End Semester Examination: MA 201 Mathematics III

Department of Mathematics, IIT Guwahati

Date: November 23, 2022 **Time: 9:00–12:00 hours** **Max Marks: 50**

Full Name: **Roll No:**

Instructions:

(I) Write your name and roll number above as soon as you receive the question paper.

(II) Make sure that you are in the designated room as per the seating plan.

(III) Enter the Question number and the corresponding page number(s) in the front page of the booklet as instructed.

(IV) Use of calculator is not permitted.

(V) Not adhering to Instructions above will invite **penalty**.

(VI) The question paper contains 15 questions with question numbers 1, 14 and 15 having sub-parts (a) and (b).

(VII) Meanings of the symbols used in the question paper are the same as followed in the course.

(VIII) Questions are self-explanatory. No query will be entertained by the invigilators.

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1. Use residue theorem to evaluate the following integrals:
(a) $\displaystyle\int_0^{2\pi} \frac{\cos(3\theta)}{5-4\cos(\theta)}\,d\theta$, **(b)** $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin(\pi x)}{x^2+2x+5}\,dx$. **[3+4]**

Solution:
(a) Putting $z = e^{i\theta}$ we have

$$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = -\frac{1}{2i}\int_{|z|=1} f(z)\,dz,$$

where $f(z) = \dfrac{z^6+1}{z^3(2z-1)(z-2)}$. Now $Res(f,0) = \dfrac{21}{8}$ and $Res(f,\frac{1}{2}) = -\dfrac{65}{24}$.

So $\displaystyle\int_{|z|=1} f(z)\,dz = -\frac{1}{2i}\cdot 2\pi i\left(\frac{21}{8}-\frac{65}{24}\right) = \frac{\pi}{12}$.

(b) Consider $\displaystyle\int_{\Gamma_R} \frac{ze^{i\pi z}}{z^2+2z+5}\,dz$, where $\Gamma_R = [-R,R]\cup\gamma_R$ with $\gamma_R = \{Re^{i\theta} : \theta\in[0,\pi]\}$. So

$$\int_{-R}^{R} \frac{xe^{i\pi x}}{x^2+2x+5}\,dx + \int_{\gamma_R} \frac{ze^{i\pi z}}{z^2+2z+5}\,dz = \int_{\Gamma_R} \frac{ze^{i\pi z}}{z^2+2z+5}dz = 2\pi i Res\left(\frac{ze^{i\pi z}}{z^2+2z+5}, -1+2i\right)$$

Taking $R\to\infty$ and using Jordan's lemma we get

$$\int_{-\infty}^{\infty} \frac{x\cos\pi x}{x^2+2x+5}\,dx + \int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+2x+5}\,dx = \frac{\pi}{2}e^{-2\pi} - i\pi e^{-2\pi}$$

Equating the imaginary parts we get $\displaystyle\int_{-\infty}^{\infty} \frac{x\sin\pi x}{x^2+2x+5}\,dx = -\pi e^{-2\pi}$.

2. Find the image of the lower half plane $\mathrm{Im}(z) < 0$ under the map $w = \dfrac{\mathrm{i}z-1}{\mathrm{i}-z}$. **[3]**

Solution: Let $w(z) = \dfrac{iz-1}{i-z}$. Then $w(0) = i, w(-i) = 0$ and $w(-1) = -1$. Let $z = x+iy$. Then

$$|w(z)| = \left|\frac{i(i+z)}{i-z}\right| = \left|\frac{x^2+(y+1)^2}{x^2+(y-1)^2}\right| < 1,$$

for $y < 0$. So the image of the lower half plane Im(z)< 0 under the map $w = \dfrac{i+z}{i-z}$, is the open unit disc.

3. Find a first-order linear partial differential equation (PDE) whose characteristic curves are represented by a one-parameter family of circles $x^2+y^2 = R^2$. **[1]**

Solution: The general form of a linear first-oder PDE in 2D is $a(x,y)u_x + b(x,y)u_y = c(x,y)u + d(x,y)$.
The characteristic equation of this PDE is given by, $\dfrac{dx}{a} = \dfrac{dy}{b}$.
Now, the given characteristic curves are: $x^2+y^2 = R^2$.
Differentiating we get, $2xdx + 2ydy = 0 \Rightarrow \dfrac{dx}{y} = \dfrac{dy}{-x}$, the characteristic equation.
Comparing these two characteristic equations we have, $a = y$ and $b = -x$.
So, the PDE whose characteris curves are circle is, $yu_x - xu_y = c(x,y)u + d(x,y)$.

