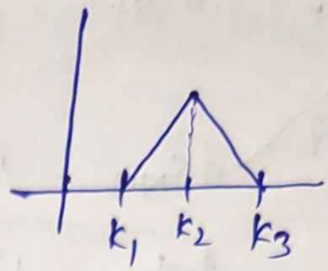
$$f(T) = \begin{cases} 0 & \text{if } S(T) \le K_1 \\ S(T) - K_1 & \text{if } K_1 < S(T) \le K_2 \\ K_3 - S(T) & \text{if } K_2 < S(T) < K_3 \\ 0 & \text{if } S(T) \ge K_3 \end{cases}$$



Straddles: $f(T) = |S(T) - K|$

~~Americative +~~

American Derivatives :- $a_0 \to (1+r)\, a_0$.

N-period binomial model with up factor u and down factor d, interest rate r.

$$S_0 \begin{cases} \to uS_0 \begin{cases} u^2 S_0 \\ udS_0 \;\cdots\cdots \end{cases} \\ \to dS_0 \begin{cases} \\ d^2 S_0 \end{cases} \end{cases}$$

Condition: $0 < d < 1+r < u$

European option: payoff $= g(S_N)$

Then the price v_n at time $n \le N$ is given by

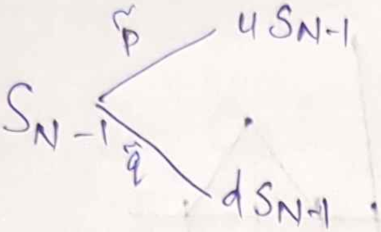
$$v_N(S_N) = \tilde{E}\left[g(S_N)\right] = g(S_N)$$

$$v_{N-1}(S_{N-1}) = \frac{1}{(1+r)} \tilde{E}_{N-1}\left[g(S_N)\right]$$

$$= \frac{1}{1+r}\left[\tilde{p}\, g(uS_{N-1}) + \tilde{q}\, g(dS_{N-1})\right] = \cancel{\frac{1}{1+r}}$$

$$v_n(s) = \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)\right]$$

$n = N-1 \ldots\ldots, 0$



$$\boxed{\tilde{p} = \frac{1+r-d}{u-d}} \quad \boxed{\tilde{q} = \frac{u-1-r}{u-d}}$$

$$\boxed{\Delta_n = \frac{v_{n+1}(us_n) - v_{n+1}(ds_n)}{(u-d)\,s_n}} \quad n = 0, \ldots\ldots, N-1$$

American call is same as the European call.

time-n Derivative payoff is $g(S_n)$ put payoff $(K-S_n)^+$

$n \leq N$

The price of American derivative is given by $v_N(S_N) = \max(g(S_N), 0)$

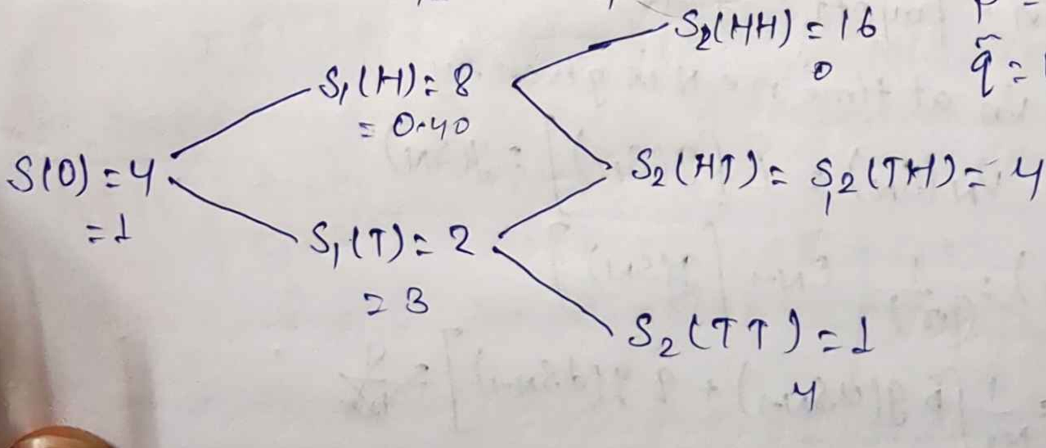
$$v_{N-1}(S_{N-1}) = \max\left\{ g(S_{N-1}), \frac{1}{1+r}\left[\tilde{p}\, v_N(uS_{N-1}) + (1-\tilde{p})\, v_N(dS_{N-1})\right]\right\}$$

$$v_n(s) = \max\left\{ g(s), \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)\right]\right\}$$

