Statistical Inference and Multivariate Analysis (MA324)

Lecture Slides
Lecture 14

Maximum Likelihood Estimator: Examples

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Maximum Likelihood Estimator (MLE): More examples

Example 1: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, 1)$, $\mu \leq 0$. Thus, the parametric space is $\Theta = (-\infty, 0]$. Find the MLE of μ.

Example 2: Let X_1 be a sample of size one from $Bernoulli(\frac{1}{1+e^{\theta}})$, where $\theta \geq 0$. What is the MLE of θ?

Example 3: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} U(0, \theta)$, $\theta > 0$. What is the MLE of θ?

Examples:

Example 4: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} U\left(\theta - \frac{1}{2}, \theta + \frac{1}{2}\right)$, $\theta \in \mathbb{R}$. The likelihood function is

$$\begin{aligned} L(\theta) &= 1 \text{ if } \theta - \frac{1}{2} \leq x_1, \ldots, x_n \leq \theta + \frac{1}{2} \\ &= 1 \text{ if } x_{(n)} - \frac{1}{2} \leq \theta \leq x_{(1)} + \frac{1}{2}, \end{aligned}$$

where $x_{(n)} = \max\{x_1, \ldots, x_n\}$ and $x_{(1)} = \min\{x_1, \ldots, x_n\}$. What is the MLE of θ?

Examples:

Theorem (Invariance Property of MLE)

If $\hat{\boldsymbol{\theta}}$ is MLE of $\boldsymbol{\theta}$, then for any function $\tau(\cdot)$ defined on Θ, the MLE of $\tau(\boldsymbol{\theta})$ is $\tau(\hat{\boldsymbol{\theta}})$.

Example 5: Let $X_1, X_2, \ldots, X_n \stackrel{i.i.d.}{\sim} P(\lambda), \lambda > 0$. To find the MLE of $P(X_1 = 0)$, we can proceed as follows. Note that $P(X_1 = 0) = e^{-\lambda}$ and we know that the MLE of λ is $\bar{X}$. Hence, the MLE of $P(X_1 = 0)$ is $e^{-\overline{X}}$.

Theorem

Let $\boldsymbol{T}$ be a sufficient statistics for $\boldsymbol{\theta}$. If a unique MLE exist for $\boldsymbol{\theta}$, it is a function of $\boldsymbol{T}$. If MLE of $\boldsymbol{\theta}$ exist but is not unique, then one can find a MLE that is a function of $\boldsymbol{T}$ only.

Examples:

Example 6: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} U(0, \theta)$, $\theta > 0$. We know that the MLE is unique and $X_{(n)}$, which is also sufficient. Thus, the unique MLE is a function of sufficient statistic in this case.

Example 7: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} U(\theta - \frac{1}{2}, \theta + \frac{1}{2})$, $\theta \in \mathbb{R}$. Note that (can be shown) a sufficient statistic for θ is $\boldsymbol{T} = (X_{(1)}, X_{(n)})$. Also, we have seen in Example 4 that MLE exists but is not unique. Any point in the interval $\left[X_{(n)} - \frac{1}{2}, X_{(1)} + \frac{1}{2}\right]$ is a MLE of θ. Hence, $\frac{1}{2}\left(X_{(1)} + X_{(n)}\right)$ is a MLE and it is also a function of $\boldsymbol{T}$. On the other hand,
$Q = \left(\sin^2 X_1\right)\left(X_{(n)} - \frac{1}{2}\right) + \left(1 - \sin^2 X_1\right)\left(X_{(1)} - \frac{1}{2}\right)$ is also a MLE but not a function of $\boldsymbol{T}$ only.