Post Midterm

26/9/22

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Jacobi (iterative method) :-

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right) \longleftrightarrow 1$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k)} - a_{23}x_3^{(k)}\right)$$

Gauss siedel :- $x_1^{(k+1)}$ = same as 1

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right)$$

$Ax = b$ ~~(If A's~~

$Mx = (M - A)x + b$

M - pre conditioner
- non-singular

$x = (I - M^{-1}A)x + M^{-1}b$

Use fixed pt iterat^n

$$x^{(k+1)} = \underbrace{(I - M^{-1}A)}_{B\text{ - iterative matrix}} x^{(k)} + M^{-1}b$$

$x^{(k)}$ will converge if $\rho(B) < 1$ — spectral radius — max of eigen value of B

$$\boxed{\rho(B) < 1 \Leftrightarrow \|B\|_2 < 1 \Leftrightarrow \|B\|_\infty = 1}$$

Let

$A = D - L - U$

For Jacobi :- $M = D$

$$x^{(k+1)} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{a_{11}}a_{11} & \frac{1}{a_{11}}a_{12} & \frac{1}{a_{11}}a_{13} \\ \frac{1}{a_{22}}a_{21} & & \\ & & \end{bmatrix}\right) x^{(k)}$$

$$+ \begin{bmatrix} \frac{b_1}{a_{11}} \\ \frac{b_2}{a_{22}} \\ \frac{b_3}{} \end{bmatrix}$$

(OR) $\because A = D - L - U$

$$M - A = D - A = L + U$$

$$Dx^{(k+1)} = (L+U)x^{(k)} + b$$

Gauss Siedel :-

$$M = D - L$$

In order to speed up the convergence of Gauss-Siedel, we can introduce the parameter ω_R

$$M = \frac{1}{\omega_R} D - L$$ — Successive Over Relaxat^n (SOR)

S!ly there is Successive Under Relaxation

$$M - A = \left(\frac{1}{\omega_R} - 1\right) D + U$$

$$\left(\frac{1}{\omega_R} D - L\right) x^{(k+1)} = \left(\left(\frac{1}{\omega_k} - 1\right) D + U\right) x^{(k)} + b$$

$$x^{(k+1)} = \left(\frac{1}{\omega_R} D - L\right)^{-1} \left(\left(\frac{1}{\omega_k} - 1\right) D + U\right) x^{(k)} + b$$

Done using Row Oriented Forward Substitut^n

Stopping criteria :-

i) $\|x^n - x\| < Tol$

ii) $r^{(n)} = Ax^{(n)} - b \quad \|r^{(n)}\| < tol$

iii) $\|x^n - x^{n-1}\| < tol$

Optimal ω_R (for speedy convergence) :-

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(B_J)^2}}$$

$B_J = D^{-1}(L+U)$

In Case of American, we have to impose the side condit^n $y \geq g$ at each iterat^n.

Correction vector $:- x^{(k)} - x^{(k-1)}$

$$b - Ax^{(k)},\ r_i^{(k)} = b - \sum_{j=1}^{i} a_{ij} x_j^{(k)} - a_{ii} x_i^{(k-1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k-1)}$$

$$x_i^{(k)} = x_i^{(k-1)} + \omega_k \frac{r_i^{(k)}}{a_{ii}}$$ — $r_i^{(k)}$ also has $x^{(k)}$ term

Projected
PSOR

$$x_i^{(k)} = \max\left\{0,\ x_i^{(k-1)} + \omega_k \frac{r_i^{(k)}}{a_{ii}}\right\}$$ — to accomodate $y \geq q$

$$y_i = -r_i^{(k)} + a_{ii}\left(x_i^{(k)} - x_i^{(k-1)}\right)$$

$\because$ Cryer's problem has a unique min

22/9/24 – Friday

$-u'' = f$

$u(0) = 0 = u(1)$

$v \in C_0^\infty(\Omega)$

$-\int u'' v\, dx = \int f v.\, dx$

$$-\int v\, d(u') = -(uv)_0^1 + \int_0^1 u'v'\, dx = \int f v.\, dx$$

u', v' – weak sense

$u \in L^p(\Omega)$ Lebesgue

u', v' – weak sense

$$L^p(\Omega) = \left\{ u \in \Omega : \int_\Omega (u(x))^p dx < \infty \right\}$$

If $u \in L^2, u' \in L^2 \Leftrightarrow u \in H^1(\Omega)$ $\longrightarrow$ this in appendix C2 in book

$H^1(\Omega) \subseteq C_0^\infty(\Omega)$ — f's here need to be infinitely differentiable

$u \in C^k(\Omega),\ v \in C_0^\infty(\Omega)$:

$$\int_\Omega D^\alpha u(x)\, v(x).dx = (-1)^{|\alpha|} \int_\Omega u(x)\, D^\alpha v(x).dx, \quad |\alpha| \leq k,$$ 1

$\forall v \in C_0^\infty(\Omega)$

(Pf by using by parts)

Let $u \in L^1(\Omega)$

$$\int_\Omega w_\alpha(x)\, v(x)\,.dx = (-1)^{|\alpha|} \int_\Omega u(x)\, D^\alpha v(x)\,.dx \quad \forall v \in C_0^\infty(\Omega) \longrightarrow (2)$$

w_α is the weak α^{th} derivative (See eqn (1) or (2) RHS of)