4. Obtain the surface that is orthogonal to the one-parameter family of surfaces $u = cxy\left(x^2+y^2\right)$ (c is a non-zero parameter) and passes through the hyperbola $x^2-y^2 = a^2$, $u = b$, $(a,b>0)$. **[4]**

Solution: Any surface passing through the integral curves of $f_x p + f_y q = f_u$ is the surface that will be orthogonal to $f(x,y,u) = c$.
For the given surface, $f(x,y,u) = \dfrac{xy\left(x^2+y^2\right)}{u} = \dfrac{1}{c}$.
So, $f_x = \dfrac{y\left(3x^2+y^2\right)}{u}$, $f_y = \dfrac{x\left(x^2+3y^2\right)}{u}$ and $f_u = \dfrac{-xy\left(x^2+y^2\right)}{u^2}$.
The auxiliary equations of $f_x p + f_y q = f_u$ is $\dfrac{dx}{f_x} = \dfrac{dy}{f_y} = \dfrac{du}{f_u}$
implies, $\dfrac{udx}{y\left(3x^2+y^2\right)} = \dfrac{udy}{x\left(x^2+3y^2\right)} = \dfrac{u^2du}{-xy\left(x^2+y^2\right)}$
implies, $\dfrac{dx}{yu\left(3x^2+y^2\right)} = \dfrac{dy}{xu\left(x^2+3y^2\right)} = \dfrac{du}{-xy\left(x^2+y^2\right)}$

For the first surface:

$$\frac{xdx+ydy}{4xyu\left(x^2+y^2\right)} = \frac{-4udu}{4xyu\left(x^2+y^2\right)}$$

implies $x^2+y^2+4u^2 = C_1$.
For the second surface:

$$\frac{xdx+ydy}{4xyu\left(x^2+y^2\right)} = \frac{xdx-ydy}{2xyu\left(x^2-y^2\right)}$$

implies $\dfrac{\left(x^2-y^2\right)^2}{\left(x^2+y^2\right)} = C_2$.
The general solution of $f_x p + f_y q = f_u$ is $F(\phi,\psi) = 0$, where

$$\phi(x,y,u) = x^2+y^2+4u^2 \text{ and } \psi(x,y,u) = \frac{\left(x^2-y^2\right)^2}{\left(x^2+y^2\right)}.$$

The integral surface passes through the hyperbola: $x^2 - y^2 = a^2$, $u = b$, $(a, b > 0)$.

From the second surface we get, $\dfrac{a^4}{(x^2+y^2)} = C_2 \Rightarrow x^2 + y^2 = \dfrac{a^4}{C_2}$.

From the first surface, $\dfrac{a^4}{C_2} + 4b^2 = C_1 \Rightarrow \left(C_1 - 4b^2\right) C_2 = a^4$.
So, the desired surface is

$$\left(x^2 + y^2 + 4u^2 - 4b^2\right)\left(x^2 - y^2\right)^2 = a^4\left(x^2 + y^2\right).$$

5. Find a second-order PDE arising from the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{u^2}{c^2} = 1$. Here, x and y are the independent variables, $u = u(x, y)$ and a, b, c are arbitrary non-zero constants. **[2]**

Solution: The given surface is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{u^2}{c^2} = 1$.
Differentiating the given equation with respect to x one gets, $\dfrac{2x}{a^2} + \dfrac{2uu_x}{c^2} = 0$.

Further, differentiating the above equation with respect to y we have,
$\dfrac{2u_xu_y}{c^2} + \dfrac{2uu_{xy}}{c^2} = 0$.

So, the desired PDE is $uu_{xy} + u_xu_y = 0$.

