30/7/24 Matrix Computations

Frobenius norm $= \|A_F\| = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2} = \left(tr(A^TA)\right)^{1/2}$

$\|A_F\|$ is not induced by an vector norm bcuz $\|I\|_F = \sup_{x \neq 0} \frac{\|Ix\|}{\|x\|} = 1$

but $\|I\|_F = \sqrt{2}$.

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M_i are called Gauss transforms

A Lower triangular matrix with all ones in the diagonal is called a unit lower Δ^{lar} matrix (like L in $A=LU$)

$$A^{(k)} = A^{(k-1)} - \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & \underbrace{\hat{M}_k}_{n-k} \end{array}\right] \qquad \hat{M}_k = \begin{bmatrix} m_{k+1,k} \\ \\ m_{nk} \end{bmatrix} \left[a_{kk}^{(k-1)} \cdots a_{kn}^{(k)}\right]$$

$$= (n-k) \times (n-k+1)$$

$$= \begin{array}{c} k \\ n-k \end{array}\left[\begin{array}{c|c} \overbrace{A_{11}^{(k-1)}}^{k} & \overbrace{A_{12}^{(k-1)}}^{n-k} \\ \hline A_{21}^{(k-1)} & A_{22}^{(k-1)} \end{array}\right] - \left[\begin{array}{cc|c} & 0 & \\ \hline 0 & 0 & \\ 0 \cdots & 0 & \\ 0 & 0 & \end{array}\right]$$

~~$\leq 1 + 2(n-1)$~~

~~$n-1 + 2n(n-1)$~~

$+1$

obtained by shifting the column partition to the right

j^{st} column of M

$$= \begin{bmatrix} a_{k+1,k}^{(k-1)} \\ \vdots \\ a_{nk}^{(k-1)} \end{bmatrix}$$

$$Ax = b$$
$$LUx = b$$

$A = LU$ till now requires diagonal entries are non-zero

the diagonal entries are all 1 otherwise it won't be a unique decomposit^n $A = \left(\frac{1}{k}L\right)(kU)$

f. of leading principal matrices

$$A = LU \iff \begin{array}{c} k \\ n-k \end{array}\left[\begin{array}{c|c} \overset{k}{A_{11}} & \overset{n-k}{A_{12}} \\ \hline A_{21} & A_{22} \end{array}\right] = \begin{array}{c} k \\ n-k \end{array}\left[\begin{array}{c|c} \overset{k}{L_{11}} & \overset{n-k}{} \\ \hline L_{21} & L_{22} \end{array}\right] \left[\begin{array}{c|c} \overset{k}{U_{11}} & \overset{n-k}{U_{12}} \\ \hline & U_{22} \end{array}\right]\begin{array}{c} k \\ n-k \end{array}$$

$A_{11} =$

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Orthogonal $P^T = P^{-1}$

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$$A = LU$$

$$= L \underbrace{\begin{bmatrix} u_{11} & & 0 \\ & \ddots & \\ 0 & & u_{nn} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} u_{11}^{-1} & & 0 \\ & \ddots & \\ 0 & & u_{nn}^{-1} \end{bmatrix}}_{D^{-1}} U$$

$$= LDV \quad \boxed{V = D^{-1}U}$$

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For complex case (in def^n for positive definite),

$x^* A x > 0 \Rightarrow x^* A x$ is a real (not complex)

A is positive definite (PD) $\Rightarrow$ non singular

Pf: Suppose A is singular. $A = [A_1 \; A_2 \; \ldots \; A_n]$

$\exists \; x_1, \ldots, x_n$ (not all zero)

$$\sum x_i A_i = 0 \Rightarrow Ax = 0$$

Principal submatrix :- Remove rows & corresponding columns (say row 3, 5, 6). All principal submatrices of a PD matrix are PD matrix.

