QR Decomposition Method

Case 1: A is full rank, i.e., $\operatorname{rank} A = m$.

Let $A = Q \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$ be a QR decomposition of A and Q_1 be the isometry formed by the first m columns of Q.

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Then $Q_1 Q_1^T$ is the orthogonal projection onto $R(A)$ or $Col(A)$. Hence, $b_1 = Q_1 Q_1^T b$ and the solution x_0 of the LSP associated with $Ax = b$ satisfies

$$
\begin{aligned}
& Ax_0 = Q_1 Q_1^T b \\
\Rightarrow & Q \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x_0 = Q[e_1 \ \cdots \ e_m][e_1 \ \cdots \ e_m]^T \underbrace{Q^T b}_{:=c} \\
\Rightarrow & \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x_0 = \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}_{n \times n} c \\
\Rightarrow & R_1 x_0 = c_1
\end{aligned}
$$

where c_1 is the vector of first m rows of c.

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$Q = [\boxed{Q_1} \; Q_2]$, $\underbrace{Q_1}_{m}$, $\underbrace{Q_2}_{n-m}$

Ae_i ↑ i-th col of A

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\begin{aligned}
Ax_0 &= Q_1 Q_1^T b \\
\Rightarrow Q \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x_0 &= \underbrace{Q[e_1 \; \cdots \; e_m]}_{=Q_1}\underbrace{[e_1 \; \cdots \; e_m]^T}_{Q_1^T} \underbrace{Q^T b}_{:=c} \\
\Rightarrow \begin{bmatrix} R_1 \\ 0 \end{bmatrix} x_0 &= \begin{bmatrix} I_m & 0 \\ 0 & 0 \end{bmatrix}_{n\times n} c \\
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$\begin{bmatrix} R_1 x_0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} I_m c_1 + 0 c_2 \\ 0 c_1 + 0 c_2 \end{bmatrix}$

where $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \begin{matrix} \}m \\ \}n-m \end{matrix}$

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Case 2: $\operatorname{rank} A = r < m$.

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Exercise: Let $AP = Q \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}$ be a rank revealing QR decomposition of A where $R_1 \in \mathbb{R}^{r \times r}$, $R_2 \in \mathbb{R}^{r \times m-r}$, R_1 being upper triangular and nonsingular. Let $Q_1 = [q_1 \ \cdots \ q_r]$ be the isometry formed by the first r columns of Q. Then prove that

$$Col(A)(= R(A)) = \operatorname{span}\{APe_1, \cdots, APe_r\} = \operatorname{span}\{q_1, \cdots, q_r\} = Col(Q_1).$$

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$$\Rightarrow A = Q\begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}P^T$$

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$$[APe_1 \cdots APe_r \, APe_{r+1} \cdots APe_m] = [\underbrace{Q_1}_{r} \ \underbrace{Q_2}_{n-r}]\begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix} = [Q_1R_1 \ Q_1R_2] \therefore [APe_1 \cdots APe_r] = Q_1R_1$$
and thus is a condensed QR decomposition of $[APe_1 \cdots APe_r]$

Therefore, $Q_1Q_1^T$ is the orthogonal projection onto $Col(A)$ and any solution x_0 of the LSP associated with $Ax = b$ satisfies

$$Ax_0 = Q_1Q_1^Tb \Rightarrow Q\begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}\underbrace{P^Tx_0}_{:=y} = Q[e_1 \ \cdots \ e_r][e_1 \ \cdots \ e_r]^T\underbrace{Q^Tb}_{:=c}$$

$$\Rightarrow \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}_{n\times m}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}_{m\times 1} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}_{n\times n}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}_{n\times 1}$$

c_1, y_1 being the vectors of first r rows of c and y respectively and c_2, y_2 the vector formed by the last $m-r$ rows of y and last $n-r$ rows of c respectively

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For ease of computation choose $y_2 = 0$. Then one solution $x_0 \in \mathbb{R}^m$ satisfying

$$\|b - Ax_0\|_2^2 = \min_{x \in \mathbb{R}^m} \|b - Ax\|_2^2$$

is given by

$$x_0 = P \begin{bmatrix} y_1 \\ 0 \end{bmatrix}$$

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Exercise: If $r = b - Ax_0$ prove that $\|r\|_2 = \|c_2\|_2$.

QR Decomposition Method

Pseudocode for solving the LSP associated with $Ax = b$ via QR decomposition method:

1. Find $R = \begin{bmatrix} R_1 & R_2 \\ 0 & 0 \end{bmatrix}$ of a rank revealing QR decomposition of A and suppose $Q_1, \ldots, Q_r$ are the reflectors and $P_1, \ldots P_r$ are the permutations required in the process. Here $r \leq m$.
   (costs $2nm^2 - \frac{2}{3}m^3 + O(nm) + O(m^2)$ flops if $r = m$.)

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2. Compute $c = Q_r \cdots Q_1 b$ and extract vectors c_1, c_2 from its first entries.
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   (costs $O(nm)$ flops if $r = m$.)
4. Solve $R_1 y = c_1$.
   (costs $O(m^2)$ flops if $r = m$.)
5. Set $x_0 = P_1 \cdots P_r \begin{bmatrix} y \\ 0 \end{bmatrix}_{m \times 1}$.
   (Here the zeroes are necessary only if $r < m$.)