Integration

$K=3$

$$\int_a^b f(x)\,dx$$

$$FP(k) = P_3(x) + f[x_0\, x_1\, x_2\, x_3 . x]\psi_3(x)$$

$$x_0 = x_1 = a$$
$$x_2 = x_3 = b$$

$$\psi_3(x) = (x-a)^2(x-b)^2$$

$$E(f) = \frac{1}{4!} f^{(4)}(\xi) \int_a^b (x-a)^2(x-b)^2\,dx$$

$$= \frac{f^4(\xi)(b-a)^5}{720}$$

$$p_3(x) = f(a) + f[a\,a](x-a) + f[a\,a\,b](x-a)^2 + f[a\,a\,b\,b](x-a)^2(x-b)$$

$$I(p_3(x)) = f(a)(b-a) + f[a\,a]\frac{(b-a)^2}{2} + f[a\,a\,b]\frac{(b-a)^3}{3} + f[a\,a\,b\,b]\left\{\frac{(b-a)^4}{4} - \frac{(b-a)^4}{3}\right\}$$

$$I(p_3) = \text{Corrected Trapezoidal} = \frac{(b-a)}{2}[f(a) + f(b)] + \frac{(b-a)^2}{12}[f'(a) - f'(b)]$$

upto any cubic polynomial this formula is exact.

So far Interpolate the Integrand we use newton's divided differences now we consider the lagrange Interpolating polynomial.

$p_n(x)$ $\quad n \geq 1$

$$h = \left(\frac{b-a}{n}\right) \qquad x_j = a + jh \quad j = 0, \ldots n$$

$$I(f) = \int_a^b f(x)dx = I_n(f) = \int_a^b P_n(x)dx$$

$$I_n(f) = \int_a^b \sum_{i=0}^{n} l_{j,n} f(x_j)dx = \sum_{j=0}^{n} w_{j,n}(x) f(x_j)$$

where $w_{j,n}(x) = \int l_{j,n}(x)dx$

By taking polynomial of degree 1

- $l_0 = \dfrac{(x - x_1)}{x_0 - x_1}$ $\qquad l_1 = \dfrac{(x - x_0)}{(x_1 - x_0)}$

$$w_0 = \int_a^b l_0 = \int_a^b \left(\frac{x - x_1}{x_0 - x_1}\right) dx = \left[\frac{(x - x_1)^2}{2(x_0 - x_1)}\right]_a^b$$

$$w_1 = \int_a^b l_1 = \int_a^b \left(\frac{x - x_0}{x_1 - x_0}\right) dx = \left[\frac{(x - x_0)^2}{2(x_0 - x_1)}\right]_a^b$$

consider the case $n = 3$

$$\therefore \quad w_0 = \int_a^b l_0(x)\,dx = \int_a^b \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx$$

rewrite $l_0(x)$ as

$$x = x_0 + \mu h$$

$$w_0 = \frac{-1}{6h^3}\int_a^b (\mu-1)(\mu-2)(\mu-3)\,h^3\,d\mu$$

$$w_1 = \int_a^b \mu(\mu-2)(\mu-3)\,h^3\,d\mu$$

$$I_n(f) = w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3)$$

$$I_n(f) = \frac{3h}{8}\left[f(x_0) + 2f(x_1) + 3f(x_2) + f(x_3)\right]$$ → Known as 3/8th rule.

Error formula

1) for even n

$f(x)$ is $(n+2)$ times continuously differentiable on $[a\ b]$ then the error

$$E(f) = I(f) - I_n(f) = C_n h^{n+3} f^{n+2}(\eta)$$

$$C_n = \frac{1}{(n+2)!}\int_0^n \mu^2(\mu-1)\cdots(\mu-n)\,d\mu$$

ii) n is odd

$$E_n(f) = C_n h^{n+2} f^{(n+1)}(\eta)$$

$$C_n = \frac{1}{(n+1)!}\int_0^n \mu(\mu-1)\dots(\mu-n)\,d\mu$$

These were Newton-Cotes formula

Composite Quadrature formulas

Always it is advisable to divide the interval $[a\ b]$ $a = x_0 < x_1 \dots < x_{n-1} < x_n = b$ into ~~not~~ equally smaller intervals $[x_i, x_{i+1}]$ and apply all those quadrature formula.

$$I(f) = \int_a^b f(x)dx = \sum_{i=1}^{N}\int_{x_{i+1}}^{x_i} f(x)dx$$

Now one can replace integrand $f(x)$ by polynomial of 2d

$$h = \frac{b-a}{N}$$

$$x_i = a + ih$$

$$\int_{x_{i-1}}^{x_i} f(x)dx = \int_{x_{i-1}}^{x_i} P_{1,k}(x)dx = \text{constant} \times [x_i - x_{i-1}]$$

when $K=0$ ~~to~~ $f(x)$ is replaced by constant interpolant

the error will be $E(R) = \sum_{i=0}^{n} \frac{f'(\eta_i)}{2!} h^2$

here the error will be less comparison to original quadrat

Consider

$$\int_a^b f(x)dx = \int_a^b w(x)g(x)dx$$

$$I(g) \simeq A_0 g(x_0) + A_1 g(x_1) + \dots\dots + A_n g(x_n)$$

where A_i's are the weights which are independent of particular function $g(x)$ so far we have chosen the nodal points x_0 $x_1 \dots x_k$ equi-spaced (uniformly distributed) and

consider $\int_a^b g(x)w(x)dx$

where $w(x)$ is non negative integrable function

~~for~~ $\frac{g}{w}$ $g(x) = \frac{f(x)}{w(x)}$ is some function

$$w(x) = (x-a)^\alpha$$

suppose nodal points $x_1, \dots x_k$ in (a,b)

$$g(x) = p_k(x) + g[x_0 \dots x_n, x]\psi_k(x)$$

$$\psi_k(x) = (x-x_0)\dots\dots(x-x_k)$$

$$I(g) = I(P_k) + \int_a^b g[x_0 \dots x_k]\psi_k(x)dx$$

$$p_k(x) = \sum_{i=0}^{k} g(x_i)\, l_i(x)$$

where $l_i(x_j) = \delta_{ij}$

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{(x-x_j)}{(x_i-x_j)}$$

$$I(p_k) = \int_a^b \sum g(x_i) l_i(x) dx = \sum g(x_i) \int_a^b \cancel{l_i(x)dx}\, w_i(x) dx$$

$$= A_0 g(x_0) + \ldots\ldots\ldots + A_n g(x_n)$$

In general we know that the error in this Quadrature will be (K+1)th derivative of G or (K+2) derivative of G_1 is ...

$$E(g) = I(g) - I(p_k)$$

$$= \int_a^b g[x_0 \ldots x_{k+1}, x]\, \psi_k(x) \cancel{dx}\, w(x) dx$$

consider

$$\int_a^b \psi_k(x) w(x) dx = 0$$

then we know that

$$E(g) = \int_a^b g[x_0 \ldots\ldots x_{K+1}, x]\, \psi_{K+1}(x) w(x) dx$$

Suppose

$$\int_0^b \psi_{K+1}(x) w(x) dx = 0 \quad \text{then}$$

$$E(g) = \int_0^b g[x_0 \ldots x_{K+2}, x]\, \psi_{K+2}(x) w(x) dx$$

$x_k ; x_{k+1} \ldots\ldots x_{k+m}$

$$\int_a \psi_k(x)(x - x_{k+1}) \ldots (x - x_{k+i+1})\, w(x) dx = 0$$

$i = 0, \ldots m-1$

$$\varepsilon(g) = I(g) - I(P_k)$$

$$= \int_a^b g[x_0 \ldots x_{k+m+1}, x]\, \psi_{k+m+1}(x)\, w(x)\, dx$$

for several choices of $w(x)$ we can find a polynomial p_{k+1} such that $\int_a^b p_{k+1}(x)\, q(x)\, w(x)\, dx = 0$

where $q(x)$ is of degree $\leq k$ which tells that the polynomial are orthogonal to each other w.r.t weight so we can express the orthogonal polynomial $p_{k+1}(x) = \alpha_{k+1}(x-\xi_0) \ldots (x-\xi_k)$ where $\xi_0 \ldots \xi_k$ are distinct points in $[a\ b]$ where p_{k+1} vanishes (i.e roots)

suppose $x_j = \xi_j$ $j = 0 \ldots k$

Hence if we set $x_j = \xi_j$ $j = 0 \ldots k$ and and let x_{k+j} are arbitrary points in $[a\ b]$ $j \neq$

$$q(x) = \frac{(x - x_{k+1})(x - x_{k+2}) \ldots \ldots (x - x_{k+1+i})}{\alpha_{k+1}}$$

$$i = 0, \ldots \ldots m-1$$

therefore your error

$$\varepsilon(g) = \int_a^b g[x_0 \ldots, x_{k+m+1}, x]\, \psi_{k+1}(x)\, w(x)\, dx$$

