MA 322: Scientific Computing

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CHAPTER 2: ROOT FINDINGS

Fixed-point method

Theorem

Assume that $g(x) \in C([a, b])$, *that* $g([a, b]) \subset [a, b]$ *(We say,* g *sends* $[a, b]$ *onto* $[a, b]$*). Then* $x = g(x)$ *has at least one solution in* $[a, b]$.

Proof.

Apply intermediate value theorem on $f(x) = g(x) - x$. Note that $f \in C([a, b])$. $\square$

Fixed-point method

Theorem (Contraction Mapping Theorem)

Assume that $g(x) \in C([a,b])$, *that* $g([a,b]) \subset [a,b]$. *Furthermore, assume there is a constant* $0 < \lambda < 1$, *with*

$$|g(x) - g(y)| \leq \lambda |x - y|, \qquad \forall x, y \in [a,b].$$

Then $x = g(x)$ *has a solution* $\alpha \in [a,b]$. *Also, the iterates*

$$x_n = g(x_{n-1}) \qquad n \geq 1$$

will converge to α *for any choice of* $x_0 \in [a,b]$, *and*

$$|\alpha - x_n| \leq \frac{\lambda^n}{1-\lambda} |x_1 - x_0|.$$

Fixed-point method

Theorem

Assume that $g(x) \in C'([a,b])$, *that* $g([a,b]) \subset [a,b]$, *and that*

$$\lambda := \max_{a \le x \le b} |g'(x)| < 1.$$

Then

1. $x = g(x)$ *has a unique solution* α *in* $[a,b]$.
2. *For any choice of* $x_0 \in [a,b]$, *with* $x_{n+1} = g(x_n)$, $n \ge 0$,
$$\lim_{n\to\infty} x_n = \alpha.$$
3. $$|\alpha - x_n| \le \lambda^n |\alpha - x_0| \le \frac{\lambda^n}{1-\lambda}|x_1 - x_0| \quad \textit{and} \quad \lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

Fixed-point method

Theorem

Assume α is a root of $x = g(x)$, and suppose that $g(x)$ is continuously differentiable in some neighbouring interval of α with $|g'(\alpha)| < 1$. Then the results of the previous theorem are still true, provided x_0 is chosen sufficiently close to α.

Higher order one-point method

Theorem

Assume α is a root of $x = g(x)$, and that $g(x)$ is p times continuously differentiable for all x near α, for some $p \geq 2$. Furthermore, assume

$$g'(\alpha) = \cdots = g^{(p-1)}(\alpha) = 0.$$

Then if the initial guess x_0 is chosen sufficiently close to α, the iteration

$$x_{n+1} = g(x_n) \qquad n \neq 0$$

will have order of convergence p, and

$$\lim_{n \to \infty} \frac{\alpha - x_{n+1}}{(\alpha - x_n)^p} = (-1)^{p-1} \frac{g^{(p)}(\alpha)}{p!}.$$

Multiple roots

Part of Lab - 2 assignment

Zeros of Polynomials: Stability Problem

- Let
$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \qquad a_n \neq 0$$
$$q(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$
and define a perturbation of $p(x)$ by $p(x;\epsilon) := p(x) + \epsilon q(x)$.
- Let $z_j(\epsilon)$, $1 \leq j \leq n$ denotes the zeros of $p(x;\epsilon)$, *repeated according to their multiplicity*, and let $z_j = z_j(0)$, $1 \leq j \leq n$ denote the corresponding n zeros of $p(x) = p(x;0)$.
- The following approximation holds,

$$z_j(\epsilon) \approx z_j - \gamma\epsilon \qquad \text{where}$$

$$\gamma = -\frac{q(z_j)}{p'(z_j)} \quad \text{for simple zeros,} \qquad \gamma^m = -\frac{m!\,q(z_j)}{p^{(m)}(z_j)} \quad \text{for zeros of multiplicity } m.$$

System of nonlinear equations: Fixed point theory

The iteration formula is

$$\mathbf{x}_{n+1} = \mathbf{g}(\mathbf{x}_n)$$

and the error formula is

$$\boldsymbol{\alpha} - \mathbf{x}_{n+1} = \mathbf{G}_n(\boldsymbol{\alpha} - \mathbf{x}_n),$$

where

$$\mathbf{G}_n = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{pmatrix}$$

is the Jacobian of $\mathbf{g}$ computed at some point lying on the line segment joining $\boldsymbol{\alpha}$ and $\mathbf{x}_n$.

System of nonlinear equations: Newton's method

The iteration formula is

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \mathbf{F}(\mathbf{x}_n)^{-1}\mathbf{f}(\mathbf{x}_n)$$

where

$$\mathbf{F}(x_n) = \begin{pmatrix} \frac{\partial f_1(\mathbf{x}_n)}{\partial x_1} & \frac{\partial f_1(\mathbf{x}_n)}{\partial x_2} \\ \\ \frac{\partial f_2(\mathbf{x}_n)}{\partial x_1} & \frac{\partial f_2(\mathbf{x}_n)}{\partial x_2} \end{pmatrix}$$

is the Jacobian of $\mathbf{f}$ computed at $\mathbf{x}_n$.

Chapter 3: Interpolation

Motivation and Preliminaries

- Laboratory data
- Satellite data
- Historical data
- Vandermonde matrix

$$\mathbb{V} = \begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{bmatrix}_{(n+1)\times(n+1)}$$

Polynomial interpolation: Newton's divided-difference formula

- Two-points (linear) interpolation

$$p_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

- Three-points (quadratic) interpolation

$$\begin{aligned} p_2(x) = f(x_0) &+ \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_1 - x_0) \\ &+ \frac{(x_1 - x_0)f(x_2) + (x_2 - x_0)f(x_1) + (x_2 - x_1)f(x_0)}{(x_2 - x_0)(x_1 - x_0)(x_2 - x_1)}(x - x_0)(x - x_1) \end{aligned}$$

Polynomial interpolation: Lagrange's formula

- Lagrange's formula for polynomial interpolation is

$$p_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x) \qquad \text{where}$$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j} \right)$$

Theorem

Let $x_0, x_1, \ldots, x_n$ be distinct real numbers, and let f be a given real valued function with $n+1$ continuous derivatives on the interval $I_t = \mathcal{H}\{t; x_0, \ldots, x_n\}$ (i.e., $f \in C^{(n+1)}(I_t)$), with t some given real number. Then $\exists\ \xi \in I_t$ with

$$f(t) - \sum_{i=0}^{n} f(x_i) l_i(x) = \frac{(t - x_0)(t - x_1) \cdots (t - x_n)}{(n+1)!} f^{(n+1)}(\xi).$$

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