Other two solutions are:
$xuu_{xx} + xu_x^2 - uu_x = 0$. (differentiang two times w.r.t. x)

$yuu_{yy} + yu_y^2 - uu_y = 0$. (differentiang two times w.r.t. y)

6. Write the general form of a second-order linear homogeneous PDE in two dimensions involving one dependent variable only. Considering the PDE as a hyperbolic one, reduce it to a normal form which will be free from mixed derivatives. **[1+3]**

Solution: The general form of a second-order linear homogeneous PDE in 2D is

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0,$$

coefficients are functions of x and y only.
Introducing new variables ξ and η by substitution, $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ [such that the Jacobian $\xi_x\eta_y - \xi_y\eta_x \neq 0$], the transformed equation can be written as,

$$Au_{\xi\xi} + Bu_{\xi\eta} + Cu_{\eta\eta} + Du_\xi + Eu_\eta + Fu = 0,$$

where,

$$A = a\xi_x^2 + b\xi_x\xi_y + c\xi_y^2,$$
$$B = 2a\xi_x\eta_x + b\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2c\xi_y\eta_y,$$
$$C = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2,$$
$$D = a\xi_{xx} + b\xi_{xy} + c\xi_{yy} + d\xi_x + e\xi_y,$$
$$E = a\eta_{xx} + b\eta_{xy} + c\eta_{yy} + d\eta_x + e\xi_y,$$
$$F = f.$$

The equation is of hyperbolic type, and let ξ =constant and η =constant be the two families of charcteristics. This implies that $A = 0$ and $C = 0$.
So the cannonical form of the equation will be

$$u_{\xi\eta} + Du_{\xi} + Eu_{\eta} + Fu = 0.$$

By a further linear transformation, $\alpha = \xi + \eta$ and $\beta = \xi - \eta$, one can get the final form of the equation, which is free from mixed derivatives, as,

$$u_{\alpha\alpha} - u_{\beta\beta} + \hat{D}u_{\alpha} + \hat{E}u_{\beta} + \hat{F}u = 0,$$
$$\text{where, } \hat{D} = D + E,\ \hat{E} = D - E,\ \hat{F} = F.$$

7. Find the general solution of the PDE $y^2 u_{xx} - 2yu_{xy} + u_{yy} = u_x + 6y$ by reducing to its canonical form. **[3]**

Solution: Here, $b^2 - 4ac = 4y^2 - 4y^2 = 0 \Rightarrow$ Parabolic.
The characteristic is $\dfrac{dy}{dx} = \dfrac{-2y}{2y^2} = -\dfrac{1}{y} \Rightarrow -\dfrac{y^2}{2} + c = x.$
Let $\xi(x, y) = \dfrac{y^2}{2} + x$. The other non-parallel function can be chosen as $\eta(x, y) = y$.
The Jacobian $\xi_x \eta_y - \xi_y \eta_x = 1 \neq 0$.
Substituting these functions into the given PDE, it becomes, $u_{\eta\eta} = 6y$.
The cannonical form is $u_{\eta\eta} = 6\eta$.
Solving this we get, $u = \eta^3 + \eta f(\xi) + g(\xi)$.
So, the general solution is

$$u(x, y) = y^3 + yf\left(\frac{y^2}{2} + x\right) + g\left(\frac{y^2}{2} + x\right).$$

Note: In questions 8, 9, 10 and 12 on next page, writing/finding the IBVP/BVP and the corresponding solution for the general problem will not carry any marks. Solutions to the specific problem as asked will only carry marks.

8. Consider the following initial boundary value problem for $u(x, t)$ governed by one-dimensional wave equation for a flexible string of unit length:

$$\begin{aligned}
&u_{tt} = u_{xx}, \quad 0 < x < 1, \quad t > 0;\\
&u(0, t) = 0, \ \ u(1, t) = 0, \quad t \geq 0;\\
&u(x, 0) = \sin(\pi x), \ \ u_t(x, 0) = 0, \quad 0 < x < 1.
\end{aligned}$$

Find the displacement of the string at location $x = 1/2$ and time $t = 2$. **[3]**
Solution: By using the given boundary conditions and the second initial condition,

$$u(x, t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cos(n\pi t),$$

with $A_n = 2\displaystyle\int_0^1 \sin(\pi x)\sin(n\pi x)\mathrm{d}x$. Here $\lambda_n = n\pi$. Using orthogonality of sine function over $[0, 1]$, $A_n = 0$ for all $n \neq 1$. Therefore, the only non-zero coefficient is $A_1 = 1$. The solution, therefore, is

$$u(x, t) = \sin(\pi x)\cos(\pi t).$$

Therefore, $u(0.5, 2) = 1$.