For $u \in C^\alpha$, $w_\alpha = u$ (the weak α^{th} derivative of u is u^α itself)

The 1st weak derivative of $|x| = \text{sgn}(x)$

For L^p fn's,

$$W_p^k(\Omega) = \{ u \in L^p(\Omega) : D^\alpha(u) \in L^p(\Omega),\ |\alpha| \le k \}$$ — Sobolev space

$$\|u\|_{W_p^k(\Omega)} = \left(\sum_{|\alpha|\le k} \|D^\alpha u\|_{L^p(\Omega)} \right)^{1/p} , 1 \le p < \infty$$

$$H^k(\Omega) := W_2^k(\Omega) = \{ u \in L^2, D^\alpha u \in L^2 \}$$

Hilbert space

$$(u,v) = \sum_{|k|\le k} (D^\alpha u, D^\alpha v)$$

norm of u,v — inner product

$$H^1(\Omega) = \left\{ u \in L^2(\Omega) : \frac{\partial u}{\partial x_j} \in L^2(\Omega),\ j = 1, \dots, n \right\}$$

weak derivatives — $u \in L^2(\Omega)$ may not be diff

$$\|u\|_{H^1(\Omega)} = \left(\|u\|^2_{L^2_p} + \sum_{j=1}^n \left\| \frac{\partial u}{\partial x_j} \right\|^2 \right)$$

H_E^1 $\quad -u'' = f$

$u(0) = 0$, $u(1) = 1$ $\quad$ We want u from H_E^1

- $H_E^1(\Omega) = \{ u \in H^1(\Omega) : u(0) = 0, u(1) = 1 \}$, "Whatever condit^n's on u are applied on v also" — So $v \in H_E^1$

$$-\int u'' v\,.dx = \int f v\,.dx$$

$$-(vu')\Big|_0^1 + \int v' u'\, dx = \int f v\,.dx$$

$$\int v' u'\,.dx = \int f v\,.dx + v(1)u'(1) - u'(0)v(0)$$
$$= \int f v\,.dx + u'(1)$$

3/10/24

Dirichlet BVP

$$-u'' = f$$
$$-u'(0) = \alpha$$
$$u'(1) = \beta$$

$$U_h(x) = \sum_{i=1}^{N} U_i \phi_i(x)$$

Neumann BVP

$$U_h(x) = \sum_{i=0}^{N+1} U_i \phi_i(x)$$

Robin

$$-u'' = f$$
$$u(0) - u'(0) = \alpha$$
$$u(1) + u'(1) = \beta$$

$$H_R^1(\Omega) = \{ u \in \Omega : u(0) - u'(0) = \alpha,\ u(1) + u'(1) = \beta \}$$

find $u \in H_R^1(\Omega)$ s.t

$$\int u'v'\,dx = \int f v\,dx + (\text{boundary cond}^n\text{s}) \quad \forall v \in H_R^1(\Omega)$$

Consider the obstacle problem. In order to solve using FEM, we define

$K = \{ v \in C^0[-1,1] : v(-1) = v(1) = 0,\ v(x) \geq g(x)\ \forall x \in \bar{\Omega}$ v is piecewise $C^1 \}$

$\bar{\Omega} = \Omega \cup \underbrace{\Omega^0}_{\text{boundary pts}}$

The sol^n u of obstacle problem

find $u \in C_0^1(x)$ s.t

$$\left. \begin{array}{l} u''(u-g) = 0,\ -u'' \geq 0,\ u - g \geq 0 \\ u(-1) = u(1) = 0 \end{array} \right\} (1),$$

$u \subset K$ and $\forall v \in K$ $v - g \geq 0$

$\therefore$ The FEM tells

$$\int_{-1}^{1} -u''(v-g)\,dx \geq 0 \quad \forall v \in K \rightarrow (2)$$

As $-u'' \geq 0$ & $v \in K \Rightarrow v - g \geq 0$

From (1), $$-\int_{-1}^{1} u''(u-g)\,dx = 0 \rightarrow (3)$$

$(2) - (3) \Rightarrow$ $$-\int u''(v-u)\,dx \geq 0 \longrightarrow (4)$$

④ doesn't contain q explicity whereas it is inside the space K.

④ —

$$\int_{-1}^{1} -d(u')(v-u).\,dx \geq 0$$

$$\left[-u'(v-u)\right]_{-1}^{1} + \int_{-1}^{1} u'(v-u)'.\,dx \geq 0$$

$\because v \in K$, $v(-1) = u(-1) = v(1) = u(1) = 0$

Weak formulat^n

find $u \in K \ni$

$$\int u'(v-u)'.\,dx \geq 0 \quad \forall\, v \in K \quad \Big\}\ ⑤$$

Since ⑤ is known as variat^{n}al inequality

If we consider an approximat^n $w (\in K)$ for $u (\in K)$

$$\int_{-1}^{1} w'(v-w)'.\,dx \geq 0 \quad \forall v \in K$$

$$\sum_{i=1}^{N} \int_{-1}^{1} \phi_i'(x)\left(\phi_j(x) - \phi_i(x)\right)'.\,dx \geq 0 \quad \forall j$$

$$w_n(x) = \sum_{i=0}^{N+1} w_i \phi_i(x)$$

In ⑤, when $v = u$, the integral vanishes.

