1 Stochastic Differential Equations

Consider a SDE of the form

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad X(0) = Z\,.$$

Question 1: Does there exist a solution? And if there is a solution then is it unique?
Question 2: How to solve such a SDE?
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $W(\cdot)$ be a Brownian motion defined on it. Let $\mathcal{F}_t$ be the filtration generated by $W(t)$ and Z, i.e., $\mathcal{F}_t = \sigma\{Z, W(s), s \leq t\}$.

Definition 1.1. *A solution of the SDE above is a continuous stochastic process $X(t), 0 \leq t \leq T$ with the following properties:*

1. *$X(t)$ is adapted to the filtration $\mathcal{F}_t$.*
2. *$\mathbb{P}(X(0) = Z) = 1$.*
3. *$\int_0^T \mathbb{E}(|b(t, X(t))|)dt < \infty$, $\int_0^T \mathbb{E}(|\sigma(t, X(t))|^2)dt < \infty$.*
4. *$X(t) = X(0) + \int_0^t b(s, X(s))ds + \int_0^t \sigma(s, X(s))dW(s)$, $0 \leq t \leq T$ almost surely.*

Definition 1.2. *The SDE above is said to have a unique solution, if X and $\tilde{X}$ are two solutions, then $\mathbb{P}(X(t) = \tilde{X}(t), 0 \leq t \leq T) = 1$.*

Theorem 1.3. *(Existence and Uniqueness) Suppose the co-efficients $b(t, x)$ and $\sigma(t, x)$ satisfy the global lipschitz and linear growth conditions*

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq K|x - y| \quad \textit{and}$$

$$|b(t, x)| + |\sigma(t, x)| \leq K(1 + |x|)$$

for some positive constant K. Further suppose that $\mathbb{E}(Z^2) < \infty$. Then the SDE has a unique solution. Further $X(t)$ satisfies $\mathbb{E} \int_0^T |X(t)|^2 dt < \infty$.

Consider the deterministic differential equations

$$dX(t) = X^2(t)dt \quad \text{and} \quad dX(t) = 3X^{2/3}(t)dt\,.$$

For the first one $b(t, x) = x^2$ does not satisfy the linear growth condition. For the second one, $b(t, x) = 3x^{2/3}$ does not satisfy the lipschitz condition. In the first case, the solution (unique) is $X(t) = \frac{1}{1-t}$. But this “explodes” as $t \uparrow 1$. Thus linear growth condition ensures that the solution does not “explode” in finite time.
For the second case, there are infinitely many solutions, in fact for any $a > 0$,

$$X(t) = \begin{cases} 0 \quad \text{for } t \leq a \\ (t-a)^3 \quad \text{for } t \geq a\,, \end{cases}$$

is a solution. Thus lipschitz property ensures uniqueness.

Lemma 1.4. *(Gronwall's Inequality) Let $f(\cdot)$ be a continuous function such that*

$$f(t) \leq C + K \int_0^t f(s)ds \quad \textit{for } t \in [0,T]\,,$$

where C is a constant and K is a positive constant. Then $f(t) \leq Ce^{Kt}$ for $t \in [0,T]$.

Proof: Define $W(t) = C + K\int_0^t f(s)ds$. Then $W(t) \geq f(t)$ for all $t \in [0,T]$. Now bt Fundamental Theorem of Calculus,

$$\begin{aligned}
&W'(t) = Kf(t) \leq KW(t) \\
\Rightarrow& e^{-Kt}W'(t) - Ke^{-Kt}W(t) \leq 0 \\
\Rightarrow& \frac{d}{dt}(e^{-Kt}W(t)) \leq 0 \\
\Rightarrow& e^{-Kt}W(t) - W(0) \leq 0 \\
\Rightarrow& W(t) \leq Ce^{Kt} \\
\Rightarrow& f(t) \leq Ce^{Kt}\,.
\end{aligned}$$

