Note: This document is a part of the lectures given during the Winter 2024 Semester

Risk-Neutral Pricing

Recall that for the binomial model:

(A) Under the risk-neutral probabilities $\widetilde{p}$ and $\widetilde{q}$:

$$S_0 = \frac{1}{1+r}\left[\widetilde{p}S_0u + \widetilde{q}S_0d\right].$$

(B) Under actual probabilities p and q:

$$S_0 < \frac{1}{1+r}\left[pS_0u + qS_0d\right].$$

Let us more generally consider a finite sample space Ω on which we have two probability measures $\mathbb{P}$ and $\widetilde{\mathbb{P}}$. Let us assume that $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ both give positive probability to every element of Ω, so we can form the quotient:

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}.$$

Because it depends on the outcome ω of a random experiment, Z is a random variable. It is called the *Radon-Nikodym derivative of* $\widetilde{\mathbb{P}}$ *with respect to* $\mathbb{P}$, although in this context of a finite sample space Ω, it is really a quotient rather than a derivative.

Theorem

Let $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ be probability measure on a finite sample space Ω. Assume that $\mathbb{P}(\omega) > 0$ and $\widetilde{\mathbb{P}}(\omega) > 0$ for every $\omega \in \Omega$, and define the random variable Z by:

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}.$$

Then we have the following:

(A) $\mathbb{P}(Z > 0) = 1$.

(B) $EZ = 1$.

(C) For any random variable Y, $\widetilde{E}Y = E[ZY]$.

Theorem

Consider the binomial model with N periods. Let $\Delta_0, \Delta_1, \ldots, \Delta_{N-1}$ be an adapted portfolio process, let X_0 be a real number, and let the wealth process $X_1, X_2, \ldots, X_N$ be generated recursively by:

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - \Delta_n S_n), n = 0, 1, \ldots, N-1.$$

Then the discounted wealth process $\frac{X_n}{(1+r)^n}$, $n = 0, 1, \ldots, N$, is a martingale under the risk-neutral measure, i.e.,

$$\frac{X_n}{(1+r)^n} = \widetilde{E}_n\left[\frac{X_{n+1}}{(1+r)^{n+1}}\right], n = 0, 1, \ldots, N-1.$$

Change of Measure

Recall the positive random variable Z to change the probability measure on a space Ω. We need to do this when we change from the actual probability measure $\mathbb{P}$ to the risk-neutral probability measure $\widetilde{\mathbb{P}}$ in models of financial markets.

Theorem

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and Z be an almost surely non-negative random variable with $EZ = 1$. For $A \in \mathcal{F}$, we define:

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega).$$

Then $\widetilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a non-negative random variable, then $\widetilde{E}X = E[XZ]$. If Z is almost surely positive, we also have:

$$EY = \widetilde{E}\left[\frac{Y}{Z}\right].$$

for any non-negative random variable Y.

In this Theorem, we began with a probability $(\Omega, \mathcal{F}, \mathbb{P})$ and a non-negative random variable Z satisfying $EZ = 1$. We then defined a new probability measure, $\widetilde{\mathbb{P}}$ by the formula:

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}.$$

Any random variable X now has two expectations, one under the original probability measure $\mathbb{P}$, which we denote by EX, and the other under the new probability measure $\widetilde{\mathbb{P}}$, which we denote by $\widetilde{E}X$. They are related by $\widetilde{E}X = E[XZ]$. If $\mathbb{P}[Z > 0] = 1$, then $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ agree which sets have probability zero and we have the companion formula, $EX = \widetilde{E}\left[\frac{X}{Z}\right]$. We say that Z is a *Radon-Nikodym derivative* of $\widetilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write:

$$Z = \frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}.$$

This is supposed to remind us that Z is like a ratio of these two probability measure. In case of a finite probability model we actually have:

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}.$$

If we multiply both sides by $\mathbb{P}(\omega)$ and then sum over ω in a set A, we obtain:

$$\widetilde{\mathbb{P}}(A) = \sum_{\omega \in A} Z(\omega)\mathbb{P}(\omega) \text{ for all } A \subset \Omega.$$