$\therefore$ ⑤ is a minimizat^n problem

$$\left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g) = 0$$

$$\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0, \quad y - g \geq 0$$

$$y(x,0) = g(x,0)$$

$$y(x_{min},\tau) = g(x_{min},\tau)$$

$$y(x_{max},\tau) = g(x_{max},\tau)$$

K – competing / admissible f^{ns}

$K = \{ v \in C_0^0 : \frac{\partial v}{\partial x}$ is piecewise C_0^0, $v(x,\tau) \geq g(x,\tau)\ \forall\ x,\tau$

$v(x,0) = g(x,0),\ v(x_{min},\tau) = g(x_{min},\tau),$

$v(x_{max},\tau) = g(x_{max},\tau) \}$

9/10/24

$v \geq g$; $\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2} \geq 0$

$$① \leftarrow \int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-g)\cdot dx \geq 0 \qquad y, v \in K$$

$$② \leftarrow \int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g)\cdot dx = 0$$

From ① & ②,

$$\int_{x_{min}}^{x_{max}} \left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(v-y)\cdot dx \geq 0$$

By parts

$$\int_{x_{min}}^{x_{max}} \left[\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right]\cdot dx - \cancel{\left[\frac{\partial y}{\partial x}(v-y)\right]_{x_{min}}^{x_{max}}} \geq 0$$

$\leftarrow 0$

$$③ \leftarrow \int_{x_{min}}^{x_{max}} \underbrace{\left[\frac{\partial y}{\partial \tau}(v-y) + \frac{\partial y}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial y}{\partial x}\right)\right]}_{S}\cdot dx \geq 0$$

~~If~~ $v = y$

③ becomes a minimizat^n problem

$$I(y;v) = \int_{x_{min}}^{x_{max}} S\cdot dx \geq 0 \quad \forall v \in K$$

$$\min_{v \in K} I(y;v) = I(y,y) = 0$$

For American opt^n, we require C_0^2, $y \in K$

and we expect the sol^n $\hat{y}$

$$\inf_{v \in K} I(\hat{y}; v) = 0$$

$K = \text{span}\{\phi_i\}$

ϕ_i's are hat f^{n}s

(a) $$\begin{cases} \hat{y} = \sum_i w_i(\tau)\phi_i(x) \\ v = \sum v_i(\tau)\phi_i(x) \end{cases}$$

$$\int_{x_{min}}^{x_{max}} \left(\sum \frac{\partial w}{\partial \tau_i}\phi_i\right) \sum (v_j - w_j)\phi_j(x)$$

$$+ \sum_i w_i(\tau)\phi_i'(x)\left(\sum_j (v_j(\tau) - w_j(\tau))\phi_j'(x)\right) dx$$

$$- \sum_i \sum_j \frac{\partial w_i}{\partial \tau}(v_j - w_j) \int \underbrace{\phi_i(x)\phi_j(x)}_{B} \, dx + \sum_i \sum_j w_i(v_j - w_j) \int \underbrace{\phi_i'\phi_j'}_{A} \, dx \geq 0$$

$$\geq 0$$

$$\left(\frac{\partial w}{\partial \tau}\right)^T B(v - w) + w^T A(v - w) \geq 0 \Leftrightarrow (v - w)^T \left[B\frac{dw}{d\tau} + Aw\right] \geq 0 \rightarrow (f)$$

$B = \int \phi_i \phi_j \, dx$ ~~$A = \int \phi_j \phi_i' \, dx$~~

$B^T = B$ $A = \int \phi_i' \phi_j' \, dx$

(f) can be discretized by $\theta \in [0, 1]$ method

$$(v^{(n+1)} - w^{(n+1)})\left[B \cdot \frac{1}{\Delta\tau}(w^{(n+1)} - w^{(n)}) + \theta A w^{(n+1)} + (1 - \theta) A w^{(n)}\right] \geq 0$$

$\theta = 0 \Rightarrow$ explicit

$\theta = \frac{1}{2} \Rightarrow$ CN

rewritten as

$$(v^{(n+1)} - w^{(n+1)})\left[(B + \Delta\tau\theta A)w^{(n+1)} - (B - \Delta\tau(1 - \theta)A)w^{(n)}\right] \geq 0$$

$r = (B - \Delta\tau(1 - \theta)A)w^{(n)}$

$C = B + \Delta\tau\theta A$

$$(v^{(n+1)} - w^{(n+1)})^T(Cw^{(n+1)} - r) \geq 0$$

$$y(x, \tau) \geq g(x, \tau)$$

$$\sum w_i(\tau_i)\phi_i(x) \geq g(x, \tau) \rightarrow (*)$$

In order to incorporate the side condit^n $w_j(\tau) \geq g(x_j, \tau)$

Suppose statement holds for $\leq n$

Let $q_1 = v/\|v\|_2$ then $\|q_1\|_2 = 1$ and $Aq_1 = \lambda q_1$

Let $\{q_1, w_2, w \dots w_n\}$ be L.I. in $\mathbb{C}^n$.

