Cantor's Intersection Theorem:

$A \subset (X,d)$

$dia(A) = \sup\{d(x,y) : x,y \in A\}$.

A is called bounded if $dia(A)$ is finite.

(X,d) — complete metric space.

$\{F_n\}$ be a sequence of non-empty closed subset in X s.t.

① $F_{n+1} \subset F_n$

② $dia(F_n) \longrightarrow 0$ as $n \to \infty$

then $\bigcap F_n$ is a singleton set.

Proof:- Take $x_n \in F_n$.

$d(x_n, x_m) \leq dia(F_m) \longrightarrow 0$ as $m,n \to \infty$ $[F_n \subset F_m]$.

$\Rightarrow \{x_n\}$ is a Cauchy sequence.

$\therefore \{x_n\}$ is convergent.

$x_n \longrightarrow x_0$ (as X is complete).

$x_0 \in \overline{F_n} \ \forall n$

$= F_n$.

$\therefore x_0 \in \bigcap F_n$

Lets say $x_0' \in \bigcap F_n$.

$\Rightarrow d(x_0, x_0') \leq dia(F_n) \ \forall n$.

$\Rightarrow x_0 = x_0'$

$\Rightarrow \bigcap F_n = \{x_0\}$.

P.T. $[0,1]$ is a compact set.

Proof :- let, $\{G_\alpha\}$ be a open cover of $[0,1]$ let $\{G_\alpha\}$ has no finite subcover.

$I_0 = [0,1]$.

$I_1 = [\frac{1}{2}, 1]$ (Any 1 of the 2 halves).

$I_2 = [\frac{1}{2}, \frac{3}{4}]$

$\vdots$

$\{I_n\} \rightarrow$ ① I_n has no finite subcover

② $l(I_n) = \frac{1}{2} l(I_{n-1}) = \frac{1}{2^n}$

③ $I_{n+1} \subset I_n$.

$\therefore \bigcap I_n = \{x_0\}$.

$\because x_0 \in I_0$

$\Rightarrow \exists\, G_{\alpha_0}$ st. $x_0 \in G_{\alpha_0}$.

$\therefore (x_0-\varepsilon, x_0+\varepsilon) \subset G_{\alpha_0}$.

$\therefore \exists\, n_0,\ I_n \subset (x_0-\varepsilon, x_0+\varepsilon) \subset G_{\alpha_0} \quad \forall\, n \geq n_0$

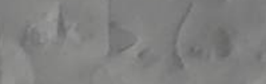
$\therefore$ finite subcover exits.

$\therefore \{G_\alpha\}$ has a finite subcover.

$K \subset (X,d)$ given $\longrightarrow$ K is compact.

K is closed $\longleftarrow$ P.T.

$x \in K,\ y \in X \setminus K$.



$B(x, r_x)$ $B(y, r_x)$.

$\exists\, r_x > 0$ st.

$\{B(x, r_x) : x \in K\}$.

$\bigcup_{x \in K} B(x, r_x) \supset K$.

$\because K$ is compact

$\exists\, x_1, \ldots, x_n$ st.

$\boxed{\bigcup_{i=1}^{n} B(x_i, r_i) \supset K}$

$r = \min\{r_i, 1 \le i \le n\}$.

$B(y,r) \subset B(y,r_i) \; \forall i = 1,2,\dots,n$

$B(y,r) \cap B(x,r_i) = \phi \; \forall i = 1,2,\dots,n$

$\Rightarrow B(y,r) \cap (\cup B(x,r_i)) = \phi$

$\Rightarrow B(y,r) \cap K = \phi$.

$\Rightarrow B(y,r) \subset X \setminus K$.

$\therefore$ $X \setminus K$ is open.

K is compact.

$x \in K \subset (X,d)$

$B(x,n)$, $n \in \mathbb{N}$.

$\{B(x,n) : n \in \mathbb{N}\}$ covers K.

$\Rightarrow B(x,n_1) \dots B(x,n_k)$ covers K $\leftarrow$ finite subcover

$N_0 = \max\{n_1, n_2, \dots, n_k\}$.

