Some Linear Algebra Insights

The representation of a vector with respect to a basis

Let V be a real or complex finite dimensional vector space and $B = \{v_1, \ldots, v_n\}$ be a basis of V.

For $v \in V$, there exist unique scalars $a_1, \ldots, a_n$ such that $v = \sum_{i=1}^{n} a_i v_i$. The vector

$$[v]_B := \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{F}^n$$

is defined to be the representation of v with respect to the basis B.

Example: The representation of $p(x) = 1 + 2x - x^2$ with respect to the standard basis $B = \{1, x, x^2\}$ of the vector space $P_2(\mathbb{R})$ of real polynomials of degree at most 2 is

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$$[p]_B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

The representation of a vector with respect to a basis

Let $B = \{e_1, \ldots, e_n\}$ be the standard basis in $\mathbb{F}^n$.
For $x = [x_1 \ \cdots \ x_n]^T \in \mathbb{F}^n$, $[x]_B = x$ as $x = \sum_{i=1}^n x_i e_i$.

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For another basis $B' = \{v_1, \ldots, v_n\}$ of $\mathbb{F}^n$, $[x]_{B'} = [y_1 \ \cdots \ y_n]^T$ if

$$x = \sum_{i=1}^n v_i y_i.$$

Let $V = [v_1 \ v_2 \ \cdots \ v_n]$. Then $x = Vy \Rightarrow y = V^{-1}x$. Thus

$$[x]_{B'} = V^{-1}x.$$

Matrix of a linear map with respect to a basis

Consider the linear map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 - x_3 \\ x_1 + x_2 \\ 2x_1 + 2x_3 + 3x_3 \end{bmatrix}.$$

Evidently, $Tx = Ax$ for all $x \in \mathbb{R}^3$ where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}.$$

Observe that for the standard basis $B = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$,

$$[Te_1]_B = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, [Te_2]_B = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, [Te_3]_B = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}.$$

A is defined to the matrix of T with respect to the basis B.

Matrix of a linear map with respect to a basis

Let V be a real or complex finite dimensional vector space and $T : V \mapsto V$ be a linear transformation.

Given a basis $B = \{v_1, \ldots, v_n\}$, of V, the matrix of T with respect to B, is defined to be the $n \times n$ matrix whose i^{th} column is $[Tv_i]_B$ for each $i = 1, \ldots, n$.

Consider the basis $B' = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For the linear map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ in the previous slide,

$$Tv_1 = T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

so that $[Tv_1]_{B'} = \begin{bmatrix} \frac{1}{2} \\ \frac{5}{2} \\ 4 \end{bmatrix}$. Likewise,

$$[Tv_2]_{B'} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix}, \ [Tv_3]_{B'} = \begin{bmatrix} -\frac{3}{2} \\ \frac{5}{2} \\ 5 \end{bmatrix}.$$

Therefore,

$$[T]_{B'} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{3}{2} \\ \frac{5}{2} & -\frac{1}{2} & \frac{5}{2} \\ 4 & 0 & 5 \end{bmatrix}.$$

Observe that

$$[T]_{B'} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = V^{-1}[T]_B V$$

where $V = [v_1 \ v_2 \ v_3]$.

Taking similarity transformations is equivalent to changing bases

Theorem Let $T : \mathbb{F}^n \mapsto \mathbb{F}^n$ be linear. Let $B = \{e_1, \ldots, e_n\}$ be the standard basis and $B' = \{v_1, \ldots, v_n\}$ be any other basis of $\mathbb{F}^n$. Then for $V = [v_1 \ \cdots \ v_n]$,

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In general, if $B_1 = \{w_1, \dots, w_n\}$ and $B_2 = \{u_1, \dots, u_n\}$ are any two bases of $\mathbb{F}^n$, then

$$[T]_{B_2} = S^{-1}[T]_{B_1} S$$

where $S = [[u_1]_{B_1} \ \cdots \ [u_n]_{B_1}]$.

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where $S = [[u_1]_{B_1} \ \cdots \ [u_n]_{B_1}]$.

Corollary Let $T : \mathbb{F}^n \mapsto \mathbb{F}^n$ be linear and $B_1 = \{w_1, \ldots, w_n\}$ and $B_2 = \{u_1, \ldots, u_n\}$ be any two orthonormal bases of $\mathbb{F}^n$. Then $Q = [[u_1]_{B_1} \ \cdots \ [u_n]_{B_1}]$ is a unitary matrix such that $[T]_{B_2} = Q^*[T]_{B_1} Q$.

In particular if B_1 is the standard orthonormal basis $\{e_1, \ldots, e_n\}$, then $Q = [u_1 \ \cdots \ u_n]$.

