$$= \left(\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \hat{Q}_1 \end{array} \right] \left[\begin{array}{c|c} a_{11} & \pm\|b\|_2 \; 0 \cdots 0 \\ \hline c & \hat{A}^T \hat{Q}_1 \end{array} \right] \right)^T$$

$$= \left[\begin{array}{c|c} a_{11} & \pm\|b\|_2 \; 0 \cdots 0 \\ \hline \hat{Q}_1 c & \hat{Q}_1 \hat{A}^T \hat{Q}_1 \end{array} \right]^T$$

$\hat{A}(i,j) \leftarrow \hat{A}(i,j) + w_i u_j + u_i w_j$

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Cost of finding $\hat{A}_1$ after finding w :- $4 \left[\underbrace{n-1 + n-2 + \cdots + 1}_{\text{only updating upper } \Delta^{lar} \text{ part as it is symmetric}} \right]$

$= 2n^2 + O(n)$

Now use this H for Rayleigh Quotient method

In complex reflector, finding

$$x = \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{C}^n \quad y = Qx = \begin{bmatrix} -\tau e^{i\theta} \\ 0 \\ \vdots \\ \vdots \end{bmatrix}$$ where $\tau = \|x\|_2$,

We want $x^* y$ to be real.

$$-\tau x_1 e^{i\theta} = -\tau |x_1| \cancel{e^{i\theta}} \cancel{e^{i\phi}}$$

$$\therefore \cancel{\theta = \frac{x}{}} \; \cancel{e^{-i\theta} = \frac{x}{|x_1|}} \; (-\tau e^{i\theta}) x_1 \in \mathbb{R}$$

$$-\tau e^{i\theta} |x_1| e^{i\theta_x} \in \mathbb{R}$$

$$\theta = -\theta_x = \frac{1}{i} \ln\left(\frac{x_1}{|x_1|}\right)$$

QR Algo:-

$$\cancel{A_j = Q_{j-1}^* A_{j-1} Q_{j-1}} \quad A_j = R_{j-1} Q_{j-1}$$

$$= Q_{j-1}^* \underline{Q_{j-1} R_{j-1}} Q_{j-1}$$

$$= Q_{j-1}^* A_{j-1} Q_{j-1}$$

$[u, H] = \text{hess}(A)$, $A_0 = H$

$O(n^3)$ process

Step:-

a) Find reflectors $Q_1, \dots, Q_{n-1}$ s.t $Q_{n-1} \cdots Q_1 H = R$

b) set $A_1 = R Q_1 \cdots Q_{n-1}$

$O(n^2)$ process

$O(n)$ if $A^T = A$ bcuz H will be tridiagonal

$H \to$ upper Hessenberg $\Rightarrow Q^* H Q$ is upper Hessenberg for unitary/orthogonal Q coming from QR decomposition of H

(Example in book)

$$A_0 = \begin{bmatrix} a_{11} & - & - & - & \cdots & a_{1n} \\ a_{21} & - & - & - & \cdots & a_{2n} \\ & \ddots & & & & \vdots \\ 0 & & \ddots & & & \vdots \\ & & & & a_{n-1n} & a_{nn} \end{bmatrix} \quad Q_1 = \left[\begin{array}{c|c} \hat{Q}_1 & \\ \hline & I_{n-2} \end{array}\right]$$

where $\hat{Q}_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} = \begin{bmatrix} -\tau \\ 0 \end{bmatrix}$

$$Q_2 = \left[\begin{array}{c|c|c} 1 & & \\ \hline & \hat{Q}_2 & \\ \hline & & I_{n-3} \end{array}\right]$$

2×2 reflector $\quad 2 \times n-1$ matrix $\Rightarrow$

$4 \times 2 \times (n-1)$ flop count for applying reflector

$$RQ_1 = \begin{bmatrix} r_{11} & \cdots & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ & & & \\ & & & \end{bmatrix} \begin{bmatrix} \hat{Q}_1 & \\ & I_{n-2} \end{bmatrix}$$

