ME101: Engineering Mechanics (3 1 0 8)

2021-2022 (II Semester)

KINETICS OF A PARTICLE

LECTURE: 26-29

Kinetics of a Particle

Kinetics:

Study of the relations between unbalanced forces and the resulting changes in motion.

Newton's Second Law of Motion : The acceleration of a particle is proportional to the resultant force acting on it and is in the direction of this force.

→ **A particle will accelerate when it is subjected to unbalanced forces**

Three approaches to solution of Kinetics problems:

1. Force-Mass-Acceleration method (direct application of Newton's Second Law)
2. Use of Work and Energy principles
3. Impulse and Momentum methods

Limitations of the topic:

- Motion of bodies that can be treated as particles (motion of the mass centre of the body)
- Forces are concurrent through the mass center (action of non-concurrent forces on the motion of bodies will be discussed in Kinetics of rigid bodies).

Kinetics of a Particle

Force-Mass-Acceleration method

Newton's Second Law of Motion

Subject a mass particle to a force $\mathbf{F}_1$ and measure accln of the particle $\mathbf{a}_1$. Similarly, $\mathbf{F}_2$ and $\mathbf{a}_2$… → The ratio of magnitudes of force and resulting acceleration will remain constant.

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \cdots = \frac{F}{a} = C$$

Constant C is a measure of some invariable property of the particle → Inertia of the particle

Inertia = Resistance to rate of change of velocity

Mass m is used as a quantitative measure of Inertia.

$C = km$ where k is a constant introduced to account for the units used.

→ $F = kma$

F = magnitude of the resultant force acting on the particle of mass m

a = magnitude of the resulting acceleration of the particle.

Accln is always in the direction of the applied force → Vector Relation: $\mathbf{F} = km\mathbf{a}$

In **Kinetic System of units**, k is taken as unity → $\mathbf{F} = m\mathbf{a}$

→ units of force, mass and acceleration are not independent

→ Absolute System since the force units depend on the absolute value of mass.

Values of *g* at Sea level and 45^0 latitude**:**

For measurements relative to rotating earth: Relative g → 9.80665 ~ 9.81 m/s^2

For measurements relative to non-rotating earth: Absolute g → 9.8236 m/s^2

Kinetics of a Particle

Force-Mass-Acceleration method

Equation of Motion

Particle of mass *m* subjected to the action of concurrent forces $\mathbf{F}_1$, $\mathbf{F}_2$,… whose vector sum is $\sum\mathbf{F}$:

→ Equation of motion: $\sum\mathbf{F} = m\mathbf{a}$

→ Force-Mass-Acceleration equation

Equation of Motion gives the instantaneous value of the acceleration corresponding to the instantaneous value of the forces.

- The equation of motion can be used in scalar component form in any coordinate system.
- For a 3 DOF problem, all three scalar components of equation of motion will be required to be integrated to obtain the space coordinates as a function of time.
- All forces, both applied or reactive, which act on the particle must be accounted for while using the equation of motion.

Free Body Diagrams:

In Statics: Resultant of all forces acting on the body = 0

In Dynamics: Resultant of all forces acting on the body = *ma* → Motion of body

Kinetics of a Particle: Force-Mass-Acceleration method

Rectilinear Motion

Motion of a particle along a straight line

For motion along *x*-direction, accelerations along *y*- and *z*-direction will be zero

→ $\sum F_x = ma_x$

$\sum F_y = 0$

$\sum F_z = 0$

For a general case:

→ $\sum F_x = ma_x$

$\sum F_y = ma_y$

$\sum F_z = ma_z$

The acceleration and resultant force are given by:

$$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$$

$$a = \sqrt{a_x^{\,2} + a_y^{\,2} + a_z^{\,2}}$$

$$\Sigma\mathbf{F} = \Sigma F_x\mathbf{i} + \Sigma F_y\mathbf{j} + \Sigma F_z\mathbf{k}$$

$$|\Sigma\mathbf{F}| = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2 + (\Sigma F_z)^2}$$

Kinetics of a Particle: Force-Mass-Acceleration method

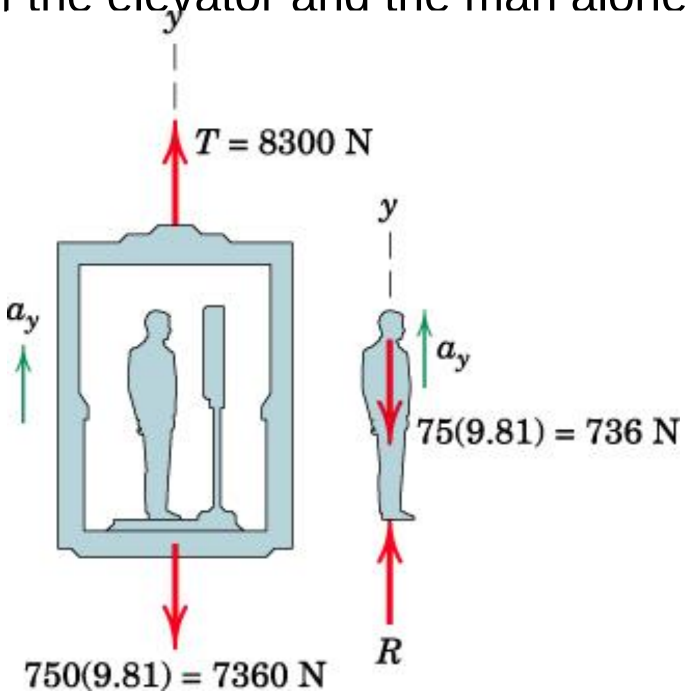
Rectilinear Motion

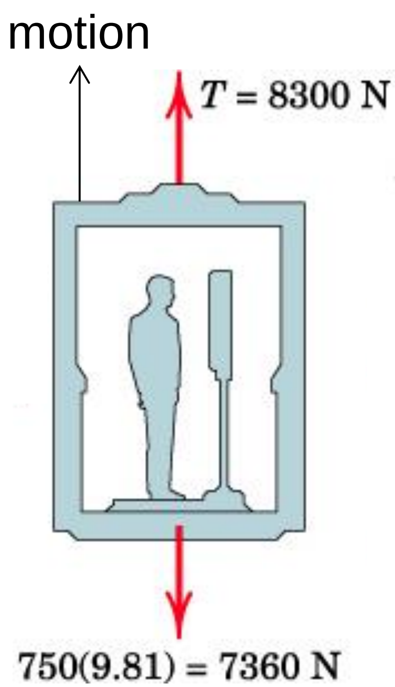
Example

A 75 kg man stands on a spring scale in an elevator. During the first 3 seconds of motion from rest, the tension *T* in the hoisting cable is 8300 N. Find the reading *R* of the scale in Newton during this interval and the upward velocity v of the elevator at the end of the 3 seconds. Total mass of elevator, man, and scale is 750 kg.

Solution

Draw the FBD of the elevator and the man alone





Kinetics of a Particle: Force-Mass-Acceleration method

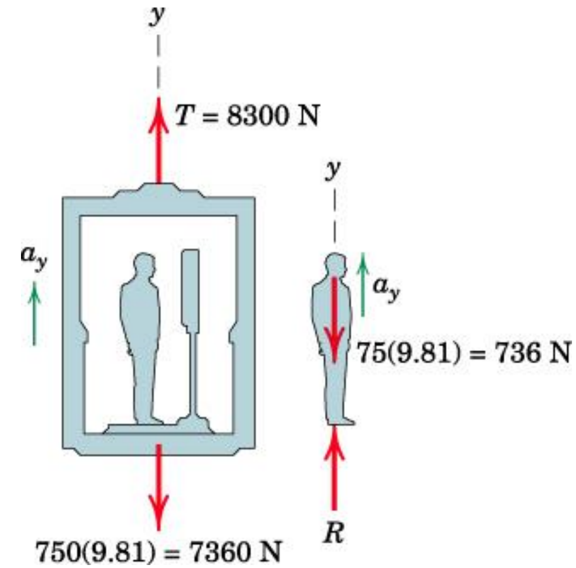
Rectilinear Motion

Example

Solution

During first 3 seconds, the forces acting on the elevator are constant. Therefore, the acceleration a_y will also remain constant during this time.

Force registered by the scale and the velocity of the elevator depend on the acceleration a_y



From FBD of the elevator, scale, and man taken together:

$\sum F_y = ma_y \rightarrow 8300\text{-}7360 = 750a_y \rightarrow a_y = 1.257\ \text{m/s}^2$

From FBD of the man alone:

$\sum F_y = ma_y \rightarrow R\text{-}736 = 75a_y \rightarrow R = 830\ \text{N}$

Velocity reached at the end of the 3 sec:

$\Delta v = \int a\, dt \rightarrow$ $v - 0 = \int_0^3 1.257\, dt$

$v = 3.77$ m/s

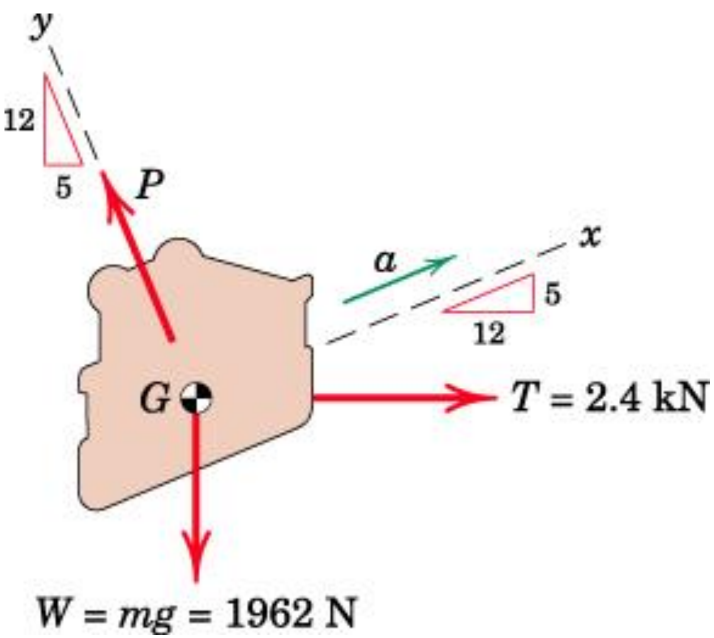
Kinetics of a Particle: Force-Mass-Acceleration method

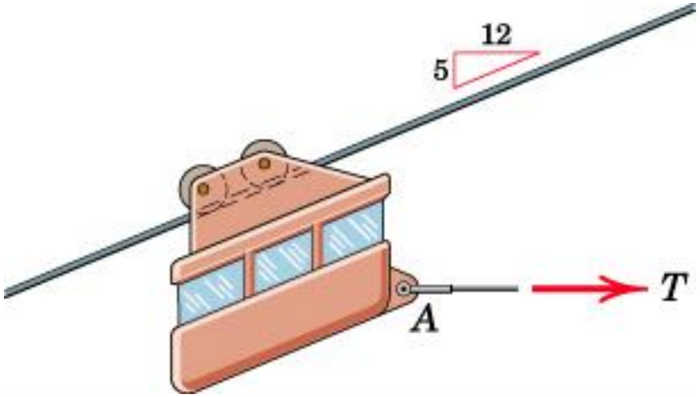
Rectilinear Motion

Example

A small inspection car with a mass of 200 kg runs along the fixed overhead cable and is controlled by the attached cable at A. Determine the acceleration of the car when the control cable is horizontal and under a tension $T = 2.4$ kN. Also find the total force P exerted by the supporting cable on the wheels.

Solution: Draw the FBD of the system





Choosing the x-y coordinate system such that the axes are along and normal to the motion (acceleration)
→ calculations simplified

Kinetics of a Particle: Force-Mass-Acceleration method

Rectilinear Motion

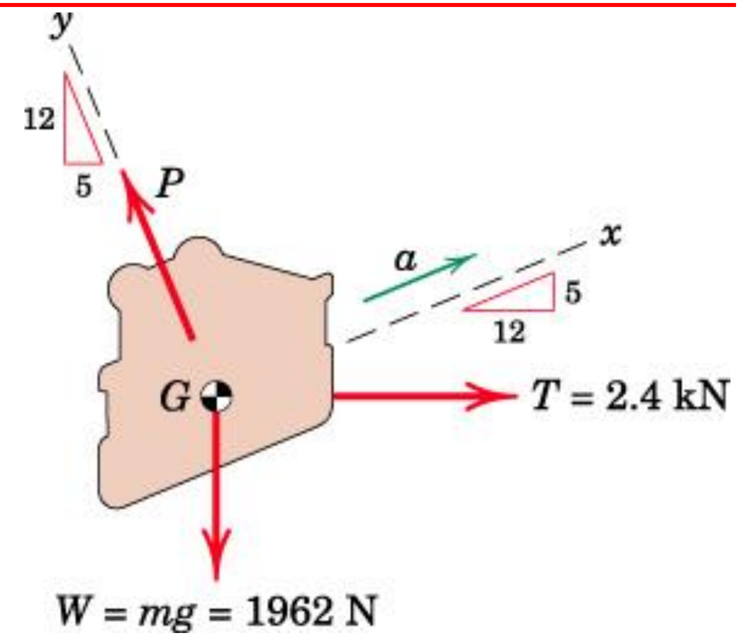
Example

Solution:

No accln in the *y*-dirn

→ The car is in equilibrium in the *y*-dirn

$\sum F_y = 0$ → $P = 2.73$ kN



Along the x-direction, equation of motion:

$\sum F_x = ma_x$

→ $a = 7.3$ m/s^2

Both equations were solved independently because of the choice of the coordinate axes.

Kinetics of a Particle: Force-Mass-Acceleration method

Curvilinear Motion: Particles move along plane curvilinear paths.

Rectangular Coordinates

$$\sum F_x = ma_x \qquad a_x = \ddot{x}$$

$$\sum F_y = ma_y \qquad a_y = \ddot{y}$$

Normal and Tangential Coordinates

$$\sum F_n = ma_n$$

$$\sum F_t = ma_t$$

$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$

$$a_t = \dot{v} = \ddot{s}$$

$$v = \rho\dot{\beta} \qquad \dot{\beta} = \omega = angular\ velocity$$

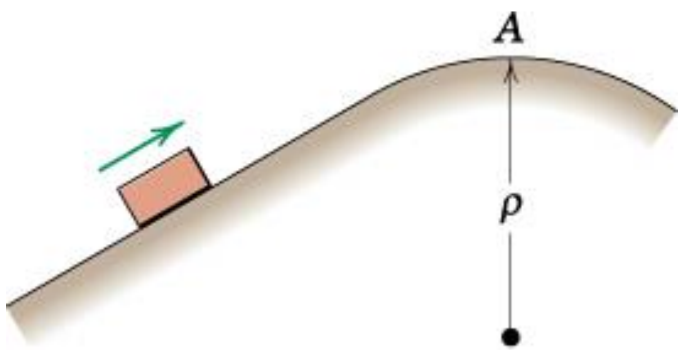
Polar Coordinates

$$\sum F_r = ma_r \qquad a_r = \ddot{r} - r\dot{\theta}^2$$

$$\sum F_\theta = ma_\theta \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Example (1) on curvilinear motion

Determine the maximum speed v which the sliding block may have as it passes point A without losing contact with the surface.

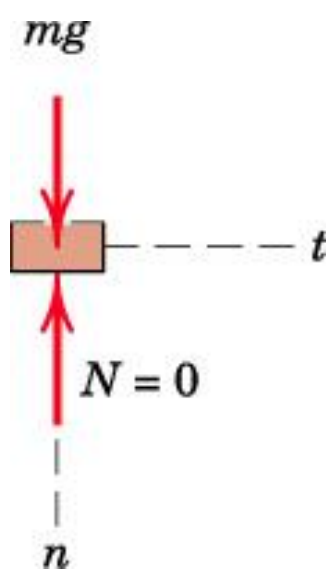


Solution:

The condition for loss of contact: Normal force N exerted by the surface on the block is equal to zero.

Draw the FBD of the block and using n-t coordinate system

Let m be the mass of the block.



Along n-direction:

$\sum F_n = ma_n$

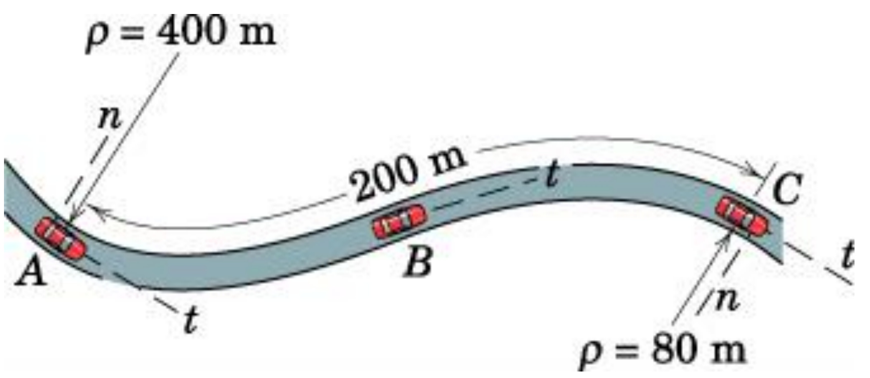
$mg - N = ma_n$

$mg = m(v^2/\rho)$

→ $\boxed{v = \sqrt{g\rho}}$

Example (2) on curvilinear motion

A 1500 kg car enters a section of curved road in the horizontal plane and slows down at a uniform rate from a speed of 100 km/h at *A* to 50 km/h at *C*. Find the total horz force exerted by the road on the tires at positions *A*, *B*, and *C*. Point *B* is the inflection point where curvature changes sign.



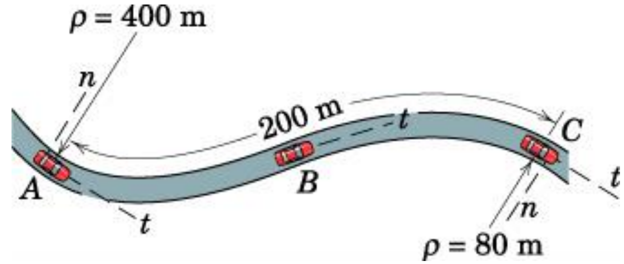
Solution:

The car will be treated as a particle → all the forces exerted by the road on tires will be treated as a single force.

Normal and tangential coordinates will be used to specify the acceleration of the car since the motion is described along the direction of the road.

Forces can be determined from the accelerations.

Example (2) on curvilinear motion



Solution:

The acceleration is constant and its direction will be along negative *t*-direction.

Magnitude of acceleration:

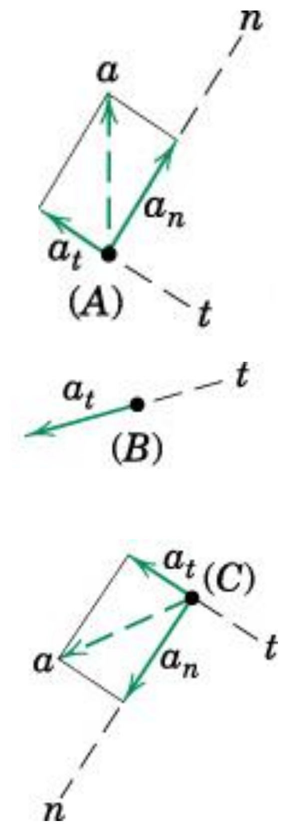
$$[v_C{}^2 = v_A{}^2 + 2a_t\,\Delta s] \qquad a_t = \left|\frac{(50/3.6)^2 - (100/3.6)^2}{2(200)}\right| = 1.447\ \text{m/s}^2$$

Normal components of the acceleration at *A*, *B*, and *C*:

$$[a_n = v^2/\rho] \qquad \text{At } A, \quad a_n = \frac{(100/3.6)^2}{400} = 1.929\ \text{m/s}^2$$

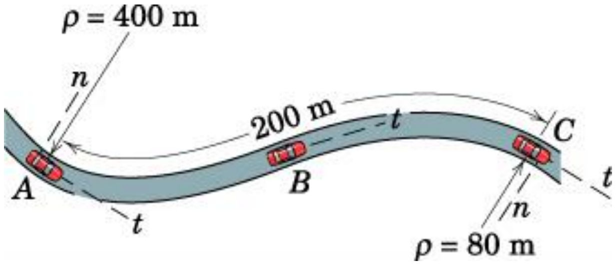
$$\text{At } B, \quad a_n = 0$$

$$\text{At } C, \quad a_n = \frac{(50/3.6)^2}{80} = 2.41\ \text{m/s}^2$$



Forces can be found out by using equation of motion along *n*- and *t*-directions

Example (2) on curvilinear motion



Solution:

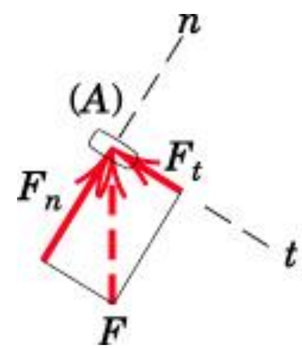
Applying equation of motion to the FBD of the car along *n*- and *t*-directions

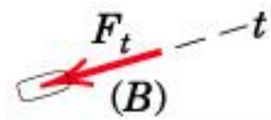
$$[\Sigma F_t = ma_t] \qquad F_t = 1500(1.447) = 2170 \text{ N}$$

$$[\Sigma F_n = ma_n] \qquad \text{At } A, \quad F_n = 1500(1.929) = 2890 \text{ N}$$

$$\text{At } B, \quad F_n = 0$$

$$\text{At } C, \quad F_n = 1500(2.41) = 3620 \text{ N}$$



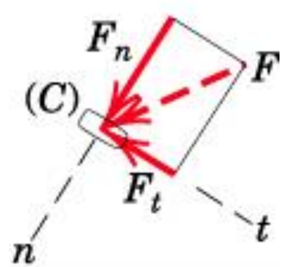


Total horz force acting on tires of car:

$$\text{At } A, \quad F = \sqrt{F_n{}^2 + F_t{}^2} = \sqrt{(2890)^2 + (2170)^2} = 3620 \text{ N}$$

$$\text{At } B, \quad F = F_t = 2170 \text{ N}$$

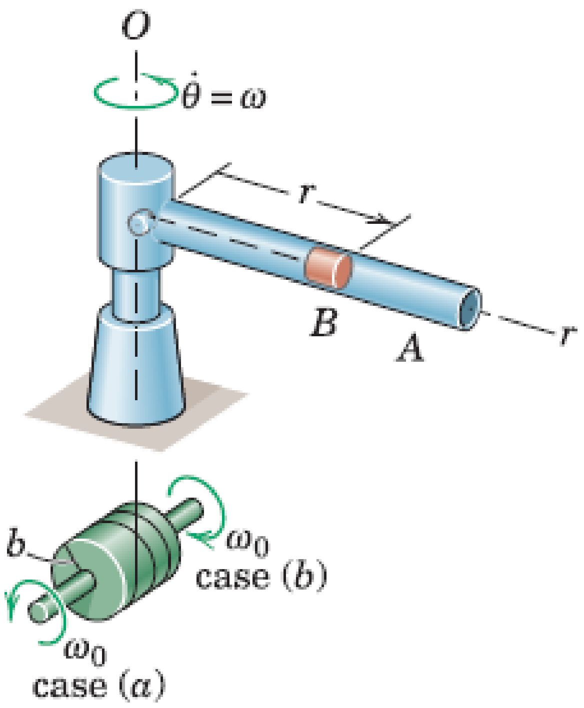
$$\text{At } C, \quad F = \sqrt{F_n{}^2 + F_t{}^2} = \sqrt{(3620)^2 + (2170)^2} = 4220 \text{ N}$$



Directions of forces will match with those of accelerations.

Example (3) on curvilinear motion

Tube A rotates about the vertical O-axis with a constant angular rate $\dot{\theta} = \omega$ and contains a small cylindrical plug B of mass m whose radial position is controlled by the cord which passes freely through the tube and shaft and is wound around the drum of radius b. Determine the tension T in the cord and the horizontal component F_θ of force exerted by the tube on the plug if the constant angular rate of rotation of the drum is ω_0 first in the direction for case (a) and second in the direction for case (b). Neglect friction.



