(\prod_{k=0}^j s_k\right)\Big/\lambda_1^j$, $j = 0, 1, \dots$ Then $q_j = \dfrac{\tilde{q}_j}{\mu_j}$, $j = 0, 1, \dots \longrightarrow (*)$

Let $i_j = \min\{1, \dots, n : |A\tilde{q}_{j-1}(i)| = \|Aq_{j-1}\|_\infty\} \Rightarrow i_j = \min\{1, \dots, n : q_j(i) = 1\}$

$(*) \Rightarrow \dfrac{\tilde{q}_j(i_j)}{\mu_j} = q_j(i_j) = 1 \Rightarrow \mu_j = \tilde{q}_j(i_j) = \left[\dfrac{A^jx}{\lambda_1^j}\right](i_j) = \left[c_1v_1 + \sum_{k=2}^m\left(\dfrac{\lambda_k}{\lambda_1}\right)^j c_kv_k\right](i_j)$

$$\therefore \underset{j\to\infty}{Lt}\,\mu_j = \underset{j\to\infty}{Lt}\,c_1v_1(i_j) \qquad \therefore \underset{j\to\infty}{Lt}\,q_j = \frac{\underset{j\to\infty}{Lt}\,\tilde{q}_j}{\underset{j\to\infty}{Lt}\,\mu_j} = \underset{j\to\infty}{Lt}\,\frac{c_1v_1}{c_1v_1(i_j)}$$

$$\underset{j\to\infty}{Lt}\left\|\frac{v_1}{v_1(i_j)}\right\|_\infty = \underset{j\to\infty}{Lt}\|q_j\|_\infty = 1 \cdot \underset{j\to\infty}{Lt} Aq_j = A\underset{j\to\infty}{Lt} q_j = A\underset{j\to\infty}{Lt}\frac{v_1}{v_1(i_j)}$$

$$= \underset{j\to\infty}{Lt}\frac{Av_1}{v_1(i_j)}$$

$$= \lambda_1 \underset{j\to\infty}{Lt}\frac{v_1}{v_1(i_j)} = \lambda_1 \underset{j\to\infty}{Lt} q_j$$

$\therefore \underset{j\to\infty}{Lt} Aq_j = \lambda_1 \underset{j\to\infty}{Lt} q_j$

$\therefore$ For large enough j,

$$Aq_j \simeq \lambda_1 q_j$$

$$\Rightarrow \|Aq_j\|_\infty \simeq |\lambda_1|, \longrightarrow ①$$

$$Aq_j(i_j) \simeq \lambda_1 q_j(i_j) = \lambda_1, \longrightarrow ②$$

$$|Aq_j(i)| \simeq |\lambda_1 q_j(i)| \le |\lambda_1| \simeq \|Aq_j\|_\infty \quad \forall\ i=1,\dots,n \longrightarrow ③$$

From ① & ②,

$$|Aq_j(i_j)| \simeq \|Aq_j\|_\infty$$

& ③ $\Rightarrow$

$$|Aq_j(i)| \le |\lambda_1| \simeq \|Aq_j\|_\infty \quad \underline{\forall\ i \ne i_j}$$

If $i_j \ne 1$, by def^n of i_j,

$$|Aq_j(i)| \simeq |\lambda_1||q_j(i)| < |\lambda_1| \quad \forall\ i=1,\dots,i-1$$

$\therefore i_{j+1} = i_j$ for large enough j.

Then $\underset{j\to\infty}{Lt} q_j = \underset{j\to\infty}{Lt}\frac{v_1}{v_1(i_j)} = \frac{v_1}{v_1(i_0)}$

ii) $\underset{j\to\infty}{Lt} q_j = \hat{v}_1 \Rightarrow$

$$\underset{j\to\infty}{Lt} Aq_j = A\underset{j\to\infty}{Lt} q_j = A\hat{v}_1 = \lambda_1\hat{v}_1$$

$$\Rightarrow \underset{j\to\infty}{Lt}\underbrace{\frac{Aq_j}{s_j}}_{q_{j+1}} = \underset{j\to\infty}{Lt}\frac{\lambda_1\hat{v}_1}{s_j} \Rightarrow \underbrace{\underset{j\to\infty}{Lt} q_{j+1}}_{\hat{v}_1} = \frac{\lambda_1\hat{v}_1}{\underset{j\to\infty}{Lt} s_j} \Leftrightarrow \hat{v}_1 = \frac{\lambda_1\hat{v}_1}{\underset{j\to\infty}{Lt} s_j}$$

$$\cancel{q_j = \frac{A^{j}x}{s_0 s_1 \cdots s_j}}$$

$$\cancel{Aq_j = \frac{A^{j+1}x}{s_0 s_1 \cdots s_j} = \frac{c_1\lambda_1^{j+1} v_1 + c_2\lambda_2^{j+1} v_2 + \cdots}{s_0 s_1 \cdots s_j}}$$

