MA 322: Scientific Computing

Department of Mathematics
Indian Institute of Technology Guwahati

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Chapter 4: Numerical Integrations or Quadratures

Classical theory of numerical integration in dimension $d = 1$

Newton-Cotes quadrature rule

$$\int_0^1 f(x)\mathrm{d}x \approx \sum_{j=0}^{n} w_j f(x_j),$$

where $x_j \in [0,1]$ are the nodes and $w_j \in \mathbb{R}$, $j = 0, 1, \ldots, n$, the quadrature weights satisfying $\sum_{j=0}^{n} w_j = 1$.

Quadrature error

$$E_n(f) = \int_0^1 f(x)\mathrm{d}x - \sum_{j=0}^{n} w_j f(x_j).$$

Classical theory of numerical integration in dimension $d = 1$

- Left rectangle rule with equally-spaced points $x_j = \dfrac{j-1}{n}$ and weights $w_j = \dfrac{1}{n}$, $j = 0, 1, \ldots, n$, with error $E_n(f) \leq \dfrac{\|f'\|_\infty}{2n}$ if $f \in C^1$.
- Trapeziodal rule with error $E_n(f) = \mathbf{O}\left(\dfrac{1}{n^2}\right)$ if $f \in C^2$.
- Simpson's rule with error $E_n(f) = \mathbf{O}\left(\dfrac{1}{n^4}\right)$ if $f \in C^4$.
- Gaussian quadrature rules with nodes being zeros of certain polynomials are exact for all polynomials of degree $2n-1$.

Quadrature rule in dimension $d > 1$

Theorem

$$\lim_{n\to\infty} E_n(f) = 0,\ \forall f \in C([0,1]) \quad \textit{iff} \quad \sup_{n\in\mathbb{N}} \sum_{j=0}^{n} |w_j| < \infty$$

The result carries over to $[0,1]^d,\ d > 1$, and to more general domains.

How to extend the ideas to higher dimension $d > 1$?

An obvious way is the product rule in $[0,1]^d$.

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Curse of dimensionality

- Take d one-dimensional quadrature rules with weights $w_{j_i} \in \mathbb{R}$ and points $x_{j_i} \in [0,1],\ i = 1, 2, \ldots d$ and consider

$$\int_0^1 \cdots \int_0^1 f(x_1, \ldots x_d)\, \mathrm{d}x_1 \cdots \mathrm{d}x_d \approx \sum_{j_1=1}^{m_1} \cdots \sum_{j_d=1}^{m_d} \prod_{i=1}^{d} w_{j_i} f(x_{j_1}, \ldots x_{j_d}).$$

- The number of quadrature formula is $n = \prod_{i=1}^{d} m_i$. For $m_i = m,\ i = 1, 2, \ldots d$, the total number is $n = m^d$; hence, it grows exponentially.
- (Curse of dimensionality) For example, the product rectangular rule has order order $\mathbf{O}(m^{-1}) = \mathbf{O}(n^{-1/d})$.