# MA 322: Scientific Computing

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# Chapter 5: Numerical Differentiations and Initial Value Problems for ODEs

## Theorem

*Assume that the solution $Y(x)$ of IVP has a bounded second derivative on $[x_0, b]$. Then the solution $\{y_h(x_n) : x_0 \le x \le b\}$ obtained by Euler's method satisfies*

$$\max_{x_0 \le x \le b} |Y(x_n) - y_h(x_n)| \le e^{(b-x_0)K}|e_0 + \left[\frac{2^{(b-x_0)K} - 1}{K}\right]\tau(h),$$

*where $\tau(h) = \frac{h}{2}\|Y''\|_\infty$ and $e_0 = Y_0 - y_h(x_0)$. If in addition to the conditions to the above theorem,*

$$|Y_0 - y_h(x_0)| \le c_1 h \qquad as \;\; h \to 0$$

*for some $c_0 \ge 0$, then there is a constant $B \ge 0$ for which*

$$\max_{x_0 \le x \le b} |Y(x_n) - y_h(x_n)| \le Bh.$$

# Stability analysis

We consider the numerical method

$$z_{n+1} = z_n + h[f(x_n, z_n) + \delta(x_n)] \qquad 0 \leq n \leq N(h) - 1$$

with $z_0 = y_0 + \epsilon$. We compare two numerical solutions $\{z_n\}$ and $\{y_n\}$ as $h \to 0$. Let $e_n = z_n - y_n$, $n \geq 0$. Then $e_0 = \epsilon$, and subtracting $y_{n+1} = y_n + hf(x_n, y_n)$ from the above equation, we get $e_{n+1} = e_n + f[f(x_n, z_n) - f(x_n, y_n)] + h\delta(x_n)$. Using the previous theorem we can show that

$$\max_{0 \leq n \leq N(h)} |z_n - y_n| \leq e^{(b-x_0)K}|\epsilon| + \left[\frac{e^{(b-x_0)K} - 1}{K}\right] \|\delta\|_\infty.$$

Consequently, there are constants $k_1$, $k_2$, independent of $h$, with

$$\max_{0 \leq n \leq N(h)} |z_n - y_n| \leq k_1|\epsilon| + k_2\|\delta\|_\infty.$$