MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 1

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1. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| \le \|\mathbf{x} - \mathbf{y}\|$.

   Solution: We have $\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{y} + \mathbf{y}\| \le \|\mathbf{x} - \mathbf{y}\| + \|\mathbf{y}\|$ and so $\|\mathbf{x}\| - \|\mathbf{y}\| \le \|\mathbf{x} - \mathbf{y}\|$. Similarly $\|\mathbf{y}\| - \|\mathbf{x}\| \le \|\mathbf{y} - \mathbf{x}\| = \|\mathbf{x} - \mathbf{y}\|$. Therefore $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| \le \|\mathbf{x} - \mathbf{y}\|$.

2. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x} + \mathbf{y}\|\, \|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

   Solution: We have $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\mathbf{x}\cdot\mathbf{y}$ and $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\mathbf{x}\cdot\mathbf{y}$. Hence $\|\mathbf{x} + \mathbf{y}\|^2\, \|\mathbf{x} - \mathbf{y}\|^2 = (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^2 - 4\,(\mathbf{x}\cdot\mathbf{y})^2 \le (\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)^2$.
   Therefore $\|\mathbf{x} + \mathbf{y}\|\, \|\mathbf{x} - \mathbf{y}\| \le \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$.

3. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x}\| \le \max\{\|\mathbf{x} + \mathbf{y}\|, \|\mathbf{x} - \mathbf{y}\|\}$.

   Solution: If possible, let $\|\mathbf{x}\| > \max\{\|\mathbf{x}+\mathbf{y}\|, \|\mathbf{x}-\mathbf{y}\|\}$. Then $\|\mathbf{x}\| > \|\mathbf{x}+\mathbf{y}\|$ and $\|\mathbf{x}\| > \|\mathbf{x}-\mathbf{y}\|$ and so $2\|\mathbf{x}\| = \|(\mathbf{x} + \mathbf{y}) + (\mathbf{x} - \mathbf{y})\| \le \|\mathbf{x} + \mathbf{y}\| + \|\mathbf{x} - \mathbf{y}\| < \|\mathbf{x}\| + \|\mathbf{x}\| = 2\|\mathbf{x}\|$, which is a contradiction. Hence $\|\mathbf{x}\| \le \max\{\|\mathbf{x} + \mathbf{y}\|, \|\mathbf{x} - \mathbf{y}\|\}$.

4. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, then show that $\|\mathbf{x} + \alpha\mathbf{y}\| \ge \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$ iff $\mathbf{x} \cdot \mathbf{y} = 0$.

   Solution: We first assume that $\mathbf{x} \cdot \mathbf{y} = 0$. If $\alpha \in \mathbb{R}$, then we have
   $\|\mathbf{x} + \alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\,\mathbf{x}\cdot\alpha\mathbf{y} + \|\alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\alpha\,\mathbf{x}\cdot\mathbf{y} + |\alpha|^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + |\alpha|^2\|\mathbf{y}\|^2 \ge \|\mathbf{x}\|^2$ and hence $\|\mathbf{x} + \alpha\mathbf{y}\| \ge \|\mathbf{x}\|$.
   Conversely, let $\|\mathbf{x} + \alpha\mathbf{y}\| \ge \|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$. If possible, let $\mathbf{x}\cdot\mathbf{y} \ne 0$. Then $\mathbf{y} \ne \mathbf{0}$ and so $\|\mathbf{y}\| \ne 0$. If $\alpha = -\frac{\mathbf{x}\cdot\mathbf{y}}{\|\mathbf{y}\|^2}$, then $\alpha \in \mathbb{R}$ and we have
   $\|\mathbf{x} + \alpha\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\alpha\,\mathbf{x}\cdot\mathbf{y} + |\alpha|^2\|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 - \frac{(\mathbf{x}\cdot\mathbf{y})^2}{\|\mathbf{y}\|^2} < \|\mathbf{x}\|^2$. Thus $\|\mathbf{x} + \alpha\mathbf{y}\| < \|\mathbf{x}\|$, which is a contradiction. Therefore $\mathbf{x}\cdot\mathbf{y} = 0$.

5. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $\alpha > 0$. Show that $|\mathbf{x}\cdot\mathbf{y}| \le \alpha\|\mathbf{x}\|^2 + \frac{1}{4\alpha}\|\mathbf{y}\|^2$.

   Solution: We have
   $|\mathbf{x}\cdot\mathbf{y}| \le \|\mathbf{x}\|\,\|\mathbf{y}\| = 2\sqrt{\alpha}\|\mathbf{x}\|\,\frac{1}{2\sqrt{\alpha}}\|\mathbf{y}\| \le \big(\sqrt{\alpha}\|\mathbf{x}\|\big)^2 + \big(\frac{1}{2\sqrt{\alpha}}\|\mathbf{y}\|\big)^2 = \alpha\|\mathbf{x}\|^2 + \frac{1}{4\alpha}\|\mathbf{y}\|^2$.

6. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Show that $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \|\mathbf{x} - \mathbf{y}\|$ iff $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha, \beta \ge 0$ with $(\alpha, \beta) \ne (0, 0)$.

   Solution: We first assume that $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \|\mathbf{x} - \mathbf{y}\|$. Then $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big|^2 = \|\mathbf{x} - \mathbf{y}\|^2$, which gives $\|\mathbf{x}\|\,\|\mathbf{y}\| = \mathbf{x}\cdot\mathbf{y}$. So $\|\mathbf{x}\|\,\|\mathbf{y}\| = |\mathbf{x}\cdot\mathbf{y}|$ and hence by the equality condition in Cauchy-Schwarz inequality, we get $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = t\mathbf{y}$ for some $t \in \mathbb{R}$. If $\mathbf{y} = \mathbf{0}$, then by taking $\alpha = 0$, $\beta = 1$, we find that $\alpha\mathbf{x} = \beta\mathbf{y}$ and $\alpha, \beta \ge 0$ with $(\alpha, \beta) \ne (0, 0)$. Again, if $\mathbf{y} \ne \mathbf{0}$ and $\mathbf{x} = t\mathbf{y}$, then since we have $\|\mathbf{x}\|\,\|\mathbf{y}\| = \mathbf{x}\cdot\mathbf{y}$, we obtain $\|t\mathbf{y}\|\,\|\mathbf{y}\| = t\mathbf{y}\cdot\mathbf{y}$, *i.e.* $|t|\,\|\mathbf{y}\|^2 = t\|\mathbf{y}\|^2$. Since $\|\mathbf{y}\| \ne 0$, we get $|t| = t$ and hence $t \ge 0$. Taking $\alpha = 1$, $\beta = t$, we find that $\alpha\mathbf{x} = \beta\mathbf{y}$ and

$\alpha,\ \beta \geq 0$ with $(\alpha, \beta) \neq (0,0)$.

Conversely, let $\alpha\mathbf{x} = \beta\mathbf{y}$ for some $\alpha,\ \beta \geq 0$ with $(\alpha, \beta) \neq (0,0)$. Then $\alpha \neq 0$ or $\beta \neq 0$. We first assume that $\alpha \neq 0$. Then $\mathbf{x} = t\mathbf{y}$, where $t = \frac{\beta}{\alpha} \geq 0$. Now, $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \big|\, \|t\mathbf{y}\| - \|\mathbf{y}\| \,\big| = |t-1|\, \|\mathbf{y}\|$ and $\|\mathbf{x}-\mathbf{y}\| = \|t\mathbf{y}-\mathbf{y}\| = |t-1|\,\|\mathbf{y}\|$. Therefore $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \|\mathbf{x}-\mathbf{y}\|$. Similarly we obtain $\big|\, \|\mathbf{x}\| - \|\mathbf{y}\| \,\big| = \|\mathbf{x}-\mathbf{y}\|$ if we assume that $\beta \neq 0$.

7. Let $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^m$ and $r > 0$ such that $\mathbf{y} \cdot \mathbf{z} = 0$ for all $\mathbf{z} \in B_r(\mathbf{x})$. Show that $\mathbf{y} = \mathbf{0}$.

Solution: If possible, let $\mathbf{y} \neq \mathbf{0}$. Then $\|\mathbf{y}\| \neq 0$. If $\mathbf{z} = \mathbf{x} + \frac{r}{2\|\mathbf{y}\|}\mathbf{y}$, then $\mathbf{z} \in \mathbb{R}^m$ and since $\|\mathbf{z}-\mathbf{x}\| = \frac{r}{2} < r$, $\mathbf{z} \in B_r(\mathbf{x})$. Hence $\mathbf{y} \cdot \mathbf{z} = 0$ and so $\mathbf{y} \cdot \mathbf{x} + \frac{r}{2\|\mathbf{y}\|}\|\mathbf{y}\|^2 = 0$. Since $\mathbf{x} \in B_r(\mathbf{x})$, $\mathbf{y} \cdot \mathbf{x} = 0$ and so from above, we get $\|\mathbf{y}\| = 0$, which is a contradiction. Therefore $\mathbf{y} = \mathbf{0}$.

8. If $\mathbf{x}_0 \in \mathbb{R}^m$ and $r > 0$, then determine $\sup\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x},\ \mathbf{y} \in B_r(\mathbf{x}_0)\}$ with justification.

Solution: For all $\mathbf{x},\ \mathbf{y} \in B_r(\mathbf{x}_0)$, $\|\mathbf{x}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{x}_0\| + \|\mathbf{x}_0-\mathbf{y}\| < r + r = 2r$ and so $2r$ is an upper bound of $\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x}, \mathbf{y} \in B_r(\mathbf{x}_0)\}$. Let $\varepsilon > 0$ such that $\varepsilon < r$. Then $\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1,\ \mathbf{x}_0 - (r - \frac{\varepsilon}{3})\mathbf{e}_1 \in \mathbb{R}^m$ and since $\|\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1 - \mathbf{x}_0\| = r - \frac{\varepsilon}{3} < r$, $\|\mathbf{x}_0 - (r - \frac{\varepsilon}{3})\mathbf{e}_1 - \mathbf{x}_0\| = r - \frac{\varepsilon}{3} < r$, we have $\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1,\ \mathbf{x}_0 - (r - \frac{\varepsilon}{3})\mathbf{e}_1 \in B_r(\mathbf{x}_0)$. Also, $\big\|\big(\mathbf{x}_0 + (r - \frac{\varepsilon}{3})\mathbf{e}_1\big) - \big(\mathbf{x}_0 - (r - \frac{\varepsilon}{3})\mathbf{e}_1\big)\big\| = 2r - \frac{2\varepsilon}{3} > 2r - \varepsilon$ and hence $2r - \varepsilon$ is not an upper bound of $\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x},\ \mathbf{y} \in B_r(\mathbf{x}_0)\}$. Therefore $\sup\{\|\mathbf{x}-\mathbf{y}\| : \mathbf{x},\ \mathbf{y} \in B_r(\mathbf{x}_0)\} = 2r$.

9. Let $S \subseteq \mathbb{R}^m$ such that $S \subseteq B_r[\mathbf{x}_0]$ for some $\mathbf{x}_0 \in \mathbb{R}^m$ and for some $r > 0$. Show that S is a bounded set.

Solution: If $\mathbf{x} \in S$, then $\mathbf{x} \in B_r[\mathbf{x}_0]$ and hence $\|\mathbf{x}\| = \|\mathbf{x}-\mathbf{x}_0+\mathbf{x}_0\| \leq \|\mathbf{x}-\mathbf{x}_0\|+\|\mathbf{x}_0\| \leq r+\|\mathbf{x}_0\|$. Therefore S is a bounded set in $\mathbb{R}^m$.

10. Let $\alpha \in (0,1)$ and let $\mathbf{x}_n = \big(n^3\alpha^n,\ \frac{1}{n}[n\alpha]\big)$ for all $n \in \mathbb{N}$. (For each $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding x.) Examine whether the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$. Also, find $\lim\limits_{n\to\infty} \mathbf{x}_n$ if the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$.

Solution: Let $x_n = n^3\alpha^n$ and $y_n = \frac{1}{n}[n\alpha]$ for all $n \in \mathbb{N}$.
Since $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim\limits_{n\to\infty} (1 + \frac{1}{n})^3\alpha = \alpha < 1$, the sequence (x_n) converges in $\mathbb{R}$ to 0. Again, since $[n\alpha] \leq n\alpha < [n\alpha] + 1$ for all $n \in \mathbb{N}$, we have $n\alpha - 1 < [n\alpha] \leq n\alpha$ for all $n \in \mathbb{N}$ and so it follows that $\alpha - \frac{1}{n} < y_n \leq \alpha$ for all $n \in \mathbb{N}$. Hence by sandwich theorem, the sequence (y_n) converges in $\mathbb{R}$ to α. Therefore the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^2$ and $\lim\limits_{n\to\infty} \mathbf{x}_n = (0, \alpha)$.

11. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that the series $\sum\limits_{n=1}^{\infty} n^2\|\mathbf{x}_n\|^2$ is convergent. Show that the series $\sum\limits_{n=1}^{\infty} \|\mathbf{x}_n\|$ is convergent.

Solution: For all $n \in \mathbb{N}$, using Cauchy-Schwarz inequality, we have
$\sum\limits_{k=1}^{n} \|\mathbf{x}_k\| = \sum\limits_{k=1}^{n} k\,\|\mathbf{x}_k\|\,\frac{1}{k} \leq \Big(\sum\limits_{k=1}^{n} k^2\|\mathbf{x}_k\|^2\Big)^{\frac{1}{2}}\Big(\sum\limits_{k=1}^{n} \frac{1}{k^2}\Big)^{\frac{1}{2}} \leq \Big(\sum\limits_{k=1}^{\infty} k^2\|\mathbf{x}_k\|^2\Big)^{\frac{1}{2}}\Big(\sum\limits_{k=1}^{\infty} \frac{1}{k^2}\Big)^{\frac{1}{2}} < \infty$. This shows that the sequence $\Big(\sum\limits_{k=1}^{n} \|\mathbf{x}_k\|\Big)$ of partial sums of the series $\sum\limits_{n=1}^{\infty} \|\mathbf{x}_n\|$ of non-negative real

numbers is bounded above and hence the sequence $\left(\sum_{k=1}^{n} \|\mathbf{x}_k\|\right)$ converges in $\mathbb{R}$. Consequently the series $\sum_{n=1}^{\infty} \|\mathbf{x}_n\|$ is convergent in $\mathbb{R}$.

12. Let $(\mathbf{x}_n)$ and $(\mathbf{y}_n)$ be sequences in $\mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y}_n \to \mathbf{y} \in \mathbb{R}^m$. Show that $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$ and $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$.

    Solution: Since $\mathbf{x}_n \to \mathbf{x}$ and $\mathbf{y}_n \to \mathbf{y}$, $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ and $\|\mathbf{y}_n - \mathbf{y}\| \to 0$. Hence $\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x} + \mathbf{y})\| \le \|\mathbf{x}_n - \mathbf{x}\| + \|\mathbf{y}_n - \mathbf{y}\| \to 0$. Therefore $\|(\mathbf{x}_n + \mathbf{y}_n) - (\mathbf{x} + \mathbf{y})\| \to 0$ and so $\mathbf{x}_n + \mathbf{y}_n \to \mathbf{x} + \mathbf{y}$.
    Again, $|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x}_n \cdot \mathbf{y} + \mathbf{x}_n \cdot \mathbf{y} - \mathbf{x} \cdot \mathbf{y}| = |\mathbf{x}_n \cdot (\mathbf{y}_n - \mathbf{y}) + (\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{y}|$
    $\le |\mathbf{x}_n \cdot (\mathbf{y}_n - \mathbf{y})| + |(\mathbf{x}_n - \mathbf{x}) \cdot \mathbf{y}| \le \|\mathbf{x}_n\| \, \|\mathbf{y}_n - \mathbf{y}\| + \|\mathbf{x}_n - \mathbf{x}\| \, \|\mathbf{y}\|$ for all $n \in \mathbb{N}$. Since $(\mathbf{x}_n)$ is a convergent sequence in $\mathbb{R}^m$, $(\mathbf{x}_n)$ is bounded in $\mathbb{R}^m$. Hence there exists $r > 0$ such that $\|\mathbf{x}_n\| \le r$ for all $n \in \mathbb{N}$. Therefore $|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}_n\| \, \|\mathbf{y}_n - \mathbf{y}\| + \|\mathbf{x}_n - \mathbf{x}\| \, \|\mathbf{y}\| \to 0$ and so $|\mathbf{x}_n \cdot \mathbf{y}_n - \mathbf{x} \cdot \mathbf{y}| \to 0$. Hence $\mathbf{x}_n \cdot \mathbf{y}_n \to \mathbf{x} \cdot \mathbf{y}$.

13. Let $\mathbf{x} \in \mathbb{R}^m$ and let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$ such that $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$ and $\mathbf{x}_n \cdot \mathbf{x} \to \mathbf{x} \cdot \mathbf{x}$. Show that $(\mathbf{x}_n)$ is convergent.

    Solution: Since $\|\mathbf{x}_n - \mathbf{x}\|^2 = \|\mathbf{x}_n\|^2 - 2\,\mathbf{x}_n \cdot \mathbf{x} + \|\mathbf{x}\|^2 \to \|\mathbf{x}\|^2 - 2\,\mathbf{x} \cdot \mathbf{x} + \|\mathbf{x}\|^2 = 2\|\mathbf{x}\|^2 - 2\|\mathbf{x}\|^2 = 0$, we have that $\|\mathbf{x}_n - \mathbf{x}\| \to 0$ and hence $\mathbf{x}_n \to \mathbf{x}$. Therefore $(\mathbf{x}_n)$ is convergent in $\mathbb{R}^m$.

14. State TRUE or FALSE with justification: If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \ne \mathbf{y}$ and $\|\mathbf{x}\| = 1 = \|\mathbf{y}\|$, then it is necessary that $\|\mathbf{x} + \mathbf{y}\| < 2$.

    Solution: We have $\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\,\mathbf{x} \cdot \mathbf{y} = 2 + 2\,\mathbf{x} \cdot \mathbf{y}$ and $\|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\,\mathbf{x} \cdot \mathbf{y} = 2 - 2\,\mathbf{x} \cdot \mathbf{y}$. Hence $\|\mathbf{x} + \mathbf{y}\|^2 = 2 + 2 - \|\mathbf{x} - \mathbf{y}\|^2 < 4$, since $\|\mathbf{x} - \mathbf{y}\| > 0$. So $\|\mathbf{x} + \mathbf{y}\| < 2$. Therefore the given statement is TRUE.

15. State TRUE or FALSE with justification: If $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^m$ such that for each $\mathbf{x} \in \mathbb{R}^m$, $\lim\limits_{n\to\infty} \mathbf{x}_n \cdot \mathbf{x}$ exists (in $\mathbb{R}$), then $\lim\limits_{n\to\infty} \|\mathbf{x}_n\|^2$ must exist (in $\mathbb{R}$).

    Solution: For each $n \in \mathbb{N}$, let $\mathbf{x}_n = \left(x_1^{(n)}, ..., x_m^{(n)}\right)$.
    By the given condition, $\lim\limits_{n\to\infty} x_j^{(n)} = \lim\limits_{n\to\infty} \mathbf{x}_n \cdot \mathbf{e}_j$ exists (in $\mathbb{R}$) for $j = 1, \dots, m$. Consequently $\lim\limits_{n\to\infty} \|\mathbf{x}_n\|^2 = \lim\limits_{n\to\infty} \left((x_1^{(n)})^2 + \cdots + (x_m^{(n)})^2\right)$ exists (in $\mathbb{R}$). Therefore the given statement is TRUE.

16. State TRUE or FALSE with justification: There exists an unbounded sequence (x_n) of distinct real numbers such that the sequence $\left((x_n, \cos x_n)\right)$ in $\mathbb{R}^2$ has a convergent subsequence.

    Solution: The sequence $(x_n) = \left(1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, \dots\right)$ in $\mathbb{R}$ is unbounded and its subsequence $(x_{2n}) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$ converges in $\mathbb{R}$. By continuity of the cosine function, the sequence $(\cos x_{2n})$ also converges in $\mathbb{R}$. Hence the subsequence $\left((x_{2n}, \cos x_{2n})\right)$ of the sequence $\left((x_n, \cos x_n)\right)$ converges in $\mathbb{R}^2$. Therefore the given statement is TRUE.

17. Let $S = \{(x,y) \in \mathbb{R}^2 : x \neq y\}$ and let $f : S \to \mathbb{R}$ be defined by $f(x,y) = \frac{x+y}{x-y}$ for all $(x,y) \in S$. Show by using the definition of continuity that f is continuous at $(1,2)$.

Solution: Let $\varepsilon > 0$. For all $(x,y) \in S$, we have $|f(x,y) - f(1,2)| = \left|\frac{x+y}{x-y} + 3\right| = 2\left|\frac{2x-y}{x-y}\right|$. If $(x,y) \in S$ and $\|(x,y)-(1,2)\| = \sqrt{(x-1)^2 + (y-2)^2} < \frac{1}{4}$, then $|x-1| < \frac{1}{4}$ and $|y-2| < \frac{1}{4}$, and so $|x-y| = |1-((2-y)+(x-1))| \geq 1-|(2-y)+(x-1)| \geq 1-(|2-y|+|x-1|) \geq 1-(\frac{1}{4}+\frac{1}{4}) = \frac{1}{2}$. Again, if $r > 0$ and $(x,y) \in S$ such that $\|(x,y) - (1,2)\| = \sqrt{(x-1)^2 + (y-2)^2} < r$, then $|x-1| < r$ and $|y-2| < r$, and so $|2x-y| = |2(x-1)+2-y| \leq 2|x-1| + |y-2| < 3r$. Hence if we choose $\delta = \min\{\frac{1}{4}, \frac{\varepsilon}{12}\}$, then $\delta > 0$ and for all $(x,y) \in S$ satisfying $\|(x,y) - (1,2)\| < \delta$, we have $|f(x,y) - f(1,2)| < 12\delta \leq \varepsilon$. Therefore f is continuous at $(1,2)$.

18. If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and $f(x,y) = x^2 + y^2$ for all $x \in \mathbb{Q}$ and for all $y \in \mathbb{R} \setminus \mathbb{Q}$, then determine $f(\sqrt{2}, 2)$.

Solution: We know that there exist sequences (x_n) in $\mathbb{Q}$ and (y_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to \sqrt{2}$ and $y_n \to 2$. Hence $(x_n, y_n) \to (\sqrt{2}, 2)$. Since f is continuous at $(\sqrt{2}, 2)$, we have
$f(\sqrt{2}, 2) = \lim\limits_{n\to\infty} f(x_n, y_n) = \lim\limits_{n\to\infty} (x_n^2 + y_n^2) = \lim\limits_{n\to\infty} x_n^2 + \lim\limits_{n\to\infty} y_n^2 = 2 + 4 = 6.$

19. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} xy & \text{if } xy \geq 0, \\ -xy & \text{if } xy < 0. \end{cases}$

Solution: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. We have $|f(x_n, y_n)| = |x_n y_n| \to 0$ and hence $f(x_n, y_n) \to 0 = f(0,0)$. Therefore f is continuous at $(0,0)$.

20. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^4} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

Solution: Let $\varepsilon > 0$. Then for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have
$|f(x,y) - f(0,0)| = \frac{|x|y^2}{x^2+y^4}|y| \leq \frac{1}{2}|y| \leq \frac{1}{2}\sqrt{x^2+y^2}$.
Since $f(0,0) = 0$, we get $|f(x,y) - f(0,0)| \leq \frac{1}{2}\sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let $\delta = 2\varepsilon$. Then $\delta > 0$ and for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y)-(0,0)\| = \sqrt{x^2+y^2} < \delta$, we have $|f(x,y)-f(0,0)| < \varepsilon$. Therefore f is continuous at $(0,0)$.

21. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$, where for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} 1 & \text{if } x > 0 \text{ and } 0 < y < x^2, \\ 0 & \text{otherwise.} \end{cases}$

Solution: Since $\left(\frac{1}{n}, \frac{1}{2n^2}\right) \to (0,0)$ but $f\left(\frac{1}{n}, \frac{1}{2n^2}\right) = 1 \to 1 \neq 0 = f(0,0)$, f is not continuous at $(0,0)$.

22. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} \frac{xy}{x-y} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$

Solution: If $\varphi(x,y) = xy$ and $\psi(x,y) = x - y$ for all $(x,y) \in \mathbb{R}^2$, then as polynomial functions,

φ, $\psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\psi(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ with $x \neq y$. Hence f is continuous at each $(x,y) \in \mathbb{R}^2$ with $x \neq y$.

Let $x \in \mathbb{R} \setminus \{0\}$. Then $(x + \frac{1}{n}, x) \to (x,x)$ but $f(x+\frac{1}{n}, x) = nx^2 + x \not\to 0 = f(x,x)$. So f is not continuous at (x,x).

Again, $(\frac{1}{n} + \frac{1}{n^2}, \frac{1}{n}) \to (0,0)$ but $f(\frac{1}{n} + \frac{1}{n^2}, \frac{1}{n}) = 1 + \frac{1}{n} \to 1 \neq 0 = f(0,0)$. So f is not continuous at $(0,0)$.

Therefore the set of points of continuity of f is $\{(x,y) \in \mathbb{R}^2 : x \neq y\}$.

23. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, if for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} xy & \text{if } xy \in \mathbb{Q}, \\ -xy & \text{if } xy \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$

Solution: Let $(x,y) \in \mathbb{R}^2$ such that $xy = 0$ and let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x,y)$. Then $x_n \to x$ and $y_n \to y$. We have $|f(x_n,y_n)| = |x_n y_n| \to |xy| = 0$ and so $f(x_n, y_n) \to 0 = f(x,y)$. Hence f is continuous at (x,y).

Again, let $(x,y) \in \mathbb{R}^2$ such that $xy \neq 0$. We consider the following two possible cases.

Case (i): $xy \in \mathbb{R} \setminus \mathbb{Q}$.

We can find two sequences (x_n) and (y_n) in $\mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$. Then $\big((x_n, y_n)\big)$ is a sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x,y)$ but $f(x_n, y_n) = x_n y_n \to xy \neq -xy = f(x,y)$. Hence f is not continuous at (x,y).

Case (ii): $xy \in \mathbb{Q}$.

Since $x \neq 0$, we can find a sequence (x_n) in $\mathbb{Q} \setminus \{0\}$ and a sequence (y_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $x_n \to x$ and $y_n \to y$. Then $\big((x_n, y_n)\big)$ is a sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x,y)$ but $f(x_n, y_n) = -x_n y_n \to -xy \neq xy = f(x,y)$. Hence f is not continuous at (x,y).

Therefore the set of points of continuity of f is $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$.

24. Let α, β be positive real numbers and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Show that f is continuous iff $\alpha + \beta > 2$.

Solution: Let $\alpha + \beta > 2$ and let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. For all $n \in \mathbb{N}$ for which $(x_n, y_n) \neq (0,0)$, we have
$0 \leq f(x_n, y_n) \leq \frac{(x_n^2+y_n^2)^{\frac{\alpha}{2}} (x_n^2+y_n^2)^{\frac{\beta}{2}}}{\frac{1}{2}(x_n^2+y_n^2)} = 2(x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-2)}$ and since $f(0,0) = 0$,
we have $0 \leq f(x_n, y_n) \leq 2(x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-2)}$ for all $n \in \mathbb{N}$. Since $2(x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-2)} \to 0$, we get $f(x_n, y_n) \to 0 = f(0,0)$. This shows that f is continuous at $(0,0)$. Also, it is clear (by similar arguments given in other examples) that f is continuous at each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Therefore f is continuous.

Conversely, let f be continuous and if possible, let $\alpha + \beta \leq 2$. We have $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{3} n^{2-(\alpha+\beta)} \not\to 0 = f(0,0)$ (because for $\alpha + \beta = 2$, $f(\frac{1}{n}, \frac{1}{n}) \to \frac{1}{3}$ and for $\alpha + \beta < 2$, the sequence $\big(f(\frac{1}{n}, \frac{1}{n})\big)$ is unbounded). Hence f is not continuous at $(0,0)$, which is a contradiction. Therefore $\alpha + \beta > 2$.

25. Let S be a nonempty subset of $\mathbb{R}^m$ and let $f_j : S \to \mathbb{R}$ for each $j \in \{1, \ldots, k\}$. If $f(\mathbf{x}) = (f_1(\mathbf{x}), \ldots, f_k(\mathbf{x}))$ for all $\mathbf{x} \in S$, then show that $f : S \to \mathbb{R}^k$ is continuous at $\mathbf{x}_0 \in S$ iff f_j is continuous at $\mathbf{x}_0$ for each $j \in \{1, \ldots, k\}$.

Solution: We first assume that f is continuous at $\mathbf{x}_0$ and let $(\mathbf{x}_n)$ be any sequence in S such that $\mathbf{x}_n \to \mathbf{x}_0$. Then $\big(f_1(\mathbf{x}_n), \ldots, f_k(\mathbf{x}_n)\big) = f(\mathbf{x}_n) \to f(\mathbf{x}_0) = \big(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0)\big)$ and hence $f_j(\mathbf{x}_n) \to f_j(\mathbf{x}_0)$ for each $j \in \{1, \ldots, k\}$. Consequently f_j is continuous at $\mathbf{x}_0$ for each $j \in \{1, \ldots, k\}$.
Conversely, let f_j be continuous at $\mathbf{x}_0$ for each $j \in \{1, \ldots, k\}$ and let $(\mathbf{x}_n)$ be any sequence in S such that $\mathbf{x}_n \to \mathbf{x}_0$. Then $f_j(\mathbf{x}_n) \to f_j(\mathbf{x}_0)$ for each $j \in \{1, \ldots, k\}$ and hence
$f(\mathbf{x}_n) = \big(f_1(\mathbf{x}_n), \ldots, f_k(\mathbf{x}_n)\big) \to \big(f_1(\mathbf{x}_0), \ldots, f_k(\mathbf{x}_0)\big) = f(\mathbf{x}_0)$. Therefore f is continuous at $\mathbf{x}_0$.

26. Examine the continuity of $f : \mathbb{R}^2 \to \mathbb{R}^2$ at $(0,0)$, where for all $(x, y) \in \mathbb{R}^2$,
$$f(x,y) = \begin{cases} \left(\frac{x^3}{x^2+y^2}, \sin(x^2+y^2)\right) & \text{if } (x,y) \neq (0,0), \\ (0,0) & \text{if } (x,y) = (0,0). \end{cases}$$

Solution: For all $(x, y) \in \mathbb{R}^2$, let $\varphi(x,y) = \sin(x^2+y^2)$ and $\psi(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Since $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a composition of a polynomial function and the sine function, both of which are continuous, φ is continuous at $(0,0)$.
Again, let $\varepsilon > 0$. Then for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have
$|\psi(x,y) - \psi(0,0)| = \frac{x^2}{x^2+y^4}|x| \le |x| \le \sqrt{x^2+y^2}$.
Since $\psi(0,0) = 0$, we get $|\psi(x,y) - \psi(0,0)| \le \sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2$. Let $\delta = \varepsilon$. Then $\delta > 0$ and for all $(x,y) \in \mathbb{R}^2$ with $\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$, we have $|\psi(x,y) - \psi(0,0)| < \varepsilon$. Therefore ψ is continuous at $(0,0)$.
Consequently (by Ex.17 of Practice Problem Set - 1) f is continuous at $(0,0)$.

27. If $f, g : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ are continuous at $\mathbf{x}_0 \in S$ and if $\varphi(\mathbf{x}) = f(\mathbf{x}) \cdot g(\mathbf{x})$ for all $\mathbf{x} \in S$, then show that $\varphi : S \to \mathbb{R}$ is continuous at $\mathbf{x}_0$.

Solution: Let $(\mathbf{x}_n)$ be any sequence in S such that $\mathbf{x}_n \to \mathbf{x}_0$. Since f and g are continuous at $\mathbf{x}_0$, $f(\mathbf{x}_n) \to f(\mathbf{x}_0)$ and $g(\mathbf{x}_n) \to g(\mathbf{x}_0)$. Hence (by Ex.9 of Practice Problem Set - 1) $\varphi(\mathbf{x}_n) = f(\mathbf{x}_n) \cdot g(\mathbf{x}_n) \to f(\mathbf{x}_0) \cdot g(\mathbf{x}_0) = \varphi(\mathbf{x}_0)$. Therefore φ is continuous at $\mathbf{x}_0$.

28. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous at $\mathbf{x}_0 \in S^0$ and let $f(\mathbf{x}_0) \neq \mathbf{0}$. Show that there exists $r > 0$ such that $f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in B_r(\mathbf{x}_0)$.

Solution: Since $\mathbf{x}_0 \in S^0$, there exists $s > 0$ such that $B_s(\mathbf{x}_0) \subseteq S$. Again, since $f(\mathbf{x}_0) \neq \mathbf{0}$, $\frac{1}{2}\|f(\mathbf{x}_0)\| > 0$. By the continuity of f at $\mathbf{x}_0$, there exists $\delta > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \frac{1}{2}\|f(\mathbf{x}_0)\|$ for all $\mathbf{x} \in S$ satisfying $\|\mathbf{x} - \mathbf{x}_0\| < \delta$. Taking $r = \min\{s, \delta\} > 0$, we find that $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \frac{1}{2}\|f(\mathbf{x}_0)\|$ for all $\mathbf{x} \in B_r(\mathbf{x}_0)$. If possible, let $f(\mathbf{x}) = \mathbf{0}$ for some $\mathbf{x} \in B_r(\mathbf{x}_0)$. Then from above, we get $\|f(\mathbf{x}_0)\| < \frac{1}{2}\|f(\mathbf{x}_0)\|$, which is not true. Therefore $f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in B_r(\mathbf{x}_0)$.

29. Let S be an open subset of $\mathbb{R}^m$ and let $f : S \to \mathbb{R}^k$ and $g : S \to \mathbb{R}^k$ be continuous at $\mathbf{x}_0 \in S$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

Solution: By the given condition, for each $n \in \mathbb{N}$, there exist $\mathbf{x}_n$, $\mathbf{y}_n \in B_{\frac{1}{n}}(\mathbf{x}_0)$ such that $f(\mathbf{x}_n) = g(\mathbf{y}_n)$. So $\|\mathbf{x}_n - \mathbf{x}_0\| < \frac{1}{n} \to 0$ and $\|\mathbf{y}_n - \mathbf{x}_0\| < \frac{1}{n} \to 0$. Hence $\mathbf{x}_n \to \mathbf{x}_0$ and $\mathbf{y}_n \to \mathbf{x}_0$. Since f and g are continuous at $\mathbf{x}_0$, $f(\mathbf{x}_n) \to f(\mathbf{x}_0)$ and $g(\mathbf{y}_n) \to g(\mathbf{x}_0)$. Therefore $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

30. If $S = \{(x, y) \in \mathbb{R}^2 : x + y \geq 2\}$, then determine (with justification) S^0.

Solution: Let $(x_0, y_0) \in S$ with $x_0 + y_0 > 2$. Let $r = \frac{x_0+y_0-2}{\sqrt{2}} > 0$ and let $(x, y) \in B_r\big((x_0, y_0)\big)$. Then $\|(x, y)-(x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < r$. By Cauchy-Schwarz inequality, we have $x_0 - x + y_0 - y \leq \sqrt{(x_0 - x)^2 + (y_0 - y)^2}\,\sqrt{1^2 + 1^2} < \sqrt{2}r = x_0 + y_0 - 2$. Hence $x + y > 2$ and so $(x, y) \in S$. Thus $B_r\big((x_0, y_0)\big) \subseteq S$ and therefore $(x_0, y_0) \in S^0$.
Now, let $(x_0, y_0) \in S$ such that $x_0 + y_0 = 2$ and if possible, let $(x_0, y_0) \in S^0$. Then there exists $r > 0$ such that $B_r\big((x_0, y_0)\big) \subseteq S$. Since $\|(x_0 - \frac{r}{2}, y_0) - (x_0, y_0)\| = \|(-\frac{r}{2}, 0)\| = \frac{r}{2} < r$, $(x_0 - \frac{r}{2}, y_0) \in B_r\big((x_0, y_0)\big)$. However, $(x_0 - \frac{r}{2}, y_0) \notin S$, since $x_0 - \frac{r}{2} + y_0 = 2 - \frac{r}{2} < 2$. Thus we get a contradiction. Hence $(x_0, y_0) \notin S^0$.
Therefore $S^0 = \{(x, y) \in \mathbb{R}^2 : x + y > 2\}$.

31. If $S = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m = 1\}$, then determine (with justification) S^0.

Solution: If possible, let $S^0 \neq \emptyset$. Then there exists $\mathbf{x} = (x_1, \ldots, x_m) \in S^0$ and hence there exists $r > 0$ such that $B_r(\mathbf{x}) \subseteq S$. If $\mathbf{y} = (x_1, \ldots, x_{m-1}, x_m + \frac{r}{2})$, then $\|\mathbf{y} - \mathbf{x}\| = \frac{r}{2} < r$ and so $\mathbf{y} \in B_r(\mathbf{x})$. But $\mathbf{y} \notin S$, because $x_m + \frac{r}{2} = 1 + \frac{r}{2} \neq 1$. Thus we get a contradiction. Therefore $S^0 = \emptyset$.

32. If $\mathbf{x} \in \mathbb{R}^m$ and $r > 0$, then determine (with justification) all the interior points of $B_r[\mathbf{x}]$.

Solution: Let $\mathbf{y} \in B_r(\mathbf{x})$. Then $\|\mathbf{y} - \mathbf{x}\| < r$. If $s = r - \|\mathbf{y} - \mathbf{x}\|$, then $s > 0$. Let $\mathbf{z} \in B_s(\mathbf{y})$. Then $\|\mathbf{z} - \mathbf{y}\| < s$ and so $\|\mathbf{z} - \mathbf{x}\| = \|\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{x}\| \leq \|\mathbf{z} - \mathbf{y}\| + \|\mathbf{y} - \mathbf{x}\| < s + \|\mathbf{y} - \mathbf{x}\| = r$. Hence $\mathbf{z} \in B_r[\mathbf{x}]$ and so $B_s(\mathbf{y}) \subseteq B_r[\mathbf{x}]$. Therefore $\mathbf{y} \in \big(B_r[\mathbf{x}]\big)^0$.
Again, let $\mathbf{y} \in B_r[\mathbf{x}]$ such that $\|\mathbf{y} - \mathbf{x}\| = r$. If possible, let $\mathbf{y} \in \big(B_r[\mathbf{x}]\big)^0$. Then there exists $s > 0$ such that $B_s(\mathbf{y}) \subseteq B_r[\mathbf{x}]$. Now, $\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) \in B_s(\mathbf{y})$, since
$\|\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) - \mathbf{y}\| = \frac{s}{2r}\|\mathbf{y} - \mathbf{x}\| = \frac{s}{2} < s$, but $\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) \notin B_r[\mathbf{x}]$, because
$\|\mathbf{y} + \frac{s}{2r}(\mathbf{y} - \mathbf{x}) - \mathbf{x}\| = (1 + \frac{s}{2r})\|\mathbf{y} - \mathbf{x}\| = r + \frac{s}{2} > r$. Thus we get a contradiction. Hence $\mathbf{y} \notin \big(B_r[\mathbf{x}]\big)^0$.
Therefore $\big(B_r[\mathbf{x}]\big)^0 = B_r(\mathbf{x})$.

33. Examine whether $\{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in $\mathbb{R}^2$.

Solution: Let $S = \{(x, y) \in \mathbb{R}^2 : 0 < x < y\}$ and let $(x_0, y_0) \in S$. If $r = \min\big\{x_0, \frac{y_0-x_0}{\sqrt{2}}\big\}$, then $r > 0$. Let $(x, y) \in B_r\big((x_0, y_0)\big)$. Then $\|(x, y) - (x_0, y_0)\| = \sqrt{(x - x_0)^2 + (y - y_0)^2} < r$. Hence $x_0 - x \leq |x - x_0| < r \leq x_0$ and so $x > 0$. Also, using Cauchy-Schwarz inequality, we have $x - x_0 + y_0 - y \leq \sqrt{(x - x_0)^2 + (y_0 - y)^2}\,\sqrt{1^2 + 1^2} < \sqrt{2}r \leq y_0 - x_0$ and hence $x - y < 0$, *i.e.*

$x < y$. Thus $(x, y) \in S$ and so $(x_0, y_0) \in S^0$. Since $(x_0, y_0) \in S$ is arbitrary, it follows that S is an open set in $\mathbb{R}^2$.

MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 2

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1. Examine whether the set $\{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have already shown in Ex.25 of Practice Problem Set - 1 that
   $S = \{(x,y) \in \mathbb{R}^2 : 0 < x < y\}$ is an open set in $\mathbb{R}^2$.
   Again, since $\left(\frac{1}{2n}, \frac{1}{n}\right) \in S$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{2n}, \frac{1}{n}\right) \to (0,0) \notin S$, S is not a closed set in $\mathbb{R}^2$.

2. Examine whether the set $\{(x,x) : x \in \mathbb{R}\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $(0,0) \in S = \{(x,x) : x \in \mathbb{R}\}$. If possible, let $(0,0) \in S^0$. Then there exists $r > 0$ such that $B_r\big((0,0)\big) \subseteq S$. Since $(\frac{r}{2}, 0) \in B_r\big((0,0)\big)$ but $(\frac{r}{2}, 0) \notin S$, we get a contradiction. Hence $(0,0) \notin S^0$. Therefore S is not an open set in $\mathbb{R}^2$.
   Again, let $\big((x_n, x_n)\big)$ be any sequence in S such that $(x_n, x_n) \to (x,y) \in \mathbb{R}^2$. Then $x_n \to x$ and $x_n \to y$. Hence $x = y$ and so $(x,y) \in S$. Therefore S is a closed set in $\mathbb{R}^2$.

3. Examine whether the set $\{(x,y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $(0,0) \in S = \{(x,y) \in \mathbb{R}^2 : y \in \mathbb{Z}\}$. If possible, let $(0,0) \in S^0$. Then there exists $r > 0$ such that $B_r\big((0,0)\big) \subseteq S$. If $s = \min\{\frac{1}{2}, \frac{r}{2}\}$, then $(0,s) \in B_r\big((0,0)\big)$ but $(0,s) \notin S$. Thus we get a contradiction. Hence $(0,0) \notin S^0$ and therefore S is not an open set in $\mathbb{R}^2$.
   Again, let $\big((x_n, y_n)\big)$ be any sequence in S such that $(x_n, y_n) \to (x,y) \in \mathbb{R}^2$. Then $y_n \to y$. Hence there exists $n_0 \in \mathbb{N}$ such that $|y_n - y| < \frac{1}{2}$ for all $n \geq n_0$ and hence
   $|y_n - y_{n_0}| \leq |y_n - y| + |y - y_{n_0}| < 1$ for all $n \geq n_0$. Since $y_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$, we get $y_n = y_{n_0}$ for all $n \geq n_0$ and so $y_n \to y_{n_0}$. Consequently $y = y_{n_0} \in \mathbb{Z}$ and so $(x,y) \in S$. Therefore S is a closed set in $\mathbb{R}^2$.

4. Examine whether the set $(0,1) \times \{0\}$ is (a) open (b) closed in $\mathbb{R}^2$.

   Solution: We have $(\frac{1}{2}, 0) \in (0,1) \times \{0\}$. If possible, let $(\frac{1}{2}, 0) \in \big((0,1) \times \{0\}\big)^0$. Then there exists $r > 0$ such that $B_r\big((\frac{1}{2}, 0)\big) \subseteq (0,1) \times \{0\}$. Since $(\frac{1}{2}, \frac{r}{2}) \in B_r\big((\frac{1}{2}, 0)\big)$ but $(\frac{1}{2}, \frac{r}{2}) \notin (0,1) \times \{0\}$, we get a contradiction. Hence $(\frac{1}{2}, 0) \notin \big((0,1) \times \{0\}\big)^0$. Therefore $(0,1) \times \{0\}$ is not an open set in $\mathbb{R}^2$.
   Again, since $\left(\frac{1}{n+1}, 0\right) \in (0,1) \times \{0\}$ for all $n \in \mathbb{N}$ and $\left(\frac{1}{n+1}, 0\right) \to (0,0) \notin (0,1) \times \{0\}$, $(0,1) \times \{0\}$ is not a closed set in $\mathbb{R}^2$.

5. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ is an open set in $\mathbb{R}^m$.

   Solution: Let $(\mathbf{x}_n)$ be any sequence in $\mathbb{R}^m \setminus S$, where $S = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) > 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \leq 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \leq 0$. Thus $\mathbf{x} \in \mathbb{R}^m \setminus S$ and therefore $\mathbb{R}^m \setminus S$ is a closed set in $\mathbb{R}^m$. Consequently S is an open set in $\mathbb{R}^m$.

6. If $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, then show that $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and $\{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ are closed sets in $\mathbb{R}^m$.

   Solution: Let $(\mathbf{x}_n)$ be any sequence in $S_1 = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) \geq 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_1$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) \geq 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) \geq 0$. Thus $\mathbf{x} \in S_1$ and therefore S_1 is a closed set in $\mathbb{R}^m$.
   Again, let $(\mathbf{x}_n)$ be any sequence in $S_2 = \{\mathbf{x} \in \mathbb{R}^m : f(\mathbf{x}) = 0\}$ and let $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$. Since f is continuous at $\mathbf{x}$, $f(\mathbf{x}_n) \to f(\mathbf{x})$. Also, since $\mathbf{x}_n \in S_2$ for all $n \in \mathbb{N}$, $f(\mathbf{x}_n) = 0$ for all $n \in \mathbb{N}$ and hence it follows that $f(\mathbf{x}) = 0$. Thus $\mathbf{x} \in S_2$ and therefore S_2 is a closed set in $\mathbb{R}^m$.

7. Using Ex.2 in the Practice Problem Set - 2, show that $\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\,\}$ is an open set in $\mathbb{R}^3$ and $\{(x, y, z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\,\}$ is a closed set in $\mathbb{R}^3$.

   Solution: If $f(x, y, z) = 3|y| - x^2 - 2z$ and $g(x, y, z) = \sin(xyz) - |xy|$ for all $(x, y, z) \in \mathbb{R}^3$, then we know that both $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ are continuous. Hence by Ex.2(a) of Practice Problem Set - 2, $\{(x, y, z) \in \mathbb{R}^3 : x^2 + 2z < 3|y|\,\} = \{(x, y, z) \in \mathbb{R}^3 : f(x, y, z) > 0\}$ is an open set in $\mathbb{R}^3$ and by Ex.2(b) of Practice Problem Set - 2,
   $\{(x, y, z) \in \mathbb{R}^3 : \sin(xyz) = |xy|\} = \{(x, y, z) \in \mathbb{R}^3 : g(x, y, z) = 0\}$ is a closed set in $\mathbb{R}^3$.

8. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{\sin(xy)}{xy} & \text{if } xy \neq 0, \\ 1 & \text{if } xy = 0. \end{cases}$
   Show that f is continuous.

   Solution: If $\varphi(x, y) = xy$ and $\psi(x, y) = \sin(xy)$ for all $(x, y) \in \mathbb{R}^2$, then we know that φ, $\psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\varphi(x, y) \neq 0$ for all $(x, y) \in \mathbb{R}^2$. Hence it follows that f is continuous at each point $(x, y) \in \mathbb{R}^2$ for which $xy \neq 0$.
   Let $(x, y) \in \mathbb{R}^2$ such that $xy = 0$ and let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x, y)$. Then $x_n \to x$, $y_n \to y$ and so $x_n y_n \to xy = 0$. Now $f(x_n, y_n) = \frac{\sin(x_n y_n)}{x_n y_n}$ if $x_n y_n \neq 0$ and $f(x_n, y_n) = 1$ if $x_n y_n = 0$. Since $\lim\limits_{t \to 0} \frac{\sin t}{t} = 1$, it follows that $f(x_n, y_n) \to 1 = f(x, y)$ and consequently f is continuous at (x, y).
   Therefore f is continuous.