Example (3) on curvilinear motion

Solution. With r a variable, we use the polar-coordinate form of the equations of motion, Eqs. 3/8. The free-body diagram of B is shown in the horizontal plane and discloses only T and F_θ. The equations of motion are

$$[\Sigma F_r = ma_r] \qquad -T = m(\ddot{r} - r\dot{\theta}^2)$$

$$[\Sigma F_\theta = ma_\theta] \qquad F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

Case (a). With $\dot{r} = +b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \qquad F_\theta = 2mb\omega_0\omega \qquad \textit{Ans.}$$

Case (b). With $\dot{r} = -b\omega_0$, $\ddot{r} = 0$, and $\ddot{\theta} = 0$, the forces become

$$T = mr\omega^2 \qquad F_\theta = -2mb\omega_0\omega \qquad \textit{Ans.}$$

Kinetics of a Particle

Work and Energy

•Second approach to solution of Kinetics problems

Work and Kinetic Energy

•Previous discussion: instantaneous relationship between the net force acting on a particle and the resulting acceleration of the particle.

- Change in velocity and corresponding displacement of the particle determined by integrating the computed accelerations using kinematic equations

•Cumulative effects of unbalanced forces acting on a particle

→Integration of the forces wrt displacement of the particle

→ leads to equations of work and energy

→ Integration of the forces wrt time they are applied

→ leads to equations of impulse and momentum

Introduction

- Newton's first and third laws are sufficient for the study of bodies at rest (statics) or bodies in motion with no acceleration.

- When a body accelerates (changes in velocity magnitude or direction), Newton's second law is required to relate the motion of the body to the forces acting on it.

- Newton's second law:
  - A particle will have an acceleration proportional to the magnitude of the resultant force acting on it and in the direction of the resultant force.
  - The resultant of the forces acting on a particle is equal to the rate of change of linear momentum of the particle.
  - The sum of the moments about *O* of the forces acting on a particle is equal to the rate of change of angular momentum of the particle about *O*.

Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

Work

Work done by the force **F** during the displacement $d\mathbf{r}$

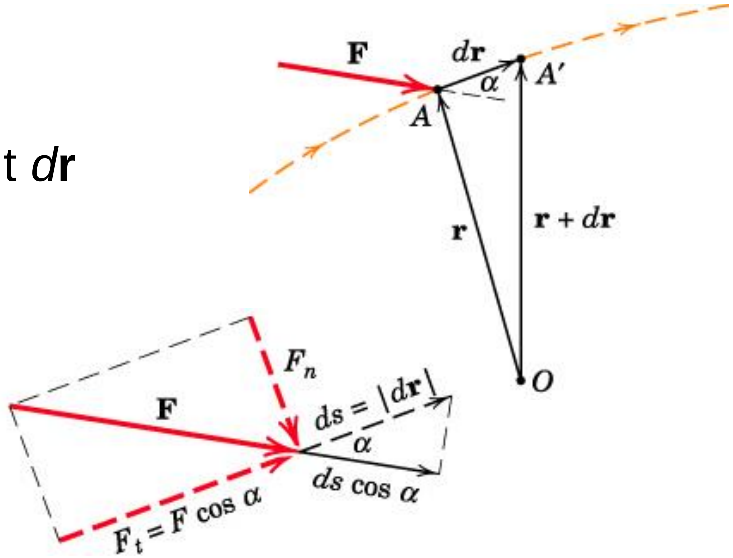
$dU = \mathbf{F}\cdot d\mathbf{r}$

$dU = F\, ds\, \cos\alpha$

The normal component of the force: $F_n = F \sin\alpha$ does no work.

→ $dU = F_t\, ds$

Units of Work: Joules (J) or Nm

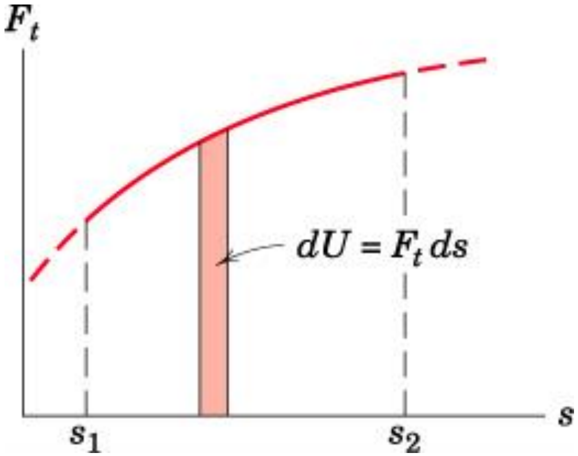


Calculation of Work:

$$U = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \left(F_x dx + F_y dy + F_z dz\right)$$

or

$$U = \int_{s_1}^{s_2} F_t\; ds$$



Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

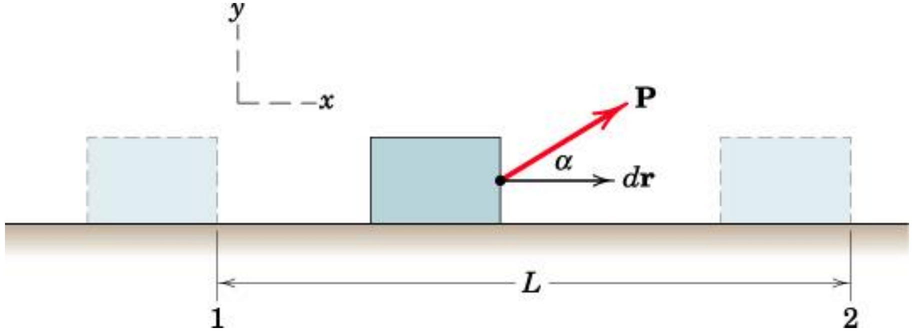
Examples of Work

Computing the work associated with three frequently associated forces:

Constant Forces, Spring Forces, and Weight

(a) Work associated with a constant external force

Work done by the constant force *P* on the body while it moves from position 1 to 2:



$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 \left[(P\cos\alpha)\mathbf{i} + (P\sin\alpha)\mathbf{j} \right] \cdot dx\mathbf{i}$$

$$= \int_{x_1}^{x_2} P\cos\alpha \, dx = P\cos\alpha \left(x_2 - x_1 \right)$$

$$= PL\cos\alpha$$

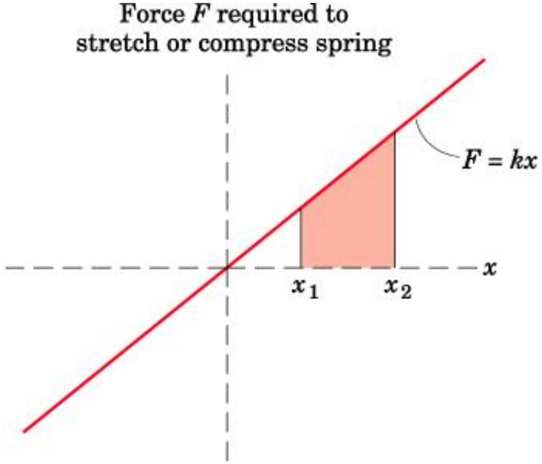
- The normal force *Psinα* does no work.
- Work done will be negative if *α* lies between 90^0 to 270^0

Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

Examples of Work (b) Work associated with a spring force

Force required to compress or stretch a linear spring of stiffness k is proportional to the deformation x. Work done by the spring force on the body while the body moves from initial position x_1 to final position x_2:



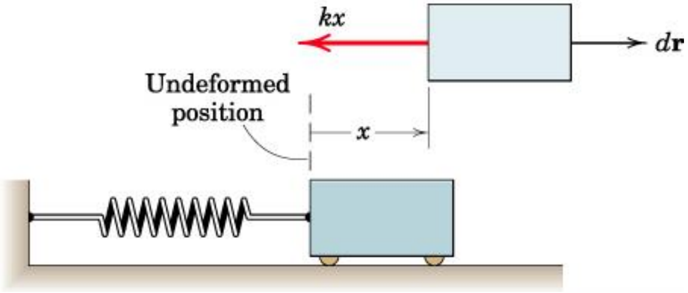
Force exerted by the spring on the body:

F = - kx**i** (this is the force exerted on the body)

$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (-kx\mathbf{i}) \cdot dx\mathbf{i} = -\int_1^2 kx\,dx$$

$$= \frac{1}{2} k\left(x_1^2 - x_2^2\right)$$

- If the initial position x_1 is zero (zero spring deformation), work done is –ve for any final position $x_2 \neq 0$.
- If we move from an arbitrary initial posn $x_1 \neq 0$ to the undeformed final position $x_2 = 0$, work done will be positive (same dirn of force & disp)



Mass of the spring is assumed to be small compared to the masses of other accelerating parts of the system → no appreciable error in using the linear static relationship $F=kx$.

Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

Examples of Work

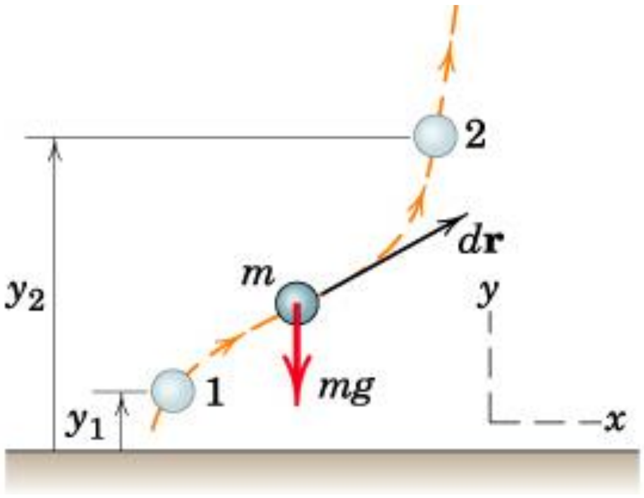
(c) Work associated with weight

Case (i) g = constant → altitude variation is sufficiently small

Work done by weight mg of the body as it is displaced from y_1 to final altitude y_2:

$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 (-mg\mathbf{j}) \cdot (dx\mathbf{i} + dy\mathbf{j})$$

$$= -mg \int_{y_1}^{y_2} dy = -mg(y_2 - y_1)$$



- Horz movement does not contribute to this work
- If the body rises (y_2-y_1 > 0) → Negative Work (opposite direction of force and displacement)
- If the body falls (y_2-y_1 < 0) → Positive Work (same direction of force and displacement)

Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

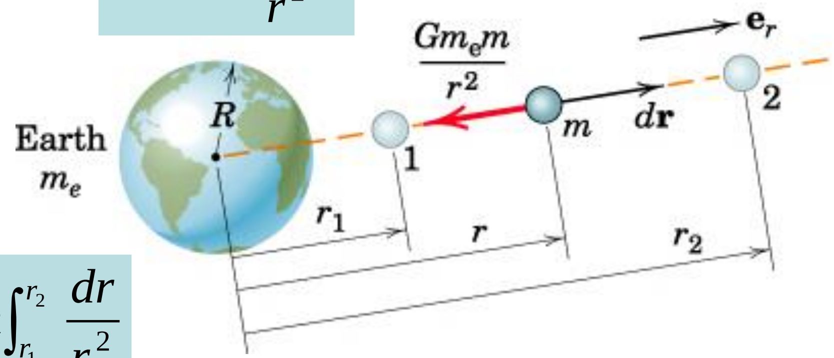
Examples of Work (c) Work associated with weight

Case (ii) $g \neq$ constant → large changes in altitude

Using Law of Gravitation and expressing weight as a variable force of magnitude:

$$F = G\frac{m_1 m_2}{r^2}$$

Using the radial coordinate system, work done by the weight during motion from r_1 to r_2 (measured from center of the earth):



$$U_{1-2} = \int_1^2 \mathbf{F}\cdot d\mathbf{r} = \int_1^2 \frac{-Gm_1 m_2}{r^2}\mathbf{e}_r \cdot dr\,\mathbf{e}_r = -Gm_e m\int_{r_1}^{r_2}\frac{dr}{r^2}$$

$$= Gm_e m\left(\frac{1}{r_2} - \frac{1}{r_1}\right) = mgR^2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$g = Gm_e/R^2$ = accln due to gravity at earth's surface

R = radius of the earth

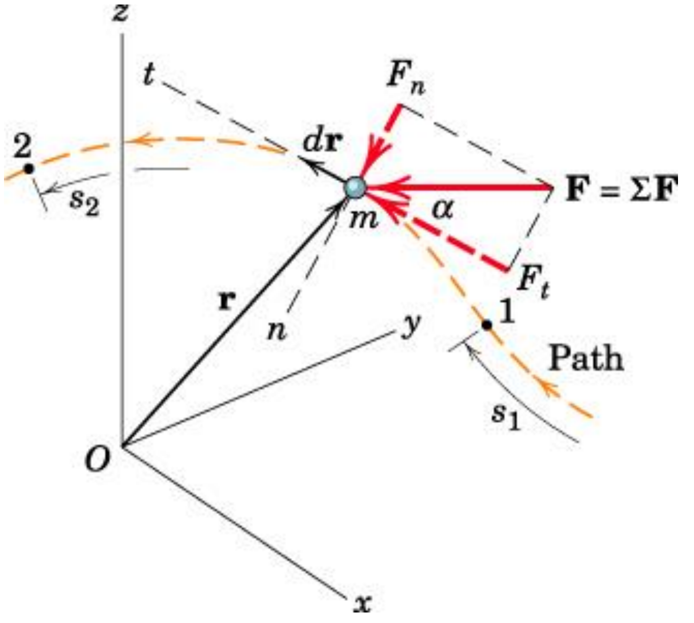
- Transverse movement does not contribute to this work
- If the body rises ($r_2 > r_1$) → Negative Work
- If the body falls ($r_2 < r_1$) → Positive Work

Kinetics of a Particle: Work and Energy

Work and Curvilinear Motion

Work done on a particle of mass *m* moving along a curved path (from 1 to 2) under the action of **F**:

- Position of m specified by position vector **r**
- Disp of *m* along its path during *dt* represented by the change *d***r** in the position vector.



$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_{s_1}^{s_2} F_t \, ds$$

Substituting Newton's Second law **F** = *m***a**:

$$U_{1-2} = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = \int_1^2 m\mathbf{a} \cdot d\mathbf{r}$$

a·*d***r** = $a_t\,ds$ a_t is the tangential component of acceleration of mass

Also, $a_t\,ds = v\,dv$

$$U_{1-2} = \int_1^2 m\mathbf{a} \cdot d\mathbf{r} = \int_{v_1}^{v_2} mv\,dv = \frac{1}{2} m\left(v_2^2 - v_1^2\right)$$

v_1 and v_2 are the velocities at points 1 and 2, respectively.

Kinetics of a Particle: Work and Energy

Principle of Work and Kinetic Energy

The Kinetic Energy T of the particle is defined as:

$$T = \frac{1}{2}mv^2$$

Scalar quantity with units of Work (Joules or Nm)
T is always positive regardless of direction of velocity

Which is the total work required to be done on the particle to bring it from a state of rest to a velocity *v*.

Rewriting the equation for Work done: $U_{1-2} = \frac{1}{2}m\left(v_2^2 - v_1^2\right)$

→ $U_{1-2} = T_2 - T_1 = \Delta T$ May be positive, negative, or zero

→ Work Energy equation for a particle

"*Total Work Done by all forces acting on a particle as it moves from point 1 to 2 equals the corresponding change in the Kinetic Energy of the particle*"

→ Work always results in change in Kinetic Energy

Alternatively, the work-energy equation may be expressed as:

$$T_1 + U_{1-2} = T_2$$

→ Corresponds to natural sequence of events

Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

Advantages of Work Energy Method

- No need to compute acceleration; leads directly to velocity changes as functions of forces, which do work.
- Involves only those forces, which do work, and thus, produces change in magnitudes of velocities.
- Two or more particles connected by rigid and frictionless members can be analyzed without dismembering the system.
  - the internal forces in the connection will be equal and opposite
  - net work done by the internal forces = 0
  - the total kinetic energy of the system is the sum of the kinetic energies of both elements of the system

Method of Analysis:

- Isolate the particles of the system
- For a single particle, draw FBDs showing all externally applied forces
- For a system of particles connected without springs, draw Active Force Diagrams showing only those external forces which do work.

Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

Power

Capacity of a machine is measured by the time rate at which it can do work or deliver energy → Power (= time rate of doing work)

Power P developed by a force **F**, which does an amount of work U:

$P = dU/dt = \mathbf{F} \cdot d\mathbf{r}/dt$

$d\mathbf{r}/dt$ is the velocity **v** at the point of application of the force

→ $P = \mathbf{F} \cdot \mathbf{v}$

- Power is a scalar quantity
- Units: Nm/s = J/s

  Special unit: Watt (W) [US customary unit: Horsepower (hp)]

  1 W = 1 J/s

  1 hp = 746 W = 0.746 kW

Kinetics of a Particle: Work and Energy

Work and Kinetic Energy

Efficiency

Mechanical Efficiency of machine (e_m) = Ratio of the work done by a machine to the work done on the machine during the same interval of time

- Basic assumption: machines operates uniformly → no accumulation or depletion of energy within it.
- Efficiency is always less than unity due to loss of energy and since energy cannot be created within the machine.
- In mechanical devices, loss of energy due to negative work done by kinetic friction forces.

At any instant of time, mechanical efficiency and mechanical power are related by:

$$e_m = \frac{P_{output}}{P_{input}}$$

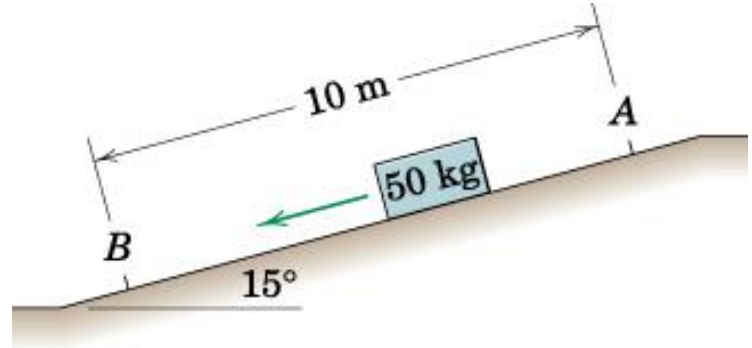
- Other energy losses are: electrical energy loss and thermal energy loss
  → electrical efficiency e_e and thermal efficiency e_t should also be considered

Overall Efficiency: $e = e_m e_e e_t$

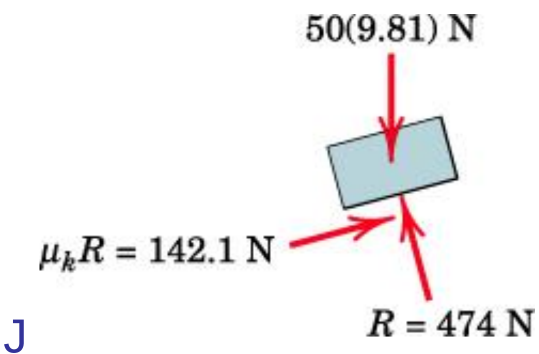
Kinetics of a Particle: Work and Kinetic Energy

Example

Calculate the velocity of the 50 kg box when it reaches point B if it is given an initial velocity of 4 m/s down the slope at A. μ_k = 0.3. Use the principle of work.



Solution: Draw the FBD of the box



Normal reaction R = 50(9.81)cos15 = 474 N
Friction Force: $\mu_k R$ = 0.3x474 = 142.1 N
Work done by the weight will be positive and
Work done by the friction force will be negative.
Total work done on the box during the motion:
$U = Fs \rightarrow U_{1\text{-}2}$ = 50(9.81)(10sin15) – 142.1(10) = -151.9 J
Using work-energy equation:

$$\boxed{T_1 + U_{1-2} = T_2}$$

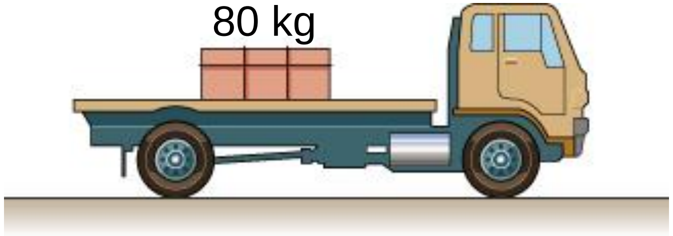
$$\frac{1}{2}mv_1^2 + U_{1-2} = \frac{1}{2}mv_2^2$$

$$\frac{1}{2}(50)(4)^2 - 151.9 = \frac{1}{2}(50)(v_2)^2 \quad \Rightarrow v_2 = 3.15\text{m/s}$$

Work done is negative → velocity reduces → Kinetic Energy reduces

Kinetics of a Particle: Work and Kinetic Energy

Example: A flatbed truck, which carries an 80kg crate, starts from rest and attains a speed of 72km/h in a distance of 75m on a level road with constant acceleration. Calculate the work done by the friction force acting on the crate during this interval if μ_s and μ_k between the crate and the truck bed are (a) 0.3 and 0.28, and (b) 0.25 and 0.2.

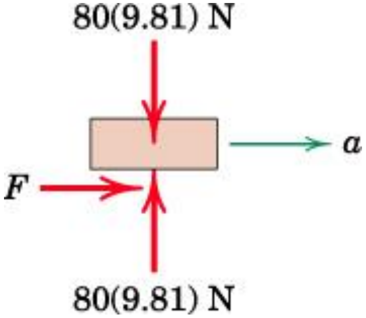


Solution: Draw the FBD of the crate

If the crate does not slip on the flatbed, accln of the crate will be equal to that of the truck:

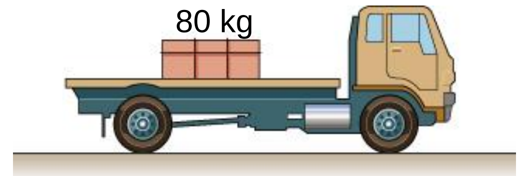
$$[v^2 = 2as]$$

$$a = \frac{v^2}{2s} = \frac{(72/3.6)^2}{2(75)} = 2.67 \text{ m/s}^2$$



Kinetics of a Particle: Work and Kinetic Energy

Example: **Solution**: Acceleration of the crate = 2.67 m/s^2



Case (a): $\mu_s = 0.3$, $\mu_k = 0.28$

The accln of the crate requires a force (friction force) on the flatbed: $F = ma = (80)2.67 = 213$ N

Maximum possible value of frictional force (limiting friction for impending motion):

$F_{lim} = \mu_s N = 0.3(80)(9.81) = 235$ N which is more than F.

→ The crate does not slip and work done by the actual static friction force (213 N):

$U = Fs = 213(75) = 16000$ J = 16 kJ

Case (b): $\mu_s = 0.25$, $\mu_k = 0.20$

The accln of the crate requires a force (friction force) on the flatbed:

$F = ma = (80)2.67 = 213$ N

Maximum possible value of frictional force (limiting friction for impending motion):

$F_{lim} = \mu_s N = 0.25(80)(9.81) = 196.2$ N which is less than F required for no slipping.

→ The crate slips, and the actual friction force is: $F = \mu_k N = 0.2(80)(9.81) = 157$ N

→ And the actual accln of the crate becomes: a = $F/m = 157/80 = 1.962$ m/s^2

The distances travelled by the crate and the truck are in proportion to their acclns.

→ Crate has a displacement of: (1.962/2.67)75 = 55.2 m.

→ Work done by the kinetic friction: $U = Fs = 157(55.2) = 8660$ J = 8.66 kJ

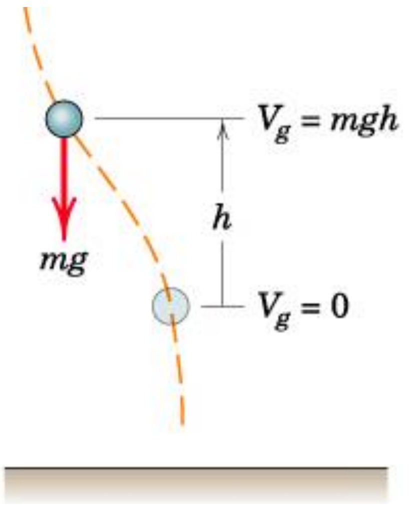
Kinetics of a Particle: Work and Energy

Potential Energy

- In work energy method, work done by gravity forces, spring forces, and other externally applied forces was determined by isolating particles.
- Potential Energy approach can be used to specifically treat the work done by gravity forces and spring forces → Simplify analysis of many problems.