Q_i affects only only i & $i+1$ row ~~column~~ & will create a non-zero in $r_{i+1,i}$

$$= \begin{bmatrix} \hat{r}_{11} & \cdots & \cdots & \hat{r}_{1n} \\ * & & & \\ 0 & & \ddots & \\ & & & \ddots \end{bmatrix}$$

~~Product of upper Δ^{lar} & upper hessenberg matrix is upper Δ^{lar} (any ord er)~~
~~$HA = QR \Rightarrow Q = HR^{-1}$ (Assuming H is non-singular)~~
~~$R = Q^* H$~~

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The Harm in producing R as in QR decomposit^n in QR algo (R does not necessarily have $+^{ve}$ diagonal entries):–

$A_0 = A$, $A_0 = Q_0 R_0$ ($R_0(i,i) > 0 \ \forall\ i = 1:n$), $A_1 := R_0 Q_0 = Q_0^* A_0 Q_0$

$A_0 = \tilde{Q}_0 \tilde{R}_0$

$A_1 = \tilde{Q}_1 \tilde{R}_1$ $\quad \tilde{A}_1 = \tilde{Q}_0^* A_0 \tilde{Q}_0$

$\tilde{A}_2 = \tilde{Q}_0^* A_1 \tilde{Q}_1$

If A, Q, R are all real, in all iterat^{n}s,

$(\tilde{A}_k)_{ij} = \pm (A_k)_{ij}$

& if complex,

$|\tilde{A}_1| = |A_1|$

The QR algo doesn't work on unitary matrices (U) bcuz $\lambda_U \cdot \bar{\lambda}_U = 1$ & $\therefore |\lambda_1| = |\lambda_2| = \cdots = |\lambda_n|$

ρ is taken as a_{nn} (Rayleigh quotient shift bcuz $a_{nn} = \dfrac{e_n^* A e_n}{e_n^* e_n}$)

For $A = [e_2 \ \cdots \ e_n \ e_1]$ wilkinson shifting strategy doesn't works. (bcuz $\lambda = 0, 0$ for $\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}$)

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In shifted QR, don't follow the slides to make A_{j+1} from A_j.

Do $\quad A_{j+1} = Q_j^* A_j Q_j$

Thm:- If A is singular properly upper Hessenberg, ... QR algo is 0.

~~Ref~~ ~~Corollary~~ $A = QR$ $\quad \underbrace{rank(R) = rank(A)}_{\text{as } Q \text{ is non singular}} = n-1$

Pf:-

$$\underbrace{[A_1 \ldots A_{n-1}}_{l.i} A_n] = Q[\underbrace{R_1 \ldots R_{n-1}}_{l.i} R_n]$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n-1} & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n-1} & a_{2n} \\ & \ddots & & \vdots & \vdots \\ & & \ddots & & \end{bmatrix}_{n\times n}$$

all are l.i columns

$$= Q\begin{bmatrix} r_{11} & \cdots & r_{1n-1} & \\ & 0 \ddots & \vdots & \\ & & r_{n-1\,n-1} & \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$ — this matrix has rank $n-1$

Now $A_1 = RQ = \begin{bmatrix} & \times & \\ 0 & 0\,0\,0\,0 & 0 \end{bmatrix} Q = \begin{bmatrix} & & \\ 0 & 0\,0\,0 & 0 \end{bmatrix}$ $\Rightarrow r_{ii} \neq 0 \ \forall\ i = 1, \ldots, n-1$

$\Rightarrow r_{nn} = 0$ (as A is singular, R is singular, $\det(R) = 0$, $\prod_{i=1}^{n} r_{ii} = 0$)

Double shift:-

$$A - \rho I = Q_\rho R_\rho \to ② \ R_\rho Q_\rho + \rho I =: \hat{A} \Rightarrow \hat{A} = Q_\rho^* A Q_\rho \longrightarrow ③$$

$$\hat{A} - \tau I = Q_\tau R_\tau \to ② \ R_\tau Q_\tau + \tau I =: \check{A} \Rightarrow \check{A} = Q_\tau^* \hat{A} Q_\tau \longrightarrow ④$$