Then let $S := [q_1 \; w_1 \; w_2 \dots w_n] = \tilde{Q}R$

$$\tilde{Q} = [q_1 \; q_1 \dots q_n]$$

$$\tilde{Q}^* A \tilde{Q} = \begin{bmatrix} q_1^* \\ \vdots \\ q_n^* \end{bmatrix} A [q_1 \dots q_n]$$

$$= \begin{bmatrix} q_1^* A q_1 & q_1^* A q_2 & \cdots \\ q_2^* A q_1 & & \\ \vdots & & \\ q_n^* A q_1 & & \end{bmatrix}$$

$$\tilde{Q}^* A \tilde{Q} = \left[\begin{array}{c|c} \lambda q_1^* q_1 & * \\ \hline \lambda q_2^* q_1 & \\ \vdots & \tilde{A} \\ \lambda q_n^* q_1 & \end{array}\right] = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{A} \end{array}\right] = \left[\begin{array}{c|c} \lambda & * \\ \hline 0 & \tilde{Q}_1 \tilde{T} \tilde{Q}_1^* \end{array}\right]$$

$$= \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q}_1 \end{array}\right] \underbrace{\left[\begin{array}{c|c} \lambda & * \\ \hline & \tilde{T} \end{array}\right]}_{:= T} \left[\begin{array}{c|c} 1 & \\ \hline & \tilde{Q}_1^* \end{array}\right]$$

If A is hermitian $\rightarrow T$ is real diagonal.

If T is real diagonal $\rightarrow A$ is hermitian

Hermitian matrix have only real eigen values.

$Av = \lambda v$

FEM

$r = (B - \Delta T(1-\theta)A) w^{(m)}$

$C := B + \Delta \tau \theta A$

for all $v \geq g$

$(v-w)^T (cw - r) \geq 0, w \geq g$

$(v-w)^T (cw-r) \geq 0$

$(v^T - w^T)(cw - r) \geq 0$

$v^T cw - v^T r - w^T cw + w^T r \geq 0$

$v^T cw - v^T r \geq w^T cw - w^T r$

FDM

$A = c, b = r$

$cw - r \geq 0, w \geq g$

$(cw - r)^T (w-g) = 0$

Assume w satisfies FDM.

$$(v-w)^T (cw-r) = (v-g)^T (cw-r) - \underbrace{(w-g)^T (cw-r)}_{\geq 0} \geq 0$$

Assume w satisfies FEM

$w \geq g$

$(v-w)^T (cw-r) \geq 0$

$v^T (cw-r) \geq w^T (cw-r)$

Take $v = g$

$(g-w)^T (cw-r) \geq 0$

$\Rightarrow (cw-r)^T (w-g) \leq 0$

We know $w \geq g$. If we show $cw - r \geq 0$ then $(cw-r)^T (w-g) = 0$

Suppose k^{th} comp. of $cw - r$ is -ve.

If I take k^{th} comp. of v is as large as possible.

then k^{th} comp. of $v^T (cw-r)$ is too small which is a contradiction.?

$\therefore (cw - r) \geq 0$.

FEM and FDM are giving the same sol^n for the basic elements.

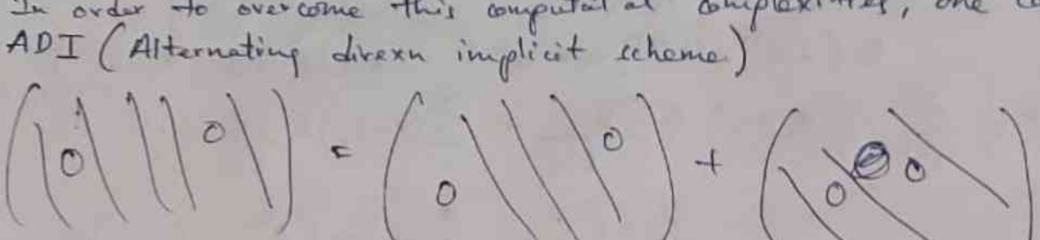
The implementation of FEM PSOR.

ADI Alternating direction implicit.

If we discretize the PDEs corresponding to exotic op^{t^n} given in ① or the 2-D heat conduxn given in ② by any implicit scheme like BTCS or the CN then the bandwidth of the matrix A $(AU = F)$ will be large.

∴ It takes enormous CPU time & memory to solve the sys of linear algebraic $eq^{n}s$.

In order to overcome this computat^{n}al complexities, one can use ADI (Alternating direxn implicit scheme)



$$t^n \longrightarrow t^{n+1} \Rightarrow t^n \rightarrow t^{n+1/2} \rightarrow t^{n+1}$$

When we progress from t^n to t^{n+1}, we introduce $t^{n+1/2}$ and the whole scheme will be returned in the following manner

i) $t^n \rightarrow t^{n+1/2}$:- x implicit & y explicit
ii) $t^{n+1/2} \rightarrow t^{n+1}$:- y " x " or vice-versa

$$\frac{U_{lm}^{n+1/2} - U_{lm}^{n}}{\left(\frac{\delta t}{2}\right)} = \left(\frac{U_{lm}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l-1m}^{n+1/2}}{h^2}\right) + \left(\frac{U_{lm+1}^{n} - 2U_{lm}^{n} + U_{lm-1}^{n}}{h^2}\right) \quad ③$$

$$\frac{U_{lm}^{n+1} - U_{lm}^{n+1/2}}{\left(\frac{\delta t}{2}\right)} = \left(\frac{U_{l+1m}^{n+1/2} - 2U_{lm}^{n+1/2} + U_{l+1m}^{n+1/2}}{h^2}\right) + \left(\frac{U_{lm+1}^{n+1} - 2U_{lm}^{n+1} + U_{lm-1}^{n+1}}{h^2}\right)$$

Starting from the given initial condit^{n}s, solve ③. By using sol^{n} of ③, solve ④.

