Remark:

If $X \sim N_2(\mu, \Sigma)$ and $Cov(X_1, X_2) = 0$, then X_1 and X_2 are independent.

$Cov(X_1, X_2) = 0$

$$\Sigma = \begin{bmatrix} \sigma_{11} & 0 \\ 0 & \sigma_{22} \end{bmatrix}$$

So, $M_X(t) = e^{t'\mu + \frac{1}{2}t'\Sigma t}$

$$= e^{t_1\mu_1 + \frac{1}{2}t_1^2\sigma_{11}} \, e^{t_2\mu_2 + \frac{1}{2}t_2^2\sigma_{22}}$$

$$= M_{X_1}(t_1)\, M_{X_2}(t_2)$$

So, X_1, X_2 are independent.

$$t'\mu = (t_1\ t_2)\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix} = t_1\mu_1 + t_2\mu_2$$

$$(t_1\ t_2)\begin{pmatrix}\sigma_{11} & 0 \\ 0 & \sigma_{22}\end{pmatrix}\begin{pmatrix}t_1 \\ t_2\end{pmatrix} = t_1^2\sigma_{11} + t_2^2\sigma_{22}$$

$X \sim N_2(\mu, \Sigma)$

BVN $\begin{pmatrix}X_1 \\ X_2\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_1 \\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2\end{pmatrix}\right)$

MVN $\begin{pmatrix}X_1 \\ X_2 \\ \vdots \\ X_n\end{pmatrix} \sim N\left(\begin{pmatrix}\mu_1 \\ \mu_2 \\ \vdots \\ \mu_n\end{pmatrix}, \begin{pmatrix}\sigma_{11} & \sigma_{12} & \cdots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2n} \\ \vdots & & & \\ \sigma_{n1} & \sigma_{n2} & \cdots & \sigma_{nn}\end{pmatrix}\right)$

Multi-Linear regression

$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \varepsilon_i \qquad \varepsilon_i \sim N(0,1) \quad i = 1, \cdots, n$

$\begin{pmatrix}Y_1 \\ Y_2 \\ \vdots \\ Y_n\end{pmatrix} \sim$ MVN

classmate

Theorem:

Let $X \sim N_2(\mu, \Sigma)$ be such that Σ is invertible, then for all $x \in \mathbb{R}^2$, X has joint PDF given by

$$f(x) = \frac{1}{2\pi |\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)}$$

$$= \frac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}} e^{A(x, y, \mu_x, \mu_y, \sigma_x, \sigma_y, \rho)}$$

$$A(x, y, \mu_x, \mu_y, \sigma_x, \sigma_y, \rho) = \frac{1}{2(1-\rho^2)} \left\{ \left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho \left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \right\}$$

$$A = -\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)$$

$$\Sigma = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}. \quad \Sigma^{-1} = \frac{1}{\sigma_x^2\sigma_y^2(1-\rho^2)} \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix}$$

$$|\Sigma| = \sigma_x^2\sigma_y^2(1-\rho^2)$$

$$|\Sigma|^{1/2} = \sigma_x\sigma_y\sqrt{1-\rho^2}$$

$$A = -\frac{1}{2} \frac{[x-\mu_x, y-\mu_y]}{\sigma_x^2\sigma_y^2(1-\rho^2)} \begin{bmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_x^2 \end{bmatrix} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}$$

classmate

$\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}$

Theorem:

Let $X \sim N_2(\mu, \Sigma)$ be such that Σ is invertible, then

1) for all $y \in \mathbb{R}$, Conditional PDF of X given $Y=y$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$= \frac{\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\left\{\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right\}\right\}}}{\frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left(\frac{x-\mu_x}{\sigma_x} - \rho\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x\sqrt{1-\rho^2}} e^{\frac{-1}{2\sigma_x^2(1-\rho^2)}\left(x-\mu_x - \rho\frac{\sigma_x}{\sigma_y}(y-\mu_y)\right)^2}$$

$\sigma_{X|Y} = \sigma_x^2(1-\rho^2)$ $\quad \mu_{x|y} = \mu_x + \rho\left(\frac{\sigma_x}{\sigma_y}\right)(y-\mu_y)$

conditional variance
independent of y

Theorem: Let $X_1, X_2, \dots, X_n$ be iid $N(0,1)$ RVs Then

$$\sum X_i^2 \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right) \equiv \chi_n^2$$

MGF of X_1^2 $\quad M_{X_1^2}(t) = E(e^{tX_1^2}) = \int_{-\infty}^{\infty} e^{tx^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx$

classmate

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(1-2t)x^2} dx$$

$$= \frac{1}{\sqrt{1-2t}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2} dy \qquad y = \sqrt{1-2t}\, x$$

$$= \left(1 - \frac{t}{1/2}\right)^{-1/2}$$

MGF of $\sum_{i=1}^{n} x_i^2 = \left(1 - \frac{t}{1/2}\right)^{-n/2} \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$

