**Niels Henrik Abel**
**1802-1829**

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**Abel's Theorem:** There are no formulas for finding the roots of generic polynomial of degree greater than 4.

# Power Method and its Variations

# Power Method

Let $A \in \mathbb{C}^{n\times n}$ be diagonalizable with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$ satisfying

$$|\lambda_1| \gneq |\lambda_2| \geq \cdots \geq |\lambda_n|$$

and let $v_1, \ldots, v_n \in \mathbb{C}^n \setminus \{0\}$ such that $Av_i = \lambda_i v_i, i = 1, 2, \ldots n$. $\lambda_1$ is called the dominant eigenvalue of $A$ and $v_1$ a corresponding dominant eigenvector.

Let $x \in \mathbb{C}^n$ such that $x = c_1 v_1 + \cdots + c_n v_n$ with $c_1 \neq 0$. Then,

$$\left\| A^j(x)/\lambda_1^j - c_1 v_1 \right\| \to 0 \text{ as } j \to \infty.$$

$$x \quad \frac{Ax}{\lambda_1}, \quad \frac{A^2 x}{\lambda_1^2}, \ldots$$

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$$\left\| A^j(x)/\lambda_1^j - c_1 v_1 \right\| \to 0 \text{ as } j \to \infty.$$

Moreover, if $c_2 \neq 0$ and $|\lambda_2| > |\lambda_3|$, then the convergence is linear at the rate $|\lambda_2|/|\lambda_1|$, i.e.,

$$\lim_{j\to\infty} \frac{\left\| A^{(j+1)}(x)/\lambda_1^{(j+1)} - c_1 v_1 \right\|}{\left\| A^j(x)/\lambda_1^j - c_1 v_1 \right\|} = \frac{|\lambda_2|}{|\lambda_1|}$$

(Ex: Prove the above limit!)

# Power Method

Let $x = [x_1, x_2, \cdots x_n]^T \in \mathbb{C}^n \setminus \{0\}$ be arbitrarily chosen. Set $q_0 = x/s_0$ where $s_0 = x_i$ such that $|x_i| = \|x\|_\infty$.

*for* $j = 1, 2, \ldots$

*Set* $\hat{q}_j = A(q_{j-1})$

*Find* $s_j = \hat{q}_j(i)$ *such that* $|\hat{q}_j(i)| = \|\hat{q}_j\|_\infty$.

*Set* $q_j = \hat{q}_j / s_j$.

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If $x = c_1 v_1 + \cdots + c_n v_n$ with with $c_1 \neq 0$, then

(i) $\lim_{j\to\infty} q_j = \hat{v}_1$, where $A\hat{v}_1 = \lambda_1 \hat{v}_1$, with $\|\hat{v}_1\|_\infty = 1$, and $\hat{v}_1(j) = 1$ for *some* $1 \leq j \leq n$.

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(Ex: Prove these!)

# Power Method

Further, if $c_1, c_2 \neq 0$ and $|\lambda_1| > |\lambda_2| > |\lambda_3|$, then $\{q_j\}$ converges to $\hat{v}_1$ linearly at the rate $\frac{|\lambda_2|}{|\lambda_1|}$, that is,

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The Power Method is used to compute a dominant eigenvector of the massive non-negative Google Matrix in Google's PageRank Algorithm. For details see:

K. Bryan and T. Leise. *The $25,000,000,000 Eigenvector: The Linear Algebra behind Google.* SIAM Rev., 48(3), 569-581.

# Shift and Invert Method

**Best idea:** Find a permutation matrix $P$, a unit lower triangular matrix $L$ and upper triangular matrix $U$ such that

$$P(A - \rho I) = LU$$

*for* $j = 1, 2, \ldots$
  *Set* $b = Pq_{j-1}$
  *Solve* $Ly = b$ *for* $y$ *(Costs $n^2$ flops)*
  *Solve* $U\hat{q}_j = y$ *for* $\hat{q}_j$ *(Costs $n^2$ flops)*
  *Find* $s_j = \hat{q}_j(i)$ *such that* $|\hat{q}_j(i)| = \|\hat{q}_j\|_\infty$.
  *Set* $q_j = \hat{q}_j / s_j$.