Pf:- $\hat{A} = A[i_1, i_2, \ldots, i_p]$

$A^T = A \Rightarrow \hat{A}^T = \hat{A}$

Let $\hat{x} \in \mathbb{R}^p \setminus \{0\}$. Let $x \in \mathbb{R}^n \ni x_i = 0$ if $i \notin \{i_1, \ldots, i_p\}$

and $x_i = \hat{x}_i$ if $i \in \{i_1, \ldots, i_p\}$

Then $x \neq 0$ and $0 < x^T A x = \hat{x}^T \hat{A} \hat{x}$

$\rightarrow$ A is PD, $X^T A X$ is PD ($X_{n \times n}$)

A $\rightarrow$ positive definite & S is non-singular s.t $S^T A S$ is defined.

Let $B = S^T A S$.

$B^T = S^T A^T S = B$

Let $x^n \in \mathbb{R}^n \setminus \{0\}$ $\quad x^T B x = x^T S^T A \underbrace{Sx}_{y} = \underbrace{y^T A y}_{> 0}$ where $y \neq 0$

[$\because S^{-1}$ exists & $x \neq 0$]

→ A is PD ⇒ all the diagonal entries are $+^{ve}$.

Pf:- $e_i^T A e_i = a_{ii} > 0$

Inner product formulat^n algo:-

(i^th row is found then (i+1)^th row)

Square root is 8 flops.

$\sum_{k=1}^{j-1} g_{kj}^2$ & $\sum_{i=1}^{j-1} g_{ij} g_{ik}$ are inner products of rows/columns hence the name.

Flop count:- $\frac{n^3}{3} + O(n^2)$ flops ($\frac{1}{2}$ of LU decomposit^n!)

Outer product formulat^n algo:-

(i^th col is found, then (i+1)^th col)

Bordered form:-

i^th row, i^th col ~~&~~ & then (i+1)^th row, (i+1)^th col

Sensitivity:-

The choice of norm (even for frobenius norm) doesn't matter.

19/8/24 Sensitivity analysis:-

minmag(A) = 0 when A is singular.

invhilb does not use hilbert matrices to invert.

Dealing with ill condit^n ing when one column is larger than other:-

$Ax = b$ $\quad A = \begin{bmatrix} 1 & 2000 \\ 3 & -1000 \end{bmatrix}$

$A \begin{bmatrix} 1 & 0 \\ 0 & - \end{bmatrix} x = b$

see this matrix ~~properly~~ correctly in slides.

$\begin{bmatrix} 1 & \\ & \end{bmatrix}$

Ill conditioning when row/cols are "nearly linearly dependent"

$K(A) = \ldots \qquad B = A/\|A\|$

$K(B) = \frac{\text{maxmag}(A)}{\text{minmag}(A)} = \frac{1}{\text{minmag}(B)} \rightarrow \text{minmag}(B) << 1$

$\rightarrow \exists\, x_0 \neq 0 \ni \frac{\|Bx_0\|}{\|x_0\|} << 1$

~~Let~~

Let $\hat{x_0} = \frac{x_0 \cancel{\|A\|}}{\|x_0\|\,\|A\|}$

$Bx_0 = B(\|A\|\,\|x_0\|\,\hat{x_0})$

$= A(\|x_0\|\,\hat{x_0})$

$\frac{\|Bx_0\|}{\|x_0\|} = \frac{\|A(\|x_0\|\,\hat{x_0})\|}{\|x_0\|}$

$= \|A\hat{x_0}\|$

$\therefore \|A\hat{x_0}\| << 1$

$\Rightarrow A\hat{x_0} \approx 0$

→ However, $\det A \approx 0$ is not always indicative of ill conditioning.

bcuz we can multiply cA to make $\det(cA)$ large or small.

20/8/24 Finite precision systems

$x_+ = x_- + \beta^{-(p-1)} \times \beta^E$

$\frac{|fl(x) - x|}{|x|} \leq \frac{|x_+ - x_-|}{2|x|} = \frac{\beta^{1-p} \times \beta^E}{2 \times s \times \beta^E} \leq \frac{\beta^{1-p}}{2}$

☆ Note that N_{min} isn't the smallest no that can be represented in IEEE. In single precision, $N_{min} = 2^{-126}$ but smallest no represented is $0.\underbrace{00\ldots0}_{22}1 \times 2^{-126} = 2^{-149}$

