Present discounted value

- Suppose 100 rupees is available to a man today. He invests it in a business which fetches 20% return per year.
- After 5 years the money will accumulate to $100(1+0.2)^5 = 100(1.2)^5 = 100(2.49) = 249$ rupees.
- Thus any given amount of money today is equivalent to more money at a future date. This is a fundamental principle of financial economics.
- In the above example 100 rupees is the **present value** of 249 rupees five years later, at 20% rate of return per year.
- It is also called the **present discounted value (PDV)** of 249 rupees because, after all, 100 rupees is less than 249 rupees.

Total Present Value

- Suppose a businessman has to make four payments in four different intervals.
- Rupees 100 after 1 year
- Rupees 200 after 2 years
- Rupees 300 after 3 years
- Rupees 400 after 4 years
- 20% per year is the rate of return he earns.
- How much money must he invest today to make these payments in future dates?

- We can generalise this as follows. Suppose a man needs to meet n payments after the next n years in the following way.
- After year 1, a_1
- After year 2, a_2
- ...
- After year n, a_n
- The rate of return or interest rate on bank deposits, is $p\%$ per year. Let, $p/100 = r$.

- Then the present value of these *n* installments is given by,

$$P_n = \frac{a_1}{(1+r)} + \frac{a_2}{(1+r)^2} + \cdots + \frac{a_n}{(1+r)^n}$$

Or, $P_n = \sum_{i=1}^{n} \frac{a_i}{(1+r)^i}$

- **Annuities:** An **annuity** is a sequence of equal payments made at fixed periods of time over some time span. If the payments to be made in each year are equal, then, $a_1 = a_2 = .. = a_n = a$, say.

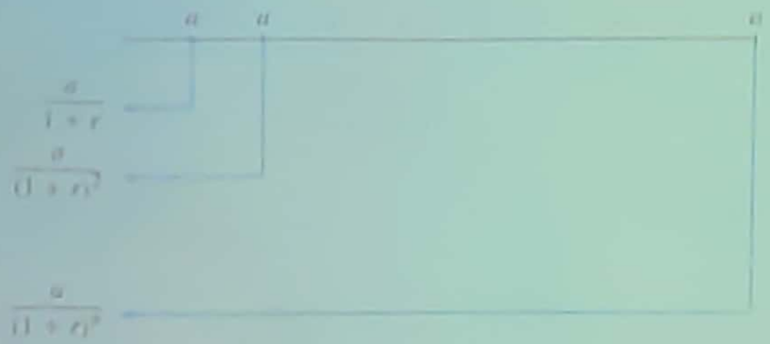
Therefore, $P_n = \frac{a}{(1+r)} + \frac{a}{(1+r)^2} + \cdots + \frac{a}{(1+r)^n}$

This geometric series has the summation,

$$= \frac{a}{1+r} \frac{1-\frac{1}{(1+r)^n}}{1-\frac{1}{1+r}}$$

Or, $P_n = \frac{a}{r}(1 - \frac{1}{(1+r)^n})$

This is the **present value** of an annuity of *a* rupees per payment period, for *n* periods, where the first payment has to be made one year from now, and the remaining amounts at intervals of one year, the rate of interest is *r*.



$\frac{a}{r}(1 - \frac{1}{(1+r)^n})$ gives the PV of *n* future claims of *a* each (as shown above).

On the other hand, one can look at the total value after the last payment has been made. This is called the **future value**:

$$F_n = a + a(1 + r) + a(1 + r)^2 + \cdots a(1 + r)^{n-1}$$

Or $F_n = \dfrac{a((1+r)^n - 1)}{r}$

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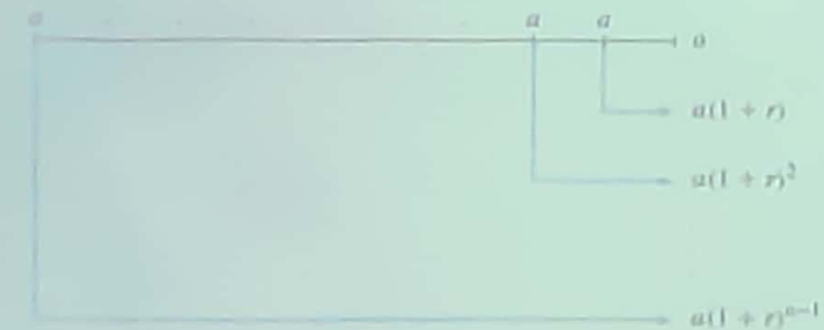
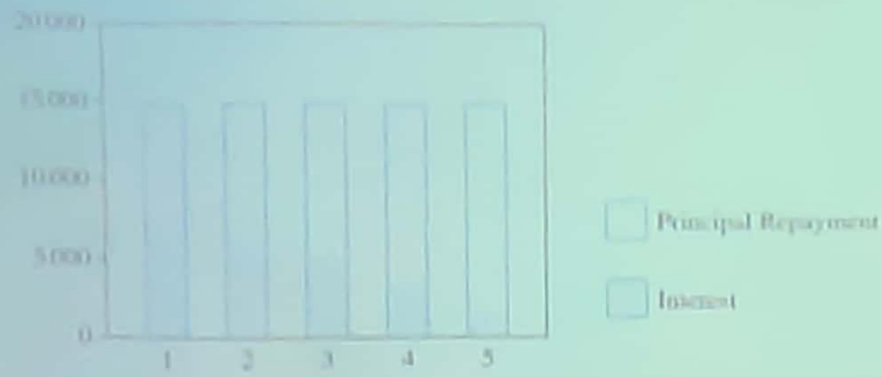


