Eckart-Young Theorem

Theorem[Schmidt, 1907], [Eckart & Young, 1936]
Let $A \in \mathbb{F}^{n\times m}$ with $\operatorname{rank} A = r$. Let $A = U\Sigma V^*$ be an SVD of A. For $k = 1, \dots r-1$, define

$$A_k = U\Sigma_k V^*$$

where $\Sigma_k = \operatorname{diag}(\sigma_1, \dots, \sigma_k, 0, \dots, 0) \in \mathbb{R}^{n\times m}$ is a diagonal matrix. Then $\operatorname{rank} A_k = k$ and

$$\|A - A_k\|_2 = \min\{\|A - B\|_2 : B \in \mathbb{F}^{n\times m} \text{ with } \operatorname{rank} B \leq k\} = \sigma_{k+1}.$$

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Corollary Let $A \in \mathbb{F}^{n \times n}$ be nonsingular. Let $A = U\Sigma V^*$ be an SVD of A. Then,

$$\sigma_n = \min\{\|A - B\|_2 : B \in \mathbb{F}^{n \times n} \text{ is singular}\}.$$

(Exercise!)

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Corollary Let $A \in \mathbb{F}^{n\times n}$ be nonsingular. Then,

$$\frac{1}{\kappa_2(A)} = \min\left\{\frac{\|\Delta A\|_2}{\|A\|_2} : A + \Delta A \text{ is singular}\right\}$$

(Exercise!)

Numerical rank determination via SVD

Let $A = U\Sigma V^*$ be an SVD of an $n \times m$ real or complex matrix A with

$$\Sigma := \mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_p) \in \mathbb{R}^{n\times m}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \sigma_p \geq 0$ for $p = \min\{n, m\}$. If $\mathrm{rank}\, A = r$, then $r \leq p$. In particular if $r < p$, then

$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_r > 0 = \sigma_{r+1} = \cdots = \sigma_p.$$

However, due to rounding error, the computed singular values of A are likely to satisfy

$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_k > \epsilon >> \sigma_{k+1} \geq \cdots \geq \sigma_p \geq 0$$

for some $1 \leq k \leq p$, where $0 < \epsilon << 1$.

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In such cases, we may set $\sigma_j = 0$, for $j = k+1, \dots, p$, and state that the *numerical rank* of A is k.

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If the entries of A are affected only by rounding error, then we may set $\epsilon = 2\max\{n, m\}u\|A\|_2$. This is the default threshold for Matlab's `rank` command which can be modified by the user.