In order to obtain the desired error we can choose

$$x_{k+j} = \xi_{j-1} \quad i = 1 \ldots k+1$$

$$\psi_{2K+1}(x) = (x-x_0)\ldots\ldots(x-x_{2K+1})$$

$$= (x-\xi_0)\ldots(x-\xi_K)(x-\xi_0)\ldots(x-\xi_K)$$

$$= \left(\frac{P_{K+1}(x)}{\alpha_{K+1}}\right)^2$$

$\therefore$ $\psi_{2K+1}(x)$ is of 1 sign (i.e non-negative) on [a b]

$\therefore$ By using mean value theorem for integral

$$E(g) = g[x_0 \ldots x_{2K+1}\ \eta]\int_a^b \left(\frac{1}{\alpha_{K+1}} P_{K+1}(x)\right)^2 w(x)\,dx$$

$$\therefore = \frac{1}{(2K+2)!} g^{2K+2}(\eta)\ \frac{S_{K+1}}{\alpha_{K+1}^2}$$ where S_{K+1} is integral

$$S_{K+1} = \int P_{K+1}^2\, w(x)\,dx$$

to summarize the gaussian quaduature we have to choose the nodal points or zeroes of polynomial of degree K+1 orthogonal to itself w.r.t to weight $w(x)$ on [a b] which tells the gaussian quaduature is exact on all polynomial of degree $\le 2K+21$

Assume p_{K+1} legendre polynomial

Suppose $w(x) = 1$

P_{K+1} = legendarge polynomial

$P_1(x) = x \qquad \xi_0 = 0$

$P_2(x) = \frac{3}{2}(x^2 - \frac{1}{3}) \qquad \xi_0 = \xi_1 = \pm\frac{1}{\sqrt{3}}$

$P_3(x) = \frac{5}{2}(x^3 - \frac{3}{5}x) \qquad \xi_0 = -\sqrt{\frac{3}{5}} \quad \xi_1 = 0 \quad \xi_2 = \sqrt{\frac{3}{5}}$

$$P_{n+1} = xP_n(x) - \frac{n^2}{4n^2-1} P_{n-1}(x)$$

$$\int_a^b f(x)dx = \int_{-1}^{1} f(x(t))\, x'(t)dt$$

where $x = (b-a)t + b$

Assume $n = 1$ then the quadrature will be

$\Rightarrow x_0 = \xi_0 = -\frac{1}{\sqrt{3}}$

$x_1 = \xi_1 = \frac{1}{\sqrt{3}}$

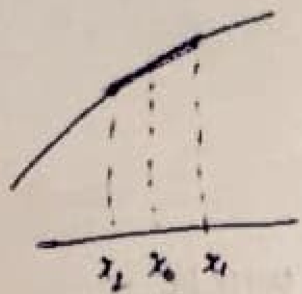
$$\int_{-1}^{1} g(x)dx = A_0 g\left(-\frac{1}{\sqrt{3}}\right) + A_1 g\left(\frac{1}{\sqrt{3}}\right)$$

$$A_1 = \int_{-1}^{1} \frac{x - (-1/\sqrt{3})}{\left(\frac{1}{\sqrt{3}}\right) - \left(-\frac{1}{\sqrt{3}}\right)} dx \qquad A_1 = 1$$

$$\int_{-1}^{1} g(x)dx = g\left(-\frac{1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

$$E = \frac{1}{135} g^{(iv)}(\eta)$$

Differentiation



$$f'(x_i) = \frac{f_{i+1} - f_i}{h} = \frac{f_i - f_{i+1}}{h} = \frac{f_{i+1} - f_{i-1}}{2h}$$

By taylor series

$$f_{i+1} = f(x_{i+1}) = f(x_i) + h f'(x_i) + \frac{h^2}{2!} f''(\xi) + \ldots$$

$$\frac{f_{i+1} - f_i}{h} = f'(x_i) = \underbrace{\frac{h}{2!} f''(\xi)}_{\text{error}}$$

$$f_{i+1} = f_i + h f'(x_i) + \frac{h^2}{2!} f_i'' + \frac{h^3}{3!} f_i''' + \frac{h^4}{4} f_i''''$$

$$f_{i-1} = f(x_i) - h f'(x_i) + \frac{h^2}{2!} f_i'' - \frac{h^3}{3!} f'''\xi + \frac{h^4}{4!} f_i''''$$

~~Ad~~ Subtracting above two

$$f_{i+1} - f_{i-1} = +2h f_i' + \frac{2h^3 f_i'''}{3!}$$

$$\frac{f_{i+1} - f_{i-1}}{2h} = f_i' + \frac{h^2 f_i'''}{3!} \qquad \underline{O(h^4)}$$

→ central difference is average of forward and backward differences.

Suppose we have given the tabulated values and we want to determine the derivative of this data

if for example

given displacement at different time interval then we can calculate velocity (first derivative) and acceleration (second derivative)

Let $f(x)$ be a continuously differentiable on $[c,d]$ which contains $[a\ b]$ into equally distributed subinterval $x_0\ x_1 \dots x_n$

Since x_i's are distinct we can approximate a polynomial

$$f[x_0\ x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

$$\frac{d}{dx} f[x_0\ x] = \frac{(x-x_0) f'(x) - (f(x) - f(x_0))}{(x-x_0)^2}$$

$$\frac{f'(x) - f'(x_0)}{(x-x_0)} = f''(x_0)$$

$$\frac{d}{dx} f[x_0 \dots x_{k+1}, x] = f[x_0, \dots x_k\ x . x]$$

$$f'(x) = p_k'(x) + f[x_0 \dots x_{k-1}\ x, x] \psi_k(x) +$$
$$f'(x_0 \dots x_k, x] \psi_k'(x)$$

Let $D = f'(a) \quad a \in [c\ d]$

$$D(f) = p_k'(x)$$

$$E(f) = D(f) - D(P_k)$$
$$= f[x_0 \dots x_k, a, a] \psi_k'(a) + f[x_0 \dots x_k, a, a] \psi_k(a)$$

Since the error is containing two terms by choosing $a = x_i$ (some nodal point) then your $\psi_k(x_i) = 0$

$\psi_k(a) = 0$ and the first term will be dropped out whereas if u choose a such that $\psi_k'(a) = 0$ then the second term will become zero.

Suppose $\psi_k'(a) = q(a)$

where $q(x) = \dfrac{\psi_k(x)}{(x - x_{ki})} = (x - x_0)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x$

<u>Case 1</u> : $x = x_i$ for some i

$$\varepsilon(f) = \frac{1}{(k+1)!} f^{(k+1)}(\eta) \prod_{\substack{j=0 \\ j \neq i}}^{k} (x - x_j) \qquad f \in C^{k+1}(c,d)$$

Let us consider second case

when $\psi_k'(a) = 0$

then in the error the second term will vanish. Suppose k is an odd number then we can achieve this by placing the x_j's symmetrically around a.

$$x_{k-j} - a = a - x_j \qquad j = 0 \ldots (k-1)/2$$

$$(x - x_j)(x - x_{k-j}) = (x - a + a - x_j)(x - a + a - x_{k-j})$$

$$= (x - a + x_{k-j} - a)(x - a + a - x_{k-j})$$

$$= (x - a)^2 - (x_{k-j} - a)^2$$

$$\psi_k(x) = \prod_{j=0}^{k}(x-x_j) = \prod_{j=0}^{(k-1)/2} (x-a)^2 - (a-x_j)^2$$

therefore $\psi_k'(x) = \frac{d}{h} = 0$

$$\therefore \text{the } E(f) = \frac{1}{(K+2)!} f^{(K+2)} \prod_{j=0}^{(k-1)/2} (\cdot - (a-x_j)^2$$

If $K=0$ then (constant Interpolant)

derivative $D(P_k) = 0$

If $k=1$ (linear polynomial)

then $P_1 = f(x_0) + f[x_0\ x_1](x-x_0)$

$D(P_1) = f[x_0\ x_1]$

$D(f) = D(P_1) = f[x_0\ x_1]$

Now a can be x_0, x_1 or $\frac{x_0+x_1}{2}$

One day left
Using Taylor's expansion

$$y(t_{i+1}) = y(t_i+h)$$
$$= y(t_i) + h y'(t_i) + \frac{h^2}{2!} y''(\xi)$$
$$= y(t_i) + h f(t_i, y_i) + \frac{h^2}{2!} y''(\xi) \qquad \xi \in (t_i, t_{i+1})$$

$$\boxed{y_{i+1} = y_i + h f(t_i, y_i)}$$

1) consistency (TE)
2) stability
3) convergence

Truncation error $- \frac{1}{h}(LHS - RHS)$

$$y_{i+1} \; \frac{=1}{h}\left(y_i + h f_i + \frac{h^2}{2!} y_i'' + \right) - y_i + h f(t_i, y_i)$$

$$= \frac{h}{2!} y''$$

$$= O(h)$$

when truncation error goes to zero as $h \to 0$ then we say the numerical scheme is consistent

Stability is related to Round off.