9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f(x, y) = e^{-\frac{x^2 - 2xy + y^2}{|x-y|}}$ for all $(x, y) \in \mathbb{R}^2$ with $x \neq y$. If $x \in \mathbb{R}$, then find $f(x, x)$ such that f is continuous on $\mathbb{R}^2$.

   Solution: Since $x^2 - 2xy + y^2 = |x - y|^2$ for all x, $y \in \mathbb{R}$, we find that $f(x, y) = e^{-|x-y|}$ for all $(x, y) \in \mathbb{R}^2$ with $x \neq y$. If $x \in \mathbb{R}$, then $(x + \frac{1}{n}, x) \to (x, x)$ and for f to be continuous at (x, x), we must have $f(x, x) = \lim\limits_{n \to \infty} f(x + \frac{1}{n}, x) = \lim\limits_{n \to \infty} e^{-\frac{1}{n}} = 1$. So, let $f(x, x) = 1$ for all $x \in \mathbb{R}$. If $g(x, y) = -|x - y|$ for all $(x, y) \in \mathbb{R}^2$ and $\varphi(t) = e^t$ for all $t \in \mathbb{R}$, then $f(x, y) = \varphi(g(x, y))$ for all $(x, y) \in \mathbb{R}^2$. Since we know that $g : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ are continuous, hence $f = \varphi \circ g$ is also continuous.

10. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}^k$ be continuous and let $g : \mathbb{R}^m \to \mathbb{R}^k$ be such that $g(\mathbf{x}) = f(\mathbf{x})$ for all $\mathbf{x} \in S$.
    (a) Show that g need not be continuous on S.

(b) If S is an open set in $\mathbb{R}^m$, then show that g is continuous on S.

Solution: (a) Let $f(x, y) = 1$ for all $(x, y) \in S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ and $g(x, y) = \begin{cases} 1 & \text{if } (x, y) \in S, \\ 2 & \text{if } (x, y) \in \mathbb{R}^2 \setminus S. \end{cases}$
Then $f : S \to \mathbb{R}$ is continuous (as a constant function) and $f(x, y) = g(x, y)$ for all $(x, y) \in S$. However, g is not continuous at $(1, 0) \in S$, since $\left(1 + \frac{1}{n}, 0\right) \to (1, 0)$ but
$g\left(1 + \frac{1}{n}, 0\right) = 2 \to 2 \neq 1 = g(1, 0)$.
(b) Let $\mathbf{x}_0 \in S$ and $\varepsilon > 0$. Since S is an open set in $\mathbb{R}^m$, there exists $r > 0$ such that $B_r(\mathbf{x}_0) \subseteq S$. Since f is continuous at $\mathbf{x}_0$, there exists $s > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{x}_0)\| < \varepsilon$ for all $\mathbf{x} \in S \cap B_s(\mathbf{x}_0)$. If $\delta = \min\{r, s\} > 0$, then $B_\delta(\mathbf{x}_0) \subseteq B_r(\mathbf{x}_0) \subseteq S$ and $B_\delta(\mathbf{x}_0) \subseteq B_s(\mathbf{x}_0)$. Hence for all $\mathbf{x} \in B_\delta(\mathbf{x}_0)$, we have $g(\mathbf{x}) = f(\mathbf{x})$ and $\|g(\mathbf{x}) - g(\mathbf{x}_0)\| < \varepsilon$. Therefore g is continuous at $\mathbf{x}_0$. Since $\mathbf{x}_0 \in S$ is arbitrary, g is continuous on S.

11. Let $S_1 = \{(x, y) \in \mathbb{R}^2 : (x-1)^2 + y^2 < 4\}$ and $S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + (y-1)^2 < 9\}$. Does there exist a continuous function from S_1 onto S_2? Justify.

    Solution: Let $\mathbf{u} = (1, 0)$, $\mathbf{v} = (0, 1)$ and let $f(\mathbf{x}) = \mathbf{v} + \frac{3}{2}(\mathbf{x} - \mathbf{u}) = \left(\frac{3x}{2} - \frac{3}{2}, 1 + \frac{3y}{2}\right)$ for all $\mathbf{x} = (x, y) \in S_1$. If $\mathbf{x} \in S_1$, then $\|f(\mathbf{x}) - \mathbf{v}\| = \frac{3}{2}\|\mathbf{x} - \mathbf{u}\| < 3$ and so $f(\mathbf{x}) \in S_2$. Thus f maps S_1 to S_2 and clearly f is continuous (since both the component functions of f are continuous). Again, if $\mathbf{y} \in S_2$, then $\mathbf{x} = \mathbf{u} + \frac{2}{3}(\mathbf{y} - \mathbf{v}) \in \mathbb{R}^2$ and $\|\mathbf{x} - \mathbf{u}\| = \frac{2}{3}\|\mathbf{y} - \mathbf{v}\| < 2$, *i.e.* $\mathbf{x} \in S_1$, and also $f(\mathbf{x}) = \mathbf{y}$. Thus $f : S_1 \to S_2$ is onto. Therefore there exists a continuous function from S_1 onto S_2.

12. If $S = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| < 1\}$, then does there exist a non-constant continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 5$ for all $\mathbf{x} \in S$? Justify.

    Solution: There exists such a function as is shown by the following example.
    Let $f(\mathbf{x}) = \begin{cases} 5 & \text{if } \mathbf{x} \in S, \\ 5\|\mathbf{x}\| & \text{if } \mathbf{x} \in \mathbb{R}^m \setminus S. \end{cases}$
    If $(\mathbf{x}_n)$ is any sequence in $\mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x} \in \mathbb{R}^m$, then using Ex.1(a) of Practice Problem Set - 1, we get $\big|\, \|\mathbf{x}_n\| - \|\mathbf{x}\| \,\big| \leq \|\mathbf{x}_n - \mathbf{x}\| \to 0$ and hence $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$. It follows that $f : \mathbb{R}^m \to \mathbb{R}$ is continuous. Clearly f is a non-constant function and $f(\mathbf{x}) = 5$ for all $\mathbf{x} \in S$.

13. Let $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{x} \neq \mathbf{y}$. Find a continuous function $f : \mathbb{R}^m \to \mathbb{R}$ such that $f(\mathbf{x}) = 1$, $f(\mathbf{y}) = 0$ and $0 \leq f(\mathbf{z}) \leq 1$ for all $\mathbf{z} \in \mathbb{R}^m$.

    Solution: Let $f(\mathbf{z}) = \frac{\|\mathbf{z} - \mathbf{y}\|}{\|\mathbf{z} - \mathbf{x}\| + \|\mathbf{z} - \mathbf{y}\|}$ for all $\mathbf{z} \in \mathbb{R}^m$. If $(\mathbf{z}_n)$ is any sequence in $\mathbb{R}^m$ such that $\mathbf{z}_n \to \mathbf{z} \in \mathbb{R}^m$, then using Ex.1(a) of Practice Problem set - 1, we find that $\|\mathbf{z}_n - \mathbf{x}\| \to \|\mathbf{z} - \mathbf{x}\|$ and $\|\mathbf{z}_n - \mathbf{y}\| \to \|\mathbf{z} - \mathbf{y}\|$. Also, $\|\mathbf{v} - \mathbf{x}\| + \|\mathbf{v} - \mathbf{y}\| \neq 0$ for all $\mathbf{v} \in \mathbb{R}^m$. Hence it follows that $f(\mathbf{z}_n) \to f(\mathbf{z})$ and consequently $f : \mathbb{R}^m \to \mathbb{R}$ is continuous. Clearly $f(\mathbf{x}) = 1$, $f(\mathbf{y}) = 0$ and $0 \leq f(\mathbf{z}) \leq 1$ for all $\mathbf{z} \in \mathbb{R}^m$.

14. Let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous such that $\lim\limits_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 1$. Show that f is bounded on $\mathbb{R}^m$.

    Solution: Since $\lim\limits_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = 1$, there exists $r > 0$ such that $|f(\mathbf{x}) - 1| < 1$ for all $\mathbf{x} \in \mathbb{R}^m$ with

$\|\mathbf{x}\| > r$. Hence $|f(\mathbf{x})| = |f(\mathbf{x})-1+1| \le |f(\mathbf{x})-1|+1 < 2$ for all $\mathbf{x} \in \mathbb{R}^m$ with $\|\mathbf{x}\| > r$. Again, since $S = \{\mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\| \le r\}$ is a closed and bounded subset of $\mathbb{R}^m$ and since $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, $f(S)$ is a bounded subset of $\mathbb{R}$. Hence there exists $K > 0$ such that $|f(\mathbf{x})| \le K$ for all $\mathbf{x} \in S$. If $M = \max\{2, K\}$, then $M > 0$ and $|f(\mathbf{x})| \le M$ for all $\mathbf{x} \in \mathbb{R}^m$. Consequently f is bounded on $\mathbb{R}^m$.

15. State TRUE or FALSE with justification: There exists $r > 0$ such that $\sin(xy) < \cos(xy)$ for all $x,\, y \in [-r,\, r]$.

    Solution: If $f(x,y) = \sin(xy) - \cos(xy)$ for all $(x,y) \in \mathbb{R}^2$, then we know that $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$ and $f(0,0) = -1 < 0$. Hence there exists $\delta > 0$ such that $f(x,y) < 0$, *i.e.* $\sin(xy) < \cos(xy)$ for all $(x,y) \in B_\delta\big((0,0)\big)$. If $r = \frac{\delta}{2} > 0$, then $[-r,r] \times [-r,r] \subseteq B_\delta\big((0,0)\big)$ and hence for all $x,\, y \in [-r,r]$, we have $(x,y) \in B_\delta\big((0,0)\big)$ and consequently $\sin(xy) < \cos(xy)$. Therefore the given statement is TRUE.

16. State TRUE or FALSE with justification: There exists a continuous function $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.

    Solution: Since $(\cos n)$ is a bounded sequence in $\mathbb{R}$, by Bolzano-Weierstrass theorem in $\mathbb{R}$, there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ and $\alpha \in \mathbb{R}$ such that $\cos n_k \to \alpha$. If $f : \mathbb{R} \to \mathbb{R}^2$ is continuous, then $\big(n_k, \frac{1}{n_k}\big) = f(\cos n_k) \to f(\alpha)$ in $\mathbb{R}^2$ and consequently the sequence (n_k) converges in $\mathbb{R}$, which is not true, since (n_k) is unbounded. Hence it follows that no continuous function $f : \mathbb{R} \to \mathbb{R}^2$ can exist satisfying $f(\cos n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$. Therefore the given statement is FALSE.

17. State TRUE or FALSE with justification: There exists a continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ onto $\mathbb{R}^2$.

    Solution: We know that $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\} = B_1\big[(0,0)\big]$ is a closed and bounded set in $\mathbb{R}^2$ and $\mathbb{R}^2$ is not bounded. Hence there cannot exist any continuous function from $B_1\big[(0,0)\big]$ onto $\mathbb{R}^2$.

18. State TRUE or FALSE with justification: There exists a one-one continuous function from $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$.

    Solution: Let $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and let $f(\mathbf{x}) = \frac{1}{1-\|\mathbf{x}\|}\mathbf{x} = \Big(\frac{x}{1-\sqrt{x^2+y^2}}, \frac{y}{1-\sqrt{x^2+y^2}}\Big)$ for all $\mathbf{x} = (x,y) \in S$. If $\mathbf{x} \in S$ and $(\mathbf{x}_n)$ is any sequence in S such that $\mathbf{x}_n \to \mathbf{x}$, then using Ex.1(a) of Practice Problem Set - 1, we get $\big|\, \|\mathbf{x}_n\| - \|\mathbf{x}\| \,\big| \le \|\mathbf{x}_n - \mathbf{x}\| \to 0$ and so $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$. Hence $1 - \|\mathbf{x}_n\| \to 1 - \|\mathbf{x}\|$ and since $1 - \|\mathbf{x}\| \ne 0$ and $1 - \|\mathbf{x}_n\| \ne 0$ for all $n \in \mathbb{N}$, it follows that $f(\mathbf{x}_n) \to f(\mathbf{x})$. Therefore $f : S \to \mathbb{R}^2$ is continuous at $\mathbf{x}$ and since $\mathbf{x} \in S$ is arbitrary, f is continuous.

    Let $\mathbf{x}_1,\, \mathbf{x}_2 \in S$ such that $f(\mathbf{x}_1) = f(\mathbf{x}_2)$. Then $\|f(\mathbf{x}_1)\| = \|f(\mathbf{x}_2)\|$, *i.e.* $\frac{\|\mathbf{x}_1\|}{1-\|\mathbf{x}_1\|} = \frac{\|\mathbf{x}_2\|}{1-\|\mathbf{x}_2\|}$, which gives $\|\mathbf{x}_1\| = \|\mathbf{x}_2\|$. Consequently from $\frac{1}{1-\|\mathbf{x}_1\|}\mathbf{x}_1 = \frac{1}{1-\|\mathbf{x}_2\|}\mathbf{x}_2$, we get $\mathbf{x}_1 = \mathbf{x}_2$. Hence f is one-one.

Again, if $\mathbf{y} \in \mathbb{R}^2$, then taking $\mathbf{x} = \frac{1}{1+\|\mathbf{y}\|}\mathbf{y}$, we find that $\|\mathbf{x}\| < 1$, *i.e.* $\mathbf{x} \in S$ and $f(\mathbf{x}) = \mathbf{y}$. Hence f is onto.

Thus $f : S \to \mathbb{R}^2$ is the required function and therefore the given statement is TRUE.

19. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is continuous, then does there exist a sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$? Justify.

Solution: If possible, let there exist a sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$. Then $\|(x_n, y_n)\| = \sqrt{x_n^2 + y_n^2} = \frac{1}{\sqrt{2}}$ for all $\mathbb{N}$ and so $\big((x_n, y_n)\big)$ is a bounded sequence in $\mathbb{R}^2$. Hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^2$, there exist $(x, y) \in \mathbb{R}^2$ and a convergent subsequence $\big((x_{n_k}, y_{n_k})\big)$ of $\big((x_n, y_n)\big)$ such that $\big(x_{n_k}, y_{n_k}\big) \to (x, y)$. Since f is continuous at (x, y), $\big(n_k, \frac{1}{n_k}\big) = f\big(x_{n_k}, y_{n_k}\big) \to f(x, y) \in \mathbb{R}^2$. Consequently the sequence (n_k) converges in $\mathbb{R}$, which is not true, since (n_k) is unbounded. Hence it follows that there cannot exist any sequence $\big((x_n, y_n)\big)$ in $\mathbb{R}^2$ such that $x_n^2 + y_n^2 = \frac{1}{2}$ and $f(x_n, y_n) = (n, \frac{1}{n})$ for all $n \in \mathbb{N}$.

20. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$ and $y_n \to 0$. Since $\left|\frac{x_n^3y_n}{x_n^4+y_n^2}\right| = \left|\frac{x_n^2y_n}{x_n^4+y_n^2}\right||x_n| \le \frac{1}{2}|x_n| \to 0$, it follows that $\frac{x_n^3y_n}{x_n^4+y_n^2} \to 0$. Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^4+y^2} = 0$.

21. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then $x_n \to 0$, $y_n \to 0$ and hence $\left|\frac{x_n^3-y_n^3}{x_n^2+y_n^2}\right| \le \frac{x_n^2}{x_n^2+y_n^2}|x_n| + \frac{y_n^2}{x_n^2+y_n^2}|y_n| \le |x_n| + |y_n| \to 0$. Consequently $\frac{x_n^3-y_n^3}{x_n^2+y_n^2} \to 0$ and therefore $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2} = 0$.

22. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{|x|}{y^2}e^{-|x|/y^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x, y) = \frac{|x|}{y^2}e^{-|x|/y^2}$ for all $(x, y) \in \mathbb{R}^2$ with $y \ne 0$. We have $(0, \frac{1}{n}) \to (0, 0)$ and $(\frac{1}{n^2}, \frac{1}{n}) \to (0, 0)$. Also, $f(0, \frac{1}{n}) \to 0$ and $f(\frac{1}{n^2}, \frac{1}{n}) \to \frac{1}{e}$. Since $\lim\limits_{n\to\infty} f\big(0, \frac{1}{n}\big) \ne \lim\limits_{n\to\infty} f\big(\frac{1}{n^2}, \frac{1}{n}\big)$, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist (in $\mathbb{R}$).

23. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{x^3+y^2}{x^2+y}$ exists (in $\mathbb{R}$) and find its value if it exists 9in $\mathbb{R}$).

Solution: Let $f(x, y) = \frac{x^3+y^2}{x^2+y}$ for all $(x, y) \in \mathbb{R}^2$ with $x^2 + y \ne 0$. We have $\big(\frac{1}{n}, 0\big) \to (0, 0)$ and $\big(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}\big) \to (0, 0)$. Also, $f\big(\frac{1}{n}, 0\big) = \frac{1}{n} \to 0$ and $f\big(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}\big) = 1 + \frac{1}{n}(\frac{1}{n} - 1)^2 \to 1$. Since $f\big(\frac{1}{n}, 0\big) \ne f\big(\frac{1}{n}, \frac{1}{n^3} - \frac{1}{n^2}\big)$, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist (in $\mathbb{R}$).

24. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$ exist (in $\mathbb{R}$) and find its values if it exists (in $\mathbb{R}$).

Solution: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0, 0)$. Then

$x_n \to 0$ and $y_n \to 0$. Since $0 \leq \frac{\sqrt{x_n^2 y_n^2+1}-1}{x_n^2+y_n^2} = \frac{x_n^2 y_n^2}{(x_n^2+y_n^2)\left(\sqrt{x_n^2 y_n^2+1}+1\right)} \leq \frac{x_n^2 y_n^2}{x_n^2+y_n^2} \leq y_n^2 \to 0$, it follows that $\frac{\sqrt{x_n^2 y_n^2+1}-1}{x_n^2+y_n^2} \to 0$. Therefore $\lim\limits_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$.

25. Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2+y^6}{x^6+y^4}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y) = \frac{x^3y^2+y^6}{x^6+y^4}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. We have $\left(\frac{1}{n},0\right) \to (0,0)$ and $\left(\frac{1}{\sqrt[3]{n}}, \frac{1}{\sqrt{n}}\right) \to (0,0)$. Also, $f\left(\frac{1}{n},0\right) \to 0$ and $f\left(\frac{1}{\sqrt[3]{n}}, \frac{1}{\sqrt{n}}\right) \to \frac{1}{2}$.
Since $\lim\limits_{(x,y)\to(0,0)} f\left(\frac{1}{n},0\right) \neq \lim\limits_{(x,y)\to(0,0)} f\left(\frac{1}{\sqrt[3]{n}}, \frac{1}{\sqrt{n}}\right)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

26. Examine whether $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y,z) = \frac{(x+y+z)^2}{x^2+y^2+z^2}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. We have $\left(\frac{1}{n},0,0\right) \to (0,0,0)$ and $\left(\frac{1}{n},\frac{1}{n},0\right) \to (0,0,0)$. Also, $f\left(\frac{1}{n},0,0\right) = 1 \to 1$ and $f\left(\frac{1}{n},\frac{1}{n},0\right) = 2 \to 2$.
Since $\lim\limits_{n\to\infty} f\left(\frac{1}{n},0,0\right) \neq \lim\limits_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n},0\right)$, $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z)$ does not exist (in $\mathbb{R}$).

27. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} x+y & \text{if } x \neq y, \\ 1 & \text{if } x = y. \end{cases}$
Examine whether $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exists (in $\mathbb{R}$).

Solution: We have $\left(\frac{1}{n},0\right) \to (0,0)$ and $\left(\frac{1}{n},\frac{1}{n}\right) \to (0,0)$. Also, $f\left(\frac{1}{n},0\right) = \frac{1}{n} \to 0$ and $f\left(\frac{1}{n},\frac{1}{n}\right) = 1 \to 1$. Since $\lim\limits_{n\to\infty} f\left(\frac{1}{n},0\right) \neq \lim\limits_{n\to\infty} f\left(\frac{1}{n},\frac{1}{n}\right)$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

28. Let $S \subseteq \mathbb{R}^2$, $(x_0,y_0) \in \mathbb{R}^2$ and $r > 0$ be such that $\big(B_r(x_0) \times B_r(y_0)\big) \setminus \{(x_0,y_0)\} \subseteq S$. Let $\lim\limits_{x\to x_0} f(x,y)$ exist (in $\mathbb{R}$) for each $y \in B_r(y_0) \setminus \{y_0\}$, $\lim\limits_{y\to y_0} f(x,y)$ exist (in $\mathbb{R}$) for each $x \in B_r(x_0) \setminus \{x_0\}$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell \in \mathbb{R}$.
Show that $\lim\limits_{x\to x_0}\left(\lim\limits_{y\to y_0} f(x,y)\right) = \lim\limits_{y\to y_0}\left(\lim\limits_{x\to x_0} f(x,y)\right) = \ell$.
$\left[\lim\limits_{x\to x_0}\left(\lim\limits_{y\to y_0} f(x,y)\right)\right.$ and $\lim\limits_{y\to y_0}\left(\lim\limits_{x\to x_0} f(x,y)\right)$ are called the iterated limits of f at $\left.(x_0,y_0).\right]$

Solution: Let $\varepsilon > 0$. Since $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$, there exists $\delta \in (0,r)$ such that $|f(x,y)-\ell| < \frac{\varepsilon}{2}$ for all $(x,y) \in B_\delta\big((x_0,y_0)\big) \setminus \{(x_0,y_0)\}$. Let $g(x) = \lim\limits_{y\to y_0} f(x,y)$ for all $x \in B_r(x_0) \setminus \{x_0\}$ and let $x \in B_{\frac{\delta}{2}}(x_0) \setminus \{x_0\}$.Then there exists $s \in (0, \frac{\delta}{2})$ such that $|f(x,y)-g(x)| < \frac{\varepsilon}{2}$ for all $y \in B_s(y_0) \setminus \{y_0\}$. We choose an $y \in B_s(y_0) \setminus \{y_0\}$. Then $0 < \|(x,y)-(x_0,y_0)\| = \sqrt{(x-x_0)^2+(y-y_0)^2} < \sqrt{\frac{\delta^2}{4}+s^2} < \delta$,
i.e. $(x,y) \in B_\delta\big((x_0,y_0)\big) \setminus \{(x_0,y_0)\}$ and hence
$|g(x)-\ell| \leq |g(x)-f(x,y)| + |f(x,y)-\ell| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$. Therefore $\lim\limits_{x\to x_0} g(x) = \ell$,
i.e. $\lim\limits_{x\to x_0}\left(\lim\limits_{y\to y_0} f(x,y)\right) = \ell$.
Similarly we can show that $\lim\limits_{y\to y_0}\left(\lim\limits_{x\to x_0} f(x,y)\right) = \ell$.

29. Show that $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\frac{x^2}{x^2+y^2}\right)\neq\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\frac{x^2}{x^2+y^2}\right)$ and hence conclude that $\lim\limits_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$ does not exist (in $\mathbb{R}$).

Solution: For each $x\in\mathbb{R}\backslash\{0\}$, $\lim\limits_{y\to 0}\frac{x^2}{x^2+y^2}=\frac{x^2}{x^2}=1$ and for each $y\in\mathbb{R}\backslash\{0\}$, $\lim\limits_{x\to 0}\frac{x^2}{x^2+y^2}=\frac{0}{y^2}=0$.
Hence $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\frac{x^2}{x^2+y^2}\right)=\lim\limits_{x\to 0}1=1\neq 0=\lim\limits_{y\to 0}0=\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\frac{x^2}{x^2+y^2}\right)$.
Using Ex.15 of Practice Problem Set - 2, we can conclude that $\lim\limits_{(x,y)\to(0,0)}\frac{x^2}{x^2+y^2}$ does not exist (in $\mathbb{R}$).