<u>Example:</u> $u = 2$ $d = 1/2$ $r = \frac{1}{4}$, $K = 5$

$\tilde{p} = 1/2$

$\tilde{q} = 1/2$



Consider an american put with $N=2$ and

$$g(s) = (5-s)^+$$

$$v_2(s) = \max\{5-s, 0\}$$

$$v_n(s) = \max\left\{ g(s), \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(us) + \tilde{q}\, v_{n+1}(ds)\right] \right\}$$

$v_2(16) = 0 \qquad v_2(4) = 1 \qquad v_2(1) = 4$

$$v_1(8) = \max\left\{ \frac{1}{\frac{5}{4}} \left\{ \frac{1}{2}\cdot 0 + 1\cdot\frac{1}{2} \right\}, 0 \right\} = \frac{4}{5} \times \frac{1}{2} = \frac{2}{5} = 0.40$$

$$v_1(2) = \left\{ \frac{1}{\frac{5}{4}} \left\{ \frac{1}{2} + 2 \right\}, 3 \right\} = \max\left\{ \frac{4}{5} \times \frac{5}{2}, 3 \right\} = 3.$$

$$v_0(4) = \max\left\{ \frac{4}{5}\left(\frac{0.40}{2} + \frac{3}{2} \right), 1 \right\} = 1.36.$$

$$\begin{tikzcd}[column sep=small]
 & & V_2(16) = 0 \\
 & V_1(8) = 0.40 \arrow[ur] \arrow[dr] & \\
V_0(4) = 1.36 \arrow[ur] \arrow[dr] & & V_2(4) = 1 \\
 & v_1(2) = 3 \arrow[ur] \arrow[dr] & \\
 & & V_2(1) = 4
\end{tikzcd}$$

$$0.40 = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0)$$

$$= \Delta_0 \cdot 8 + \frac{5}{4}(1.36 - 4\cdot\Delta_0)$$

$$\boxed{\Delta_0^{(H)} = -0.43}$$

$$3 = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0$$

$$3 = 2\Delta_0 + \frac{5}{4}(1.36 - 4\Delta_0$$

$$-3\Delta_0 = 3 - \frac{5}{4} \times (1.36)$$

$$\boxed{\Delta_0^{(T)} = -0.43}$$

$Y_n = \tilde{V}_n = \frac{1}{(1+r)^n} V_n$ $\boxed{r = 1/4}$

$Y_0(4) = 1.36$

$\to Y_1(H) = 0.32$

$\to Y_1(T) = 2.40$

$Y_2(HH) = 0$

$Y_2(HT) = Y_2(TH) = 0.64$

$Y_2(TT) = 2.56$

$\tilde{p} = \tilde{q} = 1/2$

$\tilde{E}_n[Y_{n+1}] = Y_n \to$ If it is martingale it should follow this.

$2.4 = Y_1(T) \neq \frac{1}{2}[Y_2(TH) + Y_2(TT)]$

$= \frac{1}{2}[2.56 + 0.64] = 1.6.$

Hence something is wrong.

<u>Theorem</u>: consider a N-period binomial model with $0<d<1+r<u$

$\tilde{p} = \frac{1+r-d}{u-d}$ $\tilde{q} = \frac{u-(1+r)}{u-d}$. Let the payoff fn. $g(S)$ be given and define $V_N(s)\, V_{N-1}(s), \dots V_0(s)$ by

$$V_N(s) = \max\{g(s), 0\}$$

$$V_n(s) = \max\left\{g(s), \frac{1}{(1+r)}\left[\tilde{p}\, V_{n+1}(us) + \tilde{q}\, V_{n+1}(ds)\right]\right\}$$

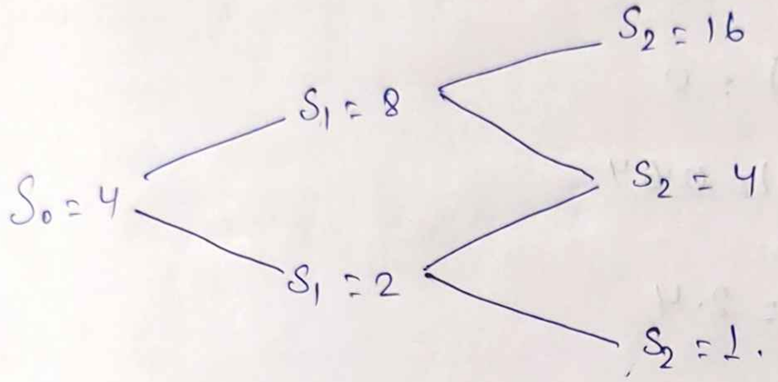
Define $\Delta_n = \frac{V_{n+1}(uS_n) - V_{n+1}(dS_n)}{(u-d)\,S_n}$ $\boxed{C_n = V_n(S_n) - \frac{1}{1+r}\left[\tilde{p}\, V_{n+1}(uS_n) + \tilde{q}\, V_{n+1}(dS_n)\right]}$

We have $C_n \geq 0$ If we set $X_0 = V_0(S_0)$ and define n.

