

MA 201: Partial Differential Equations
One-dimensional Wave Equation (Contd.)
Lecture - 12

Finite Vibrating String with no External Force

- Recall the finite string problem in a computational domain $(x, t) \in [0, L] \times [0, \infty)$ (Lecture 11)

► The governing equation

$$u_{tt} = c^2 u_{xx}, \quad (x, t) \in (0, L) \times (0, \infty). \quad (1)$$

► The boundary conditions for all $t \geq 0$:

$$u(0, t) = 0, \quad u(L, t) = 0. \quad (2)$$

► The initial conditions for $0 \leq x \leq L$:

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x). \quad (3)$$

Finite Vibrating String with no External Force

The solution is given by

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] \quad (4)$$

with

$$A_n = \frac{2}{L} \int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots$$

$$B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots$$

Example:

For a string of length L stretched between the points $x = 0$ and $x = L$, find the vibration in the string subject to the following initial conditions:

$$u(x, 0) = \sin(\pi x/L) + 1/2 \sin(3\pi x/L), \quad u_t(x, 0) = 0.$$

Solution of the Finite Vibrating String Problem: Example

Solution:

Here, initial conditions are

$$\phi(x) = \sin(\pi x/L) + 1/2 \sin(3\pi x/L), \quad \psi(x) = 0.$$

Therefore, $B_n = 0$.

and

$$A_n = \frac{2}{L} \int_0^L \left[\sin\left(\frac{\pi x}{L}\right) + \frac{1}{2} \sin\left(\frac{3\pi x}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right) dx.$$

Due to the orthogonality of the set $\{\sin(\frac{n\pi x}{L}) : n = 1, 2, \dots\}$, only A_1 and A_3 are non-zero, and

$$A_1 = 1, A_3 = 1/2.$$

Therefore, the solution is (since $A_n = 0$ for $n \neq 1, 3$)

$$u(x, t) = \sin\left(\frac{\pi x}{L}\right) \cos\left(\frac{\pi ct}{L}\right) + \frac{1}{2} \sin\left(\frac{3\pi x}{L}\right) \cos\left(\frac{3\pi ct}{L}\right).$$

Finite Vibrating String with Gravity

We now consider an external force due to the gravitational acceleration g only (consider a string oriented horizontally). Then the one-dimensional wave equation becomes

$$u_{tt} = c^2 u_{xx} - g, \quad 0 < x < L, \quad t > 0. \quad (5)$$

We seek to find

the displacement of the string at any position and at any time subject to the following boundary conditions (for $t \geq 0$) and initial conditions ($0 \leq x \leq L$):

$$u(0, t) = 0, \quad (6a)$$

$$u(L, t) = 0, \quad (6b)$$

and

$$u(x, 0) = \phi(x), \quad (7a)$$

$$u_t(x, 0) = \psi(x). \quad (7b)$$

Finite Vibrating String with Gravity (Contd.)

Due to the presence of the term g in equation (5), which has made the equation non-homogeneous,

the direct application of the method of separation of variables will not work.

Now seek a solution in the form:

$$u(x, t) = v(x, t) + h(x). \quad (8)$$

We intend to split the given problem into two problems – one IBVP in $v(x, t)$ and another BVP in $h(x)$.

The idea is to have

- $v(x, t)$ giving the solution for a string without the external force.
- $h(x)$ taking care of the non-homogeneous term in the equation.

Now, substituting (8) in equation (5), we obtain

$$v_{tt} = c^2[v_{xx} + h''(x)] - g. \quad (9)$$

Finite Vibrating String with Gravity (Contd.)

We choose $h(x)$ such that

$$c^2 h''(x) - g = 0, \quad (10)$$

in order to allow $v(x, t)$ to satisfy the homogeneous wave equation

$$v_{tt} = c^2 v_{xx}, \quad 0 < x < L, \quad t > 0. \quad (11)$$

Both functions $v(x, t)$ and $h(x)$ together satisfy the boundary conditions

$$v(0, t) + h(0) = 0, \quad t \geq 0, \quad (12a)$$

$$v(L, t) + h(L) = 0, \quad t \geq 0, \quad (12b)$$

and also the initial conditions

$$v(x, 0) + h(x) = \phi(x), \quad 0 < x < L, \quad (13a)$$

$$v_t(x, 0) = \psi(x), \quad 0 < x < L. \quad (13b)$$

Finite Vibrating String with Gravity (Contd.)