30. Show that $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right)=0=\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\frac{x^2y^2}{x^2y^2+(x-y)^2}\right)$ but that $\lim\limits_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}$ does not exist (in $\mathbb{R}$).

Solution: Let $f(x,y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$ for all $(x,y)\in\mathbb{R}^2\setminus\{(0,0)\}$.
Then $\lim\limits_{y\to 0}f(x,y)=\frac{0}{x^2}=0$ for each $x\in\mathbb{R}\setminus\{0\}$ and $\lim\limits_{x\to 0}f(x,y)=\frac{0}{y^2}=0$ for each $y\in\mathbb{R}\setminus\{0\}$.
Consequently $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}f(x,y)\right)=0=\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}f(x,y)\right)$.
Again, we have $\left(\frac{1}{n},0\right)\to(0,0)$ and $\left(\frac{1}{n},\frac{1}{n}\right)\to(0,0)$. Also, $f\left(\frac{1}{n},0\right)=0\to 0$ and $f\left(\frac{1}{n},\frac{1}{n}\right)=1\to 1$. Since $\lim\limits_{n\to\infty}f\left(\frac{1}{n},0\right)\neq\lim\limits_{n\to\infty}f\left(\frac{1}{n},\frac{1}{n}\right)$, $\lim\limits_{(x,y)\to(0,0)}f(x,y)$ does not exist (in $\mathbb{R}$).

31. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\begin{cases}x\sin\frac{1}{y} & \text{if } y\neq 0\\ 0 & \text{if } y=0.\end{cases}$

Show that $\lim\limits_{(x,y)\to(0,0)}f(x,y)=0$ and $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}f(x,y)\right)=0$ but that $\lim\limits_{y\to 0}f(x,y)$ does not exist (in $\mathbb{R}$) if $x\in\mathbb{R}\setminus\{0\}$ and so $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}f(x,y)\right)$ is not defined.

Solution: If $\big((x_n,y_n)\big)$ is any sequence in $\mathbb{R}^2\setminus\{(0,0)\}$ such that $(x_n,y_n)\to(0,0)$, then $x_n\to 0$ and hence $|f(x_n,y_n)|\leq|x_n|\to 0$. Therefore $f(x_n,y_n)\to 0$ and so $\lim\limits_{(x,y)\to(0,0)}f(x,y)=0$.
Again, for each $y\in\mathbb{R}\setminus\{0\}$, $\lim\limits_{x\to 0}f(x,y)=\lim\limits_{x\to 0}x\sin\frac{1}{y}=0$ and so $\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}f(x,y)\right)=\lim\limits_{y\to 0}0=0$.
If $x\in\mathbb{R}\backslash\{0\}$, then $\lim\limits_{y\to 0}f(x,y)=\lim\limits_{y\to 0}x\sin\frac{1}{y}$, which does not exist (in $\mathbb{R}$) and so $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}f(x,y)\right)$ is not defined.

32. Show that $\lim\limits_{(x,y)\to(0,0)}\frac{1}{3x^2+y^4}=\infty$.

Solution: Let $\big((x_n,y_n)\big)$ be any sequence in $\mathbb{R}^2\setminus\{(0,0)\}$ such that $(x_n,y_n)\to(0,0)$. Then $x_n\to 0$, $y_n\to 0$ and hence $3x_n^2+y_n^4\to 0$. If $r>0$, then there exists $n_0\in\mathbb{N}$ such that $3x_n^2+y_n^4<\frac{1}{r}$ for all $n\geq n_0$ and so $\frac{1}{3x_n^2+y_n^4}>r$ for all $n\geq n_0$. Therefore $\frac{1}{3x_n^2+y_n^4}\to\infty$ and consequently $\lim\limits_{(x,y)\to(0,0)}\frac{1}{3x^2+y^4}=\infty$.

33. Let I be an open interval in $\mathbb{R}$ and let $F:I\to\mathbb{R}^m$ be a differentiable function such that $F(t)\cdot F'(t)=0$ for all $t\in I$. Show that $\|F(t)\|$ is constant for all $t\in I$.

Solution: Since F is differentiable, the function $t\mapsto\|F(t)\|^2=F(t)\cdot F(t)$ from I to $\mathbb{R}$ is also differentiable and $\frac{d}{dt}\big(\|F(t)\|^2\big)=F'(t)\cdot F(t)+F(t)\cdot F'(t)=2\,F(t)\cdot F'(t)=0$ for all $t\in I$. Hence there exists $c\in\mathbb{R}$ such that $\|F(t)\|^2=c$ for all $t\in I$. Clearly $c\geq 0$ and so $\|F(t)\|=\sqrt{c}$ for all $t\in I$.

MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 3

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1. If $f(x,y) = e^x(x\cos y - y\sin y)$ for all $(x,y) \in \mathbb{R}^2$, then show that $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.

   Solution: For all $(x,y) \in \mathbb{R}^2$, we have $f_x(x,y) = e^x(x\cos y - y\sin y) + e^x \cos y$ and $f_y(x,y) = e^x(-x\sin y - y\cos y - \sin y)$. Hence $f_{xx}(x,y) = e^x(x\cos y - y\sin y) + 2e^x \cos y$ and $f_{yy}(x,y) = e^x(-x\cos y - 2\cos y + y\sin y)$ for all $(x,y) \in \mathbb{R}^2$. Therefore $f_{xx}(x,y) + f_{yy}(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$.

2. If $f(x,y) = x^2 \tan^{-1}\left(\frac{y}{x}\right)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(x,y) \in \mathbb{R} : x \neq 0\}$, then find $\frac{\partial^2 f}{\partial x \partial y}(1,1)$.

   Solution: For all $(x,y) \in S = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$, we have $\frac{\partial f}{\partial y}(x,y) = \frac{x^3}{x^2+y^2}$ and hence $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{x^4+3x^2y^2}{(x^2+y^2)^2}$. Therefore $\frac{\partial^2 f}{\partial x \partial y}(1,1) = 1$.

3. If $f(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$, then show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$ at each point of $\mathbb{R}^3 \setminus \{(0,0,0)\}$.

   Solution: We have $\frac{\partial f}{\partial x}(x,y,z) = -x(x^2+y^2+z^2)^{-\frac{3}{2}}$ and $\frac{\partial^2 f}{\partial x^2}(x,y,z) = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3x^2(x^2+y^2+z^2)^{-\frac{5}{2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. Similarly, we find that $\frac{\partial^2 f}{\partial y^2}(x,y,z) = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3y^2(x^2+y^2+z^2)^{-\frac{5}{2}}$ and $\frac{\partial^2 f}{\partial z^2}(x,y,z) = -(x^2+y^2+z^2)^{-\frac{3}{2}} + 3z^2(x^2+y^2+z^2)^{-\frac{5}{2}}$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. Therefore $\frac{\partial^2 f}{\partial x^2}(x,y,z) + \frac{\partial^2 f}{\partial y^2}(x,y,z) + \frac{\partial^2 f}{\partial z^2}(x,y,z) = 0$ for all $(x,y,z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$.

4. If $f(x,y) = \sqrt{|x^2-y^2|}$ for all$(x,y) \in \mathbb{R}^2$, then find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$).

   Solution: If $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then $D_{\mathbf{u}}f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{|t|\sqrt{|u_1^2-u_2^2|}}{t}$ exists (in $\mathbb{R}$) iff $u_1^2 = u_2^2$. Since $\|\mathbf{u}\| = \sqrt{u_1^2+u_2^2} = 1$, $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm\frac{1}{\sqrt{2}}$ and $u_2 = \pm\frac{1}{\sqrt{2}}$. Therefore $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $\mathbf{u} \in \left\{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})\right\}$.

5. If $f(x,y) = \big||x|-|y|\big| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$, then find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$).

   Solution: If $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then $D_{\mathbf{u}}f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{|t|}{t}\left(\big||u_1|-|u_2|\big| - |u_1| - |u_2|\right)$ exists (in $\mathbb{R}$) iff $\big||u_1|-|u_2|\big| = |u_1|+|u_2|$, *i.e.* iff $\big||u_1|-|u_2|\big|^2 = (|u_1|+|u_2|)^2$ and hence $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $u_1u_2 = 0$, *i.e.* $u_1 = 0$ or $u_2 = 0$. Since $u_1^2+u_2^2 = 1$, $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm 1$ or else $u_2 = \pm 1$. Therefore $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $\mathbf{u} \in \{(1,0),(-1,0),(0,1),(0,-1)\}$.

6. Find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$), if for all $(x, y) \in \mathbb{R}^2$, $f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

   Solution: If $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2} = 1$, then $D_{\mathbf{u}}f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{u_1u_2}{t}$ exists (in $\mathbb{R}$) iff $u_1u_2 = 0$, *i.e.* iff $u_1 = 0$ or $u_2 = 0$. Since $u_1^2 + u_2^2 = 1$, $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm 1$ or else $u_2 = \pm 1$. Therefore $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $\mathbf{u} \in \{(1,0), (-1,0), (0,1), (0,-1)\}$.

7. Find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$), if for all $(x, y) \in \mathbb{R}^2$, $f(x,y) = \begin{cases} \frac{x}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$

   Solution: Let $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$.
   If $u_2 = 0$, then $D_{\mathbf{u}}f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{0}{t} = 0$.
   Again, if $u_2 \neq 0$, then $D_{\mathbf{u}}f(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{u_1}{tu_2}$ exists (in $\mathbb{R}$) iff $u_1 = 0$.
   Thus combining the two cases, we find that $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $u_2 = 0$ or else $u_1 = 0$. Since $u_1^2 + u_2^2 = 1$, $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $u_1 = \pm 1$ or else $u_2 = \pm 1$. Therefore $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$) iff $\mathbf{u} \in \{(1,0), (-1,0), (0,1), (0,-1)\}$.

8. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that $f_x(0,0)$ exists (in $\mathbb{R}$), then $f_y(0,0)$ must exist (in $\mathbb{R}$).

   Solution: Let $f(x,y) = |y|$ for all $(x,y) \in \mathbb{R}^2$. If $(x,y) \in \mathbb{R}^2$ and $\big((x_n, y_n)\big)$ is any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x,y)$, then $y_n \to y$ and hence $f(x_n, y_n) = |y_n| \to |y| = f(x,y)$. Therefore f is continuous at (x,y) and since $(x,y) \in \mathbb{R}^2$ is arbitrary, $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous. Also, $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0}{t} = 0$ but $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{|t|}{t}$, which does not exist (in $\mathbb{R}$). Therefore the given statement is FALSE.

9. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is such that for each $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, the directional derivative of f at $(0,0)$ along $\mathbf{u}$ is 0, then f must be continuous at $(0,0)$.

   Solution: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$
   We have $(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}) \to (0,0)$ but $f(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}) = 1$ for all $n \in \mathbb{N}$, so that $f(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}) \to 1 \neq 0 = f(0,0)$. Hence f is not continuous at $(0,0)$.
   Again, let $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$. We have $f'_{\mathbf{u}}(0,0) = \lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u})-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{0}{t} = 0$. (The inequalities $tu_2 < t^2u_1^2 < 2tu_2$ are equivalent to the inequalities (i) $u_2 < tu_1^2 < 2u_2$ if $t > 0$ and (ii) $u_2 > tu_1^2 > 2u_2$ if $t < 0$. We can make $|tu_1^2|$ arbitrarily small for sufficiently small $|t| > 0$ and hence for such t, at least one inequality in each of (i) and (ii) cannot be satisfied. Thus we get $f(tu_1, tu_2) = 0$ for sufficiently small $|t| > 0$.) Therefore the given statement is FALSE.

10. Let the height $H(x, y)$ of a hill from the ground (considered as the xy-plane) at each point $(x, y) \in (-300, 300) \times (-200, 200)$ be given by $H(x, y) = 1000 - 0.005x^2 - 0.01y^2$. We assume that the positive x-axis points east and the positive y-axis points north. Consider a person situated at the point $(60, 40, 966)$ on the hill.
   (a) If the person starts walking due south, then will (s)he start to ascend or descend the hill?
   (b) If the person starts walking north-west, then will (s)he start to ascend or descend the hill?
   (c) If the person starts climbing further, in which direction will (s)he find it most difficult to climb?

   Solution: Let $S = (-300, 300) \times (-200, 200)$. Since $H_x(x, y) = -0.01x$ and $H_y(x, y) = -0.02y$ for all $(x, y) \in S$, $H_x : S \to \mathbb{R}$ and $H_y : S \to \mathbb{R}$ are continuous. Hence $H : S \to \mathbb{R}$ is differentiable and so $D_{\mathbf{u}}H(60, 40) = \nabla H(60, 40) \cdot \mathbf{u} = H_x(60, 40)u_1 + H_y(60, 40)u_2 = -0.6u_1 - 0.8u_2$ for all $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$.
   (a) The direction of south corresponds to $\mathbf{u} = (0, -1)$ and since $D_{\mathbf{u}}H(60, 40) = 0.8 > 0$, H increases in this direction and hence the person will ascend the hill if he starts walking due south.
   (b) The direction of north-west corresponds to $\mathbf{u} = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and since $D_{\mathbf{u}}H(60, 40) = -\frac{0.2}{\sqrt{2}} < 0$, H decreases in this direction and hence the person will descend the hill if he starts walking north-west.
   (c) Since H increases fastest in the direction of $\mathbf{u} = \nabla H(60, 40) = (-0.6, -0.8)$, the person will find it most difficult to climb the hill in the direction of $(-0.6, -0.8)$.

11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{x^2y(x-y)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   Examine whether $f_{xy}(0, 0) = f_{yx}(0, 0)$.

   Solution: We have $f_{xy}(0, 0) = \lim\limits_{k\to 0} \frac{f_x(0,k)-f_x(0,0)}{k}$ and $f_{yx}(0, 0) = \lim\limits_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h}$.
   Now, $f_x(0, 0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \frac{0-0}{h} = 0$ and $f_y(0, 0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \frac{0-0}{k} = 0$. Also, if $h \in \mathbb{R} \setminus \{0\}$, then $f_y(h, 0) = \lim\limits_{k\to 0} \frac{f(h,k)-f(h,0)}{k} = \lim\limits_{k\to 0} \frac{h^2(h-k)}{h^2+k^2} = h$ and if $k \in \mathbb{R} \setminus \{0\}$, then $f_x(0, k) = \lim\limits_{h\to 0} \frac{f(h,k)-f(0,k)}{h} = \lim\limits_{h\to 0} \frac{hk(h-k)}{h^2+k^2} = 0$. Hence $f_{xy}(0, 0) = \lim\limits_{k\to 0} \frac{0-0}{k} = 0$ and $f_{yx}(0, 0) = \lim\limits_{h\to 0} \frac{h-0}{h} = 1$. Therefore $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x, y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   Determine all the points of $\mathbb{R}^2$ where $f_{xy} : \mathbb{R}^2 \to \mathbb{R}$ and $f_{yx} : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

   Solution: For all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we have $f_x(x, y) = \frac{x^4y-y^5+4x^2y^3}{(x^2+y^2)^2}$ and $f_{xy}(x, y) = \frac{x^6-y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3}$. Similarly, for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, we have $f_y(x, y) = \frac{x^5-xy^4-4x^3y^2}{(x^2+y^2)^2}$ and $f_{yx}(x, y) = \frac{x^6-y^6+9x^4y^2-9x^2y^4}{(x^2+y^2)^3}$.
   Also, we have shown in an example in lectures that $f_{xy}(0, 0) = -1$ and $f_{yx}(0, 0) = 1$.
   Clearly $f_{xy} : \mathbb{R}^2 \to \mathbb{R}$ and $f_{yx} : \mathbb{R}^2 \to \mathbb{R}$ are continuous at each point of $\mathbb{R}^2 \setminus \{(0, 0)\}$. Again, since $(\frac{1}{n}, 0) \to (0, 0)$ and $(0, \frac{1}{n}) \to (0, 0)$ but $\lim\limits_{n\to\infty} f_{xy}(\frac{1}{n}, 0) = 1 \neq f_{xy}(0, 0)$ and

$\lim_{n\to\infty} f_{yx}(0, \frac{1}{n}) = -1 \neq f_{yx}(0,0)$, f_{xy} and f_{yx} are not continuous at $(0,0)$.

13. Let $f(x,y) = x + y^2 + xy$ for all $(x,y) \in \mathbb{R}^2$. Using directly the definition of differentiability, show that $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and also find $f'(x_0, y_0)$, where $(x_0, y_0) \in \mathbb{R}^2$.

    Solution: Let $(x_0, y_0) \in \mathbb{R}^2$. For all $(h,k) \in \mathbb{R}^2$, we have
    $f((x_0,y_0)+(h,k)) - f(x_0,y_0) = f(x_0+h, y_0+k) - f(x_0,y_0)$
    $= x_0 + h + (y_0+k)^2 + (x_0+h)(y_0+k) - x_0 - y_0^2 - x_0y_0 = h + 2y_k + k^2 + y_0h + x_0k + hk$. Let $\alpha = (1+y_0, x_0+2y_0)$. Then $\alpha \in \mathbb{R}^2$ and $\lim_{(h,k)\to(0,0)} \frac{|f((x_0,y_0)+(h,k)) - f(x_0,y_0) - \alpha \cdot (h,k)|}{\|(h,k)\|}$
    $= \lim_{(h,k)\to(0,0)} \frac{|k^2+hk|}{\sqrt{h^2+k^2}} = 0$, since for all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have
    $\frac{|k^2+hk|}{\sqrt{h^2+k^2}} \leq \frac{|k|}{\sqrt{h^2+k^2}}|k| + \frac{|h|}{\sqrt{h^2+k^2}}|k| \leq 2|k|$ and since $2|k| \to 0$ as $(h,k) \to (0,0)$.
    Therefore f is differentiable at (x_0,y_0) and $f'(x_0,y_0) = [1+y_0 \quad x_0+2y_0]$. Since $(x_0,y_0) \in \mathbb{R}^2$ is arbitrary, f is differentiable.

14. Let S be a nonempty open subset of $\mathbb{R}^m$ and let $g : S \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in S$. If $f : S \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = g(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)$ for all $\mathbf{x} \in S$, then show that f is differentiable at $\mathbf{x}_0$.

    Solution: For all $\mathbf{h} \in \mathbb{R}^m$ with $\mathbf{x}_0 + \mathbf{h} \in S$, we have $f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) = g(\mathbf{x}_0+\mathbf{h}) \cdot \mathbf{h}$. Now, $g(\mathbf{x}_0) \in \mathbb{R}^m$ and for all $\mathbf{h} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ with $\mathbf{x}_0+\mathbf{h} \in S$, using Cauchy-Schwarz inequality, we have $\frac{|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - g(\mathbf{x}_0)\cdot\mathbf{h}|}{\|\mathbf{h}\|} = \frac{|(g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0))\cdot\mathbf{h}|}{\|\mathbf{h}\|} \leq \frac{\|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0)\|\,\|\mathbf{h}\|}{\|\mathbf{h}\|} = \|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0)\|$. Since g is continuous at $\mathbf{x}_0$, $\lim_{\|\mathbf{h}\|\to 0} \|g(\mathbf{x}_0+\mathbf{h}) - g(\mathbf{x}_0)\| = 0$ and hence we get $\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{x}_0+\mathbf{h}) - f(\mathbf{x}_0) - g(\mathbf{x}_0)\cdot\mathbf{h}|}{\|\mathbf{h}\|} = 0$.
    Therefore f is differentiable at $\mathbf{x}_0$.

15. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0,0)$ in the directions of $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ are 1 and 2 respectively. Find $f_x(0,0)$ and $f_y(0,0)$.

    Solution: Since f is differentiable at $(0,0)$, $D_{\mathbf{u}}f(0,0) = \nabla f(0,0) \cdot \mathbf{u} = f_x(0,0)u_1 + f_y(0,0)u_2$ for all $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$. Hence taking $\mathbf{u} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\mathbf{u} = \left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ respectively, we get $\frac{1}{\sqrt{5}}f_x(0,0) + \frac{2}{\sqrt{5}}f_y(0,0) = 1$ and $\frac{2}{\sqrt{5}}f_x(0,0) + \frac{1}{\sqrt{5}}f_y(0,0) = 2$. Solving these two equations, we get $f_x(0,0) = \sqrt{5}$ and $f_y(0,0) = 0$.

16. If $f : \mathbb{R}^m \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \leq \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^m$, then examine whether f is differentiable at $\mathbf{0}$.

    Solution: Since $|f(\mathbf{0})| \leq \|\mathbf{0}\|^2 = 0$, we have $f(\mathbf{0}) = 0$. If $\alpha = \mathbf{0}$, then $\alpha \in \mathbb{R}^m$ and for all $\mathbf{h} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$, we have $\frac{|f(\mathbf{h}) - f(\mathbf{0}) - \alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} \leq \|\mathbf{h}\|$. Hence it follows that $\lim_{\mathbf{h}\to\mathbf{0}} \frac{|f(\mathbf{h}) - f(\mathbf{0}) - \alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} = 0$.
    Therefore f is differentiable at $\mathbf{0}$.

17. Let $f(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$. Examine whether $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable at $\mathbf{0}$.

    Solution: Since $\lim_{t\to 0} \frac{f(\mathbf{0}+t\mathbf{e}_1) - f(\mathbf{0})}{t} = \lim_{t\to 0} \frac{\|t\mathbf{e}_1\|}{t} = \lim_{t\to 0} \frac{|t|}{t}$ does not exist (in $\mathbb{R}$), $\frac{\partial f}{\partial x_1}(\mathbf{0})$ does not exist (in $\mathbb{R}$). Consequently f is not differentiable at $\mathbf{0}$.

18. If $f(x,y) = \sqrt{|xy|}$ for all $(x,y) \in \mathbb{R}^2$, then examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0}{t} = 0$ and $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0}{t} = 0$.
Now $\lim\limits_{(h,k)\to(0,0)} \frac{f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} \neq 0$, since $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $\lim\limits_{n\to\infty} \frac{\sqrt{\frac{1}{n^2}}}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} = \frac{1}{\sqrt{2}} \neq 0$. Therefore f is not differentiable at $(0,0)$.

19. If $f(x,y) = \big||x| - |y|\big| - |x| - |y|$ for all $(x,y) \in \mathbb{R}^2$, then examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$ and $f_y(0,0) = \lim\limits_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = 0$. Now $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)|}{\sqrt{h^2+k^2}} \neq 0$, since $(\frac{2}{n}, \frac{1}{n}) \to (0,0)$ but $\lim\limits_{n\to\infty} \frac{|f(\frac{2}{n},\frac{1}{n})|}{\sqrt{\frac{4}{n^2}+\frac{1}{n^2}}} = \frac{2}{\sqrt{5}} \neq 0$. Hence f is not differentiable at $(0,0)$.

20. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Examine whether f is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{t^3}{t^3} = 1$ and $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$. Now, $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}}$ $= \lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-h|}{\sqrt{h^2+k^2}} \neq 0$, since $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $\lim\limits_{n\to\infty} \frac{|f(\frac{1}{n},\frac{1}{n})-\frac{1}{n}|}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} = \frac{1}{2\sqrt{2}} \neq 0$. Therefore f is not differentiable at $(0,0)$.