Schur's Theorem: A fresh perspective

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Spectral Theorem for Hermitian Matrices $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if there exists a unitary matrix Q and a real diagonal matrix D such that $Q^* A Q = D$.

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Spectral Theorem for Normal Matrices $A \in \mathbb{C}^{n\times n}$ is normal if and only if there exists a unitary matrix Q and a diagonal matrix D such that $Q^* AQ = D$.

Matrix of a linear map when domain and range space may be different

Let $T : \mathbb{F}^n \mapsto \mathbb{F}^m$ be a linear transformation.

Suppose $B_1 = \{q_1, \ldots, q_n\}$ and $B_2 = \{q'_1, \ldots, q'_m\}$ are bases of $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively. The matrix of T with respect to B_1 and B_2 is defined as

$$[T]_{B_1,B_2} = \begin{bmatrix} [Tq_1]_{B_2} & \cdots & [Tq_n]_{B_2} \end{bmatrix} \in \mathbb{F}^{m\times n}.$$

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Consider $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ defined by

$$T \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ x_1 - x_2 \end{bmatrix}.$$

Let $B_1 = \{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$ and $B_2 = \{v_1, v_2, v_3\}$ be a basis of $\mathbb{R}^3$ where

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Matrix of a linear map when domain and range spaces may be different

Then,

$$Te_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 0.v_1 + v_2 + v_3$$

$$Te_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2v_1 + 0.v_2 - v_3$$

so that $[T]_{B_1,B_2} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}.$

Change of basis when range and domain may be different

Theorem Let $T : \mathbb{F}^n \mapsto \mathbb{F}^m$ be linear. Let B_1, B_1' be two bases of $\mathbb{F}^n$ and B_2, B_2' be two bases of $\mathbb{F}^m$ where $B_1' = \{v_1, \ldots, v_n\}$, $B_2' = \{u_1, \ldots, u_m\}$ and B_1 and B_2 are the standard bases. Then for $V = [v_1 \ \cdots \ v_n]$ and $U = [u_1 \ \cdots \ u_m]$,

$$[T]_{B_1', B_2'} = U^{-1} [T]_{B_1, B_2} V.$$

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$$[T]_{B_1', B_2'} = U^{-1}[T]_{B_1, B_2} V.$$

In general, if $B_1 = \{w_1, \dots, w_n\}$ and $B_2 = \{q_1, \dots, q_m\}$ are any two bases of $\mathbb{F}^n$ and $\mathbb{F}^m$ respectively, then

$$[T]_{B_1', B_2'} = M_2[T]_{B_1, B_2} M_1$$

where $M_1 = [[v_1]_{B_1} \ \cdots \ [v_n]_{B_1}]$ and $M_2 = \left[[q_1]_{B_2'} \ \cdots \ [q_m]_{B_2'}\right]$.

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Corollary Let $T : \mathbb{F}^n \mapsto \mathbb{F}^m$ be linear where $m \neq n$. Let B_1, B_1' be two orthonormal bases of $\mathbb{F}^n$ and B_2, B_2' be two orthonormal bases of $\mathbb{F}^m$ where $B_1' = \{v_1, \ldots, v_n\}$, $B_2' = \{u_1, \ldots, u_m\}$ and B_1 and B_2 are the standard bases. Then for $V = [v_1 \ \cdots \ v_n]$ and $U = [u_1 \ \cdots \ u_m]$,

$$[T]_{B_1',B_2'} = U^*[T]_{B_1,B_2}V.$$

The SVD: An alternative view

Singular Value Decomposition: The Singular Value Decomposition (SVD) of a matrix $A \in \mathbb{F}^{n \times m}$ is a decomposition of the form

$$A = U\Sigma V^*$$

where $U \in \mathbb{F}^{n \times n}$ and $V \in \mathbb{F}^{m \times m}$ are unitary matrices and $\Sigma := \mathrm{diag}(\sigma_1, \sigma_2, \ldots \sigma_p) \in \mathbb{R}^{n \times m}$ is a diagonal matrix with

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$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_p \geq 0$$

for $p = \min\{n, m\}$.

Equivalently, given any $A \in \mathbb{F}^{n \times m}$, there exist suitable choices of orthonormal bases in $\mathbb{F}^n$ and $\mathbb{F}^m$ with respect to which the matrix of the linear transformation defined by A is a diagonal matrix $\Sigma := \mathrm{diag}(\sigma_1, \sigma_2, \dots \sigma_p) \in \mathbb{R}^{n \times m}$ with

$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_p \geq 0$$

for $p = \min\{n, m\}$.