In order to incorporate the side condit^n $w_j(\tau) \geq g(x_j, \tau)$

7/10/24 — Absent

17/10/24 — Thur

$V(S,t) \longrightarrow \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV = 0 \longrightarrow$ Called 1-D only although S & t both are there

$V(S_1, S_2, t)$?

$$dS_1 = \mu_1 S_1 . dt + \sigma_1 S_1 \, dW_t$$
$$dS_2 = \mu_2 S_2 . dt + \sigma_2 S_2 \, dW_{2t}$$
$$E\left(dW^{(1)}, dW^{(2)}\right) = r . dt$$

$Cor\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right) = E(\sigma_1 dW^{(1)}, \sigma_2 dW^{(2)}) = r\sigma_1\sigma_2 \, dt$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j S_i S_j \frac{\partial^2 V}{\partial S_i \partial S_j} + \sum_{i=1}^{n}(r-\delta_i) S_i \frac{\partial V}{\partial S_i} - rV = 0$$

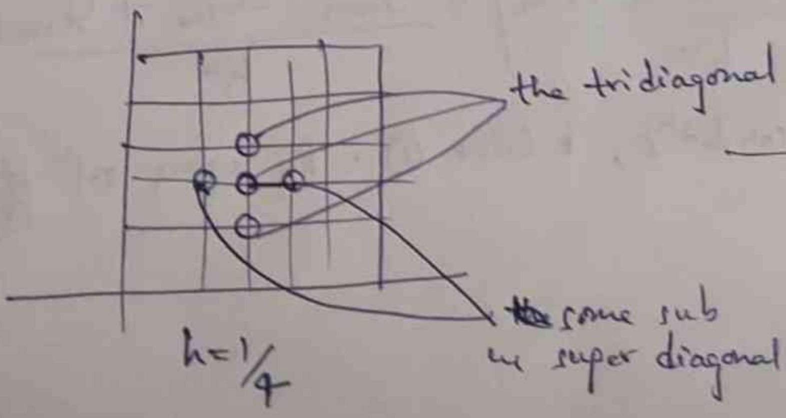
$$\Omega = (S^1_{min}, S^1_{max}) \times (S^2_{min}, S^2_{max}) \times \dots \times [0, T]$$

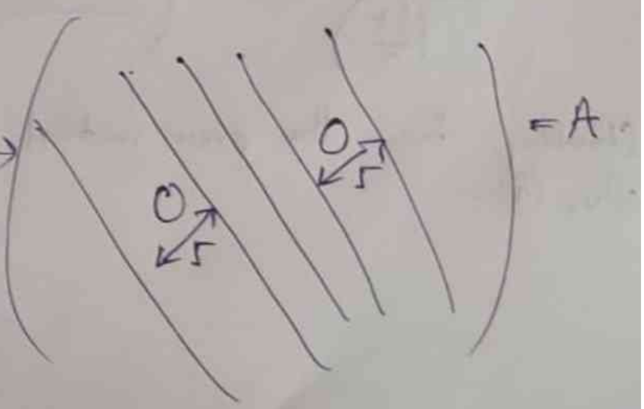
$$\frac{\partial V}{\partial t} + \frac{1}{2}\left(\rho_{12}\sigma_1\sigma_2 S_1 S_2 \left(\frac{\partial^2 V}{\partial S_1^2} + \frac{\partial^2 V}{\partial S_2^2} + \cancel{\frac{\partial^2 V}{\partial S_1 \partial S_2}}^{0}\right)\right) \longrightarrow (1)$$

2D Parabolic PDE

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \longrightarrow (2) \quad (x,y) \in (0,1) \times (0,t), \quad t \in (0,T]$$

$$\frac{U^{n+1}_{lm} - U^{n}_{lm}}{\delta t} = \left(\frac{U^{n+1}_{l+1\,m} - 2U^{n+1}_{lm} + U^{n+1}_{l-1\,m}}{h^2}\right) + \left(\frac{U^{n+1}_{l\,m+1} - 2U^{n+1}_{lm} + U^{n+1}_{l\,m+1}}{h^2}\right)$$





24/10/24

Tranformat^n

$$V(S,A,t) = \tilde{V}(S,R,t) = S H(R,t)$$
$$R = \frac{A}{S}$$

$$V_t = S H_t$$
$$V_S = H + S \frac{\partial H}{\partial R}\left(-\frac{A}{S^2}\right) = H + \cancel{S}\frac{\partial H}{\partial R}\left(-\frac{A}{S}\right)$$
$$V_{SS} = \frac{\partial H}{\partial R}\cdot\left(-\frac{A}{S^2}\right) + \cancel{S} H_{RR}\left(-\frac{A}{S}\right)\left(-\frac{A}{S^2}\right) + \left(+\frac{A}{S^2}\right)H_R$$
$$V_A = \cancel{S H_R \frac{\partial R}{\partial S}} \; S H_R \cdot \frac{\partial R}{\partial A} = \cancel{S} H_R \cdot \frac{1}{\cancel{S}}$$
$$S H_t + \frac{\sigma^2 S^2}{2} H_{RR}\cdot\frac{A^2}{S^3} + rS\left(H - H_R\left(\frac{A}{S}\right)\right) + S H_R - rSH = 0$$
$$H_t + \frac{\sigma^2}{2} R^2 H_{RR} - H_R R r + H_R = 0$$
$$\boxed{\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR) H_R = 0}$$

$$H(R,t) = \left(1 - \frac{1}{T} R_T\right)^+$$

As $R_T \to \infty$, ~~$H \to 0$~~ $H(R_T, T) = 0$
$$R_t = \frac{1}{S_t}\int_0^t S_\theta \, d\theta$$

The integral R_t is bdd. $\therefore$ When $S \to 0$, $R \to \infty$

This is the European call option.

