- We know q^* is a stationary point, $\pi'(q^*) = 0$

Or, $\frac{d}{dq}\left(R(q^*) - C(q^*)\right) = 0$

Or, $R'(q^*) = C'(q^*)$

As the producer produces more output, the instantaneous change in the revenue and cost are given by $R'(q^*)$ and $C'(q^*)$ respectively.

The above condition says, the producer has to set his output at a level where the instantaneous change in revenue and cost are equal. If the profit maximizing output exists at a stationary point, that optimal point is characterized by this condition.

- There could be multiple points which satisfy this condition.
- In that case, we evaluate $f(x)$ at those q's and choose that point (q) which gives the maximum value.
- The condition $R'(q^*) = C'(q^*)$ is called **marginal revenue equals to marginal cost**.
- In the special case where marginal revenue = price of the good, p (in a perfect competition market) the condition becomes,

$$p = C'(q^*)$$

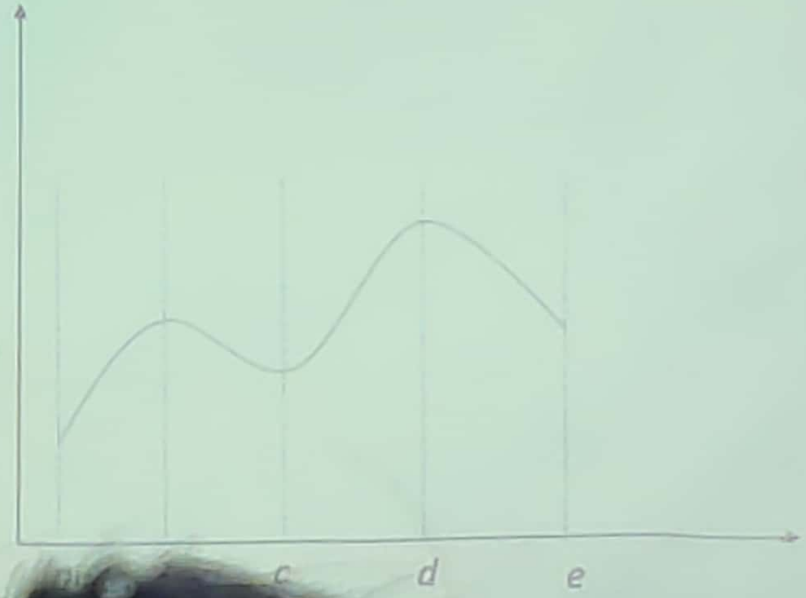
- Optimal output is where marginal cost is equal to the price per unit.

- Suppose, the government imposes a tax t per unit of output on producers.
- The cost of production changes to, $C(q) + tq$
- If q^* is the optimal output (interior), then the necessary condition becomes,

  $R'(q^*) = C'(q^*) + t$
- Since taxes add to the cost, the new condition has the tax rate added to the marginal cost term.
- This new expression $C'(q^*) + t$ has to be equated to marginal revenue.

Local maximum and minimum

- So far we dealt with **global maximum or minimum**. These are extreme values for all points belonging to the domain.
- Sometimes we may be interested in local extreme values as well.
- These are called, **local maximum** and **local minimum**.
- In this diagram, *a, b, c, d, e* are local extreme points. *b* and *d* are local maximum points. *a, c, e* are local minimum points. *d* is global maximum, *a* is global minimum.



- A function f has a **local maximum** at c if there is an interval (a, b) about c such that $f(x) \leq f(c)$ for all those x in the domain that also lie in (a, b).

- A function f has a **local minimum** at c if there is an interval (a, b) about c such that $f(x) \geq f(c)$ for all those x in the domain that also lie in (a, b).

- Correspondingly, $f(c)$ is called **local extreme point / local extreme value**.

- Like before, for local extreme points, following three cases are possible.

1. In the interior of I, where $f'(x) = 0$
2. Endpoints of I.
3. Points in I where $f'(x)$ does not exist (kink points).

Here I is a small interval around the point in question.

- $f'(x) = 0$ is a necessary condition, not a sufficient condition to know if a stationary point is maximum/minimum/neither.
- For global maximum, we compared the value of the function at the stationary points and values at end points.