Gravitational Potential Energy

- Motion in close proximity to earth's surface → g constant
- The gravitational potential energy of a particle V_g = work done (mgh) against the gravitational field to elevate the particle a distance h above some arbitrary reference plane, where V_g is taken as zero → $V_g = mgh$



This work is called potential energy because it may be converted into energy if the particle is allowed to do work on supporting body while it returns to its lower original datum.

In going from one level h_1 to higher level h_2, change in potential energy: $\Delta V_g = mg(h_2 - h_1) = mg\Delta h$

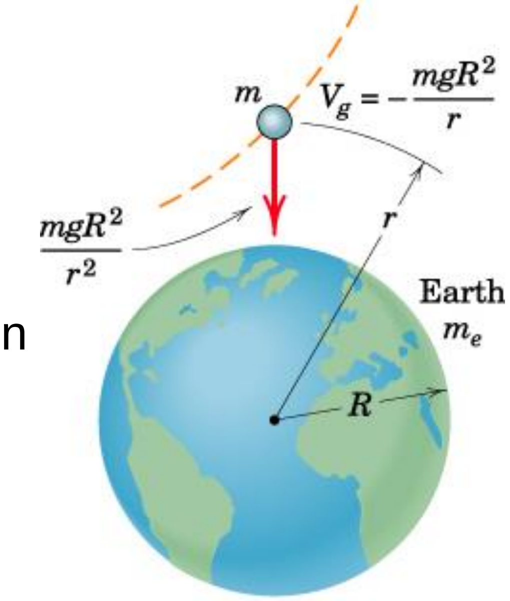
The corresponding work done by the gravitational force on particle is $-mg\Delta h$ → work done by the gravitational force is the negative of the change in V_g.

Kinetics of a Particle: Work and Energy

Potential Energy

Gravitational Potential Energy

For large changes in altitude in the field of the earth, the gravitational force $Gmm_e/r^2 = mgR^2/r^2$ is not constant.



The work done against this force to change the radial position of the particle from r_1 to r_2 is the change $(V_g)_2 - (V_g)_1$ in the gravitational potential energy →

$$\int_{r_1}^{r_2} mgR^2 \frac{dr}{r^2} = mgR^2\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = (V_g)_2 - (V_g)_1$$

$= \Delta V_g$ = negative of the work done by the gravitational force

When $r_2 = \infty$, $(V_g)_2 = 0$ → $\boxed{V_g = -\frac{mgR^2}{r}}$

Potential energy of a particle depends only on its position and not on the particular path it followed in reaching that position.

Kinetics of a Particle: Work and Energy

Potential Energy

Elastic Potential Energy

- Work done on linear elastic spring to deform it is stored in the spring and is called its elastic potential energy V_e.
- Recoverable energy in the form of work done by the spring on the body attached to its movable end during release of the deformation of spring.

Elastic potential energy of the spring = work done on it to deform at an amount x:

$$\boxed{V_e = \int_0^x kx\,dx = \frac{1}{2}kx^2}$$

k is the spring stiffness

If the deformation of the spring increases from x_1 to x_2:
Change in potential energy of the spring is final value minus initial value:

$$\Delta V_e = \frac{1}{2}k(x_2^{\,2} - x_1^{\,2})$$

Always positive as long as deformation increases

If the deformation of spring decreases during the motion interval → negative Δv_e

Force exerted on spring by moving body is equal and opposite to the force exerted by the spring on the body → work done on the spring is the negative of the work done on the spring

→ Replace work done U by the spring on the body by $-V_e$, negative of the potential energy change for the spring → the spring will be included in the system

Kinetics of a Particle: Work and Energy

Potential Energy: **Work-Energy Equation**

Total work done is given by: $U_{1-2} = T_2 - T_1 = \Delta T$

Modifying this eqn to account for the potential energy terms:

$U'_{1\text{-}2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T$ → $U'_{1-2} = \Delta T + \Delta V$

$U'_{1\text{-}2}$ is work of all external forces other than the gravitational and spring forces (Gravitational and spring forces are also known as Conservative Forces and all other external forces that do work are also known as Non-Conservative Forces)

ΔT is the change in kinetic energy of the particle

ΔV is the change in total potential energy

- The new work-energy equation is often far more convenient to use because only the end point positions of the particle and end point lengths of elastic spring are of significance.

Further, following the natural sequence of events: $T_1 + V_1 + U'_{1-2} = T_2 + V_2$

If the only forces acting are gravitational, elastic, and nonworking constraint forces → $U'_{1\text{-}2}$ term will be zero, and the energy equation becomes:

$$T_1 + V_1 = T_2 + V_2 \quad or \quad E_1 = E_2$$

$E = T+V$ is the total mechanical energy of the particle and its attached spring

→ This equation expresses the "Law of Conservation of Dynamical Energy"

Kinetics of a Particle: Work and Energy

Conservation of Energy $T_1 + V_1 = T_2 + V_2 \quad or \quad E_1 = E_2$

- During the motion sum of the particle's kinetic and potential energies remains constant. For this to occur, kinetic energy must be transformed into potential energy, and vice versa.

A ball of weight *W* is dropped from a height *h* above the ground (datum)

- PE of the ball is maximum before it is dropped, at which time its KE is zero.
  Total mechanical energy of the ball in its initial position is:

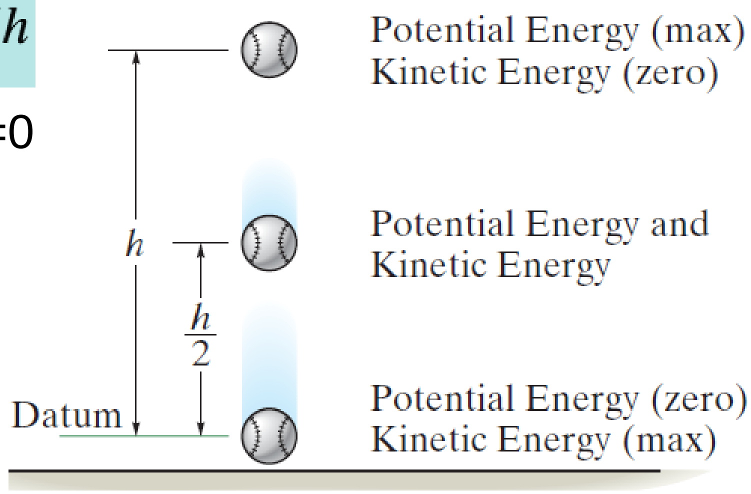
$$E = T_1 + V_1 = 0 + Wh = Wh$$

- When the ball has fallen a distance *h*/2, its speed is: $v^2 = v_0^2 + 2a_c(y - y_0)$
  $v = \sqrt{2g(h/2)} = \sqrt{gh}$
  Energy of the ball at mid-height position:

$$E = T_2 + V_2 = \frac{1}{2}\frac{W}{g}(\sqrt{gh})^2 + W\left(\frac{h}{2}\right) = Wh$$

- Just before the ball strikes the ground, its PE=0 and its speed is: $v = \sqrt{2gh}$
  The total mechanical energy of the ball:

$$E = T_3 + V_3 = \frac{1}{2}\frac{W}{g}(\sqrt{2gh})^2 + 0 = Wh$$



Kinetics of a Particle: Work and Energy

Total work done is given by: $U_{1-2} = T_2 - T_1 = \Delta T$

Modifying this eqn to account for the potential energy terms:

$U'_{1-2} + (-\Delta V_g) + (-\Delta V_e) = \Delta T$ → $U'_{1-2} = \Delta T + \Delta V$

U'_{1-2} is work of all external forces other than the gravitational and spring forces

ΔT is the change in kinetic energy of the particle
ΔV is the change in total potential energy

•*More convenient form because only the end point positions of the particle and end point lengths of elastic spring are of significance*.

$$T_1 + V_1 + U'_{1-2} = T_2 + V_2$$

If the only forces acting are gravitational, elastic, and *nonworking constraint* forces

$$T_1 + V_1 = T_2 + V_2 \quad or \quad E_1 = E_2$$

$E = T+V$ is the total mechanical energy of the particle and its attached spring

→ This equation expresses the "Law of Conservation of Dynamical Energy"

Kinetics of a Particle: Work and Energy

Conservation of Energy $T_1 + V_1 = T_2 + V_2 \quad or \quad E_1 = E_2$

•During the motion, only transformation of KE into PE occurs and it can be vice versa

A ball of weight *W* is dropped from a height *h* above the ground (datum)

•PE of the ball is maximum before it is dropped, at which time its KE is zero. Total mechanical energy of the ball in its initial position is:

$$E = T_1 + V_1 = 0 + Wh = Wh$$

•When the ball has fallen a distance *h*/2, its speed is:

$$v^2 = v_0^2 + 2a_c(y - y_0)$$

$$v = \sqrt{2g(h/2)} = \sqrt{gh}$$

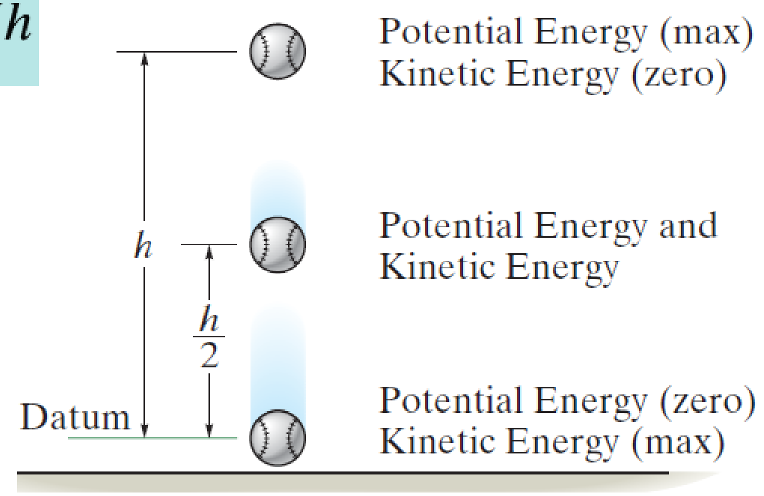
Energy of the ball at mid-height position:

$$E = T_2 + V_2 = \frac{1}{2}\frac{W}{g}\left(\sqrt{gh}\right)^2 + W\left(\frac{h}{2}\right) = Wh$$

•Just before the ball strikes the ground, its PE=0 and its speed is: $v = \sqrt{2gh}$

The total mechanical energy of the ball:

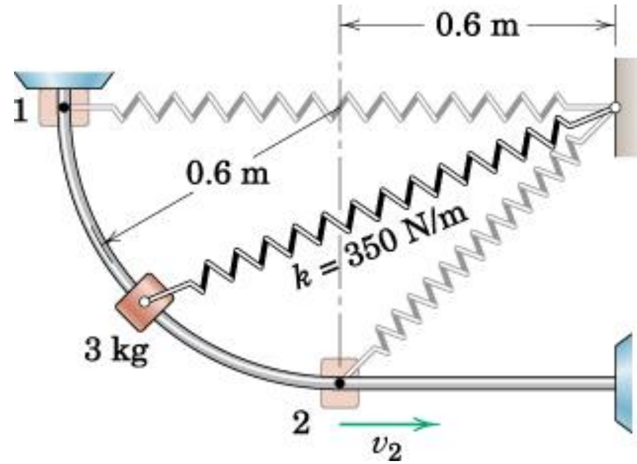
$$E = T_3 + V_3 = \frac{1}{2}\frac{W}{g}\left(\sqrt{2gh}\right)^2 + 0 = Wh$$



Kinetics of a Particle: Work and Energy

Potential Energy

Example: A 3 kg slider is released from rest at position 1 and slides with negligible friction in vertical plane along the circular rod. Determine the velocity of the slider as it passes position 2. The spring has an unstretched length of 0.6 m.



Solution:

•**Reaction** of rod on slider is **normal** to the **motion** → **does no work** → $U'_{1\text{-}2} = 0$

Defining the datum to be at the level of position 1

Kinetic Energy: $T_1 = 0$ and $T_2 = ½ (3)(v_2)^2$

Gravitational Potential Energies:

$V_1 = 0$ and $V_2 = -mgh = -3(9.81)(0.6) = -17.66$ J

Initial and final elastic potential energies:

$V_1 = ½kx_1^2 = 0.5(350)(0.6)^2 = 63$J and $V_2 = ½kx_2^2 = 0.5(350)(0.6\sqrt{2} – 0.6)^2 = 10.81$J

$$T_1 + V_1 + U'_{1-2} = T_2 + V_2$$

→$0 + (0+63) + 0 = ½ (3)(v_2)^2 + (-17.66 + 10.81)$

→$v_2 = 6.82$ m/s

Kinetics of a Particle: Impulse and Momentum

Third approach to solution of Kinetics problems

•***Integrate the equation of motion with respect to time*** (rather than disp.)

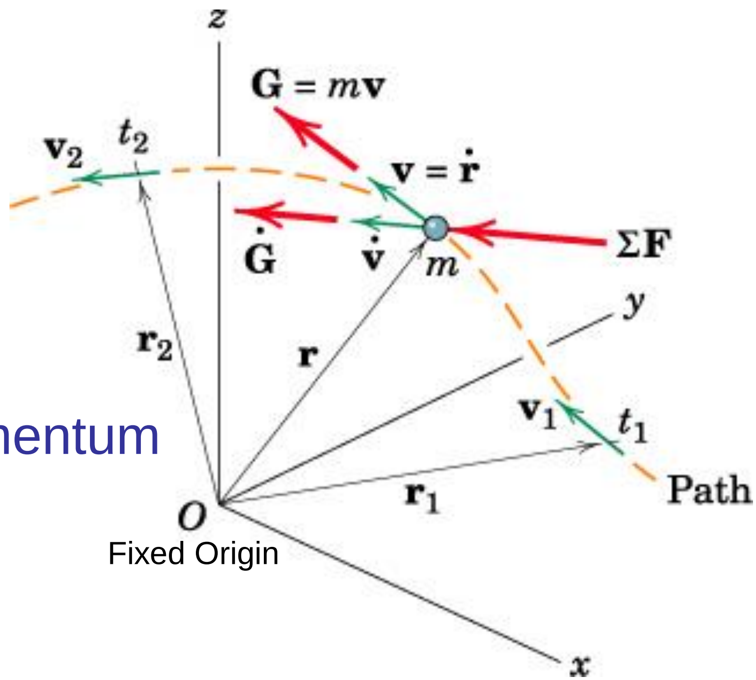
•Cases where the **applied forces** act for a **very short period of time (e.g., Impact loads)** or over **specified intervals of time**

Linear Impulse and Linear Momentum

$$\Sigma \mathbf{F} = m\dot{\mathbf{v}} = \frac{d}{dt}(m\mathbf{v}) \rightarrow \boxed{\Sigma \mathbf{F} = \dot{\mathbf{G}}}$$

→Resultant of all forces acting on a particle equals its time rate of change of linear momentum

Invariability of mass with time!!!



Kinetics of a Particle

Linear Impulse and Linear Momentum

$$\boxed{\Sigma \mathbf{F} = \dot{\mathbf{G}}}$$

Three scalar components of the eqn: $\Sigma F_x = \dot{G}_x \qquad \Sigma F_y = \dot{G}_y \qquad \Sigma F_z = \dot{G}_z$

Linear Impulse-Momentum Principle

•Describes the effect of resultant force on linear momentum of the particle over a finite period of time

Multiplying the eqn by dt → $\sum \mathbf{F}\, dt = d\mathbf{G}$ and integrating from t_1 to t_2

$$\int_{t_1}^{t_2} \Sigma \mathbf{F}\, dt = \mathbf{G}_2 - \mathbf{G}_1 = \Delta \mathbf{G}$$

$\mathbf{G}_1$ = linear momentum at $t_1 = m\mathbf{v}_1$
$\mathbf{G}_2$ = linear momentum at $t_1 = m\mathbf{v}_2$

The product of force and time is defined as Linear Impulse of the Force.

Alternatively:

$$\boxed{\mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma \mathbf{F}\, dt = \mathbf{G}_2}$$

Initial linear momentum of the body plus the linear impulse applied to it equals its final linear momentum

Impulse integral is a vector!!

Kinetics of a Particle

Linear Impulse and Momentum

Impulse-Momentum Equation →

$$\mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma\mathbf{F}\, dt = \mathbf{G}_2$$

It is necessary to write this eqn in component form and then combine the integrated components:

$$m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x\, dt = m(v_2)_x$$

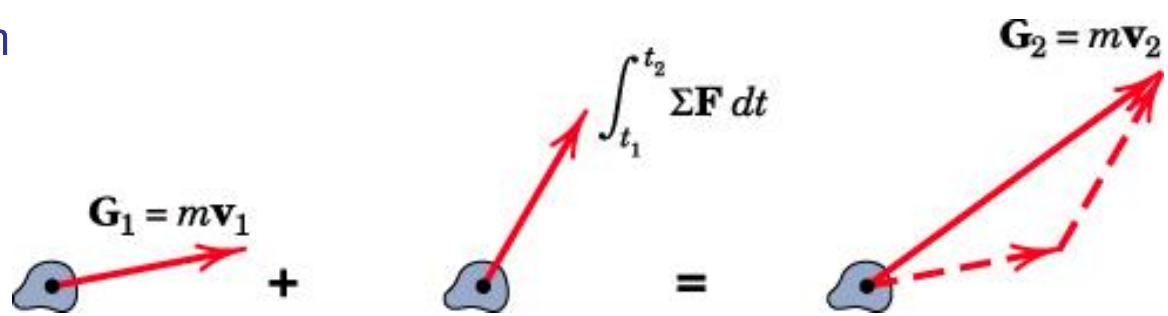
$$m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y\, dt = m(v_2)_y$$

$$m(v_1)_z + \int_{t_1}^{t_2} \Sigma F_z\, dt = m(v_2)_z$$

The three scalar impulse-momentum eqns are completely independent

Impulse-Momentum Diagram

In the middle drawing linear impulses due to all external forces should be included (except for those forces whose magnitudes are negligible)



Impulse-Momentum diagrams can also show the components

Kinetics of a Particle

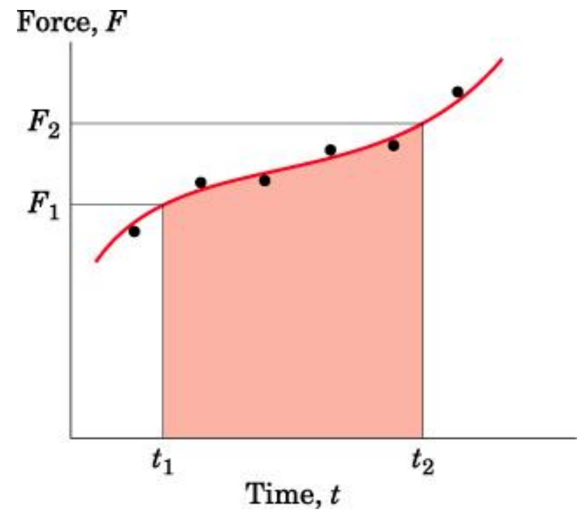
$$\mathbf{G}_1 + \int_{t_1}^{t_2} \Sigma \mathbf{F} \, dt = \mathbf{G}_2$$

Linear Impulse and Linear Momentum

Impulsive Forces: Large forces of short duration *(e.g., hammer impact)*

•In some cases Impulsive forces constant over time → they can be brought outside the linear impulse integral.

Non-impulsive Forces: can be neglected in comparison with the impulsive forces (e.g., weight of small bodies)



In few cases, graphical or numerical integration is required to be performed.

→The impulse of this force from t_1 to t_2 is the shaded area under the curve

Conservation of Linear Momentum

If resultant force acting on a particle is zero during an interval of time, the impulse momentum equation requires that its linear momentum **G** remains constant.

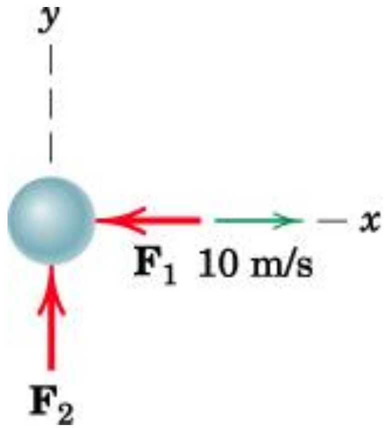
→ The linear momentum of the particle is said to be conserved (in any or all dirn).

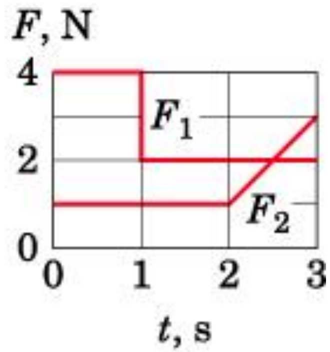
$$\Delta \mathbf{G} = \mathbf{0} \qquad \text{or} \qquad \mathbf{G}_1 = \mathbf{G}_2$$

This principle is also applicable for motion of two interacting particles with equal and opposite interactive forces

Example

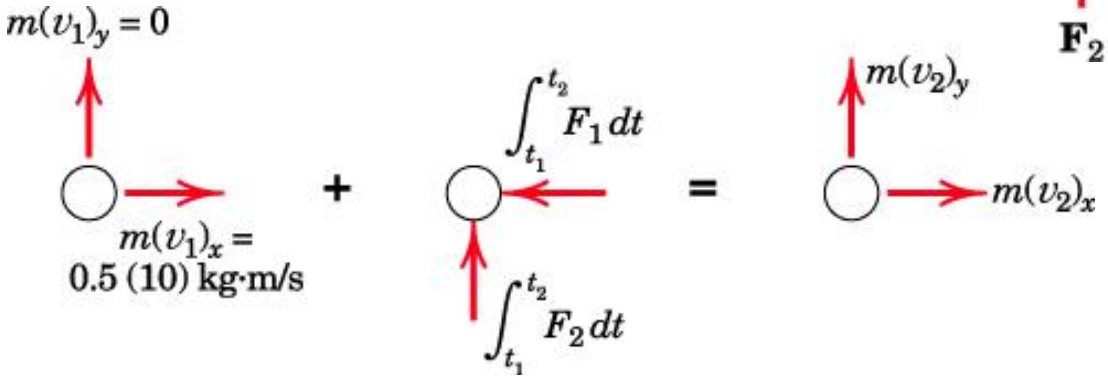
A particle with a mass of 0.5 kg has a velocity of 10 m/s in the x-direction at time $t = 0$. Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on the particle, and their magnitudes change with time according to the graphical schedule shown. Determine the velocity $\mathbf{v}_2$ of the particle at the end of the 3-s interval. The motion occurs in the horizontal x-y plane.





