① Let A be the event that a person has COVID-19 infection.

And, let B_i be the event that i^{th} test is positive for $i = 1, 2$.

So,

$$P(A) = 0.1$$

and $P(B_1 | A) = 0.9$, $P(B_2 | A) = 0.9$

$$P(B_i | A^c) = 1 - P(B_i^c | A^c) = 1 - 0.9 = 0.1$$

Let $B = B_1 B_2$, where B denote both tests are positive.

So, $P(B|A) = P(B_1|A) \cdot P(B_2|A)$ (independent test given disease status)

$$= 0.9 \times 0.9$$
$$= 0.81$$

Similarly, $P(B|A^c) = P(B_1|A^c) \, P(B_2|A^c)$

$$= 0.1 \times 0.1$$
$$= 0.01$$

Now, the probability that the person has the COVID-19 infection given both the tests are positive, using Bayes Theorem, is

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)}$$

$$= \frac{0.81 \times 0.1}{0.81 \times 0.1 + 0.01 \times 0.9}$$

$$= \frac{0.081}{0.081 + 0.009}$$

$$= \frac{0.081}{0.090}$$

$$= \frac{9}{10}$$

$$= 0.9$$

(2)

(a) Recurrence relation betⁿ p_n and p_{n-1} for $n \geq 1$.

Let A_n denote the events that even no. of ~~heads~~ ~~tosses~~ heads has occurred after n tosses.

So,
$$p_n = P(A_n)$$
$$= P(A_n | A_{n-1}) P(A_{n-1}) + P(A_n | A_{n-1}^c) P(A_{n-1}^c)$$
$$= (1-p) p_{n-1} + p(1-p_{n-1})$$
$$= p + (1-2p) p_{n-1}$$

Hence $p_n = p + (1-2p) p_{n-1}$.

(b) Find the value of p_{21} in terms of p.

$$p_{21} = p + (1-2p) p_{20}$$
$$= p + (1-2p)p + (1-2p)^2 p_{19}$$
$$\vdots$$
$$= p + (1-2p)p + \cdots + (1-2p)^{20} p + (1-2p)^{21}$$
$$= \frac{1}{2} + \frac{1}{2}(1-2p)^{21}.$$

Hence $p_{21} = \frac{1}{2} + \frac{1}{2}(1-2p)^{21}$.

(3) $\Omega = [0,1] \times [0,1]$.

$X(\omega)$: distance betn $\omega \in \Omega$ and nearest edge of the unit square.

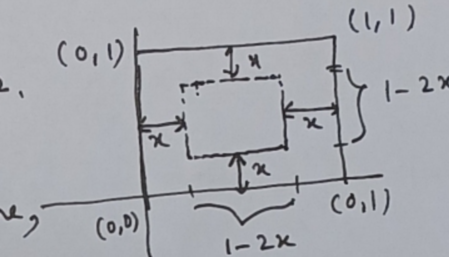
The CDF, $F_X(x) = P(X \leq x)$.

For $x < 0$, $P(X \leq x) = 0$ as the distance cannot be negative.

It is clear that, $0 \leq X \leq 1/2$.

For $x \geq \frac{1}{2}$, $P(X \leq x) = 1$ because the greatest value of X is $1/2$.

$X = \frac{1}{2}$, when we select the center of the unit square.



Now, for $0 \leq x < \frac{1}{2}$, from the Figure, it is clear that,

$$P(X > x) = (1-2x)^2$$

$$\Rightarrow P(X \leq x) = 1 - (1-2x)^2, \text{ for } 0 \leq x < 1/2$$

Thus, the CDF of X is

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-(1-2x)^2 & \text{if } 0 \leq x < 1/2 \\ 1 & \text{if } x \geq 1/2. \end{cases}$$