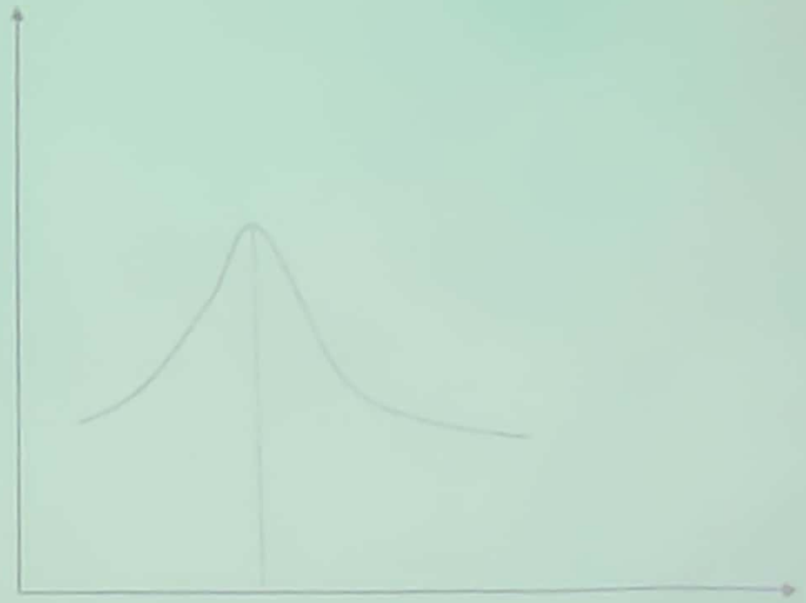
- Suppose $f(x)$ is differentiable in an interval I and suppose $f(x)$ has only one stationary point at c.
- If $f'(x) \geq 0$, for all $x \in I$, such that $x \leq c$, and $f'(x) \leq 0$, for all $x \in I$, such that $x \geq c$, then $f(x)$ is increasing to the left of c and decreasing to the right of c.
- Thus, $f(x) \leq f(c)$ for all $x \leq c$ and $f(c) \geq f(x)$ for all $x \geq c$.
- Therefore, $x = c$ is a maximum point for f in I.
- The diagram in the next slide illustrates this.

- $f'(x) \geq 0$ for $x \leq c$,
- $f'(x) \leq 0$ for $x \geq c$

$\rightarrow f(x)$ has a maximum at $x = c$



- $f'(x) \leq 0$ for $x \leq c$,
- $f'(x) \geq 0$ for $x \geq c$

$\rightarrow f(x)$ has a minimum at $x = c$

- These two can be called the **first derivative test** for maximum and minimum.
- Example: Suppose a producer uses labour to produce output, which is sold in the market for p rupees per unit. The production function is given by, $Q = f(L)$, where Q is the units of output produced, L is the amount of labour used in the production. The labour is paid wage rate of w per unit. There is no other input of production.
- Here the profit of the producer is given by,

$\pi(L) = p.Q - w.L$

$= pf(L) - wL$

This is a function in L, labour, alone. It is assumed that p and w are given.

- Let there exist an amount of labour L^* such that, $\pi'(L) \geq 0$ for $L \leq L^*$, and $\pi'(L) \leq 0$ for $L \geq L^*$.
- In that case at $L = L^*$, the profit function $\pi(L)$ has a maximum.
- The maximum point is found by the condition,

$\pi'(L^*) = 0$

Or, $\frac{d}{dL}(pf(L*) - wL*) = 0$

Or, $pf'(L*) - w = 0$

Or, $pf'(L*) = w$

- What is the interpretation of this condition?
- As one more unit of labour is employed (labour is the only input), the producer gains something, and loses something.
- This condition states that additional gains and losses must be equal.
- $pf'(L*)$ is the gain of employing one additional labour unit. Because $f'(L*)$ is the increment of output as an additional labour is hired, this is sold at price p.
- The loss is w, the wage paid to an additional unit of labour.
- At the point of maximum profit the additional gain and loss balance each other. In other words, the producer will decide the hiring level such that this condition is satisfied.

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Example: A man lives for two periods, 1 and 2. The incomes are given by y_1 and y_2. He plans to consume c_1 and c_2 during these two periods. Suppose his utility function (which he wants to maximize) is given by,

$U(c_1, c_2) = \ln c_1 + \frac{1}{1+\alpha} \ln c_2$, implying that α is his rate of discounting. He can borrow and lend in the market at r rate of interest. We want to find the man's saving-consumption plan.