$\square$

Proof of Uniqueness: Suppose there exists two solutions $X_1(t)$ and $X_2(t)$. Thus

$$X_1(t) = Z + \int_0^t b(s, X_1(s))ds + \int_0^t \sigma(s, X_1(s))dW(s)\,,$$

$$X_2(t) = Z + \int_0^t b(s, X_2(s))ds + \int_0^t \sigma(s, X_2(s))dW(s)\,.$$

Thus,

$$\begin{aligned}
&\mathbb{E}|X_1(t) - X_2(t)|^2 = \mathbb{E}\left(\int_0^t [b(s,X_1(s)) - b(s,X_2(s))]ds + \int_0^t [\sigma(s,X_1(s)) - \sigma(s,X_2(s))]dW(s)\right)^2 \\
&\leq 2\left[\mathbb{E}\left(\int_0^t [b(s,X_1(s)) - b(s,X_2(s))]ds\right)^2 + \mathbb{E}\left(\int_0^t [\sigma(s,X_1(s)) - \sigma(s,X_2(s))]dW(s)\right)^2\right] \\
&\leq 2\left[t\int_0^t \mathbb{E}\left(b(s,X_1(s)) - b(s,X_2(s))\right)^2 ds + \int_0^t \mathbb{E}\left(\sigma(s,X_1(s)) - \sigma(s,X_2(s))\right)^2 ds\right] \\
&\leq 2K^2(1+t)\int_0^t \mathbb{E}|X_1(s) - X_2(s)|^2 ds\,.
\end{aligned}$$

Thus

$$\mathbb{E}|X_1(t) - X_2(t)|^2 \leq 2K^2(1+T)\int_0^t \mathbb{E}|X_1(s) - X_2(s)|^2 ds\,,$$

for all $t \in [0,T]$. So by Gronwall's inequality,

$$\mathbb{E}|X_1(t) - X_2(t)|^2 = 0\,.$$

Thus $\mathbb{P}(X_1(t) = X_2(t)) = 1$ for each $t \in [0,T]$. Thus $\mathbb{P}(X_1(t) = X_2(t)\ \forall\ t \in \mathbb{Q} \cap [0,T]) = 1$. By continuity, $\mathbb{P}(X_1(t) = X_2(t)\ \forall\ t \in [0,T]) = 1$. Hence we have the uniqueness. $\square$

Consider the SDE

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \quad S(0) = S_0\,,$$

where $\mu(\cdot)$ and $\sigma(\cdot)$ are deterministic functions satisfying $|\mu(t)| + |\sigma(t)| \leq K$. Thus $b(t,x) = \mu(t)x$ and $\sigma(t,x) = \sigma(t)x$. Both the conditions for uniqueness and existence are satisfied. Now by Ito's formula,

$$
\begin{aligned}
d(\log(S(t)) &= \frac{1}{S(t)} dS(t) - \frac{1}{2S^2(t)} dS(t)dS(t) \\
&= \frac{1}{S(t)} dS(t)[\mu(t)S(t)dt + \sigma(t)S(t)dW(t)] - \frac{1}{2S^2(t)}[\sigma^2(t)S^2(t)] \\
&= \sigma(t)dW(t) + (\mu(t) - \frac{1}{2}\sigma^2(t))dt\,.
\end{aligned}
$$

Thus

$$
\log \frac{S(t)}{S_0} = \int_0^t \sigma(s)dW(s) + \int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s))ds
$$

$$
\Rightarrow S(t) = S_0 \exp\{\int_0^t \sigma(s)dW(s) + \int_0^t (\mu(s) - \frac{1}{2}\sigma^2(s))ds\}\,.
$$

Consider a first order ordinary differential equation

$$
\frac{dx(t)}{dt} = f(t)x(t) + g(t), \quad x(0) = x\,,
$$

where f is a continuous function. Then we know the integrating factor method for solving the above equation. The integrating factor is given by $h(t) = e^{-\int_0^t f(s)ds}$. The solution is given by,

$$
x(t) = (h(t))^{-1}x + (h(t))^{-1}\int_0^t h(s)g(s)ds\,.
$$

By linear SDE, we mean a SDE of the form,

$$
dX(t) = \{\phi(t)X(t) + \theta(t)\}dW(t) + \{f(t)X(t) + g(t)\}dt \quad X(0) = Z\,.
$$