$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n)$ then $X_n = V_n$ & $X_n \geq g(S_n)$ $\forall n$.

Stopping time: (N- period binomial). $\Omega_N = \{(\omega_1, \ldots \omega_N), \omega_i \in H \text{ or } T\}$

A random variable $\tau: \Omega_N \to \{0, 1, 2, \ldots N, \infty\}$ satisfies condition if $\tau(\omega_1, \omega_2, \ldots \omega_n, \omega_{n+1}, \omega_{n+2}, \ldots \omega_N) = n$ then $\tau(\omega_1, \omega_2, \ldots \omega_n, \omega'_{n+1}, \ldots \omega'_N) = n$



$g_n = 5 - S_n$

$\tau: \{HH, HT, TH, TT\} \to \{0, 1, 2, \infty\}$.

$\tau(HH) = \infty$

$\tau(HT) = 2$

$\tau(TH) = 1$

$\tau(TT) = 1$. stopping time.

$\rho: \Omega \to \{0, 1, 2, \infty\}$.

$\rho(HH) = 0$

$\rho(HT) = 0$ → Not Stopping time.

$\rho(TH) = 1$.

$\rho(TT) = 2$

$Y_{n\wedge\tau} \to$ Martingale.

$$\begin{tikzcd}[column sep=small]
 & & Y_2(HH) = 0 \\
 & Y_1(H) = 0.32 \arrow[ur] \arrow[dr] & \\
Y_0 = 1.36 \arrow[ur] \arrow[dr] & & Y_2(HT) = Y_2(TH) = 0.64 \\
 & Y_1(T) = 2.40 \arrow[ur] \arrow[dr] & \\
 & & Y_2(TT) = 2.56.
\end{tikzcd}$$

$Y_{0\wedge\tau} = Y_0 = 1.36.$

$Y_{1\wedge\tau} = Y_1 =$

$Y_{2\wedge\tau}(HH) = Y_2(HH) = 0$

$Y_{2\wedge\tau}(HT) = Y_2(HT) = 0.64$

$Y_{2\wedge\tau}(TH) = Y_1(T) = 2.4$

$Y_{2\wedge\tau}(TT) = Y_1(T) = 2.4$

$$\begin{tikzcd}[column sep=small]
 & & 0 \\
 & 0.32 \arrow[ur] \arrow[dr] & \\
1.36 \arrow[ur] \arrow[dr] & & 0.64 \\
 & & 2.4 \\
 & 2.4 \arrow[ur] \arrow[dr] & \\
 & & 2.4
\end{tikzcd}$$

Now $\boxed{Y_{n\wedge\tau} \to \text{Martingale}}$

$\mathcal{S}_n = \{\tau$ - stopping time $: \Omega_N \to \{n, n+1, \ldots N, \infty\}\}$

Consider a american derivative with intrinsic value at n is G_n.

$G_n = \max\limits_{0\le i\le n} S_i - \min\limits_{0\le i\le n} S_i$ → Particular example.

Stock price

Defn: $V_n = \max\limits_{\tau\in\mathcal{S}_n} \tilde{E}_n\left[\mathbb{1}_{\{\tau\le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau\right]$

$n = 0, 1, \ldots N$

General American derivatives:-

$S_n = \{\tau\text{ - stopping time } \tau : \Omega_N \to \{n, n+1, \dots, N, \infty\}\}$

S_0 - set of all stopping time.

An american derivative with intrinsic value at time n is G_n

$$\boxed{V_n = \max_{\tau \in S_n} \tilde{E}_n\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau\right]} \quad \text{for } n = 0, 1, \dots N.$$

$\tau \in S_N$

$V_N = \max\{G_N, 0\}$

$G_n = 5 - S_n$

$G_0 = 1$

$G_1(H) = -3$ $G_2(HH) = -11$ $V_2(HH) = 0$

$G_2(TH) = G_2(HT) = 1$

$V_2(TH) = V_2(HT) = 1$

$G_1(T) = 3$ $G_2(TT) = 4$ $V_2(TT) = 4$

$\tilde{p} = 1/2 = \tilde{q}$

$\tau(TH) = 1 = \tau(TT)$

If $\omega_1 = T$

$\tau_1(TH) = 2 \quad \tau_1(TT) = 2$

$v_1^{\tau}(T) = \tilde{E}_1\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-1}} G_\tau\right]$

$\tau_2(TH) = 2 \quad \tau_2(TT) = \infty$

$\tau_3(TT) = 2 \quad \tau_3(TH) = \infty$

$\tau = 1.$ $= \tilde{E}_1[G_1] = 3$

Theorem: If: $V_n = \max_{\tau \in S_n} \tilde{E}_n \left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau - n}} G_\tau \right]$

then

(i) $V_n \geq \max\{G_n, 0\} \ \forall n$. $\rightarrow$ for seller.