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$
Examine whether f is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$ and $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{t\sqrt{t^2}}{t|t|} = 1$. Now $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}}$ $= \lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-k|}{\sqrt{h^2+k^2}} \neq 0$, since $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $\lim\limits_{n\to\infty} \frac{|f(\frac{1}{n},\frac{1}{n})-\frac{1}{n}|}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} = 1 - \frac{1}{\sqrt{2}} \neq 0$. Hence f is not differentiable at $(0,0)$.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x,y) = \begin{cases} \sqrt{x^2+y^2} & \text{if } y > 0, \\ x & \text{if } y = 0, \\ -\sqrt{x^2+y^2} & \text{if } y < 0. \end{cases}$
Examine whether f is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0} \frac{x}{x} = 1$. Also, since $\lim\limits_{y\to 0+} \frac{f(0,y)-f(0,0)}{y} = \lim\limits_{y\to 0+} \frac{y}{y} = 1$ and $\lim\limits_{y\to 0-} \frac{f(0,y)-f(0,0)}{y} = \lim\limits_{y\to 0-} \frac{-(-y)}{y} = 1$, we get $f_y(0,0) = 1$.
Now, $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-h-k|}{\sqrt{h^2+k^2}} \neq 0$, since $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but

$\frac{|f(\frac{1}{n},\frac{1}{n})-\frac{1}{n}-\frac{1}{n}|}{\sqrt{\frac{1}{n^2}+\frac{1}{n^2}}} \to \sqrt{2}-1$. Hence f is not differentiable at $(0,0)$.

23. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$
Examine whether f is differentiable at $(0,0)$.

Solution: We have $(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}) \to (0,0)$ but $f(\frac{1}{\sqrt{n+1}}, \frac{1}{n+2}) = 1 \to 1 \neq 0 = f(0,0)$. Hence f is not continuous at $(0,0)$ and consequently f is not differentiable at $(0,0)$.

24. For all $(x,y) \in \mathbb{R}^2$, let $f(x,y) = \begin{cases} x & \text{if } |x| < |y|, \\ -x & \text{if } |x| \geq |y|. \end{cases}$
Examine whether $f:\mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{-t-0}{t} = -1$ and $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$. Now $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)+h|}{\sqrt{h^2+k^2}} \not\to 0$, since $(\frac{1}{n}, \frac{2}{n}) \to (0,0)$ but $\frac{|f(\frac{1}{n},\frac{2}{n})+\frac{1}{n}|}{\sqrt{\frac{1}{n^2}+\frac{4}{n^2}}} = \frac{2/n}{\sqrt{5}/n} \to \frac{2}{\sqrt{5}} \neq 0$. Therefore f is not differentiable at $(0,0)$.

25. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{\sin(x^2y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Examine whether f is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim\limits_{x\to 0} \frac{0-0}{x} = 0$ and $f_y(0,0) = \lim\limits_{y\to 0} \frac{f(0,y)-f(0,0)}{y} = \lim\limits_{y\to 0} \frac{0-0}{y} = 0$. For all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $\varepsilon(h,k) = \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \frac{|\sin(h^2k^2)|}{(h^2+k^2)^{3/2}} \leq \frac{h^2k^2}{(h^2+k^2)^{3/2}} \leq \frac{(h^2+k^2)^2}{(h^2+k^2)^{3/2}} \leq \sqrt{h^2+k^2}$. This implies that $\lim\limits_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$ and so f is differentiable at $(0,0)$.

26. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \sin^2 x + x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
Examine whether f is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0}(\sin t\, \frac{\sin t}{t} + t\sin\frac{1}{t}) = 0$ and $f_y(x,y) = 0$ for all $(x,y) \in \mathbb{R}^2$. Since $f_y:\mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$, it follows that g is differentiable at $(0,0)$.

27. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Show that f is differentiable at $(0,0)$ although neither $f_x:\mathbb{R}^2 \to \mathbb{R}$ nor $f_y:\mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,0)$.

Hint: Here $f_x(0,0) = f_y(0,0) = 0$. For all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$,
$\varepsilon(h,k) = \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} \leq \sqrt{h^2+k^2}$, so that $\lim\limits_{(h,k)\to(0,0)} \varepsilon(h,k) = 0$. Hence f is differentiable at $(0,0)$.
Again, $f_x(x,y) = 2x\sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) - \frac{x}{\sqrt{x^2+y^2}}\cos\left(\frac{1}{\sqrt{x^2+y^2}}\right)$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Now $\left((\frac{1}{2n\pi}, 0)\right)$ is a sequence in $\mathbb{R}^2$ converging to $(0,0)$ but $f_x(\frac{1}{2n\pi}, 0) = -1$ for all $n \in \mathbb{N}$ and so $f_x(\frac{1}{2n\pi}, 0) \to -1 \neq f_x(0,0)$. This shows that f_x is not continuous at $(0,0)$. Similarly f_y is not

continuous at $(0,0)$.

28. Let $f(x,y) = \begin{cases} (x^2+y^2)\cos\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}, \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Examine whether $f : \mathbb{R}^2 \to \mathbb{R}$ is continuously differentiable.

Solution: For all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$, we have $f_x(x,y) = 2x\cos\left(\frac{1}{x^2+y^2}\right) + \frac{2x}{x^2+y^2}\sin\left(\frac{1}{x^2+y^2}\right)$. Now $\left(\frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}, 0\right) \to (0,0)$ but $f_x\left(\frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}, 0\right) = \sqrt{2(4n+1)\pi} \to \infty$. Hence $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exist (in $\mathbb{R}$) and consequently f_x is not continuous at $(0,0)$. Therefore f is not continuously differentiable.

29. Let $\alpha \in \mathbb{R}$ and $\alpha > 0$. If $f(x,y) = |xy|^\alpha$ for all $(x,y) \in \mathbb{R}^2$, then determine all values of α for which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$.

Solution: We have $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$ and
$f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = \lim\limits_{t\to 0} \frac{0-0}{t} = 0$. For all $(h,k) \in \mathbb{R}^2 \setminus \{(0,0)\}$, let
$\varphi(h,k) = \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \frac{|hk|^\alpha}{\sqrt{h^2+k^2}}$. If $\alpha > \frac{1}{2}$, then
$\varphi(h,k) \le \frac{(h^2+k^2)^{\alpha/2}(h^2+k^2)^{\alpha/2}}{\sqrt{h^2+k^2}} = (h^2+k^2)^{\alpha-\frac{1}{2}}$ and so $\lim\limits_{(h,k)\to(0,0)} \varphi(h,k) = 0$. Consequently f is differentiable at $(0,0)$.
Again, if $\alpha \le \frac{1}{2}$, then $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $\varphi(\frac{1}{n}, \frac{1}{n}) = \frac{1}{\sqrt{2}} n^{1-2\alpha} \not\to 0$ (for $\alpha = \frac{1}{2}$, $\varphi(\frac{1}{n}, \frac{1}{n}) \to \frac{1}{\sqrt{2}}$ and for $\alpha < \frac{1}{2}$, the sequence $\left(\varphi(\frac{1}{n}, \frac{1}{n})\right)$ is unbounded). Hence $\lim\limits_{(h,k)\to(0,0)} \varphi(h,k) \neq 0$ and so f is not differentiable at $(0,0)$.

30. Let $f(x,y) = |xy|$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Solution: Let $S_1 = \{(x,y) \in \mathbb{R}^2 : xy > 0\}$ and $S_2 = \{(x,y) \in \mathbb{R}^2 : xy < 0\}$. Then $f(x,y) = xy$ for all $(x,y) \in S_1$ and $f(x,y) = -xy$ for all $(x,y) \in S_2$. Since $f_x(x,y) = y$ and $f_y(x,y) = x$ for all $(x,y) \in S_1$, we find that both $f_x : S_1 \to \mathbb{R}$ and $f_y : S_1 \to \mathbb{R}$ are continuous. Hence f is differentiable at every point of S_1. By a similar argument, we can show that f is differentiable at every point of S_2. If $\alpha (\neq 0) \in \mathbb{R}$, then $f_y(\alpha, 0) = \lim\limits_{t\to 0} \frac{f(\alpha,t)-f(\alpha,0)}{t} = \lim\limits_{t\to 0} \frac{|\alpha||t|}{t}$ does not exist (in $\mathbb{R}$) and similarly $f_x(0,\alpha)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at any point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for which $xy = 0$. Again, $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$, $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$ and $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|h||k|}{\sqrt{h^2+k^2}} = 0$ (since $|h||k| \le h^2+k^2$ for all $(h,k) \in \mathbb{R}^2$). Hence f is differentiable at $(0,0)$. Therefore the set of all points of $\mathbb{R}^2$ at which f is differentiable is $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\} \cup \{(0,0)\}$.

31. Let $f(x,y) = (xy)^{\frac{2}{3}}$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ at which $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Solution: Let $S = \{(x,y) \in \mathbb{R}^2 : xy \neq 0\}$. Since $f_x(x,y) = \frac{2}{3}x^{-\frac{1}{3}}y^{\frac{2}{3}}$ and $f_y(x,y) = \frac{2}{3}x^{\frac{2}{3}}y^{-\frac{1}{3}}$ for all $(x,y) \in S$, we find that both $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are continuous. Hence f is differentiable at every point of S. If $\alpha (\neq 0) \in \mathbb{R}$, then $f_y(\alpha, 0) = \lim\limits_{t\to 0} \frac{f(\alpha,t)-f(\alpha,0)}{t} = \lim\limits_{t\to 0} \frac{\alpha^{\frac{2}{3}}}{t^{\frac{1}{3}}}$

does not exist (in $\mathbb{R}$) and similarly $f_x(0,\alpha)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at any point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ for which $xy = 0$. Again, $f_x(0,0) = \lim\limits_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = 0$, $f_y(0,0) = \lim\limits_{t\to 0} \frac{f(0,t)-f(0,0)}{t} = 0$ and $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k)-f(0,0)-hf_x(0,0)-kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{|h|^{\frac{2}{3}}|k|^{\frac{2}{3}}}{\sqrt{h^2+k^2}} = 0$ (since $|h|^{\frac{2}{3}}|k|^{\frac{2}{3}} \le (h^2+k^2)^{\frac{2}{3}}$ for all $(h,k) \in \mathbb{R}^2$). Hence f is differentiable at $(0,0)$. Therefore the set of all points of $\mathbb{R}^2$ at which f is differentiable is $\{(x,y) \in \mathbb{R}^2 : xy \neq 0\} \cup \{(0,0)\}$.

32. Let $f(x,y) = |x|\sin(x^2+y^2)$ for all $(x,y) \in \mathbb{R}^2$. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Hint: Clearly f is differentiable at all $(x,y) \in \mathbb{R}^2$ for which $x \neq 0$. Let $y_0 \in \mathbb{R}$. Then $f_x(0,y_0) = \lim\limits_{x\to 0} \frac{f(x,y_0)-f(0,y_0)}{x} = \lim\limits_{x\to 0} \frac{|x|\sin(x^2+y_0^2)}{x}$, which exists in $\mathbb{R}$ (and equals 0) iff $y_0 = \pm\sqrt{n\pi}$ for some $n \in \mathbb{N}\cup\{0\}$. Also, $f_y(x,y) = 2|x|y\cos(x^2+y^2)$ for all $(x,y) \in \mathbb{R}^2$. So f_y is continuous at each point of $\mathbb{R}^2$. Therefore f is differentiable at $(0,y_0)$ iff $y_0 = \pm\sqrt{n\pi}$ for some $n \in \mathbb{N}\cup\{0\}$.

33. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} x^2+y^2 & \text{if both } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

Solution: Since $|f(x,y)| \le x^2+y^2 = \|(x,y)\|^2$ for all $(x,y) \in \mathbb{R}^2$, by Ex.12(a) of Practice Problem Set - 3, f is differentiable at $(0,0)$.

Let $(x_0,y_0) \in \mathbb{R}^2 \setminus \{(0,0)\}$. If $(x_0,y_0) \in \mathbb{Q}\times\mathbb{Q}$, then $(x_0+\frac{\sqrt{2}}{n}, y_0) \to (x_0,y_0)$ but $f(x_0+\frac{\sqrt{2}}{n}, y_0) = 0 \to 0 \neq x_0^2+y_0^2 = f(x_0,y_0)$. Again if $(x_0,y_0) \notin \mathbb{Q}\times\mathbb{Q}$, then we choose rational sequences (x_n) and (y_n) such that $x_n \to x_0$ and $y_n \to y_0$. Then $(x_n,y_n) \to (x_0,y_0)$ but $f(x_n,y_n) = x_n^2+y_n^2 \to x_0^2+y_0^2 \neq 0 = f(x_0,y_0)$. Hence f is not continuous at (x_0,y_0) and consequently f is not differentiable at (x_0,y_0).

34. State TRUE or FALSE with justification: If $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ and if $f(x,y) = |xy|$ for all $(x,y) \in S$, then $f : S \to \mathbb{R}$ is differentiable.

Solution: Clearly $(\frac{1}{2},0) \in S$. Since $\lim\limits_{t\to 0} \frac{f(\frac{1}{2},t)-f(\frac{1}{2},0)}{t} = \lim\limits_{t\to 0} \frac{|t|}{2t}$ does not exist (in $\mathbb{R}$), $f_y(\frac{1}{2},0)$ does not exist (in $\mathbb{R}$). Hence f is not differentiable at $(\frac{1}{2},0)$ and so f is not differentiable. Therefore the given statement is FALSE.

35. State TRUE or FALSE with justification: There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ which is differentiable only at $(1,0)$.

Solution: For all $(x,y) \in \mathbb{R}^2$, let $f(x,y) = \begin{cases} (x-1)^2+y^2 & \text{if } x, y \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

Taking $\alpha = (0,0) \in \mathbb{R}^2$, we find that
$\lim\limits_{(h,k)\to(0,0)} \frac{|f(1+h,k)-f(1,0)-\alpha\cdot(h,k)|}{\sqrt{h^2+k^2}} \le \lim\limits_{(h,k)\to(0,0)} \frac{h^2+k^2}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \sqrt{h^2+k^2} = 0$. Hence f is differentiable at $(1,0)$.

Again let $(x,y) \in \mathbb{R}^2 \setminus \{(1,0)\}$. Then $f(x,y) \neq 0$. We can find a sequence (x_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $x_n \to x$. So $(x_n,y) \to (x,y)$ but $f(x_n,y) = 0$ for all $n \in \mathbb{N}$ and so $f(x_n,y) \to 0 \neq f(x,y)$. Hence f is not continuous at (x,y) and so f is not differentiable at (x,y). Thus $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable only at $(1,0)$. Therefore the given statement is TRUE.

36. Let $f:\mathbb{R}^2\to\mathbb{R}$ be differentiable at $(0,0)$ and let $\lim\limits_{x\to 0}\frac{f(x,x)-f(x,-x)}{x}=1$. Find $f_y(0,0)$.

Solution: Since f is differentiable at $(0,0)$, we have $\lim\limits_{t\to 0}\frac{|f(t,t)-f(0,0)-tf_x(0,0)-tf_y(0,0)|}{\sqrt{2t^2}}=0$ and $\lim\limits_{t\to 0}\frac{|f(t,-t)-f(0,0)-tf_x(0,0)+tf_y(0,0)|}{\sqrt{2t^2}}=0$. Using the triangle inequality of $|\cdot|$, we get $\lim\limits_{t\to 0}\frac{|f(t,t)-f(t,-t)-2tf_y(0,0)|}{\sqrt{2}|t|}=0$ and so $\lim\limits_{t\to 0}\left|\frac{f(t,t)-f(t,-t)}{t}-2f_y(0,0)\right|=0$. Hence $2f_y(0,0)=\lim\limits_{t\to 0}\frac{f(t,t)-f(t,-t)}{t}=1$ and therefore $f_y(0,0)=\frac{1}{2}$.

37. Let $f:\mathbb{R}^m\to\mathbb{R}$ be differentiable at $\mathbf{0}$ and let $f(\alpha\mathbf{x})=\alpha f(\mathbf{x})$ for all $\mathbf{x}\in\mathbb{R}^m$ and for all $\alpha\in\mathbb{R}$. Show that $f(\mathbf{x}+\mathbf{y})=f(\mathbf{x})+f(\mathbf{y})$ for all $\mathbf{x},\mathbf{y}\in\mathbb{R}^m$.

Solution: We have $f(\mathbf{0})=f(0\,\mathbf{0})=0f(\mathbf{0})=0$. Since f is differentiable at $\mathbf{0}$, there exists $\mathbf{a}\in\mathbb{R}^m$ such that $\lim\limits_{\mathbf{h}\to\mathbf{0}}\frac{|f(\mathbf{h})-\mathbf{a}\cdot\mathbf{h}|}{\|\mathbf{h}\|}=\lim\limits_{\mathbf{h}\to\mathbf{0}}\frac{|f(\mathbf{0}+\mathbf{h})-f(\mathbf{0})-\mathbf{a}\cdot\mathbf{h}|}{\|\mathbf{h}\|}=0$. If $\mathbf{x}\in\mathbb{R}^m\setminus\{\mathbf{0}\}$, then from above, we get $\lim\limits_{t\to 0}\frac{|f(t\mathbf{x})-\mathbf{a}\cdot t\mathbf{x}|}{\|t\mathbf{x}\|}=0$, which gives $\lim\limits_{t\to 0}\frac{|tf(\mathbf{x})-t\,\mathbf{a}\cdot\mathbf{x}|}{\|t\mathbf{x}\|}=0$ and so $\lim\limits_{t\to 0}\frac{|t|\,|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|}{|t|\,\|\mathbf{x}\|}=0$. Thus $\lim\limits_{t\to 0}\frac{|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|}{\|\mathbf{x}\|}=0$ and so $\frac{|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|}{\|\mathbf{x}\|}=0$, which gives $|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|=0$ and hence $f(\mathbf{x})=\mathbf{a}\cdot\mathbf{x}$. Since $f(\mathbf{0})=0=\mathbf{a}\cdot\mathbf{0}$, we have $f(\mathbf{x})=\mathbf{a}\cdot\mathbf{x}$ for all $\mathbf{x}\in\mathbb{R}^m$. Now, if $\mathbf{x}$, $\mathbf{y}\in\mathbb{R}^m$, then $f(\mathbf{x}+\mathbf{y})=\mathbf{a}\cdot(\mathbf{x}+\mathbf{y})=\mathbf{a}\cdot\mathbf{x}+\mathbf{a}\cdot\mathbf{y}=f(\mathbf{x})+f(\mathbf{y})$.

38. Let $f:\mathbb{R}^m\to\mathbb{R}$ be differentiable at $\mathbf{0}$ and $f(\mathbf{0})=0$. Show that there exist $\alpha>0$ and $r>0$ such that $|f(\mathbf{x})|\le\alpha\|\mathbf{x}\|$ for all $\mathbf{x}\in\mathbb{R}^m$ with $\|\mathbf{x}\|<r$.

Solution: Since f is differentiable at $\mathbf{0}$ and $f(\mathbf{0})=0$, there exists $\mathbf{a}\in\mathbb{R}^m$ such that $\lim\limits_{\mathbf{x}\to\mathbf{0}}\frac{|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|}{\|\mathbf{x}\|}=\lim\limits_{\mathbf{x}\to\mathbf{0}}\frac{|f(\mathbf{x})-f(\mathbf{0})-\mathbf{a}\cdot\mathbf{x}|}{\|\mathbf{x}\|}=0$. Hence there exists $r>0$ such that $\frac{|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|}{\|\mathbf{x}\|}<1$ for all $\mathbf{x}\in\mathbb{R}^m$ with $0<\|\mathbf{x}\|<r$. Therefore if $\mathbf{x}\in\mathbb{R}^m$ with $\|\mathbf{x}\|<r$, then $|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|\le\|\mathbf{x}\|$ and so $|f(\mathbf{x})|\le|f(\mathbf{x})-\mathbf{a}\cdot\mathbf{x}|+|\mathbf{a}\cdot\mathbf{x}|\le\|\mathbf{x}\|+\|\mathbf{a}\|\,\|\mathbf{x}\|=\alpha\,\|\mathbf{x}\|$, where $\alpha=1+\|\mathbf{a}\|>0$.

39. Let $f:\mathbb{R}^2\to\mathbb{R}$ be such that f_x exists (in $\mathbb{R}$) at all points of $B_\delta\big((x_0,y_0)\big)$ for some $(x_0,y_0)\in\mathbb{R}^2$ and $\delta>0$, f_x is continuous at (x_0,y_0) and $f_y(x_0,y_0)$ exists (in $\mathbb{R}$). Show that f is differentiable at (x_0,y_0).

Solution: For all $(h,k)\in B_\delta\big((0,0)\big)$, we have $f(x_0+h,y_0+k)-f(x_0,y_0)=f(x_0+h,y_0+k)-f(x_0,y_0+k)+f(x_0,y_0+k)-f(x_0,y_0)$. Now, by the mean value theorem for single real variable, we get $f(x_0+h,y_0+k)-f(x_0,y_0+k)=hf(x_0+\theta h,y_0+k)$ for some $\theta\in(0,1)$. Again, if $\varepsilon(k)=\frac{f(x_0,y_0+k)-f(x_0,y_0)}{k}-f_y(x_0,y_0)$ for all $k\in\mathbb{R}\setminus\{0\}$ with $|k|<\delta$ and $\varepsilon(0)=0$, then $f(x_0,y_0+k)-f(x_0,y_0)=kf_y(x_0,y_0)+k\varepsilon(k)$ for all $k\in\mathbb{R}$ with $|k|<\delta$ and $\varepsilon(k)\to 0$ as $k\to 0$. Now, $\lim\limits_{(h,k)\to(0,0)}\frac{|f(x_0+h,y_0+k)-f(x_0,y_0)-hf_x(x_0,y_0)-kf_y(x_0,y_0)|}{\sqrt{h^2+k^2}}$
$\le\lim\limits_{(h,k)\to(0,0)}\left(\frac{|h|}{\sqrt{h^2+k^2}}|f_x(x_0+\theta h,y_0+k)-f_x(x_0,y_0)|+\frac{|k|}{\sqrt{h^2+k^2}}|\varepsilon(k)|\right)$
$\le\lim\limits_{(h,k)\to(0,0)}(|f_x(x_0+\theta h,y_0+k)-f_x(x_0,y_0)|+|\varepsilon(k)|)=0$ (since f_x is continuous at (x_0,y_0)).
Therefore f is differentiable at (x_0,y_0).

40. Let $f,\, g : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in S^0$. Show that $f + g : S \to \mathbb{R}$ is differentiable at $\mathbf{x}_0$ and $\nabla(f+g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$.

Solution: Since f and g are differentiable at $\mathbf{x}_0$, $\nabla f(\mathbf{x}_0),\, \nabla g(\mathbf{x}_0) \in \mathbb{R}^m$ and by increment theorem, there exist $\delta_1,\, \delta_2 > 0$ and functions $\varepsilon_1 : B_{\delta_1}(\mathbf{0}) \to \mathbb{R}$, $\varepsilon_2 : B_{\delta_2}(\mathbf{0}) \to \mathbb{R}$ such that $\lim\limits_{\mathbf{h}\to\mathbf{0}} \varepsilon_1(\mathbf{h}) = \lim\limits_{\mathbf{h}\to\mathbf{0}} \varepsilon_2(\mathbf{h}) = 0$ and $f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\cdot\mathbf{h} + \|\mathbf{h}\|\varepsilon_1(\mathbf{h})$ for all $\mathbf{h} \in B_{\delta_1}(\mathbf{0})$ and $g(\mathbf{x}_0 + \mathbf{h}) = g(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)\cdot\mathbf{h} + \|\mathbf{h}\|\varepsilon_2(\mathbf{h})$ for all $\mathbf{h} \in B_{\delta_2}(\mathbf{0})$. Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and $(f+g)(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0+\mathbf{h}) + g(\mathbf{x}_0+\mathbf{h}) = (f+g)(\mathbf{x}_0) + (\nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0))\cdot\mathbf{h} + \|\mathbf{h}\|\varepsilon(\mathbf{h})$ for all $\mathbf{h} \in B_\delta(\mathbf{0})$, where $\varepsilon : B_\delta(\mathbf{0}) \to \mathbb{R}$ is defined by $\varepsilon(\mathbf{h}) = \varepsilon_1(\mathbf{h}) + \varepsilon_2(\mathbf{h})$ for all $\mathbf{h} \in B_\delta(\mathbf{0})$ and so $\lim\limits_{\mathbf{h}\to\mathbf{0}} \varepsilon(\mathbf{h}) = \lim\limits_{\mathbf{h}\to\mathbf{0}} \varepsilon_1(\mathbf{h}) + \lim\limits_{\mathbf{h}\to\mathbf{0}} \varepsilon_2(\mathbf{h}) = 0$. Therefore by increment theorem, $f+g$ is differentiable at $\mathbf{x}_0$ and $\nabla(f+g)(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) + \nabla g(\mathbf{x}_0)$.

