MA 322: Scientific Computing

Department of Mathematics
Indian Institute of Technology Guwahati

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Chapter 4: Numerical Integrations or Quadratures

Newton-Cotes integration: Boole's rule

- Newton-Cotes formula

$$\int_a^b f(x)\mathrm{d}x \approx I_n(f) = \sum_{j=0}^{n} w_j f(x_j),$$

where

$$w_j = \int_{x_0}^{x_n} \prod_{\substack{i=0\\ i\neq j}}^{n} \left(\frac{x - x_i}{x_j - x_i}\right) \mathrm{d}x.$$

- Newton-Cotes formula for $n = 4$,

$$\int_{x_0}^{x_4} f(x)\mathrm{d}x = \frac{2h}{45}\left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)\right]$$

$$-\frac{8h^7}{945} f^{(6)}(\eta), \quad x_0 \leq \eta \leq x_4.$$

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Newton-Cotes integration: Error Analysis

Theorem

1. *For n even, assume $f(x)$ is $n+2$ times continuously differentiable on $[a, b]$. Then*

$$I(f) - I_n(f) = C_n h^{n+3} f^{(n+2)}(\eta), \quad \eta \in [a, b]$$

with

$$C_n = \frac{1}{(n+2)!} \int_0^n \mu^2(\mu - 1) \cdots (\mu - n) \mathrm{d}\mu.$$

2. *For n odd, assume $f(x)$ is $n+1$ times continuously differentiable on $[a, b]$. Then*

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Newton-Cotes integration: Convergence

Theorem

Let $I_n(f) = \sum_{j=0}^{n} w_j f(x_j), \quad n \geq 1$ *be a sequence of numerical integration formula that approximate* $I(f) = \int_a^b f(x)\mathrm{d}x$. *Let* $\mathcal{F}$ *be a family dense in* $C[a,b]$.

Then $I_n(f) \to I(f)$ *for all* $f \in C[a,b]$ *if and only if*

1. $I_n(f) \to I(f), \ \forall f \in \mathcal{F}$, *and*
2. $B \equiv \sup_{n \geq 1} \sum_{j=0}^{n} |w_j| < \infty.$

(Closed) Newton-Cotes integration: Convergence

Lemma

The set of all polynomials is dense in $C[a, b]$.

Definition

Let $\mathcal{F}$ be a family of continuous functions on a given interval $[a, b]$. We say $\mathcal{F}$ is dense in $C[a, b]$ if for every $f \in C[a, b]$ and every $\epsilon > 0$, there is a function f_ϵ in $\mathcal{F}$ for which

$$\max_{a \leq x \leq b} |f(x) - f_\epsilon(x)| \leq \epsilon.$$

Theorem (Weierstrass)

Let $f(x)$ be a continuous function on $[a, b]$ and let $\epsilon > 0$. Then there is a polynomial $p(x)$ for which

$$|f(x) - p(x)| \leq \epsilon, \quad a \leq x \leq b.$$

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Newton-Cotes integration: Convergence

$$I = \int_{-4}^{4} \frac{1}{1+x^2} \mathrm{d}x = 2\tan^{-1}(4) \approx 2.6516.$$

n	I_n
2	5.4902
4	2.2776
6	3.3288
8	1.9411
10	3.5956

Newton-Cotes formula does not converge in this case.

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Newton-Cotes integration: Convergence

- Newton-Cotes formula for $n = 8$,

$$\int_{x_0}^{x_8} f(x)\mathrm{d}x \approx I_8(f) = \frac{4h}{14175}[989(f(x_0)+f(x_8)) + 5888(f(x_1)+f(x_7)) \\ -928(f(x_2)+f(x_6)) + 10496(f(x_3)+f(x_5)) + 4540f(x_4)].$$

- Such formulas can cause loss-of-significance errors, although it is unlikely to be a serious problem until n is large.
- People have generally avoided using Newton-Cotes formulas for $n \geq 8$, even in forming composite formulas.
- The most serious problem of the Newton-Cotes method is that it may not converge for perfectly well-behaved integrands.

(Open) Newton-Cotes integration: Error Analysis

Theorem

Suppose $I_n(f) = \sum_{j=0}^{n} w_j f(x_j)$ *denotes the* $(n+1)$*-point open Newton-Cotes formula with* $x_{-1} = a, x_{n+1} = b$ *and* $h = (b-a)/(n+2)$. *Then* $\exists\ \eta \in (a,b)$ *such that*

1. *For* n *even, assume* $f(x)$ *is* $n+2$ *times continuously differentiable on* $[a,b]$. *Then*

$$I(f) - I_n(f) = C_n \frac{h^{n+3} f^{(n+2)}(\eta)}{(n+2)!} \int_{-1}^{n+1} \mu^2(\mu-1)\cdots(\mu-n)\mathrm{d}\mu.$$

2. *For* n *odd, assume* $f(x)$ *is* $n+1$ *times continuously differentiable on* $[a,b]$. *Then*

$$I(f) - I_n(f) = \frac{h^{n+2} f^{(n+1)}(\eta)}{(n+1)!} \int_{-1}^{n+1} \mu(\mu-1)\cdots(\mu-n)\mathrm{d}\mu.$$

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