9. Consider solving the one-dimensional heat conduction equation $u_t = u_{xx}$ for a thin metal rod of length π with an initial temperature distribution $u(x,0) = \sin^2 x,\ 0 < x < \pi$. Further, homogeneous Neumann conditions are prescribed at the ends $x = 0,\ \pi$ for $t \geq 0$. Obtain the solution of the corresponding IBVP. **[4]**
Solution: The IBVP is

$$\begin{aligned}&u_t = u_{xx},\ 0 < x < \pi,\ t > 0,\\&u_x(0,t) = 0 = u_x(\pi,t),\ t \geq 0,\\&u(x,0) = \sin^2 x.\end{aligned}$$

By looking at the boundary conditions, the solution will be

$$u(x,t) = \frac{A_0}{2} + \sum_{n=1}^{\pi} A_n e^{-n^2 t}\cos nx.$$

where

$$A_n = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \cos nx\ dx.$$

Evaluating the integral to find A_0 and A_n, we get $A_0 = 1$ and $A_2 = -1/2$. All other $A_n = 0$ for $n \neq 0, 2$.

Hence the solution of the IBVP is

$$u(x,t) = \frac{1}{2} - \frac{1}{2}e^{-4t}\cos 2x.$$

10. By considering the one-dimensional wave equation for finite spatial domain along with suitable boundary conditions and initial conditions, discuss the basic idea behind Duhamel's principle. **[2]**
Solution: Suppose the one-dimensional wave equation for string problem is given as

$$\begin{aligned}&u_{tt} = u_{xx} + F(x,t), \quad 0 < x < L, \quad t > 0;\\&u(0,t) = 0,\ \ u(L,t) = 0, \quad t \geq 0;\\&u(x,0) = 0,\ \ u_t(x,0) = 0, \quad 0 < x < L.\end{aligned}$$

According to Duhamel's principle, the solution can be found as

$$u(x,t) = \int_0^t v(x, \tau - t, t)d\tau,$$

in which $v(x,t,s)$ is the solution of the IBVP

$$\begin{aligned}&v_{tt} = v_{xx}, \quad 0 < x < L, \quad t > 0;\\&v(0,s,t) = 0,\ \ v(L,s,t) = 0, \quad t \geq 0;\\&v(x,s,0) = 0,\ \ v_t(x,s,0) = F(x,s), \quad 0 < x < L.\end{aligned}$$

In other words, the source term in the problem for u has been shifted to the second initial condition in the problem for v.

11. Suppose it is required to solve the steady-state heat conduction equation (without source) in a rectangular plate $0 \leq x \leq a, 0 \leq y \leq b$ subject to two homogeneous Dirichlet boundary conditions along $x = 0, x = a$ and two non-homogeneous Dirichlet conditions along $y = 0, y = b$. Write the resulting boundary value problem clearly. Show how you would restructure the problem so that the method of separation of variables can be appropriately used. **[1+2]**

(**You need not solve.**)
Solution: The BVP is:

$$\begin{aligned}
&u_{xx} + u_{yy} = 0, \;\; 0 < x < a, \; 0 < y < b, \\
&u(0, y) = 0, \; u(a, y) = 0, \; u(x, 0) = f_1(x), \; u(x, b) = f_2(x).
\end{aligned}$$

(The last two conditions can be taken as Neumann conditions also.)
Split $u(x, y) = v(x, y) + w(x, y)$ to get the following BVPs:

$$\begin{aligned}
&v_{xx} + v_{yy} = 0, \;\; 0 < x < a, \; 0 < y < b, \\
&v(0, y) = 0, \; v(a, y) = 0, \; v(x, 0) = f_1(x), \; v(x, b) = 0; \\
&\text{and} \\
&w_{xx} + w_{yy} = 0, \;\; 0 < x < a, \; 0 < y < b, \\
&w(0, y) = 0, \; w(a, y) = 0, \; w(x, 0) = 0, \; w(x, b) = f_2(x).
\end{aligned}$$

12. Consider a circular planar disk of unit radius. Solve the corresponding BVP outside the disk with the given boundary condition: **[3]**

$$\begin{aligned}
&u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0, \;\; r > 1, \;\; -\pi < \theta < \pi, \\
&u(1, \theta) = \theta, \;\; -\pi < \theta < \pi.
\end{aligned}$$