$\therefore \boxed{B(x,N_0) \supset K} \rightarrow \boxed{dia(K) \le 2N_0}$ bounded.

$(X,d) \longrightarrow$ compact in space.

$F \longrightarrow$ closed subset of X. Is F compact?

Proof:- Let $\{G_\alpha\}$ be a open cover of F.

$\{X \setminus F\} \cup \{G_\alpha\}_{\alpha\in\Lambda}$ is a open cover of X.

$\hookrightarrow$ has finite subcover (As X is compact).

$\therefore \{G_\alpha\}_{\alpha\in\Lambda}$ has a finite subcover.

$= [a,b]$, $d(x,y) = |x-y|$). <u>Heine-Borel Theorem</u>

F is closed and bdd subset of $(\mathbb{R}^n, d_u)$

$F \subset [a,b] \Leftrightarrow$ F is compact.

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f is cts f^n <u>$f: A \subset \mathbb{R} \longrightarrow \mathbb{R}$.</u>

① A is bounded $\Rightarrow$ f(A) is bdd? X.

② A is closed & bdd. $\Rightarrow$ f(A) is <u>closed</u> and <u>bdd</u>?

<u>Proof ②:-</u> ~~f(A) is~~ Let, f(A) is not bdd X

For each n, $\exists x_n \in A$ st. $|f(x_n)| > n$

$\{x_n\}$ seq in A, $x_{n_k} \longrightarrow x_0 \in A$.

$f(x_{n_k}) \longrightarrow f(x_0)$ [As f is cts].

Let us assume f(A) is not closed.

x_0 be a limit point of f(A) s.t. $x_0 \notin f(A)$.

$\exists \{y_n\}$ in f(A) s.t. $y_n \longrightarrow x_0$.

$\therefore \exists \{x_n\}$ in A : $y_n = f(x_n)$.

$x_{n_k} \longrightarrow x'$ in A.

$\Rightarrow f(x_{n_k}) \longrightarrow f(x')$, $f(x_{n_k}) \to x_0$

$\therefore x_0 \in f(A) \Rightarrow\!\Leftarrow$.

In $\mathbb{R}/\mathbb{R}^n$, cts. image of closed and bdd subset is closed and bounded ie, cts image of compact set is compact

$f: (X,d) \to (Y,\rho)$ be cts. map. K be a compact subset of (X,d). Then f(K) is also compact.

Proof:- Let, $\{G_\alpha\}_{\alpha\in\Lambda}$ be a open cover of $f(K)$.

$\Rightarrow \{f^{-1}(G_\alpha)\}$ is a open cover of K.

$\therefore \exists\ \alpha_1, \ldots, \alpha_n$ s.t. $\left(\bigcup_{i=1}^{n} f^{-1}(G_i)\right) \supset K$

$\Rightarrow f^{-1}\left(\bigcup_{i=1}^{n} G_{\alpha_i}\right) \supset K$ ← $\boxed{\text{same (prove) H/W}}$

$\Rightarrow f(K) \subset \bigcup_{i=1}^{n} G_{\alpha_i}$

$\Rightarrow f(K)$ is compact.

Subresult:-

$f:(X,d) \to (Y,\rho)$ cts.

G is open in $Y \Rightarrow f^{-1}(G)$ is open in X.

$\boxed{f^{-1}(A) = \{x \in X : f(x) \in A\}}$.

$x \in f^{-1}(G)$.

$\Rightarrow f(x) \in G$.

$\Rightarrow B(f(x), \varepsilon) \subset G$ for some $\varepsilon > 0$.

f is cts at $x \Rightarrow \delta > 0$ st.

$d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$

$f(B(x,\delta)) \subset B(f(x),\varepsilon) \subset G$.

$\Rightarrow B(x,\delta) \subset f^{-1}(G)$.

G is closed in $Y \Rightarrow f^{-1}(G)$ is closed in X H/W

Continuous function from a compact set is uniformly continuous.

Proof:- $\{B(x,1) : x \in X\}$ is a cover of X.

$x_{11}, x_{21}, x_{31}, \ldots, x_{n1} \in$ finite subcovers.