Define $H(t)$ as follows:

$$
H(t) = e^{-Y(t)}, \quad Y(t) = \int_0^t \phi(s)dW(s) + \int_0^t f(s)ds - \frac{1}{2}\int_0^t \phi^2(s)ds\,.
$$

Now $d(H(t)X(t)) = X(t)dH(t) + H(t)dX(t) + dH(t)dX(t)$

$$
\begin{aligned}
dH(t) &= -e^{-Y(t)}dY(t) + \frac{1}{2}e^{-Y(t)}dY(t)dY(t) \\
&= -H(t)[f(t)dt + \phi(t)dW(t) - \frac{1}{2}\phi^2(t)] + \frac{1}{2}H(t)\phi^2(t)dt \\
&= -H(t)[f(t)dt + \phi(t)dW(t) - \phi^2(t)dt]\,.
\end{aligned}
$$

Thus

$$
dX(t)dH(t) = -H(t)\phi(t)[\phi(t)X(t) + \theta(t)]dt\,.
$$

So we get,

$$
\begin{aligned}
d(X(t)H(t)) = H(t)[dX(t) - X(t)f(t)dt - X(t)\phi(t)dW(t) + \\
\phi^2(t)X(t)dt - \phi^2(t)X(t)dt - \theta(t)\phi(t)dt] \\
= H(t)[\theta(t)dW(t) + g(t)dt - \theta(t)\phi(t)dt]\,.
\end{aligned}
$$

Thus

$$H(t)X(t) = Z + \int_0^t H(s)\theta(s)dW(s) + \int_0^t H(s)\{g(s) - \theta(s)\phi(s)\}ds\,.$$

$$\Rightarrow X(t) = Ze^{Y(t)} + \int_0^t e^{Y(t)-Y(s)}\theta(s)dW(s) + \int_0^t e^{Y(t)-Y(s)}\{g(s) - \theta(s)\phi(s)\}ds\,.$$

Example:

$$dX(t) = \mu X(t)dt + \sigma dW(t), \quad X(0) = Z\,.$$

Thus $f(t) = \mu$, $g(t) = 0$, $\phi(t) = 0$, $\theta(t) = \sigma$. Thus $Y(t) = \mu t$. Thus the solution is given by

$$X(t) = Ze^{\mu t} + \int_0^t e^{\mu(t-s)}\sigma dW(s)\,.$$

Exercise:

i) $d(X)(t) = -X(t)dt + e^{-t}dW(t)$, $X(0) = Z$.
ii) $d(X)(t) = rdt + \alpha X(t)dW(t)$, $X(0) = Z$.
iii) $d(X)(t) = (m - X(t))dt + \sigma dW(t)$, $X(0) = Z$.
iv) $d(X)(t) = \frac{1}{2}X(t)dt + X(t)dW(t)$, $X(0) = 1$.
v) $d(X)(t) = \frac{b-X(t)}{1-t}dt + dW(t)$, $X(0) = a$.

Consider the SDE,

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t)\,.$$

Let $h(\cdot)$ be a Borel measurable function. Define the function

$$g(t, x) = \mathbb{E}[h(X(T))|X(t) = x] = \mathbb{E}_{t,x}[h(X(T))]\,. \tag{1}$$

Theorem 1.5. *Let $X(u), u \geq 0$, be a solution to the SDE above with some initial condition at 0. Then for any $0 \leq t \leq T$,*

$$\mathbb{E}[h(X(T))|\mathcal{F}_t] = g(t, X(t))\,.$$

Corollary 1.6. *Solutions to stochastic differential equations are Markov processes.*

The following theorem relates SDEs and PDEs.