(ii) $\frac{1}{(1+r)^n} V_n$ is a Supermartingale.

(iii) If Y_n is another process satisfying (i) & (ii) then

$V_n \leq Y_n \ \forall n$. $\rightarrow$ for buyer

Proof: (i) $\hat{\tau} = n$, $\hat{\tau} \subset S_n$

$$\tilde{E}_n \left[\mathbb{1}_{\{\hat{\tau} \le N\}} \frac{1}{(1+r)^{\hat{\tau} - n}} G_{\hat{\tau}} \right] = G_n$$

$\tilde{\tau} = \infty$ then

$$\tilde{E}_n \left[\mathbb{1}_{\{\hat{\tau} \le N\}} \frac{1}{(1+r)^{(\tilde{\tau} - n)}} G_{\tilde{\tau}} \right] = 0$$

$$Y_n \geq \max\{G_n, 0\} \leftarrow (i)$$

proved.

(ii) Let n be given and suppose τ^* attains the maximum in the definition of V_{n+1}

$$Y_{n+1} = \tilde{E}_n \left[\mathbb{1}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^* - (n+1)}} G_{\tau^*} \right]$$

$\tau^* \in S_{n+1}$

$\Rightarrow \tau^* \in S_n$

$$Y_n \geq \tilde{E}_n \left[\mathbb{1}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^* - n}} G_{\tau^*} \right]$$

$$= \tilde{E}_n \left[\tilde{E}_{n+1} \left[\mathbb{1}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n}} G_{\tau^*} \right] \right]$$

$$= \tilde{E}_n \left[\frac{1}{(1+r)} \tilde{E}_{n+1} \left[\mathbb{1}_{\{\tau^* \le N\}} \frac{1}{(1+r)^{\tau^*-n-1}} G_{\tau^*} \right] \right]$$

$$= \tilde{E}_n \left[\frac{V_{n+1}}{(1+r)} \right]$$

$$\frac{V_n}{(1+r)^n} \geq \tilde{E}_n \left[\frac{V_{n+1}}{(1+r)^{n+1}} \right]$$ supermartingale. (2) proved.

(iii) Suppose Y_n is another process satisfying (I) & (II)

$$Y_k \geq \max\{G_k ; 0\} \ \forall k.$$

$$\mathbb{1}_{\{\tau \le N\}} G_{(\tau)} \le \mathbb{1}_{\{\tau \le N\}} \max\{G_\tau, 0\}$$

$$\le \mathbb{1}_{\{\tau \le N\}} \max\{G_{\tau \wedge N}, 0\}$$

$$+ \mathbb{1}_{\{\tau = \infty\}} \max\{G_{\tau \wedge N}, 0\}$$

$$= \max\{G_{\tau \wedge N}, 0\}$$

$n, \tau \in S_n$

$$\le Y_{\tau \wedge N}$$

$$V_n = \max_{\tau \in S_n} \tilde{E}_n \left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau - n}} G_\tau \right]$$

$$\tilde{E}_n \left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^\tau} G_\tau \right] = \tilde{E}_n \left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau \wedge N}} G_\tau \right]$$

$$\le \tilde{E}_n \left[\frac{1}{(1+r)^{\tau \wedge N}} Y_{\tau \wedge N} \right]$$

$$\le \frac{1}{(1+r)^{\tau \wedge n}} Y_{\tau \wedge n} = \frac{Y_n}{(1+r)^n}$$

$$\Rightarrow \tilde{E}\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right] \le Y_n$$

$$\Rightarrow \boxed{V_n \le Y_n} \quad \forall\ \tau \in S_n$$

(1) $$V_n = \max_{\tau \in S_n} \tilde{E}\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau-n}} G_\tau \right]$$

<u>Theorem</u>: we have the following American pricing algorithm for the path dependent derivative.

(2) $$V_N(\omega_1, \ldots \omega_N) = \max\{G_N(\omega_1, \ldots \omega_N), 0\}$$

$$V_n(\omega_1, \ldots \omega_n) = \max\left\{ G_n(\omega_1, \ldots \omega_n), \frac{1}{(1+r)}\left[\tilde{p}\, v_{n+1}(\omega_1, \ldots \omega_n, H) + \tilde{q}\, v_{n+1}(\omega_1, \ldots \omega_n, T)\right] \right\}$$

for $n = N-1, N-2, \ldots 0$

$$\Delta_n(\omega_1, \ldots \omega_n) = \frac{V_{n+1}(\omega_1, \ldots \omega_n, H) - V_{n+1}(\omega_1, \ldots \omega_n, T)}{S_{n+1}(\omega_1, \ldots \omega_n, H) - S_{n+1}(\omega_1, \ldots \omega_n, T)}$$

$$C_n(\omega_1, \ldots, \omega_n) = V_n(\omega_1, \ldots \omega_n) - \frac{1}{1+r}\left[\tilde{p}\, v_{n+1}(\omega_1, \ldots \omega_n, H) + \tilde{q}\, v_{n+1}(\omega_1, \ldots \omega_n T)\right]$$

for $n = N-1, \ldots, 0$

We have $C_n \geq 0$ $\forall$ if we set $V_0 = X_0$ and

$$X_{n+1} = \Delta_n S_{n+1} + (1+r)(X_n - C_n - \Delta_n S_n) \ \forall n.$$

then $X_n \geq G_n$ $\forall n$.

