QR Decompositions of Matrices

QR decomposition of matrices

QR Decomposition: Given any matrix $A \in \mathbb{R}^{n \times m}$, $n \geq m$, there exists an orthogonal matrix $Q \in \mathbb{R}^{n \times n}$ and an upper triangular matrix $R \in \mathbb{R}^{n \times m}$ such that

$$A = QR. \tag{2}$$

The decomposition (2) is called a QR decomposition of A.

If $n > m$, then $R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ where $R_1 \in \mathbb{R}^{m \times m}$ is upper triangular.

In particular if $n = m$, then (2) takes the form $A = QR$ where R is a square upper triangular matrix.

If $A \in \mathbb{C}^{n \times m}$, $n \geq m$, then (2) holds with $\mathbb{R}$ replaced by $\mathbb{C}$, Q being a unitary matrix.

Condensed QR decomposition

Given $A \in \mathbb{R}^{n \times m}$ with $n > m$, if $A = QR$ be a QR decomposition of A with $R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$, then partitioning $Q = [Q_1 \ \ Q_2]$ where $Q_1 \in \mathbb{R}^{n \times m}$, gives,

$$A = QR = [Q_1 \ \ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1 .$$

This motivates the following theorem.

Theorem Given any $n \times m$ matrix A with $n > m$, there exists an isometry $Q \in \mathbb{R}^{n \times m}$ and an upper triangular matrix R such that

$$A = QR. \qquad ((3))$$

If $\operatorname{rank} A = m$, then R is nonsingular.

The decomposition in (3) is called a *condensed QR decomposition of A*.

Unitary/Orthogonal matrices

Unitary matrices are square matrices such that $Q^* = Q^{-1}$.
A real unitary matrix is called an orthogonal matrix.

Unitary/Orthogonal matrices

Unitary matrices are square matrices such that $Q^* = Q^{-1}$.
A real unitary matrix is called an orthogonal matrix.

Evidently, a square matrix Q is unitary (orthogonal) if and only if its columns form an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{C}^n$).

Unitary/Orthogonal matrices

Unitary matrices are square matrices such that $Q^* = Q^{-1}$.
A real unitary matrix is called an orthogonal matrix.

Evidently, a square matrix Q is unitary (orthogonal) if and only if its columns form an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{C}^n$).

They play a very important role in matrix computations due to their nice properties. In the following let Q be an $n \times n$ unitary matrix.

Unitary/Orthogonal matrices

Unitary matrices are square matrices such that $Q^* = Q^{-1}$.
A real unitary matrix is called an orthogonal matrix.

Evidently, a square matrix Q is unitary (orthogonal) if and only if its columns form an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{C}^n$).

They play a very important role in matrix computations due to their nice properties. In the following let Q be an $n \times n$ unitary matrix.

- $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{C}^n$.
- $\|Qx\|_2 = \|x\|_2$.
- $\|QB\|_2 = \|B\|_2$ for any $B \in \mathbb{C}^{n\times m}$.
- $\|Q\|_2 = 1$ and $\|Q\|_F = \sqrt{n}$.
- $\kappa_2(Q) = 1$.
- Q^*AQ is Hermitian if A is Hermitian.
- If A is real symmetric and Q is orthogonal, then Q^TAQ is also real symmetric.
- In the presence of rounding errors, $fl(QA) = Q(A + E)$ where $\|E\|_2/\|A\|_2$ is $O(u)$.

Unitary/Orthogonal matrices

Unitary matrices are square matrices such that $Q^* = Q^{-1}$.
A real unitary matrix is called an orthogonal matrix.

Evidently, a square matrix Q is unitary (orthogonal) if and only if its columns form an orthonormal basis of $\mathbb{R}^n$ ($\mathbb{C}^n$).

They play a very important role in matrix computations due to their nice properties. In the following let Q be an $n \times n$ unitary matrix.

- $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{C}^n$.
- $\|Qx\|_2 = \|x\|_2$.
- $\|QB\|_2 = \|B\|_2$ for any $B \in C^{n \times m}$.
- $\|Q\|_2 = 1$ and $\|Q\|_F = \sqrt{n}$.
- $\kappa_2(Q) = 1$. Prove these properties!
- $Q^* AQ$ is Hermitian if A is Hermitian.
- If A is real symmetric and Q is orthogonal, then $Q^T AQ$ is also real symmetric.
- In the presence of rounding errors, $fl(QA) = Q(A + E)$ where $\|E\|_2 / \|A\|_2$ is $O(u)$.

Isometry

A matrix $Q \in \mathbb{C}^{n \times m}$, or $\mathbb{R}^{n \times m}$ with $n > m$, is said to be an isometry if $Q^* Q = I_m$.

Isometry

A matrix $Q \in \mathbb{C}^{n\times m}$, or $\mathbb{R}^{n\times m}$ with $n > m$, is said to be an isometry if $Q^*Q = I_m$.

Clearly a matrix Q with more rows than columns is an isometry if and only if its columns form an orthonormal set in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Isometry

A matrix $Q \in \mathbb{C}^{n\times m}$, or $\mathbb{R}^{n\times m}$ with $n > m$, is said to be an isometry if $Q^*Q = I_m$.

Clearly a matrix Q with more rows than columns is an isometry if and only if its columns form an orthonormal set in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Isometries have properties very similar to that of unitary matrices. Given an $n \times m$ isometry $Q = [q_1 \cdots q_m]$,

- $\langle Qx, Qy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$.
- $\|Qx\|_2 = \|x\|_2$.
- $\|QB\|_2 = \|B\|_2$ for any $B \in \mathbb{C}^{n\times m}$.
- $\|Q\|_2 = 1$ and $\|Q\|_F = \sqrt{m}$.
- $\kappa_2(Q) = 1$.
- In the presence of rounding errors, $fl(QA) = Q(A + E)$ where $\|E\|_2/\|A\|_2$ is $O(u)$.
- QQ^* is the orthogonal projection onto $\mathrm{span}\,\{q_1, \ldots, q_m\}$, that is, $QQ^*v = v$ for all $v \in \mathrm{span}\,\{q_1, \ldots, q_m\}$ and $QQ^*w = 0$ for all $w \in \{q_1, \ldots, q_m\}^\perp$. Prove this!