In exercise, it says IEEE hidden bit = 0

& $p = 24$ ~~(not 23)~~

$\cancel{Ax = b}$

$\cancel{(A+\delta A)x = b+\delta b}$

→ If A is non singular and $\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)}$ then $A+\delta A$ is non-singular

Pf:- Suppose $A+\delta A$ is singular

$\Rightarrow (A+\delta A)y = 0$ for some y

$$Ay + \delta A y = 0$$

$$y = -A^{-1}\delta A y$$

$$\|y\| = \|A^{-1}\delta A y\| \leq \|A^{-1}\| \, \|\delta A\| \, \|y\|$$

$$1 \leq \|A^{-1}\| \, \|\delta A\|$$

→ Let A be non singular. Let $\delta A \in \mathbb{F}^{n\times n}$ ∋ $\frac{\|\delta A\|}{\|A\|} < \frac{1}{\kappa(A)}$.

Consider $Ax = b$ & $(A+\delta A)(x+\delta x) = b+\delta b$

Then $$\frac{\|\delta x\|}{\|x\|} \leq \kappa(A)\left(\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right) \Big/ \left(1 - \kappa(A)\cdot\frac{\|\delta A\|}{\|A\|}\right)$$

Pf:-

$$\delta A x + A\delta x + \delta A \cdot \delta x = \delta b \qquad = \|\delta A\| \, \|A^{-1}\| + \frac{\|\delta b\|}{\|b\|}\|A\| \, \|A^{-1}\|$$

$$\cancel{\delta A \cdot \delta x = \delta b - A\delta x - \delta A x}$$

$$A\cdot\delta x = \delta b - \delta A\cdot\delta x - \delta A\cdot x$$

$$\delta x = A^{-1}\delta b - A^{-1}\delta A\cdot\delta x - A^{-1}\delta A\cdot x$$

$$\|\delta x\| \leq \|A^{-1}\| \, \|\delta b\| + \|A^{-1}\| \, \|\delta A\| \, \|\delta x\| + \|A^{-1}\| \, \|\delta A\| \, \|x\|$$

$$\|\delta x\|\left(1 - \|A^{-1}\| \, \|\delta A\|\right) \leq \|A^{-1}\| \, \|\delta b\| + \|A^{-1}\| \, \|\delta A\| \, \|x\|$$

$$\frac{\|\delta x\|}{\|x\|} \leq \frac{\|A^{-1}\| \, \|\delta b\| / \|x\| + \|A^{-1}\| \, \|\delta A\|}{1 - \|A^{-1}\| \, \|\delta A\|}$$

$Ax = b$

$\|b\| \leq \|A\| \, \|x\|$

$$\leq \frac{\|A^{-1}\| \, \|\delta b\| \frac{\|A\|}{\|b\|} + \|A^{-1}\| \, \|\delta A\|}{1 - \|A^{-1}\| \, \|\delta A\|}$$

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Rounding errors:-

$$fl\left(\frac{x}{y}\right) = fl\left(\frac{fl(x)}{fl(y)}\right) = \left(\frac{x(1+\varepsilon_1)}{y(1+\varepsilon_2)}\right)(1+\varepsilon_3) = \frac{x}{y}(1+\varepsilon_1)(1+\varepsilon_3)(1+\varepsilon_2)^{-1}$$

$|\varepsilon_i| \le u$

$$= \frac{x}{y}\Big(1+\underbrace{\varepsilon_1+\varepsilon_3-\varepsilon_2+\varepsilon_1\varepsilon_3-\varepsilon_1\varepsilon_2-\varepsilon_3\varepsilon_2-\varepsilon_1\varepsilon_2\varepsilon_3+\dots}_{\varepsilon}\Big)$$