Theorem: (optimal Exercise time)

The stopping time $\tau^* = \min_n \{n : G_n = V_n\}$ maximises (1) and

$$V_0 = \tilde{E}\left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{\tau^*}} G_{\tau^*}\right]$$

Proof: Claim $\frac{1}{(1+r)^{\tau^* \wedge n}} V_{\tau^* \wedge n} = Y_n$ is a martingale under $\tilde{P}$.

If along the path as $\omega_1 \ldots \omega_n$ (first n coin toss)

$\tau^* \geq n+1$ $\boxed{G_n < V_n}$

$$V_{n \wedge \tau^*}(\omega_1 \ldots \omega_n)$$

$$= V_n(\omega_1 \ldots \omega_n) = \frac{1}{1+r}\left[\tilde{p} V_{n+1}(\omega_1 \ldots \omega_n H) + \tilde{q} V_{n+1}(\omega_1 \ldots \omega_n T)\right]$$

$$= \frac{1}{(1+r)}\left[\tilde{p} v_{n+1 \wedge \tau^*}(\omega_1 \ldots \omega_n, H) + \tilde{q} V_{n+1 \wedge \tau^*}(\omega_1 \ldots \omega_n T)\right]$$

let $\tau^* \leq n$

$$V_{n \wedge \tau^*}(\omega_1, \omega_2 \ldots \omega_n) = V_{\tau^*}(\omega_1, \ldots \omega_n)$$

$$= \tilde{p}\, v_{\tau^*}(\omega_1 \ldots \omega_n H) + \tilde{q}\, v_{\tau^*}(\omega_1, \ldots \omega_n, T)$$

$$= \tilde{p}\, v_{(n+1) \wedge \tau^*}(\omega_1, \ldots \omega_n, H) + \tilde{q}\, v_{\tau^* \wedge (n+1)}(\omega_1, \ldots \omega_n, T)$$

$$V_0 = \tilde{E}\left[\frac{1}{(1+r)^{N \wedge \tau^*}} V_{N \wedge \tau^*}\right]$$

$$= \tilde{E}\left[\mathbb{1}_{\{\tau^* \leq N\}} \frac{1}{(1+r)^{N \wedge \tau^*}} V_{N \wedge \tau^*}\right] + E\left[\mathbb{1}_{\{\tau^* = \infty\}} \frac{V_{N \wedge \tau^*}}{(1+r)^{N \wedge \tau^*}}\right]$$

If $\tau^* = \infty$ at N, $V_N > G_N$ $\therefore V_N = 0$

$$V_N = \max\{G_N, 0\} \Rightarrow G_N < 0$$

$$\tilde{E}\left[\mathbb{1}_{\{\tau^* = \infty\}} \frac{1}{(1+r)^{N \wedge \tau^*}} V_{N \wedge \tau^*}\right] = 0$$

Call options:

Let $g : [0, \infty) \to \mathbb{R}$ be a convex function with $g(0) = 0$

$$s_1 \geq 0 \quad s_2 \geq 0 \quad 0 \leq \lambda \leq 1$$

$$g(\lambda s_1 + (1-\lambda) s_2) \leq \lambda g(s_1) + (1-\lambda) g(s_2)$$

$$g(x) = (x-k)^+$$

Theorem: Consider a American derivative with convex Payoff function $g(s)$ satisfying $g(0) = 0$

The value of derivative at time $t = 0$

$$V_0^A = \max_{\tau \in S_0} \tilde{E}\left[\mathbb{1}_{\{\tau \leq N\}} \frac{1}{(1+r)^\tau} g(S_\tau)\right]$$

is same as the value of European derivative with payoff $g(S_N)$

$$V_0^E = \tilde{E}\left[\frac{1}{(1+r)^N} \max\{g(S_N), 0\}\right]$$

$$g^+(s) = \max\{g(s), 0\}$$

$$g(\lambda s_1 + (1-\lambda) s_2) \leq \lambda g(s_1) + (1-\lambda) g(s_2)$$
$$\leq \lambda g^+(s_1) + (1-\lambda) g^+(s_2)$$

$$g^+(\lambda s_1 + (1-\lambda) s_2) \leq \lambda g^+(s_1) + (1-\lambda) g^+(s_2)$$

