MA 201 Complex Analysis
Lecture 13:
Identity Theorem and Maximum Modulus Theorem

Zeros of analytic functions

Suppose that $f : D \to \mathbb{C}$ is analytic on an open set $D \subset \mathbb{C}$.

- A point $z_0 \in D$ is called zero of f if $f(z_0) = 0$.
- The z_0 is a zero of multiplicity/order m if there is an analytic function $g : D \to \mathbb{C}$ such that

$$f(z) = (z - z_0)^m g(z), \;\; g(z_0) \neq 0.$$

- In this case $f(z_0) = f'(z_0) = f''(z_0) = \cdots = f^{(m-1)}(z_0) = 0$ but $f^m(z_0) \neq 0$.

- **Understanding of multiplicity via Taylor's series:** If f is analytic function in D, then f has a Taylor series expansion around z_0

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^n, \quad |z - z_0| < R.$$

- If f has a zero of order m at z_0 then

$$f(z) = (z - z_0)^m \sum_{n=m}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^{n-m}$$

- Define $g(z) = \sum_{n=m}^{\infty} \frac{f^n(z_0)}{n!}(z - z_0)^{n-m}$, then

$$f(z) = (z - z_0)^m g(z).$$

Zeros of analytic functions

Zeros of a non-constant analytic function are isolated: If $f : D \to \mathbb{C}$ is non-constant and analytic at $z_0 \in D$ with $f(z_0) = 0$, then there is an $R > 0$ such that $f(z) \neq 0$ for $z \in B(z_0, R) \setminus \{z_0\}$.

Proof.

- Assume that f has a zero at z_0 of order m. Then
$$f(z) = (z - z_0)^m g(z)$$
where $g(z)$ is analytic and $g(z_0) \neq 0$.
- Since g is continuous at z_0 thus for $\epsilon = \dfrac{|g(z_0)|}{2} > 0$, we can find a $\delta > 0$ such that
$$|g(z) - g(z_0)| < \frac{|g(z_0)|}{2},$$
whenever $|z - z_0| < \delta$.
- Therefore whenever $|z - z_0| < \delta$, we have $0 < \dfrac{|g(z_0)|}{2} < |g(z)| < \dfrac{3|g(z_0)|}{2}$. Take $R = \delta$.

Identity Theorem: Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ is analytic. If there exists an infinite sequence $\{z_k\} \subset D$, such that $f(z_k) = 0$, $\forall k \in \mathbb{N}$ and $z_k \to z_0 \in D$, $f(z) = 0$ for all $z \in D$.

Proof.

- **Case I:** If $D = \{z \in \mathbb{C} : |z - z_0| < r\}$ then

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ for all } z \in D.$$

- We will show that $f^n(z_0) = 0$ for all n. If possible assume that $f^n(z_0) \neq 0$ for some $n > 0$.
- Let n_0 be the smallest positive integer such that $f^{n_0}(z_0) \neq 0$. Then

$$f(z) = \sum_{n=n_0}^{\infty} a_n (z - z_0)^n = (z - z_0)^{n_0} g(z),$$

where $g(z_0) = a_{n_0} \neq 0$.

Identity Theorem

- Since g is continuous at z_0, there exist $\epsilon > 0$ such that $g(z) \neq 0$ for all $z \in B(z_0, \epsilon)$.
- There exists some k such that $z_0 \neq z_k \in B(z_0, \epsilon)$ and $f(z_k) = 0$. This forces $g(z_k) = 0$ which is a contradiction.
- **Case II:** If D is a domain.
- Since $z_0 \in D$ therefore there exists $\delta > 0$ such that $B(z_0, \delta) \subset D$.
- By Case I, $f(z) = 0, \ \forall \ z \in B(z_0, \delta)$.
- Now take $z \in D$ join z and z_0 by a line segment. Cover the line segments by open balls in such a way that center of a ball lies in the previous ball. Apply the above argument to get $f(z) = 0$ for all $z \in D$.

Uniqueness Theorem

Uniqueness Theorem: Let $D \subset \mathbb{C}$ be a domain and $f, g : D \to \mathbb{C}$ is analytic. If there exists an infinite sequence $\{z_n\} \subset D$, such that $f(z_n) = g(z_n)$, $\forall n \in \mathbb{N}$ and $z_n \to z_0 \in D$, $f(z) = g(z)$ for all $z \in D$.

- Find all entire functions f such that $f(r) = 0$ for all $r \in Q$.
- Find all entire functions f such that $f(x) = \cos x + i \sin x$ for all $x \in (0, 1)$.
- Find all analytic functions $f : B(0, 1) \to \mathbb{C}$ such that $f(\frac{1}{n}) = \sin(\frac{1}{n})$, $\forall n \in \mathbb{N}$.
- There does not exists an analytic function f defined on $B(0, 1)$ such that $f(x) = |x|^3$ for all $x \in (-1, 1)$?

Maximum Modulus Theorem

Maximum Modulus Theorem: Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ is analytic. If there exists a point $z_0 \in D$, such that $|f(z)| \leq |f(z_0)|, \ \forall z \in D$, then f is constant on D.

Proof. Choose a $r > 0$ such that $\overline{B(z_0, r)} \subset D$. Let $\gamma(t) = z_0 + re^{it}$ for $0 \leq t \leq 2\pi$. By Cauchy integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt.$$

Hence

$$|f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{it})|\, dt \leq |f(z_0)|.$$

This gives

$$\int_0^{2\pi} \left[|f(z_0)| - |f(z_0 + re^{it})| \right] dt = 0.$$

It follows that $|f(z_0)| = |f(z_0 + re^{it})|$ for all t. Now f analytic and $|f|$ is constant gives f is constant on $B(z_0, r)$. Applying identity theorem we get f is constant through out the domain D.

Consequences of Maximum Modulus Theorem

- If f is analytic in a bounded domain D and continuous on ∂D then $|f(z)|$ attains its maximum at some point on the boundary ∂D.
- Define $f(z) = e^{e^z}$ for $z \in D = \{z \in \mathbb{C} : |\text{Im } z| < \frac{\pi}{2}\}$. Then for $a + ib \in \partial D = \{\zeta \in \mathbb{C} : |\text{Im } \zeta| = \frac{\pi}{2}\}$,

$$f(a+ib) = \left| e^{e^{a \pm i\frac{\pi}{2}}} \right| = \left| e^{\pm ie^a} \right| = 1.$$

Again if $x \in \mathbb{R} \subset D$ then, $f(x) = e^{e^x} \to \infty$ as $x \to \infty$.

- **Minimum Modulus Theorem** Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ is analytic. If there exists a point $z_0 \in D$, such that $|f(z)| \geq |f(z_0)|$ for all $z \in D$, then either f is constant function or $f(z_0) = 0$.

Hint. Apply maximum modulus theorem on $1/f$.