Since, $h(x)$ is a user-defined function, we can set

$$h(0) = 0 \text{ \& } h(L) = 0. \quad (14)$$

so that $v(x, t)$ satisfies zero boundary conditions at $x = 0$ and $x = L$ allowing us to use separation of variables method for the IBVP in $v(x, t)$.

Subsequently, the original non-homogeneous IBVP can be conveniently split into the following two problems:

Problem I (BVP):

$$\begin{aligned} c^2 h''(x) &= g, \\ h(0) &= 0 = h(L). \end{aligned}$$

Problem II (IBVP):

$$\begin{aligned} v_{tt} &= c^2 v_{xx}, \\ v(0, t) &= 0 = v(L, t), \\ v(x, 0) &= \phi(x) - h(x), \quad v_t(x, 0) = \psi(x). \end{aligned}$$

Finite Vibrating String with Gravity (Contd.)

The solution for Problem I can be easily found by integrating $h''(x)$ twice:

$$h(x) = \frac{gx^2}{2c^2} + Ax + B.$$

Upon using the conditions $h(0) = 0 = h(L)$, we get

$$B = 0 \quad \& \quad A = -gL/(2c^2)$$

and hence

$$h(x) = -g \frac{(L-x)x}{2c^2}. \tag{15}$$

Finite Vibrating String with Gravity (Contd.)

The solution of Problem II is known to us, which is

$$v(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right], \quad (16)$$

where A_n and B_n are given, respectively, by

$$A_n = \frac{2}{L} \int_0^L [\phi(x) - h(x)] \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots, \quad (17)$$

$$B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots \quad (18)$$

Hence the solution $u(x, t)$ for our IBVP is given by the sum of (15) and (16).

Finite Vibrating String with Gravity (Contd.)

In other words,

$$u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right] - g \frac{(L-x)x}{2c^2}, \quad (19)$$

where A_n and B_n are given, respectively, by

$$A_n = \frac{2}{L} \int_0^L \left(\phi(x) + g \frac{(L-x)x}{2c^2} \right) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots, \quad (20)$$

$$B_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots \quad (21)$$

Remark: Clearly, this splitting method would be applicable only when the non-homogeneous term in the governing equation is a constant or a function of x .

For problems in which the non-homogeneous term is a function of x and t , Laplace transform method is considered to be the most appropriate one. However, Duhamel's Principle can also be applied to solve non-homogeneous problems.

Duhamel's Principle: Finite String Problem

Assume that $v(x, t, s)$ is the solution of the problem

$$v_{tt} - c^2 v_{xx} = 0, \quad (x, t) \in (0, L) \times (0, \infty), \quad (22)$$

$$\text{with BCs } v(0, t, s) = 0, \quad v(L, t, s) = 0, \quad t \geq 0, \quad s > 0, \quad (23)$$

$$\text{with ICs } v(x, 0, s) = 0, \quad v_t(x, 0, s) = f(x, s), \quad s > 0. \quad (24)$$

Then, $u(x, t)$ defined by

$$u(x, t) = \int_0^t v(x, t - \tau, \tau) d\tau \quad (25)$$

is the solution to the non-homogeneous problem

$$u_{tt} - c^2 u_{xx} = f(x, t), \quad (x, t) \in (0, L) \times (0, \infty), \quad (26)$$

$$\text{with BCs } u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0, \quad (27)$$

$$\text{with ICs } u(x, 0) = 0, \quad u_t(x, 0) = 0. \quad (28)$$

Duhamel's Principle: Finite String Problem

It is to be noted that

the vibration of the string takes place due to one of the following:

- At least one of the initial conditions is non-zero (with no source),
- There is a source term present in the governing equation (when both initial conditions are zero). This condition leads to the consideration of a non-homogeneous governing equation.

Further

- Duhamel's principle allows the shifting of the non-homogeneous term (the source) from the original equation to an initial condition for the modified problem.
- Since the source term is connected to some sort of force, therefore it gets shifted to the initial condition connected to the velocity, i.e., $u_t(x, 0)$, not to $u(x, 0)$.