Take $\delta_1 = \delta$ and $\delta_2 = 0$

$g^+(\lambda \delta_1) \le \lambda g^+(\delta)$

$\frac{1}{(1+r)^n} S_n$ is a martingale under $\tilde{P}$.

$$S_n = \tilde{E}_n \left[\frac{1}{(1+r)} S_{n+1} \right]$$

$$g^+(S_n) = g^+\left(\tilde{E}_n \left[\frac{1}{(1+r)} S_{n+1} \right] \right) \le \tilde{E}_n \left[g^+\left(\frac{1}{(1+r)} S_{n+1} \right) \right]$$

$$\le \frac{1}{(1+r)} \tilde{E}_n \left[g^+(S_{n+1}) \right]$$

$$\frac{1}{(1+r)^n} g^+(S_n) \le \frac{1}{(1+r)^{n+1}} \hat{E}_n \left[g^+(S_{n+1}) \right]$$

$\Rightarrow \frac{1}{(1+r)^n} g^+(S_n)$ is a Submartingale.

$\tau \in \mathcal{S}_0$

then we have $\tilde{E} \left[\frac{1}{(1+r)^{\tau \wedge N}} g^+(S_{N \wedge \tau}) \right]$

$$\le \tilde{E} \left[\frac{1}{(1+r)^N} g^+(S_N) \right] = V_0^E$$

If $\tau \le N$ then

$$\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau}} g(S_\tau) = \frac{1}{(1+r)^{N \wedge \tau}} g(S_{\tau \wedge N})$$

$$\le \frac{1}{(1+r)^{N \wedge \tau}} g^+(S_{\tau \wedge N})$$

If $\tau = \infty$

$$0 \frac{1}{(1+r)^{\tau}} g(S_\tau) \le \frac{1}{(1+r)^{N \wedge \tau}} g^+(S_{N \wedge \tau})$$

$$C_0^A = \tilde{E}\left[\mathbb{1}_{\{\tau \le N\}} \frac{1}{(1+r)^{\tau}} g(S_\tau)\right] \le \tilde{E}\left[\frac{1}{(1+r)^{\tau \wedge N}} g^+(S_{N\wedge\tau})\right]$$

$$= \underline{V_0^E}$$

$$\cancel{\le \tilde{E}\left[\frac{1}{\ }\right.}$$

18/3/24

Stopping time: (Continuous time American Derivatives)

Defn: A random variable τ taking values in $[0,\infty]$ is called a stopping time if $\{\tau \le t\} \in \mathcal{F}(t)$ $\forall t$

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$$

Define $\tau_m = \min\{t \ge 0 : S(t) = m\}$ If $S(t)$ never reaches level m then we interpret $\tau_m = \infty$.

$X(t)$ is a martingale (sub/super) then $X(t\wedge\tau_m)$ is also a martingale (sub/super martingale)

$$X(t\wedge\tau_m) = \begin{cases} X(t) & t \le \tau_m \\ X(\tau_m) & t > \tau_m \end{cases}$$

Perpetual put: The underlying asset price is given by

$$dS(t) = rS(t)\,dt + \sigma S(t)\,d\tilde{W}(t)$$

Where $\tilde{W}(t)$ is a B.M. under risk neutral measure $\tilde{P}$.

$$S(t) = S(0)\exp\left\{\sigma\tilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\}$$

The american put payoff $k - S(t)$ if it is exercised at time t.

Defn: let $S = \mathcal{I}$ be the set of all stopping time.

$$v_*(x) = \max_{\tau \in S} E\left[(k - S(\tau))e^{-r\tau}\right] \rightarrow \text{Independent of time.} \qquad S(0) = x.$$

If $\tau = \infty$ we interpret $e^{-r\tau} = 0$.

Lemma: Let $\tilde{w}(t)$ be a B.M. under the probability measure $\tilde{P}$, let μ be a real number and let m be a positive no.

Set $X(t) = \mu t + \tilde{w}(t)$ and

$$\tau_m = \min\{t \geq 0 : X(t) = m\}.$$

If $X(t)$ never reaches the level m then $\tau_m = \infty$

$$\tilde{E}\left[e^{-\lambda \tau_m}\right] = e^{-m(-\mu + \sqrt{\mu^2 + 2\lambda})} \quad \forall \lambda > 0$$

$$e^{-\lambda \tau_m} = 0 \text{ if } \tau_m = \infty.$$

Set a level L.

