(S1) The joint pdf of (X, Y) is

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1} y^{\alpha_2-1} e^{-\beta(x+y)} & \text{if } x>0,\ y>0 \\ 0 & \text{o.w.} \end{cases}$$

Let $U = \dfrac{X}{X+Y}$ and $V = X+Y$

Here

~~$g(\ldots) = \ldots$~~

~~$(\ldots) = g(g_1(x,y), g_2(x,y))$~~

$(u, v) = g(x, y) = (g_1(x,y), g_2(x,y))$

$= \left(\dfrac{x}{x+y}, x+y \right)$

and

$(x, y) = h(u, v) = (h_1(u,v), h_2(u,v))$

$= (uv, v(1-u))$

✓ clearly g is one-to-one

✓ Both g and its inverse exists and they ~~are~~ are continuous.

✓ partial derivatives exists and continuous as,

$$\frac{\partial x}{\partial u} = v, \quad \frac{\partial x}{\partial v} = u$$

$$\frac{\partial y}{\partial u} = -v, \quad \frac{\partial y}{\partial v} = 1-u$$

✓ Jacobian, $J = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v > 0.$

Hence all the four remained conditions hold.

Therefore, the joint PDF of ~~...~~ (U, V) is,

~~$f_{U,V}(u,v) =$~~

$$f_{U,V}(u,v) = \begin{cases} \dfrac{\beta^{\alpha_1+\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot (uv)^{\alpha_1-1} ((1-u)v)^{\alpha_2-1} e^{-\beta v} \, |v| & 0<u<1, v>0 \\ 0 & \text{o.w.} \end{cases}$$

So, the marginal PDF of U is,

$$f_U(u) = \int_0^\infty \frac{\beta^{\alpha_1+\alpha_2}(1-u)^{\alpha_2-1}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} u^{\alpha_1-1} v^{\alpha_1+\alpha_2-1} e^{-\beta v} \, dv$$

$$= \frac{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} \cdot \int_0^\infty \beta (\beta v)^{\alpha_1+\alpha_2-1} e^{-\beta v} \, dv$$

Take $\beta v = z$

$$= \frac{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} \cdot \int_0^\infty z^{\alpha_1+\alpha_2-1} e^{-z} \, dz$$

$$= \frac{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}{\Gamma(\alpha_1)\,\Gamma(\alpha_2)} \cdot \Gamma(\alpha_1+\alpha_2)$$

$$= \frac{u^{\alpha_1-1}(1-u)^{\alpha_2-1}}{B(\alpha_1,\alpha_2)}.$$

Hence, the PDF of $U = \frac{X}{X+Y}$, is

$$f_U(u) = \begin{cases} \dfrac{1}{B(\alpha_1,\alpha_2)} u^{\alpha_1-1}(1-u)^{\alpha_2-1}, & \text{if } 0<u<1 \\ 0 & \text{o.w.} \end{cases}$$

Q2

(a) The marginal PDF of X is,

$$f_X(x) = \begin{cases} 4x\int_0^1 y\,dy & \text{if } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 2x & \text{if } 0<x<1 \\ 0 & \text{o.w.} \end{cases}$$

Similarly, the marginal PDF of Y is,

$$f_Y(y) = \begin{cases} 2y & \text{if } 0<y<1 \\ 0 & \text{o.w.} \end{cases}$$

(b) From above, it is clear that,

$$f_{X,Y}(x,y) = f_X(x)\,f_Y(y) \text{ for all } (x,y)\in\mathbb{R}^2.$$

Hence, X and Y are independent.

(c) $P(0<X<0.5,\ 0.25<Y<1)$

$= P(0<X<\frac{1}{2})\,P(\frac{1}{4}<Y<1)$. (X & Y are independent)

$= \left(\frac{1}{2}\right)^2 \times \left(1-\left(\frac{1}{4}\right)^2\right)$

$= \frac{1}{4} \times \frac{15}{16} = \frac{15}{64}$.

Also, $P(X+Y<1)$

$= \iint_{x+y<1} f_{X,Y}(x,y)\,dx\,dy$

$= \int_0^1\int_0^1 4xy\,dx\,dy$ ($x+y<1$)

$= 4\int_0^1 y\int_0^{1-y} x\,dx\,dy$

$= \frac{4}{2}\int_0^1 y(1-y)^2\,dy$

$= \frac{4}{2}B(2,3)$

$= 2\cdot\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)}$

$= 2\cdot\frac{2}{24}$

$= \frac{4}{24}$

$= \frac{1}{6}$.

Q3 (a) $\int_0^\infty P(X>x)\,dx$

$= \int_0^\infty \int_x^\infty f_X(t)\,dt\,dx$, $0<x<t<\infty$.

$= \int_0^\infty f_X(t) \int_0^t 1 \cdot dx\,dt$

$= \int_0^\infty t\, f_X(t)\,dt$

$= E(X)$

$\therefore E(X) = \int_0^\infty P(X>x)\,dx.$

(Corresponding r.v. is X).
$= \int_0^\infty x\, f_X(x)\,dx$ by $t=x$
$= E(X)$

(b) Now, $E(X) = \int_0^\infty P(X>x)\,dx$

$= \int_0^1 P(X>x)\,dx + \int_1^2 P(X>x)\,dx + \cdots \quad \text{--- } ①$

From ①,

$E(X) \leq \int_0^1 P(X>0)\,dx + \int_1^2 P(X>1)\,dx + \int_2^3 P(X>2)\,dx + \cdots$

$= P(X>0)\int_0^1 dx + P(X>1)\int_1^2 dx + \cdots$

$= P(X>0) + P(X>1) + \cdots$

$= \sum_{n=0}^{\infty} P(X>n)$

Again from ①, $E(X) \geq \int_0^1 P(X>1)\,dx + \int_1^2 P(X>2)\,dx + \cdots$

$= P(X>1) + P(X>2) + \cdots$

$= \sum_{n=1}^{\infty} P(X>n).$

Hence, $\sum_{n=1}^{\infty} P(X>n) \leq E(X) \leq \sum_{n=0}^{\infty} P(X>n).$

(Q4) $P(A) = \liminf_{n\to\infty} \frac{\#(A \cap \{1,2,\dots,n\})}{n}$, $A \subseteq \mathbb{N}$.

If possible, let $P(A)$ is a prob. measure ... (1) on $\mathcal{F}$.

For $A = \{k\}$, the set $A \cap \{1,2,\dots,n\}$ is empty if $n < k$ and has just one element k if $n \geq k$.

Thus, the sequence in the formula (1), for $P(\{k\})$ begin with $(k-1)$ zeros followed by $\frac{1}{k}, \frac{1}{k+1}, \dots,$ which converges to 0.

So, $P(\{k\}) = 0$.

Now, $P(\Omega) = P(\mathbb{N}) = \sum_{k=1}^{\infty} P(\{k\})$ using countable additivity, $\{k\}$ for $k=1,2,\dots$ are disjoint.

$= 0$

But $P(\Omega) = 1$.

So, it is a contradiction.

Hence, $P(A)$ is not a probability measure on $\mathcal{F}$.