MA 373 : Financial Engineering II

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Exercises 2

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1. Consider a zero-strike Asian call option whose payoff at time T is

$$V(T) = \frac{1}{T}\int_0^T S(u)du.$$

(i) Suppose at time t we have $S(t) = x \geq 0$ and $\int_0^t S(u)du = y \geq 0$. Use the fact that $e^{-ru}S(u)$ is a martingale under the risk neutral measure $\tilde{\mathbb{P}}$ to compute

$$e^{-r(T-t)}\tilde{\mathbb{E}}\Big[\frac{1}{T}\int_0^T S(u)du|\mathcal{F}(t)\Big].$$

Call your answer $v(t,x,y)$.
(ii) Verify that $v(t,x,y)$ satisfies the Black-Scholes-Merton equation

$$v_t(t,x,y)+rxv_x(t,x,y)+xv_y(t,x,y)+\frac{1}{2}\sigma^2x^2v_{xx}(t,x,y) = rv(t,x,y)0 \leq t < T, x \geq 0, y \geq 0.$$

and the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}\frac{y}{T}$$

and

$$v(T,x,y) = \frac{y}{T}$$

(iii) Determine explicitly the process $\Delta(t) = v_x(t,x,y)$, and observe that it is not random.
(iv) Use the Ito-Doeblin formula to show that if you begin with initial capital $X(0) = v_x(0,S(0),0)$ and at each time hold $\Delta(t)$ shares of the underlying asset, investing or borrowing at the interest rate r in order to do this, then at time T the value of your portfolio will be

$$X(T) = \frac{1}{T}\int_0^T S(u)du.$$

2. Consider the continuously sampled a derivative security with payoff function

$$V(T) = \frac{1}{T}\int_0^T S(u)du - K,$$

but assume now that the interest rate is $r = 0$. Find an initial capital $X(0)$ and a nonrandom function $\gamma(t), 0 \leq t \leq T$, which will be the number of shares of risky asset held by our portfolio so that

$$X(T) = \frac{1}{T}\int_0^T S(u)du - K$$

still holds. Give the formula for the resulting process $X(t), 0 \leq t \leq T$, in term of underlying asset price and K.

3. Consider a new derivative, the Mean with effective period given by $[T_1, T_2]$ the holder of a Mean contract will, at the date of maturity T_2, obtain the amount

$$\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} S(u)du$$

Determine the arbitrage free price, at time t, of the Mean contract where $t < T_1$.

4. Let $X(t) = W(t) - tW(1),\ 0 \leq t \leq 1$ be a Brownian bridge fixed at 0 and 1. Let $Y(t) = X^2(t)$. Find $E[Y(t)]$ and $Var(Y(t))$.

5. Let X be the solution of the SDE, for $t < 1$

$$dX(t) = -\frac{1}{2(1-t)} X(t)dt + \sqrt{1-t}\, dW(t),\ X(0) = x_0$$

   (i) Find the solution $X(t)$ of this equation.

   (ii) Is $\{X(t), t \geq 0\}$ a Gaussian process?

   (iii) Compare the variance of $X(t)$ with the corresponding variance of a Brownian bridge at time t.

   (iv) Is $X(t)$ a Brownian bridge?

6. The stochastic average of stock prices between 0 and t is defined by

$$X(t) = \frac{1}{t} \int_0^t S(u)dW(u),$$

where $\{W(t)\}_{t \geq 0}$ is Brownian motion.
(a) Find $dX(t), E[X(t)]$ and $Var(X(t))$
(b) Show that $\sigma X(t) = R(t) - \alpha A(t)$, where $R(t) = \frac{S(t)-S(0)}{t}$ is the raw average of the stock price and

$$A(t) = \frac{1}{t} \int_0^t S(u)du$$

is the continuous arithmetic average.

7. Let

$$S(t) = S(0) \exp\{\alpha t + \sigma \tilde{W}(t)\},\ \alpha = (r - \frac{\sigma^2}{2})$$

be the geometric Brownian motion, where $\tilde{W}(t), 0 \leq t \leq T$ is a Brownian motion under the risk neutral measure $\tilde{\mathbb{P}}$. Let $0 < K_1 < K_2$. Find the price at time t of a derivative which pays at maturity

$$V(T) = \begin{cases} 1 & \text{if } K_1 \leq S(T) \leq K_2 \\ 0 & \text{otherwise.} \end{cases}$$