$\therefore |\varepsilon| \le 3u + O(u^2)$

$\therefore fl\left(\frac{x}{y}\right) = \left(\frac{x}{y}\right)(1+\varepsilon)$

$$fl(x+y) = fl(fl\,x + fl\,y) = \big(x(1+\varepsilon_1)+y(1+\varepsilon_2)\big)(1+\varepsilon_3)$$

$$= (x+y)\left(1+(\varepsilon_1+\varepsilon_3+\varepsilon_1\varepsilon_3)\frac{x}{x+y}+(\varepsilon_2+\varepsilon_3+\varepsilon_2\varepsilon_3)\frac{y}{x+y}\right)$$

$$|\varepsilon| = \left|(\varepsilon_1+\varepsilon_3+\varepsilon_1\varepsilon_3)\frac{x}{x+y}+(\varepsilon_2+\varepsilon_3+\varepsilon_2\varepsilon_3)\frac{y}{x+y}\right|$$

$$\le \left(\frac{|x|}{|x+y|}+\frac{|y|}{|x+y|}\right)\left[2u+O(u^2)\right]$$

Ex:- $(10,3,-2,2)$ sys Doing $(99.93+0.026)-(100+0.5)$

Swamping $\rightarrow$ Catastrophic cancellatn

$fl(99.93) = 9.99\times 10$ $fl(0.026) = 0.026$ $\therefore fl(99.93+0.026) = 9.99\times 10$

$fl(100.5) = fl(1.005\times 10^2) = 100$

Backward analysis:-

$$fl(x+y) = fl(fl(x)+fl(y)) = \big(x(1+\varepsilon_1)+y(1+\varepsilon_2)\big)(1+\varepsilon_3)$$

$$= x(1+\varepsilon_1)(1+\varepsilon_3)+y(1+\varepsilon_2)(1+\varepsilon_3)$$

$$= x(1+\varepsilon)+y(1+\hat{\varepsilon})$$

$$= \hat{x}+\hat{y}$$

where $|\varepsilon| \le 2u + O(u^2)$

$|\hat{\varepsilon}| \le 2u + O(u^2)$

Slly, All 4 ops are backward stable

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Thm:- Let w_i, $i=1,\dots,n$ be floating pt nos. Then there exist γ_i, $i=1,\dots,n$ satisfying $|\gamma_i| \le (n-1)u + O(u^2)$ ∋

$$fl\left(\sum_{i=1}^n w_i\right) = \sum_{i=1}^n w_i(1+\gamma_i)$$

Pf:- $n=2$ $\quad fl(w_1+w_2) = \hat{w}_1 + \hat{w}_2$ where $\dfrac{|\hat{w}_1 - w_1|}{|w_1|}$ & $\dfrac{|\hat{w}_2 - w_2|}{|w_2|} \le u + O(u^2)$

Let $\gamma_i = \dfrac{w_i - \hat{w}_i}{w_i}$. Then $\hat{w}_i = w_i(1+\gamma_i)$ where $\gamma_i \le u + O(u^2)$ $\forall\, i=1,2$

$$\therefore fl(w_1+w_2) = \sum_{i=1}^2 w_i(1+\gamma_i) \text{ where } |\gamma_i| \le u + O(u^2) \;\forall\, i=1,2$$

Suppose $1 \le k \le n-1$. $fl\left(\sum_{i=1}^n w_i\right) = fl\left(fl\left(\sum_{i=1}^k w_i\right) + fl\left(\sum_{i=k+1}^n w_i\right)\right) \rightarrow ③$

$$fl\left(\sum_{i=1}^k w_i\right) = \sum_{i=1}^k w_i(1+\delta_i),\quad |\delta_i| \le (k-1)u + O(u^2) \rightarrow ①$$

$$fl\left(\sum_{i=k+1}^n w_i\right) = \sum_{i=1}^n w_i(1+\hat{\delta}_i),\quad |\delta_i| \le (n-k-1)u + O(u^2) \rightarrow ②$$

$$\therefore fl\left(\sum_{i=1}^n w_i\right) = fl\left(\sum_{i=1}^k w_i(1+\delta_i) + \sum_{i=k+1}^n w_i(1+\hat{\delta}_i)\right)$$

$$= \left(\sum_{i=1}^k w_i(1+\delta_i) + \sum_{i=k+1}^n w_i(1+\hat{\delta}_i)\right)(1+\varepsilon) \text{ where } |\varepsilon| \le u$$

$$= \sum_{i=1}^k w_i(1+\gamma_i)$$

$\gamma_i = \delta_i + \varepsilon + \delta_i\varepsilon$ for $i=1,\dots,k$

$\gamma_i = \hat{\delta}_i + \varepsilon + \hat{\delta}_i\varepsilon$ for $i=k+1,\dots,n$