15/10/24 Shift n Invert Method

$$Av_i = \lambda_i v_i$$

$$(A-\rho I)v_i = (\lambda_i-\rho)v_i$$

$$(A-\rho I)^{-1}(A-\rho I)v_i = (A-\rho I)^{-1}(\lambda_i-\rho)v_i$$

$$\Rightarrow \frac{\hat{v}_i}{\lambda_i-\rho} = (A-\rho I)^{-1}\hat{v}_i$$

Rayleigh Quotient method

$\mu q = Aq$ μ is unknown
   q, A are known

$q\mu = Aq \longrightarrow (1)$

A, b A, b in $Ax=b$

From normal eq^{n}s for LSP, $\mu = \dfrac{q^*Aq}{q^*q}$

Thm:- $Av = \lambda v$, $\|v\|_2 = 1$, $\rho := q^*Aq$ with $\|q\|_2 = 1$

$v^*Av = \lambda$

$$\begin{aligned}
|\lambda-\rho| = |v^*Av - q^*Aq| &= |v^*Av - v^*Aq + v^*Aq - q^*Aq| \\
&= |v^*A(v-q) + (v-q)^*Aq| \\
&\leq |v^*A(v-q)| + |((v-q)^*Aq)^*| \\
&= |v^*A(v-q)| + |q^*A^*(v-q)| \\
&\leq \|v\|_2 \|A(v-q)\|_2 + \|q\|_2 \|A^*(v-q)\|_2 \\
&\leq \|A\|_2 \|v-q\|_2 + \|A^*\|_2 \|v-q\|_2
\end{aligned}$$

Upper Hessenberg Matrices:-

Ex:- U will have to 2×2 reflectors only.

Example where Rayleigh quotient method convergence doesn't happen:-

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $q_0 = x$

$$\rho_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

Solve $(A - \rho_0 I)\hat{q_1} = q_0$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{q_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{q_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = q_1$$

(This happened bcuz $\rho = 0$ is halfway b/n $\lambda_0 = -1$ & $\lambda_1 = 1$)

$\rho_1 = q_1^* A q_1 = 0$

Solve $(A - \rho I)\hat{q_2} = q_1$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \hat{q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\hat{q_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

quadratic convergence $\rightarrow$ $e_{n+1} = c e_n^2$

16/10/24

$\rightarrow$ Exam questⁿ :- $\exists$ unique householder reflector $\ni Qx = y$,

Sol) $\langle x+y, x-y \rangle = \|x\|_2^2 - \|y\|_2^2 \underbrace{- y^* x + x^* y}_{=0}$ ... $x^* y \in \mathbb{R}$.

Uniqueness :-

Suppose $\hat{Q} = I - \frac{2\hat{u}\hat{u}^*}{\|\hat{u}\|^2}$ be another reflector

$$\hat{Q} x = y$$

$$x - \frac{2}{\|\hat{u}\|_2^2} \hat{u} (\hat{u}^* x) = y$$

$$\underbrace{\frac{2}{\|\hat{u}\|_2^2} (\hat{u}^* x)}_{=\alpha} \hat{u} = x - y$$

$x \neq y \Rightarrow \hat{u}^* x \neq 0$

$\Rightarrow \alpha \hat{u} = x - y$

$$Q = I - \frac{2(x-y)(x-y)^*}{\|x-y\|_2^2} = I - \frac{2\hat{u}\hat{u}^*}{\|\hat{u}\|_2^2} = \hat{Q}$$

$\rightarrow A = QR \ni R(i,i) > 0 \ \forall \ i = 1:n$ is unique

Pf:- $R^* R = A^* A \leftarrow$ +ve definite as A is full rank

$R^* R$ is cholesky decomposit^n $\longrightarrow$ unique as $R(i,i) > 0$

$$A = Q_1 R = Q_2 R$$

$$Q_1 R R^{-1} = Q_2$$

$$\rightarrow Q_1 = Q_2$$

Another pf:-

$$A = Q_1 R_1 = Q_2 R_2$$

$$\underbrace{Q_2^* Q_1}_{} = R_2 R_1^{-1}$$

unitary but not both Q_2 & Q_1 are unitary as Q_2, Q_1 may be tall.