Error $= |y(t_i) - y_i| \leq c h^p$

if the error goes to 0 as the step size $h \to 0$

Lemma 1: for all $x \geq -1$ and any positive m we have $0 \leq (1+x)^m \leq \exp(mx)$

~~since x→~~

By taylor's expansion

$$e^x = 1 + x + \frac{x^2 e^\xi}{2!} \quad \xi \in (0,x)$$

since $x > -1$

$\therefore \; 1+x \geq 0$

$$(1+x) < e^x$$

Lemma 2: Let s and t be two positive real numbers and a_i be a sequence satisfying $\{a_i\}$ $a_0 \geq 0$ and

$$a_{i+1} \leq (1+s)a_i + t, \; i = 0, \dots k$$

then

$$a_{i+1} \leq \exp((i+1)s)\left[a_0 + \frac{t}{s}\right] - \frac{t}{s}$$

use previous lemma

$$a_{i+1} \leq (1+s)a_i + t \;\Big|\; a_i < (1+s)a_{i-1} + t \; \dots \;\Big|\; a_1 < (1+s)a_0 + t$$

$$a_{i+1} \leq (1+s)^2 a_{i-1} + t(1+s)$$

$$\vdots$$

$$a_{i+1} \leq (1+s)^{i+1} a_0 + t\left[1 + (1+s) + \dots + (1+s)^i\right]$$

$$\leq (1+s)^{i+1}\underbrace{\left(a_0 + \frac{t}{s}\right)}_{\text{this quantity is non negative}} - \frac{t}{s}$$

using previous lemma.

Convergence of explicit Euler Scheme

Suppose f is continuous and satisfy the Lipschitz condition with constant L on domain $D = \{(t,y): a \le t \le b, -\infty < y < \infty\}$ and $\exists$ $M>0$ such that $|y''(t)| < M$ $\quad t \in [a,b]$

Now

Let $y(t)$ be the unique solution of IVP

$$\begin{cases} y'(t) = f(t,y) \\ y(a) = \alpha \end{cases}$$

$w_0, w_1, \ldots \ldots w_N$

$$w_{n+1} = w_n + hf(t,n) \qquad n = 0, \ldots N-1$$

then Error $= |y(t_i) - w_i|$

$$\le \frac{hM}{2L}\left[\exp(L(t_i - a)) - 1\right] \le ch$$

$$y(t_{i+1}) = y(t_i) + hy'(t) + \frac{h^2}{2!} y''(\xi)$$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

$$y(t_{i+1}) - w_{i+1} = (y(t_i) - w_i) + h\left(f(t_i, y_i) - f(t_i, w_i)\right) + \frac{h^2}{2!} y''(\xi)$$

$$|y(t_{i+1}) - w_{i+1}| \le |y_i - w_i|(1 + hL) + \frac{h^2}{2!} M$$

$$a_{i+1} \le (1+s)a_i + t$$

we can choose our s and t.

$$w_0 = \alpha + \delta$$

$$w_{i+1} = w_i + hf(t_i, w_i)$$

the previous theorem we assume all the calculations are free from error but it is not true in practice therefore if we consider the round off error then the scheme will become.

$$w_0 = \alpha + \delta$$

$$w_{i+1} = w_i + hf(t_i, w_i) + \delta_{i+1} \qquad i = 0, \ldots N-1$$

then the Euler scheme becomes this

Assume $|\delta_i| < \delta$

then the error $|y(t_i) - w_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right) \exp(L$

$$\exp(L(t_i - a)) - 1)$$

$$+ |\delta_0| \exp(L(t_i - a))$$

$$\forall i = 0 \ldots N$$

$$\lim_{h \to 0} \left(\frac{hM}{2} + \frac{\delta}{h}\right) = \infty$$

Let $\varepsilon(h) = \frac{hM}{2} + \frac{\delta}{h}$

$$\varepsilon'(h) = 0$$

$$\Downarrow$$

1) if $h < \sqrt{\frac{\delta}{m}}$ then your ε' will be negative

that means ε is decreasing

then ε is increasing.

$$\int_{t_n}^{t_{n+1}} y'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$$

$$\therefore y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t_n))\,ds$$

$$= f(t_{n+1}, y(t_n))h$$

$$\Rightarrow \boxed{w_{n+1} = w_n + hf(t, w_n)}$$

$$\therefore \int_{t_n}^{t_{n+1}} [\theta f(t_n, y_n) + (1-\theta) f[t_{n+1}, y_{n+1}]\,ds$$

$$\Rightarrow y_{n+1} = y_n + h[\theta f(t_n, y_n) + (1-\theta) f(t_{n+1}, y_{n+1})]$$

weighted average of slope at point t_n and t_{n+1}.

(i) $\theta = 0$
(ii) $\theta = 1$ $\Big\} \rightarrow$ order $= 1$
(iii) $\theta = \frac{1}{2}$ $\} \rightarrow$ order $= 2$

Truncation error $= \frac{1}{2}(L.H.S - R.H.S)$

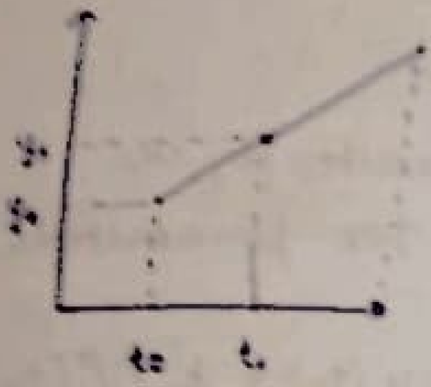
the truncation error of $\theta = \frac{1}{2}$ show that 1st dominat term is order of 2

Runge - Kutta methods

the underlying idea of this Runge Kutta (RK) method is to take some weighted average of slope of tangent in order to increase or enhance order of truncation error.

$$y(t_{n+1}) = y(t_n) + h\phi(t_n, y(t_n), h)$$
$$y_0 = \alpha$$

where this ϕ is continuous on all of its arguments



pictorial view of Euler starting from some point y_0.

→ All these methods categroise as single-step methods because at point t_{n+1} we need only t_n

R-Stage R-K

$$y_{n+1} = y_n + \Phi(t_n, y_n, h)$$

where $\Phi(t_i, y_i, h) = \sum_{r=1}^{R} c_r k_r$

$$K_1 = hf(t, y)$$

$$K_r = h\, f\left(t + hc_r,\ y + \sum_{s=1}^{r-1} a_{rs} k_s\right)$$

$$r = 2, \ldots R$$

$$c_r = \sum_{s=1}^{r-1} a_{rs}, \quad r = 2, \ldots R$$

Consider $R = 2$ (two stage Runge Kutta method)

$$y_{n+1} = y_n + \omega_1 K_1 + \omega_2 K_2 \quad —①$$

$$② \begin{cases} V_{11} = hf(K, y_n) \\ V_{12} = h f(t_n + c_2 h, y_n + a_{21} V_{11}) \end{cases}$$

we have four unknown the parameter ω_1, ω_2
in order to determine the values for parameters, we use taylor series expansion.

