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Stock Under the Risk-Neutral Measure

Let $W(t), 0 \le t \le T$, be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{F}(t), 0 \le t \le T$, be a filtration for this Brownian motion. Here T is a fixed final time. Consider a stock price process whose differential is:

$$dS(t) = \alpha(t)S(t)dt + \sigma(t)S(t)dW(t), 0 \le t \le T.$$

The mean rate of return $\alpha(t)$ and the volatility $\sigma(t)$ are allowed to be adapted processes. We assume that, for all $t \in [0, T]$, $\sigma(t)$ is almost surely not zero. This stock price is a generalized Brownian motion, and an equivalent way of writing it is:

$$S(t) = S(0) \exp\left\{ \int_0^t \left(\alpha(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_0^t \sigma(s)dW(s) \right\}.$$

In addition, suppose we have an adapted interest rate process $R(t)$. We define the *discounted process*

$$D(t) = e^{-\int_0^t R(s)ds}$$

and note that

$$dD(t) = -R(t)D(t)dt.$$

To obtain this, we can define $I(t) = \int_0^t R(s)ds$ so that $dI(t) = R(t)dt$ and $dI(t)dI(t) = 0$. We introduce the function $f(x) = e^{-x}$ for which $f'(x) = -f(x)$, $f''(x) = f(x)$ and then use the Ito-Doeblin formula to write:

$$dD(t) = df((I(t)) = f'(I(t))dI(t) + \frac{1}{2}f''(I(t))dI(t)dI(t) = -f(I(t))R(t)dt = -R(t)D(t)dt.$$

The discounted stock price process is:

$$D(t)S(t) = S(0) \exp\left\{ \int_0^t \left(\alpha(s) - R(s) - \frac{1}{2}\sigma^2(s) \right) ds + \int_0^t \sigma(s)dW(s) \right\},$$

and its differential is:

$$d(D(t)S(t)) = (\alpha(t) - R(t))D(t)S(t)dt + \sigma(t)D(t)S(t)dW(t) = \sigma(t)D(t)S(t)\left[\Theta(t)dt + dW(t)\right],$$

where we define the *market price of risk* to be:

$$\Theta(t) = \frac{\alpha(t) - R(t)}{\sigma(t)}.$$

Note that the mean rate of return of the discounted stock price is $\alpha(t) - R(t)$, which is the mean rate $\alpha(t)$ of the undiscounted stock price, reduced by the interest rate $R(t)$. The volatility of the discounted stock price is the same as the volatility of the undiscounted stock price. We introduce the probability measure $\widetilde{\mathbb{P}}$ defined in Girsanov's Theorem which uses the market price of risk $\Theta(t) = \dfrac{\alpha(t) - R(t)}{\sigma(t)}$. In terms of the Brownian motion $\widetilde{W}(t)$ of that Theorem we may now write:

$$d(D(t)S(t)) = \sigma(t)D(t)S(t)d\widetilde{W}(t)$$

We call $\widetilde{\mathbb{P}}$, the measure defined in Girsanov's Theorem, the risk-neutral measure because it is equivalent to the original measure $\mathbb{P}$ and it renders the discounted stock price $D(t)S(t)$ into a martingale. Indeed:

$$D(t)S(t) = S(0) + \int_0^t \sigma(u)D(u)S(u)d\widetilde{W}(u),$$

and under $\widetilde{\mathbb{P}}$ the process $\int_0^t \sigma(u)D(u)S(u)d\widetilde{W}(u)$ is an Ito integral and hence a martingale.

The undiscounted stock price $S(t)$ has mean rate of return equal to the interest rate under $\widetilde{\mathbb{P}}$, as one can verify making the replacement $dW(t) = -\Theta(t)dt + d\widetilde{W}(t)$ in:

$$\begin{aligned} dS(t) &= \alpha(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t) \\ &= \alpha(t)S(t)dt + \sigma(t)S(t)\left(-\frac{\alpha(t) - R(t)}{\sigma(t)}dt + d\widetilde{W}(t)\right) \\ &= R(t)S(t)dt + \sigma(t)S(t)d\widetilde{W}(t). \end{aligned}$$

We can either solve this equation for $S(t)$ or simply replace the Ito integral $\int_0^t \sigma(s)dW(s)$ by its equivalent $\int_0^t \sigma(s)d\widetilde{W}(s) - \int_0^t (\alpha(s) - R(s))ds$ to obtain the formula:

$$S(t) = S(0)\exp\left\{\int_0^t \left(R(s) - \frac{1}{2}\sigma^2(s)\right)ds + \int_0^t \sigma(s)d\widetilde{W}(s)\right\}.$$

In discrete time, the change of measure does not change the binomial tree, only the probabilities on the branches of the tree. In continuous time, the change from the actual measure $\mathbb{P}$ to the risk-neutral measure $\widetilde{\mathbb{P}}$ changes the mean rate of return of the stock but not the volatility.