For $i=1:k$, $|\gamma_i| \le |\delta_i| + |\varepsilon| + |\delta_i||\varepsilon| \le (k-1)u + u + O(u^2) \le (n-1)u + O(u^2)$

Ily for $i=k+1:n$

Backward stability analysis for forward/backward substitutn:-

$$\hat{y}_i = fl\left[\left(b_i - \sum_{j=1}^{i-1} g_{ij}\hat{y}_j\right)/g_{ii}\right] \quad \forall\, i=1,\dots,n$$

$$fl\left(b_i - \sum_{j=1}^{i-1} g_{ij}\hat{y}_j\right) = fl\left[fl(b_i) - \underbrace{fl\left(\sum_{j=1}^{i-1} g_{ij}\hat{y}_j\right)}_{=\sum_{j=1}^{i-1} g_{ij}\hat{y}_j(1+\alpha_{ij})}\right]$$

where $\alpha_{ij} \le (i-1)u + O(u^2)$

$$= b_i(1+\gamma_{ii}) - \sum_{i=1}^{i-1} q_{ij}\hat{y}_j(1+\alpha_{ij})(1+\gamma_{ij}) \quad \text{where } |\gamma_{ij}| \le (i-1)u + O(u^2)$$

$$\therefore \hat{y}_i = \left[\frac{b_i(1+\gamma_{ii}) - \sum_{i=1}^{i-1} q_{ij}\hat{y}_j(1+\alpha_{ij})(1+\gamma_{ij})}{q_{ii}}\right](1+\epsilon_i) \quad \text{where } |\epsilon_i| \le u$$

($\because$ i no's are being added/subtracted)

$$= \frac{b_i - \sum_{i=1}^{i-1} q_{ij}\hat{y}_j \underbrace{\left(\frac{(1+\alpha_{ij})(1+\gamma_{ij})}{1+\gamma_{ii}}\right)}_{=1+\delta_{ij}}}{\underbrace{\dfrac{q_{ii}}{(1+\gamma_{ii})(1+\epsilon_i)}}_{=1+\delta_{ii}}}$$

bound these using the prev bounds given

$$\boxed{\therefore\ \hat{y}_n = b_i - \sum_{j=1}^{i-1} q_{ij}(1+\delta_{ij})\hat{y}_j}$$

<u>Altern pf</u>:-

$$\hat{y}_i = fl\left[\left(b_i - \sum_{j=1}^{i-1} q_{ij}\hat{y}_j\right)/q_{ii}\right]$$

it is already a float $\therefore$ you don't need to write float again.

$$= fl\left[fl\left[b_i - \sum_{j=1}^{i-1} q_{ij}\hat{y}_j\right]/q_{ii}\right]$$

is an inner product $b_i \cdot 1 - q_{1j}\hat{y}_1 - q_{2j}\hat{y}_2 - \dots$

where $|\gamma_{ij}| \le iu + O(u^2) \ \forall\, j = 1, \dots, i$

$$= fl\left[\frac{b_i(1+\gamma_{ii}) - \sum_{j=1}^{i-1} q_{ij}\hat{y}_j(1+\gamma_{ij})}{q_{ii}}\right]$$

$$= \frac{b_i(1+\gamma_{ii}) - \sum_{j=1}^{i-1} q_{ij}\hat{y}_j(1+\gamma_{ij})}{q_{ii}}(1+\epsilon_i) \quad \text{where } |\epsilon_i| \le u$$

$$= \frac{b_i - \sum_{j=1}^{i-1} q_{ij}\hat{y}_j(1+\gamma_{ij})(1+\gamma_{ii})^{-1}}{q_{ii}(1+\gamma_{ii})^{-1}(1+\epsilon_i)^{-1}}$$

Let $1+\delta_{ij} = \begin{cases}(1+\gamma_{ij})(1+\gamma_{ii})^{-1} & \text{for } i = j, \dots, i-1 \\ (1+\gamma_{ii})^{-1}(1+\epsilon_i)^{-1} & \text{for } j = i\end{cases}$

