Statistical Inference and Multivariate Analysis (MA324)

LECTURE SLIDES

Lecture 23

Interval Estimation: Method of Finding CI

Indian Institute of Technology Guwahati

Jan-May 2023

Method of Finding CI

There are several ways of construction of CI. Here, we will discuss the **construction of CI based on pivot**. The definition of pivot in given below.

Def: A random variable $T = T(\boldsymbol{X}, \theta)$ is **called a pivot** (or a pivotal quantity) if the **distribution of T does not involve any unknown parameters**.

Remark: Pivot is a **function of random sample and unknown parameters**, but its' **distribution is independent** of all unknown parameters. Hence, pivot is not a statistic in general.

Remark: In general, we want to find a pivot that is a function of **minimal sufficient statistic.**

Example 1: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, 1)$. Then $\bar{X} - \mu$ is a pivot as $\bar{X} - \mu \sim N(0, 1/n)$.

Example 2: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$ and μ and σ both are unknown. Then $\bar{X} - \mu$ is not a pivot as $\bar{X} - \mu \sim N(0, \sigma^2/n)$. However, $\frac{\sqrt{n}}{\sigma}\left(\bar{X} - \mu\right) \sim N(0, 1)$ and $\frac{\sqrt{n}}{S}\left(\bar{X} - \mu\right) \sim t_{n-1}$. Therefore, these are pivots.

Example 3: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} Exp(\lambda)$. Then $2\lambda \sum_{i=1}^{n} X_i \sim \chi^2_{2n}$ (why?), and hence, is a pivot.

- Once an appropriate pivot is found, the CI for a parameter θ can be obtained as follows. Let T be a pivot. **Find two real numbers** a and b such that

$$P_\theta\left(a \le T(\boldsymbol{X},\, \theta) \le b\right) \ge 1 - \alpha.$$

- Note that a and b are independent of all unknown parameters as the **distribution** of T does **not involve any unknown parameter**. Let us denote the set

$$C(\boldsymbol{x}) = \left\{\theta \in \Theta : a \le T(\boldsymbol{x}; \theta) \le b\right\}.$$

  Then, $C(\boldsymbol{X})$ is a $100(1-\alpha)\%$ CI for θ.

- Note that $C(\boldsymbol{x})$ does not involve any unknown parameters as a and b are independent of all unknown parameters. Also notice that if $T(\boldsymbol{x}; \theta)$ is **monotone** in $\theta \in \Theta$ for each $\boldsymbol{x}$, then $C(\boldsymbol{x})$ is an interval. Otherwise it could be a general set.

One-sample Problems

Example 4: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ is unknown and $\sigma > 0$ is known. We are interested in μ. A pivot based on minimal sufficient statistics is $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \sim N(0, 1)$. Let z_α be the upper α-point of the standard normal distribution. We can take $a = z_{1-\alpha/2} = -z_{\alpha/2}$ (as $N(0, 1)$ distribution is symmetric about zero) and $b = z_{\alpha/2}$. Now,

$$P\left(-z_{\frac{\alpha}{2}} \leq \frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \leq z_{\frac{\alpha}{2}}\right) = 1-\alpha \implies P\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}} \leq \mu \leq \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}\right)$$

Hence, a $100(1-\alpha)\%$ symmetric CI for μ is

$$C(\boldsymbol{X}) = \left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}},\, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{\frac{\alpha}{2}}\right].$$

Note that the choice of $a = -z_{\frac{\alpha}{2}}$ and $b = z_{\frac{\alpha}{2}}$ corresponds to symmetric CI, as we leave $\frac{\alpha}{2}$ probability on both sides and take the middle part of the probability distribution. Of course, there are infinite number of choices for a and b. For example, let $\alpha_1 > 0$, $\alpha_2 > 0$ are to real numbers such that $\alpha_1 + \alpha_2 = \alpha$. Then, $a = z_{1-\alpha_1}$ and $b = z_{\alpha_2}$ can be considered (see the right panel of Figure below).

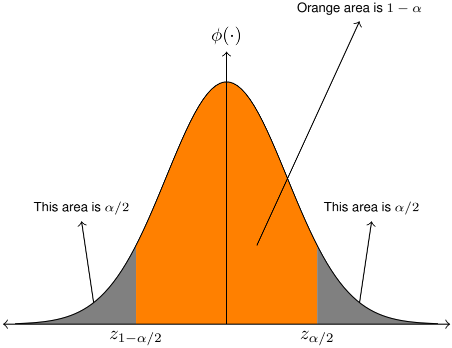

Figure: Symmetric and asymmetric CIs

Example 5: Let $X_1,\, X_2,\, \dots,\, X_n \stackrel{i.i.d.}{\sim} N(\mu,\, \sigma^2)$, where $\mu \in \mathbb{R}$ is known and $\sigma > 0$ is unknown. We are interested in CI of σ^2. A pivot based on minimal sufficient statistics is $\frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \sim \chi^2_n$. Let $\chi^2_{n,\alpha}$ be the upper α-point of a χ^2-distribution with degrees of freedom n. We can take $a = \chi^2_{n,1-\alpha/2}$ and $b = \chi^2_{n,\alpha/2}$. Hence, a $100(1-\alpha)\%$ symmetric CI for σ^2 is

$$C(\boldsymbol{X}) = \left[\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{n,\alpha/2}},\, \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\chi^2_{n,1-\alpha/2}} \right].$$

Example 6: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. We are interested in CI of μ. A pivot based in minimal sufficient statistic is $\frac{\sqrt{n}(\overline{X}-\mu)}{S} \sim t_{n-1}$, where $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$. Let $t_{n,\alpha}$ be the upper α-point of a t-distribution with degrees of freedom n. Then, we can take $a = t_{n-1,1-\alpha/2} = -t_{n-1,\alpha/2}$ (as t-distribution is symmetric about zero) and $b = t_{n-1,\alpha/2}$. Hence, a $100(1-\alpha)\%$ symmetric CI for μ is

$$C(\boldsymbol{X}) = \left[\overline{X} - \frac{S}{\sqrt{n}}t_{n-1,\alpha/2},\ \bar{X} + \frac{S}{\sqrt{n}}t_{n-1,\alpha/2}\right].$$

Example 7: Let $X_1, X_2, \ldots, X_n \stackrel{i.i.d.}{\sim} N(\mu, \sigma^2)$, where $\mu \in \mathbb{R}$ and $\sigma > 0$ are unknown. We are interested in CI for σ^2. A pivot based on minimal sufficient statistic is $\frac{1}{\sigma^2} \sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2 \sim \chi^2_{n-1}$. We can take $a = \chi^2_{n-1,1-\alpha/2}$ and $b = \chi^2_{n-1,\alpha/2}$. Hence, a $100(1-\alpha)\%$ symmetric CI for σ^2 is

$$C(\boldsymbol{X}) = \left[\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\chi^2_{n-1,\frac{\alpha}{2}}}, \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{\chi^2_{n-1,1-\frac{\alpha}{2}}}\right].$$