Indian Institute of Technology Guwahati

QUIZ -1, September 1, 2022

MA201: Mathematics III

Answers without proper justifications will fetch zero marks.

Time: 50 Minutes (8.00AM -8.50 AM) Marks: 10

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1. Let $f : B(0,1) \to \mathbb{C}$ defined by [2]

$$f(z) = \begin{cases} e^{-\dfrac{1}{\sin z}} & \text{if } z \neq 0, \\ 0 & \text{if } z = 0. \end{cases}$$

Check the continuity of f at $z = 0$.

Answer: Given function f is not continuous at $z = 0$.

Path-I(Limit along Real axis) Take $z_n = \frac{1}{n} \to 0$ an $n \to \infty$. Then

$$\lim_n f(z_n) = \lim_n e^{-\frac{1}{\sin(1/n)}} = 0.$$

Path-II(Limit along Imaginary axis) Take $z_n = \arcsin(\frac{i}{n}) \to 0$ an $n \to \infty$. Then

$$\lim_n f(z_n) = \lim_n e^{in} \neq 0.$$

So the limit is different along different paths and hence limit does not exists. So f is not continuous at $z = 0$.

2. Find all entire functions $f = u + iv$ satisfying $u_x v_y - u_y v_x + 1 = 2u_x$. [3]

Answer: If we use C-R equations in the given equation we will get $|f'(z) - 1| = 0$. This implies that $f'(z) = 1$ for all $z \in \mathbb{C}$. Define $g(z) = f(z) - z$. Then $g'(z) = 0$ for all $z \in \mathbb{C}$. So $g(z) = \alpha$ for all $z \in \mathbb{C}$ and $f(z) = z + \alpha$ for all $z \in \mathbb{C}$.

3. Let f be an entire function such that the image of f lies in $L = \{\log(1+|x|)+iy : x, y \in \mathbb{R}\}$. If $f(0) = 1$, then find the value of $f(1)$. Justify your answer. [3]

Answer: Let $f = u + iv$ be an entire function such that image of f lies in $L = \{\log(1 + |x|) + iy : y \in \mathbb{R}\}$ i.e Re(f)= $u \geq 0$. Consider

$$|e^{-f}| = |e^{-u-iv}| = |e^{-u}| \leq 1.$$

So by Liouville's theorem e^{-f} is constant. So $\frac{d}{dz}e^{-f(z)} = -f'(z)e^{-f(z)} = 0$. That would imply $f(z) = c$ for all $z \in \mathbb{C}$. Since $f(0) = 1$, $f(z) = 1$ for all $z \in \mathbb{C}$.

4. Evaluate the integral [2]

$$\int_{|z|=1} e^{i \sin z} d\bar{z}.$$

Answer: Notice that $|z|^2 = z\bar{z} = 1$. So $\bar{z} = \frac{1}{z}$ and $d\bar{z} = -\frac{1}{z^2}dz$. So by Cauchy integral formula

$$\int_{|z|=1} e^{i \sin z} d\bar{z} = \int_{|z|=1} e^{i \sin z} \left(-\frac{1}{z^2}\right) dz = -2\pi i e^{i \sin 0} i \cos 0 = 2\pi.$$

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