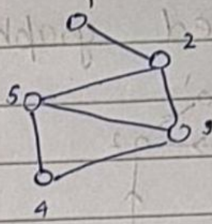
Graphs:



deg(1) = 1
deg(2) = 3

$G = (V, E)$

$E = \{ \langle x, y \rangle / x, y \in V \}$

$\phi : E \to V \times V$

$V = \langle 1, 2, 3, 4, 5 \rangle$

$E = \{ \langle 1, 2 \rangle, \langle 2, 3 \rangle, \langle 2, 5 \rangle, \langle 3, 4 \rangle, \langle 3, 5 \rangle, \langle 4, 5 \rangle \}$

degree of a vertex: The ~~no. of~~ #edges incident on it

$$\boxed{\sum_{v \in V} d(v) = 2|E|}$$ Handshaking lemma:

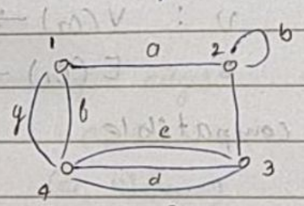
- Every edge contribute 2 deg.
- $S = \{ (v, e) \ / \ v \in V, \ e \in E \}$

  $|S| = \sum_{v \in V} d(v)$ // # define v θ

  $\Rightarrow |S| = \sum_{e \in E} 2 = 2|E|$

Multigraphs:

Simple graph with multiple edges & loops



$$A = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 0 & 2 \\ 2 & 1 & 2 & 1 & 0 \\ 3 & 0 & 1 & 0 & 3 \\ 4 & 2 & 0 & 3 & 0 \end{array}$$

~~Adjancy M~~

Adjacency matrix: is an $n \times n$ matrix $A = (a_{ij})$

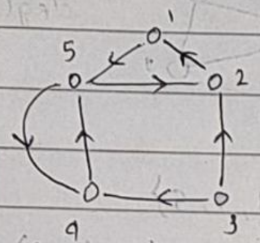
$n = |V|$ $\quad a_{ij} = \begin{cases} 1, & \langle i, j \rangle \in E \\ 0, & \text{ow} \end{cases}$

Incidence matrix:

$|V| \times |E|$ matrix, $B = (b_{ij})$

Pg[6]...

Directed graph:



$V = \{1,2,3,4,5\}$

$E = \{(2,1), (1,5), (3,2), (3,4), (4,5), (5,2)$
$(5,4)\}$

$\sum$ indegree $= |E|$

outdegree $= |E|$

Complete graph: K_n

- n vertices and all possible edges

$\binom{n}{2}$ edges

How many simple graphs on 'n' vertices? $2^{\binom{n}{2}}$

$\{\{u,v\} \mid u,v \in V,\ u \neq v\}$

for directed ~~$\binom{n}{2}$~~ $4^{\binom{n}{2}}$

Isomorphic graphs:

$G \cong H$ if $\exists$ a bijection $\nu : V(G) \to V(H)$

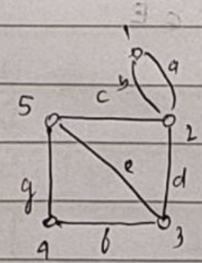
$\epsilon : E(G) \to E(H)$

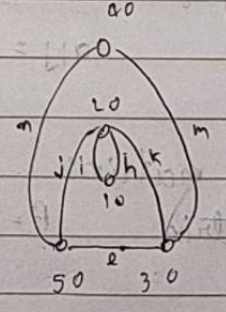
that are ~~compita~~ compatible

undirected: Every edge $e = \{x,y\}$ in G has $\epsilon(e) = \{\nu(x), \nu(y)\}$ in H

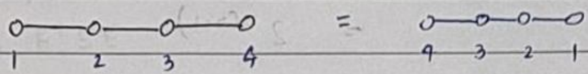
directed: " " $e = (x,y)$ in " $\epsilon(e) = (\nu(x), \nu(y))$ in H

Eg: G:



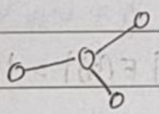


Path graph: P_4



#ways to label $\frac{n!}{2}$

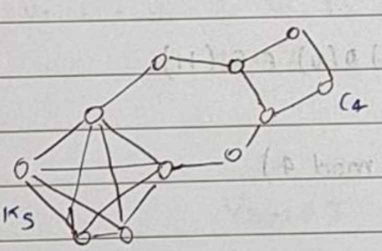
Star graph: S_4

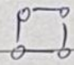


n

Group action, orbit stablizers:

subgraph:





$G = (V(G), E(G))$

$H \subseteq G$

$H = (V(H), E(H))$

Walks:

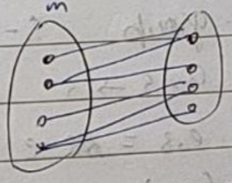
Sequence of edges

$e_1, e_2, \ldots\ldots\ e_k$

$(u, v)(v, w)$

no repetition of edges: Trail

no rep of edges, vertex: path.



cycle: closed path

$K_{m,n}$ complete bipartate graph

~~Biparte~~

Bipartite graph: $G = (V, E)$

$V = V_1 \cup V_2 \qquad V_1 \cap V_2 = \phi$

There are no edges b/w any two vertices in the same partition.