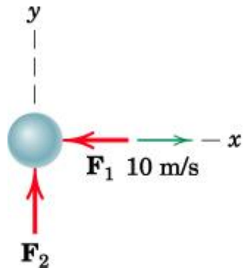
Solution:

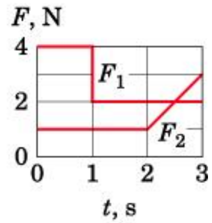
Construct the impulse-momentum diagram

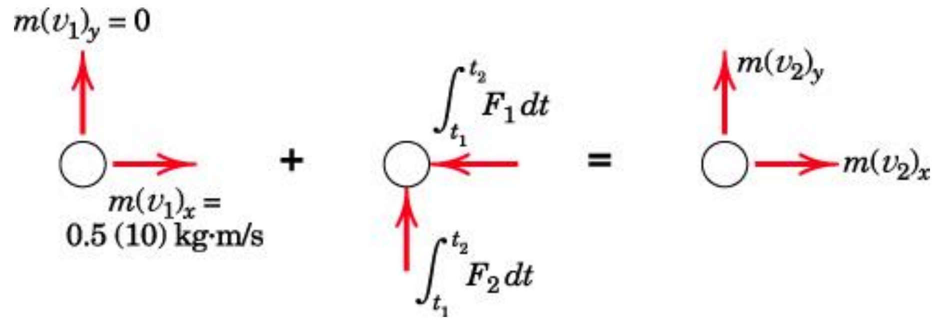


Kinetics of a Particle: Linear Impulse and Linear Momentum

Example Solution:







Using the impulse-momentum eqns:

$$m(v_1)_x + \int_{t_1}^{t_2} \Sigma F_x\,dt = m(v_2)_x$$

$$0.5(10) - [4(1) + 2(3-1)] = 0.5(v_2)_x$$

$$(v_2)_x = -6 \text{ m/s}$$

$$m(v_1)_y + \int_{t_1}^{t_2} \Sigma F_y\,dt = m(v_2)_y$$

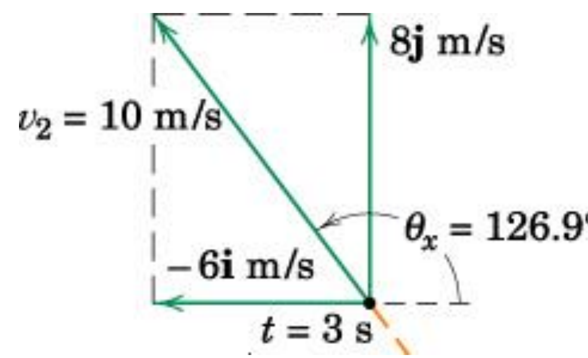
$$0.5(0) + [1(2) + 2(3-2)] = 0.5(v_2)_y$$

$$(v_2)_y = 8 \text{ m/s}$$

The velocity $\mathbf{v}_2$:

$$\mathbf{v}_2 = -6\mathbf{i} + 8\mathbf{j} \text{ m/s} \qquad \text{and} \qquad v_2 = \sqrt{6^2 + 8^2} = 10 \text{ m/s}$$

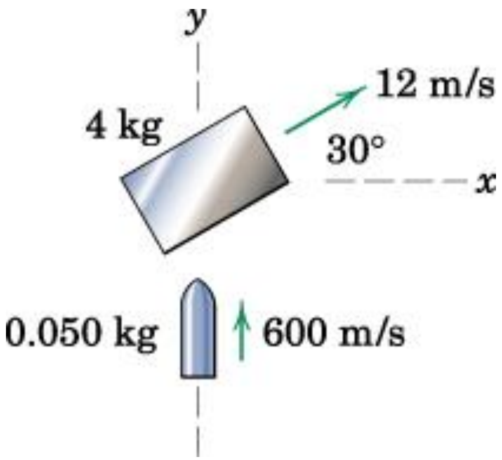
$$\theta_x = \tan^{-1}\frac{8}{-6} = 126.9^\circ$$



Kinetics of a Particle: Linear Impulse and Linear Momentum

Example

The 50-g bullet traveling at 600 m/s strikes the 4-kg block centrally and is embedded within it. If the block slides on a smooth horizontal plane with a velocity of 12 m/s in the direction shown prior to impact, determine the velocity $\mathbf{v}_2$ of the block and embedded bullet immediately after impact.



Solution:

The force of impact is internal to the system composed of the block and the bullet. Further, no other external force acts on the system in the plane of the motion.

→ Linear momentum of the system is conserved

→ $\mathbf{G}_1 = \mathbf{G}_2$

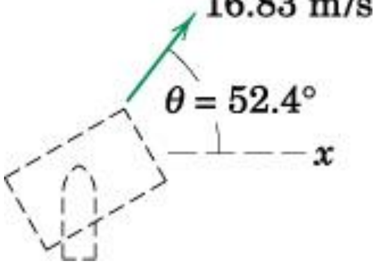
$$0.050(600\mathbf{j}) + 4(12)(\cos 30^\circ\mathbf{i} + \sin 30^\circ\mathbf{j}) = (4 + 0.050)\mathbf{v}_2$$

$$\mathbf{v}_2 = 10.26\mathbf{i} + 13.33\mathbf{j} \text{ m/s}$$

Final velocity and direction:

$$v_2 = \sqrt{(10.26)^2 + (13.33)^2} = 16.83 \text{ m/s}$$

$$\tan\theta = \frac{13.33}{10.26} = 1.299 \qquad \theta = 52.4^\circ$$

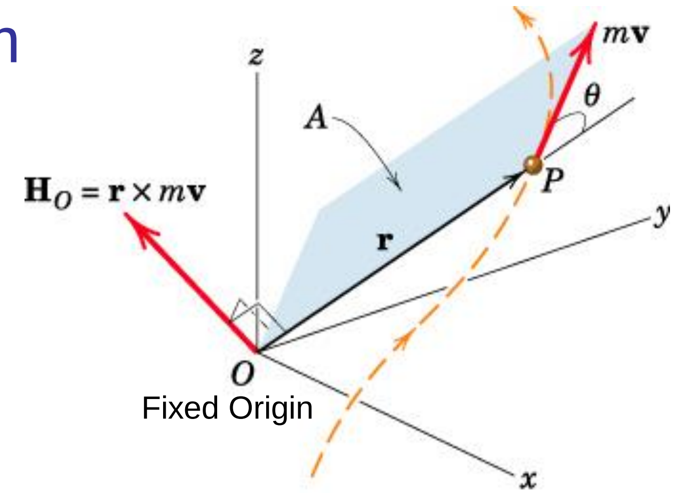


Kinetics of a Particle

Angular Impulse and Angular Momentum

Velocity of the particle is $\mathbf{v} = \dot{\mathbf{r}}$

Momentum of the particle: $\mathbf{G} = m\mathbf{v}$



Moment of the linear momentum vector $m\mathbf{v}$ about the origin O is defined as Angular Momentum $\mathbf{H}_O$ of P about O and is given by:

$$\boxed{\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}}$$

Scalar components of angular momentum:

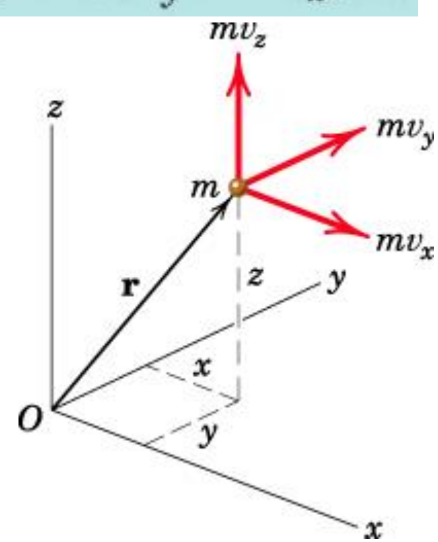
$$\mathbf{H}_O = \mathbf{r} \times m\mathbf{v} = m(v_z y - v_y z)\mathbf{i} + m(v_x z - v_z x)\mathbf{j} + m(v_y x - v_x y)\mathbf{k}$$

$$\mathbf{H}_O = m\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix}$$

$$H_x = m(v_z y - v_y z)$$

$$H_y = m(v_x z - v_z x)$$

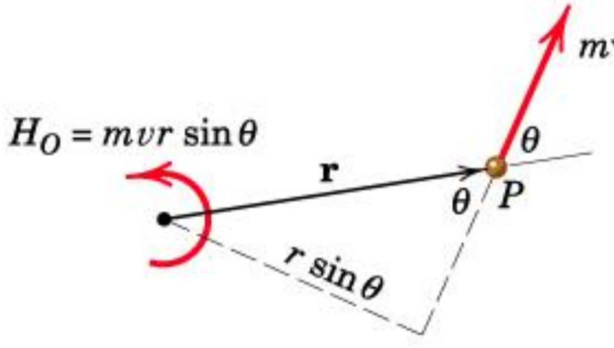
$$H_z = m(v_y x - v_x y)$$



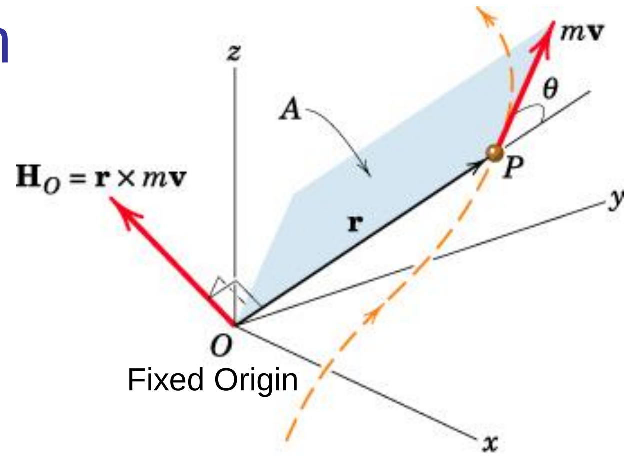
Kinetics of a Particle

Angular Impulse and Angular Momentum

A 2-D representation of vectors in plane A is shown:



View in plane A



Magnitude of the moment of $m\mathbf{v}$ @ O = linear momentum mv times the moment arm

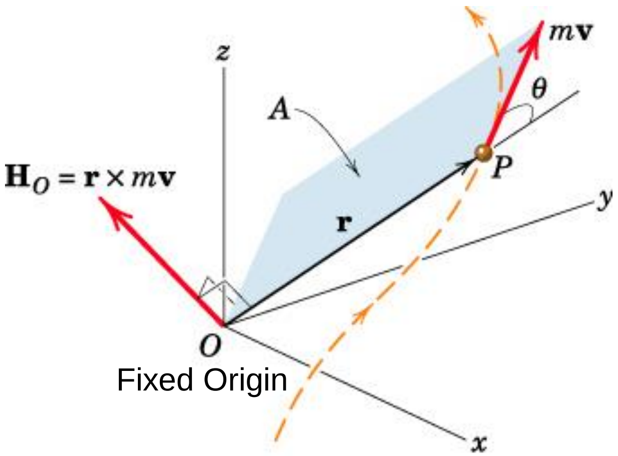
→ $H_O = mvr \sin\theta$

→ This is the magnitude of the cross product $\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$

Units of Angular Momentum: kg.(m/s).m = kg.m^2/s or N.m.s

Kinetics of a Particle

Angular Impulse and Angular Momentum



Rate of Change of Angular Momentum

- To relate moment of forces and angular momentum

Moment of resultant of all forces acting on *P* @ origin:

$$\Sigma \mathbf{M}_O = \mathbf{r} \times \Sigma \mathbf{F} = \mathbf{r} \times m\dot{\mathbf{v}}$$

Differentiating with time:

$$\dot{\mathbf{H}}_O = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}}$$

v and *m***v** are parallel vectors → **v** x *m***v** = 0

→ $\boxed{\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O}$ Using this vector equation, moment of forces and angular momentum are related

→ Moment of all forces @ O = time rate of change of angular momentum

Scalar components: $\Sigma M_{O_x} = \dot{H}_{O_x} \qquad \Sigma M_{O_y} = \dot{H}_{O_y} \qquad \Sigma M_{O_z} = \dot{H}_{O_z}$

Kinetics of a Particle

Angular Impulse and Angular Momentum

Angular Impulse-Momentum Principle $\boxed{\Sigma \mathbf{M}_O = \dot{\mathbf{H}}_O}$

-This eqn gives the instantaneous relation between moment and time rate of change of angular momentum

$$\Sigma \mathbf{M}_O \, dt = d\mathbf{H}_O$$

Integrating: $$\int_{t_1}^{t_2} \Sigma \mathbf{M}_O \, dt = (\mathbf{H}_O)_2 - (\mathbf{H}_O)_1 = \Delta \mathbf{H}_O$$

$$(\mathbf{H}_O)_2 = \mathbf{r}_2 \times m\mathbf{v}_2$$

$$(\mathbf{H}_O)_1 = \mathbf{r}_1 \times m\mathbf{v}_1$$

The product of moment and time is defined as the angular impulse.

→*The total angular impulse on m about the fixed point O equals the corresponding change in the angular momentum of m about O.*

Alternatively:

$$\boxed{(\mathbf{H}_O)_1 + \int_{t_1}^{t_2} \Sigma \mathbf{M}_O \, dt = (\mathbf{H}_O)_2}$$

Initial angular momentum of the particle plus the angular impulse applied to it equals the final angular momentum.

Kinetics of a Particle

$$(\mathbf{H}_O)_1 + \int_{t_1}^{t_2} \Sigma \mathbf{M}_O \, dt = (\mathbf{H}_O)_2$$

Angular Impulse and Angular Momentum

Angular Impulse-Momentum Principle

In the component form:

The *x*-component of this eqn:

Similarly other components can be written

$$(H_{O_x})_1 + \int_{t_1}^{t_2} \Sigma M_{O_x} \, dt = (H_{O_x})_2$$

$$m(v_z y - v_y z)_1 + \int_{t_1}^{t_2} \Sigma M_{O_x} \, dt = m(v_z y - v_y z)_2$$

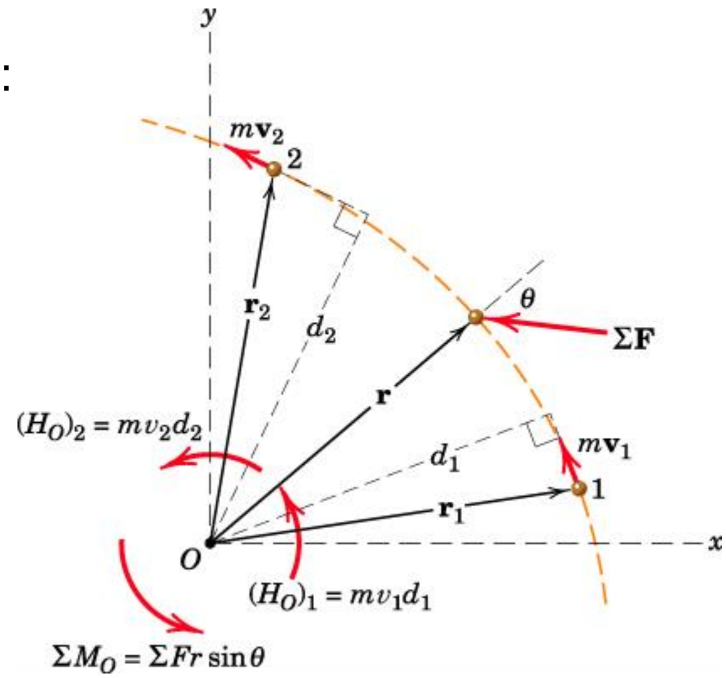
Plane Motion Applications

- In most applications, plane motions are encountered instead of 3-D motion.
- Simplifying the eqns

Using the scalar form of the principle betn 1 and 2:

$$(H_O)_1 + \int_{t_1}^{t_2} \Sigma M_O \, dt = (H_O)_2$$

→ $$m v_1 d_1 + \int_{t_1}^{t_2} \Sigma F r \sin\theta \, dt = m v_2 d_2$$



Kinetics of a Particle

Angular Impulse and Angular Momentum

Conservation of Angular Momentum

If the resulting moment @ a fixed point *O* of all forces acting on a particle is zero during an interval of time, the angular momentum of the particle about that point remain constant.

$$\Delta \mathbf{H}_O = \mathbf{0} \quad \text{or} \quad (\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

→ Principle of Conservation of Angular Momentum

→ Also valid for motion of two interacting particles with equal and opposite interacting forces

Kinetics of a Particle

Angular Impulse and Angular Momentum

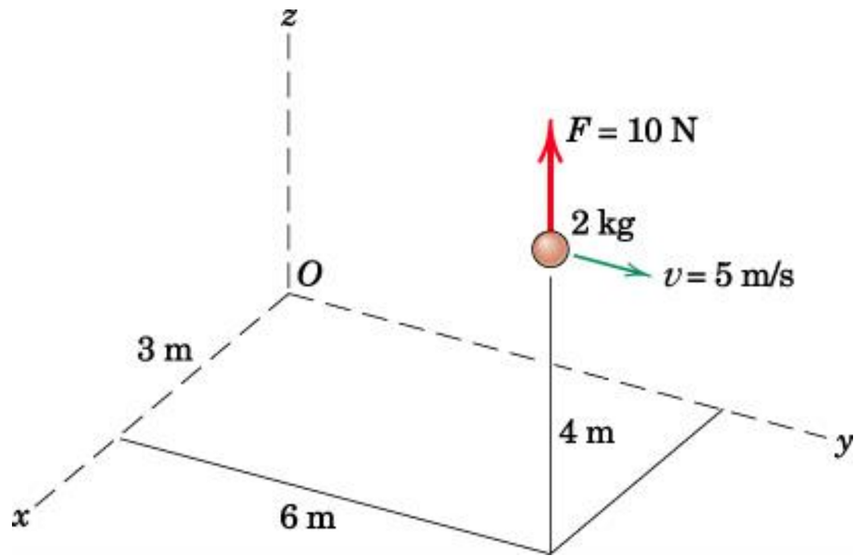
Example

A small sphere has the position and velocity indicated in the figure and is acted upon by the force F. Determine the angular momentum $\mathbf{H}_O$ about point O and the time derivative $\dot{\mathbf{H}}_O$.

Solution

$$\begin{aligned}\mathbf{H}_O &= \mathbf{r} \times m\mathbf{v} \\ &= (3\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}) \times 2(5\mathbf{j}) \\ &= -40\mathbf{i} + 30\mathbf{k}\ \mathrm{N\cdot m/s}\end{aligned}$$

$$\begin{aligned}\dot{\mathbf{H}}_O &= \mathbf{M}_O \\ &= \mathbf{r} \times \mathbf{F} \\ &= (3\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}) \times 10\mathbf{k} \\ &= 60\mathbf{i} - 30\mathbf{j}\ \mathrm{N\cdot m}\end{aligned}$$



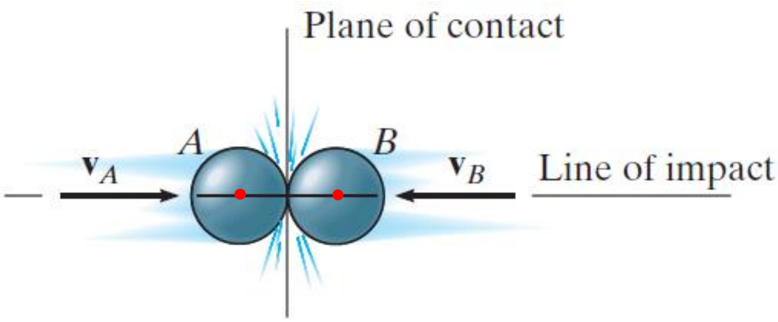
Kinetics of a Particle

Impact

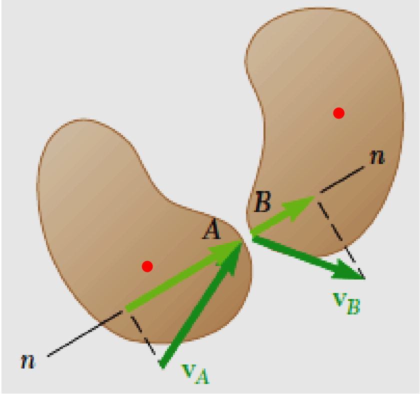
- Collision between two bodies during a very short period of time.
- Generation of large contact forces (impulsive) acting over a very short interval of time.
- Complex phenomenon (material deformation and recovery, generation of heat and sound)
- Line of Impact is the common normal to the surfaces in contact during impact.

Impact primarily classified as two types:

- Central Impact: Mass centers of two colliding bodies are located on line of impact → Current chapter deals with central impact of two particles.
- Eccentric Impact: Mass centers are not located on line of impact.



Central Impact



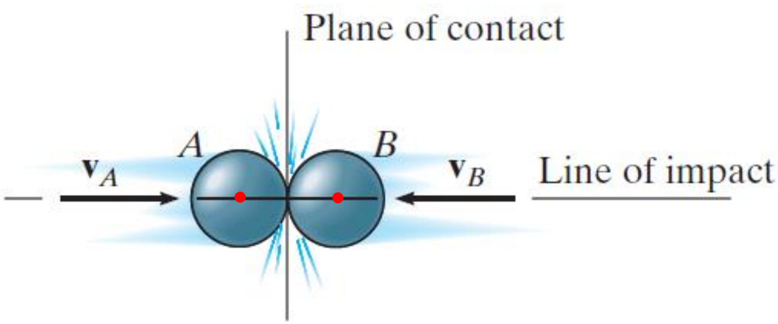
Eccentric Impact

Kinetics of a Particle

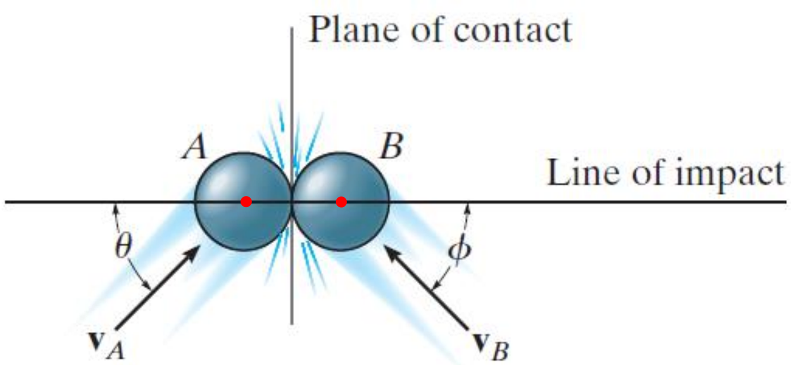
Central Impact

- Can be classified into two types

- Direct Central Impact (or Direct Impact)
  - Velocities of the two particles are directed along the line of impact
  - Direction of motion of the particles will also be along the line of impact
- Oblique Central Impact (or Oblique Impact)
  - Velocity and motion of one or both particles is at an angle with the line of impact.
  - Initial and final velocities are not parallel.



Direct Central Impact



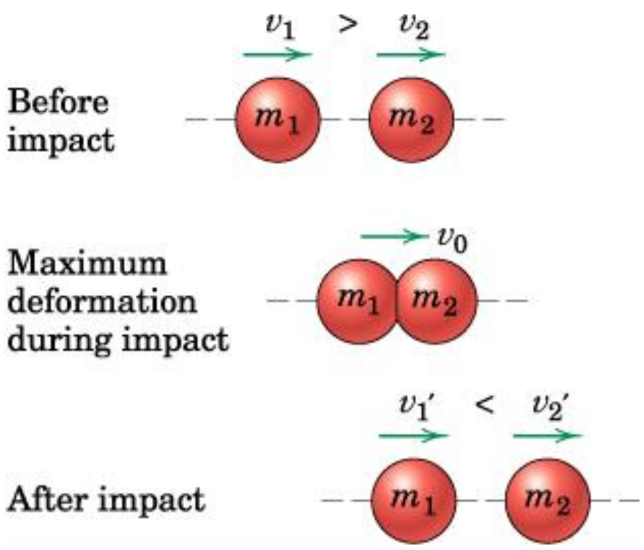
Oblique Central Impact

Kinetics of a Particle: Impact

Direct Central Impact

Collinear motion of two spheres ($v_1 > v_2$)

- Collision occurs with contact forces directed along the line of impact (line of centers)
- Deformation of spheres increases until contact area ceases to increase. Both spheres move with the same velocity.
- Period of restoration during which the contact area decreases to zero
- After the impact, spheres will have different velocities (v'_1 & v'_2) with $v'_1 < v'_2$



During impact, contact forces are equal and opposite. Further, there are no impulsive external forces → linear momentum of the system remains unchanged. Applying the law of conservation of linear momentum:

$$m_1 v_1 + m_2 v_2 = m_1 v_1{}' + m_2 v_2{}'$$

Assumptions

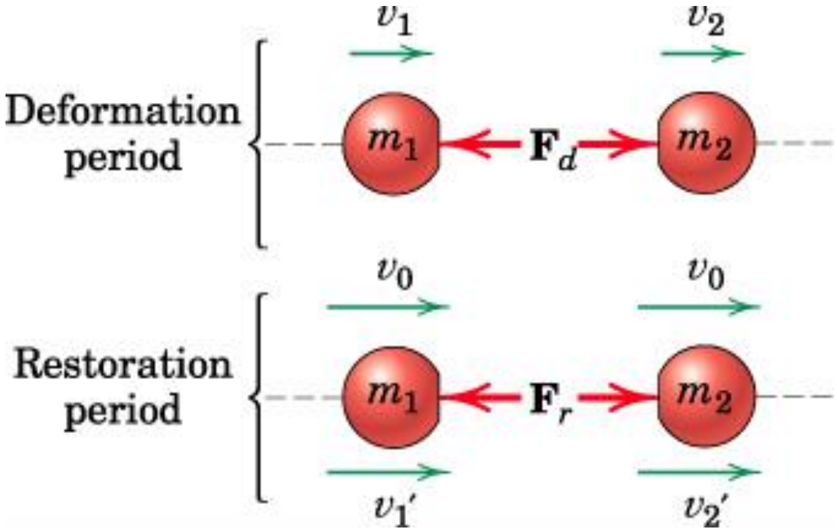
- Particles are perfectly smooth and frictionless.
- Impulses created by all forces (other than the internal forces of contact) are negligible compared to the impulse created by the internal impact force.
- No appreciable change in position of mass centers during the impact.