$X = \bigcup_{i=1}^{n} B(x_{i1}, 1)$.

$A_1 = \{x^1_1, x^1_2, \ldots, x^1_{n_1}\}$

$A_2 = \{x^2_1, x^2_2, \ldots, x^2_{n_2}\}$.

$\vdots$

$A_K = \{x^K_1, x^K_2, \ldots, x^K_{n_K}\}$

$X = \bigcup_{i=1}^{n} B(x^K_i, 1/K)$

$\boxed{A = \bigcup_{K=1}^{\infty} A_K}$ H/W

Prove that A is dense in X.

$\boxed{X \text{ is compact} \Rightarrow X \text{ is separable}}$.

$\therefore X$ is separable.

$f:[a,b] \xrightarrow{\text{cts}} \mathbb{R} \Rightarrow f$ is u.c.

Let, $f:(X,d) \to (Y,\rho)$ be a cts f^n. If (X,d) is compact. then f is uniformly cts.

Proof:- For given $\varepsilon > 0$, for each $x \in X$, $\exists\, \delta_x > 0$

s.t. $d(x, y) < \delta_x \Rightarrow \rho(f(x), f(y)) < \varepsilon/2$.

$\{B(x, \delta_{x}/2) : x \in X\}$ is an open cover of X.

$\therefore \exists\, x_1, x_2, \ldots, x_k$ st. $\bigcup_{i=1}^{k} (x_i, \delta_{x_i}/2) = X$.

Let, $\delta = \min_{1 \le i \le k} \{\delta_{x_i}\}$

x, y in X s.t. $d(x, y) < \delta$.

Claim:- $\rho(f(x), f(y)) < \varepsilon$.

Proof:- $x \in X \Rightarrow \exists\, j$ s.t. $x \in B(x_j, \delta_{j}/2)$.

$$d(x_j, x) < \delta_{x_j}/2.$$

$$\Rightarrow \rho(f(x), f(x_j)) < \varepsilon/2$$

Also, $d(y, x_j) \le d(y, x) + d(x, x_j)$

$$< \delta + \delta_{x_j}/2 < \delta_{x_j}$$

$$\Rightarrow \rho(f(y), f(x_j)) < \varepsilon/2.$$

$$\rho(f(x), f(y)) \le \rho(f(x), f(x_j)) + \rho(f(x_j), f(y))$$

$$< \varepsilon/2 + \varepsilon/2$$

$$\therefore \rho(f(x), f(y)) < \varepsilon.$$

Closed interval is compact (H/W)

(X, d)

① **Sequentially compact**: If every sequence in X has a convergent subsequence.

$\Longleftrightarrow$

② **Bolzano-Weierstrass (BW) property**: Every infinite set has a limit point in X.

Proof of equivalence:- ① $\Rightarrow$ ②

A is an infinite seq in X. $\{x_n\}$ in A ($x_n \ne x_m\ \forall\, n \ne m$)

x_0 be a limit point of $\{x_n\}$. $B(x_0, \varepsilon) \ni x_n \in A$

$\Rightarrow x_0 \in A'$

② $\Rightarrow$ ①

$\{x_n\} = A$. If A is finite, some x_i is repeting infinitely. Take that as the subsequence.

If A is infinite, limit point exists.

$\boxed{\text{A is closed and bdd in } \mathbb{R} \Leftrightarrow \text{every sequence in A has a conv. subseq. (H/W).}}$

Every compact set follows B.W. property

Proof:- (X,d) is a compact metric space.

$A \subset X$, infinite set. Aim:- $A' \neq \phi$.

If possible, assume $A' = \phi \Rightarrow$ each $x \in X$,

$\exists\, r_x : B(x, r_x) \setminus \{x\} \cap A = \phi$.

$\therefore \{B(x, r_x) : x \in X\}$ is an open cover and it has a finite subcover.

$\Rightarrow X \subset A = B(x_1, r_{x_1}) \cup B(x_2, r_{x_2}) \cup \ldots \cup B(x_n, r_{x_n})$

$\Rightarrow\Leftarrow$ because A is infinite.

$\therefore A' \neq \phi$.

