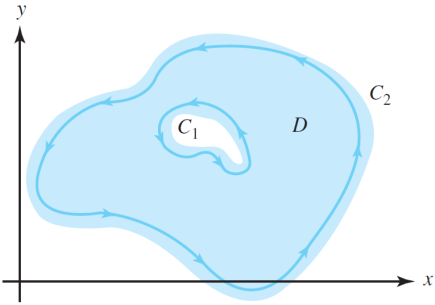
MA 201 Complex Analysis
Lecture 10: Cauchy Integral Formula

Deformation of contours

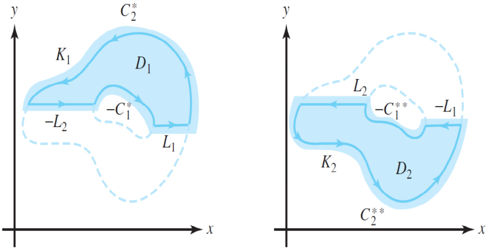
Theorem Let C_1 and C_2 be two simple closed positively oriented contours such that C_1 lies interior to C_2. If f is analytic in a domain D that contains both C_1 and C_2 and the region between them, then

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$



Deformation of contours

Proof. Assume that both C_1 and C_2 have positive (counterclockwise) orientation. We construct two disjoint contours or cuts L_1 and L_2 that join C_1 to C_2. The contour C_1 is cut into two contours C_1^* and C_1^{**} and the C_2 is cut into two contours C_2^* and C_2^{**}.



$$K_1 = -C_1^* + L_1 + C_2^* - L_2 \quad \text{and} \quad K_2 = -C_1^{**} + L_2 + C_2^{**} - L_1.$$

The function f will be analytic on a simply connected domain D_1, D_2 containing K_1, K_2 respectively.

By Cauchy's theorem,

$$\int_{K_1} f(z)dz = \int_{k_2} f(z)dz = 0.$$

Also $K_1 + K_2 = C_2^* + C_2^{**} - C_1^* - C_1^{**} = C_2 - C_1$. Thus

$$\int_{K_1+K_2} f(z)dz = \int_{C_2-C_1} f(z)dz = 0$$

this implies that

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz.$$

Example: If C is a positively oriented simple closed contour surrounding the origin then

$$\int_C \frac{1}{z}dz = 2\pi i$$

Deformation of contours

Theorem Let $C, C_1, C_2 \cdots, C_n$ be simple closed positively oriented contours such that C_k lies interior to C for $k = 1, 2, \cdots, n$ and C_k has no point in common with the interior of C_j if $k \neq j$. Let f be analytic on a domain D that contains all the contour and the region between C and $C_1 + C_2 + \cdots + C_n$. Then

$$\int_C f(z)dz = \sum_{k=1}^{n} \int_{C_k} f(z)dz.$$

Cauchy Integral Formula

Theorem Let f be analytic on a simply connected domain D. Suppose that $z_0 \in D$ and C is a simple closed curve oriented in the counterclockwise in D that encloses z_0. Then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \qquad (\textbf{Cauchy Integral Formula}).$$

Proof. Let $C(z_0, r)$ denotes the circle of radius r around z_0 for a sufficiently small $r > 0$ then

$$\begin{aligned}
\left| \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz - f(z_0) \right| &= \left| \frac{1}{2\pi i} \int_{C(z_0, r)} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \\
&= \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta}) - f(z_0)}{re^{i\theta}} ire^{i\theta} d\theta \right| \\
&\leq \frac{1}{2\pi} 2\pi \times \sup_{\theta \in [0, 2\pi]} |f(z_0 + re^{i\theta}) - f(z_0)| \\
&\qquad (\text{ by } ML \text{ inequality}).
\end{aligned}$$

As f is continuous it follows that the righthand side goes to zero as r tends to zero.

Cauchy Integral Formula

- $\displaystyle\int_{|z-4|=5} \frac{\cos z}{z} dz = 2\pi i.$
- $\displaystyle\int_{|z-i|=1} \frac{z^2}{z^2+1} dz = -\pi.$
- Can we use Cauchy's integral formula to evaluate the following?

$$I = \int_{|z|=2} \frac{e^z}{z(z-1)} dz$$

Yes ! Write

$$I = \int_{C(0,2)} \frac{e^z}{z-1} dz - \int_{C(0,2)} \frac{e^z}{z} dz$$

now apply Cauchy's integral formula to get the value of the integral as $2\pi i(e-1)$.

Cauchy Integral Formula for higher derivatives

Theorem If f is analytic on a simply connected domain D then f has derivatives of all orders in D (which are then analytic in D). For any $z_0 \in D$ one has

$$f^n(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

where C is a simple closed contour (oriented counterclockwise) around z_0 in D.

Proof: By Cauchy's integral formula

$$\begin{aligned}
f'(z_0) &= \lim_{h\to 0} \frac{f(z_0+h) - f(z_0)}{h} \\
&= \lim_{h\to 0} \frac{1}{2\pi i h} \int_C \left(\frac{f(z)}{z - z_0 - h} - \frac{f(z)}{z - z_0} \right) dz \\
&\qquad (C \text{ is so chosen that the point } z_0 + h \text{ is enclosed by } C) \\
&= \lim_{h\to 0} \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0 - h)(z - z_0)} dz.
\end{aligned}$$

Cauchy Integral Formula for higher derivatives

So we need to prove that

$$\begin{aligned}
&\left| \int_C \frac{f(z)}{(z-z_0-h)(z-z_0)} dz - \int_C \frac{f(z)}{(z-z_0)^2} dz \right| \\
= &\left| \int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2} dz \right| \to 0, \text{ as } \quad h \to 0.
\end{aligned}$$

We will use ML inequality to prove this.

- Let $|f(z)| \leq M$ for all $z \in C$.
- Let $\alpha = \min\{|z-z_0| : z \in C\}$, then $|z-z_0|^2 \geq \alpha^2$.
- $\alpha \leq |z-z_0| = |z-z_0-h+h| \leq |z-z_0-h| + |h|$.
- Hence for $|h| \leq \frac{\alpha}{2}$ we have $|z-z_0-h| \geq \alpha - |h| \geq \frac{\alpha}{2}$.

Therefore

$$\left| \int_C \frac{f(z)h}{(z-z_0-h)(z-z_0)^2} dz \right| \leq \frac{M|h|l}{\frac{\alpha}{2}\alpha^2} = \frac{2M|h|l}{\alpha^3} \to 0,$$

as $h \to 0$.

By repeating exactly the same technique we get

$$f^2(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^3} dz$$

and so on.

Cauchy Integral Formula for higher derivatives

- $\displaystyle\int_{|z|=1} e^z z^{-3} dz = i\pi.$
- $\displaystyle\int_{|z-1|=5/2} \frac{1}{(z-4)(z+1)^4} dz = \frac{2\pi i}{3!} \frac{d^3}{dz^3}\left(\frac{1}{z-4}\right)\Big|_{z=-1}.$

Summary Let C be a simple closed curve contained in a simply connected domain D and f is an analytic function defined on D. Then

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \begin{cases} 2\pi i f(z_0), & \text{if } n = 0 \text{ and } z_0 \text{ is enclosed by } C. \\ \frac{2\pi i}{n!} f^n(z_0), & \text{if } n \geq 1 \text{ and } z_0 \text{ is enclosed by } C. \\ 0, & z_0 \text{ lies out side the region enclosed by } C. \end{cases}$$