Kinetics of a Particle: Impact

Coefficient of Restitution

The momentum eqn contains 2 unknowns v'_1 & v'_2 (assuming that v_1 & v_2 are known)

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$



→ Another equation is required
→ Coefficient of Restitution (e)

$$e = \frac{\text{magnitude of the Restoration Impulse}}{\text{magnitude of the Deformation Impulse}}$$

Using the definition of impulse:

$$e = \frac{\int_{t_0}^{t} F_r \, dt}{\int_{0}^{t_0} F_d \, dt}$$

F_r and F_d = magnitudes of the contact forces during the restoration and deformation periods.
t_0 = time for the deformation
t = total time of contact

Further using Linear Impulse-Momentum Principle, we can write this equation in terms of change in momentum

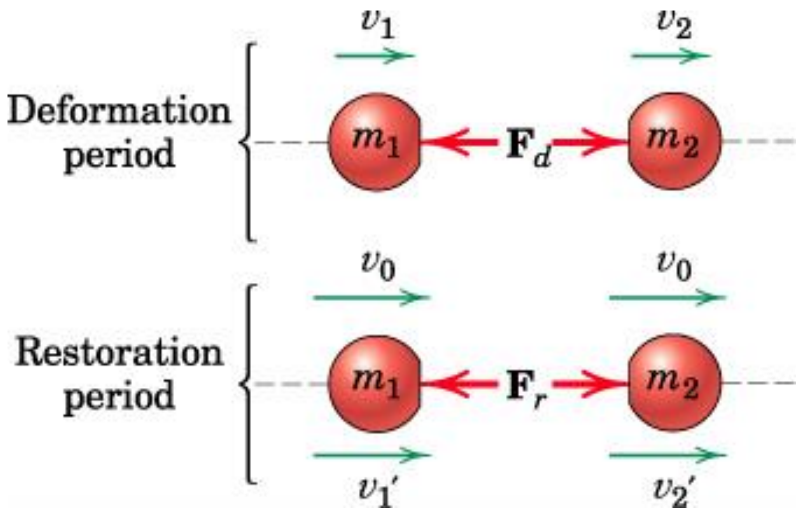
$$\int_{t_1}^{t_2} \Sigma \mathbf{F} \, dt = \mathbf{G}_2 - \mathbf{G}_1 = \Delta \mathbf{G}$$

Kinetics of a Particle: Impact

Coefficient of Restitution

For Particle 1:

$$e = \frac{\int_{t_0}^{t} F_r\, dt}{\int_{0}^{t_0} F_d\, dt} = \frac{m_1[-v_1' - (-v_0)]}{m_1[-v_0 - (-v_1)]} = \frac{v_0 - v_1'}{v_1 - v_0}$$



For Particle 2:

$$e = \frac{\int_{t_0}^{t} F_r\, dt}{\int_{0}^{t_0} F_d\, dt} = \frac{m_2(v_2' - v_0)}{m_2(v_0 - v_2)} = \frac{v_2' - v_0}{v_0 - v_2}$$

Change in momentum (and thus the velocity) is expressed in the same direction as impulse (and thus the force)

Eliminating v_0 between the two expressions:

$$e = \frac{v_2' - v_1'}{v_1 - v_2} = \frac{|\text{relative velocity of separation}|}{|\text{relative velocity of approach}|}$$

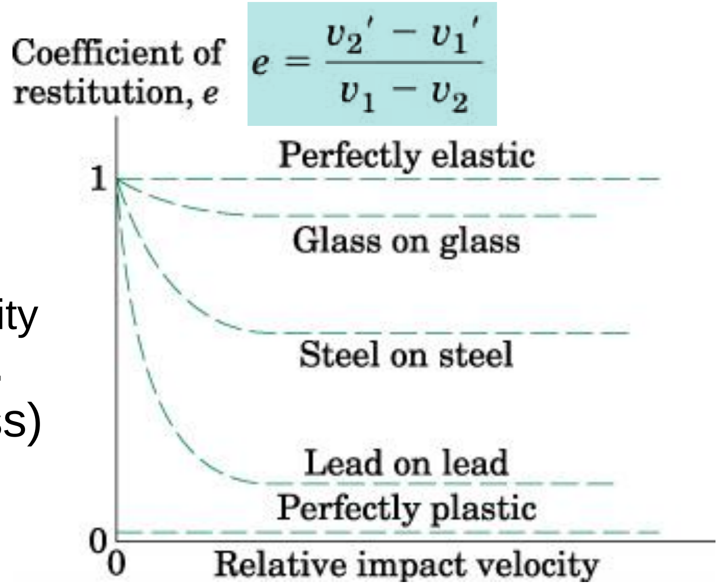
If v_1, v_2, and e are known, the final velocities v'_1 & v'_2 can be obtained using the two eqns → eqn for e and momentum eqn $m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$

Kinetics of a Particle: Impact

Coefficient of Restitution

Energy Loss during impact

- Impact always associated with energy loss (heat, inelastic deformation, etc)
- Energy loss may be determined by finding the change in the KE of the system before and after the impact.



Classical theory of impact:

$e = 1$ → Elastic Impact (no energy loss)

- Put $e = 1$ in the eqn → $v'_2 - v'_1 = v_1 - v_2$
- Relative velocities before & after impact are equal
- Particles move away after impact with the same velocity with which they approached each other before impact.

$e = 0$ → Inelastic or Plastic Impact (max energy loss)

- Put $e = 0$ in the eqn → $v'_1 = v'_2$
- The particles stick together after collision and move with a common velocity

Real conditions lie somewhere betn these extremes

- ❖ e varies with impact velocity, and size and shape of the colliding bodies.
- ❖ e is considered constant for given geometries and a given combination of contacting materials.
- ❖ e approaches unity as the impact velocity approaches zero.

Kinetics of a Particle: Impact

Oblique Impact

- In-plane initial and final velocities are not parallel
- Choosing the *n*-axis along the line of impact, and the *t*-axis along the common tangent.
- Directions of velocity vectors measured from *t*-axis.

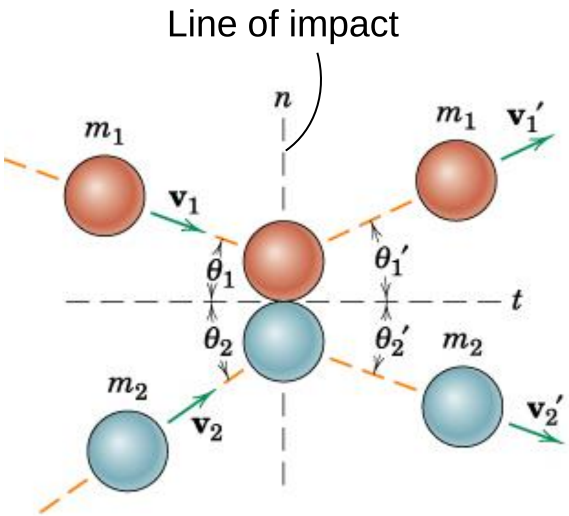
Initial velocity components are:

$(v_1)_n = -v_1 sin\theta_1$, $(v_1)_t = v_1 cos\theta_1$

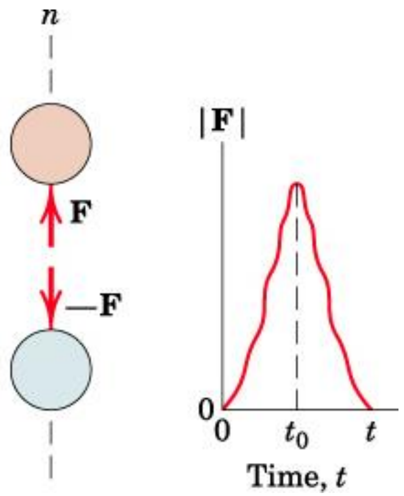
$(v_2)_n = v_2 sin\theta_2$, $(v_2)_t = v_2 cos\theta_2$

→ Four unknowns $(v'_1)_n$, $(v'_1)_t$, $(v'_2)_n$, $(v'_2)_t$

→ Four equations are required.



Particles are assumed to be perfectly smooth and frictionless → the only impulses exerted on the particles during the impact are due to the internal forces directed along the line of impact (*n*-axis).

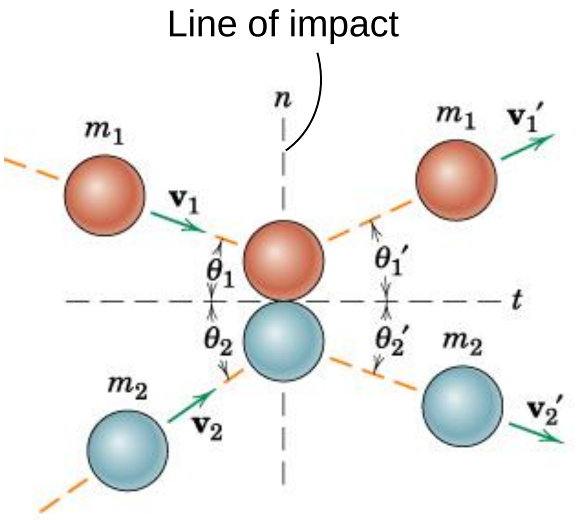


Impact forces acting on the two particles are equal and opposite: **F** and –**F** (variation during impact is shown).

→ Since there are no impulsive external forces, total momentum of the system (both particles) is conserved along the *n*-axis

Kinetics of a Particle: Impact

Oblique Impact:



- total momentum of the system is conserved along the *n*-axis

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n \qquad (1)$$

- Component along *t*-axis of the momentum of each particle, considered separately, is conserved (since there is no impulse on particles along *t*-direction)

$$m_1(v_1)_t = m_1(v_1')_t$$
$$m_2(v_2)_t = m_2(v_2')_t \qquad \text{(2 and 3)}$$

→ *t*-component of velocity of each particle remains unchanged.

- Coefficient of the Restitution is the positive ratio of the recovery impulse to the deformation impulse → the eqn will be applied to the velocity components along *n*-direction.

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \qquad (4)$$

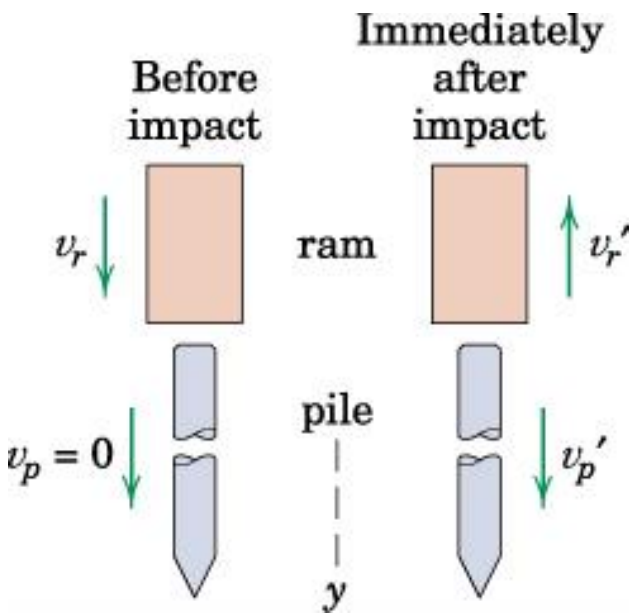
Once the four final velocities are determined, angles θ_1' and θ_2' can be easily determined.

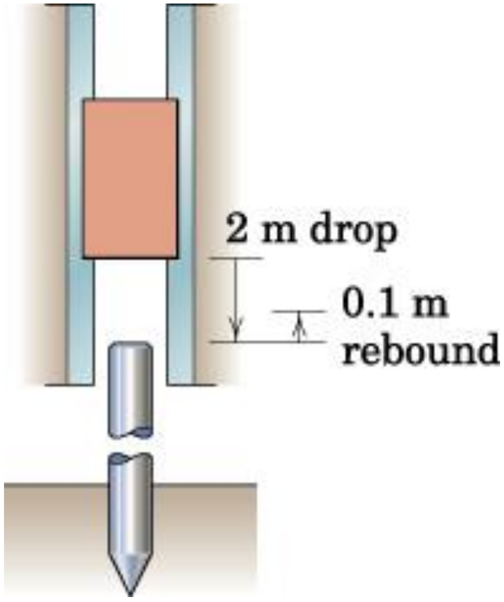
Kinetics of a Particle: Impact

Example:

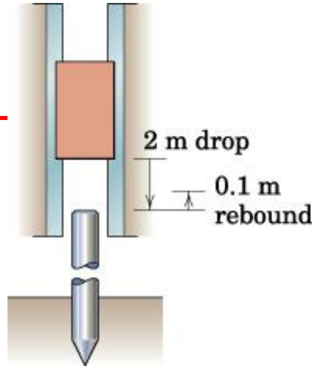
The ram of a pile driver has a mass of 800 kg and is released from rest 2 m above the top of the 2400-kg pile. If the ram rebounds to a height of 0.1 m after impact with the pile, calculate (*a*) the velocity v_p' of the pile immediately after impact, (*b*) the coefficient of restitution *e*, and (*c*) the percentage loss of energy due to the impact.

Solution:
Direct Central Impact





Kinetics of a Particle: Impact



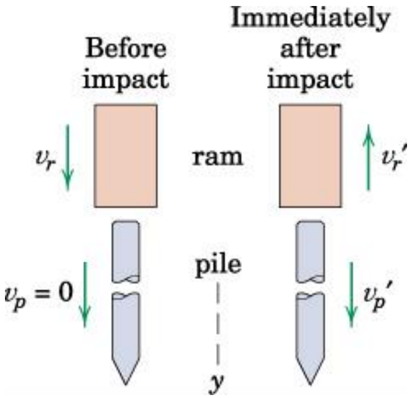
Example:

Energy is conserved during free fall: $T_1 + V_1 = T_2 + V_2 \quad or \quad E_1 = E_2$

Initial and final velocities of the ram can be calculated from:

$$v^2 = v_0^2 + 2a_c(y - y_0) \quad \rightarrow \quad v = \sqrt{2gh}$$

$$v_r = \sqrt{2(9.81)(2)} = 6.26 \text{ m/s} \qquad v_r' = \sqrt{2(9.81)(0.1)} = 1.401 \text{ m/s}$$



Conservation of the momentum of the system of ram and pile:

$G_1 = G_2$ $\qquad 800(6.26) + 0 = 800(-1.401) + 2400v_p'$

→ $v_p' = 2.55$ m/s [impulses of the weights are neglected]

Coefficient of Restitution $\quad e = \dfrac{|\text{rel. vel. separation}|}{|\text{rel. vel. approach}|} \qquad e = \dfrac{2.55 + 1.401}{6.26 + 0} = 0.631$

Loss of energy due to impact can be calculated from difference in KE of system

KE just before impact: T = ½ (800)(6.26)^2 = 15675 J

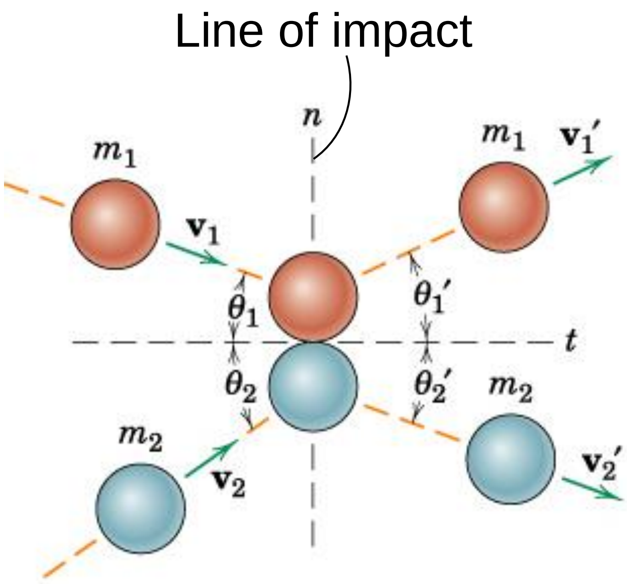
Since the energy is conserved this can also be calculated from PE=*mgh*

KE just after impact: T' = ½ (800)(1.401)^2 + ½ (2400)(2.55)^2 = 8588 J

% loss of energy: [(15675-8588)/15675]x(100) = 45.2%

Kinetics of a Particle: Oblique Central Impact

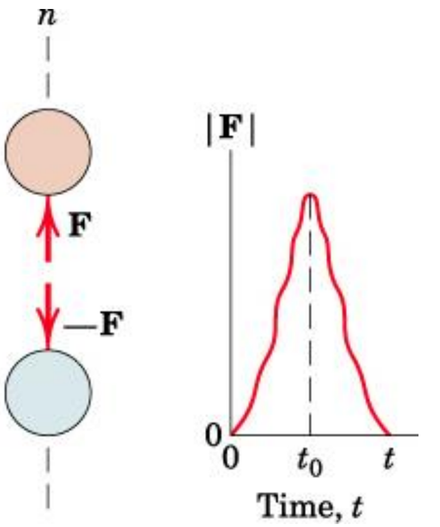
- In-plane initial and final velocities are not parallel
- Directions of velocity vectors measured from *t*-axis.



Initial velocity components are:
$(v_1)_n = -v_1 sin\theta_1$, $(v_1)_t = v_1 cos\theta_1$
$(v_2)_n = v_2 sin\theta_2$, $(v_2)_t = v_2 cos\theta_2$

→ Four unknowns $(v'_1)_n$, $(v'_1)_t$, $(v'_2)_n$, $(v'_2)_t$
→ Four equations are required.

:: The only impulses are due to the internal forces directed along the line of impact (*n*-axis)

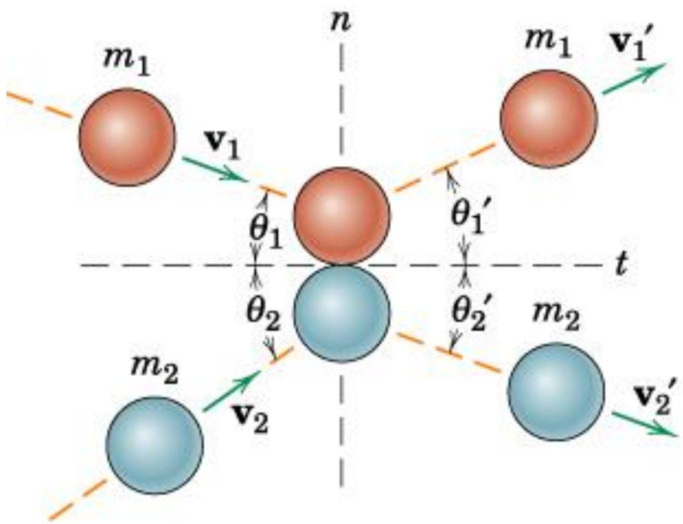


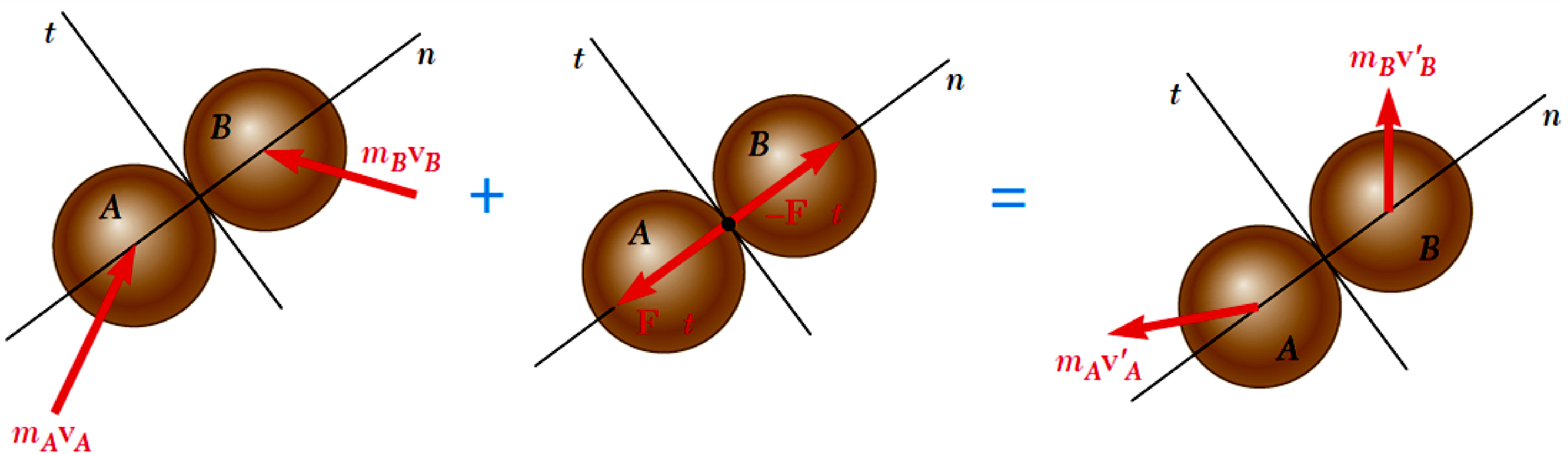
No impulsive external forces
:: **total momentum** of the system (*both particles*) is **conserved** along the ***n*-axis**

Kinetics of a Particle: Oblique Central Impact

- Example :: Billiards pool

Kinetics of a Particle: Oblique Central Impact

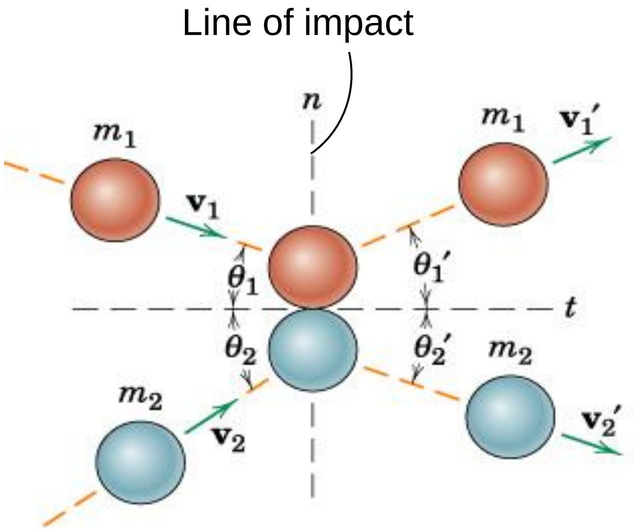




Kinetics of a Particle: Oblique Central Impact

Conservation of total momentum along the *n*-axis

$$m_1(v_1)_n + m_2(v_2)_n = m_1(v_1')_n + m_2(v_2')_n \quad (1)$$



•Conservation of momentum component along *t*-axis for each particle (*no impulse on particles along t-direction*)

$$m_1(v_1)_t = m_1(v_1')_t$$
$$m_2(v_2)_t = m_2(v_2')_t \quad \text{(2 and 3)}$$

→ *t*-component of velocity of each particle remains unchanged.