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2!} y''(t_n) + \ldots$$

$$y' = f(t, y)$$

$$y'' = f_t + f_y y' = f_t + f_y f$$

$$y''' = \cancel{f_t + g f_{yt}} = f_{tt} + f_y f_t + f_{yt} f + f_{ty} + f_{yy} f + f_y f_y$$

$=$

Substitute these values in above equation

$$K_2 = hf(t_n + c_2 h, y_n + a_{21} k_1)$$

$$= h\left(f(t_n, y_n) + \right.$$

$$\omega_1 + \omega_2 = 1$$

$$c_2 \omega_2 = 1/2$$

$$a_{11} \omega_2 = 1/2$$

Since we have only three equations we can choose c_2 arbitrarily

$$\omega_1 = 1 - \omega_2 \quad 1 - \frac{1}{2c_2}$$

$$a_{21} = c_2$$

$$\omega_2 = \frac{1}{2c_2}$$

for different values of c_2 we can obtain family of schemes

$$y_{n+1} = y_n + h f_n + \frac{h^2}{2}(f_x + f_n f_y) + \frac{c_2 h^3}{4}(f_{xx} + 2 f_n f_{xy} + f_n^2 f_{yy}) + $$

Butcher's table

$$\begin{array}{c|cc} c_2 & a_{21} & \\ \hline & w_1 & w_2 \end{array}$$

$$\begin{array}{c|cc} 1/2 & 1/2 & \\ \hline & 0 & 1 \end{array} \qquad \begin{array}{c|cc} 2/3 & 2/3 & \\ \hline & 1/4 & 3/4 \end{array}$$

optimal method

Improved tangent

$$\begin{array}{c|cc} 1 & 1 & \\ \hline & 1/2 & 1/2 \end{array}$$

← Euler-Cauchy

R = 3, 4

3. stage RK

$$\begin{array}{c|ccc} c_2 & a_{21} & & \\ c_3 & a_{31} & a_{32} & \end{array}$$

Nystrom method

$$\begin{array}{c|ccc} 2/3 & 2/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 2/8 & 3/8 & 3/8 \end{array}$$

$$\begin{array}{c|ccc} 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 4/6 & 1/6 \end{array}$$

← classical Method

$$\begin{array}{c|ccc} 1/2 & 1/2 & & \\ 3/4 & 0 & 3/4 & \\ \hline & 2/9 & 3/9 & 4/9 \end{array}$$

Nearly optimal.

4- stage R-K

$$
\begin{array}{c|cccc}
1/2 & 1/2 & & & \\
1/2 & 0 & 1/2 & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 2/6 & 2/6 & 1/6
\end{array}
\qquad\qquad
\begin{array}{c|cccc}
1/3 & 1/3 & & & \\
2/3 & -1/3 & 1 & & \\
1 & 1 & -1 & 1 & \\
\hline
 & 1/8 & 3/8 & 3/8 & 1/8
\end{array}
$$

classical method $\qquad\qquad$ Kutta method

Truncation Error:

$$TE = \frac{1}{h}(LHS - RHS)$$

Stability

$$\begin{cases} y' = \lambda y & \lambda < 0 \in R \\ y(0) = y_0 \end{cases}$$

$$y(t) = y_0 e^{\lambda t} \to 0$$
$$t \to \infty$$

$$y_{n+1} = y_n + hf(t_n, y_n)$$
$$= y_n + h\lambda y_n$$
$$y_{n+1} = (1+\lambda h) y_n$$
$$= (1+\lambda h)^{n+1} y_0$$

$$y_n = (1+\lambda h)^n y_0$$

$$|1+\lambda h| < 1 \qquad\qquad -1 < 1+\lambda h < 1$$
$$-2 < \lambda h < 0$$
$$\boxed{h \in \left(0, \frac{-2}{\lambda}\right)}$$ restrict steps for stability

take diff values of λ and interval (0,6)

&

experiment violating restriction on h what is result.

for implicit euler

$\Rightarrow (1-\lambda h)y_{n+1} = y_n$

$$y_n = \left(\frac{1}{1-\lambda h}\right)^n y_0$$

what restriction on h??

as $n \to \infty$ does $y_n \to 0$ or not.

$$\left|\frac{1}{1-\lambda h}\right| < 1$$

$$-1 < \frac{1}{1-\lambda h} < 1$$ → its always less than since λ is negative

since this quantity is always less than 1 $y_n \to 0$ as $n \to \infty$ without any condition on step size h (unconditionally stable)

<u>for runge-kutta</u>

$y_{n+1} = y_0 + \omega_1 K_1 + \omega_2 K_2$

$\Rightarrow y_{n+1} = \left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right) y_n$

$$y_n = \left(1 + \lambda h + \frac{(\lambda h)^2}{2}\right)^n y_0$$

$y_n \to 0$ as $n \to \infty$

$$\left|1 + \lambda h + \frac{(\lambda h)^2}{2}\right| < 1$$

$$h \in \left(0, \frac{2}{|\lambda|}\right)$$

the 3-4 stage Runge Kutta Method

for 3 order:

$\left(0, \frac{2.51}{1.11}\right)$, (

for 4 order

$\left(0, \frac{2.78}{1.11}\right)$

Region of absolute Stability

Multistep Schemes

The Runge-Kutta methods we have to evaluate slopes K_1 K_2 K_3 which are computationally expensive in order to overcome the computational cost one can use the multistep methods which make use of interpolating integrand $f(t,y)$ at more than one previous point nodal point.

Let the domain $T(t_0, t_1, \ldots t_n)$ discritise by nodal points then integrating the differ eq we get this

$$y_{n+1} - y_n = \int_{t_n}^{t_{n+1}} f(t,y)\, dt \quad -①$$

In order to evaluate this Integral we approximate by a polynomial which interpolates $f(t, y(t))$ at $t_0, t_1 \ldots t_n$ by using newton's backward difference formula

$$p_m(t) = \sum_{k=0}^{m} (-1)^k \, {}^{-s}C_k \, \Delta^k f_{n-k} \quad -② \qquad s = \frac{(t - t_n)}{h}$$

using ② in ①, we get

$$y_{n+1} = y_n + \int_0^{1} \sum_{k=0}^{m} (-1)^k \binom{-s}{k} \Delta^k f_{n-k}\, dt$$

$$= y_n + h \sum_{k=0}^{m} (-1)^k \int_0^{1} \binom{-s}{k} ds\, \Delta^k f_{n-k}\, dt$$

$$y_{n+1} = y_n + h\left(\gamma_0 f_n + \gamma_1 \Delta f_{n-1} + \dots + \gamma_3 \Delta^3 f_{n-3}\right)$$

(4.1)

$$\gamma_k = (-1)^k \int_0^1 \binom{-s}{k} ds = \frac{1}{k}$$

$$\gamma_1 = \frac{1}{2}$$

$$\gamma_0 = 1, \quad \gamma_1 = \frac{1}{2}, \quad \gamma_2 = \frac{5}{12}, \quad \gamma_3 = \frac{3}{8}, \quad \gamma_4 = \frac{251}{720}$$

equation 1 is known as Adams Bashforth formula

$n = 3$

t_{n-3}	y_{n-3}	f_{n-3}	Δf_{n-3}	$\Delta^2 f_{n-3}$	$\Delta^3 f_{n-3}$
t_{n-2}	y_{n-2}	f_{n-2}	Δf_{n-2}	$\Delta^2 f_{n-2}$	
t_{n-1}	y_{n-1}	f_{n-1}	Δf_{n-1}		
t_n	y_n	f_n			

$$y_{n+1} = y_n + \frac{h}{24}\left(55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}\right)$$

Error in Newton Backward diff

$$h^4 f^{(4)}(\eta) \binom{-s}{4}$$

TE. $$E_{AB} = h \int_0^1 h^4 f^{(4)}(\eta) \binom{-s}{4} ds$$

$$= \frac{251}{720} h^5 y^{(5)}(\xi)$$

Adams - Moulton

Here to interpolate the integrand f at the point for some integer $m>0$

$t_{n-m} \ldots t_n, t_{n+1}$ $(m+2)$ points

the newton's backward difference formula which interpolates the function f at these $(m+2)$ points.

$$p_{m+1}(s) = \sum_{k=0}^{m+1} (-1)^k \binom{1-s}{k} \Delta^k f_{n+1-k} \quad -②$$

$$s = \frac{(t-t_n)}{h}$$

$$\int_{t_n}^{t_{n+1}} y'(s)\,ds = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds \quad -①$$

Putting ② in ①

$$y_{n+1} = y_n + h\left(\nu_0' f_{n+1} + \nu_1' \Delta f_n + \ldots \nu_{n+1}' \Delta^{m+1} f_{n-m}\, p_{m+1}(s)\right.$$

$$\nu_k' = (-1)^k \int_0^1 \binom{1-s}{k} ds \qquad k = 0 \ldots m+1$$

$$\nu_0' = 1 \quad \nu_1' = \frac{-1}{2} \quad \nu_2' = -\frac{1}{12} \quad \nu_3' = \frac{-1}{24} \quad \nu_4' = \frac{-19}{72}$$

Remainder E_{AM} = $\nu_{m+1}' h^{m+3} y^{(m+3)}(\xi)$
of taylor expansion

$$y_{n+1} = y_n + \frac{h}{24}\left(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}\right)$$

$$E_{AM} = -\frac{19}{720} h^5 y^{(5)}(\xi)$$

they are not self starting and one has to use any fourth order formula to evaluate approximate values the computational cost of multi step scheme are less compared to Runge-Kutta method of order 4 where we have to evaluate the slopes K_1, K_2, K_3, K_4 at each point but in multi step scheme we use existing info at previous nodal points.