χ_n^2

n degrees of freedom

$\text{MGF} = \left(1 - \frac{t}{\beta}\right)^{-\alpha}$

$$f_X(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & x > 0 \\ 0 & \text{ow} \end{cases}$$

<u>Theorem</u>: Let $X_1, X_2, X_3, \ldots, X_n$ be iid $N(\mu, \sigma^2)$ RVs. Then

$\overline{X} \sim N(\mu, \sigma^2/n)$, $\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2$ and $\overline{X}$ & S^2 are indendently distributed.

$$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2$$

Let A be an orthogonal matrix with first row as $\left(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}\right)$ It can be shown that such matrix exist ~~It can be show~~ $A'A = I$, $A^{-1} = A'$

Now consider $Y = AX$

$X = (X_1, X_2, \ldots, X_n)$

$Y = (Y_1, Y_2, \ldots, Y_n)$

classmate

Here $g(x) = Ax$ is one-one as A^{-1} exists.
The Jacobian of inverse transformation $J = \det(A^{-1})$
$|J| = 1$, as A is orthogonal.
→ The JPDF of X, for $x \in \mathbb{R}^n$ $x = (x_1, x_2, \ldots x_n)$.

$$f_X(x) = \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2}$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}(x-\mu)'(x-\mu)}$$

$\mu = (\mu_1, \mu_2 \ldots \mu_n)$
$= (\mu\, \mu \ldots \mu)$

Now, JPDF of Y for $y \in \mathbb{R}^n$ is

$$f_Y(y) = f_X(A'y)\, |J|$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}(A'y-\mu)'(A'y-\mu)}$$

$$= \frac{1}{(\sigma\sqrt{2\pi})^n} e^{-\frac{1}{2\sigma^2}(y-\eta)'(y-\eta)}.$$

$$(A'y-\mu)'(A'y-\mu) = (A'(y-A\mu))'A'(y-A\mu)$$

$$= (y-A\mu)'AA'(y-A\mu)$$

$$= (y-\eta)'(y-\eta) \qquad \eta = A\mu$$

$\eta = (\eta_1\, \eta_2\, \eta_3 \ldots \eta_n)$ & $\eta_1 = \sqrt{n}\mu$

$$\eta'\eta = \mu'A'A\mu = \mu'\mu = n\mu^2$$

$$\Rightarrow \sum_{i=1}^{n}\eta_i^2 = n\mu^2$$

$$\Rightarrow \sum_{i=2}^{n}\eta_i^2 = n\mu^2 - \eta_1^2 = n\mu^2 - n\mu^2$$

$$= 0$$

So, all others are zero

classmate

JPDF of Y for $y \in \mathbb{R}^n$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2\sigma^2}(y_1 - \sqrt{n}\mu)^2} \prod_{i=2}^{n} \left(\frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-y_i^2}{2\sigma^2}} \right).$$

So, $Y_1, Y_2 \ldots Y_n$ are independents RV's with $Y_1 \sim N(\sqrt{n}\mu, \sigma^2)$ & $Y_i \sim N(0, \sigma^2)$ $i = 2, 3, \ldots n$

$$\Rightarrow \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \sim N(\sqrt{n}\mu, \sigma^2)$$

$$\Rightarrow \sqrt{n}\,\bar{X} \sim N(\sqrt{n}\mu, \sigma^2)$$

$$E(\sqrt{n}\bar{X}) = \sqrt{n}\mu \qquad V(\sqrt{n}\bar{X}) = \sigma^2$$

$$\cancel{\sqrt{n}} E(\bar{X}) = \cancel{\sqrt{n}}\mu \qquad \Rightarrow n V(\bar{X}) = \sigma^2$$

$$V(\bar{X}) = \sigma^2/n$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$Y'Y = (AX)'(AX) = X'X$$

$$\Rightarrow \sum_{i=1}^{n} Y_i^2 = \sum_{i=1}^{n} X_i^2$$

$$\Rightarrow \sum_{i=2}^{n} Y_i^2 = \sum_{i=1}^{n} X_i^2 - Y_1^2 = \sum_{i=1}^{n} X_i^2 - n\bar{X}^2 = (n-1)S^2$$

$$(n-1)S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$$

$$= \sum_{i=1}^{n} (X_i^2 - 2\bar{X}X_i + \bar{X}^2)$$

$$= \sum_{i=1}^{n} X_i^2 - 2n\bar{X}^2 + n\bar{X}^2$$

$$= \sum_{i=1}^{n} X_i^2 - n\bar{X}^2$$

$$\Rightarrow \frac{(n-1)S^2}{\sigma^2} = \sum_{i=2}^{n} \left(\frac{y_i}{\sigma}\right)^2$$

$$\Rightarrow \frac{y_i}{\sigma} \sim N(0,1)$$

$$\sim \chi^2_{n-1}$$

classmate

Multinomial distribution

Def: Consider n independent trials, each of which results in one of the outcomes 1, 2, .., r with respective probabilities $p_1, p_2, .., p_r$, where $\sum_{i=1}^{r} p_i = 1$. let N_i be no of trials that outcome i.
Then

$(N_1, .., N_r)$ is said to have Multinomial distribution.