We cannot exercise the call option.

$H \to 0$ as $R \to \infty$

$$\left.\frac{\partial H}{\partial R}\right|_{R=0} = \frac{3H_0^n + 4H_1^n - H_2^n}{2\delta R} + O(\delta R^2)$$

Once we obtain H, We can obtain $V = SH$

When Avg is on ~~discrete~~ discrete time intervals, then

$$A_{t_k} = \frac{1}{k}\sum_{i=1}^{k} S_{t_i} \qquad k = 1:M$$

$$A_{t_k} = \frac{(k-1)A_{t_{k-1}} + S_{t_k}}{k} = A_{t_{k-1}} + \frac{(S_{t_k} - A_{t_{k-1}})}{k}$$

$$A_{t_{k-1}} = \frac{kA_{t_k} - S_t}{k-1} = A_{t_k} - \frac{(S_{t_k} - A_{t_k})}{k-1}$$

We can observe that A_t is const in b/n time intervals and it jumps at t_k with $+\frac{(A_{t_k} - S_{t_k})}{k-1}$.

~~From no arbitrage~~

Jump at k^{th} step:

$$A^-(S) = A^+(S) + \frac{1}{k-1}\left(A^+(S) - S\right), \quad S = S_{t_k}$$

From no arbitrage principle we can, from the continuity of value of that opt^n at t_k for any realizat^n of the random walk:-

$$V(S, A^+, t_k) = V(S, A^-, t_k)$$

For any fixed S & A, the equat^n defines the jump at t_k.

For the numerical calculat^n of this jump condit^n, if we discretize the A axis into $A_1, A_2, \ldots A_J$, then for each time period b/n two consecutive samplings, $t_{k+1} \to t_k$, the option value is independent of A bcz in the discretion of A_t, A_t is piecewise const. $\therefore \frac{\partial V}{\partial A} = 0$. So for each j, we get 1-D BS PDE. $(j = 1:J)$ & each of them are independent of one another.

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$$dR_t = \left(1 + (\sigma^2 - \mu) R_t\right) \cdot dt + \sigma R_t \cdot dW_t$$

$$dR\Big|_{t=0} = dt \qquad \cancel{R_T = 0}$$

For the left hand bdry condit^{n}s, we face some difficulties. For ex; if $R_0 = 0$, $dR_0 = dt$ ∴ R_t won't remain at 0.
∴ We can't expect R_T to be zero.

∴ To obtain the bdary cond^n, we use the PDE directly.
~~∴ ...~~ At $R=0$,

$$\frac{\partial H}{\partial t} + \frac{\sigma^2}{2} \cancel{R^2 \frac{\partial^2 H}{\partial R^2}}^{\to 0} + \underbrace{(1 - rR)\frac{\partial H}{\partial R}}_{= \frac{\partial H}{\partial R}} = 0$$

$\to 0$ iff $\frac{\partial^2 H}{\partial R^2}$ is bdd (or H is bdd)

~~Suppose~~ Claim :- $R^2 \frac{\partial^2 H}{\partial R^2}$ is bdd as $R \to 0$

Pf :- Suppose $R^2 \frac{\partial^2 H}{\partial R^2} = c \Rightarrow \frac{\partial^2 H}{\partial R^2} = O\left(\frac{1}{R^2}\right)$

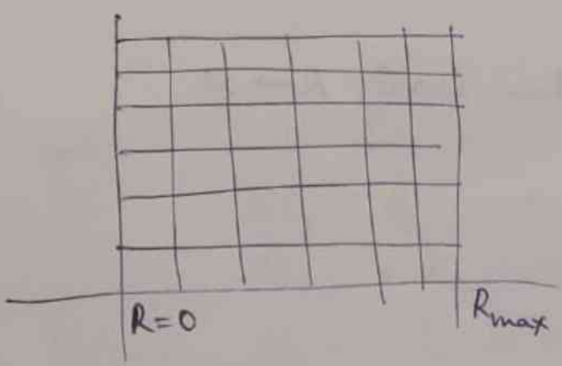
$$\frac{\partial^2 H}{\partial R^2} = \frac{c}{R^2}$$

$$H = -c \ln R + dR + e$$

As $R \to 0$, H is unbdd. $\Rightarrow\Leftarrow$ (we assumed H is bdd)