Mortgage (housing loan) Repayment

- When someone takes a loan and pays back on equal instalments of a per year over n number of years, then the payments are like an annuity. The same formula applies.
- Each year, a part of a goes to pay the interest on the outstanding loan. And the rest to pay back a fraction of the principal.
- With time, the first component falls (since the outstanding loan falls), while the second fraction (principal repayment) rises.
- A man takes a mortgage of 50000 rupees at 15% annual rate of interest to be paid back in 5 equal annual instalments of a each.
- Using, $P_n = \frac{a}{r}(1 - \frac{1}{(1+r)^n})$, $50000 = \frac{a}{0.15}(1 - \frac{1}{(1.15)^5})$ which gives,
- $a \approx 14{,}915.78$



- If K is the loan amount, we can use $K = \frac{a}{r}(1 - \frac{1}{(1+r)^n})$ to get,

$$a = \frac{rK}{1-(1+r)^{-n}}$$

Example: a house loan has the value of 1,000,000 rupees today. The borrower will pay back with equal annual amount over the next 10 years, the first payment after the first year from now. The rate of interest is 12% per year. What is going to be the amount of annual payment?

- These were cases of **ordinary annuities**, where payment is made at the end of a period. In **annuity due** the payment is made in the beginning of a period. So, PDV = $a + \frac{a}{(1+r)} + \frac{a}{(1+r)^2} + \cdots + \frac{a}{(1+r)^{n-1}}$

$= a + \frac{a}{r}(1 - \frac{1}{(1+r)^{n-1}})$

- In some case the lender specifies the amount she will pay a year, and let the payments continue till the full payment is done. This is also a case of annuities.
- In such cases one may want to estimate **the time** it would take to make the full payment.
- Using, $K = \frac{a}{r}(1 - \frac{1}{(1+r)^{n}})$ we get,

- $\frac{rK}{a} = 1 - \frac{1}{(1+r)^n}$ which simplifies to,

$(1 + r)^n = \frac{a}{a-rK}$. Taking log natural and manipulating terms gives,

$n = \frac{lna-\ln(a-rK)}{\ln(1+r)}$

- This gives the number of years in fractions. Hence some adjustment payment has to be made.
- Example: A loan of 50000 rupees is to be repaid with 20000 rupees annuity at the end of each coming year, with 15% annual interest rate.
- Here, $K = 50000, a = 20000, r = 0.15$. Using $n = \frac{lna-\ln(a-rK)}{\ln(1+r)}$ we get,

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Deposit within an interest period

- Banks usually have an interest period of one year. If a depositor deposits within an interest period, the bank will use simple interest, not compound interest.
- After the period the principal rises to, **$P(1+rt)$**, where r is the annual interest rate, t is the remaining fraction of the interest period.

Example: A depositor pays 100 rupees on 31st December, 2000; 100 rupees on 30th June, 2001; 100 rupees on 31st December, 2001. The annual interest rate is 10%. This pattern is repeated in 2002. What is the total deposit amount on 31st December 2002?

Bonds

- Someone invests P amount of money on a bond, which pays an amount of a each year for n years (**coupon payment**). a/P is called the **coupon rate**. P is the loan of the buyer to the seller of the bond. a is the annual interest.
- At the end of the period, the borrower pays that year's a and the principal, P (**face value**). Suppose the opportunity cost is interest rate r (the return than can be obtained elsewhere). The present value of the cash flows at r is called the bond price: PV = P_b, price of bond = $\frac{a}{1+r} + \frac{a}{(1+r)^2} + \cdots + \frac{a+P}{(1+r)^n}$

$$= \frac{a}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{P}{(1+r)^n}$$

- One can alternatively ask: if PV is the price of bond, then what rate does the investor expect. Here it is r. r is also called the **yield to maturity.**

$$P_b = \frac{a}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{P}{(1+r)^n}$$

Thus, $P_b - P = \frac{a}{r}\left(1 - \frac{1}{(1+r)^n}\right) + \frac{P}{(1+r)^n} - P$

$$= \frac{a}{r}\left(1 - \frac{1}{(1+r)^n}\right) - P\left(1 - \frac{1}{(1+r)^n}\right)$$

$$= \left(\frac{a}{r} - P\right)\left(1 - \frac{1}{(1+r)^n}\right)$$

If, $\frac{a}{r} = P$, then, $\frac{a}{P} = r$, coupon rate = yield to maturity. Here $P_b = P$

If, coupon rate > yield to maturity, $P_b > P$, price of bond will be sold at higher than the face value, at a premium. This happens because the bond fetches better return than market. Hence greater demand for the bond, which adjusts the price to the level where both are at par.

On the other hand, if coupon rate < yield to maturity, $P_b < P$, price of bond will be sold at lower than the face value, at a discount.

- Bond price is usually expressed of a percentage of the face value. That is, (PV/P) x 100%

- The price of the bond and the rate of interest *r* are related inversely. As interest rate (i.e., yield) falls, the price of bond rises. As $r \uparrow \rightarrow P_b \downarrow$
- The higher the interest rate that investors demand, the less that they will be prepared to pay for the bond.
- This inverse relation has an implication for the macroeconomic analysis of the asset market. As the government sells its bonds to borrow from the market, price of bond falls (excess supply). By the above relation rate of interest in the market tends to rise.

Evaluating Investment projects

- Suppose there is an investment project which involves the following profits or incomes over a total life time of *n* periods.
- Period 0 $\rightarrow a_0$
- Period 1 $\rightarrow a_1$
- Period 2 $\rightarrow a_2$
- ...
- Period n - 1 $\rightarrow a_{n-1}$
- All these returns are in terms of money. Typically, the amount a_0

would be a large, negative quantity, because the costs of setting up the facility and fixed costs are borne in the initial years.