Sequence, Limit and Continuity

Functions of a complex variable

- Let $S \subseteq \mathbb{C}$. A **complex valued function** f is a rule that assigns to each complex number $z \in S$ a unique complex number w.
- We write $w = f(z)$. The set S is called the **domain of** f and the set $\{f(z) : z \in S\}$ is called **range** of f.
- For any complex function, the independent variable and the dependent variable can be separated into real and imaginary parts:

$$z = x + iy \quad \text{and} \quad w = f(z) = u(x, y) + iv(x, y),$$

  where $x, y \in \mathbb{R}$ and $u(x, y), v(x, y)$ are **real-valued** functions.
- In other words, the components of the function $f(z)$, $u(x, y) = \text{Re}\,(f(z))$ and $v(x, y) = \text{Im}\,(f(z))$ can be interpreted as **real-valued functions** of the two real variables x and y.

Complex Sequences

- **Complex Sequences:** A **complex sequence** is a function whose domain is the set of natural numbers and range is a subset of complex numbers.
- **In other words, a sequence can be written as** $f(1), f(2), f(3) \ldots$**. Usually, we will denote such a sequence by the symbol** $\{z_n\}$**, where** $z_n = f(n)$**.**
- A sequence $\{z_n\} = \{z_1, z_2, \ldots\}$ of complex numbers is said to **converge** to $l \in \mathbb{C}$ if

$$\lim_{n\to\infty} |z_n - l| = 0 \quad \text{and we write} \quad \lim_{n\to\infty} z_n = l.$$

- In other words, $l \in \mathbb{C}$ is called the **limit** of a sequence $\{z_n\}$, if for every $\epsilon > 0$, there exists a $N_\epsilon > 0$ such that $|z_n - l| < \epsilon$ whenever $n \geq N_\epsilon$.
- If the limit of the sequence exists we say that the sequence is **convergent**; otherwise it is called divergent.
- A convergent sequence has a **unique** limit.

Algebra of sequence

- Let $\{z_n\}, \{w_n\}$ be sequences in $\mathbb{C}$ with $\lim\limits_{n\to\infty} z_n = z$ and $\lim\limits_{z_n\to\infty} w_n = w$. Then,
  - $\lim\limits_{n\to\infty} [z_n \pm w_n] = \lim\limits_{n\to\infty} z_n \pm \lim\limits_{n\to\infty} w_n = z \pm w.$
  - $\lim\limits_{n\to\infty} [z_n \cdot w_n] = \lim\limits_{n\to\infty} z_n \cdot \lim\limits_{n\to\infty} w_n = zw.$
  - $\lim\limits_{n\to\infty} \dfrac{z_n}{w_n} = \dfrac{\lim\limits_{n\to\infty} z_n}{\lim\limits_{n\to\infty} w_n} = \dfrac{z}{w} \quad (\text{if} \quad w \neq 0).$
  - $\lim\limits_{n\to\infty} Kz_n = K \lim\limits_{n\to\infty} f(z) = Kz \quad \forall \quad K \in \mathbb{C}.$
- If $z_n = x_n + iy_n$ and $l = \alpha + i\beta$ then

$$\lim_{n\to\infty} z_n = l \Longleftrightarrow \lim_{n\to\infty} x_n = \alpha \quad \text{and} \quad \lim_{n\to\infty} y_n = \beta.$$

Complex Sequences

- A sequence $\{z_n\}$ is said to be a **Cauchy Sequence** (or simply **Cauchy**) if $|z_n - z_m| \to 0 \quad$ as $\quad n, m \to \infty$.
- In other word, a sequence $\{z_n\}$ is said to be a **Cauchy** if for every $\epsilon > 0$, there exists a $N_\epsilon > 0$ such that $|z_n - z_m| < \epsilon$ for all $n, m \geq N_\epsilon$.
- **Theorem:** A sequence $\{z_n\}$ in $\mathbb{C}$ is convergent if and only if $\{z_n\}$ is Cauchy.
- Given a sequence $\{z_n\}$, consider a sequence n_k of $\mathbb{N}$ such that $n_1 < n_2 < n_3 < \cdots$. Then the sequence z_{n_k} is called **subsequence** of z_n.
- A sequence $\{z_n\}$ is said to be a **bounded** if $\exists\ k > 0$ such that $|z_n| \leq k$ for all $n = 1, 2, 3, \ldots$.
- Every convergent sequence is **bounded**.
- **But** every bounded sequence may not converge.
- **Example:** (a) $z_n = i^n$, (b) $\cos(n\pi) + i\cos(n\pi)$
- Every bounded sequence has a convergent subsequence.

