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Corollary Let ρ be an eigenvalue of A and B be the matrix obtained after one step of shifted QR with shift ρ. Then the last row of B is $[0,\ 0, \cdots\ \rho]$.

Double Shift QR

Let $A \in \mathbb{C}^{n \times n}$ and $\rho, \tau \in \mathbb{C}$. Consider one iteration of Shifted QR with shift ρ followed by another with shift τ :

$$\begin{aligned} A - \rho I &= Q_\rho R_\rho, \hat{A} = R_\rho Q_\rho + \rho I \\ \hat{A} - \tau I &= Q_\tau R_\tau, \tilde{A} = R_\tau Q_\tau + \tau I \end{aligned}$$

Let $Q = Q_\rho Q_\tau$ and $R = R_\tau R_\rho$. Then,

$$(A - \rho I)(A - \tau I) = QR \text{ and } \tilde{A} = Q^* A Q.$$

Also if A is real and $\tau = \bar{\rho}$, then $(A - \rho I)(A - \tau I)$ and $\tilde{A}$ are real.

Additionally, if ρ and τ are not eigenvalues of A, then given *any* QR decomposition $(A - \rho I)(A - \tau I) = Q_1 R_1$ of $(A - \rho I)(A - \tau I)$, if $A_1 := Q_1^* A Q_1$, then there exists diagonal matrix D with $D(i,i) = \pm 1$, $i = 1, \ldots, n$, such that $A_1 = \bar{D} \tilde{A} D$.

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Exercise: Prove the above statements!

Double Shift QR

Let $A_0 = A$ and $\rho_j, \tau_j \in \mathbb{F}$ for $j = 0, 1, \ldots$.

for j = 1,2, . . .

(i) Form $M = (A_{j-1} - \rho_{j-1} I)(A_{j-1} - \tau_{j-1} I)$.

(ii) Find a reflectors $Q_{j-1}^{(1)}, Q_{j-1}^{(2)}, \ldots, Q_{j-1}^{(n-1)}$ *such that*

$$\mathbf{Q_{j-1}^{(n-1)}} \cdots \mathbf{Q_{j-1}^{(2)}} \mathbf{Q_{j-1}^{(1)}} M$$

is upper triangular.

(iii) Find $A_j = \mathbf{Q_{j-1}^{(n-1)}} \cdots \mathbf{Q_{j-1}^{(2)}} \mathbf{Q_{j-1}^{(1)}} A_{j-1} \mathbf{Q_{j-1}^{(1)}} \mathbf{Q_{j-1}^{(2)}} \cdots \mathbf{Q_{j-1}^{(n-1)}}$.

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Generally the shifts ρ_{j-1} and τ_{j-1} are taken to be the eigenvalues of

$$\begin{bmatrix} a_{n-1,n-1}^{(j-1)} & a_{n-1,n}^{(j-1)} \\ a_{n,n-1}^{(j-1)} & a_{nn}^{(j-1)} \end{bmatrix}.$$

This is called generalized Rayleigh Quotient shifting strategy.

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(i) Form $M = (A_{j-1} - \rho_{j-1} I)(A_{j-1} - \tau_{j-1} I)$.

(Costs $O(n^3)$ flops and may be severely affected by rounding error. Also M is not upper Hessenberg!)

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