41. Using the linearization of a suitable function at a suitable point, find an approximate value of $\left((3.8)^2 + 2(2.1)^3\right)^{\frac{1}{5}}$.

Solution: Let $S = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$ and let $f(x, y) = (x^2 + 2y^3)^{\frac{1}{5}}$ for all $(x, y) \in S$. Then $f_x(x, y) = \frac{2}{5}x(x^2 + 2y^3)^{-\frac{4}{5}}$ and $f_y(x, y) = \frac{6}{5}y^2(x^2 + 2y^3)^{-\frac{4}{5}}$ for all $(x, y) \in S$. Since $f_x,\, f_y : S \to \mathbb{R}$ are continuous, $f : S \to \mathbb{R}$ is differentiable. Hence the linearization of f at $(4, 2) \in S$ is given by $L(x, y) = f(4, 2) + f_x(4, 2)(x-4) + f_y(4, 2)(y-2) = 2 + \frac{1}{10}(x-4) + \frac{3}{10}(y-2)$ for all $(x, y) \in S$. Therefore an approximate value of $f(3.8, 2.1)$ is given by
$L(3.8, 2.1) = 2 - 0.02 + 0.03 = 2.01$.

42. Show that the maximum error in calculating the volume of a right circular cylinder is approximately $\pm\, 8\%$ if its radius can be measured with a maximum error of $\pm\, 3\%$ and its height can be measured with a maximum error of $\pm\, 2\%$.

Solution: We know that the volume of a right circular cylinder of radius r and height h is given by $V(r, h) = \pi r^2 h$. If $S = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$, then $V : S \to \mathbb{R}$ is differentiable (since $V_r,\, V_h : S \to \mathbb{R}$ are continuous) and the linearization of V at any point $(r_0, h_0) \in S$ is given by
$L(r, h) = V(r_0, h_0) + V_r(r_0, h_0)(r - r_0) + V_h(r_0, h_0)(h - h_0) = V(r_0, h_0) + 2\pi r_0 h_0 (r - r_0) + \pi r_0^2 (h - h_0)$ for all $(r, h) \in \mathbb{R}^2$. Hence the absolute value of an approximate percentage error in $V(r, h)$ at (r_0, h_0) is given by $\frac{|L(r,h) - V(r_0,h_0)|}{V(r_0,h_0)} \times 100$. Since it is given that $\frac{|r - r_0|}{r_0} \times 100 \le 3$ and $\frac{|h - h_0|}{h_0} \times 100 \le 2$, we get $\frac{|L(r,h) - V(r_0,h_0)|}{V(r_0,h_0)} \times 100 \le 2\frac{|r - r_0|}{r_0} \times 100 + \frac{|h - h_0|}{h_0} \times 100 \le 6 + 2 = 8$. Therefore the maximum error in calculating $V(r, h)$ at any $(r_0, h_0) \in S$ is approximately $\pm\, 8\%$.

MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Practice Problem Set - 4

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1. Let $f(\mathbf{x}) = \|\mathbf{x}\|^{\frac{5}{2}}$ for all $\mathbf{x} \in \mathbb{R}^m$. Using chain rule, show that $f : \mathbb{R}^m \to \mathbb{R}$ is differentiable and determine $f'(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m$.

   Solution: Let $g(\mathbf{x}) = \|\mathbf{x}\|^2$ for all $\mathbf{x} \in \mathbb{R}^m$ and let $\varphi(x) = x^{\frac{5}{4}}$ for all $x \in [0, \infty)$. Then we know that $g : \mathbb{R}^m \to \mathbb{R}$ and $\varphi : [0, \infty) \to \mathbb{R}$ are differentiable with $\nabla g(\mathbf{x}) = 2\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m$ and $\varphi'(x) = \frac{5}{4}x^{\frac{1}{4}}$ for all $x \in [0, \infty)$. Since $f(\mathbf{x}) = \varphi(g(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^m$, by chain rule, $f = \varphi \circ g$ is differentiable and for each $\mathbf{x} \in \mathbb{R}^m$, $\nabla f(\mathbf{x}) = \varphi'(g(\mathbf{x}))\nabla g(\mathbf{x}) = \frac{5}{2}\sqrt{\|\mathbf{x}\|}\,\mathbf{x}$. Therefore $f'(\mathbf{x}) = \frac{5}{2}\sqrt{\|\mathbf{x}\|}\begin{bmatrix} x_1 & \cdots & x_m \end{bmatrix}$ for all $\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be differentiable and let $u(x, y, z) = f(x - y, y - z, z - x)$ for all $(x, y, z) \in \mathbb{R}^2$. Show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ at each point of $\mathbb{R}^3$.

   Solution: Let $r(x, y, z) = x - y$, $s(x, y, z) = y - z$ and $t(x, y, z) = z - x$ for all $(x, y, z) \in \mathbb{R}^3$. Since all the partial derivatives of r, s, t are continuous on $\mathbb{R}^3$, r, s, $t : \mathbb{R}^3 \to \mathbb{R}$ are differentiable. Hence by chain rule, we get $\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t}$,
   $\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial y} = \frac{\partial f}{\partial s} - \frac{\partial f}{\partial r}$, and $\frac{\partial u}{\partial z} = \frac{\partial f}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial f}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial z} = \frac{\partial f}{\partial t} - \frac{\partial f}{\partial s}$, where each of the partial derivatives has been considered at the respective point. Therefore $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$ at each point of $\mathbb{R}^3$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable and let $u(r, \theta) = f(r\cos\theta, r\sin\theta)$ for all $r > 0$, $\theta \in \mathbb{R}$. Show that $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$ at each point $(x, y) = (r\cos\theta, r\sin\theta)$ of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

   Solution: Since all the second order partial derivatives of f are continuous on $\mathbb{R}^2$, $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$ for all $(x, y) \in \mathbb{R}^2$.
   Let $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$ for all $(r, \theta) \in \mathbb{R}^2$. Then x, $y : \mathbb{R}^2 \to \mathbb{R}$ are twice continuously differentiable. Hence by chain rule, for all $(x, y) = (r\cos\theta, r\sin\theta) \in \mathbb{R}^2$, we get
   $\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos\theta\frac{\partial f}{\partial x} + \sin\theta\frac{\partial f}{\partial y}$,
   $\frac{\partial^2 u}{\partial r^2} = \cos\theta\big(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial r}\big) + \sin\theta\big(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\big) = \cos^2\theta\frac{\partial^2 f}{\partial x^2} + \sin 2\theta\frac{\partial^2 f}{\partial x \partial y} + \sin^2\theta\frac{\partial^2 f}{\partial y^2}$,
   $\frac{\partial u}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial f}{\partial x} + r\cos\theta\frac{\partial f}{\partial y}$,
   and $\frac{\partial^2 u}{\partial \theta^2} = -r\cos\theta\frac{\partial f}{\partial x} - r\sin\theta\frac{\partial f}{\partial y} - r\sin\theta\big(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial \theta}\big) + r\cos\theta\big(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial \theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial \theta}\big)$
   $= r^2\sin^2\theta\frac{\partial^2 f}{\partial x^2} - r^2\sin 2\theta\frac{\partial^2 f}{\partial x \partial y} + r^2\cos^2\theta\frac{\partial^2 f}{\partial y^2}$,
   where the partial derivatives have been considered at the respective points.
   Therefore $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = (\cos^2\theta + \sin^2\theta)\frac{\partial^2 f}{\partial x^2} + (\cos^2\theta + \sin^2\theta)\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ at each point $(x, y) = (r\cos\theta, r\sin\theta)$ of $\mathbb{R}^2 \setminus \{(0, 0)\}$.

4. Show that a differentiable function $f : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree $\alpha \in \mathbb{R}$ (*i.e.* $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and for all $t > 0$) iff $\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$.

(The only if part of this result is known as Euler's theorem on homogeneous functions.)

Solution: We first assume that f is homogeneous of degree α. Let $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ and let $\varphi(t) = f(t\mathbf{x}) - t^\alpha f(\mathbf{x})$ for all $t > 0$. Then $\varphi(t) = 0$ for all $t > 0$ and since f is differentiable, by chain rule, we get $\varphi'(t) = \nabla f(t\mathbf{x}) \cdot \mathbf{x} - \alpha t^{\alpha-1} f(\mathbf{x}) = 0$ for all $t > 0$. Putting $t = 1$, we obtain $\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x})$.
Conversely, let $\nabla f(\mathbf{x}) \cdot \mathbf{x} = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$. Let $\mathbf{x} \in \mathbb{R}^m \setminus \{0\}$ and let $g(t) = t^{-\alpha} f(t\mathbf{x})$ for all $t > 0$. Since f is differentiable, by chain rule, $g : (0, \infty) \to \mathbb{R}$ is differentiable and $g'(t) = t^{-\alpha}\, \nabla f(t\mathbf{x}) \cdot \mathbf{x} - \alpha t^{-\alpha-1} f(t\mathbf{x}) = \alpha t^{-\alpha-1}\, \nabla f(t\mathbf{x}) \cdot \mathbf{x} - \alpha t^{-\alpha-1} f(t\mathbf{x}) = 0$ for all $t > 0$. Hence g is a constant function and so $g(t) = g(1) = f(\mathbf{x})$ for all $t > 0$. Consequently $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $t > 0$ and therefore f is a homogeneous function of degree α.

5. If $f(x, y) = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$, where $S = \{(x, x) : x \in \mathbb{R}\}$, then using Euler's theorem on homogeneous functions, show that $xf_x(x, y) + yf_y(x, y) = \sin(2f(x, y))$ for all $(x, y) \in \mathbb{R}^2 \setminus S$.

   Solution: If $g(x, y) = \tan(f(x, y)) = \frac{x^3+y^3}{x-y}$ for all $(x, y) \in \mathbb{R}^2 \setminus S$, then $g(tx, ty) = t^2 g(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$ and for all $t > 0$. Hence $g : \mathbb{R}^2 \setminus S \to \mathbb{R}$ is a homogeneous function of degree 2 and therefore by Euler's theorem on homogeneous functions, $xg(x, y) + yg(x, y) = 2g(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$. Again, by chain rule, $g_x(x, y) = \sec^2(f(x, y))f_x(x, y)$ and $g_y(x, y) = \sec^2(f(x, y))f_y(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus S$. Hence we get $xf_x(x, y) + yf_y(x, y) = 2\tan(f(x, y))\cos^2(f(x, y)) = \sin(2f(x, y))$ for all $(x, y) \in \mathbb{R}^2 \setminus S$.

6. If $f : \mathbb{R}^2 \setminus \{(0, 0)\} \to \mathbb{R}$ is a twice continuously differentiable homogeneous function of degree $n \in \mathbb{N}$, then show that $\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right)(x, y) = n(n-1)f(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

   Solution: By Euler's theorem on homogeneous functions, we get $x\frac{\partial f}{\partial x}(x, y) + y\frac{\partial f}{\partial y}(x, y) = nf(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Differentiating this partially with respect to x and y respectively, we get $x\frac{\partial^2 f}{\partial x^2}(x, y) + \frac{\partial f}{\partial x}(x, y) + y\frac{\partial^2 f}{\partial x \partial y}(x, y) = nf_x(x, y)$ and $x\frac{\partial^2 f}{\partial y \partial x}(x, y) + y\frac{\partial^2 f}{\partial y^2}(x, y) + \frac{\partial f}{\partial y}(x, y) = nf_y(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Since the second order partial derivatives of f are continuous, we have $\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ and hence by multiplying the above two relations by x and y respectively and then adding, we get $\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right)(x, y) + \left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)(x, y) = n\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right)(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$. Therefore $\left(x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2}\right)(x, y) = n(n-1)f(x, y)$ for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

   Solution: Let $(a, b) \in \mathbb{R}^2$ and let $g(t) = f(a + bt, b - bt)$ for all $t \in [0, 1]$. Then $g : [0, 1] \to \mathbb{R}$ is continuously differentiable. By the mean value theorem of single variable calculus, there exists $t_0 \in (0, 1)$ such that $g(1) - g(0) = g'(t_0) = \nabla f(a + bt_0, b - bt_0) \cdot (b, -b)$ (by chain rule) and hence $f(a + b, 0) - f(a, b) = bf_x(a + bt_0, b - bt_0) - bf_y(a + bt_0, b - bt_0) = 0$. Therefore

$f(a, b) = f(a + b, 0) > 0.$

8. Let $\alpha > 0$ and let $f : \mathbb{R}^m \to \mathbb{R}$ satisfy $|f(\mathbf{x}) - f(\mathbf{y})| \le \alpha \|\mathbf{x} - \mathbf{y}\|^2$ for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^m$. Show that f is a constant function.

   Solution: Let $\mathbf{x}_0$, $\mathbf{h} \in \mathbb{R}^m$. By the given condition $|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - \mathbf{0} \cdot \mathbf{h}| \le \alpha \|\mathbf{h}\|^2$ and so $\lim\limits_{\mathbf{h} \to \mathbf{0}} \frac{|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - \mathbf{0} \cdot \mathbf{h}|}{\|\mathbf{h}\|} = 0$. Hence f is differentiable at $\mathbf{x}_0$ and $\nabla f(\mathbf{x}_0) = \mathbf{0}$. Since $\mathbf{x}_0 \in \mathbb{R}^m$ is arbitrary, f is differentiable and $\nabla f(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^m$. If $\mathbf{x}_1$, $\mathbf{x}_2 \in \mathbb{R}^m$, then $L = \{(1 - t)\mathbf{x}_1 + t\mathbf{x}_2 : t \in [0, 1]\} \subseteq \mathbb{R}^m$ and hence by the mean value theorem, there exists $\mathbf{c} \in L$ such that $f(\mathbf{x}_2) - f(\mathbf{x}_1) = \nabla f(\mathbf{c}) \cdot (\mathbf{x}_2 - \mathbf{x}_1) = 0$. Thus $f(\mathbf{x}_1) = f(\mathbf{x}_2)$ and therefore f is a constant function.

9. Let S be a nonempty open and convex set in $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that $f_x(x, y) = 0 = f_y(x, y)$ for all $(x, y) \in S$. Show that f is a constant function.
   (A set $S \subseteq \mathbb{R}^m$ is called convex if $(1 - t)\mathbf{x} + t\mathbf{y} \in S$ for all $\mathbf{x}$, $\mathbf{y} \in S$ and for all $t \in [0, 1]$.)

   Solution: Since $f_x(x, y) = 0 = f_y(x, y)$ for all $(x, y) \in S$, f_x, $f_y : S \to \mathbb{R}$ are continuous and hence f is differentiable. If $\mathbf{x}_1$, $\mathbf{x}_2 \in S$, then $L = \{(1 - t)\mathbf{x}_1 + t\mathbf{x}_2 : t \in [0, 1]\} \subseteq S$ (since S is convex) and hence by the mean value theorem, there exists $\mathbf{c} \in L$ such that $f(\mathbf{x}_2) - f(\mathbf{x}_1) = \nabla f(\mathbf{c}) \cdot (\mathbf{x}_2 - \mathbf{x}_1) = 0$, since $\nabla f(\mathbf{c}) = (f_x(\mathbf{c}), f_y(\mathbf{c})) = (0, 0)$. Thus $f(\mathbf{x}_1) = f(\mathbf{x}_2)$ and therefore f is a constant function.

10. Find the equations of the tangent plane and the normal line to the surface given by $z = x^2 + y^2 - 2xy + 3y - x + 4$ at the point $(2, -3, 18)$.

    Solution: Let $f(x, y, z) = x^2 + y^2 - 2xy - x + 3y - z + 4$ for all $(x, y, z) \in \mathbb{R}^3$. Then $f : \mathbb{R}^3 \to \mathbb{R}$ is differentiable and $f_x(x, y, z) = 2x - 2y - 1$, $f_y(x, y, z) = 2y - 2x + 3$ and $f_z(x, y, z) = -1$ for all $(x, y, z) \in \mathbb{R}^3$. Hence the equation of the tangent plane to the given surface $f(x, y, z) = 0$ at $\mathbf{x}_0 = (2, -3, 18)$ is $f_x(\mathbf{x}_0)(x - 2) + f_y(\mathbf{x}_0)(y + 3) + f_z(\mathbf{x}_0)(z - 18) = 0$, *i.e.* $10(x - 2) - 7(y + 3) - (z - 18) = 0$, which simplifies to $10x - 7y - z = 23$.
    Again, the equation of the normal line to the given surface $f(x, y, z) = 0$ at $\mathbf{x}_0$ is $\frac{x-2}{f_x(\mathbf{x}_0)} = \frac{x-2}{f_y(\mathbf{x}_0)} = \frac{x-2}{f_z(\mathbf{x}_0)}$, *i.e.* $\frac{x-2}{10} = \frac{y+3}{-7} = \frac{z-18}{-1}$.

11. Find all points on the paraboloid $z = x^2 + y^2$ at which the tangent plane to the paraboloid is parallel to the plane $x + y + z = 1$. Also, determine the equations of the corresponding tangent planes.

    Solution: Let $(x_0, y_0, z_0) \in \mathbb{R}^3$ be a point on the paraboloid $z = x^2 + y^2$ at which the tangent plane to the paraboloid is parallel to the plane $x + y + z = 1$. If $g(x, y) = x^2 + y^2$ for all $(x, y) \in \mathbb{R}^2$, then $g : \mathbb{R}^2 \to \mathbb{R}$ is differentiable and $g_x(x, y) = 2x$, $g_y(x, y) = 2y$ for all $(x, y) \in \mathbb{R}^2$. Hence the equation of the tangent plane to the paraboloid $z = g(x, y)$ at (x_0, y_0, z_0) is $z = g(x_0, y_0) + g_x(x_0, y_0)(x - x_0) + g_y(x_0, y_0)(y - y_0)$, $z = z_0 + 2x_0(x - x_0) + 2y_0(y - y_0)$. Since this plane is parallel to the plane $z = 1 - x - y$, we must have that $2x_0 = -1$ and $2y_0 = -1$ and hence the required point is $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$.

Also, the equation of the tangent plane to the paraboloid at the point $(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2})$ is $z = \frac{1}{2} - (x + \frac{1}{2}) - (y + \frac{1}{2})$, *i.e.* $2x + 2y + 2z + 1 = 0$.

12. If $f(x, y) = x^3 + y^3 - 63x - 63y + 12xy$ for all $(x, y) \in \mathbb{R}^2$, then determine all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^2 \to \mathbb{R}$.

    Solution: We have $f_x(x, y) = 3x^2 - 63 + 12y$, $f_y(x, y) = 3y^2 - 63 + 12x$, $f_{xx}(x, y) = 6x$, $f_{yy}(x, y) = 6y$ and $f_{xy}(x, y) = 12$ for all $(x, y) \in \mathbb{R}^2$. We solve the system of equations $f_x(x, y) = 0$, $f_y(x, y) = 0$. Considering $f_x(x, y) - f_y(x, y) = 0$, we obtain $(x - y)(x + y - 4) = 0$ and hence $x = y$ or $x + y = 4$. If $x = y$, then from $f_x(x, y) = 0$, we get $x^2 + 4x - 21 = 0$ and so $x = 3, -7$. Hence in this case we get total two critical points $(3, 3)$ and $(-7, -7)$. Again, if $x + y = 4$, then $f_x(x, y) = 0$ gives $x^2 - 4x - 5 = 0$ and so $x = 5, -1$. Hence in this case we again get total two critical points $(5, -1)$ and $(-1, 5)$.
    Since $f_{xx}(3, 3)f_{yy}(3, 3) - f_{xy}(3, 3)^2 = 180 > 0$ and $f_{xx}(3, 3) = 18 > 0$, f has a local minimum at $(3, 3)$.
    Since $f_{xx}(-7, -7)f_{yy}(-7, -7) - f_{xy}(-7, -7)^2 = 1620 > 0$ and $f_{xx}(-7, -7) = -42 < 0$, f has a local maximum at $(-7, -7)$.
    Again, since $f_{xx}f_{yy} - f_{xy}^2 = -324 < 0$ for each of $(5, -1)$ and $(-1, 5)$, both $(5, -1)$ and $(-1, 5)$ are saddle points of f.

13. If $f(x, y) = 2x^4 + 2x^2y + y^2$ for all $(x, y) \in \mathbb{R}^2$, then determine all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^2 \to \mathbb{R}$.

    Solution: Solving $f_x(x, y) = 8x^3 + 4xy = 0$ and $f_y(x, y) = 2x^2 + 2y = 0$, we get $(x, y) = (0, 0)$ and hence $(0, 0)$ is the only critical point of f. Now, $f_{xx}(x, y) = 24x^2 + 4y$, $f_{yy}(x, y) = 2$ and $f_{xy}(x, y) = 4x$ for all $(x, y) \in \mathbb{R}^2$ and hence $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 0$. Therefore no definite conclusion (regarding local extremum and saddle point) of f at $(0, 0)$ can be obtained from the second order partial derivatives of f.
    However, since $f(x, y) = (x^2 + y)^2 + x^4 \geq 0 = f(0, 0)$ for all $(x, y) \in \mathbb{R}^2$, f has a local (in fact, absolute) minimum at $(0, 0)$.

14. If $f(x, y) = 4x^2 - xy + 4y^2 + x^3y + xy^3 - 4$ for all $(x, y) \in \mathbb{R}^2$, then determine all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^2 \to \mathbb{R}$.

    Solution: We have $f_x(x, y) = 8x - y + 3x^2y + y^3$, $f_y(x, y) = -x + 8y + x^3 + 3xy^2$, $f_{xx}(x, y) = 8 + 6xy$, $f_{yy}(x, y) = 8 + 6xy$ and $f_{xy}(x, y) = -1 + 3x^2 + 3y^2$ for all $(x, y) \in \mathbb{R}^2$. We solve the system of equations $f_x(x, y) = 0$, $f_y(x, y) = 0$. Considering $f_x(x, y) + f_y(x, y) = 0$, we obtain $(x + y)[(x + y)^2 + 7] = 0$ and hence $x + y = 0$. Now, $f_x(x, y) = 0$ gives $x(9 - 4x^2) = 0$ and so $x = 0, \frac{3}{2}, -\frac{3}{2}$. Hence we get total three critical points $(0, 0)$, $(\frac{3}{2}, -\frac{3}{2})$ and $(-\frac{3}{2}, \frac{3}{2})$.
    Since $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = 63 > 0$ and $f_{xx}(0, 0) = 8 > 0$, f has a local minimum at $(0, 0)$.
    Again, since $f_{xx}f_{yy} - f_{xy}^2 = -324 < 0$ for each of $(\frac{3}{2}, -\frac{3}{2})$ and $(-\frac{3}{2}, \frac{3}{2})$, both $(\frac{3}{2}, -\frac{3}{2})$ and $(-\frac{3}{2}, \frac{3}{2})$ are saddle points of f.

15. If $f(x,y,z) = x^2 + y^2 + z^2 + 2xyz - 4zx - 2yz - 2x - 4y + 4z$ for all $(x,y,z) \in \mathbb{R}^3$, then find all the points of local maximum, local minimum and all the saddle points of $f : \mathbb{R}^3 \to \mathbb{R}$.