~~Every non empty set has~~

seq. comp $\Longleftrightarrow$ B.W. property

Compactness

Every set in A has a conv. subseq. (In $\mathbb{R}^n$).

$\Rightarrow$ A is bdd and closed. $\Rightarrow$ A is compact.

Lebesgue No:- A positive real no. 'a' is called Lebesgue no. corresponding to an open cover $\{G_i\}$ of X if every subset of (X,d) with diameter less than or equal to 'a' lies in at least one G_i.

If (X,d) is seq.^{ly} compact, then X has a Lebesgue No.

Proof:- Let $\{G_\alpha\}$ be an open cover.
say, no lebesgue no.
For $a=1$, $\exists B_1$ s.t. dia $(B_1)=1$.
but B_1 doesn't lie in any of G_i's.
For $a=\frac{1}{2}$, $\exists B_2$ . . . .
For $a=\frac{1}{n}$, $\exists B_n$ s.t. dia $(B_n)\le \frac{1}{n}$.
but B_n doesn't lie in any of G_i's.
$\therefore \exists \{x_n\}: x_i \in B_i$.
$\{x_n\}$ in (X,d). As (X,d) is seq. compact
$\Rightarrow \exists \{x_{n_k}\}$ s.t. $x_{n_k} \to x_0 \in X$.
$\exists G_{i_0}$ s.t. $x_0 \in G_{i_0}$.
$\exists r>0$ s.t. $B(x_0, r) \subset G_{i_0}$.
$\therefore B(x_0, \frac{r}{2})$ contains infinitely many x_{n_k}.
Choose K_0 s.t. $\frac{1}{n_{K_0}} < r/2$, and $d(x_{n_{K_0}}, x_0) < r/2$.
$\Rightarrow B_{n_{K_0}} \subset B(x_0, r) \subset G_{i_0}$. $\Rightarrow\Leftarrow$

Totally bounded set

(X,d), ~~$G\subset X$~~ is called totally bounded if $\forall r>0$, $\exists$ finitely many open balls, say $B_1, B_2, B_3, \ldots, B_n \ni \boxed{\bigcup_{i=1}^{n} B_i = X}$
$\boxed{\text{dia}(B_i) < r}$

→ Every bounded set of $\mathbb{R}$ is totally bounded.
(a,b) r → break (a,b) into segments of length $< r$.

$\boxed{\text{Totally bounded} \Longrightarrow \text{bounded}}$ H/W

* Sequentially compact $\Rightarrow$ Totally bounded

Proof:- Let, (X,d) is not totally bounded.

(X,d) is seq compact.

Given $r>0$,

$x_1 \in X$, $B(x_1, r) \neq X$.

x_2 s.t. $x_2 \notin B(x_1, r)$ and $B(x_1, r) \cup B(x_2, r) \neq X$

$\vdots$

$x_n \notin B(x_1, r) \cup \cdots \cup B(x_n, r)$

Complete the proof. (H/W)

Let (X,d) is seq. compact

$\overset{?}{\Rightarrow}$ (X,d) is compact.

Proof:- Let, $(G_\alpha)_{\alpha \in \Lambda}$ be any open cover of (X,d).

(X,d) seq. compact $\Rightarrow \{G_\alpha\}$ has a Lebesgue number, $a>0$.

totally bounded $\Rightarrow$

Take $r = a/3$, $\Rightarrow$ finitely many balls with radius r s.t. $(x_1, x_2, \ldots, x_n)$

$\bigcup_{i=1}^{n} B(x_i, r) = X$, $dia(B(x_i, r)) < a \; \forall i$

$\Rightarrow \exists \alpha_1, \alpha_2, \ldots, \alpha_n \ni B(x_i, r) \subset G_{\alpha_i}$

$\therefore \boxed{\bigcup_{i=1}^{n} G_{\alpha_i} \text{ covers } (X,d)}$.

P.T. $B(0,1)$ is compact in $\mathbb{R}^n$.

Proof:-

(X,d) is a complete metric space.
If (X,d) is totally bounded, then it is compact?

