MA 201 Complex Analysis
Lecture 7: Complex Integration

Integral of a complex valued function of real variable:

- **Definition:** Let $f : [a, b] \to \mathbb{C}$ be a function. Then $f(t) = u(t) + iv(t)$ where $u, v : [a, b] \to \mathbb{R}$.Define,

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

If $U' = u$ and $V' = v$ and $F(t) = U(t) + iV(t)$ then by fundamental theorem of calculus $\int_a^b f(t)dt = F(b) - F(a)$.

- For $\alpha \in \mathbb{R}$, $\int_a^b e^{i\alpha t}dt = \dfrac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}$.
- $\int_0^1 (1 + it)^2\, dt = \int_0^1 (1 - t^2)\, dt + i\int_0^1 2t\, dt = \dfrac{2}{3} + i$.
- If $f : [a, b] \to \mathbb{C}$ piecewise continuous then $\int_a^b f(t)dt$ exists.

Complex integration

- $\text{Re}\left(\int_a^b f(t)dt\right) = \int_a^b \text{Re}\,(f(t))dt.$
- $\text{Im}\left(\int_a^b f(t)dt\right) = \int_a^b \text{Im}\,(f(t))dt.$
- $\int_a^b [f(t) \pm g(t)]dt = \int_a^b f(t)dt \pm \int_a^b g(t)dt.$
- $\int_a^b \alpha f(t)dt = \alpha \int_a^b f(t)dt, \quad \alpha \in \mathbb{C}$
- $\int_a^b f(t)dt = -\int_b^a f(t)dt.$
- $\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$

Complex integration

- **Orientation:** Let γ be a simple closed contour with parametrization $\gamma(t),\ t \in [a, b]$. As t moves from a to b, the curve γ moves in a specific direction called the orientation of the curve induced by the parametrization.

- **Convention:** If the interior bounded domain of γ is kept on the left as t moves from a to b, then we say the orientation is in the **positive sense** (counter clockwise or anticlockwise sense). Otherwise γ is oriented **negatively** (clockwise direction).

- Let $\gamma : [a, b] \to \mathbb{C}$ be a curve then the curve with the reverse orientation is denoted as $-\gamma$ and is defined as

$$-\gamma : [a, b] \to \mathbb{C}, ; \quad -\gamma(t) = \gamma(b + a - t).$$

- $\gamma(t) = e^{it},\ t \in [0, 2\pi]$ (Positive orientation)

  where as $\gamma(t) = e^{i(2\pi - t)},\ t \in [0, 2\pi]$ (Negative orientation)

Complex integration

- Let γ be a piecewise smooth curve defined on $[a, b]$. The length of γ is given by
$$L(\gamma) = \int_a^b |\gamma'(t)| dt.$$

- **Definition:** Let $\gamma(t)$; $t \in [a, b]$, be a contour and f be complex valued continuous function defined on a set containing γ then the **line integral or the contour integral** of f along the curve γ is defined by
$$\int_\gamma f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt.$$

Complex integration

Example: Let $f(z) = \bar{z}$.

- If $\gamma_1(t) = e^{it}, \;\; t \in [0, \pi]$ then,
$$\int_{\gamma_1} \bar{z} dz = \int_0^{\pi} \overline{\gamma_1(t)} \gamma_1'(t) dt = \int_0^{\pi} e^{-it}(i) e^{it} dt = i\pi.$$
- If $\gamma_2(t) = 1(1-t) + t.(-1) = 1 - 2t, \;\; t \in [0, 1]$ then,
$$\int_{\gamma_2} \bar{z} dz = \int_0^{1} \overline{\gamma_2(t)} \gamma_2'(t) dt = \int_0^{1} [1 - 2t](-2) dt = 0.$$
- In the above example γ_1 and γ_2 are two paths joining 1 and -1. But the line integral along the paths γ_1 and γ_2 are NOT same.
- **Question:** When a line integral of f does not depend on path?

Complex integration

- (The fundamental integral) For $a \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{Z}$

$$\int_{C_{a,r}} (z-a)^n dz = \begin{cases} 0 & \text{if} \quad n \neq -1 \\ 2\pi i & \text{if} \quad n = -1 \end{cases}$$

  where $C_{a,r}$ denotes the circle of radius r centered at a.

- Let f, g be piecewise continuous complex valued functions then

$$\int_\gamma [\alpha f \pm g](z) dz = \alpha \int_\gamma f(z) dz \pm \int_\gamma g(z) dz.$$

- Let $\gamma : [a, b] \to \mathbb{C}$ be a curve and $a < c < b$. If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ then

$$\int_\gamma f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz.$$

- $\displaystyle\int_{-\gamma} f(z) dz = -\int_\gamma f(z) dz.$

Complex integration

ML-inequality:

- Let f be a piecewise continuous function and let γ be a contour. If $|f(z)| \leq M$ for all $z \in \gamma$ and $L =$length of γ then

$$\left|\int_{\gamma} f(z)dz\right| \leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \leq M \int_a^b |\gamma'(t)|dt = ML.$$

- Let $\gamma(t) = 2e^{it}, t \in [0, \frac{\pi}{2}]$ and $f(z) = \dfrac{z+4}{z^3-1}$. Then by ML-ineuqality

$$\left|\int_{\gamma} f(z)\,dz\right| \leq \frac{6\pi}{7}.$$

Antiderivatives

- Answer to the **Question:** When a line integral of f does not depend on path?
- **Definition:** The **antiderivative or primitive** of a continuous function f in a domain D is a function F such that $F'(z) = f(z)$ for all $z \in D$. The primitive of a function is **unique** up to an additive constant.
- **Theorem:** Let f be a continuous function defined on a domain D and $f(z)$ has antiderivative $F(z)$ in D. Let $z_1, z_2 \in D$. Then for any contour C lying in D starting from z_1, and ending at z_2 the value of the integral

$$\int_C f(z)dz$$

is **independent of the contour**.

Antiderivatives

- **Proof.** Suppose that C is given by a map $\gamma : [a, b] \to \mathbb{C}$. Then $\frac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t)$. Hence

$$\begin{aligned}\int_C f(z)dz &= \int_a^b f(\gamma(t))\gamma'(t)dt \\ &= \int_a^b \frac{d}{dt}F(\gamma(t))dt \\ &= F(\gamma(a)) - F(\gamma(b)) = F(z_2) - F(z_1).\end{aligned}$$

- When such F exists we write

$$\int_C f(z)dz = \int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} F'(z)dz = F(z_1) - F(z_2).$$

- In particular, $\displaystyle\int_C f(z)dz = 0$ if C is a closed contour.

Antiderivatives

1. $$\int_{z_1}^{z_2} z^2 dz = \frac{z_2^3 - z_1^3}{3}.$$

2. $$\int_{-i\pi}^{i\pi} \cos z dz = \sin(i\pi) - \sin(-i\pi) = 2\sin(i\pi).$$

3. $$\int_{-i}^{i} \frac{1}{z} dz = \text{Log }(i) - \text{Log }(-i) = \frac{i\pi}{2} - \frac{-i\pi}{2} = i\pi.$$

4. The function $\frac{1}{z^n}$, $n > 1$ is continuous on $\mathbb{C}^*$. If γ is a contour joining nonzero complex numbers $z_1,\ z_2$ not passing through origin then
$$\int_\gamma \frac{dz}{z^n} = -(n-1)\left(\frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}}\right).$$