Value of Portfolio Process Under the Risk-Neutral Measure

Consider an agent who begins with an initial capital $X(0)$ and at each time $t, 0 \leq t \leq T$, holds $\Delta(t)$ shares of the stock, investing or borrowing at the interest rate $R(t)$ as necessary to finance this. The differential of this agent's portfolio value is given by:

$$\begin{aligned} dX(t) &= \Delta(t)dS(t) + R(t)(X(t) - \Delta(t)S(t))dt, \\ &= \Delta(t)(\alpha(t)S(t)dt + \sigma(t)S(t)dW(t)) + R(t)(X(t) - \Delta(t)S(t))dt, \\ &= R(t)X(t)dt + \Delta(t)(\alpha(t) - R(t))S(t)dt + \Delta(t)\sigma(t)S(t)dW(t), \\ &= R(t)X(t)dt + \Delta(t)\sigma(t)S(t)\left[\Theta(t)dt + dW(t)\right]. \end{aligned}$$

Ito product rule implies that:

$$\begin{aligned} d(D(t)X(t)) &= \Delta(t)\sigma(t)D(t)S(t)\left[\Theta(t)dt + dW(t)\right], \\ &= \Delta(t)d(D(t)S(t)). \end{aligned}$$

Our agent has two investment options:

(A) A money market account with rate of return $R(t)$.

(B) A stock with a mean rate of return $R(t)$ under $\widetilde{\mathbb{P}}$.

Regardless of how the agent invests, the mean of return for the portfolio will be $R(t)$ under $\widetilde{\mathbb{P}}$, and hence the discounted value of this portfolio, $D(t)X(t)$, will be a martingale.

Pricing Under the Risk-Neutral Measure

Let $V(T)$ be an $\mathcal{F}(T)$-measurable random variable. This represents the payoff at time T of a derivative security. We allow the payoff to be path dependent, which is what $\mathcal{F}(T)$-measurability means. We wish to know what initial capital $X(0)$ and portfolio process $\Delta(t), 0 \le t \le T$, an agent would need in order to hedge a short position in this derivative security, i.e., in order to have $X(T) = V(T)$ almost surely. We do not assume a constant mean rate of return, volatility, and interest rate.

Our agent wishes to choose initial capital $X(0)$ and portfolio strategy $\Delta(t), 0 \le t \le T$, such that $X(T) = V(T)$ holds.

Once it has been done, the fact that $D(t)X(t)$ is a martingale under $\widetilde{\mathbb{P}}$ implies,

$$\begin{aligned} D(t)X(t) &= \widetilde{E}[D(T)X(T)|\mathcal{F}(t)], \\ &= \widetilde{E}[D(T)V(T)|\mathcal{F}(t)]. \end{aligned}$$

The value $X(t)$ of the hedging portfolio is the capital needed at time t in order to successfully complete the hedge of the short position in the derivative security with payoff $V(T)$. Hence, we can call this the *price* $V(t)$ of the derivative security at time t and therefore:

$$D(t)V(t) = \widetilde{E}[D(T)V(T)|\mathcal{F}(t)], 0 \le t \le T.$$

Recall :This is the continuous time analogous of the formula:

$$\frac{V_n}{(1+r)^n} = \widetilde{E}_n\left[\frac{V_N}{(1+r)^N}\right],$$

for the binomial model.

Dividing the above relation by $D(t)$ which is $\mathcal{F}(t)$-measurable and recalling the definition of $D(t)$ we have:

$$V(t)\widetilde{E}\left[e^{-\int_t^T R(u)du} V(T)|\mathcal{F}(t)\right], 0 \le t \le T.$$

We shall refer to this as the *risk-neutral pricing formula* for the continuous-time model.