Proof :- → P.T. every subset of (X,d) is totally bdd.
H/W → Every seq has a Cauchy subsequence.
→ Use completeness ⇒ (X,d) is seq. compact.

$A = [0,1)$, $B = [0,\frac{1}{2}) \cup (\frac{1}{2},1)$.

$(X,d) : \to A, B$ two non-empty open subsets of X, are called separation of X if $X = A \cup B$ and $A \& B$ are disjoint.

→ (X,d) is called disconnected if (X,d) has separation.

A metric space (X,d) is called connected if it is not disconnected. (X,d) is disconnected ⇒ ∃ a subset which is both closed and open (clopen).

Theorem : E is connected subset of $\mathbb{R} \Leftrightarrow E$ is an interval

Proof :- E is connected, then E is an interval.

Let us assume, E is not interval

$\Rightarrow \exists\, x, y, z : x < z < y,\ x, y \in E$ but $z \notin E$.

Take $A = (-\infty, z) \cap E$, $B = (z, \infty) \cap E$.

∴ Separation ⇒⇐.

Conversely, E is interval

Assume disconnected, $\exists\, x, y \in E$: A, B is separation s.t. $x \in A,\ y \in B$.

$\Rightarrow \exists\, A, B$.

① $A = E \setminus B$.

② B is open, $y \in B \Rightarrow (y-\delta, y] \subset B$.

Let, $\alpha = \sup A \Rightarrow \alpha \le y - \delta$.

Case I :- $\alpha \in A$.

As A is open, $[\alpha, \alpha+\varepsilon) \subseteq A$.

$\Rightarrow \Leftarrow$.

Case II :- $\alpha \in B$.

$(\alpha-\eta, \alpha+\eta) \subset B$.

As α is sup of A, $\Rightarrow \exists \underbrace{\alpha_0 < \alpha}_{\alpha-\eta}$, $\alpha_0 \in A$. but $\alpha_0 \in B$

$\Rightarrow \Leftarrow$.

$\longrightarrow f : (X, d) \longrightarrow (Y, \rho)$

$f(X)$ is connected in Y if X is connected.

Proof :- Let, $f(X)$ is disconnected.

$\Rightarrow$ A & B are 2 non-empty disjoint open subsets of $f(X)$ s.t. $A \cup B = f(X)$.

$f^{-1}(A) \cup f^{-1}(B) = X$. $[f^{-1}(A), f^{-1}(B)$ are open$]$ ($\because f$ is cts).

If $y \in f^{-1}(A) \cap f^{-1}(B)$.

$\Rightarrow f(y) \in A \cap B . \Rightarrow \Leftarrow$

$\therefore f^{-1}(A) \cap f^{-1}(B) = \phi$.

$\therefore f^{-1}(A), f^{-1}(B)$ are separation.

$\therefore$ X is disconnected. $\Rightarrow \Leftarrow$.

$\therefore f(X)$ is connected.

$f : I \to \mathbb{R}$. (IVT)

$f(I)$ is connected subset of $\mathbb{R}$

$\Rightarrow f(I)$ is an interval.

$\alpha, \beta \in f(I)$, $\alpha = f(a)$, $\beta = f(b)$.

$\alpha < \gamma < \beta \Rightarrow \exists\, a < c < b : f(c) = \gamma$.

(X, d).

Any two points $a, b \in X$. If $\exists f : [0,1] \xrightarrow{cts} X$ s.t. $f(0) = a$ $f(1) = b \Rightarrow X$ is connected.

Proof :- Let X is disconnected.

$\Rightarrow \exists$ a separation A, B.

$a \in A$, $b \in B$, $A \cup B = X$

$\exists f : [0,1] \longrightarrow X \ni f(0) = a$, $f(1) = b$.

$0 \in f^{-1}(A)$, $1 \in f^{-1}(B)$

$\Rightarrow f^{-1}(A) \cup f^{-1}(B) = [0,1]$.

$\therefore f^{-1}(A), f^{-1}(B)$ is a separation. $\Rightarrow\Leftarrow$

$\therefore$ X is connected.

H/W

Does $\exists$ a homeomorphism from a subset of $\mathbb{R}$ onto circle