∴ when $R=0$, $\frac{\partial H}{\partial t} + \frac{\partial H}{\partial R} = 0$, $R=0$

$$\cancel{H(R_t, t)} \; H(R_T, T) = \left(1 - \frac{R_T}{T}\right)^+$$



$$\frac{H_m^{n+1} - H_m^n}{\delta t} + \frac{H_{m+1}^{n+1} - H_m^{n+1}}{\delta R} = 0$$

$$\frac{\partial H}{\partial R} = \frac{H_{m+1}^n - H_{m-1}^n}{\delta R}$$

$$= D^+ D^+ H_m^n$$

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$$\begin{cases} dX(t) = a(t, X)\cdot dt + b(t, X)\, dW_t \\ X(0) = X_0 \end{cases}$$

$$\int_0^t dX(s) = \int_0^t a(s, X)\cdot ds + \int_0^t b(s, X)\cdot dW_s$$

$$X(t) - X(0) = \text{"} \quad \text{"}$$

$$dX = \mu X\, dt + \sigma X\cdot dW_t \longrightarrow (1)$$

$$\frac{dX}{X} = \mu\cdot dt + \sigma\, dW_t$$

$$d(\ln X) = \frac{1}{X}\cdot dX - \frac{1}{2X^2}\cdot \sigma^2 X^2$$

$$= \frac{1}{X}\cdot dX - \frac{\sigma^2}{2}\cdot dt$$

$$d(\ln X) = \left(\mu - \frac{\sigma^2}{2}\right)\cdot dt + \sigma\, dW_t$$

$$\int_a^b f(x)\cdot dW_t = \sum_{i=0}^{n} f(t_{i-1})\, \delta W_i \qquad \delta W_i = W_{i+1} - W_i = Z_i\sqrt{\Delta t}$$

$$= \cancel{x_i\sqrt{\Delta t}}$$

$$\boxed{\begin{aligned} Y &= f(t, X) \\ dY &= f_t\cdot dt + f_x\cdot dX + \frac{1}{2} f_{xx}\cdot (dX)^2 \end{aligned}}$$

where $dX\cdot dX$ can be interpreted using

	dt	dW_t
dt	0	0
dW_t	0	dt

The solⁿ of BS diffusion eqⁿ (1), is

$$\therefore X = X_0 \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}$$

In principle, it is difficult to obtain closed form analytical solⁿ for SDEs. ∴ One has to seek numerical solⁿ

Glasserman 6^{th} chapter (or Seydel 3^{rd} chapter)

Euler

$y'(t) = f(t, y)$, $t \in (0, T)$

$y(0) = \alpha$

$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)$

T.E (Truncat^n error)

= LHS − RHS

Euler – Maruyama

$dX(t) = a(t, X(t))\, dt + b(t, X)\, dW_t$

$X(0) = X_0$ $\quad t \in [0, T]$

$h = \frac{T}{N}$

$\int_{t_n}^{t_{n+1}} dX(s) = \int_{t_n}^{t_{n+1}} a(s, X(s))\, ds + \int_{t_n}^{t_{n+1}} b(s, X(s))\, dW(s)$

$X(t_{n+1}) - X(t_n)$

$= a(t_n, X_n)\, \cancel{dt}\, h$

$+ b(t_n, X_n)\, \Delta W_{i+1}$

$Z_i \in N(0,1)$

$\Delta W_{i+1} = Z_i \sqrt{\Delta t}$

TE of

Bcz of GBM, the order of Euler – Maruyama scheme get diminished by ~~…~~ 1/2.

All the schemes like Runge-Kutta, Adam's etc can be done in stochastic case also.

$\theta a = \alpha e_1' \iff \alpha \theta e_1' = a \iff \theta e_1' = \frac{a}{\alpha}$

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~~$dX(t)$~~

$\because$ Euler Maruyama is $O(\sqrt{h}) < O(h)$ (of Euler-scheme) & the order get diminished due to approximatn of diffusion term. In order to enhance the order, we have to consider more no of terms in Brownian eqn.

$$X(t+h) = X(t) + a(X(t))h + \underbrace{b(X(t))\underbrace{[W(t+h) - W(t)]}_{\Delta O(\sqrt{h})}}$$

$$db(X(t)) = \cancel{b_t\, dt} + b_x . dX + \frac{1}{2} b_{xx}\, dX . dX$$

$$= b'(X(t)) . dX(t) + \frac{1}{2} b''(X(t))\, b^2(X(t))\, dt$$

$$= b'(X(t))\left[aX(t)\,dt + b(X(t))\, dW_t\right] + \frac{1}{2} b''(X(t))\, b^2(X(t)) . dt$$

$$= \underbrace{\left[ab' + \frac{1}{2} b'' b^2\right]}_{\mu_b} . dt + \underbrace{bb'}_{\sigma_b} dW_t$$

$[t, t+h]$, $t \leq u \leq t+h$

$$b(X(u)) \approx b(X(t)) + \mu_b(X(t))\underbrace{[u-t]}_{O(u-t)} + \sigma_b \underbrace{[W(u) - W(t)]}_{O(h)}$$

$\because W(u) - W(t)$ is of order of $\sqrt{u-t}$

$\because$ We want to retain the 1st order convergence of Euler-scheme, we drop $\mu_b(X(t)[u-t]$ & use $\sigma_b[W(u) - W(t)]$

$$b(X(u)) \approx b(X(t)) + \sigma_b[W(u) - W(t)]$$

$$\int_0^t dX(t) = \int_0^t aX(u) . du + \int_0^t bX(u) . dW_u$$

$\because$ The integral in $\int_t^{t+h} b(X(u)) . dW_u = \cancel{\int_t^{t+h} \left(b(X(t)) + \sigma_b[W(u) - W(t)]\right)(W(u) - W(t))}$