$|\delta_{ij}| \le (i-1)u + O(u^2)$

$\delta G_{ij} = g_{ij}\,\delta_{ij} \quad j = 1, \dots, i$

$\therefore \hat{y}_i = \left[b_i - \sum_{j=1}^{i-1} (g_{ij} + \delta g_{ij})\, \hat{y}_j \right] / \left((g_{ii} + \delta g_{ii}) \right)$

Backward error analysis of Gaussian Eliminatn:-

$a_{ij} = \sum_{k=1}^{\min\{i,j\}} l_{ik} u_{kj} = \sum_{i=1}^{j-1} l_{ik} u_{kj} + l_{ij} u_{jj}$

$l_{ij} = \left(a_{ij} - \sum_{i=1}^{j-1} l_{ik} u_{kj} \right) / u_{jj}$

$\hat{l}_{ij} = fl \Big[a_{ij} - \sum_{k=1}^{j-1} \hat{l}_{ik} u_k$

In book

in slides

(11) $\rightarrow$ $A + \delta A = L_c U_c \rightarrow ① \; |\delta A| \leq 2nu |L_c||U_c| + O(u^2)$

$(L_c + \delta L_c) y_c = b \rightarrow ② \; |\delta L_c| \leq 2nu |L_c| + O(u^2)$

$(U_c + \delta U_c) x_c = y_c \rightarrow ③ \; |\delta U_c| \leq 2nu |U_c| + O(u^2)$

② in ③ $\Rightarrow (L_c + \delta L_c)(U_c + \delta U_c) x_c = b \Rightarrow \left[L_c U_c + L_c \delta U_c + \delta L_c U_c + \delta L_c \delta U_c \right] x_c = b$

$\Big[A + \underbrace{\delta A + L_c \delta U_c + \delta L_c U_c + \delta L_c \delta U_c}_{E} \Big] x_c = b$

$|E| \leq |\delta A| + |L_c||\delta U_c| + |\delta L_c||U_c| + |\delta L_c||\delta U_c|$

$= 2nu + 2u$

2/9/24

$$|\delta A| \leq 2nu\,|G_c^T||G_c| + O(u^2)$$

$$\Rightarrow \|\delta A\|_F \leq 2nu\,\|G_c^T\|_F\|G_c\|_F + O(u^2)$$
$$= 2nu\,\|G_c\|_F^2 + O(u^2)$$

$$\|G_c\|_F = tr(G_c^T G_c) = tr(A+\delta A) = tr(A) + tr(\delta A)$$
$$= \sum a_{ii} + \sum \delta a_{ii}$$
$$\leq \sum |a_{ii}| + \sum |\delta a_{ii}|$$
$$\leq \sqrt{n}\left(\sum |a_{ii}|^2\right)^{1/2} + \sqrt{n}\left(\sum |\delta a_{ii}|^2\right)^{1/2}$$
$$\leq \sqrt{n}\left(\|A\|_F + \|\delta A\|_F\right)$$

$$\therefore \|\delta A\|_F \leq 2nu\sqrt{n}\left(\|A\|_F + \|\delta A\|_F\right) + O(u^2)$$

$$\Rightarrow \|\delta A\|_F\left(1 - 2n^{3/2}u\right) \leq 2n^{3/2}u\,\|A\|_F + O(u^2)$$

Least Squares Problem:-

$$[A_1 \cdots A_m]\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = \sum x_i A_i \in col(A) = Range(A) = \{Ay \mid y \in \mathbb{R}^m\}$$

Linear map defⁿ:- A is linear map iff
1) $A(x+y) = Ax + Ay$
2) $A(\alpha x) = \alpha Ax$

$$S^\perp = \{u \in \mathbb{R}^n, \langle u, s\rangle = 0 \ \forall\ s \in \mathbb{R}^n\}$$

~~A~~ 3/9/24

Normal Eqⁿs Method:- TFAE:-
1) rank A = m $\longrightarrow$ $Ax \neq 0\ \forall x \Rightarrow$
2) A^TA is non singular
~~3) A^TA~~

