MA 201 Complex Analysis
Lecture 11: Applications of Cauchy's Integral Formula

Cauchy's estimate

Cauchy's estimate: Suppose that f is analytic on a simply connected domain D and $\overline{B(z_0, R)} \subset D$ for some $R > 0$. If $|f(z)| \leq M$ for all $z \in C(z_0, R)$, then for all $n \geq 0$,

$$|f^n(z_0)| \leq \frac{n!M}{R^n},$$

where $C(z_0, R) = \{z : |z - z_0| = R\}$.

Proof: From Cauchy's integral formula and ML inequality we have

$$\begin{aligned} |f^n(z_0)| &= \left|\frac{n!}{2\pi i}\int_{|z-z_0|=R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| \\ &\leq \frac{n!}{2\pi}M\frac{1}{R^{n+1}}2\pi R = \frac{n!M}{R^n}. \end{aligned}$$

Liouville's Theorem

Liouville's Theorem: If f is analytic and bounded on the whole $\mathbb{C}$ then f is a constant function.

Proof: By Cauchy's estimate for any $z_0 \in \mathbb{C}$ we have,

$$|f'(z_0)| \leq \frac{M}{R}$$

for all $R > 0$. This implies that $f'(z_0) = 0$. Since z_0 is arbitrary and hence $f' \equiv 0$. Therefore f is a constant function.

- $\sin z, \cos z, e^z$ etc. can not be bounded. If so then by Liouville's theorem they are constant.

Liouville's Theorem

- Does there exists a non constant entire function f such that $e^{f(z)}$ is bounded?
- Does there exists a non constant entire function f such that $\mathrm{Re}(f)$ is bounded?
- Does there exists a non constant entire function f such that $\mathrm{Im}(f)$ is bounded?
- Does there exists a non constant entire function f such that $f(x)$ is bounded for all real x?
- Does there exists a non constant entire function f such that $|f(z)| > 1$ for all $z \in \mathbb{C}$?

Fundamental Theorem of Algebra

- **Fundamental Theorem of Algebra:** Every polynomial $p(z)$ of degree $n \geq 1$ has a root in $\mathbb{C}$.

- **Proof:** Suppose $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_0$ is a polynomial with no root in $\mathbb{C}$. Then $\frac{1}{P(z)}$ is an entire function.

- Since

$$\left|\frac{P(z)}{z^n}\right| = \left|1 + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}\right| \to 1, \quad \text{as} \; ; \quad |z| \to \infty,$$

- It follows that $|p(z)| \to \infty$ and hence $|1/p(z)| \to 0$ as $|z| \to \infty$.

- Consequently $\frac{1}{p(z)}$ is a bounded function.

- Hence by Liouville's theorem $\frac{1}{p(z)}$ is constant which is impossible.

Morera's Theorem

Morera's Theorem: If f is continuous in a simply connected domain D and if

$$\int_C f(z)dz = 0$$

for every simple closed contour C in D then f is analytic.

Proof: Fix a point $z_0 \in D$ and define

$$F(z) = \int_{z_0}^{z} f(w)dw.$$

Use the idea of proof of existence of antiderivative to show that $F' = f$. Now by Cauchy integral formula f is analytic.