Simple undirected graph on n vertices $\Big\} = 2^{\binom{n}{2}}$

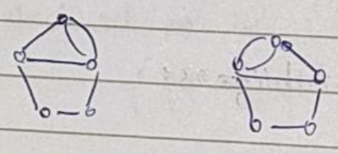
directed $= 2^{n(n-1)}$

$G \cong H$ $\quad v: V(G) \to V(H)$

$\quad \epsilon: E(G) \to E(H)$

$e_1 = (u, v_1)$ $\quad \epsilon(e_1) = \{v(u_1), v(v_1)\}$

$G \cong H$, then $|V(G)| = |V(H)|$ & $|E(G)| = |E(H)|$

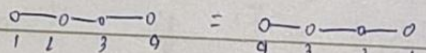


$\theta : V(G) \to V(H)$

$uv \in E(G) \iff \theta(u)\theta(v) \in E(H)$

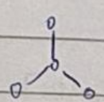
$G \cong \bar{G}$, $\quad |V(G)| \equiv 0, 1 \pmod 4$

P_n - Path vertices: $\quad n!/2$



S_n - star $\quad n$

S_4



$\frac{n!}{(n-1)!} = n$

G. group $\quad$ S-set

$G \times S \to S$ $\quad$ Satisfying $g.s$

(i) $e.s = e$

(ii) $(gh).s = g.(hs)$

orbit of $s \in S$ $\quad G.s = \{g.s \mid g \in G\}$

stablizer $\quad G_s = \{g \in G \mid g.s = e\}$

classmate

Date ______
Page ______

$G = (V, E)$

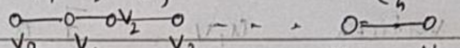
$H = (V', E')$

$V' \subseteq V$, $E' \subseteq E$

walks, trails, paths

cycle - closed path

$e_1 \quad e_2 \quad e_3 \qquad\qquad e_n$



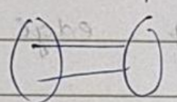
$v_0 \quad v_1 \quad v_2 \quad v_3 \qquad v_{n-1} \quad v_n$

$e_1, e_2, \ldots, e_n$

lenght of the walk = # edges

$G = (V, E)$

$V = X \cup Y$, $X \cap Y = \phi$



$X \qquad Y$

Thm A graph is bipartite iff it contains no odd cycles

Pf $v_0\ v_1\ v_2 \ldots v_n\ v_0$

$v_0 \in X$ $\qquad v_1 \in Y$

$v_2 \in X$

$\vdots$

$v_{2i} \in X$ $\qquad v_{2i+1} \in Y$

$\Rightarrow n =$ odd

$u \in V(G)$

shortest path

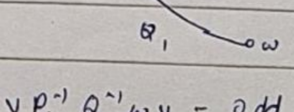
$X = \{x \in V \mid d(u, x) \text{ is even}\}$

$Y = \{y \in V \mid d(u, y) \text{ is odd}\}$

P, Q, even lenghts

$v, w \in X$

$u \qquad P_{u_1} \qquad P_1$



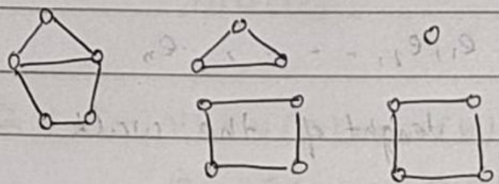
Q_1

$v P_1^{-1} Q_1^{-1} w v$ - odd cycle

P_1, Q_1 are of same parity

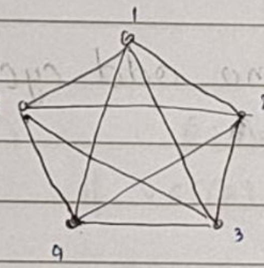
Connected graph:

$\forall$ vertices u, v, $\exists$ a u-v path



Hamilton path: path that visits every vertex exactly once.