If $S(0) = x \leq L$

$$V_L(x) = (k - x)$$

If $S(0) = x > L$

$$\tau_L = \min\{t \geq 0; S(t) = L\}.$$

$$V_L(x) = \tilde{E}\left[(k - S(\tau_L))e^{-r\tau_L}\right]$$

$$\boxed{V_L(x) = (k - L)\,\tilde{E}\left[e^{-r\tau_L}\right]}$$

$$S(t) = x \exp\left\{\sigma \tilde{W}(t) + \left(r - \frac{1}{2}\sigma^2\right)t\right\}$$

$$S(\tau_L) = L$$

$$\Leftrightarrow x \exp\left\{\sigma \tilde{W}(\tau_L) + \left(r - \frac{1}{2}\sigma^2\right)\tau_L\right\} = L$$

$$\Rightarrow \sigma \tilde{W}(\tau_L) + \left(r - \frac{1}{2}\sigma^2\right)\tau_L = \ln\left(\frac{L}{x}\right)$$

$$\Rightarrow -\tilde{W}(\tau_L) - \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)\tau_L = -\frac{1}{\sigma}\ln\left(\frac{x}{L}\right)$$

From above lemma

$$\mu = -\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right). \quad m = \frac{1}{\sigma}\ln\left(\frac{x}{L}\right)$$

$$\lambda = r.$$

$$\mu^2 + 2\lambda = \frac{1}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)^2 + 2r$$

$$= \frac{1}{\sigma^2}\left(r + \frac{1}{2}\sigma^2\right)^2$$

$$\boxed{\mu + \sqrt{\mu^2 + 2\lambda} = \frac{2r}{\sigma}}$$

$$\tilde{E}\left[e^{-r\tau_L}\right] = \exp\left\{-\frac{1}{\sigma}\ln\left(\frac{x}{L}\right)\cdot\frac{2r}{\sigma}\right\} = \left(\frac{x}{L}\right)^{-2r/\sigma^2}$$

$$V_L(x) = \begin{cases} (K - x) & \text{if } x \le L \\ (K - L)\left(\frac{x}{L}\right)^{-2r/\sigma^2} & \text{if } x \geq L \end{cases}$$

$$V_L(x) = (K-L)\left(\frac{x}{L}\right)^{-2r/\sigma^2}$$

let $g(L) = (K-L)(L)^{2r/\sigma^2}$

$$g'(L) = -(L)^{2r/\sigma^2} + (K-L)\frac{2r}{\sigma^2}(L)^{2r/\sigma^2 - 1}$$

$$= -\frac{g(L)}{(K-L)} + \left(\frac{2r}{\sigma^2}\right) \times \frac{1}{L} g(L)$$

$$= g(L)\left(\frac{2r}{\sigma^2 L} - \frac{1}{K-L}\right)$$

$$g'(L) = 0$$

$$\frac{2r}{\sigma^2} = \frac{L}{K-L} \Rightarrow \boxed{L^* = \frac{2rK}{2r+\sigma^2}}$$

$$V_{L^*}(x) = \begin{cases} (K-x) & \text{if } 0 \le x \le L^* \\ (K-L^*)\left(\frac{x}{L^*}\right)^{-2r/\sigma^2} & \text{if } x \ge L^* \end{cases}$$

$$V'_{L^*}(x) = \begin{cases} -1 & 0 \le x \le L^* \\ (K-L^*)\left[\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}-1}\right] \times \frac{1}{L^*}\left(-\frac{2r}{\sigma^2}\right) & x \ge L^* \end{cases}$$

$$V''_{L^*}(x) = \begin{cases} 0 & 0 \le x \le L^* \\ \frac{(K-L^*)}{L^{*2}}\left(\frac{2r}{\sigma^2}\right)\left(\frac{2r}{\sigma^2}+1\right)\left(\frac{x}{L^*}\right)^{-\frac{2r}{\sigma^2}-2} & x \ge L^* \end{cases}$$

for $x > L^*$

$$rv_{L^*}(x) - rx\,v'_{L^*}(x) - \frac{1}{2}\sigma^2 x^2 v''_L(x) = 0$$

if $0 \le x \le L$

$$rv_L(x) - rx\,v'_L(x) - \frac{1}{2}\sigma^2(x^2) \, v''_L(x) = r(K-x) + rx$$

$$= rK$$

$V_{L^*}(x)$ satisfies the linear complementarity conditions.

$$v(x) \geq (K-x)^+ \quad \forall\, x \geq 0 \quad \text{——} \quad (1)$$

$$r v(x) - r x\, v'(x) - \frac{1}{2}\sigma^2 x^2 v''(x) \geq 0 \quad \forall\, x \quad (2)$$

and for each $x \geq 0$ equality holds in either (1) or (2) -- (3)

Cannot be both (1) & (2)

<u>Theorem</u>: $\tau_{L^*} = \min\{t \geq 0,\ S(t) = L^*\}$ Then

$e^{-rt} v_{L^*}(S(t))$ is a Supermartingale and

$e^{-r(t\wedge \tau_{L^*})} v_{L^*}(S(t\wedge \tau_{L^*}))$ is a martingale.