=

$$\int_t^{t+h} b(X(u))\,dW_u = \int_t^{t+h} \left(b(X(t)) + \sigma_b(X(t))\right) dW_u$$

$$= b(X(t))\left[W(t+h) - W(t)\right] + \sigma_b(X(t)) \int_t^{t+h} (W(u) - W(t))\,dW_u$$

$$= \quad '' \quad \sigma_b \underbrace{\int_t^{t+h} W(u)\,dW_u}_{= Y(t)} - W(t)\left(W(t+h) - W(t)\right)$$

$$Y(t) = \int_0^t W(t)\,dW_t$$

$$Y(0) = 0$$

$$dY(t) = W(t)\,dW_t$$

$$Y(t) = \frac{1}{2}W^2(t) - \frac{1}{2}t$$

$$= \quad '' \quad + \frac{\sigma_b}{2}\left[W(t) - W(t+h)\right]^2 - \frac{\sigma_b h}{2}$$

$$= \quad '' \quad + \frac{1}{2} b b' \left[\underbrace{(W(t+h) - W(t))^2}_{h Z_i^2} - h\right]$$

$$X(t+h) = a(X(t))h + b(X(t))\left[\underbrace{W(t+h) - W(t)}_{\sqrt{h} Z_i}\right] + \frac{1}{2} b b' \left\{\underbrace{\left[W(t+h) - W(t)\right]^2}_{h Z_i^2} - h\right]$$

Convergence
~~Truncat~~ error (in random walk case):-

$$\mathbb{E}\left[\| X(T) - \hat{X}(t_n) \|\right] \leq C h^p \longrightarrow \text{strong sense}$$

$$\mathbb{E}\left[\| \mathbb{E}(f(X(T))) - \mathbb{E}(f(X_n)) \| \leq \right. \qquad \longrightarrow \text{weak sense}$$

$$\{\hat{X}(0), \hat{X}(h), \hat{X}(2h), \ldots, \hat{X}(nh)\} \quad n = \left\lfloor \tfrac{T}{h} \right\rfloor$$

$$\left.\begin{array}{l} \mathbb{E}\left[\, \| \hat{X}(nh) - X(T) \| \,\right] \leq c h^{\beta}, \quad \mathbb{E}\left(\| \hat{X}(nh) - X(T) \|^2 \right) \\ \mathbb{E}\left[\sup_{0 \leq t \leq T} \| \hat{X}\left(\left[\tfrac{t}{h}\right] h\right) - X(t) \| \right] \end{array}\right\} \text{Strong}$$

$f \in C_p^{2\beta+2}$ p - polynomial

$$\| \mathbb{E}\left(f(\hat{X}(nh)) \right) - \mathbb{E}\left(f(X(T)) \right) \| \leq c h^{\beta} \xrightarrow{\text{Weak}} (2)$$

c depends on the polynomial f

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$f \in C_p^{2\beta+2}$:= consists of $f : \mathbb{R}^d \to \mathbb{R}$ whose derivatives of order $0, 1, \ldots, 2\beta+2$ are polynomially bounded.

Lower order in strong convergence can be higher order in weaker sense.

We can extend 1-D to 2-D:-

$$dX_1(t) = a(X_1(t)) \cdot dt + b(X_1(t)) \cdot dW_t$$
$$dX_2(t) = a(X_2(t)) \cdot dt + b(X_2(t)) \cdot dW_t$$

Second-order scheme

H.W

$$\mathcal{L}^0 := a\frac{d}{dx} + \frac{1}{2}b^2\frac{d^2}{dx^2}$$

$$\mathcal{L}^1 := b\cdot\frac{d}{dx}$$

$$X(t+h) = X(t) + \underbrace{\int_{t+h}^{t} a(X(u))\cdot du}_{} + \int_t^{t+h} b(X(u))\cdot dW_u$$

~~$= X(t) +$~~

$$\circledast\ a(X(u)) = a(X(t)) + \int_t^u \mathcal{L}^0(a(X(s))\cdot ds + \int_t^u \mathcal{L}^1(a(X(s)))\cdot dW_s$$

$$= a(X(t)) + \mathcal{L}^0 a(X(t))\int_t^u ds + \mathcal{L}^1 a(X(t))\cdot\int_t^u dW_s$$

$$\circledast\ b(X(u)) = b(X(t))$$

~~$X(t+h) = X(t) + h\,a\,X(t) + \mathcal{L}^0 a(X(t)) + h$~~

$$X(t+h) = X(t) + \int_t^{t+h} a(X(t))\cdot du + \mathcal{L}^0 a(X(t))\int_t^{t+h}\int_t^u ds\cdot du$$

$$+ \mathcal{L}^1 a(X(t))\int_t^{t+h}\int_t^u dW(s)\cdot du$$

$$+ \int_t^{t+h} b(X(t))\cdot du + \int\int_t^u \mu_b\cdot du$$

4 double integrals you have to evaluate

Ex CIBOR

6.21) $dL_i(t) = L_i(t)\mu_i(L(t), t) + L_i(t)\,\sigma_i(t)^T dW(t)$

6.22) Stochastic Volatility Model