Adams - Basheth - Predicator
Adams - Moulton - corrector

stability $\xleftrightarrow[\text{related to}]{}$ Roundoff error.

Difference Equations

Consider the general difference equations

$$\Delta^N y_n = f(n, y_n, \Delta y_n, \ldots, \Delta^{N-1} y_n) \quad \text{— ①}$$

In the case of linear you will have an equation of type

$$y_{n+N} + a_{n,N-1} y_{n+N-1} + a_{n,N-2} y_{n,N-2} \ldots + a_{n,0} y_0 = b_n$$

order will be N

$$y_{n+1} - y_n = 1 \quad \forall n$$

$$y_{n+1} - y_n = n, \quad n \geq 0 \rightarrow y_n = n + c$$

$$y_{n+1} - (n+1) y_n = 0 \quad \rightarrow n \text{ factorial}$$

$$y_{n+2} - (2\cos(\theta)) y_{n+1} + y_n = 1$$

Consider the following

$$y_{n+N} + a_{N-1} y_{n,N-1} + \ldots + a_0 y_n = 0 \quad a_0 \neq 0 \text{ — ②}$$

let us assume solution of form

$$y_n = \beta^n \quad \forall n \text{ — ③}$$

$$\beta^{n+N} + a_{N-1}\beta^{n+N-1} + \dots a_0\beta^n = 0 \quad — (3)$$

$$\beta^n(\beta^N + a_{N-1}\beta^{N-1} + \dots + a_0) = 0$$

$$p(\beta) = \beta^N + a_{N-1}\beta^{N-1} + \dots a_0 = 0 \quad — (4)$$

It has N roots which may be real or distinct Imaginary

<u>real or distinct roots</u>

Say roots are $\beta_1, \beta_2, \dots \beta_N$

$$y_n = c_1\beta_1^n + c_2\beta_2^n + \dots + c_N\beta_N^n$$

the constants $c_1, c_2, \dots c_N$ can be determined uniquely

eg

$$y_{n+3} - 2y_{n+2} - y_{n+1} + 2y_n = 0$$

$$\beta^3 - 2\beta^2 - \beta + 2 = 0 \qquad \beta = \pm 1, 2$$

$$y_n = c_1(1)^n + (-1)^n c_2 + 2^n c_3$$

$y_0 = 0$
$y_1 = 1$
$y_2 = 1$

$$y_0 = c_1 + c_2 + c_3 = 0$$
$$y_1 = c_1 - c_2 + 2c_3 = 1$$
$$y_2 = c_1 + c_2 + 4c_3 = 1$$

$$c_2 = \frac{1}{2}$$

$$y_n = (-1)^n\left(-\frac{1}{3}\right) + 2^n\left(\frac{1}{3}\right)$$

If the characteristic polynomial eq (4) has a pair of conjugate complex roots then

$$\beta_1 = \alpha + i\beta \qquad \beta_2 = \alpha - i\beta$$

using the polar form

$\beta_1 = re^{i\theta} \qquad \beta_2 = re^{-i\theta}$

$c_1\beta_1^n + c_2\beta_2^n$

Since β_1, β_2 are linear independent we can write

$c_1\beta_1^n + c_2\beta_2^n = c_1 r^n e^{in\theta} + c_2 r^n e^{-in\theta}$

$= r^n(c_1 e^{\cancel{in}} \cos n\theta + c_2 \sin n\theta)$

$y_{n+2} - 2y_{n+1} - 2y_n = 0$

$\beta^2 - 2\beta + 2 = 0$

the $r = \sqrt{2}$

$\theta = \pi/4$

$= (\sqrt{2})^n (c_1 \cos \frac{n\pi}{4} + c_2 \sin \frac{n\pi}{4})$

one root β_1 is multiplicity 2 another $n\beta_1^n$

$p(\beta) = 0 \qquad p'(\beta_1) = 0$

$y_n = n\beta_1^n$ — ⊛⊛

$\beta^{n+N} + a_{N-1}\beta^{n+N-1} + \ldots + a_0\beta^n = 0$

$\beta^n \underbrace{(\beta^N + a_{N-1}\beta^{N-1} + \ldots a_0)}_{p(\beta)} = 0$

use ⊛⊛ in eq ①

$(n+N)\beta_1^{n+N} + a_{N-1}(n+N-1)\beta_1^{n+N-1} + \ldots n a_0 \beta_1^n =$

$\beta_1^n(n \underbrace{(\beta_1^N + a_{N-1}\beta_1^{N-1} + \ldots a_0)}_{p(\beta)} + \beta_1 \underbrace{(N\beta_1^{N-1} + (N-1)\beta_1^{N-2} + \ldots a_1)}_{p'(\beta_1)}]$

$\beta_1^n(n \times 0 + \beta_1 \times 0) = 0$

example

$y_{n+3} - 5y_{n+2} + 8y_{n+1} - 4y_n = 0$

$\beta^3 - 5\beta^2 + 8\beta - 4 = 0$ 1 , 2 , 2

$y_n = c_1(1)^n + c_2(2^n) + c_3(n2^n)$

$= c_1 + 2^n(c_2 + nc_3)$.

Consider the non homogeneous ~~co~~ difference equation

$a_n y_n = b_n$

then the solution can be.

$y_n = \underbrace{y_n^{G}}_{\text{General}} + y_n^{P} \rightarrow$ Particular

Consider the case $b_n = b$ (some constant independent of n)

$y_n^P = A$ (constant)

$A = \dfrac{b}{1 + a_{n-1} + \dots + a_0}$ Provided Denominator $\neq 0$

example.

$y_{n+2} - 2y_{n+1} + 2y_n = 1$

For homogeneous

$\beta_2 - 2\beta_1 + 2\beta_0 = 0$

$$y' = f(t, y)$$

$$\frac{y_{n+1} - y_{n-1}}{2h} = f(t_n, y_n)$$

$$y_{n+1} = y_{n-1} + 2h f(t, y_n)$$

$$TE = O(h^2)$$

Example

$$y' = -2y + 1$$
$$y(0) = 1.$$

$$y(t) = \frac{1}{2}\left(e^{-2t} + 1\right)$$

$$y_{n+1} = y_{n-1} + 2h(-2y_n + 1)$$

$$y_{n+1} = \cancel{y} + 4hy_n - y_{n-1} = 2h$$

$$\beta^2 + 4h\beta - 1 = 0$$

$$\beta = \frac{-4h \pm \sqrt{16h^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$\beta = \frac{-4h \pm \sqrt{16h^2 + 4}}{2}$$

$$\beta = -2h \pm \sqrt{1 + 4h^2}$$

$$y_n = c_1\beta_1^n + c_2\beta_2^n + \frac{1}{2}$$

Approximating $\sqrt{1+4h^2}$ to linear terms $\beta_1 = 1 - 2h$
$\beta_2 = -(1 + 2h)$

$$y_n = c_1(1 - 2h)^n + c_2(-1)(1 + 2h)^n + \frac{1}{2}$$

As $\lim_{\epsilon \to 0} (1 + \epsilon)^{1/\epsilon} = \exp(1)$

$$\lim_{h\to 0} (1+2h)^{?} = \lim_{h\to 0} (1+2h)^{?}$$

$$= \lim_{h\to 0} \left(\left((1+2h)^{?} \right)^{1/2h} \right)^{2t_n}$$

$$= (\exp^{?})^{2t_n}$$

$$= \exp^{2t_n}$$

$$y_n = \left(C_1 \exp(-2t_n) + \frac{1}{2} \right) + C_2 (-1)^n \exp(-2t_n) \quad \text{—— ⊙⊙}$$

exponentially inc. $C_2 \neq 0$ (due to rounding error)

$$y_{n+1} = y_n + h f(t_n, y_n)$$

$$y_{n+1} = y_n + h(-2y_n + 1)$$

$$\Rightarrow y_{n+1} - (1-2h) y_n = h$$

$$y_n = C_1 (1-2h)^n + \frac{1}{2}$$

$$y_n = C_1 \exp^{(-2t_n)} + \frac{1}{2}$$

Since $y(0) = 1$

$\therefore C_1 = 1/2$

$$y_n = \frac{1}{2}\left(\exp^{(-2t_n)} + 1 \right) \quad \text{—— ⊙}$$

Solution of diff equation corresponding to explicit euler exactly explicit exact solution whereas solution ⊙⊙ obtained from the central difference from y' containing the extra term that $C_2 (-1)^n \exp(-2t_n)$ if $C_2 = 0$ from initial condition then we don't have any problem unfortunately due to round off error $C_2 \neq 0$. this extra term is oscillatory increasing amplitude which come into existence because we replace a first order ode by a second order diff equation and this extra term is nothing to do with exact solution, which is called extraneous term.