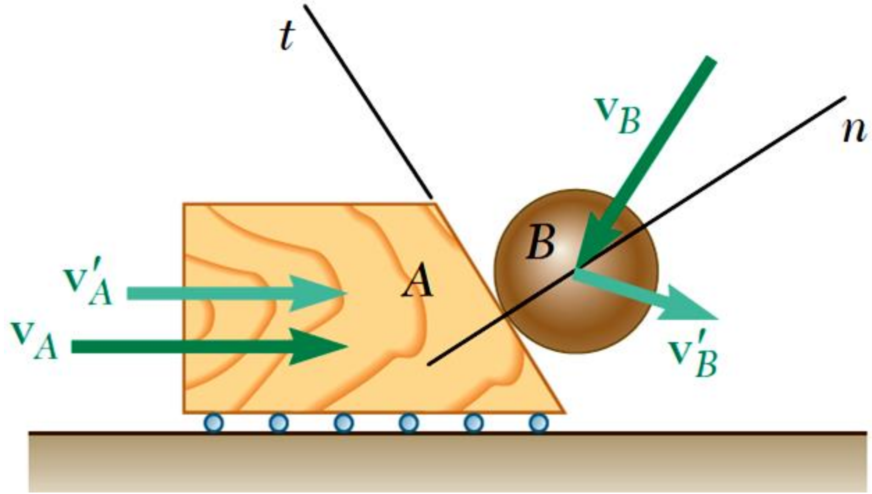
•Coefficient of Restitution
= **positive ratio** of the **recovery impulse** to the **deformation impulse**

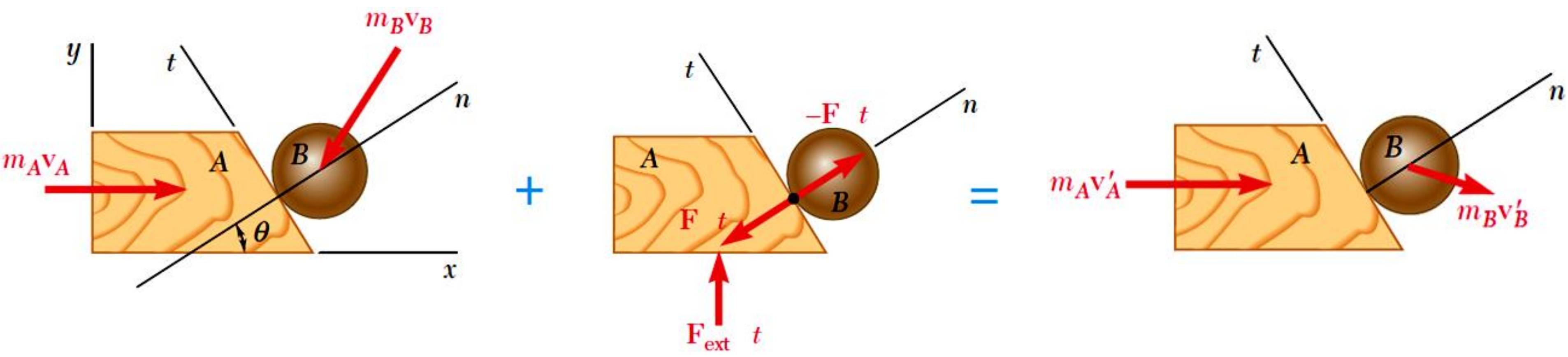
$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \quad (4)$$

:: Final velocities are determined → angles θ_1' and θ_2' can be determined

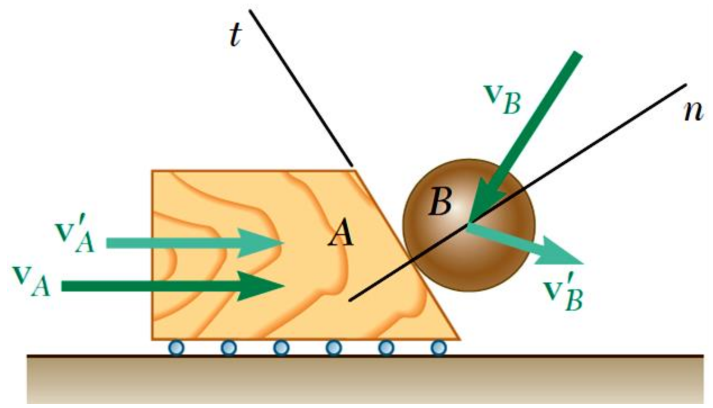
Kinetics of a Particle: Oblique Central Impact

- Constrained motion





Kinetics of a Particle: Oblique Central Impact



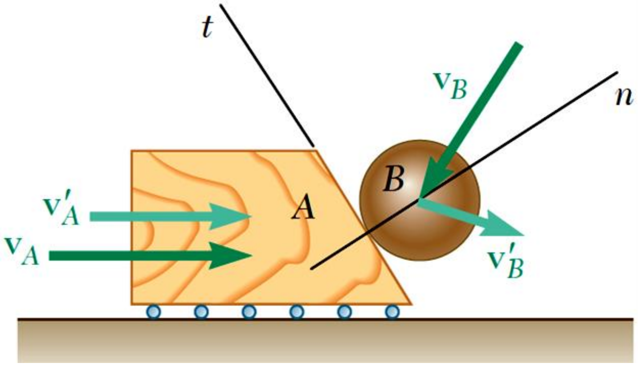
$$(v_B)_t = (v'_B)_t \tag{1}$$

$$m_A v_A + m_B (v_B)_x = m_A v'_A + m_B (v'_B)_x \tag{2}$$

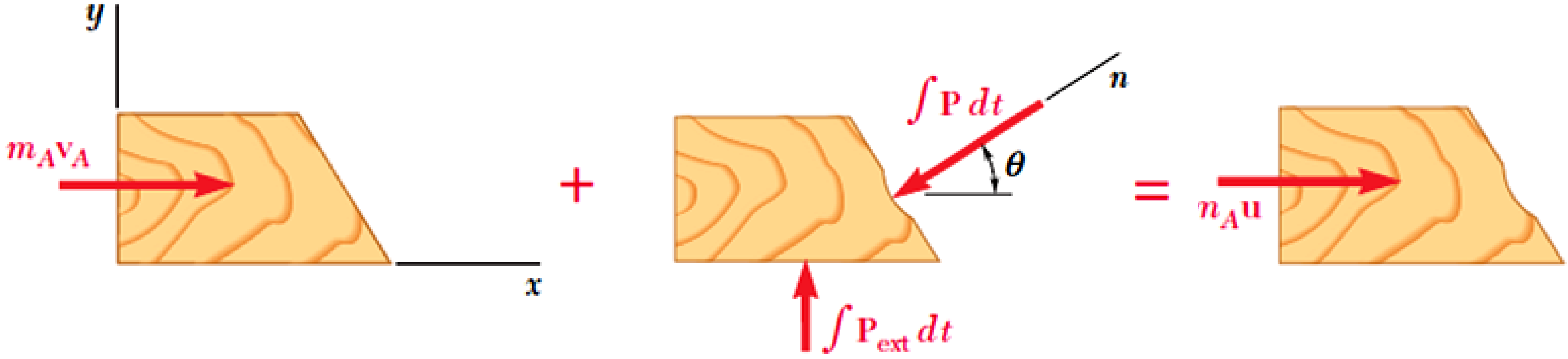
$$(v'_B)_n - (v'_A)_n = e[(v_A)_n - (v_B)_n] \tag{3}$$

Is Eq.(3) valid under external impulse??

Kinetics of a Particle: Oblique Central Impact



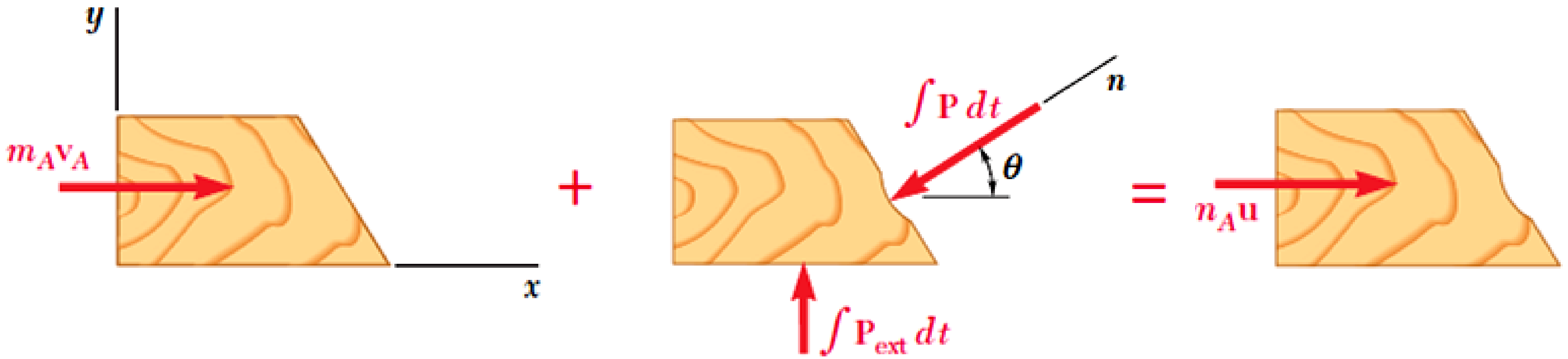
Apply impulse-momentum equation for Block A over the deformation period



$$m_A v_A - \left(\int P\,dt\right)\cos\theta = m_A u$$

$$m_A u - \left(\int R\,dt\right)\cos\theta = m_A v'_A$$

Kinetics of a Particle: Oblique Central Impact



$$m_A v_A - \left(\int P\,dt\right)\cos\theta = m_A u$$

$$m_A u - \left(\int R\,dt\right)\cos\theta = m_A v_A'$$

$$e = \frac{\int R\,dt}{\int P\,dt}$$

$$e = \frac{u - v_A'}{v_A - u}$$

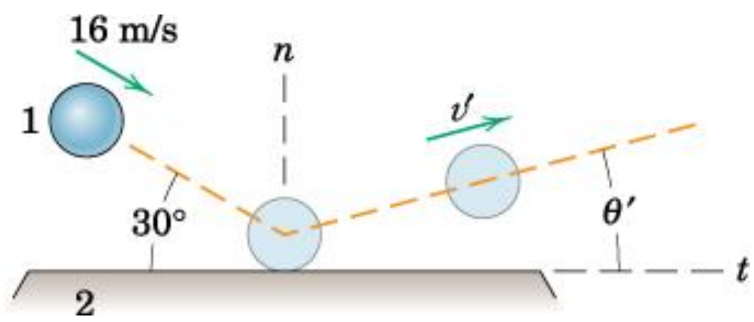
$$e = \frac{u_n - (v_A')_n}{(v_A)_n - u_n}$$

Kinetics of a Particle: Impact

Example:

A ball is projected onto a very heavy plate. If the effective coefficient of restitution is 0.5, compute the rebound velocity of the ball and its angle of rebound.

→ Oblique central impact



Solution: Since the plate is very heavy compared to that of the ball, its velocity after impact may be considered to be zero.

Applying the coefficient of restitution to the velocity comp along the *n*- direction:

$$e = \frac{(v_2')_n - (v_1')_n}{(v_1)_n - (v_2)_n} \qquad 0.5 = \frac{0 - (v')_n}{-16\sin 30 - 0}$$

→ $(v')_n = 4$ m/s

Neglecting the impulse of both the weights.

Momentum of the ball along the *t*-direction is conserved (assuming smooth surfaces, no force acts on the ball in that direction).

$m(v_1)_t = m(v'_1)_t$ → $(v_1)_t = (v'_1)_t = 13.86$ m/s

Therefore, the rebound velocity: $v' = \sqrt{(v_1')_n^{\,2} + (v_1')_t^{\,2}}$ → $v' = 14.42$ m/s

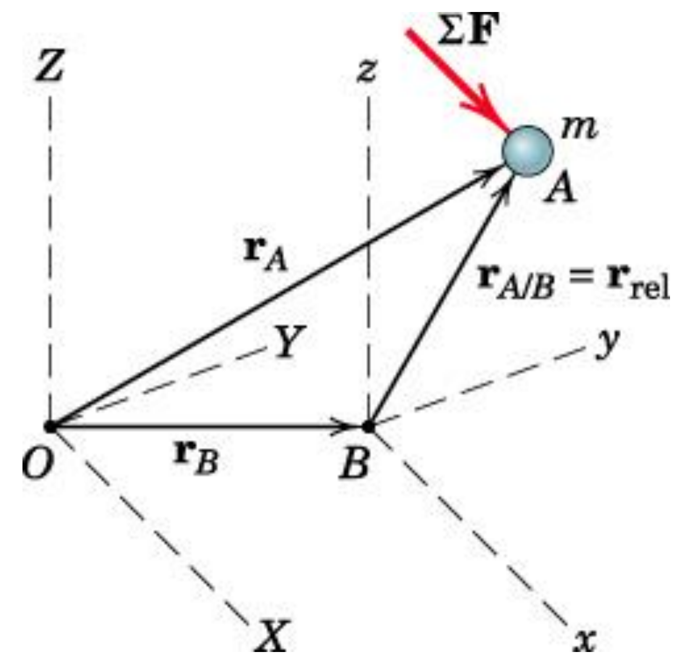
Rebound angle: $\theta' = \tan^{-1}\left(\frac{(v_1')_n}{(v_1')_t}\right)$ → $\theta' = 16.10^{o}$

Kinetics of a Particle

Relative Motion

Observing motion of A from axes *x-y-z* which translate wrt the fixed reference frame *X-Y-Z* (motion relative to rotating axes → later)

→*x-y-z* directions will always remain parallel to *X-Y-Z* directions



Acceleration of origin *B* of *x-y-z* is $\mathbf{a}_B$
Acceleration of *A* as observed from *x-y-z* (or relative to *x-y-z*) is $\mathbf{a}_{rel}$ → $\mathbf{a}_{\text{rel}} = \mathbf{a}_{A/B} = \ddot{\mathbf{r}}_{A/B}$

Absolute acceleration of *A*: $\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{rel}$
Newton's second law of motion: $\sum \mathbf{F} = m\mathbf{a}_A$
→ $\sum \mathbf{F} = m(\mathbf{a}_B + \mathbf{a}_{rel})$
→ Observation from both *x-y-z* and *X-Y-Z* will be same.
→ Newton's second law does not hold wrt an accelerating system (*x-y-z*) since $\sum \mathbf{F} \neq m(\mathbf{a}_{rel})$

Kinetics of a Particle: Relative Motion

D'Alembert's Principle

•Accln of a particle measured from fixed set of axes *X-Y-Z* is its **absolute acceleration** (**a**).

→ Newton's second law of motion can be applied ($\sum \mathbf{F} = m\mathbf{a}$)

•If the particle is observed from a moving system (*x-y-z*) attached to a particle, the **particle appears to be at rest or in equilibrium** in *x-y-z*.

→ Therefore, the observer who is accelerating with *x-y-z* would conclude that a force $-m\mathbf{a}$ acts on the particle to balance $\sum \mathbf{F}$.

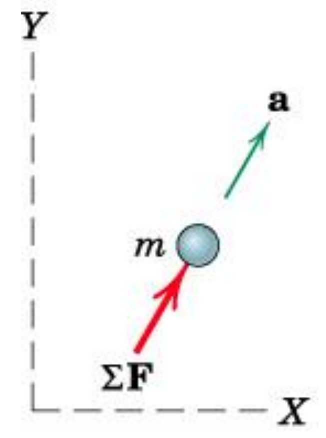
→ Treatment of dynamics problem by the method of statics → work of D'Alembert (1743)

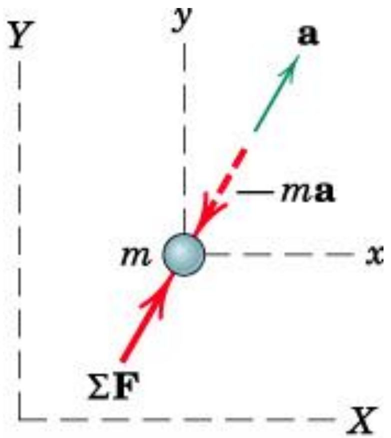
→ As per this approach, Equation of Motion is rewritten as: $\sum \mathbf{F} - m\mathbf{a} = 0$ → $-m\mathbf{a}$ is also treated as a force

→ This fictitious force is known as Inertia Force

→ The artificial state of equilibrium created is known as Dynamic Equilibrium.

→ Transformation of a problem in dynamic to one in statics is known as D'Alembert's Principle.

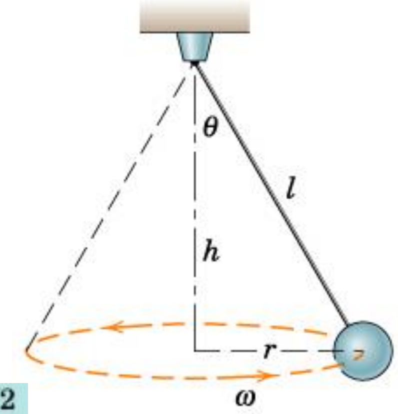




Kinetics of a Particle: Relative Motion

Application of D'Alembert's Principle

- The approach does not provide any specific advantage. It is just an alternative method.



Consider a pendulum of mass m swinging in a horizontal circle with its radial line *r* having an angular velocity *w*.

Usual Method: Applying eqn of motion $\sum \mathbf{F} = m\mathbf{a}_n$ in the direction *n* of the accln:

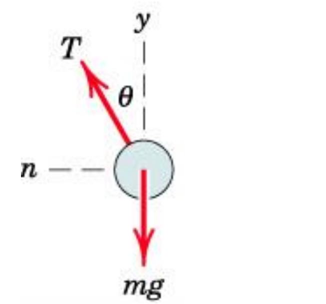
$$\rho_{xy} = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Normal accln is given by: $a_n = \rho\dot{\beta}^2 = rw^2$

From FBD: Along *n*-direction: $T\sin\theta = mrw^2$

Equilibrium along *y*-direction: $T\cos\theta - mg = 0$

→ Unknowns T and θ can be found out from these two eqns.



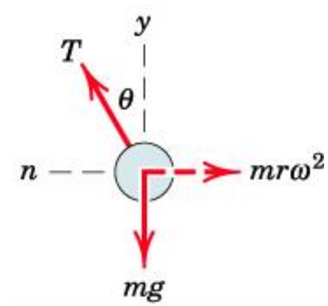
Using D'Alembert's Principle:

If the reference axes are attached to the particle, it will appear to be in equilibrium relative to these axes.

→ An imaginary inertia force $-m\mathbf{a}$ must be added to the mass.

→ Application of mrw^2 in the dirn opposite to the accln (FBD)

Force summation along *n*-dirn: $T\sin\theta - mrw^2 = 0$ → same result



For circular motion of particle, this hypothetical inertia force is known as the Centrifugal Force since it is directed away from the center & its dirn is opposite to the dirn of the accln.

Kinetics of a Particle: Relative Motion

Constant Velocity, Nonrotating Systems

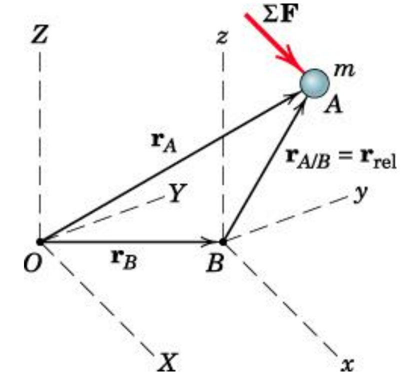
- A special case where the moving reference system *x-y-z* has a constant velocity and no rotation → $\mathbf{a}_B = 0$

→ Absolute acceleration of *A*: $\mathbf{a}_A = \mathbf{a}_{rel}$

Newton's second law of motion: $\sum \mathbf{F} = m(\mathbf{a}_{rel})$

→Newton's second law of motion holds for measurements made in a system moving with a constant velocity.

→Such system is known as: Inertial system or Newtonian Frame of Reference



Validity of work-energy eqn relative to a constant-velocity, non-rotating system

x-y-z move with a constant velocity $\mathbf{v}_B$ relative to the fixed axes. $\mathbf{v}_B = \dot{\mathbf{r}}_B$

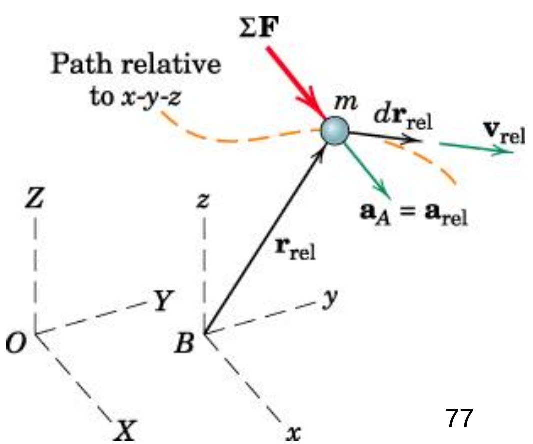
Work done by $\sum \mathbf{F}$ relative to *x-y-z* is $dU_{\text{rel}} = \Sigma\mathbf{F}\cdot d\mathbf{r}_{\text{rel}}$

$\Sigma\mathbf{F} = m\mathbf{a}_A = m\mathbf{a}_{\text{rel}}$ and $\mathbf{a}_{\text{rel}}\cdot d\mathbf{r}_{\text{rel}} = \mathbf{v}_{\text{rel}}\cdot d\mathbf{v}_{\text{rel}}$

(since $a_t ds = v dv$ for curvilinear motion)

$$dU_{\text{rel}} = m\mathbf{a}_{\text{rel}}\cdot d\mathbf{r}_{\text{rel}} = mv_{\text{rel}}\,dv_{\text{rel}} = d(\tfrac{1}{2}mv_{\text{rel}}{}^2)$$

Defining the KE relative to *x-y-z* as: $T_{\text{rel}} = \frac{1}{2}mv_{\text{rel}}{}^2$



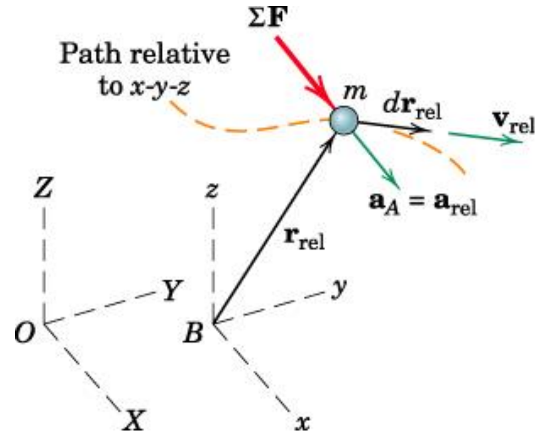
Kinetics of a Particle: Relative Motion

Constant Velocity, Nonrotating Systems

$$dU_{rel} = m\mathbf{a}_{rel} \cdot d\mathbf{r}_{rel} = mv_{rel}\, dv_{rel} = d(\tfrac{1}{2} m v_{rel}{}^2)$$

→ $\boxed{dU_{rel} = dT_{rel}}$ Or $\boxed{U_{rel} = \Delta T_{rel}}$

→ Work-energy eqn holds for measurements made relative to a constant velocity, non-rotating system.



