Complex Integration

Recall

Definition: Let $\gamma : [a, b] \to \mathbb{C}$, be a contour and $S \subset \mathbb{C}$ such that $\gamma \subset S$. If $f : S \to \mathbb{C}$ is a continuous function then the **the contour integral (or line integral)** of f along the curve γ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Example: Let $f(z) = \bar{z}$.

- If $\gamma_1(t) = e^{it}, \; t \in [0, \pi]$ then, $\int_{\gamma_1} f(z)dz = i\pi$.
- If $\gamma_2(t) = 1(1-t) + t.(-1) = 1 - 2t, \; t \in [0, 1]$ then, $\int_{\gamma_2} f(z)dz = 0$.
- In the above example γ_1 and γ_2 are two paths joining 1 and -1. But the line integral along the paths γ_1 and γ_2 are NOT same.
- **Question:** When a line integral of f does not depend on path?

Complex integration

- (The fundamental integral) For $a \in \mathbb{C}$, $r > 0$ and $n \in \mathbb{Z}$

$$\int_\gamma (z-a)^n dz = \begin{cases} 0 & \text{if} \quad n \neq -1 \\ 2\pi i & \text{if} \quad n = -1 \end{cases}$$

  where $\gamma(t) = a + re^{it}$ for $t \in [0, 2\pi]$ is the circle of radius r centered at a.

- Let f, g be piecewise continuous complex valued functions then $\displaystyle\int_\gamma [\alpha f \pm g](z)dz = \alpha \int_\gamma f(z)dz \pm \int_\gamma g(z)dz.$
- Let $\gamma : [a, b] \to \mathbb{C}$ be a curve and $a < c < b$. If $\gamma_1 = \gamma|_{[a,c]}$ and $\gamma_2 = \gamma|_{[c,b]}$ then $\displaystyle\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$
- $\displaystyle\int_{-\gamma} f(z)dz = -\int_\gamma f(z)dz.$

- Let f be a piecewise continuous function defined on a set containing a cotour γ. If $|f(z)| \leq M$ for all $z \in \gamma$ and L =length of γ then

$$\begin{aligned}\left|\int_\gamma f(z)dz\right| &\leq \left|\int_a^b f(\gamma(t)\gamma'(t)|dt\right| \\ &\leq \int_a^b |f(\gamma(t))||\gamma'(t)|dt \\ &\leq M\int_a^b |\gamma'(t)|dt = ML. \quad \textbf{(ML-inequality)}\end{aligned}$$

- Let $\gamma(t) = 2e^{it}, t \in [0, \frac{\pi}{2}]$ and $f(z) = \dfrac{z+4}{z^3-1}$. Then by ML-ineuqality

$$\left|\int_\gamma f(z)\,dz\right| \leq \frac{6\pi}{7}.$$

Antiderivatives

Definition: The **antiderivative or primitive** of a continuous function f in a domain D is a function F such that $F'(z) = f(z)$ for all $z \in D$.

- The primitive of a function is **unique** up to an additive constant.
- The following theorem is an answer to the **Question:** When a line integral of f does not depend on path?)
- **Theorem:** Let D be a domain in $\mathbb{C}$ and γ be a contour in D with initial and end points z_1 and z_2 respectively. If $f : D \to \mathbb{C}$ is a continuous function with primitive $F : D \to \mathbb{C}$, then

$$\int_\gamma f(z)dz = F(z_2) - F(z_1).$$

Proof. Let $\gamma : [a, b] \to \mathbb{C}$. Since $\dfrac{d}{dt}F(\gamma(t)) = F'(\gamma(t))\gamma'(t)$ therefore

$$\begin{aligned} \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt &= \int_a^b \frac{d}{dt}F(\gamma(t))dt \\ &= F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1). \end{aligned}$$

- **Corollary:** In particular, if γ is a closed contour then $\int_\gamma f(z)dz = 0$.

Antiderivatives

When such F exists we write

$$\int_\gamma f(z)dz = \int_{z_1}^{z_2} f(z)dz = \int_{z_1}^{z_2} F'(z)dz = F(z_2) - F(z_1).$$

1. $\displaystyle\int_{z_1}^{z_2} z^2 dz = \frac{z_2^3 - z_1^3}{3}.$
2. $\displaystyle\int_{-i\pi}^{i\pi} \cos z dz = \sin(i\pi) - \sin(-i\pi) = 2\sin(i\pi).$
3. $\displaystyle\int_{-i}^{i} \frac{1}{z} dz = \text{Log }(i) - \text{Log }(-i) = \frac{i\pi}{2} - \frac{-i\pi}{2} = i\pi.$
4. The function $\frac{1}{z^n}$, $n > 1$ is continuous on $\mathbb{C}^*$. Thus the integral of the above function on any contour joining nonzero complex numbers z_1, z_2 not passing through origin is given by
$$\int_{z_1}^{z_2} \frac{dz}{z^n} = -(n-1)\left(\frac{1}{z_2^{n-1}} - \frac{1}{z_1^{n-1}}\right).$$
In particular we have $\displaystyle\int_C \frac{dz}{z^n} = 0$ where C any closed curve not possing through origin.

- So far, we get an answer to the following question:
- **Question:** When a line integral of f does not depend on path?
- We proved that "a line integral of f does not depend on a path if f has primitive.
- Now, we will come by an answer to the following question:
- **Question:** Under what conditions on f we can guarantee the existence of g such that $g' = f$?