Solution: We have $f_x(x,y,z) = 2yz - 4z + 2x - 2$, $f_y(x,y,z) = 2zx - 2z + 2y - 4$ and $f_z(x,y,z) = 2xy - 4x - 2y + 2z + 4$ for all $(x,y,z) \in \mathbb{R}^3$. In order to solve the system of equations $f_x(x,y,z) = 0$, $f_y(x,y,z) = 0$, $f_z(x,y,z) = 0$, we add the last two equations to obtain $x(y + z - 2) = 0$, and so $x = 0$ or $y + z = 2$.
Case 1: $x = 0$
In this case $y - z = 2$ and $yz - 2z = 1$, from which we get $z = 1, -1$. Hence in this case we obtain total two critical points of f, which are $(0,3,1)$ and $(0,1,-1)$.
Case 2: $y + z = 2$
In this case $-z^2 + x - 1 = 0$ and so $(z^2 + 1)z - 2z = 0$, which gives $z = 0,\ 1,\ -1$. Hence in this case we obtain total three critical points of f, which are $(1,2,0)$, $(2,1,1)$ and $(2,3,-1)$.
Now, $f_{xx}(x,y,z) = 2$, $f_{yy}(x,y,z) = 2$, $f_{zz}(x,y,z) = 2$, $f_{xy}(x,y,z) = 2z$, $f_{yz}(x,y,z) = 2x - 2$ and $f_{zx}(x,y,z) = 2y - 4$ for all $(x,y,z) \in \mathbb{R}^3$. Hence $H_f(x,y,z) = \begin{bmatrix} 2 & 2z & 2y-4 \\ 2z & 2 & 2x-2 \\ 2y-4 & 2x-2 & 2 \end{bmatrix}$ for all $(x,y,z) \in \mathbb{R}^3$.
The leading principal minors of $H_f(1,2,0)$ are 2, 4 and 8 (all of which are positive), and therefore f has a local minimum at $(1,2,0)$.
It can also be easily seen that $\det(H_f(x,y,z)) = -32 < 0$ for each of the remaining four critical points of f and $f_{xx}(x,y,z) = 2 > 0$ for each of these points. Therefore each of these remaining four critical points of f are saddle points of f.

16. If $S = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$, then determine $\max\{x^2 + 2x + y^2 : (x,y) \in S\}$ and $\min\{x^2 + 2x + y^2 : (x,y) \in S\}$.

Solution: Let $f(x,y) = x^2 + 2x + y^2$ for all $(x,y) \in S$. Since S is a closed and bounded set in $\mathbb{R}^2$ and $f : S \to \mathbb{R}$ is continuous, both $\max\{f(x,y) : (x,y) \in S\}$ and $\min\{f(x,y) : (x,y) \in S\}$ exist (in $\mathbb{R}$).
We first look for local extrema of f in $S^0 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Solving the system of equations $f_x(x,y) = 2x + 2 = 0$, $f_y(x,y) = 2y = 0$, we get $(x,y) = (-1,0)$, which does not belong to S^0. Hence f does not have any local extremum in S^0.
Again, the boundary of S consists of all the points on the circle $x^2 + y^2 = 1$. Taking the parametric representation of the circle $x^2 + y^2 = 1$ as $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, we look for local extrema of $\varphi : [0, 2\pi] \to \mathbb{R}$, where $\varphi(t) = f(\gamma(t)) = 1 + 2\cos t$ for all $t \in [0, 2\pi]$. Clearly φ has local (in fact, absolute) maxima only at $t = 0,\ 2\pi$ and local (in fact, absolute) minimum at $t = \pi$. These points correspond to the points $(1,0)$ and $(-1,0)$ of S.
Since $f(1,0) = 3$ and $f(-1,0) = -1$, it follows that $\max\{f(x,y) : (x,y) \in S\} = 3$,
$\min\{f(x,y) : (x,y) \in S\} = -1$ and these values are attained by f at $(1,0)$ and $(-1,0)$ respectively.

17. Find the (absolute) maximum value of $f(x, y, z) = 8xyz^2 - 200(x + y + z)$ subject to the constraint $x + y + z = 100$, $x \geq 0$, $y \geq 0$, $z \geq 0$.

Solution: Let $S = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0\}$ and let $f(x, y, z) = 8xyz^2 - 200(x + y + z)$, $g(x, y, z) = x + y + z - 100$ for all $(x, y, z) \in S$. If either of x, y, or z is 0, then $f(x, y, z) = -200(x + y + z)$ and so under the constraint $x + y + z = 100$, $f(x, y, z) = -20000$, which is clearly not the maximum value of $f(x, y, z)$ under the given conditions. Hence in order to find the maximum value of $f(x, y, z)$ subject to the given constraint, we may assume that $x > 0$, $y > 0$, and $z > 0$. Clearly f, $g : S \to \mathbb{R}$ are continuously differentiable on $S^0 = \{(x, y, z) \in \mathbb{R}^3 : x > 0, y > 0, z > 0\}$ and $\nabla g(x, y, z) = (1, 1, 1) \neq (0, 0, 0)$ for all $(x, y, z) \in S^0$. Let $(x_0, y_0, z_0) \in \Omega = \{(x, y, z) \in S : g(x, y, z) = 0\}$ and let $\lambda \in \mathbb{R}$ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$. Then $(8y_0z_0^2 - 200, 8x_0z_0^2 - 200, 16x_0y_0z_0 - 200) = \lambda(1, 1, 1)$ and hence $8y_0z_0^2 - 200 = \lambda$, $8x_0z_0^2 - 200 = \lambda$, $16x_0y_0z_0 - 200 = \lambda$. So, we get $8y_0z_0^2 = 8x_0z_0^2$ and hence $x_0 = y_0$. Consequently $8x_0z_0^2 = 16x_0^2z_0$ and so $z_0 = 2x_0$. Since $x_0 + y_0 + z_0 = 100$, we get $x_0 = 25$, $y_0 = 25$, $z_0 = 50$. Hence by Lagrange multiplier method, $(25, 25, 50)$ is the only possible point in S^0 where $f\big|_\Omega$ has a local extremum. Again, since Ω is a closed and bounded set in $\mathbb{R}^3$ and since f is continuous on Ω, $\max\{f(x, y, z) : (x, y, z) \in \Omega\}$ must exist (in $\mathbb{R}$). Consequently $f(25, 25, 50) = 12480000$ is the required maximum value.

MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Tutorial Problem Set - 1

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1. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. Show that $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ iff $\mathbf{y} = \mathbf{0}$ or $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \geq 0$.

   Solution: If $\mathbf{y} = \mathbf{0}$, then $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$. Also, if $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \geq 0$, then $\|\mathbf{x}+\mathbf{y}\| = \|(\alpha+1)\mathbf{y}\| = (\alpha+1)\|\mathbf{y}\|$ and $\|\mathbf{x}\| + \|\mathbf{y}\| = \alpha\|\mathbf{y}\| + \|\mathbf{y}\| = (\alpha+1)\|\mathbf{y}\|$, so that $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$.
   Conversely, let $\|\mathbf{x}+\mathbf{y}\| = \|\mathbf{x}\| + \|\mathbf{y}\|$ and let $\mathbf{y} \neq \mathbf{0}$. Then $\|\mathbf{x}+\mathbf{y}\|^2 = (\|\mathbf{x}\| + \|\mathbf{y}\|)^2$, which gives $\|\mathbf{x}\|^2 + 2\,\mathbf{x}\cdot\mathbf{y} + \|\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + 2\|\mathbf{x}\|\,\|\mathbf{y}\| + \|\mathbf{y}\|^2$ and so $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|$. Hence $|\mathbf{x}\cdot\mathbf{y}| = \|\mathbf{x}\|\,\|\mathbf{y}\|$ and by the equality condition in Cauchy-Schwarz inequality, we get $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha \in \mathbb{R}$. Since we have $\mathbf{x}\cdot\mathbf{y} = \|\mathbf{x}\|\,\|\mathbf{y}\|$, we obtain $\alpha\mathbf{y}\cdot\mathbf{y} = \|\alpha\mathbf{y}\|\,\|\mathbf{y}\|$, *i.e.* $\alpha\|\mathbf{y}\|^2 = |\alpha|\,\|\mathbf{y}\|^2$. Since $\|\mathbf{y}\| \neq 0$, we get $\alpha = |\alpha|$ and hence $\alpha \geq 0$.

2. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$ and $r, s > 0$. Show that $B_r[\mathbf{x}] \cap B_s[\mathbf{y}] \neq \emptyset$ iff $\|\mathbf{x}-\mathbf{y}\| \leq r+s$.

   Solution: Let us first assume that $B_r[\mathbf{x}] \cap B_s[\mathbf{y}] \neq \emptyset$ Then there exists $\mathbf{z} \in B_r[\mathbf{x}] \cap B_s[\mathbf{y}]$ and so $\|\mathbf{z}-\mathbf{x}\| \leq r$, $\|\mathbf{z}-\mathbf{y}\| \leq s$. Hence $\|\mathbf{x}-\mathbf{y}\| = \|\mathbf{x}-\mathbf{z}+\mathbf{z}-\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{z}\| + \|\mathbf{z}-\mathbf{y}\| \leq r+s$.
   Conversely, let $\|\mathbf{x}-\mathbf{y}\| \leq r+s$. If $\mathbf{z} = \frac{s}{r+s}\,\mathbf{x} + \frac{r}{r+s}\,\mathbf{y}$, then $\mathbf{z} \in \mathbb{R}^m$ and
   $\|\mathbf{z}-\mathbf{x}\| = \frac{1}{r+s}\|s\mathbf{x}+r\mathbf{y}-r\mathbf{x}-s\mathbf{x}\| = \frac{r}{r+s}\|\mathbf{x}-\mathbf{y}\| \leq r$, *i.e.* $\mathbf{z} \in B_r[\mathbf{x}]$. Similarly we get $\|\mathbf{z}-\mathbf{y}\| \leq s$ and so $\mathbf{z} \in B_s[\mathbf{y}]$. Hence $\mathbf{z} \in B_r[\mathbf{x}] \cap B_s[\mathbf{y}]$ and therefore $B_r[\mathbf{x}] \cap B_s[\mathbf{y}] \neq \emptyset$.

3. Let $(\mathbf{x}_n)$ be a sequence in $\mathbb{R}^m$. Show that $(\mathbf{x}_n)$ converges in $\mathbb{R}^m$ iff for each $\mathbf{x} \in \mathbb{R}^m$, the sequence $(\mathbf{x}_n\cdot\mathbf{x})$ converges in $\mathbb{R}$.

   Solution: Let us first assume that $(\mathbf{x}_n)$ converges in $\mathbb{R}^m$ and let $\mathbf{x}_0 \in \mathbb{R}^m$ such that $\mathbf{x}_n \to \mathbf{x}_0$. If $\mathbf{x} \in \mathbb{R}^m$, then for all $n \in \mathbb{N}$, $|\mathbf{x}_n\cdot\mathbf{x} - \mathbf{x}_0\cdot\mathbf{x}| = |(\mathbf{x}_n - \mathbf{x}_0)\cdot\mathbf{x}| \leq \|\mathbf{x}_n - \mathbf{x}_0\|\,\|\mathbf{x}\|$ (by Cauchy-Schwarz inequality). Since $\mathbf{x}_n \to \mathbf{x}_0$, we have $\|\mathbf{x}_n - \mathbf{x}_0\| \to 0$ and hence $|\mathbf{x}_n\cdot\mathbf{x} - \mathbf{x}_0\cdot\mathbf{x}| \to 0$. Therefore $\mathbf{x}_n\cdot\mathbf{x} \to \mathbf{x}_0\cdot\mathbf{x} \in \mathbb{R}$ and so the sequence $(\mathbf{x}_n\cdot\mathbf{x})$ converges in $\mathbb{R}$.
   Conversely, let the sequence $(\mathbf{x}_n\cdot\mathbf{x})$ converge in $\mathbb{R}$ for each $\mathbf{x} \in \mathbb{R}^m$. Let $\mathbf{x}_n = \left(x_1^{(n)}, \ldots, x_m^{(n)}\right)$ for all $n \in \mathbb{N}$. By the given condition, for each $j \in \{1, \ldots, m\}$, the sequence $\left(x_j^{(n)}\right) = (\mathbf{x}_n\cdot\mathbf{e}_j)$ converges in $\mathbb{R}$. Therefore the sequence $(\mathbf{x}_n)$ converges in $\mathbb{R}^m$.

4. (a) State TRUE or FALSE with justification: If $(\mathbf{x}_n)$ is a sequence in $\mathbb{R}^m$ having no convergent subsequence, then it is necessary that $\lim\limits_{n\to\infty} \|\mathbf{x}_n\| = \infty$.

   Solution: Let $r > 0$ and if possible, let $S = \{n \in \mathbb{N} : \|\mathbf{x}_n\| \leq r\}$ be an infinite set. Then there exists a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $\|\mathbf{x}_{n_k}\| \leq r$ for all $k \in \mathbb{N}$. This implies that the subsequence $(\mathbf{x}_{n_k})$ of the sequence $(\mathbf{x}_n)$ is bounded in $\mathbb{R}^m$ and hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^m$, $(\mathbf{x}_{n_k})$ has a convergent subsequence. This convergent subsequence is also a convergent subsequence of $(\mathbf{x}_n)$, which is a contradiction to the given condition. Therefore S is a finite set. Let $n_0 = 1$ if $S = \emptyset$ and $n_0 = \max S + 1$ if $S \neq \emptyset$. Then $\|\mathbf{x}_n\| > r$ for all

$n \geq n_0$ and hence $\lim\limits_{n\to\infty} \|\mathbf{x}_n\| = \infty$. Therefore the given statement is TRUE.

(b) State TRUE or FALSE with justification: If $\big((x_n, y_n)\big)$ is a bounded sequence in $\mathbb{R}^2$ such that every convergent subsequence of $\big((x_n, y_n)\big)$ converges to $(0,1)$, then $\big((x_n, y_n)\big)$ must converge to $(0,1)$.

Solution: If possible, let $(x_n, y_n) \not\to (0,1)$. Then there exists $\varepsilon > 0$ such that $(x_n, y_n) \notin B_\varepsilon\big((0,1)\big)$ for infinitely many $n \in \mathbb{N}$ and hence we can find a strictly increasing sequence (n_k) in $\mathbb{N}$ such that $(x_{n_k}, y_{n_k}) \notin B_\varepsilon\big((0,1)\big)$ for all $k \in \mathbb{N}$. Since $\big((x_n, y_n)\big)$ is bounded, its subsequence $\big((x_{n_k}, y_{n_k})\big)$ is also bounded and hence by the Bolzano-Weierstrass theorem in $\mathbb{R}^2$, $\big((x_{n_k}, y_{n_k})\big)$ has a convergent subsequence $\big((x_{n_{k_l}}, y_{n_{k_l}})\big)$. Now, $\big((x_{n_{k_l}}, y_{n_{k_l}})\big)$ is also a subsequence of $\big((x_n, y_n)\big)$ and hence by the given condition $\big(x_{n_{k_l}}, y_{n_{k_l}}\big) \to (0,1)$. But this contradicts the fact that $\big(x_{n_{k_l}}, y_{n_{k_l}}\big) \notin B_\varepsilon\big((0,1)\big)$ for all $l \in \mathbb{N}$. Hence $(x_n, y_n) \to (0,1)$. Therefore the given statement is TRUE.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy}{x^2-y^2} & \text{if } x^2 \neq y^2, \\ 0 & \text{if } x^2 = y^2. \end{cases}$
Determine all the points of $\mathbb{R}^2$ where f is continuous.

Solution: If $\varphi(x,y) = xy$ and $\psi(x,y) = x^2 - y^2$ for all $(x,y) \in \mathbb{R}^2$, then as polynomial functions, $\varphi,\ \psi : \mathbb{R}^2 \to \mathbb{R}$ are continuous and $\psi(x,y) \neq 0$ for all $(x,y) \in \mathbb{R}^2$ with $x^2 \neq y^2$. Hence f is continuous at each $(x,y) \in \mathbb{R}^2$ with $x^2 \neq y^2$.
Let $(x,y) \in \mathbb{R}^2$ such that $x^2 = y^2 \neq 0$. Then $(x + \frac{x}{n}, y) \to (x,y)$ but $|f(x + \frac{x}{n}, y)| = \frac{n+1}{2+\frac{1}{n}} \to \infty$ and so $f(x + \frac{x}{n}, y) \not\to 0 = f(x,y)$. Hence f is not continuous at (x,y).
Again, $(\frac{2}{n}, \frac{1}{n}) \to (0,0)$ but $f(\frac{2}{n}, \frac{1}{n}) = \frac{2}{3}$ for all $n \in \mathbb{N}$, so that $f(\frac{2}{n}, \frac{1}{n}) \not\to 0 = f(0,0)$. Hence f is not continuous at $(0,0)$.
Therefore the set of points of continuity of f is $\{(x,y) \in \mathbb{R}^2 : x^2 \neq y^2\}$.

6. Let α, β be positive real numbers and let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
$f(x,y) = \begin{cases} \frac{|x|^\alpha |y|^\beta}{\sqrt{x^2+y^2}} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
Show that f is continuous iff $\alpha + \beta > 1$.

Solution: Let $\alpha + \beta > 1$ and let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (0,0)$. Then $x_n \to 0$ and $y_n \to 0$. For all $n \in \mathbb{N}$ for which $(x_n, y_n) \neq (0,0)$, we have
$0 \leq f(x_n, y_n) \leq \frac{(x_n^2+y_n^2)^{\frac{\alpha}{2}} (x_n^2+y_n^2)^{\frac{\beta}{2}}}{\sqrt{x_n^2+y_n^2}} = (x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)}$ and since $f(0,0) = 0$,
we have $0 \leq f(x_n, y_n) \leq (x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)}$ for all $n \in \mathbb{N}$. Since $(x_n^2 + y_n^2)^{\frac{1}{2}(\alpha+\beta-1)} \to 0$, we get $f(x_n, y_n) \to 0 = f(0,0)$. This shows that f is continuous at $(0,0)$. Also, it is clear (by similar arguments given in other examples) that f is continuous at each $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Therefore f is continuous.
Conversely, let f be continuous and if possible, let $\alpha + \beta \leq 1$. We have $(\frac{1}{n}, \frac{1}{n}) \to (0,0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{\sqrt{2}} n^{1-(\alpha+\beta)} \not\to 0 = f(0,0)$ (because for $\alpha+\beta = 1$, $f(\frac{1}{n}, \frac{1}{n}) \to \frac{1}{\sqrt{2}}$ and for $\alpha+\beta < 1$, the sequence $\big(f(\frac{1}{n}, \frac{1}{n})\big)$ is unbounded). Hence f is not continuous at $(0,0)$, which is a contradiction.

Therefore $\alpha + \beta > 1$.

7. Let $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0, y_0) \in S$. Let $A = \{x \in \mathbb{R} : (x, y_0) \in S\}$ and $B = \{y \in \mathbb{R} : (x_0, y) \in S\}$. Define $\varphi(x) = f(x, y_0)$ for all $x \in A$ and $\psi(y) = f(x_0, y)$ for all $y \in B$. If f is continuous at (x_0, y_0), then show that $\varphi : A \to \mathbb{R}$ is continuous at x_0 and $\psi : B \to \mathbb{R}$ is continuous at y_0. Is the converse true? Justify.

   Solution: Let (x_n) be a sequence in A such that $x_n \to x_0$ and let (y_n) be a sequence in B such that $y_n \to y_0$. Then (x_n, y_0), $(x_0, y_n) \in S$ for all $n \in \mathbb{N}$ and $(x_n, y_0) \to (x_0, y_0)$, $(x_0, y_n) \to (x_0, y_0)$. Since f is continuous at (x_0, y_0), $\varphi(x_n) = f(x_n, y_0) \to f(x_0, y_0) = \varphi(x_0)$ and $\psi(y_n) = f(x_0, y_n) \to f(x_0, y_0) = \psi(y_0)$. Therefore φ is continuous at x_0 and ψ is continuous at y_0.
   The converse is not true, in general. For example, let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by
   $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$
   Then f is not continuous at $(0, 0)$, since $(\frac{1}{n}, \frac{1}{n}) \to (0, 0)$ but $f(\frac{1}{n}, \frac{1}{n}) = \frac{1}{2} \to \frac{1}{2} \neq 0 = f(0, 0)$. However, $\varphi(x) = f(x, 0) = 0$ for all $x \in \mathbb{R}$ and $\psi(y) = f(0, y) = 0$ for all $y \in \mathbb{R}$. So $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are continuous at 0.

8. If $S = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3\}$, then determine (with justification) S^0.

   Solution: Let $(x_0, y_0) \in S$ and let $0 < x_0 < 3$. If $r = \min\{x_0, 3 - x_0\}$, then $r > 0$. Let $(x, y) \in B_r\big((x_0, y_0)\big)$. Then $|x - x_0| \leq \sqrt{(x - x_0)^2 + (y - y_0)^2} < r$. Hence $x - x_0 < r \leq 3 - x_0$, which gives $x < 3$, and $x_0 - x < r \leq x_0$, which gives $x > 0$. Therefore $(x, y) \in S$ and so $B_r\big((x_0, y_0)\big) \subseteq S$. Hence $(x_0, y_0) \in S^0$.
   Now, let $y \in \mathbb{R}$.
   If possible, let $(0, y) \in S^0$. Then there exists $r > 0$ such that $B_r\big((0, y)\big) \subseteq S$. Since $\|(-\frac{r}{2}, y) - (0, y)\| = \frac{r}{2} < r$, $(-\frac{r}{2}, y) \in B_r\big((0, y)\big)$ and since $-\frac{r}{2} < 0$, $(-\frac{r}{2}, y) \notin S$. Thus we get a contradiction. Hence $(0, y) \notin S^0$.
   Again, if possible, let $(3, y) \in S^0$. Then there exists $r > 0$ such that $B_r\big((3, y)\big) \subseteq S$. Since $\|(3 + \frac{r}{2}, y) - (3, y)\| = \frac{r}{2} < r$, $(3 + \frac{r}{2}, y) \in B_r\big((3, y)\big)$ and since $3 + \frac{r}{2} > 3$, $(3 + \frac{r}{2}, y) \notin S$. Thus we get a contradiction. Hence $(3, y) \notin S^0$.
   Therefore $S^0 = \{(x, y) \in \mathbb{R}^2 : 0 < x < 3\}$.

MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Tutorial Problem Set - 2

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1. Let $A = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x,y,z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is (a) an open set (b) a closed set in $\mathbb{R}^3$.

   Solution: We have $(0,0,0) \in A \cap B$. If possible, let $(0,0,0) \in (A \cap B)^0$. Then there exists $r > 0$ such that $B_r\big((0,0,0)\big) \subseteq A \cap B$. Since $(0,0,\frac{r}{2}) \in B_r\big((0,0,0)\big)$ but $(0,0,\frac{r}{2}) \notin A \cap B$, we get a contradiction. Hence $(0,0,0) \notin (A \cap B)^0$. Therefore $A \cap B$ is not an open set in $\mathbb{R}^3$.
   Again, since $(1 - \frac{1}{n}, 0, 0) \in A \cap B$ for all $n \in \mathbb{N}$ and since $(1 - \frac{1}{n}, 0, 0) \to (1,0,0) \notin A \cap B$, $A \cap B$ is not a closed set in $\mathbb{R}^3$.

2. Show that $\{\mathbf{x} \in \mathbb{R}^m : 1 < \|\mathbf{x}\| \le 2\}$ is neither an open set nor a closed set in $\mathbb{R}^m$.