- A similar method will not be effective in case of local maximum, because the value of a local maximum can in fact be less than the value of a local minimum!
- There are two main ways to determine

if a stationary point is indeed a local

maximum/minimum point.

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The first-derivative test

Suppose c is a stationary point of $y = f(x)$

1. If $f'(x) \geq 0$ throughout some interval (a, c) to the left of c and $f'(x) \leq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local maximum point of f.
2. If $f'(x) \leq 0$ throughout some interval (a, c) to the left of c and $f'(x) \geq 0$ throughout some interval (c, b) to the right of c, then $x = c$ is a local minimum point of f.
3. If $f'(x) > 0$ throughout some interval (a, c) to the left of c and throughout some interval (c, b) to the right of c, then $x = c$ is not a local extreme point of f. Same for $f'(x) < 0$ for both sides of c.

Example: $f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$, find the stationary points and determine if they are local maximum or minimum points.

$$f(x) = \frac{1}{9}x^3 - \frac{1}{6}x^2 - \frac{2}{3}x + 1$$

Or, $f'(x) = \frac{1}{3}x^2 - \frac{1}{3}x - \frac{2}{3} = \frac{1}{3}(x^2 - x - 2) = \frac{1}{3}(x^2 - 2x + x - 2)$

$= \frac{1}{3}(x-2)(x+1)$

Thus, $f'(x) = 0$ at x = -1 and 2.

The second-derivative test

Let f be a twice-differentiable function in an interval I. Suppose c is an interior point of I.

1. $f'(c) = 0$ and $f''(c) < 0$ implies that c is a strict local maximum point.
2. $f'(c) = 0$ and $f''(c) > 0$ implies that c is a strict local minimum point.
3. $f'(c) = 0$ and $f''(c) = 0$ does not allow us to conclude anything.

Example: The necessary condition for profit maximization is $R'(q^*) = C'(q^*) + t$, where q^* is the output at which profit is maximum and t is the tax per unit of output. Suppose $R''(q^*) < 0$, $C''(q^*) > 0$. Find $\frac{dq^*}{dt}$. Show that $\frac{d\pi(q^*)}{dt} = -q^*$.

We know, $\pi(q) = R(q) - C(q) - tq$

The necessary condition: $\frac{d\pi(q)}{dq} = 0$ at q^*

Or, $R'(q^*) - C'(q^*) - t = 0$ [A]

Now, $\frac{d\pi^2}{d^2q} = R''(q^*) - C''(q^*)$ at $q = q^*$

Since, $R''(q^*) < 0$, $C''(q^*) > 0$, $\frac{d\pi^2}{d^2q} = R''(q^*) - C''(q^*) < 0$ at $q = q^*$

Thus, $q = q^*$ is indeed a maximum point.

- Through implicit differentiation of [A] with respect to t we get,

$$R''(q^*)\ \frac{dq^*}{dt} - C''(q^*)\frac{dq^*}{dt} - 1 = 0$$

Or, $\frac{dq^*}{dt} = \frac{1}{R''(q^*) - C''(q^*)}$, which is negative since, $R''(q^*) - C''(q^*) < 0$

The profit maximizing output falls as tax per unit rises.

The profit function at q^* is given by, $\pi(q^*) = R(q^*) - C(q^*) - tq^*$

By implicit differentiation with respect to t we get,

$$\frac{d\pi(q^*)}{dt} = [R'(q^*) - C'(q^*)]\frac{dq^*}{dt} - t\,\frac{dq^*}{dt} - q^* \text{(using the chain rule)}$$

$$= [R'(q^*) - C'(q^*) - t]\frac{dq^*}{dt} - q^*$$

$= -q^*$, since $R'(q^*) - C'(q^*) - t = 0$

Hence, the proof.

The maximized profit falls as tax rate rises.

Example: A tree is planted at time $t = 0$, $P(t)$ is its current market value at time t. It's a differentiable function. r is the rate of interest. $P''(t) < 0$. When should the tree be cut down to maximize the present discounted value with continuous compound discounting?

Let the present value be given by $f(t)$; we know, $f(t) = P(t)e^{-rt}$