Note Title 24-09-2020

Construction of Confidence Interval

Sampling Distribution

* **Result** : $X_1, \dots, X_n \overset{i.i.d}{\sim} N(\mu, \sigma^2)$

Defn, $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$; $s^2 = \frac{1}{(n-1)} \sum_{i=1}^{n} (X_i - \bar{X})^2$

Then $\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ & $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

and $\bar{X}$ & S^2 are ind. r.v.

$$f(\underline{x}) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\sum_{i=1}^{n} \frac{(x_i-\mu)^2}{\sigma^2}}$$

χ^2_n

$\cancel{f(x) \propto e^{-x/2} x^{\alpha-1}}$

$f(x) \propto \underbrace{e^{-x/2} x^{n/2-1}}$

$$Y_1 = \underbrace{\frac{1}{\sqrt{n}}(X_1-\mu) + \frac{1}{\sqrt{n}}(X_2-\mu) + \cdots + \frac{1}{\sqrt{n}}(X_n-\mu)}$$

$$Y_i = \frac{1}{\sqrt{i(i-1)}}(X_1-\mu) + \cdots + \frac{(X_{i-1}-\mu)}{\sqrt{i(i-1)}} - \frac{(i-1)}{\sqrt{i(i-1)}}(X_i-\mu)$$

$$Y_n = \frac{1}{\sqrt{n(n-1)}}(X_1-\mu) + \cdots + \frac{(X_{n-1}-\mu)}{\sqrt{n(n-1)}} - \frac{(n-1)(X_n-\mu)}{\sqrt{n(n-1)}}$$

$$\underset{\sim}{Y} = P\underbrace{(\underset{\sim}{X} - \underline{\mu})} \qquad \Big\{ \quad \boxed{P'P = I.}$$

$$\boxed{\begin{aligned}\underline{y}'\underline{y} &= (\underline{x}-\underline{\mu})' P' P (\underline{x}-\underline{\mu}) \\ &= (\underline{x}-\underline{\mu})'(\underline{x}-\underline{\mu})\end{aligned}}$$

$$|J| = 1$$

$$f(\underset{\sim}{y}) = \frac{1}{(2\pi)^{n/2}\sigma^n} e^{-\frac{\underline{y}'\underline{y}}{\sigma^2}} \Rightarrow Y_i \overset{i.i.d}{\sim} N(0, \sigma^2)$$

$$\frac{Y_1}{\sigma} = \underbrace{\left[\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}\right]}_{} \sim N(0,1)$$

$$\Rightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$\frac{1}{\sigma^2}\underline{y}'\underline{y} = \frac{1}{\sigma^2}\sum_{i=1}^{n} y_i^2 = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu)^2$$

$$\frac{1}{\sigma^2}\sum_{i=2}^{n} Y_i^2 = \frac{1}{\sigma^2}\left[\sum_{i=1}^{n}(x_i - \mu)^2 - \frac{n(\bar{x}-\mu)^2}{\sigma^2}\right]$$

$f(y_2, \ldots, y_n)$

$$= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{(n-1)s^2}{\sigma^2}$$

Sum square ind normal(0,1)

$(n-1)$

$X \sim N(0,1)$

$Y = X^2$

$x = \pm\sqrt{y}$

$\frac{dx}{dy} = \frac{1}{2} y^{-1/2}$

$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}$

$f(y) \propto \underbrace{e^{-y/2} \cdot y^{-1/2}}$

$\underbrace{e^{-y/2} \quad y^{1/2 - 1}}$

$Y \sim \chi^2_1$

We can show

$E(e^{tY}) = \frac{1}{(1-2t)^{n/2}}$

$Y \sim \chi^2_n$

$Z_i^2 \sim \chi^2_1$

$$E\left(e^{t \sum_{i=1} Z_i^2}\right) = \prod_{i=1}^{n-1} \underbrace{E\left(e^{tZ_i^2}\right)}$$

$$= \underbrace{\frac{1}{(1-2t)^{\frac{(n-1)}{2}}}} \rightarrow \text{MGF. of } \chi^2_{n-1}$$

$$\frac{(n-1)s^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=2}^{n} Y_i^2 \sim \chi^2_{n-1} \checkmark$$

$$T = \frac{\sqrt{n}(\bar{X} - \mu)}{s} \overset{\text{distin}}{\sim} \frac{\dfrac{\sqrt{n}(\bar{X} - \mu)}{\sigma}}{\sqrt{\dfrac{(n-1)s^2}{\sigma^2} \Big/ (n-1)}}$$

$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0,1)$

$\frac{(n-1)s^2}{\sigma^2} \to \chi^2_{n-1}$

$\sim t_{n-1}$

(H.W)

2

$X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, 1)$

You want to find out the 95% C.I.