Solution: In general, the solution can be written as

$$u(r, \theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \big[A_n \sin(n\theta) + B_n \cos(n\theta)\big]\big[C_n r^n + D_n r^{-n}\big].$$

with

$$\begin{aligned}
A_n &= \frac{1}{a^n \pi} \int_{-\pi}^{\pi} u(a, \theta) \sin(n\theta) \mathrm{d}\theta, \\
B_n &= \frac{1}{a^n \pi} \int_{-\pi}^{\pi} u(a, \theta) \cos(n\theta) \mathrm{d}\theta.
\end{aligned}$$

Here $a = 1$ and the solution is to be obtained only for $r > 1$ with $u(a, \theta) = \theta$. Therefore, the solution reduces to

$$u(r, \theta) = \sum_{n=1}^{\infty} A_n \sin(n\theta) r^{-n}.$$

with

$$\begin{aligned}
A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} \theta \sin(n\theta) \mathrm{d}\theta \\
&= \frac{2}{n\pi} (-\pi \cos n\pi) \\
&= \frac{2(-1)^{n+1}}{n}.
\end{aligned}$$

Thus, the solution is

$$u(r, \theta) = 2 \sum_{n=1}^{\infty} \frac{(}{-} 1)^{n+1} n \sin(n\theta) r^{-n}.$$

13. Obtain the Fourier series expansion of the function f given by **[3]**

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x \leq 0, \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi. \end{cases}$$

Solution: Clearly f is even and hence $B_n = 0$. For a_n, we use the definition to have

$$A_0 = 0 \;\;\&\;\; A_n = \frac{4}{n^2\pi^2}\Big[1-(-1)^n\Big], \;\; n = 1,\; 2,\; 3, \ldots.$$

That is,

$$A_n = \begin{cases} \frac{8}{n^2\pi^2}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Therefore, the Fourier series for f is given by

$$\frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos((2n-1))x}{(2n-1)^2} = \frac{8}{\pi^2}\Big(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\Big).$$

14. **(a)** By considering an appropriate Fourier transform of the function $f(t) = e^{-\beta t}$, $t > 0$, where $\beta > 0$ is a constant, evaluate the improper integral $\displaystyle\int_0^\infty \frac{\sigma\sin(\sigma t)}{\beta^2+\sigma^2}\,d\sigma$. **[3]**

Solution: Consider an odd extension in $(-\infty,\infty)$ so that

$$\begin{aligned} \mathcal{F}\{f_O(t)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\sigma\tau} f_O(\tau)d\tau \\ &= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{0}(-e^{a\tau})e^{-i\sigma\tau}d\tau + \int_0^\infty (e^{-a\tau})e^{-i\sigma\tau}d\tau\right] \\ &= \frac{1}{\sqrt{2\pi}}\frac{(-2i\sigma)}{a^2+\sigma^2} = g(\sigma). \end{aligned}$$

Inverting and using oddness and evenness of the functions

$$\begin{aligned} f_O(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\sigma)e^{i\sigma t}d\sigma \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(-i\sigma)}{a^2+\sigma^2}e^{i\sigma t}d\sigma \\ &= \frac{2}{\pi}\int_0^\infty \frac{\sigma\sin\sigma t}{a^2+\sigma^2}d\sigma \end{aligned}$$

Now,

$$\begin{aligned} e^{-at} &= \frac{2}{\pi}\int_0^\infty \frac{\sigma\sin\sigma t}{a^2+\sigma^2}d\sigma \\ \Rightarrow \int_0^\infty \frac{\sigma\sin\sigma t}{a^2+\sigma^2}d\sigma &= \frac{\pi}{2}e^{-at}. \end{aligned}$$

(b) Consider the heat conduction in an infinite thin metal rod having thermal diffusivity α with an initial temperature distribution $\phi(x)$. By using Fourier transform, find the temperature distribution $u(x,t)$ at any point of the rod at any subsequent time $t > 0$. How would you approach this problem if the rod is considered to be semi-infinite? **[2+1]**

Solution: The IVP is

$$\begin{aligned} &\frac{\partial u}{\partial t} = \alpha\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,\; t > 0, \\ &u(x,0) = \phi(x). \end{aligned}$$