Simply Connected

- **Definition:** A domain D is called **simply connected** if every simple closed contour (within it) encloses points of D only.
- **Examples:**
  - The whole complex plane $\mathbb{C}$
  - Any open disc
  - The right half plane $RHP = \{z : \text{Re } z > 0\}$.
- A domain D is called **multiply connected** if it is **not** simply connected.
- **Examples:**
  - The sets $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$
  - $B(o, r) \setminus \{0\}$,
  - The annulus $A(a, b) = \{z \in \mathbb{C} : a < |z| < b\}$.

Theorem: (Cauchy's Theorem) If a function f is analytic on a simply connected domain D and C is a simple closed contour lying in D then

$$\int_C f(z)dz = 0.$$

To prove the above theorem we need the following **Green's Theorem**.

Green's Theorem *Let C be a positively orientated simple closed curve. Let R be the domain that forms the interior of C. If u and v are continuous and have continuous partial derivatives u_x, u_y, v_x and v_y at all points on C then*

$$\int_C udx + vdy = \int\int_R [v_x - u_y]dxdy.$$

Cauchy's Theorem

Proof. Let $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ and $C(t) = x(t) + iy(t)$, $a \leq t \leq b$ is the curve C. Then

$$\begin{aligned}
\int_C f(z)dz &= \int_a^b f(C(t))C'(t)dt \\
&= \int_a^b [u(x(t),y(t)) + iv(x(t),y(t))][x'(t) + iy'(t)]dt \\
&= \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt \\
&= \int_C udx - vdy + i\int_C vdx + udy \\
&= \int\int_R (-v_x - u_y)dxdy + i\int\int_R (u_x - v_y)dxdy, \\
&\quad \text{(by Green's theorem)} \\
&= 0 \quad \text{(by CR equations } u_x = v_y \text{ and } u_y = -v_x).
\end{aligned}$$

Let $C(t) = e^{it}$, $-\pi \leq t \leq \pi$, denotes the unit circle.

1. It follows from Cauchy's theorem that $\int_C f(z)dz = 0$, if $f(z) = e^{z^n}$, $\cos z$, or $\sin z$.
2. $\displaystyle\int_C f(z)dz = 0$ if $f(z) = \frac{1}{z^2}$, or $cosec^2 z$ from the fundamental theorem as $\frac{d}{dz}(-\frac{1}{z}) = \frac{1}{z^2}$ and $\frac{d}{dz}(-\cot z) = cosec^2 z$. Note that here Cauchy's theorem cannot be applied as the integrands are not analytic at zero.
3. $\displaystyle\int_C \frac{e^{iz^2}}{z^2+4}dz = 0$ by Cauchy's theorem. Note that the integrand is not analytic at $z = \pm 2$ but that does not bother us as these points are not enclosed by C.
4. If $f(z) = (\text{Im } z)^2$ then $\displaystyle\int_C f(z)dz = 0$ (**check this**). As f is not analytic anywhere in $\mathbb{C}$ Cauchy's theorem can not be applied to prove this.

Consequences of Cauchy's Theorem

- **Independence of path:** Let D be a simply connected domain and $f : D \to \mathbb{C}$ analytic. Let z_1, z_2 be two points in D. If γ_1 and γ_2 be two simple contour joining z_1 and z_2 such that the curves lie entirely in D then,
$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$
- **Proof:** If we define
$$\gamma(t) = \begin{cases} \gamma_1(2t) & \text{if } 0 \le t \le 1/2 \\ \eta(t) = \gamma_2(2(1-t)) & \text{if } 1/2 \le t \le 1 \end{cases}$$
then γ is a simple closed curve and
$$\int_{\gamma} f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\eta} f(z)dz.$$
By Cauchy's theorem
$$\int_{\gamma} f(z)dz = 0.$$
From last two equations we get
$$\int_{\gamma_1} f(z)dz = -\int_{\eta} f(z)dz = \int_{\gamma_2} f(z)dz.$$

- Following theorem is a answer to the question Under what conditions on f we can guarantee the existence of g such that $g' = f$?
- **Theorem:** If f is an analytic function on a simply connected domain D then there exists a function g, which is analytic on D such that $g' = f$.
  - **Proof.** Fix a point $z_0 \in D$ and define

$$g(z) = \int_{z_0}^{z} f(w)dw.$$

  - The integral is considered as a contour integral over any curve lying in D and joining z with z_0.
  - By the result the integral does not depend on the curve we choose and hence the function g is well defined.
  - We will show that $g' = f$.

Consequences of Cauchy's Theorem

- If $z + h \in D$ then

$$g(z+h) - g(z) = \int_{z_0}^{z+h} f(w)dw - \int_{z_0}^{z} f(w)dw = \int_{z}^{z+h} f(w)dw,$$

where the curve joining z and $z + h$ can be considered as a straight line $l(t) = z + th,\ t \in [0, 1]$. Since $\int_l f(z)dw = f(z)h$ therefore we get

$$\left| \frac{g(z+h) - g(z)}{h} - f(z) \right| = \left| \frac{1}{h} \int_{z}^{z+h} (f(w) - f(z))dw \right|.$$

- Now f is continuous at z, then for any given $\epsilon > 0$ there exist a $\delta > 0$ such that $|f(z+h) - f(z)| < \epsilon$ if $|h| < \delta$.
- Thus for $|h| < \delta$ we get from ML-inequality that

$$\left| \frac{1}{h} \int_{z}^{z+h} (f(w) - f(z))dw \right| \leq \frac{\epsilon |h|}{|h|} = \epsilon.$$

- This show that $g'(z) = \lim\limits_{h \to 0} \frac{g(z+h) - g(z)}{h} = f(z)$.