Confidence interval for μ

$[a, b] \rightarrow$ random interval $\rightarrow$ prob

$a < \mu < b$

$\cancel{a < \bar{X} < b}$

$100(1-\alpha)\%$ C.I for μ means some random interval ~~should~~ include μ with prob $(1-\alpha)$

estimate of μ; say $\bar{X}$

$\{\bar{X} - \text{error}, \bar{X} + \text{error}] \rightarrow (\mu)$

μ^* known

$X_1, \ldots, X_n \sim N(\mu, 1) = f(x; \mu)$

generate

$(X_{11}^*, \ldots, X_{1n}^*) \longrightarrow \left(\bar{X} - \frac{z_{\alpha/2}}{\sqrt{n}}, \bar{X} + \frac{z_{\alpha/2}}{\sqrt{n}}\right) = [a_1, b_1] \ni \mu$

$(X_{21}^*, \ldots, X_{2n}^*) \longrightarrow [a_2, b_2] \ni \mu$

$\vdots$

$(X_{n1}^*, \ldots, X_{nn}^*) \longrightarrow [a_n, b_n] \not\ni \mu$

$1-\alpha$

μ

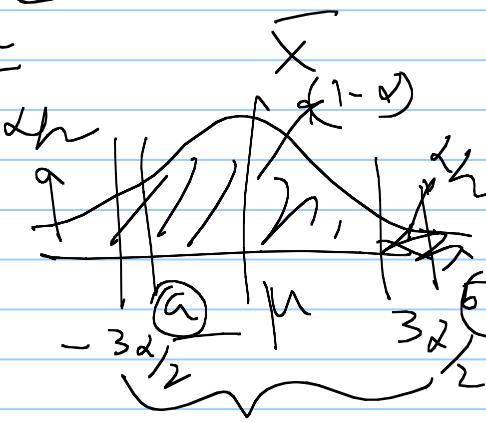
Coverage prob

simulation proportion of intervals which will include true parameter.

$X_1, \ldots, X_n \sim N(\mu, \sigma^2)$; $\left\{ \bar{X} \text{ is said to be un..., i.e. } E(\bar{X}) = \mu \right.$

Aim: 100 $(1-\alpha)\%$ C.I. for μ

$$Z = \frac{\sqrt{n}\,(\bar{X} - \mu)}{\sigma} \sim N(0,1) \quad \text{Pivotal Statistic}$$



$$P\left[-z_{\alpha/2} < \frac{\sqrt{n}\,(\bar{X} - \mu)}{\sigma} < z_{\alpha/2}\right] = (1-\alpha)$$

$P(Z > z_{\alpha/2}) = \alpha/2$

approx $P\left[\left(\bar{X} - \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}},\ \bar{X} + \frac{z_{\alpha/2}\,\sigma}{\sqrt{n}}\right) \ni \mu\right] = 1-\alpha$

n

$$\left(\bar{X} - \text{error},\ \bar{X} + \text{error}\right)$$

μ

$[-1000, 1000]$

$[-10, 10]$

$[-0.5, 0.5]$

$$\text{error} = \boxed{\sigma\,\frac{z_{\alpha/2}}{\sqrt{n}}}$$

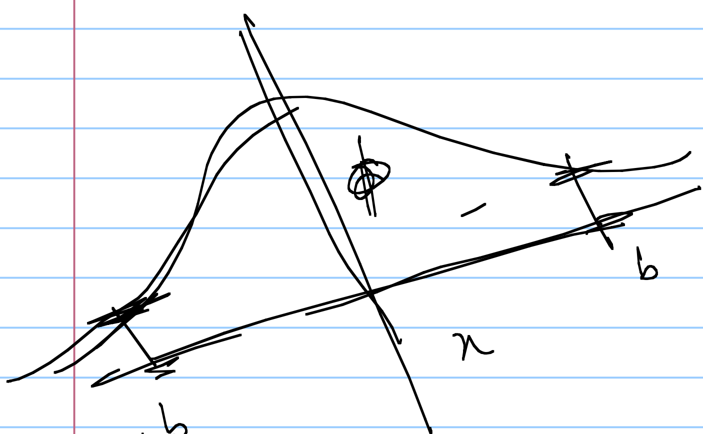
$$\min_{a,b} (b - a) \xrightarrow{\text{given the fact that}} P\left(\overset{(a,b) \ni \mu_C}{\cancel{\text{term } t\,[\ \ ]}} \right) = (1-\alpha)$$

$$L = b - a \Rightarrow \frac{dL}{da} = \frac{db}{da} - 1 = 0 \quad \text{—— (†)}$$

$$P\left[a < \underbrace{\left(\frac{\bar{X}-\mu}{\sigma}\right)\sqrt{n}}_{Z} < b \right] = (1-\alpha)$$

$$\Rightarrow \Phi(b) - \Phi(a) = (1-\alpha) \quad \text{—— (*)}$$

Take derivative of $\textcircled{\alpha}$ w.r.t a



$$\phi(b)\frac{db}{da} - \phi(a) = 0$$

$$\Rightarrow \phi(b) = \phi(a)$$

$$\Rightarrow \boxed{b = -a}\ , \quad (a \neq b)$$

$$\Phi(-a) - \Phi(a) = (1-\alpha)$$

$\Rightarrow \quad 1 - 2\,\Phi(a) = 1 - \alpha$

$\Rightarrow \quad \underline{\Phi(a) = \alpha/2} \Rightarrow a = \cancel{\Phi}^{-3}\alpha/2$