   Solution: Let $S = \{\mathbf{x} \in \mathbb{R}^m : 1 < \|\mathbf{x}\| \le 2\}$. Since $\|(2 + \frac{1}{n})\mathbf{e}_1\| = 2 + \frac{1}{n} > 2$ for all $n \in \mathbb{N}$, $(2 + \frac{1}{n})\mathbf{e}_1 \in \mathbb{R}^m \setminus S$ for all $n \in \mathbb{N}$. Also, $(2 + \frac{1}{n})\mathbf{e}_1 \to 2\mathbf{e}_1 \notin \mathbb{R}^m \setminus S$, since $\|2\mathbf{e}_1\| = 2$. Hence $\mathbb{R}^m \setminus S$ is not a closed set in $\mathbb{R}^m$ and consequently S is not an open set in $\mathbb{R}^m$.
   Again, since $\|(1 + \frac{1}{n})\mathbf{e}_1\| = 1 + \frac{1}{n} \in (1, 2]$ for all $n \in \mathbb{N}$, $(1 + \frac{1}{n})\mathbf{e}_1 \in S$ for all $n \in \mathbb{N}$. Also, $(1 + \frac{1}{n})\mathbf{e}_1 \to \mathbf{e}_1 \notin S$, since $\|\mathbf{e}_1\| = 1$. Hence S is not a closed set in $\mathbb{R}^m$.

3. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous and if S is a bounded subset of $\mathbb{R}^2$, then $f(S)$ must be a bounded subset of $\mathbb{R}$.

   Solution: Since S is a bounded subset of $\mathbb{R}^2$, there exists $r > 0$ such that $S \subseteq B_r[\mathbf{0}]$. Now, since $B_r[\mathbf{0}]$ is a closed and bounded set in $\mathbb{R}^2$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, $f(B_r[\mathbf{0}])$ is a bounded set in $\mathbb{R}$. Hence there exists $M > 0$ such that $|f(\mathbf{x})| \le M$ for all $\mathbf{x} \in B_r[\mathbf{0}]$. So, in particular, $|f(\mathbf{x})| \le M$ for all $\mathbf{x} \in S$. Hence $f(S)$ is a bounded subset of $\mathbb{R}$. Therefore the given statement is TRUE.

4. Let S be a nonempty subset of $\mathbb{R}^m$ such that every continuous function $f : S \to \mathbb{R}$ is bounded. Show that S is a closed and bounded set in $\mathbb{R}^m$.

   Solution: If possible, let S be not closed in $\mathbb{R}^m$. Then there exists $\mathbf{x}_0 \in \mathbb{R}^m \setminus S$ and a sequence $(\mathbf{x}_n)$ in S such that $\mathbf{x}_n \to \mathbf{x}_0$. The function $f : S \to \mathbb{R}$, defined by $f(\mathbf{x}) = \frac{1}{\|\mathbf{x} - \mathbf{x}_0\|}$ for all $\mathbf{x} \in S$, is continuous but not bounded (since $\|\mathbf{x}_n - \mathbf{x}_0\| \to 0$ and so $f(\mathbf{x}_n) \to \infty$), which contradicts the hypothesis. Hence S must be a closed set in $\mathbb{R}^m$.
   Again, if possible, let S be not bounded in $\mathbb{R}^m$. Then the function $g : S \to \mathbb{R}$, defined by $g(\mathbf{x}) = \|\mathbf{x}\|$ for all $\mathbf{x} \in S$, is continuous but not bounded, which contradicts the hypothesis. Hence S must be bounded in $\mathbb{R}^m$.

5. Let $S = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$ and let $f : S \to \mathbb{R}$ be continuous. Show that there exist $\alpha, \beta \in \mathbb{R}$ with $\alpha \le \beta$ such that $f(S) = [\alpha, \beta]$.

   Solution: We know that $S = B_1[\mathbf{0}]$ is a closed and bounded set in $\mathbb{R}^3$. Since $f : S \to \mathbb{R}$

is continuous, there exist $\mathbf{x}_0,\ \mathbf{y}_0 \in S$ such that $f(\mathbf{x}_0) \le f(\mathbf{x}) \le f(\mathbf{y}_0)$ for all $\mathbf{x} \in S$. Taking $\alpha = f(\mathbf{x}_0)$ and $\beta = f(\mathbf{y}_0)$, we find that $\alpha,\ \beta \in \mathbb{R}$, $\alpha \le \beta$ and $f(S) \subseteq [\alpha, \beta]$. Again, if $t \in [0,1]$, then $(1-t)\mathbf{x}_0 + t\mathbf{y}_0 \in \mathbb{R}^3$ and since $\|(1-t)\mathbf{x}_0 + t\mathbf{y}_0\| \le (1-t)\|\mathbf{x}_0\| + t\|\mathbf{y}_0\| \le 1-t+t = 1$, $(1-t)\mathbf{x}_0 + t\mathbf{y}_0 \in S$. Let $F(t) = (1-t)\mathbf{x}_0 + t\mathbf{y}_0$ and $\varphi(t) = f(F(t))$ for all $t \in [0,1]$. Since the functions $F : [0,1] \to S$ and $f : S \to \mathbb{R}$ are continuous, $\varphi = f \circ F : [0,1] \to \mathbb{R}$ is continuous. Assuming $\alpha < \beta$, let $\gamma \in (\alpha, \beta) = (\varphi(0), \varphi(1))$. Then by the intermediate value property of the continuous function φ, there exists $t_0 \in (0,1)$ such that $\gamma = \varphi(t_0) = f(F(t_0)) \in f(S)$, since $F(t_0) \in S$. Therefore $f(S) = [\alpha, \beta]$.

6. (a) Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $\big((x_n, y_n)\big)$ be any sequence in $\mathbb{R}^2 \setminus \{(0,0)\}$ such that $(x_n, y_n) \to (0,0)$. Then $x_n^2 + y_n^2 \ne 0$ for all $n \in \mathbb{N}$ and $x_n^2 + y_n^2 \to 0$ in $\mathbb{R}$. Since $\lim\limits_{t\to 0} \frac{1-\cos t}{t^2} = \lim\limits_{t\to 0} \frac{\sin t}{2t} = \frac{1}{2}$, we have $\lim\limits_{n\to\infty} \frac{1-\cos(x_n^2+y_n^2)}{(x_n^2+y_n^2)^2} = \frac{1}{2}$. It follows that $\lim\limits_{(x,y)\to(0,0)} \frac{1-\cos(x^2+y^2)}{(x^2+y^2)^2}$ exists and its value is $\frac{1}{2}$.

(b) Examine whether $\lim\limits_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \sin \frac{1}{x^2+y^2}$ exists (in $\mathbb{R}$) and find its value if it exists (in $\mathbb{R}$).

Solution: Let $f(x,y) = \frac{y}{x^2+y^2} \sin \frac{1}{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$.
Since $\Big(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\Big) \to (0,0)$ and $f\Big(0, \frac{\sqrt{2}}{\sqrt{(4n+1)\pi}}\Big) = \sqrt{2n\pi + \frac{\pi}{2}} \to \infty$, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist (in $\mathbb{R}$).

7. Let S be a nonempty open set in $\mathbb{R}$ and let $F : S \to \mathbb{R}^m$ be a differentiable function such that $\|\, F(t) \,\|$ is constant for all $t \in S$. Show that $F(t) \cdot F'(t) = 0$ for all $t \in S$.

Solution: Let $c \in \mathbb{R}$ such that $\|\, F(t) \,\| = c$ for all $t \in S$. Then $F(t) \cdot F(t) = \|\, F(t) \,\|^2 = c^2$ for all $t \in S$. Hence $\frac{d}{dt}\big(F(t) \cdot F(t)\big) = 0$ for all $t \in S$. This gives $F'(t) \cdot F(t) + F(t) \cdot F'(t) = 0$ for all $t \in S$. So $2\,F(t) \cdot F'(t) = 0$ for all $t \in S$. Therefore $F(t) \cdot F'(t) = 0$ for all $t \in S$.

MA 101 (Mathematics I)

Multivariable Calculus : Hints / Solutions of Tutorial Problem Set - 3

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1. Let S be a nonempty open subset of $\mathbb{R}^2$ and let $f : S \to \mathbb{R}$ be such that the partial derivatives f_x and f_y exist at each point of S. If $f_x : S \to \mathbb{R}$ and $f_y : S \to \mathbb{R}$ are bounded, then show that f is continuous.

   Solution: Since f_x and f_y are bounded, there exist $M_1, M_2 > 0$ such that $|f_x(x,y)| \le M_1$ and $|f_y(x,y)| \le M_2$ for all $(x,y) \in S$. Let $(x_0,y_0) \in S$. Since S is open in $\mathbb{R}^2$, there exists $r > 0$ such that $B_r\big((x_0,y_0)\big) \subseteq S$. For all $h,k \in \mathbb{R}$ with $|h| < \frac{r}{2}$, $|k| < \frac{r}{2}$, we have $|f(x_0+h,y_0+k) - f(x_0,y_0)| = |f(x_0+h,y_0+k) - f(x_0,y_0+k) + f(x_0,y_0+k) - f(x_0,y_0)|$ $\le |f(x_0+h,y_0+k) - f(x_0,y_0+k)| + |f(x_0,y_0+k) - f(x_0,y_0)|$ $= |h||f_x(x_0+\theta_1 h, y_0+k)| + |k||f_y(x_0, y_0+\theta_2 k)|$ for some $\theta_1, \theta_2 \in (0,1)$ (using Lagrange's mean value theorem of single real variable). Hence if $\varepsilon > 0$, then choosing $\delta = \min\{\frac{r}{2}, \frac{\varepsilon}{M_1+M_2}\} > 0$, we find that $|f(x_0+h,y_0+k) - f(x_0,y_0)| \le M_1|h| + M_2|k| < \varepsilon$ for all $(h,k) \in \mathbb{R}^2$ with $\|(h,k)\| = \sqrt{h^2+k^2} < \delta$. Therefore f is continuous at (x_0,y_0). Since $(x_0,y_0) \in S$ is arbitrary, f is continuous.

2. Find all $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$ for which the directional derivative $D_{\mathbf{u}}f(0,0)$ exists (in $\mathbb{R}$), if for all $(x,y) \in \mathbb{R}^2$, $f(x,y) = \begin{cases} 1 & \text{if } y < x^2 < 2y, \\ 0 & \text{otherwise.} \end{cases}$

   Solution: Let $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$. We have $\lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u}) - f(0,0)}{t} = \lim\limits_{t\to 0} \frac{f(tu_1,tu_2)}{t} = \lim\limits_{t\to 0} \frac{0}{t} = 0$. (The inequalities $tu_2 < t^2u_1^2 < 2tu_2$ are equivalent to the inequalities (i) $u_2 < tu_1^2 < 2u_2$ if $t > 0$ and (ii) $u_2 > tu_1^2 > 2u_2$ if $t < 0$. We can make $|tu_1^2|$ arbitrarily small for sufficiently small $|t| > 0$ and hence for such t, at least one inequality in each of (i) and (ii) cannot be satisfied. Thus we get $f(tu_1,tu_2) = 0$ for sufficiently small $|t| > 0$.)
   Therefore $D_{\mathbf{u}}f(0,0)$ exists (and equals 0) for each $\mathbf{u} \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$.

3. State TRUE or FALSE with justification: If $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous such that all the directional derivatives of f at $(0,0)$ exist (in $\mathbb{R}$), then f must be differentiable at $(0,0)$.

   Solution: Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{x^2y\sqrt{x^2+y^2}}{x^4+y^2} & \text{if } (x,y) \ne (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$
   We know that f is continuous at each point of $\mathbb{R}^2 \setminus \{(0,0)\}$. Let $\varepsilon > 0$. We have $|f(x,y) - f(0,0)| = \left|\frac{x^2y}{x^4+y^2}\right|\sqrt{x^2+y^2} \le \frac{1}{2}\sqrt{x^2+y^2}$ for all $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and $|f(x,y) - f(0,0)| = 0$ if $(x,y) = (0,0)$. Hence choosing $\delta = 2\varepsilon > 0$, we find that $|f(x,y) - f(0,0)| < \varepsilon$ for all $(x,y) \in \mathbb{R}^2$ satisfying $\|(x,y) - (0,0)\| = \sqrt{x^2+y^2} < \delta$. This shows that f is continuous at $(0,0)$ and therefore f is continuous.
   If $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, then $\lim\limits_{t\to 0} \frac{f((0,0)+t\mathbf{u}) - f(0,0)}{t} = \lim\limits_{t\to 0} \frac{u_1^2u_2|t|\sqrt{u_1^2+u_2^2}}{t^2u_1^4+u_2^2} = 0$, *i.e.* $D_{\mathbf{u}}f(0,0)$ exists. Hence all the directional derivatives of f at $(0,0)$ exist.
   Again, $\lim\limits_{(h,k)\to(0,0)} \frac{|f(h,k) - f(0,0) - hf_x(0,0) - kf_y(0,0)|}{\sqrt{h^2+k^2}} = \lim\limits_{(h,k)\to(0,0)} \frac{h^2k}{h^4+k^2} \ne 0$, since $(\frac{1}{n}, \frac{1}{n^2}) \to (0,0)$ but

$\frac{\frac{1}{n^2}\cdot\frac{1}{n^2}}{\frac{1}{n^4}+\frac{1}{n^4}} = \frac{1}{2} \not\to 0$. Hence f is not differentiable at $(0,0)$.
Therefore the given statement is FALSE.

4. Determine all the points of $\mathbb{R}^2$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable, if for all $(x,y) \in \mathbb{R}^2$,
$f(x,y) = \begin{cases} x^{4/3}\sin\left(\frac{y}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

   Solution: Let $E = \{(x,y) \in \mathbb{R}^2 : x \neq 0\}$. Since $f_x(x,y) = \frac{4}{3}x^{1/3}\sin\left(\frac{y}{x}\right) - \frac{y}{x^{2/3}}\cos\left(\frac{y}{x}\right)$ and $f_y(x,y) = x^{1/3}\cos\left(\frac{y}{x}\right)$ for all $(x,y) \in E$, $f_x : E \to \mathbb{R}$ and $f_y : E \to \mathbb{R}$ are continuous. Hence f is differentiable at all $(x,y) \in E$. Let $y_0 \in \mathbb{R}$ and let $\varepsilon > 0$. Then
$f_x(0,y_0) = \lim\limits_{h\to 0} \frac{f(h,y_0)-f(0,y_0)}{h} = \lim\limits_{h\to 0} h^{1/3}\sin\left(\frac{y_0}{h}\right) = 0$ (since $|h^{1/3}\sin\left(\frac{y_0}{h}\right)| \leq |h|^{1/3}$ for all $h \in \mathbb{R}\setminus\{0\}$) and $f_y(0,y_0) = \lim\limits_{k\to 0}\frac{f(0,y_0+k)-f(0,y_0)}{k} = 0$. Also, for all $(x,y) \in E$, we have $f_y(x,y) = x^{1/3}\cos\left(\frac{y}{x}\right)$, and so $|f_y(x,y) - f_y(0,y_0)| \leq |x|^{1/3} < \varepsilon$ for all $(x,y) \in B_\delta\big((0,y_0)\big)$, where $\delta = \varepsilon^3 > 0$. Thus $f_x(0,y_0)$ exists (in $\mathbb{R}$), $f_y(x,y)$ exists (in $\mathbb{R}$) for all $(x,y) \in \mathbb{R}^2$ and $f_y : \mathbb{R}^2 \to \mathbb{R}$ is continuous at $(0,y_0)$. Hence by Ex.21 of Practice Problem Set - 3, f is differentiable at $(0,y_0)$. Therefore f is differentiable at all points of $\mathbb{R}^2$.

   Alternative solution: As shown above f is differentiable at all $(x,y) \in \mathbb{R}^2$ for which $x \neq 0$. Let $y_0 \in \mathbb{R}$. Then as shown above $f_x(0,y_0) = f_y(0,y_0) = 0$. For all $(h,k) \in \mathbb{R}^2$ with $h \neq 0$, we have
$\varepsilon(h,k) = \frac{|f(h,y_0+k)-f(0,y_0)-hf_x(0,y_0)-kf_y(0,y_0)|}{\sqrt{h^2+k^2}} = \frac{h^{4/3}|\sin(\frac{y_0+k}{h})|}{\sqrt{h^2+k^2}} = |h|^{1/3}\frac{|h|}{\sqrt{h^2+k^2}}|\sin(\frac{y_0+k}{h})| \leq |h|^{1/3}$.
Also, $\varepsilon(0,k) = 0$ for all $k \in \mathbb{R}\setminus\{0\}$. Hence it follows that $\lim\limits_{(h,k)\to(0,0)}\varepsilon(h,k) = 0$. Consequently f is differentiable at $(0,y_0)$. Therefore f is differentiable at all points of $\mathbb{R}^2$.

5. Let $f : S \subseteq \mathbb{R}^m \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in S^0$ and let $f(\mathbf{x}_0) = 0$. If $g : S \to \mathbb{R}$ is continuous at $\mathbf{x}_0$, then show that $fg : S \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in S$, is differentiable at $\mathbf{x}_0$.

   Solution: Since f is differentiable at $\mathbf{x}_0$, there exists $\alpha \in \mathbb{R}^m$ such that
$\lim\limits_{\mathbf{h}\to\mathbf{0}}\frac{|f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} = 0$. For all $\mathbf{h} \in \mathbb{R}^m$ for which $\mathbf{x}_0+\mathbf{h} \in S$, we have
$(fg)(\mathbf{x}_0+\mathbf{h})-(fg)(\mathbf{x}_0)-g(\mathbf{x}_0)\alpha\cdot\mathbf{h} = (f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\alpha\cdot\mathbf{h})g(\mathbf{x}_0+\mathbf{h})+(g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0))\alpha\cdot\mathbf{h}$.
Hence for all $\mathbf{h} \in \mathbb{R}^m\setminus\{\mathbf{0}\}$ for which $\mathbf{x}_0+\mathbf{h} \in S$, we have
$\frac{|(fg)(\mathbf{x}_0+\mathbf{h})-(fg)(\mathbf{x}_0)-g(\mathbf{x}_0)\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} \leq \frac{|(f(\mathbf{x}_0+\mathbf{h})-f(\mathbf{x}_0)-\alpha\cdot\mathbf{h})|}{\|\mathbf{h}\|}|g(\mathbf{x}_0+\mathbf{h})| + |g(\mathbf{x}_0+\mathbf{h})-g(\mathbf{x}_0)|\frac{|\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|}$. Since g is continuous at $\mathbf{x}_0$, $\lim\limits_{\mathbf{h}\to\mathbf{0}} g(\mathbf{x}_0+\mathbf{h}) = g(\mathbf{x}_0)$ and since $|\alpha\cdot\mathbf{h}| \leq \|\alpha\|\,\|\mathbf{h}\|$, it follows that $\lim\limits_{\mathbf{h}\to\mathbf{0}}\frac{|(fg)(\mathbf{x}_0+\mathbf{h})-(fg)(\mathbf{x}_0)-g(\mathbf{x}_0)\alpha\cdot\mathbf{h}|}{\|\mathbf{h}\|} = 0$. Since $g(\mathbf{x}_0)\alpha \in \mathbb{R}^m$, we conclude that fg is differentiable at $\mathbf{x}_0$.

6. Show that $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(x_0,y_0) \in S^0$ iff there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0,y_0) and $f(x,y)-f(x_0,y_0) = (x-x_0)\varphi(x,y)+(y-y_0)\psi(x,y)$ for all $(x,y) \in S$.

   Solution: We first assume that f is differentiable at (x_0,y_0). Then $\alpha = f_x(x_0,y_0)$ and $\beta = f_y(x_0,y_0)$ exist (in $\mathbb{R}$). For each $(x,y) \in S$, let
$g(x,y) = f(x,y) - f(x_0,y_0) - \alpha(x-x_0) - \beta(y-y_0)$,

$$\varphi(x,y) = \begin{cases} \alpha + \frac{(x-x_0)g(x,y)}{(x-x_0)^2+(y-y_0)^2} & \text{if } (x,y) \neq (x_0,y_0), \\ \alpha & \text{if } (x,y) = (x_0,y_0), \end{cases}$$

and $\psi(x,y) = \begin{cases} \beta + \frac{(y-y_0)g(x,y)}{(x-x_0)^2+(y-y_0)^2} & \text{if } (x,y) \neq (x_0,y_0), \\ \beta & \text{if } (x,y) = (x_0,y_0). \end{cases}$

If $(x,y) \in S \setminus \{(x_0,y_0)\}$, then

$(x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y) = \alpha(x-x_0) + \beta(y-y_0) + g(x,y) = f(x,y) - f(x_0,y_0)$. Also, if $(x,y) = (x_0,y_0)$, then $(x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y) = 0 = f(x,y) - f(x_0,y_0)$. Hence $f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$ for all $(x,y) \in S$.

Again, for all $(x,y) \in S \setminus \{(x_0,y_0)\}$, we have

$|\varphi(x,y) - \varphi(x_0,y_0)| = \frac{|x-x_0|\,|g(x,y)|}{(x-x_0)^2+(y-y_0)^2} \leq \frac{|g(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$. Since f is differentiable at (x_0,y_0), $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{|g(x,y)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$ and hence it follows that $\lim\limits_{(x,y)\to(x_0,y_0)} \varphi(x,y) = \varphi(x_0,y_0)$. Therefore φ is continuous at (x_0,y_0). Similarly we can show that ψ is continuous at (x_0,y_0).

Conversely, let there exist functions $\varphi, \psi : S \to \mathbb{R}$ such that φ, ψ are continuous at (x_0,y_0) and $f(x,y) - f(x_0,y_0) = (x-x_0)\varphi(x,y) + (y-y_0)\psi(x,y)$ for all $(x,y) \in S$. Then for all $(x,y) \in S \setminus \{(x_0,y_0)\}$, we have $\frac{|f(x,y)-f(x_0,y_0)-(x-x_0)\varphi(x_0,y_0)-(y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}$

$\leq \frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}|\varphi(x,y) - \varphi(x_0,y_0)| + \frac{|y-y_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}}|\psi(x,y) - \psi(x_0,y_0)|$

$\leq |\varphi(x,y) - \varphi(x_0,y_0)| + |\psi(x,y) - \psi(x_0,y_0)|$. Since φ and ψ are continuous at (x_0,y_0), $\lim\limits_{(x,y)\to(x_0,y_0)} |\varphi(x,y) - \varphi(x_0,y_0)| = 0$ and $\lim\limits_{(x,y)\to(x_0,y_0)} |\psi(x,y) - \psi(x_0,y_0)| = 0$.

Hence $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{|f(x,y)-f(x_0,y_0)-(x-x_0)\varphi(x_0,y_0)-(y-y_0)\psi(x_0,y_0)|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$ and therefore f is differentiable at (x_0,y_0).

7. Let the temperature $T(x,y)$ at any point $(x,y) \in \mathbb{R}^2$ be given by $T(x,y) = 2x^2 + xy + y^2$. An insect is at the point $(1,1)$.
   (a) What is the best direction for the insect to move to feel cooler?
   (b) In which direction should the insect move to feel no change in temperature?

Solution: Since $T_x(x,y) = 4x + y$ and $T_y(x,y) = x + 2y$ for all $(x,y) \in \mathbb{R}^2$, $T_x : \mathbb{R}^2 \to \mathbb{R}$ and $T_y : \mathbb{R}^2 \to \mathbb{R}$ are continuous and hence $T : \mathbb{R}^2 \to \mathbb{R}$ is differentiable.

Since $\nabla T(1,1) = \big(T_x(1,1), T_y(1,1)\big) = (5,3)$, the temperature will decrease fastest in the direction of $-\frac{1}{\|\nabla T(1,1)\|}\nabla T(1,1) = \big(-\frac{5}{\sqrt{34}}, -\frac{3}{\sqrt{34}}\big)$ and so this is the best direction for the insect to start moving to feel cooler.

Again, if $\mathbf{u} = (u_1,u_2) \in \mathbb{R}^2$ with $\|\mathbf{u}\| = 1$, is the direction for the insect to feel no change in temperature, then we must have $D_{\mathbf{u}}T(1,1) = \nabla T(1,1) \cdot \mathbf{u} = 0$. This gives $5u_1 + 3u_2 = 0$. Since we also have $u_1^2 + u_2^2 = 1$, we get $\mathbf{u} = \big(\frac{3}{\sqrt{34}}, -\frac{5}{\sqrt{34}}\big)$ or $\big(-\frac{3}{\sqrt{34}}, \frac{5}{\sqrt{34}}\big)$.