If the man saves money in period 1, he can consume additionally $(y_1 - c_1)(r+1)$in period 2. So, $c_2 = y_2 + (y_1 - c_1)(r+1)$

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- On the other hand, if be borrows to finance his consumption in period 1, then he has to payback, so, $c_2 = y_2 - (c_1 - y_1)(r + 1)$
- These two condition are equivalent. They show the **budget constraint** over two periods. We substitute $c_2 = y_2 - (c_1 - y_1)(r + 1)$ in the utility function,

$$U = \ln c_1 + \frac{1}{1+\alpha} \ln(y_2 - (c_1 - y_1)(r + 1))$$

This is a function of only c_1 now.

Let there is a positive $c_1 = c_1^*$, where it is maximized. Therefore, at c_1^* the first derivative is zero, it is a stationary point.

$$\frac{dU}{dc_1} = \frac{1}{c_1^*} + \frac{1}{\alpha+1}\frac{-(1+r)}{y_2-(1+r)(c_1^*-y_1)}$$

Or, $$\frac{dU}{dc_1} = \frac{(\alpha+1)[y_2-(1+r)(c_1^*-y_1)]-c_1^*(1+r)}{c_1^*(\alpha+1)[y_2-(1+r)(c_1^*-y_1)]}$$

$$= \frac{(\alpha+1)[y_2+(1+r)y_1)]-c_1^*(1+r)-c_1^*(1+r)(1+\alpha)}{c_1^*(\alpha+1)[y_2-(1+r)(c_1^*-y_1)]}$$

$$\frac{dU}{dc_1} = \frac{(\alpha+1)[y_2+(1+r)y_1)]-c_1^*(1+r)(2+\alpha)}{c_1^*(\alpha+1)[y_2-(1+r)(c_1^*-y_1)]} \text{ [A]}$$

$\frac{dU}{dc_1} = 0$ implies, $(\alpha+1)[y_2+(1+r)y_1)] - c_1^*(1+r)(2+\alpha) = 0$

Or, $c_1^* = \frac{(\alpha+1)[y_2+(1+r)y_1)]}{(1+r)(2+\alpha)}$

This is the optimal consumption in period 1.

Two observations:

1. If $c_1 > c_1^*$, then $c_1 > \frac{(\alpha+1)[y_2+(1+r)y_1)]}{(1+r)(2+\alpha)}$

Or, $(\alpha + 1)[y_2+(1 + r)y_1)] - c_1\ (1 + r)(2 + \alpha) < 0$

From [A] above, we see, $\frac{dU}{dc_1} < 0$

Similarly, if $c_1 < c_1^*, \frac{dU}{dc_1} > 0$

So, c_1^* indeed maximizes the utility function, which was our initial assumption.

2. When does the man borrow or lend?

He lends, if $c_1^* <$ his income in period 1 = y_1

From, $c_1^* < y_1$ we get, $y_1\ (1+r)(2+\alpha) > (\alpha+1)[\ y_2 + (1+r)y_1)]$

Or, $y_1\ (1+r) > y_2(\alpha+1)$

Thus, a very high interest rate can make a man a lender.

But a high y_2 will make him a borrower.

- But finding a stationary point to find the maximum point may not work.

Example: T is the tax people pay on their income Y, $T = a(bY + c)^p + kY$, (a, b, c, p are positive constants, $p > 1$). We want to find at what income level the average tax is the highest.

- Average tax, $\theta = \frac{T}{Y}$

$= \frac{a(bY+c)^p + kY}{Y}$

$= \frac{a(bY+c)^p}{Y} + k$

Setting $\theta'(Y) = 0$, we get,

$$\frac{d}{dY}\left(\frac{a(bY + c)^p}{Y} + k\right) = 0$$

Or, $\frac{a}{Y}\frac{d}{dY}(bY+c)^p + (bY+c)^p\frac{d}{dY}\frac{a}{Y} = 0$

Or, $\frac{ab}{Y}p(bY+c)^{p-1} - (bY+c)^p\frac{a}{Y^2} = 0$

Multiplying both sides by $\frac{Y^2}{(bY+c)^{p-1}}$,

$$Yapb - a(bY+c) = 0$$

Or, $a(Ybp - Yb - c) = 0$ [*]

Or, $Y = \frac{c}{b(p-1)} = Y^*$, say

However, average tax, θ is not maximized at Y^*. It is minimized.