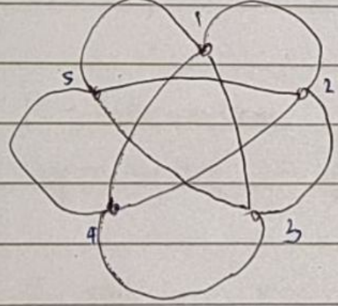
Eulerian trail. trail that ~~wit~~ uses every edge exacty once



1, 2, 3, 4, 5 Hamilton path

1, 2, 3, 4, 5, 1 Hamilton cycle

Graph is called Hamilton if it contains Hamilton cycle



Eulerian cycle

1, 2, 3, 4, 5, 1, 3, 5, 2, 4, 1

Graph is Eulerian if it contains Eulerian cycle.

(laplacian matrix)

Cayley's Formula:

trees with vertices from $[n] = n^{n-2}$

- Pruger
- Andre Joyal

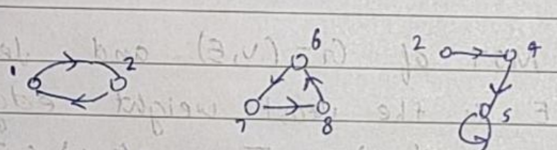
$n^n = n^2 A_n \rightarrow$ doubly rooted trees - head Tail

$\downarrow$ funtions from $[n] \rightarrow [n]$

~~A=~~ $n = 8$, $f : [8] \rightarrow [8]$

$f(1) = 3$ $f(2) = 4$ $f(3) = 1$ $f(4) = 5$

$f(5) = 5$ $f(6) = 7$ $f(7) = 8$ $f(8) = 6$

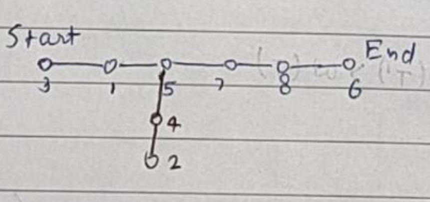


$C_1 < C_2 \ldots < C_k$

$C = \{1, 3, 5, 6, 7, 8\}$

$d_i = f(c_i)$

$D = \{3, 1, 5, 7, 8, 6\}$



$f(2) = 4$ $f(4) = 5$

$f(1) = 3$ $f(3) = 1$ $f(5) = 5$

$f(6) = 7$ $f(7) = 8$ $f(8) = 6$

Krushkal's Algorithm:

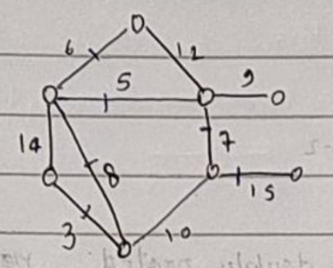
A Connected graph $G = (V, E)$ $\omega : E \rightarrow \mathbb{R}$

Q^n: min weighted Spanning tree

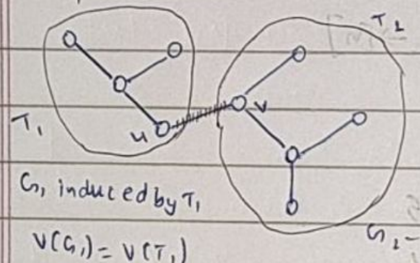
T - spanning tree

$V(T) = V(G)$

$w(T) = \sum_{e \in T} \omega(e)$



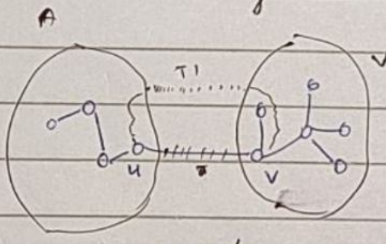
G, T- MST



$T' = T_1' \cup T_2' \cup \{(u,v)\}$

$w(T') < w(T)$

Theorem: Let T be a MST of $G = (V, E)$ and let $A \subseteq V$. Suppose $(u, v) \in E$ is the least weight edge connecting A and ~~V~~ $V \setminus A$. Then $(u,v) \in T$



$(u, v) \notin T$ $\quad w(T') > w(T)$

Matchings: $G = (V, E)$

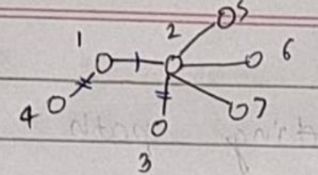
- An independent set of edges (not share
  $M \subseteq E$

$(u,v) \in M$ $\quad$ M - Saturates u & v

Maximal matching: (no more edges can be added) $\quad$ unsaturated

Maximum Matching:
- Matching of maximum cardinality

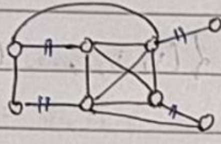
Perfect matching:
- Matching that saturates all vertices in G