Condensed QR decompositⁿ:-

$$A = QR$$

non-singular

an Isometry, obtained by taking linearly independent columns of non-singular unitary matrix so non-singular

$\therefore$ R is also non-singular

$\langle Qx, Qy \rangle = (Qy)^*(Qx) = y^* Q^* Q x = y^* x = \langle x, y \rangle$

$$\|QB\|_2 = \max_{x \neq 0} \frac{\|QBx\|}{\|x\|} = \max_{x \neq 0} \frac{\|Bx\|}{\|x\|} = \|B\|_2$$

$$\|Q\|_2 = \max_{x \neq 0} \frac{\|Qx\|}{\|x\|} = \max_{x \neq 0} \frac{\|x\|}{\|x\|} = 1$$

$$\kappa_2(Q) = \frac{\text{maxmag}(Q)}{\text{minmag}(Q)} = \frac{\max \frac{\|Qx\|_2}{\|x\|_2}}{\min \frac{\|Qx\|_2}{\|x\|_2}}$$ but $\|Qx\|_2 = \|x\|_2 \;\forall x$

$= 1$

$\rightarrow QQ^*$ is the orthogonal projexn onto span$\{q_1, \dots, q_m\}$

Pf:- $QQ^* q_i$

$$= [q_1 \; q_2 \; \dots \; q_m] \begin{bmatrix} q_1^* \\ q_2^* \\ \vdots \\ q_m^* \end{bmatrix} q_i = [q_1 q_1^* + q_2 q_2^* + \dots] q_i$$

$= q_i q_i^* q_i + 0 + 0 + \dots$

$= q_i$

$q_i^* q_i = 1$

$q_i^* q_j = 0 \; (i \neq j)$

s||y $QQ^* w = 0$

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Orthogonal Projectⁿ property :-

If $\mathbb{F}^n = U \oplus V$ & if $Px = x_1$ where - - - . Find P

Pf:- $U = [u_1 \dots u_p]$

$P = U(U^*U)^{-1}U^*$. Then $P^2 = P$ & $P^* = P$.

$\underset{i=1:p}{Pu_i} = U(U^*U)^{-1}U^*u_i = U(U^*U)^{-1}U^*Ue_i = Ue_i = u_i$

$v \in U$. Then $v = \sum \alpha_i u_i$ & $Pv = P(\sum \alpha_i u_i) = \sum \alpha_i Pu_i = \sum \alpha_i u_i = v$

$v \in U^\perp$. $Pv = U(U^*U)^{-1}\underbrace{U^*v}_{0} = 0$

In Least squares Problem, If columns of A are l.i,

$P = A(A^*A)^{-1}A^*$

$b_1 = Pb = A\underbrace{(A^*A)^{-1}A^*b}_{x_0}$

---

rank $A = m$

$\underset{n\times m}{A} = \underset{n\times m}{Q}\, R_{m\times m}$ is a condensed QR decompositⁿ

$[Q^TQ = I_m$, R is upper Δ^{lar} & $m\times m]$

$$[A_1 \dots A_m] = [q_1 \dots q_m]\begin{bmatrix} r_{11} & - & \dots & r_{1m} \\ & r_{22} & \dots & r_{2m} \\ & & \ddots & \vdots \\ & & & r_{mm}\end{bmatrix}$$

$A_1 = q_1 r_{11}$, $A_2 = r_{12}q_1 + r_{22}q_2$, ...

$\mathrm{span}\{A_1, \dots, A_k\} = \mathrm{span}\{r_1 q_1, \sum_{i=1}^{2} r_{i2}q_i, \dots, \sum_{i=1}^{k} r_{ik}q_i\}$

$\subseteq \mathrm{span}\{q_1, \dots, q_k\}$

$Q = AR^{-1} \Rightarrow$ Here now R^{-1} is upper Δ^{le}, same as above how we proved span

$\mathrm{span}\{q_1, \dots, q_k\} \subseteq \mathrm{span}\{A_1, \dots, A_k\}$

$AR = Q$

full rank

full rank

$\therefore R$ is full rank

basis

$U P = [u_1\ u_2 \dots u_p]$