MA 322: Scientific Computing



Department of Mathematics Indian Institute of Technology Guwahati

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Chapter 4: Numerical Integrations or Quadratures



Newton-Cotes integration: Boole's rule

Newton-Cotes formula

$$\int_a^b f(x) dx \approx I_n(f) = \sum_{j=0}^n w_j f(x_j),$$

where

$$w_j = \int_{x_0}^{x_n} \prod_{\substack{i=0\\i\neq j}}^n \left(\frac{x-x_i}{x_j-x_i}\right) \mathrm{d}x.$$

Newton-Cotes formula for n = 4

$$\int_{x_0}^{x_4} f(x) dx = \frac{2h}{45} \left[7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4) \right] - \frac{8h^7}{945} f^{(6)}(\eta), \quad x_0 \le \eta \le x_4.$$



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Newton-Cotes integration: Error Analysis

Theorem

1. For n even, assume f(x) is n + 2 times continuously differentiable on [a, b]. Then

$$I(f) - I_n(f) = C_n h^{n+3} f^{(n+2)}(\eta), \quad \eta \in [a, b]$$

with

$$C_n = \frac{1}{(n+2)!} \int_0^n \mu^2(\mu-1) \cdots (\mu-n) \mathrm{d}\mu.$$

2. For n odd, assume f(x) is n+1 times continuously differentiable on [a,b]. Then

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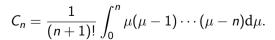
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Theorem

Let
$$I_n(f) = \sum_{j=0}^n w_j f(x_j)$$
, $n \ge 1$ be a sequence of numerical integration formula that

approximate
$$I(f) = \int_a^b f(x) dx$$
. Let \mathcal{F} be a family dense in $C[a, b]$.

Then $I_n(f) \to I(f)$ for all $f \in C[a, b]$ if and only if

1.
$$I_n(f) \rightarrow I(f), \ \forall f \in \mathcal{F}$$
, and

$$2. B \equiv \sup_{n \ge 1} \sum_{j=0}^{n} |w_j| < \infty.$$



(Closed) Newton-Cotes integration: Convergence

Lemma

The set of all polynomials is dense in C[a, b].

Definition

Let \mathcal{F} be a family of continuous functions on a given interval [a,b]. We say \mathcal{F} is dense in C[a,b] if for every $f \in C[a,b]$ and every $\epsilon > 0$, there is a function f_{ϵ} in \mathcal{F} for which

$$\max_{a \le x \le b} |f(x) - f_{\epsilon}(x)| \le \epsilon.$$

Theorem (Weierstrass)

Let f(x) be a continuous function on [a,b] and let $\epsilon > 0$. Then there is a polynomial p(x) for which

$$|f(x) - p(x)| \le \epsilon$$
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$$I = \int_{-4}^{4} \frac{1}{1+x^2} dx = 2 \tan^{-1}(4) \approx 2.6516.$$

n	I _n
2	5.4902
4	2.2776
6	3.3288
	1.9411
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Newton-Cotes formula does not converge in this case.



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▶ Newton-Cotes formula for n = 8,

$$\int_{x_0}^{x_8} f(x) dx \approx I_8(f) = \frac{4h}{14175} [989(f(x_0) + f(x_8)) + 5888(f(x_1) + f(x_7)) -928(f(x_2) + f(x_6)) + 10496(f(x_3) + f(x_5)) + 4540f(x_4)].$$

- ► Such formulas can cause loss-of-significance errors, although it is unlikely to be a serious problem until *n* is large.
- ▶ People have generally avoided using Newton-Cotes formulas for $n \ge 8$, even in forming composite formulas.
- ► The most serious problem of the Newton-Cotes method is that it may not converge for perfectly well-behaved integrands.



(Open) Newton-Cotes integration: Error Analysis

Theorem

Suppose $I_n(f) = \sum_{j=0}^n w_j f(x_j)$ denotes the (n+1)-point open Newton-Cotes formula with $x_{-1} = a, x_{n+1} = b$ and h = (b-a)/(n+2). Then $\exists \ \eta \in (a,b)$ such that

1. For n even, assume f(x) is n+2 times continuously differentiable on [a,b]. Then

$$I(f) - I_n(f) = C_n \frac{h^{n+3} f^{(n+2)}(\eta)}{(n+2)!} \int_{-1}^{n+1} \mu^2(\mu-1) \cdots (\mu-n) d\mu.$$

2. For n odd, assume f(x) is n+1 times continuously differentiable on [a,b]. Then

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