Theorem 1.7. *(Feynman-Kac) Consider the SDE,*

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t)\,.$$

Let $h(\cdot)$ be a Borel measurable function. Let $g(t, x)$ be as in (1). *Then $g(t, x)$ satisfies the PDE*

$$g_t(t, x) + b(t, x)g_x(t, x) + \frac{1}{2}\sigma^2(t, x)g_{xx}(t, x) = 0\,,$$

with terminal condition $g(T, x) = h(x)$ for all x.

Proof: Claim: $g(t, X(t))$, $0 \leq t \leq T$ is a martingale. Now by previous theorem $g(t, X(t)) = \mathbb{E}[h(X(T))|\mathcal{F}_t]$. Thus for $s < t$,

$$\mathbb{E}[g(t, X(t))|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[h(X(T))|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[h(X(T))|\mathcal{F}_s] = g(s, X(s))\,.$$

Hence the claim. Now

$$\begin{aligned}dg(t,X(t)) &= g_t(t,X(t))dt + g_x(t,X(t))dX(t) + \frac{1}{2}g_{xx}(t,X(t))dX(t)dX(t)\\ &= (g_t(t,X(t)) + b(t,X(t))g_x(t,X(t)) + \frac{1}{2}\sigma^2(t,X(t))g_{xx}(t,X(t)))dt\\ &+ g_x(t,X(t))\sigma(t,X(t))dW(t)\,.\end{aligned}$$

Since $g(t,X(t))$ is a martingale, so the the dt term must be equal to 0. Thus we must have

$$g_t(t,x) + b(t,x)g_x(t,x) + \frac{1}{2}\sigma^2(t,x)g_{xx}(t,x) = 0\,.$$

□

Theorem 1.8. *(Discounted Feynman-Kac) Consider the SDE,*

$$dX(t) = b(t,X(t))dt + \sigma(t,X(t))dW(t)\,.$$

Let $h(\cdot)$ be a Borel measurable function. Define $f(t,x) = \mathbb{E}_{t,x}[e^{-r(T-t)}h(X(T))]$ Then $f(t,x)$ satisfies the PDE

$$f_t(t,x) + b(t,x)f_x(t,x) + \frac{1}{2}\sigma^2(t,x)f_{xx}(t,x) = rf(t,x)\,,$$

with terminal condition $f(T,x) = h(x)$ for all x.

Proof: By a similar argument as in the previous theorem it can be shown that $e^{-rt}f(t,X(t))$ is a martingale. Now,

$$\begin{aligned}&d(e^{-rt}f(t,X(t)) = e^{-rt}df(t,X(t)) - re^{-rt}f(t,X(t))\\ &= e^{-rt}(-rf(t,X(t)) + f_t(t,X(t)) + b(t,X(t))f_x(t,X(t)) + \frac{1}{2}\sigma^2(t,X(t))f_{xx}(t,X(t)))dt\\ &+ e^{-rt}f_x(t,X(t))\sigma(t,X(t))dW(t)\,.\end{aligned}$$

So in order to have the dt term equal to 0 we must have,

$$f_t(t,x) + b(t,x)f_x(t,x) + \frac{1}{2}\sigma^2(t,x)f_{xx}(t,x) = rf(t,x)\,.$$

□

Application to BSM model: The risk neutral valuation of an European call is given by

$$V(t) = \tilde{\mathbb{E}}[e^{-r(T-t)}(S(T)-K)^+|\mathcal{F}_t]\,,$$

where $S(t)$ satisfies,

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t)\,.$$

Thus $S(t)$ is a Markov process. Hence $V(t) = c(t,S(t))$ where

$$c(t,x) = \tilde{\mathbb{E}}_{t,x}[e^{-r(T-t)}(S(T)-K)^+].$$

So by discounted Feynman-Kac formula, $c(t,x)$ satisfies

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2x^2c_{xx}(t,x) = rc(t,x)\,,$$

with terminal condition $c(T,x) = (x-K)^+$.