Now for the stability of multistep schemes one root of the difference equation corresponding to exact solution and the rest of β_i's are strictly less than 1 then we do not have any problem we say multistep scheme is obsolutely stable.

1 class missed

$$u'' + Ku' = 0$$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + K\frac{u_{i+1} - u_i}{h} = 0$$

$$\left(\frac{1}{h^2} + \frac{K}{h}\right) u_{i+1} - \left(\frac{2}{h^2} + \frac{K}{h}\right) u_i + \frac{1}{h^2} u_{i-1} = 0$$

$$u_n = c_1 (1)^n + c_2 \left(\frac{1}{1+Kh}\right)^n$$

No restriction $\longleftrightarrow$ $K>0$ $h>0$

Pa de' approx

If $K<0$ $h < \frac{1}{|K|}$

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + K\frac{u_i - u_{i-2}}{h} = 0$$

$$\frac{1}{h^2} u_{i+1} - \left(\frac{2}{h^2} - \frac{K}{h}\right) u_i + \left(\frac{1}{h^2} - \frac{K}{h}\right) u_{i-1} = 0$$

$$\frac{1}{h^2} \beta^2 - (\ )\beta + (\ ) = 0$$

$$\beta_1 = 1 \qquad \beta_2 = 1 - Kh$$

$U_n = c_1(1)^n + c_2(1+kh)^n$

if $k<0$: no oscillation

if $k>0$ $\quad h < \frac{1}{k}$

<u>Conclusion :</u>

One has to be careful in approximating convection term as it depends on convection coefficient k

$$u_j' = \begin{cases} D^+U_j = \dfrac{U_{j+1}-U_j}{h} & \text{if } k>0 \\[2ex] D^-U_j = \dfrac{U_j-U_{j-1}}{h} & \text{if } k<0 \end{cases}$$

PDE

$$P_x + Q_t = R$$

$$U_t + aU_x = 0$$

$$U_t - U_{xx} = 0 \longleftarrow \text{Parabolic}$$

$$U_{tt} = c^2 U_{xx}$$

$$\Delta U = U_{xx} + U_{yy} = 0$$

Steady state

$$\left.\begin{aligned} &U_t - U_{xx} = 0 \quad (x,t) \in (0,1)\times(0,T) \\ &t=0 : U(x,0) = f(x) \\ &U(0,t) = f_1(t) \quad U(1,t) = f_2(t) \end{aligned}\right\} \text{IBVP}$$

$$u(x,t) = X(x)\,T(t)$$
$$\uparrow \quad \uparrow$$

$u_m^n = u(t_n, x_m)$

<u>step 1</u>: Discretization of the domain

$$\Delta t = k = \frac{1}{N}$$

$$\Delta x = h = \frac{1}{H}$$

<u>step 2</u> Derivatives by FD

$$u_t\Big|_{x_m t_n} = \begin{cases} \dfrac{U_m^{n+1} - U_m^{n}}{k} \\ \dfrac{U_m^{n} - U_m^{n-1}}{k} \\ \dfrac{U_m^{n+1} - U_m^{n-1}}{2k} \end{cases}$$

IC : $U_m^0 = \phi(x_n)$ BCs $U_0^n = f(t_n)$

$U_N^n = f_2(t_n)$

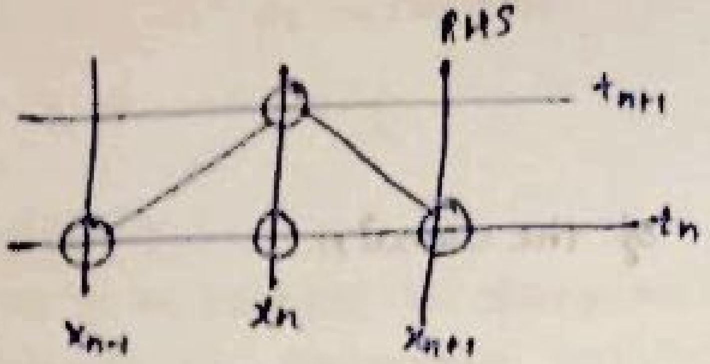
$$U_{xx}\Big|_{(x_m, t_n)} = \frac{U_{m+1}^{n} - 2U_m^{n} + U_{m-1}^{n}}{h^2}$$

:-

$$\frac{U_m^{n+1} - U_m^{n}}{k} = \frac{U_{m+1}^{n} - 2U_m^{n} + U_{m-1}^{n}}{h^2} = 0$$

$$\Rightarrow U_m^{n+1} = \lambda U_{m-1}^{n} + (1-2\lambda)U_m^{n} + \lambda U_{m+1}^{n}$$

$n=0$



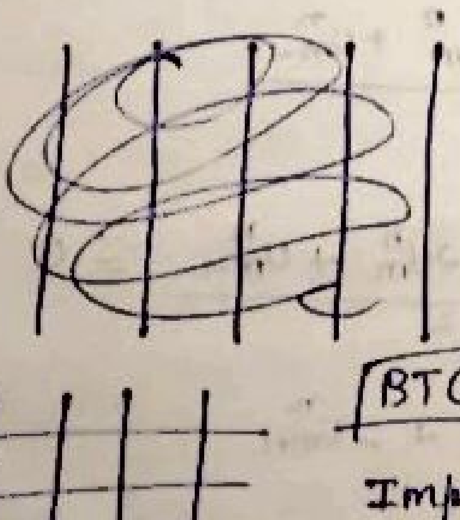
Similarly we can march $0 \to 1$ and so on
this is an explicit scheme (Euler for time) or forward
(central diff for space)
scheme is called
<u>forward time central space (FTCS)</u>.

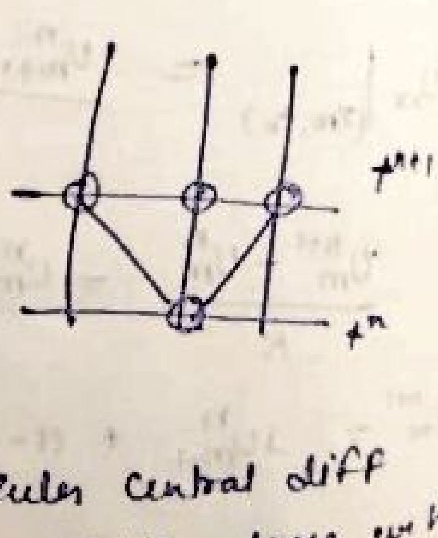
replace LHS by $\dfrac{U_m^n - U_m^{n-1}}{k}$

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2} \qquad 0 \le n \le N-1$$

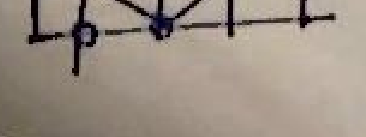
$$1 \le m < N-1$$

$$-\lambda U_{m-1}^{n+1} + (1+2\lambda) U_m^{n+1} - \lambda U_{m+1}^{n+1} = U_m^n \qquad \lambda = k/h^2$$





$\boxed{\text{BTCS}}$



Implicit Euler central diff

because of Implicit nature we have to solve system of linear algebra system at each time level.
(expensive).

average of FTCS & BTCS

addition of all these schemes $= \frac{1}{2}$ (LHS + RHS)

first order in time
second order in space
so TE $= O(\Delta t + \Delta x^2)$ for both scheme