Validity of impulse-momentum eqn rel to a constant-velocity, non-rotating system

Relative to *x-y-z* the impulse on the particle during time *dt* is:

$\Sigma\mathbf{F}\, dt = m\mathbf{a}_A\, dt = m\mathbf{a}_{rel}\, dt$ But $m\mathbf{a}_{rel}\, dt = m\, d\mathbf{v}_{rel} = d(m\mathbf{v}_{rel})$

→ $\Sigma\mathbf{F}\, dt = d(m\mathbf{v}_{rel})$

Defining the linear momentum of the particle relative to *x-y-z* as: $\mathbf{G}_{rel} = m\mathbf{v}_{rel}$

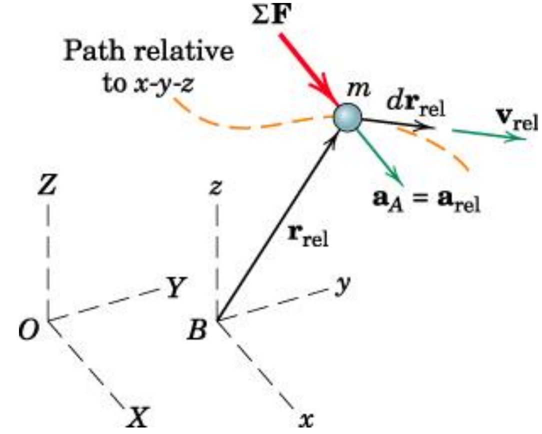
→ $\Sigma\mathbf{F}\, dt = d\mathbf{G}_{rel}$ Dividing by dt and integrating:

$\boxed{\Sigma\mathbf{F} = \dot{\mathbf{G}}_{rel}}$ and $\boxed{\int \Sigma\mathbf{F}\, dt = \Delta\mathbf{G}_{rel}}$

→ Impulse-momentum eqns for a fixed reference system also hold for measurements made relative to a constant velocity, non-rotating system.

Kinetics of a Particle: Relative Motion

Constant Velocity, Nonrotating Systems



Validity of moment-angular momentum eqn rel to a constant-velocity, non-rotating system

Defining the relative angular momentum of the particle @ point *B* in *x-y-z* as the moment of the relative linear Momentum → $(\mathbf{H}_B)_{rel} = \mathbf{r}_{rel} \times \mathbf{G}_{rel}$

Time derivative → $(\dot{\mathbf{H}}_B)_{rel} = \dot{\mathbf{r}}_{rel} \times \mathbf{G}_{rel} + \mathbf{r}_{rel} \times \dot{\mathbf{G}}_{rel}$

First term is zero since it is equal to $\mathbf{v}_{rel} \times m\mathbf{v}_{rel}$

And the second term becomes $\mathbf{r}_{rel} \times \sum\mathbf{F} = \sum\mathbf{M}_B$ = sum of moments @ *B* of all forces on *m*

→ $\boxed{\Sigma\mathbf{M}_B = (\dot{\mathbf{H}}_B)_{rel}}$ → Moment-angular momentum eqns hold for measurements made relative to a constant velocity, non-rotating system.

However, the individual eqns for work, KE, and momentum differ betn the fixed and moving systems.

$$(dU = \Sigma\mathbf{F}\cdot d\mathbf{r}_A) \neq (dU_{rel} = \Sigma\mathbf{F}\cdot d\mathbf{r}_{rel})$$

$$(T = \tfrac{1}{2}mv_A{}^2) \neq (T_{rel} = \tfrac{1}{2}mv_{rel}{}^2)$$

$$(\mathbf{G} = m\mathbf{v}_A) \neq (\mathbf{G}_{rel} = m\mathbf{v}_{rel})$$

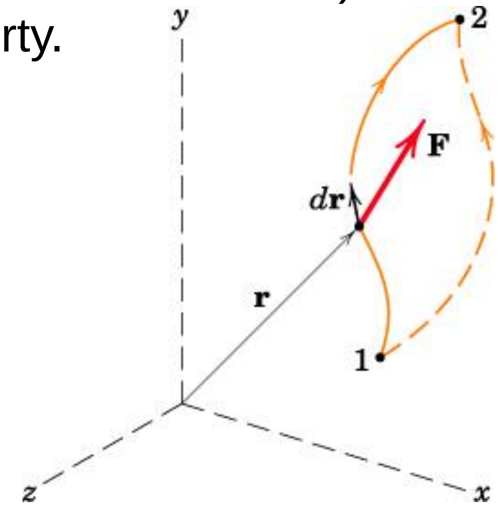
Kinetics of a Particle: Potential Energy

Conservative Force Fields

- Work done against a gravitational or elastic force depends only on net change of position and not on the particular path followed in reaching the new position.
- Forces with this characteristic are associated with **Conservative Force Fields**

  → A **conservative force** is a force with the property that the work done by it in moving a particle between two points is independent of the path taken (Ex: Gravity is a conservative force, friction is a non-conservative force)

  → these forces possess an important mathematical property.



Consider a force field where the force **F** is a function of the coordinates. Work done by **F** during displacement $d\mathbf{r}$ of its point of application: $dU = \mathbf{F}\cdot d\mathbf{r}$

→ Total work done along its path from 1 to 2:

$$U = \int \mathbf{F}\cdot d\mathbf{r} = \int (F_x\,dx + F_y\,dy + F_z\,dz)$$

- **In general**, the integral $\int \mathbf{F}\cdot d\mathbf{r}$ is a line integral that depends on the path followed between 1 and 2.
- For some forces, $\mathbf{F}\cdot d\mathbf{r}$ is an exact differential $-dV$ of some scalar function V of the coordinates

(minus sign for dV is arbitrary but agree with the sign of PE change in the gravity field of the earth)

A function $d\Phi=Pdx+Qdy+Rdz$ is an exact differential in the coordinates x-y-z if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \qquad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x} \qquad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

Kinetics of Particles: Relative Motion

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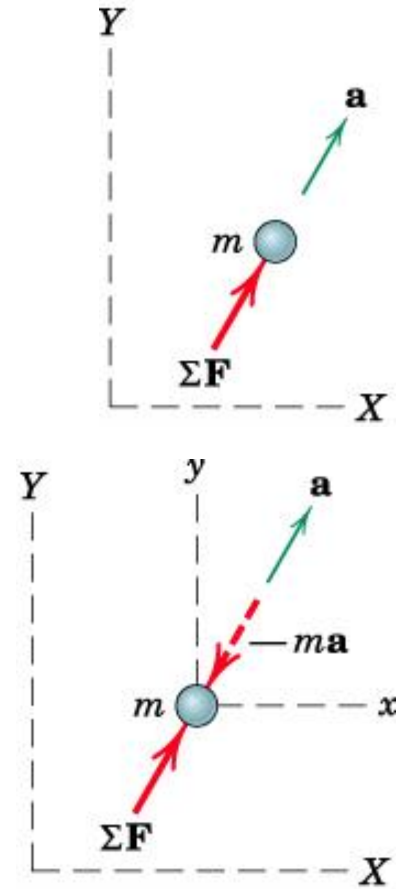

Kinetics of a Particle: Potential Energy

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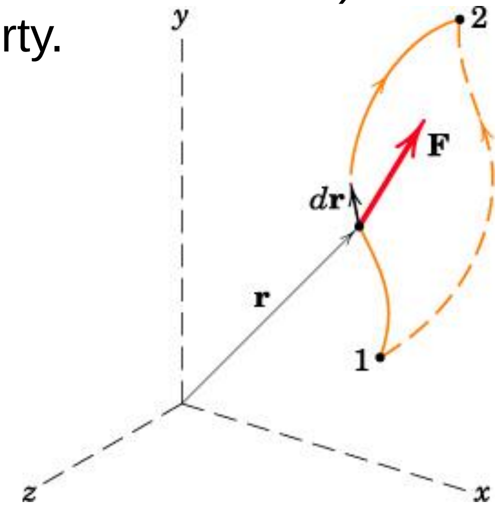

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Kinetics of a Particle: Potential Energy

Conservative Force Fields

If **F**·*d***r** is an exact differential –*dV* of some scalar function *V* of the coordinates:

$$U_{1\text{-}2} = \int_{V_1}^{V_2} -dV = -(V_2 - V_1)$$

→ The work done depends on only the end points of the motion (i.e., independent of the path followed!)

If *V* exists, differential change in *V* becomes: $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

Using $-dV = \mathbf{F}\cdot d\mathbf{r} = F_x\,dx + F_y\,dy + F_z\,dz$

→ $F_x = -\frac{\partial V}{\partial x} \qquad F_y = -\frac{\partial V}{\partial y} \qquad F_z = -\frac{\partial V}{\partial z}$

For any closed path (points 1 and 2 coincide), work done by the conservative force field **F** is zero: $\oint \mathbf{F}\cdot d\mathbf{r} = 0$ (circle on the integration sign indicates that the path is closed)

The force may also be written as the vector: $\mathbf{F} = -\nabla V$ or $\mathbf{F} = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k}$

The vector operator "del": $\boldsymbol{\nabla} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$

V is known as the Potential Function and the expression $\boldsymbol{\nabla} V$ is known as the gradient of the potential function *V* (scalar)

→ $\nabla\times\mathbf{F} = -\nabla\times\Delta V = \mathbf{0}$

Since curl of the gradient of any scalar function is a zero vector.

Kinetics of a Particle: Potential Energy

Conservative Force Fields

A force field F is said to be conservative if it meets any of the following three equivalent conditions:

1. $\mathbf{F} = -\nabla V$

→ If the force can be written as the negative gradient of a potential. When force components are derivable from a potential, the force is said to be conservative, and the work done by the force between any two points is independent of the path followed.

2. $\oint \mathbf{F} \cdot d\mathbf{r} = 0$

→ For any closed path, net work done by the conservative force field is zero.

3. $\nabla \times \mathbf{F} = \mathbf{0}$

→ If the curl of the force is zero.

Kinetics of Systems of Particles

Generalized Newton's Second Law

•*n* mass particles bounded by a closed surface in space

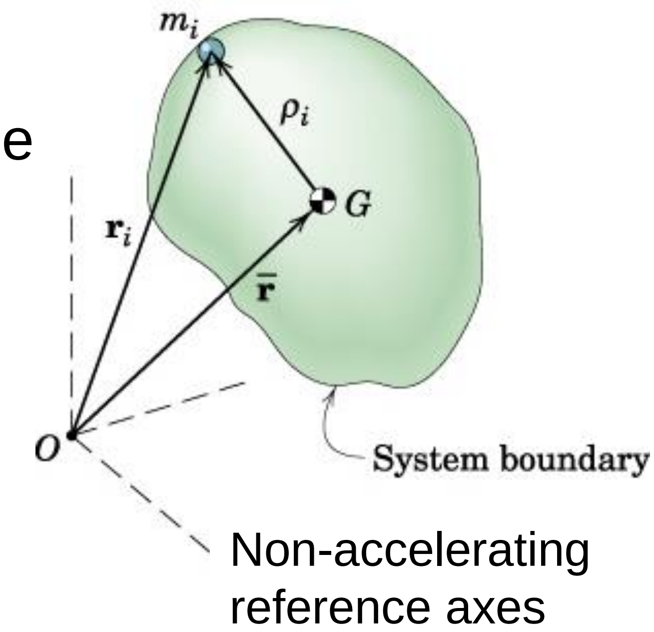
• $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$, … acting on m_i from sources external to the envelope

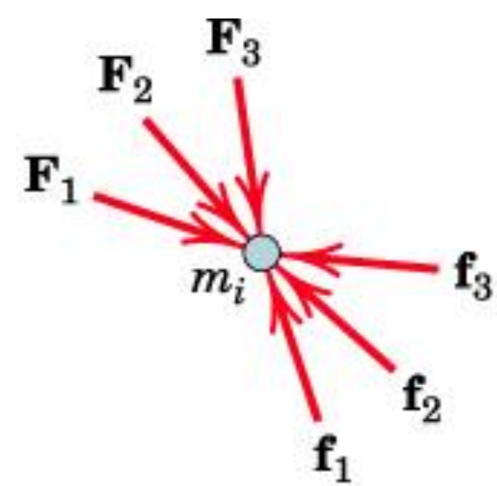
:: e.g., *contact with external bodies, gravitational*, *electric*, *magnetic* etc.)

• $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$, … acting on m_i from sources internal to the system boundary

:: *reaction forces from other mass particles within the boundary*

•Mass centre *G* can be located by $m\bar{\mathbf{r}} = \Sigma m_i \mathbf{r}_i$





Kinetics of Systems of Particles

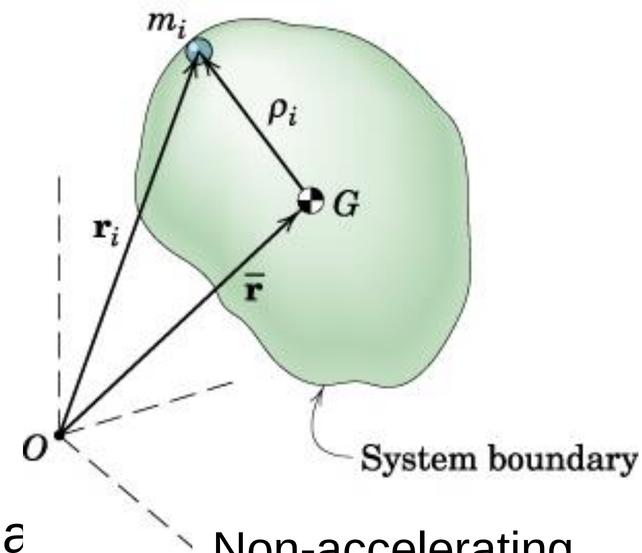
Generalized Newton's Second Law

Applying Newton's Second law to m_i:

$$\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{f}_1 + \mathbf{f}_2 + \mathbf{f}_3 + \cdots = m_i\ddot{\mathbf{r}}_i$$

For all particles of the system:

$$\Sigma\mathbf{F} + \Sigma\mathbf{f} = \Sigma m_i\ddot{\mathbf{r}}_i$$



Non-accelerating reference axes

$\sum\mathbf{f} = 0$ since all internal forces occur in pairs of equal and opposite actions and reactions.

$$m\bar{\mathbf{r}} = \Sigma m_i\mathbf{r}_i \quad \rightarrow \quad m\ddot{\bar{\mathbf{r}}} = \Sigma m_i\ddot{\mathbf{r}}_i$$

(assuming constant m)

$$\boxed{\Sigma\mathbf{F} = m\ddot{\bar{\mathbf{r}}} \quad \text{or} \quad \Sigma\mathbf{F} = m\bar{\mathbf{a}}}$$

$\bar{\mathbf{a}}$ is the accln of CM (has same direction as $\sum\mathbf{F}$)

$\sum\mathbf{F}$ does not necessarily pass through G

$$\Sigma F_x = m\bar{a}_x \qquad \Sigma F_y = m\bar{a}_y \qquad \Sigma F_z = m\bar{a}_z$$

→ Generalized Law in component form

Kinetics of Systems of Particles

Work-Energy

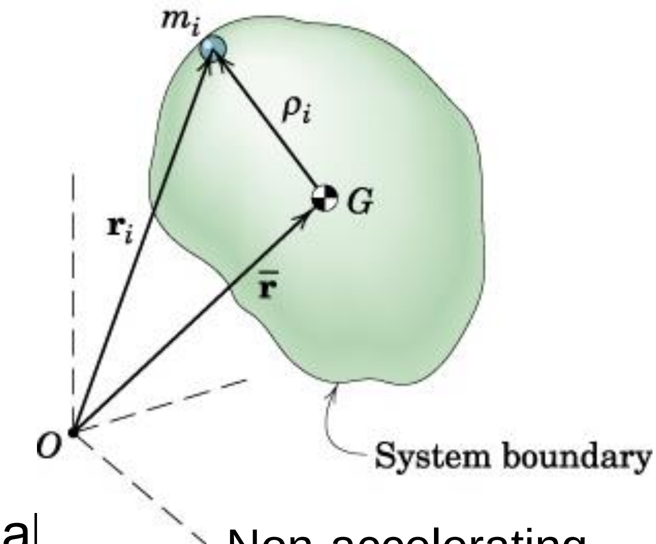
•Work-energy relation for a particle:

$$U_{1-2} = T_2 - T_1 = \Delta T$$

•Work-energy relation for m_i :

$$(U_{1-2})_i = \Delta T_i$$

$(U_{1-2})_i$ is work done by m_i during an interval of motion by all forces (external $\mathbf{F}_1$+$\mathbf{F}_2$+… and internal $\mathbf{f}_1$+$\mathbf{f}_2$+…)



Non-accelerating reference axes

•KE of m_i : T_i = ½ $m_i v_i^{\,2}$ $\mathbf{v}_i = \dot{\mathbf{r}}_i$

v_i is the magnitude of the particle velocity

For the entire sysem: $\sum(U_{1\text{-}2})_i = \sum \Delta T_i$

→ Same work-energy relation:

$$U_{1-2} = \Delta T \quad or \quad T_1 + U_{1-2} = T_2$$

Kinetics of Systems of Particles

- ## Work-Energy

$U_{1\text{-}2} = \sum(U_{1\text{-}2})_i$ is total work done by all forces on all particles.

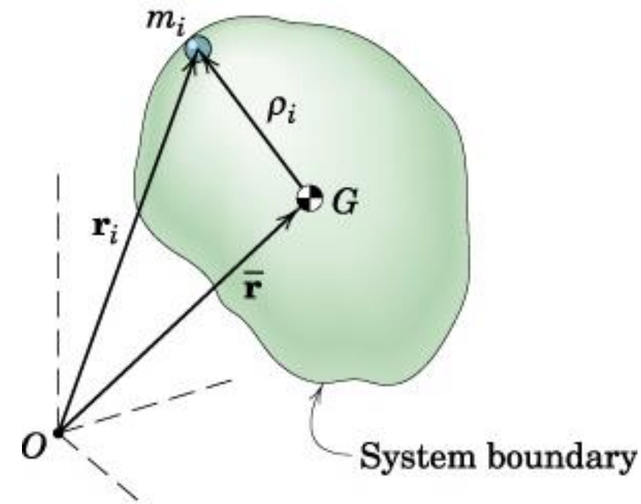
:: ΔT is the change in the total KE, $T = \sum T_i$, of the system

:: In rigid bodies, work done by all pairs of internal forces is zero
→ $U_{1\text{-}2}$ is the total work done by only the external forces on the system.

• In non-rigid systems, conversion into change in the internal elastic PE V_e.

$$U'_{1-2} = \Delta T + \Delta V \qquad T_1 + V_1 + U'_{1-2} = T_2 + V_2$$

$V = V_e + V_g$ = Total PE



Non-accelerating reference axes

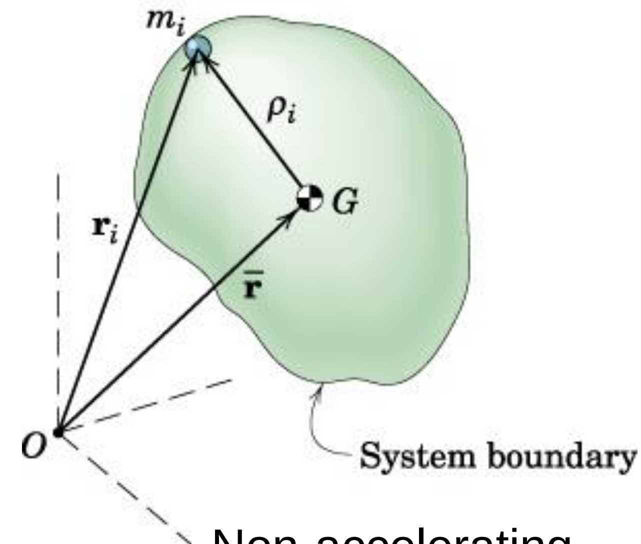
Kinetics of Systems of Particles

Work-Energy

KE of the mass system : $T = \sum \frac{1}{2} m_i v_i^{\,2}$

Relative Motion: $\mathbf{v}_i = \bar{\mathbf{v}} + \dot{\boldsymbol{\rho}}_i$

$$v_i^{\,2} = \mathbf{v}_i \cdot \mathbf{v}_i$$



Non-accelerating reference axes

KE of the system:

$$T = \Sigma \frac{1}{2} m_i \mathbf{v}_i \cdot \mathbf{v}_i = \Sigma \frac{1}{2} m_i (\bar{\mathbf{v}} + \dot{\boldsymbol{\rho}}_i) \cdot (\bar{\mathbf{v}} + \dot{\boldsymbol{\rho}}_i)$$

$$= \Sigma \frac{1}{2} m_i \bar{v}^2 + \Sigma \frac{1}{2} m_i |\dot{\boldsymbol{\rho}}_i|^2 + \Sigma m_i \bar{\mathbf{v}} \cdot \dot{\boldsymbol{\rho}}_i$$

Since ρ_i is measured from the mass center $\Sigma m_i \boldsymbol{\rho}_i = \mathbf{0}$

$\therefore$ The third term: $\bar{\mathbf{v}} \cdot \Sigma m_i \dot{\boldsymbol{\rho}}_i = \bar{\mathbf{v}} \cdot \dfrac{d}{dt} \Sigma (m_i \boldsymbol{\rho}_i) = 0$

and $\Sigma \frac{1}{2} m_i \bar{v}^2 = \frac{1}{2} \bar{v}^2 \Sigma m_i = \frac{1}{2} m \bar{v}^2$

Therefore, the total KE:

$$\boxed{T = \frac{1}{2} m \bar{v}^2 + \Sigma \frac{1}{2} m_i |\dot{\boldsymbol{\rho}}_i|^2}$$

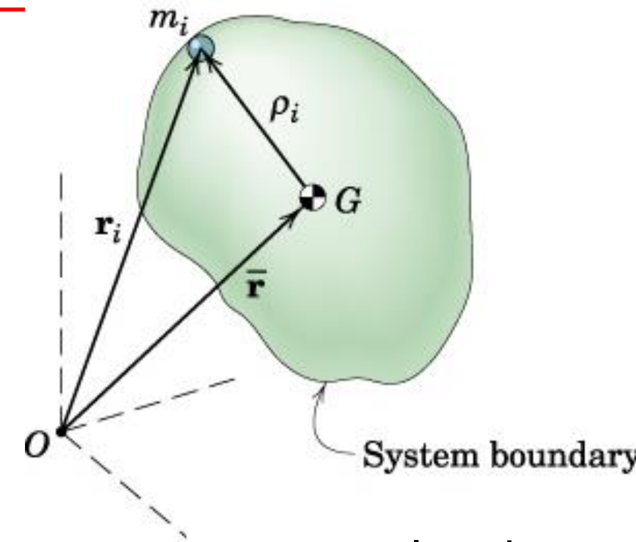
Total KE of a mass system = **KE of the mass center translation of the system** as a whole + **KE** due to **motion of all particles relative to the mass center**

Kinetics of Systems of Particles

Impulse-Momentum

Linear Momentum (for single particle: *G* = *mv*)

$\mathbf{G}_i = m_i\mathbf{v}_i \quad \mathbf{v}_i = \dot{\mathbf{r}}_i$



Linear Momentum of the system is defined as the vector sum of the LM of all its particles $\mathbf{G} = \Sigma m_i\mathbf{v}_i$

Now $\mathbf{v}_i = \bar{\mathbf{v}} + \dot{\boldsymbol{\rho}}_i \quad \Sigma m_i\boldsymbol{\rho}_i = \mathbf{0}$

Non-accelerating reference axes

→ $\mathbf{G} = \Sigma m_i(\bar{\mathbf{v}} + \dot{\boldsymbol{\rho}}_i) = \Sigma m_i\bar{\mathbf{v}} + \frac{d}{dt}\Sigma m_i\boldsymbol{\rho}_i = \bar{\mathbf{v}}\,\Sigma m_i + \frac{d}{dt}(\mathbf{0})$

→ $\boxed{\mathbf{G} = m\bar{\mathbf{v}}}$ **LM** of any **system of constant mass** is the **product** of the **mass** and the **velocity** of **its center of mass**