Thm: JPMF of $(N_1, .., N_r)$ is

$$f(n_1, n_2, \ldots, n_r) = \begin{cases} \binom{n}{n_1, n_2, \ldots, n_r} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r} & \text{for } n_1 \geq 0, \ldots, n_r \geq 0, \sum_{i=1}^{r} n_i = n \\ 0 & \text{ow} \end{cases}$$

Prob $\quad p_1 \quad p_2 \quad \ldots \quad p_r$

Mult $\quad 1 \quad 2 \quad \ldots \quad r$

$n_1 + n_2 + \ldots + n_r = n$

Bin $\quad 1 \quad 2 \quad (r=2)$

$n_1 + n_2 = n$

Notation: $\text{Mult}(n, p_1, p_2, .., p_r)$.

Theorem:

$N_i \sim \text{Bin}(n, p_i) \quad i = 1, 2, \ldots r$

$$\sum \frac{p_2 + \ldots + p_r}{1-p_1} = 1$$

$$f(n_1) = \sum_{n_2 n_3 \ldots n_r} \frac{n!}{n_1! n_2! \ldots n_r!} p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$$

$$= \frac{n! \, p_1^{n_1} (1-p_1)^{n-n_1}}{n_1! (n-n_1)!} \sum_{n_2 n_3 \ldots n_r} \frac{(n-n_1)!}{n_2! n_3! \ldots n_r!} \frac{p_2^{n_2} \cdots p_r^{n_r}}{(1-p_1)^{n-n_1}}$$

classmate

$$= \binom{n}{n_1} p_1^{n_1} (1-p_1)^{n-n_1}$$

$$N_1 \sim Bin(n, p_1)$$

$$N_i \sim Bin(n, p_i),\ i = 1, 2, \dots, r$$

Theorem:

Let $\{i_1, \dots, i_k\} \subset \{1, 2, \dots, r\}$. Then JPMF of $(N_{i_1}, N_{i_2}, \dots, N_{i_k})$ is

$$f(n_{i_1}, n_{i_2}, \dots, n_{i_k}) = \begin{cases} \dfrac{n!}{w!\, n_{i_1}!\, n_{i_2}! \cdots n_{i_k}!} \left(1 - \sum_{s=1}^{k} p_{i_s}\right)^{w} p_{i_1}^{n_{i_1}} \cdots p_{i_k}^{n_{i_k}} & \text{if } n_{i_1} \geq 0, \dots, n_{i_k} \geq 0 \text{ \& } \sum_{s=1}^{k} n_{i_s} \leq n \\ 0 & \text{o.w.} \end{cases}$$

$$w = n - \sum_{s=1}^{k} n_{i_s}$$

Theorem: k, l be natural nos s.t. $k + l = r$. Let

$A = \{i_1, \dots, i_k\}$ $B = \{j_1, \dots, j_l\}$

classmate

$\omega = n - \sum (n_s)$

$n_1 \quad n$

Theorem: $Cov(N_i, N_j) = -np_i p_j$

$$Cov(N_1, N_2) = E(N_1 N_2) - E(N_1)E(N_2)$$
$$= E(N_1 N_2) - (np_1)(np_2)$$
$$= E(N_1 N_2) - n^2 p_1 p_2$$

$$E(N_1 N_2) = \sum_{n_1 n_2} n_1 n_2 \frac{n!}{n_1! n_2! (n - n_1 - n_2)!} p_1^{n_1} p_2^{n_2} (1 - p_1 - p_2)^{n - n_1 -}$$

$$= n(n-1) p_1 p_2 \sum_{n_1, n_2} \frac{(n-2)!}{(n_1 - 1)! (n_2 - 1)! (n - 2 - (n_1 - 1) - (n_2 - 1))} p_1^{n_1 - 1} p_2^{n_2 -} (1 - p_1 - p_2)^{n - 2 - (n_1 - 1) - (}$$

$$= n(n-1) p_1 p_2$$
$$= n^2 p_1 p_2 - n p_1 p_2$$
$$\Rightarrow Cov(N_1, N_2) = -n p_1 p_2$$

characteristic function $\phi_x(t) = E(e^{itx})$