- **Theorem:** Let A be a subset of $\mathbb{C}$. If $a \in A'$ then there exists an infinite sequence $\{z_n\}$ in A such that $z_n \to a$.
- **Proof:** Let $a \in A'$, i.e. a is a limit point of A.
  - It follows from the definition of limit point that, for each $n \in \mathbb{N}$ there exists a $z_n \in A$ such that $z_n \in B(z, 1/n) \setminus \{a\}$.
  - This implies that $|z_n - a| < 1/n \to 0$.
  - This show that there exists an infinite sequence $\{z_n\}$ in A such that $z_n \to a$.
- Let A be a subset of $\mathbb{C}$.
  - Then $z \in \bar{A}$ (closure of A) if and only if exists a sequence $\{z_n\}$ in A such that $z_n \to z$. In particular, if A is closed then $z \in A$ if and only if exists a sequence $\{z_n\}$ in A such that $z_n \to z$. (In this case $A = \bar{A}$).
  - A is compact if and only if every sequence has a convergent subsequence.

Limit of a function

- **Limit of a function:** Let f be a complex valued function defined at all points z in some deleted neighborhood of z_0. We say that f has a **limit** l as $z \to z_0$ if for every $\epsilon > 0$, there is a $\delta > 0$ such that

  $$|f(z) - l| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta \quad \text{and we write} \quad \lim_{z \to z_0} f(z) = l.$$

- If the limit of a function $f(z)$ exists at a point z_0, it is **unique**.
- If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$ then,

  $$\lim_{z \to z_0} f(z) = u_0 + iv_0 \iff \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

- **Note:**
  - The point z_0 can be approached from **any direction**. If the limit $\lim_{z \to z_0} f(z)$ exists, then $f(z)$ must approach a **unique** limit, no matter how z approaches z_0.
  - If the limit $\lim_{z \to z_0} f(z)$ is different for different path of approaches then $\lim_{z \to z_0} f(z)$ does not exists.

Algebra of limit

Let f, g be complex valued functions with $\lim_{z \to z_0} f(z) = \alpha$ and $\lim_{z \to z_0} g(z) = \beta$.

Then,

- $\lim_{z \to z_0} [f(z) \pm g(z)] = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z) = \alpha \pm \beta.$
- $\lim_{z \to z_0} [f(z) \cdot g(z)] = \lim_{z \to z_0} f(z) \cdot \lim_{z \to z_0} g(z) = \alpha\beta.$
- $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \dfrac{\alpha}{\beta} \quad (\text{if} \quad \beta \neq 0).$
- $\lim_{z \to z_0} Kf(x) = K \lim_{z \to z_0} f(z) = K\alpha \quad \forall \quad K \in \mathbb{C}.$

Properties of continuous functions

- **Continuity at a point:** A function $f : D \to \mathbb{C}$ is continuous at a point $z_0 \in D$ if for for every $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta.$$

  In other words, f is continuous at a point z_0 if the following conditions are satisfied.
  - $\lim_{z \to z_0} f(z)$ exists,
  - $\lim_{z \to z_0} f(z) = f(z_0)$.
- A function f is continuous at z_0 if and only if for every sequence $\{z_n\}$ converging to z_0, the sequence $\{f(z_n)\}$ converges to $f(z_0)$.
- A function f is continuous on D if it is continuous at each and every point in D.
- A function $f : D \to \mathbb{C}$ is continuous at a point $z_0 \in D$ if and only if $u(x, y) = \text{Re}\ (f(z))$ and $v(x, y) = \text{Im}\ (f(z))$ are continuous at z_0.

Continuity

Let $f, g : D \subseteq \mathbb{C} \to \mathbb{C}$ be continuous functions at the point $z_0 \in D$. Then

- $f \pm g,\ fg,\ kf\ (k \in \mathbb{C}), \dfrac{f}{g}\ (g(z_0) \neq 0)$ are continuous at z_0.
- Composition of continuous functions is continuous.
- $\overline{f(z)}, |f(z)|, \text{Re }(f(z))$ and $\text{Im }(f(z))$ are continuous.
- If a function $f(z)$ is continuous and nonzero at a point z_0, then there is a $\epsilon > 0$ such that $f(z) \neq 0,\ \ \forall\ z \in B(z_0, \epsilon)$.
- Continuous image of a compact set (closed and bounded set) is compact.