$= \frac{1}{2}$ (FTCS + BTCS)

$(\ ) U_{m-1}^{n+1} + (\ ) U_m^{n+1} + (\ ) U_{m+1}^{n+1} = (\ ) U_{m-1}^n + (\ ) U_m^n + (\ ) U_{m+1}^n$

$U_m^{n+1} = \lambda U_{m-1}^n + (1 - 2\lambda) U_m^n + \lambda U_{m+1}^n$

$-\lambda U_{m-1}^{n+1} + (1 + 2\lambda) U_m^{n+1} - \lambda U_{m+1}^{n+1} = U_m^n$

$-\lambda U_{m-1}^{n+1} + (1+2\lambda) U_m^{n+1} - \lambda U_{m+1}^{n+1} = \lambda U_{m-1}^n + (1-2\lambda) U_m^n$

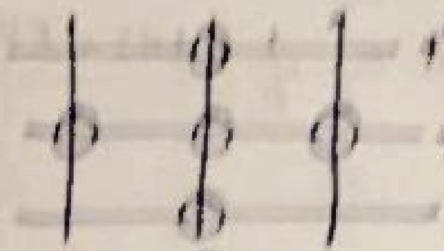
$U_m^{n+1} - \lambda U_{m-1} + (1+2\lambda) U_m^{n+1} - \lambda U_{m+1}^{n+1} = \lambda U_{m-1}^n + \ldots + \lambda U_{m+1}^n + U_m^n$

this is again an implicit scheme at each time level we have to solve system of linear algebraic system
the advantage is second order in both time and space (TE : $O(\Delta t^2 + \Delta x^2)$
and name is (Crank Nicolson
all these schemes are two level schemes

fourth scheme :

$\frac{U_m^{n+1} - U_m^{n-1}}{2k} = \frac{U_{m+1}^n - 2U_m^n + U_{m-1}^n}{h^2}$ $\quad S = \frac{k}{h^2}$

$U_m^{n+1} = U_m^{n+1} + \lambda U_{m-1}^{n} + (1-2\lambda) U_m^{n} + \lambda U_{m+1}^{n}$



$u_t - \Delta u = f(x,y,t)$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= u_{xx} + u_{yy}$$

$\Delta u = f(x,y)$

Elliptic BVPs

$\Delta u = f(x,y)$

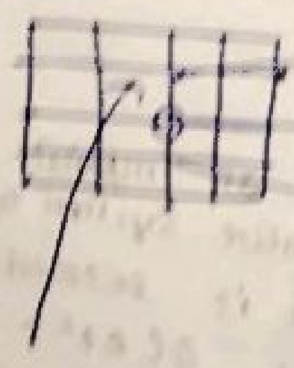
$u(x,y) = g(x,y)$

Step 1

Discretization of the domain

$h = (b-a)/N$

$K = (d-c)/M \quad h = k \quad M = N$



→ super diagonal

main diagonal

→ sub diagonal

Step 2 :

Replacing the derivative by difference quotient :

$$\left(\frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}\right) + \left(\frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2}\right) = f_{ij}$$

$1 \le i,j \le m$

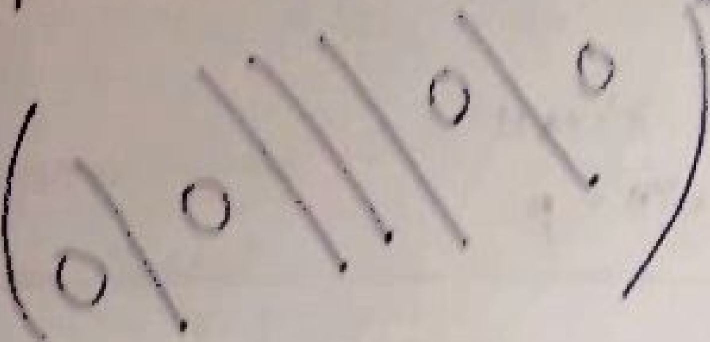
$U_{i-1j} - 4U_{ij} + U_{i+1j} + U_{ij-1} + U_{ij+1} = h^2 f_{ij}$

$k = (i-1)(N+1) + j$

for $i, j = 1, ?$

$p = (i-1)(N+1) + j$

Sparse matrix



→ 5 non zero entries parallel to main diagonal.

$A(P, P)$ $\quad A(P, P-1)$

$A(P, P+1)$ $\quad A(P-1, p)$

$A(P+1, p)$

$AV = F$
Stiffness matrix

A is a $\boxed{\begin{matrix} A^T = A \\ \text{Symmetric matrix} \end{matrix}}$ which has 5 non-zero diagonal entries ~~that~~ and it is positive definite

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{N} |a_{ij}|$$

Matrix A is diagonally dominant irreducible.

Substitute $i=1$ and $j=1$

$U'' = f(x)$ $\qquad x = (0,1)$

$U(0) = \alpha$ $\qquad U(1) = \beta$

$\dfrac{U_{j-1} - 2U_j + U_{j+1}}{h^2} = f_j$

$+\frac{1}{h^2}U_{i-1} - \frac{2}{h^2}U_i + \frac{1}{h^2}U_{i+1} = f_i$

$u(x,y) = g(x,y)$

$\frac{\partial u}{\partial x} = f(x,y)$

$u'' = f(x) \quad x \in (0,1)$

$u(0) = \alpha \quad u'(1) = \beta$

$\Delta u = f(x,y) \qquad \Omega = (0,1)^2$

$$
\begin{array}{ccccc}
 & & (i,j+1) & & \\
 & & | & & \\
(i-1,j) & \leftarrow & (i,j) & \rightarrow & (i+1,j) \\
 & & | & & \\
 & & (i,j-1) & &
\end{array}
$$

LHS - RHS = Ω

$u'' = f$

$O(h^4)$

$$U_{i,j\pm1} = U_{ij} \pm h\frac{\partial u}{\partial y} + \frac{h^2}{2!}\frac{\partial^2 u}{\partial y^2} \pm \frac{h^3}{3!}\frac{\partial^3 u}{\partial y^3} + \frac{h^4}{4!}\frac{\partial^4 u}{\partial y^4} + \dots$$

$U_{i\pm1,j}$

$u \in C^4(\Omega)$

$$\left(\frac{U_{i+1j} - 2U_{ij} + U_{i-1j}}{h^2}\right) + \left(\frac{U_{ij+1} - 2U_{ij} + U_{ij-1}}{h^2}\right)$$

$$= f_{ij}$$

$$U_{ij}^{(s+1)} = \frac{1}{4}\left(U_{i+1j}^{(s)} + U_{i-1j}^{(s)} + U_{ij+1}^{(s)} + U_{ij-1}^{(s)}\right) - \frac{h^2}{4} f_{ij}$$

<u>Poisson Solver</u> $AV = F$

this is the Jacobi

stopping criteria

$$\left| U_{ij}^{(s+1)} - U_{ij}^{(s)} \right| < Tol = 10^{-4}$$

that is called this Maximum Principle

$$\frac{U_{i+1} - 2U_{i} + U_{i-1}}{h^2}$$

this is other way of ordinary : Red black

this type of ordering will be well suited for parallel computing

<u>The discrete maximum principle</u> :

For any Mesh function V_{ij} just defined on discrete domain Ω^h the following maximum principle holds true.

$$Lu = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y)$$

If $L^h V_{ij} \geq 0 \quad \forall\, i,j$

$\Rightarrow$ the maximum $\underbrace{V_{ij}}_{\Omega^h \text{ boundary}} \geq \underbrace{\max V_{ij}}_{\Omega^h \text{ Interior}}$

<u>The Proof</u>: Assume the contrary i_0, j_0

Suppose take i_0, j_0

or interior point where maximum exist

$i_0, j_0 \quad V_{i_0,j_0} = \max\limits_{\Omega^h} V_{ij} \;\Big|\; V_{i_0,j_0} > \max\limits_{\Omega^h} V_{ij}$

Given $L^h V_{ij} \geq 0$

$$V_{i_0 j_0} \leq \frac{1}{4}\left(V_{i_0+1\, j_0} + V_{i_0-1\, j_0} + V_{i_0\, j_0+1} + V_{i_0\, j_0-1}\right)$$

But we $V_{i_0 j_0} \geq \max$ all neigbouring points

$$V_{i_0 j_0} \geq \max\{V_{x+1\, j_0},\ V_{i_0+j_0+1}\ V_{i, j+1}\ V_{i j_0 -1})$$