$\{(2,5)\}$ M

$\{(1,2)\}$ Maximal

$\{(1,4),(2,3)\}$ maximum



complete graph on $[n]$ → K_n - # perfect matchings

K_{2n} - $\frac{(2n)!}{2^n n!}$ $\quad K_{2n+1}$ - 0

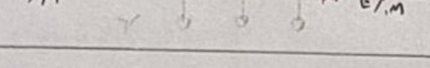
$f(n) = (2n-1) f(n-1)$

Theorem: $\left(\text{M is a maximal matching in G}\right) \iff \left(\text{G has no M-augmenting path}\right)$

M-alternating path: path, edges alternate b/w M and $E \setminus M$

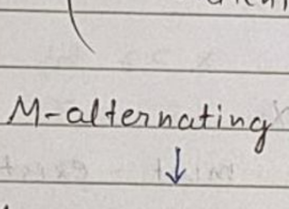
↓

M-augmenting path: M-alternating that, ~~Sata~~ If end points are unsaturated, it is called a M-augmenting path.



Lemma: M, M' - matching in $G=(V,E)$. Every component of $M \Delta M'$ is a path or an even cycle.

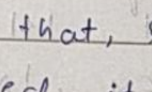
Proof:



$F = M \Delta M'$

$\deg(v) \le 2 \ \forall \ v \in F$

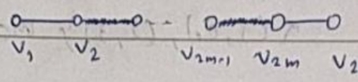
suppose not, $\exists \ v \in F \ni \deg(v) \geq 3$



$\Rightarrow\Leftarrow$

(=>) M is a max matching

Suppose G has an M-augmenting path



$M' = (M \setminus \{(v_2, v_3) \dots (v_{2m-1}, v_{2m})\}) \cup \{(v_1, v_2), \dots (v_{2m}, v_{2m+1})\}$

$|M'| = |M| + 1$

$\Rightarrow\Leftarrow$

$\exists$ a matching M' larger than M

- construct an M-augmenting path

$F = M \Delta M'$. By prev lemma

F has only paths and even cycles

$|M'| > |M|$

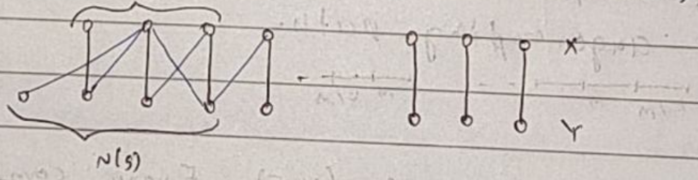
Perfect matching in a bipartite graph:

G - bipartite graph (X, Y)

M saturates every vertex in X

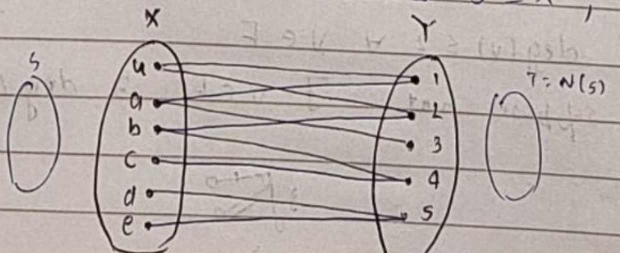
Then for every $S \subseteq X$, there must exist atleast $|S|$ edges that have neighbour in S

i.e) $\forall S \subseteq X$, $|N(S)| \geq |S|$ (Hall's cond^n)



($\Leftarrow$): contrapositive

if M is any maximum matching in G which doesn't saturates X then $\exists S \subseteq X$, $|N(S)| < |S|$.



$u \in X$ M- unsaturated vertex

$V_u = \{$ all vertices that are reachable from u by an M-alternating path $\}$

$S = V_u \cap X$

$T = V_u \cap Y$

$|T| = |S \setminus \{u\}| = |S| - 1$

$|N(S)| = |T| = |S| - 1 < |S|$

$T \subseteq N(S)$

Cor: $k > 0$, every k-regular ($\deg(v) = k \; \forall v \in V$) bipartite graph has a perfect matching.

Proof: G is k-regular

$k|X| = |E| = k|Y|$

$|X| = |Y| \quad (\because k > 0)$

Let $S \subseteq X$

$E_1 =$ Set of edges incident with vertices in S

$E_2 =$ " $N(S)$

$E_1 \subseteq E_2$

$k|N(S)| = |E_2| \geq |E_1| = k|S|$

$\therefore |N(S)| \geq |S|$

By Hall's theorem, G has a matching that saturates every vertex in X

$\because |X| = |Y|$, it is a perfect ~~macting~~. matching.