Time derivative of G: $m\dot{\bar{\mathbf{v}}} = m\bar{\mathbf{a}}$ = ∑**F**

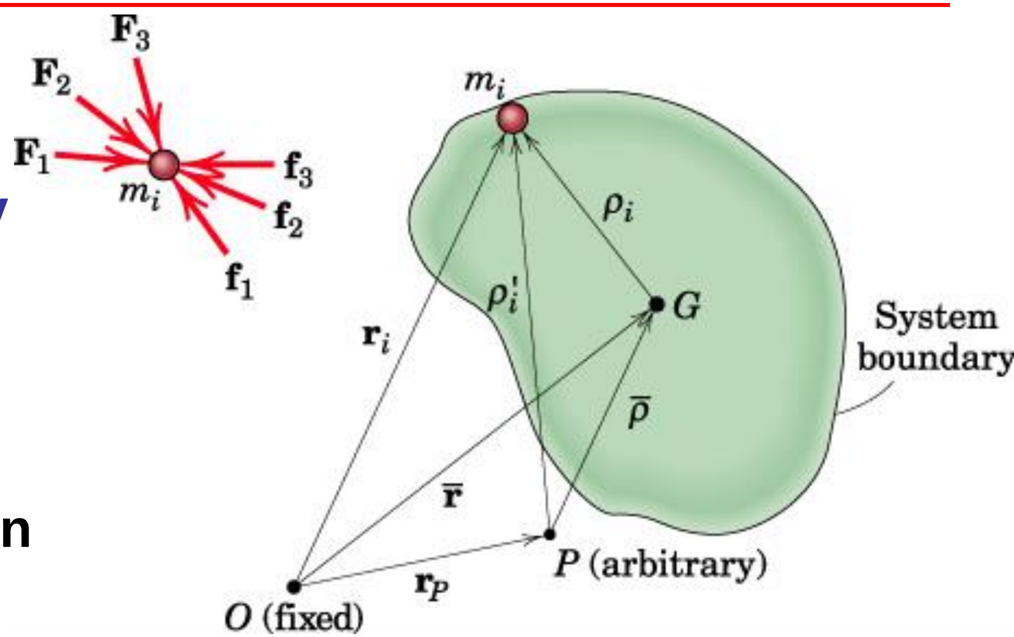
→ $\boxed{\Sigma\mathbf{F} = \dot{\mathbf{G}}}$ → Has the same form as that for a single particle.
Resultant of the **external forces** on **any mass system** = **time rate of change** of **LM of the system**

Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum

AM of a single particle: $\mathbf{H}_O = \mathbf{r} \times m\mathbf{v}$



AM of general mass system @

- **Fixed Point *O***
- **Mass Center *G***
- **An arbitrary point *P* with accln**

$$\mathbf{a}_P = \ddot{\mathbf{r}}_P$$

Angular Momentum @ a fixed point *O*

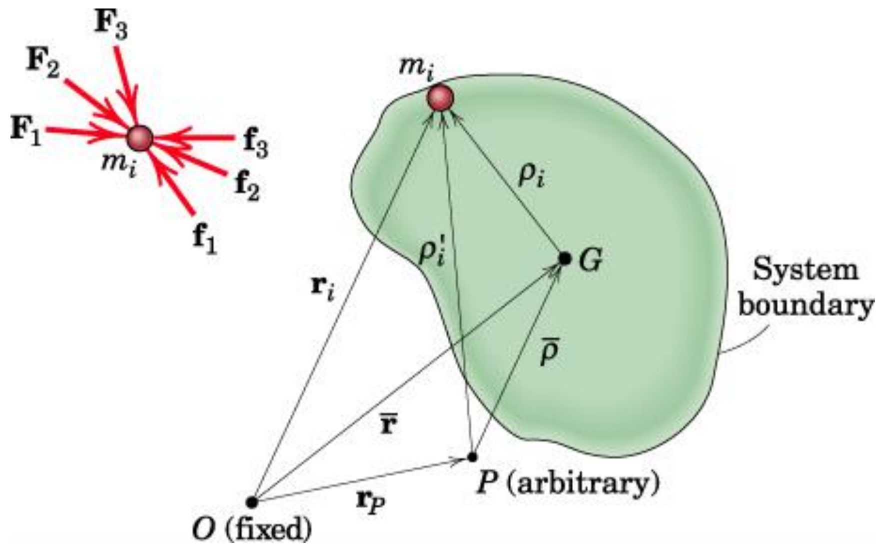
AM of the mass system @ the point *O* fixed in the Newtonian Reference System = the **vector sum of the moments** of the **LM @ *O* of all particles of the system**

$$\mathbf{H}_O = \Sigma(\mathbf{r}_i \times m_i \mathbf{v}_i)$$

Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum



The time derivative of the vector product:

$$\dot{\mathbf{H}}_O = \Sigma(\dot{\mathbf{r}}_i \times m_i\mathbf{v}_i) + \Sigma(\mathbf{r}_i \times m_i\dot{\mathbf{v}}_i)$$

First term represents cross product of two parallel vectors → **zero**

The second term: $\Sigma(\mathbf{r}_i \times m_i\mathbf{a}_i) = \Sigma(\mathbf{r}_i \times \mathbf{F}_i)$

→= $\sum \mathbf{M}_O$ that represents only moments of the external forces

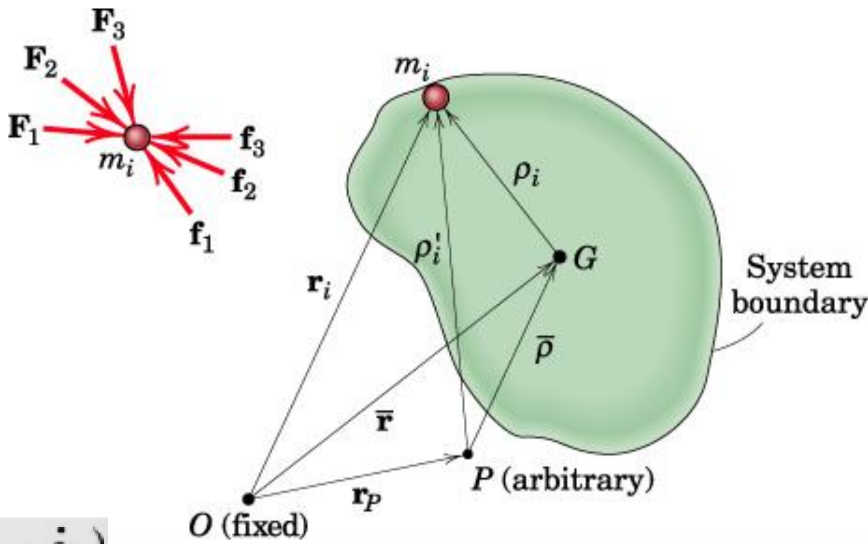
→ $\boxed{\Sigma\mathbf{M}_O = \dot{\mathbf{H}}_O}$

→ Similar to the eqn for a single particle. **Resultant vector moment** of all **ext forces @ any fixed point** = **time rate of change of AM of the system @ the fixed point**. (Eqn can also be applied to non-rigid systems)

Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum



Angular Momentum @ mass Center *G*

AM of the mass system @ *G* =
sum of the moments of the LM @ *G*
of all particles of the system

→ $\mathbf{H}_G = \Sigma \boldsymbol{\rho}_i \times m_i \dot{\mathbf{r}}_i$

We may write the absolute velocity $\dot{\mathbf{r}}_i$ as $(\dot{\bar{\mathbf{r}}} + \dot{\boldsymbol{\rho}}_i)$

→ $\mathbf{H}_G = \Sigma \boldsymbol{\rho}_i \times m_i(\dot{\bar{\mathbf{r}}} + \dot{\boldsymbol{\rho}}_i) = \Sigma \boldsymbol{\rho}_i \times m_i \dot{\bar{\mathbf{r}}} + \Sigma \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$

→ $\mathbf{H}_G = \Sigma \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$ Relative AM because relative velocity is used

If *G* is taken as the reference, the absolute and relative AM will be identical

Differentiating first eqn wrt time: $\dot{\mathbf{H}}_G = \Sigma \dot{\boldsymbol{\rho}}_i \times m_i(\dot{\bar{\mathbf{r}}} + \dot{\boldsymbol{\rho}}_i) + \Sigma \boldsymbol{\rho}_i \times m_i \ddot{\mathbf{r}}_i$

First term: $\Sigma \dot{\boldsymbol{\rho}}_i \times m_i \dot{\bar{\mathbf{r}}} + \Sigma \dot{\boldsymbol{\rho}}_i \times m_i \dot{\boldsymbol{\rho}}_i$

This is zero because both the terms are zero since the first term may be written as:

$$-\dot{\bar{\mathbf{r}}} \times \Sigma m_i \dot{\boldsymbol{\rho}}_i = -\dot{\bar{\mathbf{r}}} \times \frac{d}{dt} \Sigma m_i \boldsymbol{\rho}_i$$

Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum

Angular Momentum @ mass Center *G*

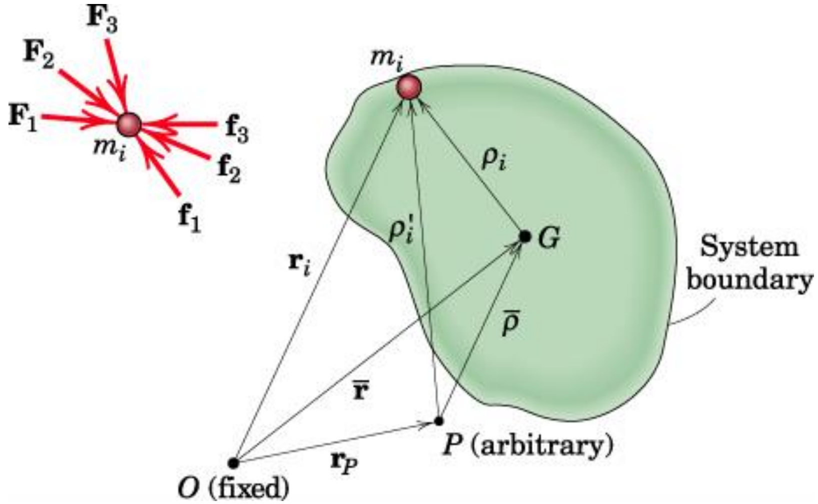
$$\dot{\mathbf{H}}_G = \Sigma \dot{\boldsymbol{\rho}}_i \times m_i(\dot{\bar{\mathbf{r}}} + \dot{\boldsymbol{\rho}}_i) + \Sigma \boldsymbol{\rho}_i \times m_i \ddot{\mathbf{r}}_i$$

The first term is seen to be zero.
Second term using Newton's second law:

$$\Sigma \boldsymbol{\rho}_i \times (\mathbf{F}_i + \mathbf{f}_i) = \Sigma \boldsymbol{\rho}_i \times \mathbf{F}_i = \Sigma \mathbf{M}_G$$

→ Sum of all external moments @ *G*

→ $\boxed{\Sigma \mathbf{M}_G = \dot{\mathbf{H}}_G}$



- We may use either the absolute or the relative momentum
- The eqn can also be applied to non-rigid systems
- Mass should remain constant

Kinetics of Systems of Particles

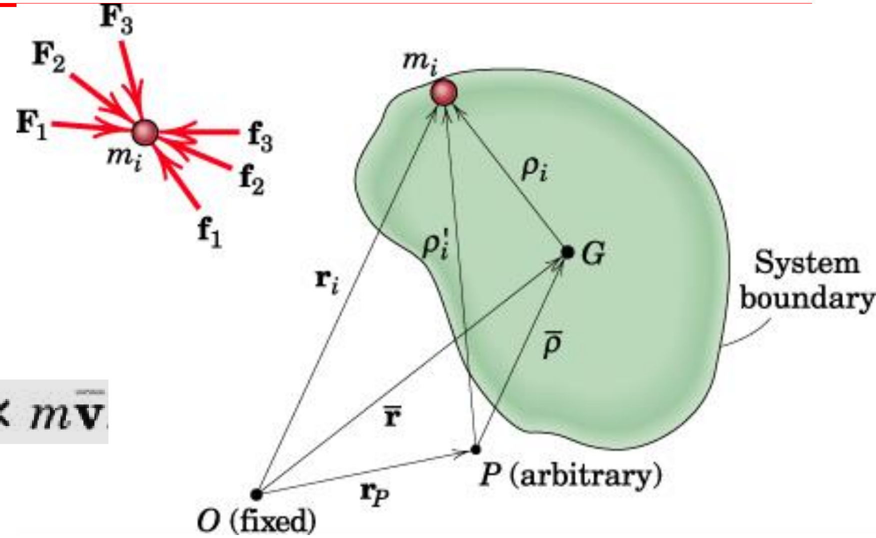
Impulse-Momentum

Angular Momentum

Angular Momentum @ an arbitrary point *P*

- Point *P* may have an accln $\mathbf{a}_P = \ddot{\mathbf{r}}_P$

$$\mathbf{H}_P = \Sigma \boldsymbol{\rho}_i' \times m_i \dot{\mathbf{r}}_i = \Sigma(\bar{\boldsymbol{\rho}} + \boldsymbol{\rho}_i) \times m_i \dot{\mathbf{r}}_i$$



The first term: $\bar{\boldsymbol{\rho}} \times \Sigma m_i \dot{\mathbf{r}}_i = \bar{\boldsymbol{\rho}} \times \Sigma m_i \mathbf{v}_i = \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{v}}$

The second term: $\Sigma \boldsymbol{\rho}_i \times m_i \dot{\mathbf{r}}_i = \mathbf{H}_G$

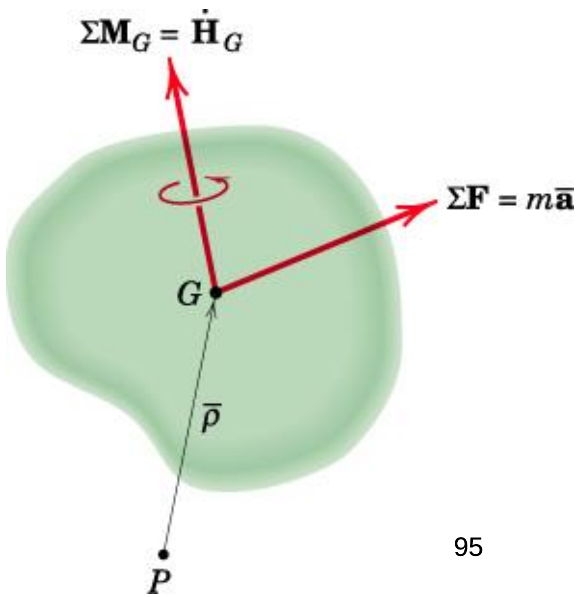
→ $\boxed{\mathbf{H}_P = \mathbf{H}_G + \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{v}}}$ Absolute AM @ any point *P* = AM @ *G* + moment @ *P* of the LM of the system considered concentrated at *G*

Using the Principle of Moments

Resultants of the ext forces acting on the system can be expressed as resultant force ∑**F** through G and the corresponding couple $\sum \mathbf{M}_G$

$$\Sigma \mathbf{M}_P = \Sigma \mathbf{M}_G + \bar{\boldsymbol{\rho}} \times \Sigma \mathbf{F}$$

→ $\boxed{\Sigma \mathbf{M}_P = \dot{\mathbf{H}}_G + \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{a}}}$



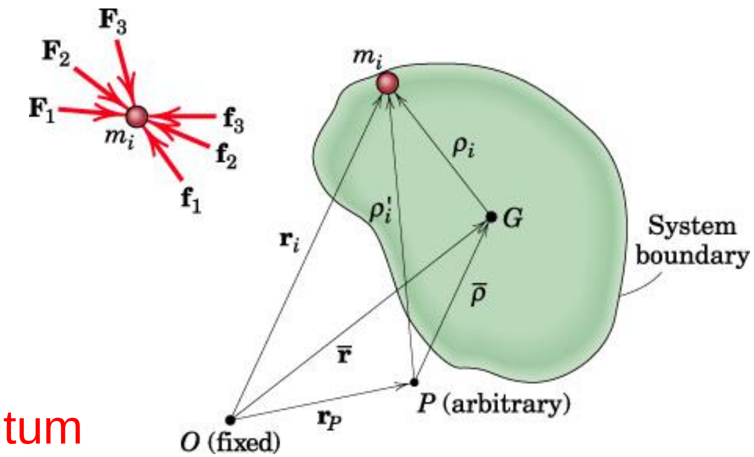
Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum

Angular Momentum @ an arbitrary point P
Using the Principle of Moments

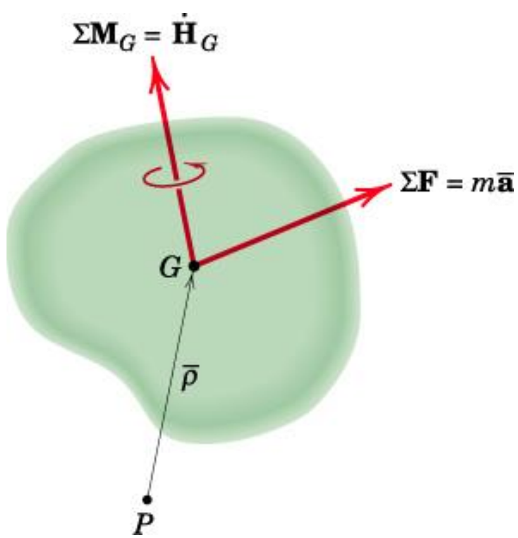
$$\boxed{\Sigma \mathbf{M}_P = \dot{\mathbf{H}}_G + \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{a}}}$$



Similar eqn can also be developed using momentum relative to P: $(\mathbf{H}_P)_{\text{rel}} = \Sigma \boldsymbol{\rho}_i' \times m_i \dot{\boldsymbol{\rho}}_i'$

Substituting $\boldsymbol{\rho}_i' = \bar{\boldsymbol{\rho}} + \boldsymbol{\rho}_i$ and $\dot{\boldsymbol{\rho}}_i' = \dot{\bar{\boldsymbol{\rho}}} + \dot{\boldsymbol{\rho}}_i$

$$(\mathbf{H}_P)_{\text{rel}} = \Sigma \bar{\boldsymbol{\rho}} \times m_i \dot{\bar{\boldsymbol{\rho}}} + \Sigma \bar{\boldsymbol{\rho}} \times m_i \dot{\boldsymbol{\rho}}_i + \Sigma \boldsymbol{\rho}_i \times m_i \dot{\bar{\boldsymbol{\rho}}} + \Sigma \boldsymbol{\rho}_i \times m_i \dot{\boldsymbol{\rho}}_i$$



The first term: $\bar{\boldsymbol{\rho}} \times m\bar{\mathbf{v}}_{\text{rel}}$

Second will be zero: $\bar{\boldsymbol{\rho}} \times \dfrac{d}{dt} \Sigma m_i \boldsymbol{\rho}_i$

Third term will be zero: $-\dot{\bar{\boldsymbol{\rho}}} \times \Sigma m_i \boldsymbol{\rho}_i$

Fourth term is: $(\mathbf{H}_G)_{\text{rel}}$ which is same as $\mathbf{H}_G$

→ $(\mathbf{H}_P)_{\text{rel}} = (\mathbf{H}_G)_{\text{rel}} + \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{v}}_{\text{rel}}$

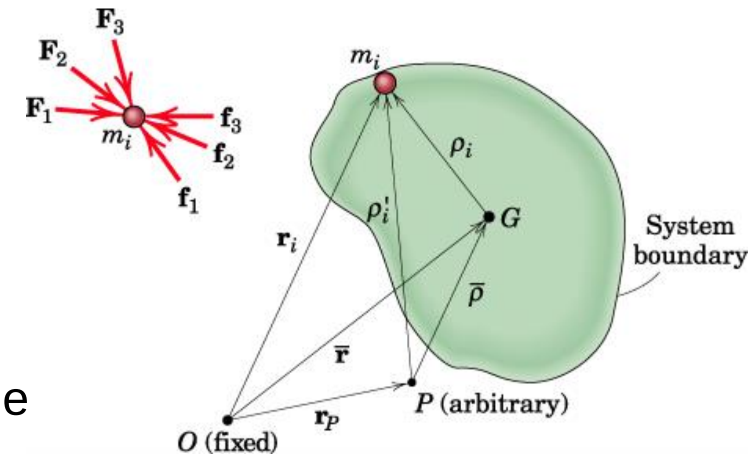
Kinetics of Systems of Particles

Impulse-Momentum

Angular Momentum

Angular Momentum @ an arbitrary point *P*

Using the Principle of Moments



$$(\mathbf{H}_P)_{\text{rel}} = (\mathbf{H}_G)_{\text{rel}} + \bar{\boldsymbol{\rho}} \times m\bar{\mathbf{v}}_{\text{rel}}$$

Differentiating defn $(\mathbf{H}_P)_{\text{rel}} = \Sigma\boldsymbol{\rho}_i' \times m_i\dot{\boldsymbol{\rho}}_i'$ wrt time and substituting $\ddot{\mathbf{r}}_i = \ddot{\mathbf{r}}_P + \ddot{\boldsymbol{\rho}}_i'$

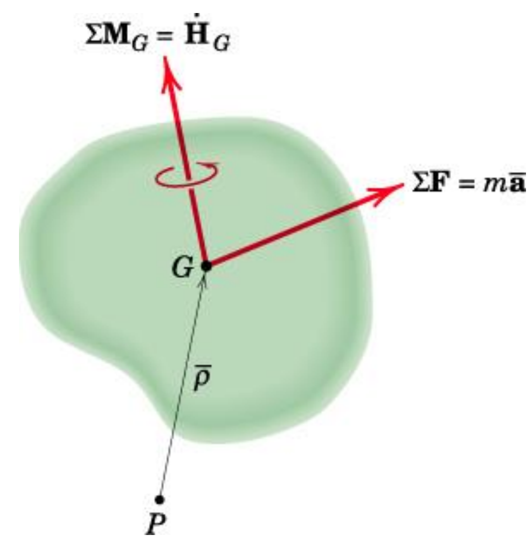
→ $(\dot{\mathbf{H}}_P)_{\text{rel}} = \Sigma\dot{\boldsymbol{\rho}}_i' \times m_i\dot{\boldsymbol{\rho}}_i' + \Sigma\boldsymbol{\rho}_i' \times m_i\ddot{\mathbf{r}}_i - \Sigma\boldsymbol{\rho}_i' \times m_i\ddot{\mathbf{r}}_P$

First term is zero. Second term is $\Sigma\mathbf{M}_P$ (moment of ext **F**)

Third term:

$$\Sigma\boldsymbol{\rho}_i' \times m_i\mathbf{a}_P = -\mathbf{a}_P \times \Sigma m_i\boldsymbol{\rho}_i' = -\mathbf{a}_P \times m\bar{\boldsymbol{\rho}} = \bar{\boldsymbol{\rho}} \times m\mathbf{a}_P$$

→ $\boxed{\Sigma\mathbf{M}_P = (\dot{\mathbf{H}}_P)_{\text{rel}} + \bar{\boldsymbol{\rho}} \times m\mathbf{a}_P}$ Convenient when a point P whose accln is known is used as a moment center



→ The eqn reduces to a simpler form $\Sigma\mathbf{M}_P = (\dot{\mathbf{H}}_P)_{\text{rel}}$

If (1) $\mathbf{a}_P = \mathbf{0}$ equivalent to first case (AM @ *O*)

(2) $\bar{\boldsymbol{\rho}} = \mathbf{0}$ equivalent to second case (AM @ *G*)

(3) $\bar{\boldsymbol{\rho}}$ and $\mathbf{a}_P$ are parallel ($\mathbf{a}_P$ directed toward or away from *G*)

Kinetics of Systems of Particles

Conservation of Energy

- A mass system is said to be conservative if it does not lose energy by virtue of internal friction forces that do negative work or by virtue of inelastic members that dissipate energy upon cycling.
- If no work is done on a conservative system during an interval of motion by external forces other than gravity or other potential forces, then energy of the system is not lost → $U'_{1\text{-}2} = 0$

$$\boxed{\Delta T + \Delta V = 0} \qquad \text{or} \qquad \boxed{T_1 + V_1 = T_2 + V_2}$$

Conservation of Momentum

In absence of external impulse (resultant external force zero)

$$\boxed{\mathbf{G}_1 = \mathbf{G}_2}$$

In absence of external angular impulse (resultant moment @ *G* or *O* of all external forces zero)

$$\boxed{(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2 \qquad \text{or} \qquad (\mathbf{H}_G)_1 = (\mathbf{H}_G)_2}$$