Lecture 15 Monday, 5/9/2022

Note Title 9/4/2022

Theorem 1: Let G be a group, and let $\{H_\alpha : \alpha \in \Delta\}$ be a collection of subgroups of G. Then, $\bigcap_{\alpha \in \Delta} H_\alpha$ is again a subgroup of G.

Proof: $e \in H_\alpha \ \forall \alpha \in \Delta \Rightarrow e \in \bigcap_{\alpha \in \Delta} H_\alpha$.

Let $a, b \in \bigcap_{\alpha \in \Delta} H_\alpha$. Then, $a, b \in H_\alpha \ \forall \alpha \in \Delta \Rightarrow ab^{-1} \in H_\alpha \ \forall \alpha \in \Delta$

$\Rightarrow ab^{-1} \in \bigcap_{\alpha \in \Delta} H_\alpha \quad \therefore \bigcap_{\alpha \in \Delta} H_\alpha \leq G$.

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Union of subgroups need not be a subgroup.

Theorem 2: Let H and K be two subgroups of a group. Then, $H \cup K \leq G \Leftrightarrow$ either $H \subseteq K$ or $K \subseteq H$.

__Proof:__ If $H \subseteq K$ or $K \subseteq H$, then $H \cup K = K$ or $H \cup K = H$.

$\therefore$ $H \cup K$ is a subgroup of G.

Conversely, suppose that $H \leq G$, $K \leq G$ and $H \cup K \leq G$.

We prove that either $H \subseteq K$ or $K \subseteq H$.

If possible, suppose that $H \nsubseteq K$. Then, $\exists\, h \in H$ s.t. $h \notin K$.

__Aim:__ $K \subseteq H$. Let $x \in K$. Then, $x, h \in H \cup K$.

$\Rightarrow xh \in H \cup K$ $(\because H \cup K \leq G)$.

__Case I:__ $xh \in H$. Then, $x = xh \cdot h^{-1} \in H$.

__Case II:__ $xh \in K$. Then, $h = x^{-1} \cdot xh \in K$, a contradiction as $h \notin K$.

Thus, $x \in K \Rightarrow x \in H$. $\therefore K \subseteq H$. This completes the proof.

__Ex:__ $U(8)=\{1,3,5,7\}$. Let $H_1=\{1,3\}$, $H_2=\{1,5\}$, $H_3=\{1,7\}$.

Clearly, $H_i \nsubseteq H_j$ if $i \neq j$ and $U(8)=H_1 \cup H_2 \cup H_3$.

Hence, Theorem 3 is not true if we consider more than two subgroups. #

§ __Subgroup generated by a subset X:__ Let G be a group and let X be a non-empty subset of G.

Let $\mathcal{H}(X)=\{H \leq G \mid X \subseteq H\}$ be the collection of all the subgroups of G containing X. Since $G \in \mathcal{H}(X)$, so $\mathcal{H}(X)$ is non-empty.

By Theorem 1, $\bigcap_{H \in \mathcal{H}(X)} H$ is a subgroup of G, which is clearly the smallest subgroup of G containing X.

<u>Definition</u>: Let X be a non-empty subset of G. The smallest subgroup of G containing X is called the subgroup generated by X, and is denoted by $\langle X \rangle$.

- If $X = \{a_1, \dots, a_n\}$ is a finite subset of G, then we write $\langle a_1, \dots, a_n \rangle$ in place of X.

- A group G is called finitely generated if $\exists\, a_1, \dots, a_n \in G$ such that $G = \langle a_1, \dots, a_n \rangle$.

Theorem 3: For a non-empty subset X of a group G, let $X^{-1} = \{x^{-1} : x \in X\}$. Then, the subgroup $\langle X \rangle$ consists of all the finite products of elements of $X \cup X^{-1}$.

That is, if $x \in \langle X \rangle$, then $x = x_1 * x_2 * \cdots * x_n$ where $n \geq 1$ and $x_i \in X \cup X^{-1}$ $\forall i$.

Proof: Let $F = \{ x_1 * \cdots * x_n \mid n \geq 1,\ x_i \in X \cup X^{-1}\ \forall i \}$

Claim: $F = \langle X \rangle$.

Clearly, F is a subgroup of G and $X \subseteq F$. Since $\langle X \rangle$ is the smallest subgroup of G containing X, so $\langle X \rangle \subseteq F$. Let H be a subgroup of G s.t. $X \subseteq H$. Then, $X^{-1} \cup X \subseteq H \Rightarrow F \subseteq H$.
$\Rightarrow F \subseteq \langle X \rangle$, by taking $H = \langle X \rangle$. Hence, $F = \langle X \rangle$. #

Ex: $SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle$. [we will prove it later]

Let $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\therefore SL_2(\mathbb{Z}) = \left\{ T^{k_1} S^{k_2} \cdots T^{k_{n-1}} S^{k_n} \,\middle|\, n \geq 1,\ k_i \in \mathbb{Z}\ \forall i \right\}$.

Here, we apply Theorem 3 with $X = \{T, S\}$ in the group $SL_2(\mathbb{Z})$.

§ Commutativity measure:

- Let G be a cyclic group. Then, $G = \langle a \rangle$.
  Let $x, y \in G$. Then, $x = a^n$ and $y = a^m$ for some $n, m \in \mathbb{Z}$.
  $\therefore$ $xy = a^{n+m} = yx$.
  This proves that every cyclic group is abelian.

Definition: Let G be a group, and $a \in G$. The centralizer of 'a' in G is the set $C_G(a) = \{x \in G \mid xa = ax\}$.

- For every $a \in G$, $C_G(a)$ is a subgroup of G.

Definition (Centre of a group): Let G be a group. Then, the centre of G is defined as

$$Z(G) = \{x \in G \mid xy = yx \quad \forall\, y \in G\}.$$

- $Z(G)$ is a subgroup of G.
- G is abelian $\iff Z(G) = G$.
- $Z(G) = \bigcap_{a \in G} C_G(a)$.

Ex: Find $Z(Q_8)$ and $Z(GL_2(\mathbb{R}))$.