If, $Y < \frac{c}{b(p-1)}$, it implies, $Ybp - Yb - c < 0$

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- By [*], $\theta'(Y) < 0$.
- Similarly, if $Y > \frac{c}{b(p-1)}$, one can show, $\theta'(Y) > 0$.
- Thus at $\frac{c}{b(p-1)} = Y^*$, average tax $\theta(Y)$ is minimized.

- In the abovementioned method we examined how the sign of the first derivative of a function changes to find its extreme points. Following is a systematic way to locate them.

- Suppose, $f(x)$ is a differentiable function defined in the interval I, and has a maximum point at $x = c$ in the interior on I, then $f'(x) = 0$ at $x = c$. It's a stationary point.
- Thus, it is a necessary condition. If the extreme points exist, stationary points are implied. Not the other way, because they can be inflection points, or points which are **local extreme points**.
- If the function is continuous over a closed, bounded interval I, then by **the extreme-value theorem**, we know that a maximum and a minimum exist in the interval. There could be three possibilities for extreme points.

1. In the interior of I, where $f'(x) = 0$
2. Endpoints of I.
3. Points in I where $f'(x)$ does not exist (kink points).

If the function in question is differentiable then the last case can be ruled out. One can have the following algorithm to find the maximum and minimum values of a differentiable function f defined on a closed, bounded interval $[a, b]$.

1. Find all stationary point of f in (a, b), i.e., all points where $f'(x) = 0$.
2. Evaluate f at the endpoints a and b of the interval and at all stationary points found in (1).
3. The largest function value in (2) is the maximum value of f in $[a, b]$.
4. The smallest function value in (2) is the minimum value of f in $[a, b]$.

Example: find the extreme points of

$f(x) = x^3 - 12x^2 + 36x + 8$ defined in the interval [1, 7].

First, to find the stationary points, set

$f'(x) = 0$

Or, $3x^2 - 24x + 36 = 0$

Or, $x^2 - 8x + 12 = 0$

Or, $(x-6)(x-2)=0$

Thus $x = 2$ and $x = 6$ are the two stationary points.

$f(2) = 40$, $f(6) = 8$.

At the endpoints of the interval:

$f(1) = 37$, $f(7) = 14$

Following the above set of steps, there is a maximum at $x = 2$ (interior point), and a minimum at $x = 6$ (also an interior point). Both are stationary points.

Example: A producer is producing a commodity whose quantity is denoted by q, the revenue earned by him is given $R(q)$. The cost of producing q is given by $C(q)$. Both these are continuous functions of q. The producer seeks to maximize his profits.

Profit function is given by, $\pi(q) = R(q) - C(q)$

The minimum possible output is $q = 0$.

Let $q = q'$ be the maximum technologically possible output level.

Since $\pi(q)$ is a continuous function in q defined over the closed and bounded interval $[0, q']$, the maximum point is in the interval.

Let $q = q^*$ be the maximum point, that is, $\pi(q^*)$ is greater than $\pi(0)$ and $\pi(q')$.

- We know q^* is a stationary point, $\pi'(q^*) = 0$

Or, $\frac{d}{dq}\left(R(q^*) - C(q^*)\right) = 0$

Or, $R'(q^*) = C'(q^*)$

As the producer produces more output, the instantaneous change in the revenue and cost are given by $R'(q^*)$ and $C'(q^*)$ respectively.

The above condition says, the producer has to set his output at a level where the instantaneous change in revenue and cost are equal. If the profit maximizing output exists at a stationary point, that optimal point is characterized by this condition.

- There could be multiple points which satisfy this condition.
- In that case, we evaluate $f(x)$ at those q's and choose that point (q) which gives the maximum value.
- The condition $R'(q^*) = C'(q^*)$ is called **marginal revenue equals to marginal cost**.
- In the special case where marginal revenue = price of the good, p (in a perfect competition market) the condition becomes,

$$p = C'(q^*)$$

- Optimal output is where marginal cost is equal to the price per unit.

- Suppose, the government imposes a tax t per unit of output on producers.
- The cost of production changes to, $C(q) + tq$
- If q^* is the optimal output (interior), then the necessary condition becomes,

  $R'(q*) = C'(q*) + t$
- Since taxes add to the cost, the new condition has the tax rate added to the marginal cost term.
- This new expression $C'(q*) + t$ has to be equated to marginal revenue.