In a general probability model, we cannot write $Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}$ because $\mathbb{P}(\omega)$ is typically zero for each individual ω, but we can write an analogous:

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F},$$

of

$$\widetilde{\mathbb{P}}(A) = \sum_{\omega \in A} Z(\omega)\mathbb{P}(\omega) \text{ for all } A \subset \Omega.$$

<u>Example</u>

We can use this change of measure idea to move the mean of a normal random variate. In particular, if X is a standard normal random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, θ is a constant and we define:

$$Z = \exp\left\{-\theta X - \frac{1}{2}\theta^2\right\},$$

then under the probability measure $\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$ for all $A \in \mathcal{F}$, the random variable $Y = X + \theta$ is normal. In particular, $\widetilde{E}Y = 0$, whereas $EY = EX + \theta = \theta$. By changing the probability measure, we have changed the expectation of Y.

We now perform a similar change in measure in order to change a mean, but this time for a whole process rather than for a single random variable. To set the stage, suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\mathcal{F}(t)$, defined for $0 \leq t \leq T$, where T is a final fixed time. Suppose further that Z is almost surely positive random variable satisfying $EZ = 1$, and we define $\widetilde{\mathbb{P}}$ by:

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F}.$$

Recall : Let us study this in discrete time.

<u>Theorem</u>

Let Z be a random variable in a N-period binomial model. Define $Z_n = E_n Z, n = 0, 1, \ldots, N$. Then $Z_n, n = 0, 1, \ldots, N$, is a martingale under $\mathbb{P}$.

<u>Definition</u>

In an N-period binomial model, let $\mathbb{P}$ be the actual probability measure, $\widetilde{\mathbb{P}}$ the risk-neutral probability measure, and assume that $\mathbb{P}(\omega) > 0$ and $\widetilde{\mathbb{P}}(\omega) > 0$ for every sequence of coin tosses ω. Define the Radon-Nikodym derivative (random variable):

$$Z(\omega) = \frac{\widetilde{\mathbb{P}}(\omega)}{\mathbb{P}(\omega)}.$$

The Radon-Nikodym derivative process is $Z_n = E_n[Z], n = 0, 1, \ldots, N$. In particular $Z_N = Z$ ad $Z_0 = 1$.

We can then define the *Radon-Nikodym derivative process:*

$$Z(t) = E[Z|\mathcal{F}(t)], 0 \leq t \leq T.$$

The Radon-Nikodym derivative process is a martingale because of iterated conditioning. For $0 \leq s \leq t \leq T$:

$$E[Z(t)|\mathcal{F}(s)] = E[E[Z|\mathcal{F}(t)]|\mathcal{F}(s)] = E[Z|\mathcal{F}(s)] = Z(s).$$

Lemma

Let t satisfying $0 \leq t \leq T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then:

$$\widetilde{E}Y = E[YZ(t)].$$

Lemma

Let s and t satisfying $0 \leq s \leq t \leq T$ be given and let Y be an $\mathcal{F}(t)$-measurable random variable. Then:

$$\widetilde{E}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)} E[YZ(t)|\mathcal{F}(s)].$$

Girsanov Theorem in One Dimension

Let $W(t), 0 \leq t \leq T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathcal{F}(t), 0 \leq t \leq T$, be a filtration for this Brownian motion. Let $\Theta(t), 0 \leq t \leq T$, be an adapted process. We define:

$$Z(t) = \exp\left\{ -\int_0^t \Theta(u) dW(u) - \frac{1}{2}\int_0^t \Theta^2(u) du \right\},$$

and

$$\widetilde{W}(t) = W(t) + \int_0^t \Theta(u) du.$$

Assume that:

$$E\int_0^T \Theta^2(u) Z^2(u) du < \infty.$$

Set $Z = Z(T)$. Then $EZ = 1$ and under the probability measure $\widetilde{\mathbb{P}}$ given by:

$$\widetilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega) \text{ for all } A \in \mathcal{F},$$

the process $\widetilde{W}(t), 0 \leq t \leq T$, is a Brownian motion.