Transformat^n to Upper Hessenberg

$$A_{1/2} Q_1 = \left((A_{1/2} Q_1)^T \right)^T$$

$$= \left(Q_1^T A_{1/2}^T \right)^T$$

$$= \left(\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{Q}_1 \end{array} \right] \left[\begin{array}{c|c} a_{11} & \pm \|b\|_2 \ 0 \ \ldots \ 0 \\ \hline c & \hat{A}^T \hat{Q}_1 \end{array} \right] \right)^T$$

$$= \left[\begin{array}{c|c} a_{11} & \pm \|b\|_2 \ 0 \ \ldots \ 0 \\ \hline \hat{Q}_1 c & \hat{Q}_1 A^T \hat{Q}_1 \end{array} \right]^T$$

SVD

a) If $A = U\Sigma V^*$ then $A^{-1} = (VF)(F\Sigma^{-1}F)(UF)^*$

$F = \begin{bmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{bmatrix}$

$F^2 = I$

$FF^* = I$

Pf:- $AA^{-1} = (U\Sigma V^*)(VF)(F\Sigma^{-1}F)(UF)^*$

$= U\Sigma \Sigma^{-1} FF^* U^*$

$= I$

b) — Rank $A = p$ — $m \geq n$ — Use SVD of A^T

— $m \leq n$ — Use SVD of A

— Rank $A \neq p$ — $\exists\, x \ni Ax = 0$ or minimag $A^T = 0$

c) $Av_i = \sigma_i u_i$

$u_i^* A = \sigma_i v_i^*$

$A^*Av_i = A^* \sigma_i u_i = \sigma_i (v_i \sigma_i)$

$A = U\Sigma V^*$

$A^+ = V\Sigma^+ U^*$

$U_{n\times n}\ \Sigma_{n\times m}\ V_{m\times m}\ \ A_{n\times m}$

$\Sigma^+_{m\times n}\ \ A^+_{m\times n}$

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_ _ E A R

a) $AA^+ = U\Sigma V^* V\Sigma^+ U^*$

$= U\Sigma I_{m\times m} \Sigma^+ U^*$

$= U\Sigma\Sigma^+_{m\times n} U^* = I$

b) $\cancel{A^+A = I_{m\times m}}$

$A^* = V\Sigma^* U^*$

$A = U\Sigma V^*$

$(A^*A) = V\Sigma^* U^* U\Sigma V^*$

$= V\Sigma^*\Sigma V^*$

$I_{m\times m}\Sigma^+_{m\times n}$

$= \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_1} & & & & 0 \\ & \frac{1}{\sigma_2} & & & \\ 0 & & \ddots & & \\ & & & \frac{1}{\sigma_r} & 0 \\ & & & & 0 \end{bmatrix}$

$= \Sigma^+_{m\times n}$

In general

$I_{m\times m} B_{m\times n} = B_{m\times n}$

$\Sigma^*\Sigma$ is invertible

$\Sigma\Sigma^*$ is not

$(A^*A)^{-1}A^*$

$= V(\Sigma^*\Sigma)^{-1} V^{-1} V\Sigma^* U^*$

$= V(\Sigma^*\Sigma)^{-1}\Sigma^* U^*_{m\times m}$

$= V \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \cdots \\ 0 & \frac{1}{\sigma_2^2} & \\ 0 & 0 & \frac{1}{\sigma_m^2} \end{bmatrix} U^* = V\Sigma U^*$

$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots\ \sigma_m \end{bmatrix}$ $\Sigma^* = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & & & \\ 0 & \cdots & 0\ \sigma_m & \cdots & 0 \end{bmatrix}$

$\cancel{(U\Sigma\Sigma^+ U^*)^*} = U(\Sigma\Sigma^+)^* U^*$ $\qquad U\Sigma\Sigma^+ U^*$

d) $\cancel{AMA} = A \quad \cancel{MAM = M}$

$M_1 = M_1 A M_1 = M_1 M_1' A' = M_1 M_1' A' M_2' A' = M_1 M_1' A' A M_2$

$= M_1 A M_1 A M_2$

$= M_1 A M_2$

$\cancel{= M_1 M_2 A M_2}$

$= M_1 A M_2 A M_2$

$= A' M_1' M_2 A M_2$

$= A' M_1' A' M_2' M_2$

$= A' M_2' M_2$

$= M_2 A M_2 = M_2$

$(AA^+)^* = AA^+$ and the other 3 statements are true becz they are true when $A = \Sigma$

f) $U(\Sigma^+)^* V^* = (A^+)^*$

$(A^*)^+ = (V\Sigma^* U^*)^+$

$= U(\Sigma^*)^+ V^*$

So f) is also true becz $(\Sigma^+)^* = (\Sigma^*)^+$