Finite String Problem: Application of Duhamel's Principle

Example: Find $u(x, t)$ such that

$$u_{tt} - u_{xx} = t \sin\left(\frac{\pi x}{L}\right), \quad (x, t) \in (0, L) \times (0, \infty), \quad (29)$$

$$\text{with ICs } u(x, 0) = 0, \quad u_t(x, 0) = 0, \quad x \in (0, L), \quad (30)$$

$$\text{with BCs } u(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0. \quad (31)$$

Solution: Suppose $v(x, t, s)$ is a solution to the user defined problem:

$$v_{tt} - v_{xx} = 0, \quad (x, t) \in (0, L) \times (0, \infty), \quad (32)$$

$$\text{with ICs } v(x, s, 0) = 0, \quad v_t(x, s, 0) = s \sin\left(\frac{\pi x}{L}\right), \quad x \in (0, s, L), \quad s > 0, \quad (33)$$

$$\text{with BCs } v(0, s, t) = 0, \quad v(L, s, t) = 0, \quad t \geq 0. \quad (34)$$

Finite String Problem: Application of Duhamel's Principle

The solution to IVBP (32)-(34) is given by

$$v(x, t, s) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left[A_n \cos\left(\frac{n\pi t}{L}\right) + B_n \sin\left(\frac{n\pi t}{L}\right) \right] \quad (35)$$

with

$$A_n = \frac{2}{L} \int_0^L \phi(x) \sin\left(\frac{n\pi x}{L}\right) dx = 0, \quad n = 1, 2, 3, \dots, \quad (\text{since initial displacement is zero}),$$

$$B_n = \frac{2}{n\pi} \int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right) dx, \quad n = 1, 2, 3, \dots$$

$$= \frac{2}{n\pi} \int_0^L s \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx.$$

Finite String Problem: Application of Duhamel's Principle

Thus, $B_1 = \frac{sL}{\pi}$ and $B_n = 0$, $n \neq 1$, and hence

$$v(x, t, s) = \frac{sL}{\pi} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi t}{L}\right).$$

Then solution $u(x, t)$ of the given problem is obtained as

$$\begin{aligned} u(x, t) &= \int_0^t v(x, t - \tau, \tau) d\tau \\ &= \int_0^t \frac{\tau L}{\pi} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi(t - \tau)}{L}\right) d\tau \\ &= \frac{L}{\pi} \sin\left(\frac{\pi x}{L}\right) \int_0^t \tau \sin\left(\frac{\pi(t - \tau)}{L}\right) d\tau. \end{aligned}$$

Evaluate the integral and find the solution.

Two-dimensional Wave Equation

We will not discuss the solution of two-dimensional wave equation. But knowing the associated IBVP will be helpful in understanding how wave propagates in a rectangular membrane.

Let the four sides of a rectangular membrane ($0 < x < a$, $0 < y < b$) be fixed along its boundaries. Then, an initial displacement and an initial velocity (or either one of them) will induce deflection $u(x, y, t)$ of the membrane.

The IBVP:

Governing Equation:

$$u_{tt} = c^2(u_{xx} + u_{yy}), \quad 0 < x < a, \quad 0 < y < b, \quad (36)$$

Boundary Conditions:

$$u(0, y, t) = 0, \quad u(a, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, b, t) = 0, \quad (37)$$

Initial Conditions:

$$u(x, y, 0) = \phi(x, y), \quad u_t(x, y, 0) = \psi(x, y). \quad (38)$$

Summary

- By associating appropriate boundary and/or initial conditions, an IBVP or IVP can be formulated and solved which gives the deflection or displacement in a string.
- For an IBVP due to a homogeneous equation, both boundary conditions can be zero but at least one of the initial conditions must be non-zero. Here the deflection is due to the initial condition(s).
- For an IBVP due to a non-homogeneous equation, both boundary conditions and both initial conditions may be zero. Here the deflection is due to the source term present in the governing equation.
- The instantaneous displacements are due to the wave created.
- For an infinite string (recall D'Alembert's solution), it is a **progressive wave** and hence the solution is known as progressive wave solution.
- For a finite string, it is a **standing or stationary wave** and hence the solution is known as stationary wave solution since it remains within the bounded domain.