If any of the four ~~boundary~~ neighbouring point is on boundary we get the condition

If the max or min exist in the interior the function remain constant similarly we can show mini principle If $L^h V_{ij} \leq 0$

then $\max_{\Omega^h} v_{ij} \le \max_{\partial\Omega^h} v_{ij}$

(boundary) (interior)

Corollary: the uniqueness of the solution. Let u_{ij} and v_{ij} be two solution of $L^h v_{ij} = f_{ij}$ then both are identical

$u_{ij} = v_{ij}$

for $\begin{cases} Lu = f, & \Omega \\ u = g, & \partial\Omega \end{cases}$

$$w_{ij} = u_{ij} - v_{ij}$$

$$L^h w_{ij} = L^h u_{ij} - L^h v_{ij}$$
$$= f_{ij} - f_{ij} = 0$$

$$w_{ij} = 0, \Gamma^h$$

$$\max_{\Gamma^h} w_{ij} \ge \max_{\Omega^h} w_{ij} \ge \min_{\Omega^h} w_{ij} \ge \min_{\Gamma^h} w_{ij}$$

The discrete solution v_{ij} satisfies the following bound

$$\max_{\Omega^h} |v_{ij}| \le \max_{\Gamma^h} |v_{ij}| + \frac{1}{4} \max |L^h v_{ij}|$$

Consider the Mesh function

$$\phi_{ij} = (ih)^2/2$$

$$\max_{\Gamma^h} |\phi_{ij}| = 1/2$$

$$L^h \phi_{ij} = 1$$

$$M = \max_{\Omega^h} |L^h v_{ij}|$$

define the mesh function such that

$w_{ij}^{\pm}$, w_{ij} $\quad w_{ij}^{\pm} = \pm v_{ij} + M\phi_{ij}$

$L^h w_{ij}^{\pm} \geq 0$?

from maximum principle

$$\max_{\Omega^h} w_{ij} \leq \max_{\beta^h} w_{ij}$$

$$V_{ij} = U_{ij}$$

$$\max_{\Omega^h} |v_{ij}| \leq \max_{\Omega^h} | g_{ij}) + \frac{1}{2} \max | f_{ij} |$$

Round off

$$\max w_{ij}^{\pm} \leq \max w_{ij}^{\pm}$$

$$\max_{\Omega^h} (\pm v_{ij} + M\varrho) \leq \max_{\beta} (\quad)$$

$$\pm \max |v_{ij}| \leq \max |v_{ij}| + M \max | f_{ij} |$$

$$e_{ij} = U(x_i, y_i) - v_{ij}$$

$$L^h e_{ij} = | L^h U(x_i, y_i) - L^h e_{ij} = \frac{\partial^2 U}{\partial x}$$

apriori error estimate

2D PDE

$$U_t = -\Delta U = f(x, y, t)$$

$$-U'' = f$$

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f$$

Consider the two dimension heat conduction

$$\frac{U^{n+1}_{lm} - U^{n}_{lm}}{k} = \left(\frac{U^{n}_{l+1m} - 2U^{n}_{lm} + U^{n}_{l-1m}}{h^2}\right)$$

$$+ \left(\frac{U^{n}_{lm+1} - 2U^{n}_{lm} + U^{n}_{lm-1}}{h^2}\right)$$

$$= f(x_l, y_m, t_n) \qquad \text{(FTCS) FTCS}$$

BTCS is an implicit scheme at each time level to we have to solve one elliptic boundary value problem

average of FTCS + BTCS will give crank nikolson scheme

$O(\Delta t + \Delta x^2 + \Delta y^2)$

$O(k + h^2)$

C-N $O(k^2 + h^2)$

PDE 1D heat eq
PDE 2D heat eq
steady st eq

$U_t + aU_x = 0$

$x \in (0,1)$
$t \in (0,T]$

$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$

$U_t = U_{xx}$

$U(x,0) = \sin \pi x$

$U(0,t) = \cdot$

~~the soluti~~

We know that solution is constant along any characteristic curve, which completely determines the solution at any point (x,t) by projecting along the characteristic back to the x-axis that is $t=0$.

Similarly for $a<0$ characteristic are other direction.
One of the major properties of the solution of hyperbolic PDE is that of wave propagation which can be seen from exact solution.

<u>1 class missed</u>

Von-Neuman Stability

$$|g(\theta)| < 1$$

one-step

$$\exists\ K, \theta, k, h$$

$$|g(\theta,k,h)| \leq 1 + Kk$$

$$\forall (k,h) \in \Omega$$

if g is independs k & h

$$|g(\theta)| \leq 1$$

$$U_t = U_{xx}$$

<u>FTCS</u>

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}}{h^2}$$

$$\Rightarrow U_m^{n+1} = \lambda U_{m+1}^n + (1-2\lambda) U_m^n + \lambda U_{m-1}^n$$

$$U_m^n = g^n e^{im\theta}$$

$$g^{n+1}e^{im\theta} = d g^n e^{i(m+1)\theta} + (1-2d) g^n e^{im\theta} + d g^n e^{i(m-1)\theta}$$

$$g(\theta) = d e^{i\theta} + (1-2d) + d e^{-i\theta}$$

$$|g(\theta)| = \left|1 - 4d\sin^2(\theta/2)\right| \le 1$$

$$\Rightarrow -1 \le \quad \le 1$$

BTCS

$$g(\theta) = \frac{1}{1 + 4d\sin^2(\theta/2)}$$

since $d > 0$

we do not require condition

$0 < d < 1/2$

$k \le h^2/2$

$\Rightarrow$ unconditionally stable

Crank Nikolson (exam *)

(Average of FTCS & BTCS)

$$g(\theta) = \frac{1 - 2d\sin^2(\theta/2)}{1 + 2d\sin^2(\theta/2)}$$

Unconditionally stable

It is the beauty of implicit schemes.

Study the stability of FTCS & BTCS scheme for two dimensional parabolic schemes.

Hyperbolic PDE

$$u_t = \Delta u$$

$$u_t + a u_x = 0$$

FTFS

$$\frac{U_m^{n+1} - U_m^n}{k} + a\frac{U_{m+1}^n - U_m^n}{h} = 0$$

$$R = \frac{ak}{h}$$

$$g(\theta) = (1+R) - Re^{i\theta}$$

If $|R| \le 1$ is it stable ??

If $|R| \le 1$
then difference scheme is stable

$$|g(\theta)|^2 = (1+R)^2 - 2R(1+R)\cos\theta + R^2$$

Here we have to use some alternate way to study the stability that determine the max and min for $|g(\theta)|^2$ for $\theta \in (-\pi,$

take $\theta = -\pi, 0, \pi$

$$|g(0)|^2 = 1$$

$$|g(\pm\pi)| = |1 + 2R|$$

critical for g

$$|g(0)| = 1$$

$$|g(\pm\pi)| = |1+2R|$$

$$|g(\pm\pi)| \le 1 \Rightarrow -1 \le 1 + 2R \le 1$$

$$\Rightarrow -2 \le 2R \le 0$$

$$\Rightarrow -1 \le R \le 0$$

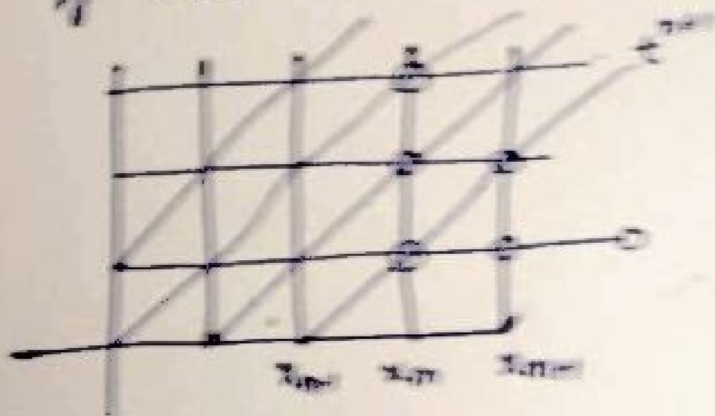
If a is -ve then R is -ve

Case I: If a is -ve then R is -ve
and the scheme is conditionally stable
FTBS

condition is ~~under~~ $-1 \leq R < 0$

Case II: If a is +ve then R is +ve ie $R > 0$
so that is FTBS is unconditionally
stable

If $a = 0 \Rightarrow$ then ODE



If we take $R = 1$