$$dS(t) = r S(t)\, dt + \sigma S(t)\, d\tilde{W}(t)$$

<u>Proof</u>: $d\left(e^{-rt} v_{L^*}(S(t))\right)$

$$= -r e^{-rt} v_{L^*}(S(t))\, dt + e^{-rt} d(v_{L^*}(S(t)))$$

$$= -r e^{-rt} v_{L^*}(S(t))\, dt + e^{-rt}\left(v'_{L^*}(S(t))\, dS(t) + \frac{1}{2} v''(S(t))\, dS(t)\, dS(t)\right)$$

$$= e^{-rt}\left[-r v_{L^*}(S(t)) + r S(t)\, v'_{L^*}(S(t)) + \frac{1}{2}\sigma^2 S^2(t)\, v''_{L^*}(S(t))\right] dt + e^{-rt}\sigma S(t)\, d\tilde{W}(t)\, v'_{L^*}(S(t))$$

$$= -e^{-rt}\left(r K\, \mathbb{1}_{\{S(t) < L^*\}}\right) dt + e^{-rt}\sigma S(t)\, d\tilde{W}(t)\, v'_{L^*}(S(t))$$

Exercise. $dX(t) = r(t)dt + \sigma(t)dW(t)$ $r(t) \leq 0 \Rightarrow$ Supermartingale
$r(t) \geq 0 \Rightarrow$ submartingale

from eq.
$\Rightarrow e^{-rt} v_{L^*}(S(t))$ is a Supermartingale.

Applying Integration

$$\int_0^{u \wedge \tau_{L^*}} d\left(e^{-rt} v_{L^*}(S(t))\right) = -\int_0^{u \wedge \tau_{L^*}} e^{-rt} rK \mathbb{1}_{\{S(t) < L^*\}} dt + \int_0^{u \wedge \tau_{L^*}} e^{-rt} \sigma S(t) v_{L^*}'(S(t)) d\tilde{W}(t)$$

$$e^{-r(u \wedge \tau_{L^*})} v_{L^*}(S(u \wedge \tau_{L^*})) - v_{L^*}(S(0)) = \int_0^{u \wedge \tau_{L^*}} e^{-rt} \sigma S(t) v_{L^*}'(S(t)) d\tilde{W}(t).$$

$e^{-r(t \wedge \tau_{L^*})} v_{L^*}(S(t \wedge \tau_{L^*}))$ is a martingale.

<u>Corollary</u>: $v_{L^*}(x) = \max_{\tau \in S} \tilde{E}\left[e^{-r\tau}(K - S(\tau))\right]$ where $S(0) = x$.

<u>Proof</u>: $e^{-rt} v_{L^*}(S(t))$ is a Supermartingale.

$$v_{L^*}(x) = v_{L^*}(S(0)) \geq \tilde{E}\left[e^{-rt} v_{L^*}(S(t))\right]$$

let $\tau \in S$ then $e^{-r(t \wedge \tau)} v_{L^*}(S(t \wedge \tau))$ is also Super-martingale.

$$v_{L^*}(x) \geq \tilde{E}\left[e^{-r(t \wedge \tau)} v_{L^*}(S(t \wedge \tau))\right]$$

$t \to \infty$ by Dominated convergence theorem since v_{L^*} is bounded.

$$v_{L^*}(x) \geq \tilde{E}\left[e^{-r\tau} v_{L^*}(S(\tau))\right]$$

$$\geq \tilde{E}\left[e^{-r\tau}(K - S(\tau))\right] \text{ (by 1)}$$

1 $\boxed{v_{L^*}(x) \geq (K-x)^+}$

$$\boxed{V_{L^*}(x) \geq \max_{\tau \in \mathcal{S}} \tilde{E}\left[e^{-r\tau}(K - S(\tau))\right]} \to a$$

- .. for other direction

$e^{-r(t \wedge \tau_{L^*})} v_{L^*}(S(t \wedge L^*_*))$ is a martingale.

by martingale property

$$V_{L^*}(x) = \tilde{E}\left[e^{-r(t \wedge \tau_{L^*})} v_{L^*}(S(t \wedge \tau^*_{L}))\right]$$

when $t \to \infty$ by DCT

$$v_{L^*}(x) = \tilde{E}\left[e^{-r\tau_{L^*}} v_{L^*}(S(\tau_{L^*}))\right]$$

$$= \tilde{E}\left[e^{-r\tau_{L^*}}(K - S(\tau_{L^*}))\right]$$

$$\boxed{V_{L^*}(x) \leq \max_{\tau \in \mathcal{S}} \tilde{E}\left[e^{-r\tau}(K - S(\tau))\right]} \to b$$

from a & b

$$\boxed{V_{L^*}(x) = \max_{\tau \in \mathcal{S}} \tilde{E}\left[e^{-r\tau}(K - S(\tau))\right]}$$