Applying Fourier transform in x variable, we get, $(\mathcal{F}\{u(x,t)\} = \bar{u}(\sigma,t))$ $\frac{d}{dt}\bar{u}(\sigma,t) = -\alpha\sigma^2\bar{u}(\sigma,t)$ and $\mathcal{F}\{u(x,0)\} = \bar{u}(\sigma,0) = \bar{f}(\sigma)$. On solving the ODE in t variable, we get $\bar{u}(\sigma,t) = Ce^{-\alpha\sigma^2 t}$. So

$$\bar{u}(\sigma,t) = \bar{f}(\sigma)e^{-\alpha\sigma^2 t}.$$

Taking inverse Fourier transform, we get

$$u(x,t) = \frac{1}{2\pi} \int_0^\infty \bar{f}(\sigma) e^{-\alpha\sigma^2 t} e^{\mathrm{i}\sigma x}\, d\sigma.$$

If the rod is of semi-infinite length, we would have a boundary condition at $x = 0$. Depending on if the boundary condition is Dirichlet or Neumann, we will use sine or cosine transform, respectively.

15. **(a)** Find the Laplace transform of $f(t) = 2H(\sin \pi t) - 1$ where H is Heaviside unit step function. **[2]**

Solution: From definition of unit step function

$$H(\sin \pi t) = \begin{cases} 1, & \sin \pi t > 0, \\ 0, & \sin \pi t < 0. \end{cases}$$

This will give the given function as $+1$ between 0 and 1, 2 and 3, and so on whereas it will be -1 between 1 and 2, 3 and 4, and so on. That is

$$\begin{aligned}
\mathcal{L}\{2H(\sin \pi t) - 1\} &= \int_0^1 e^{-st} dt - \int_1^2 e^{-st} dt + \int_2^3 e^{-st} dt - \int_3^4 e^{-st} dt + \cdots \\
&= \left| \frac{e^{-st}}{-s} \right|_0^1 - \left| \frac{e^{-st}}{-s} \right|_1^2 + \left| \frac{e^{-st}}{-s} \right|_2^3 - \left| \frac{e^{-st}}{-s} \right|_3^4 + \cdots \\
&= \frac{1}{s} \left[(1 - e^{-s}) + (e^{-2s} - e^{-s}) + (e^{-2s} - e^{-3s}) + (e^{-4s} - e^{-3s}) + \cdots \right] \\
&= \frac{1}{s} [1 - 2e^{-s}(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots)] \\
&= \frac{1}{s} \left[1 - 2e^{-s} \frac{1}{1 + e^{-s}} \right] \\
&= \frac{1}{s} \frac{1 - e^{-s}}{1 + e^{-s}} \\
&= \frac{1}{s} \tanh \frac{s}{2}
\end{aligned}$$

(b) By using Laplace transform, solve the second-order ODE $\dfrac{d^2x}{dt^2} + 2p\dfrac{dx}{dt} + qx = f(t), t > 0$ subject to the initial conditions $x(0) = a$, $\dfrac{dx}{dt}(0) = b$ for the case $q - p^2 > 0$, where p, q, a and b are constants, and f is piecewise continuous and of exponential order. **[3]**

Solution: Notation: $\mathcal{L}\{x\} = X(s)$, $\mathcal{L}\{f(t)\} = \overline{f}(s)$. Taking Laplace transform

$$\begin{aligned}
& s^2 X(s) - s x(0) - \frac{dx}{dt}(0) + 2p(sX(s) - x(0)) + qX(s) = \overline{f}(s) \\
\Rightarrow\ & (s^2 + 2ps + q) X(s) = as + 2pa + b + \overline{f}(s).
\end{aligned}$$

Therefore

$$\begin{aligned}
X(s) &= \frac{as + 2pa + b + \overline{f}(s)}{(s^2 + 2ps + q)} \\
&= \frac{a(s+p) + ap + b + \overline{f}(s)}{(s+p)^2 + q - p^2} \\
&= \frac{a(s+p)}{(s+p)^2 + n^2} + \frac{ap + b}{(s+p)^2 + n^2} + \frac{\overline{f}(s)}{(s+p)^2 + n^2}, \quad n = \sqrt{q - p^2}.
\end{aligned}$$

Taking inverse

$$x(t) = ae^{-pt} \cos nt + \frac{ap + b}{n} e^{-pt} \sin nt + \frac{1}{n} \int_0^t f(t - \tau) e^{-p\tau} \sin n\tau \, d\tau.$$