~~EXTRA SPACE FOR ROUGH WORK~~

(5) X: Exam 1 Score. Y: Exam 2 Score.

(a) We need to find $P(Y > 75 \mid X = 80)$.

Note that the conditional distn of Y given $X = x$ is normal with mean $\mu_Y + \rho \frac{\sigma_Y}{\sigma_X}(x - \mu_X)$ and variance $\sigma_Y^2(1-\rho^2)$.

Here $x = 80$ and $Y \mid X = 80 \sim N(69, 144)$.

$$\sigma_Y^2(1-\rho^2) = 15^2 \times (1 - 0.36) = 225 \times 0.64 = 144.$$

$$\mu_Y + \rho \cdot \frac{\sigma_Y}{\sigma_X}(x - \mu_X) = 60 + 0.6 \times \frac{15}{10}(80 - 70) = 60 + 0.6 \times 15 = 69$$

Here the required probability is,

$$P(Y > 75 \mid X = 80)$$
$$= P\left(\frac{Y - 69}{12} > \frac{75 - 69}{12} \,\middle|\, X = 80\right)$$
$$= P\left(\frac{Y-69}{12} > 0.5 \,\middle|\, X = 80\right)$$
$$= 1 - \Phi(0.5)$$
$$= \Phi(-0.5).$$

(b) To find $P(X + Y > 150)$.

Note that, $X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y)$
$$\equiv N(130, 505)$$

Hence the required probability is

$$P(X + Y > 150) = P\left(\frac{X + Y - 130}{\sqrt{505}} > \frac{20}{\sqrt{505}}\right)$$
$$= 1 - \Phi(0.89)$$
$$= \Phi(-0.89)$$

$$\boxed{\text{OR } \Phi\left(-\frac{20}{\sqrt{505}}\right).}$$

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Q2 X_i and $Y_i \overset{iid}{\sim} U(0,1)$. ; $N_n = \#\{k : 1 \le k \le n, X_k^2 + Y_k^2 \le 1\}$.

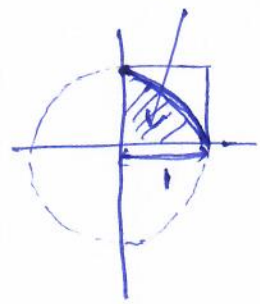
To show: $\frac{4N_n}{n} \longrightarrow \pi$ almost surely.

Define Z_i, for $i = 1, 2, 3, \ldots$

$$Z_i = \begin{cases} 1 & \text{if } X_i^2 + Y_i^2 \le 1 \\ 0 & \text{o.w} \end{cases}$$

So, we have $N_n = \sum_{i=1}^{n} Z_i$. Also Z_i's are iid RVs. So $\{Z_n\}_{n\ge1}$ is a seq. of iid RVs.

Therefore, $E(Z_i) = P(X_i^2 + Y_i^2 \le 1) = \frac{\pi}{4}$



using strong Law of Large numbers (SLLN),

$\bar{Z}_n \longrightarrow \frac{\pi}{4}$ almost surely.

$\Rightarrow$ $4\bar{Z}_n \longrightarrow \pi$ almost surely.

$\Rightarrow$ $\frac{4N_n}{n} \longrightarrow \pi$ almost surely.

EXTRA SPACE FOR ROUGH WORK

Q3) Let T be the time (units) until state 3 occurs.

TPM: $\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$

So, $k_i = E(T \mid X_0 = i)$, $i = 1, 2, 3$

with $k_3 = 0$.

So, $k_2 = E(T \mid X_0 = 2)$

$$= \sum_{i=1}^{3} E(T \mid X_1 = i, X_0 = 2)\, P(X_1 = i \mid X_0 = 2)$$

$$= (1+k_1) p_{21} + (1+k_2) p_{22} + 1 \cdot p_{23}$$

$$= (p_{21} + p_{22} + p_{23}) + k_1 p_{21} + k_2 p_{22}$$

$$= 1 + 0 + \tfrac{1}{2} k_2$$

$$\Rightarrow \tfrac{1}{2} k_2 = 1 \Rightarrow k_2 = 2.$$

Now, $k_1 = E(T \mid X_0 = 1)$

$$= \sum_{i=1}^{3} E(T \mid X_1 = i, X_0 = 1)\, P(X_1 = i \mid X_0 = 1)$$

$$= (1+k_1) p_{11} + (1+k_2) p_{12} + 1 \cdot p_{13}$$

$$= (p_{11} + p_{12} + p_{13}) + k_1 p_{11} + k_2 p_{12}$$

$$= 1 + \tfrac{1}{3} k_1 + 2 \times \tfrac{1}{3}$$

$$\Rightarrow \tfrac{2}{3} k_1 = \tfrac{5}{3}$$

$$\Rightarrow k_1 = \tfrac{5}{2} = 2.5$$