③ & ④, $\check{A} = Q_\tau^* \hat{A} Q_\tau = Q_\tau^* \underbrace{Q_\rho^* A Q_\rho}_{= \hat{A} \text{ (By ③)}} Q_\tau = Q^* A Q$

$$QR = Q_\rho Q_\tau R_\tau R_\rho = Q_\rho(\hat{A} - \tau I) R_\rho \ (\text{By } ②)$$
$$= Q_\rho [Q_\rho^* A Q_\rho - \tau Q_\rho^* Q_\rho] R_\rho \ (\text{By } ③)$$
$$= Q_\rho Q_\rho^* (A - \tau I) Q_\rho R_\rho$$
$$= (A - \tau I) Q_\rho R_\rho$$
$$= (A - \tau I)(A - \rho I) \longrightarrow ①$$

If $\tau = \bar{\rho}$, $(A - \rho I)(A - \tau I) = A^2 - 2Re(\rho) A + |\rho|^2 I$

Even in double shift also,

$$(A-\rho I)(A-\tau I) = Q_1 R_1$$
$$= QR$$

then $Q_1 = QD$ (D is unitary diagonal)

$$Q_1^* A Q_1 = (QD)^* A (QD)$$
$$= D^* Q^* A Q D$$

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$$\begin{bmatrix} \alpha \\ 0 \\ 0 \\ \vdots \end{bmatrix} = Q_{j-1}^{(1)} (A_{j-1} - \rho_{j-1} I) e_1 = Q_{j-1}^{(1)} \begin{bmatrix} a_{11}^{(j-1)} - \rho_{j-1} \\ a_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad Q_{j-1}^{(1)} = \begin{bmatrix} \tilde{Q}_{j-1}^{(1)} & 0 \\ 0 & I_{n-2} \end{bmatrix}$$

$$\underbrace{Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)}}_{2}$$

$$= \begin{bmatrix} \tilde{Q}_{j-1}^{(1)} & \\ & I_{n-2} \end{bmatrix} \begin{bmatrix} \begin{matrix} a_{11}^{(j-1)} \\ a_{21}^{(j-1)} \end{matrix} & \\ 0 & I_{n-2} \end{bmatrix}$$

$=$

X not Hessenberg matrix

Advantage of Francis algo:-

i) The $a_{nn}^{(j)}$ tends to become eigen value as ρ_j also to λ. So catastrophic cancellat^n may occur in $A-\rho I$. But in Francis, we ~~are at~~ only want 1st column of A

Francis $QR \equiv QR$

2) If A is upper Hessenberg, then $K(A, e_1)$ is upper Δ^{lar}.

Pf:- $K(A, e_1) = [e_1 \;\; Ae_1 \; \ldots \; A^{n-1} e_1]$

$$A^2 e_1 = A(Ae_1) = A \begin{bmatrix} a_{11} \\ a_{21} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = A[a_{11} e_1 + a_{21} e_2]$$
$$= a_{11} A e_1 + a_{21} A e_2 = \begin{bmatrix} * \\ * \\ a_{21} a_{32} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A^k e_1 = \begin{bmatrix} * \\ * \\ \vdots \\ \prod_{i=1}^{k} a_{i+1,i} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \leftarrow k+1$$

3) Pf by induxn:- The (k,k) entry of $K(A, e_1)$ is $\prod_{i=1}^{k-1} a_{i+1,i}$ & all entries (j,k) $j = k+1:n$ are zero $\forall$ $k = 1:n$.

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Francis QR $\equiv$ QR

Thm:- A — hessenberg . . . $p(A) = QR$

Pf:- $A \to$ properly upper hessenberg

$Q \to$ unitary

$Qe_1 = \alpha p(A) e_1$ ($\alpha \in \mathbb{C}$) ($\alpha \neq 0$ as 1^{st} col of Q can't be zero)

$\hat{A} := Q^* A Q$

$$\underbrace{K(\hat{A}, e_1)}_{\text{upper } \Delta^{lar} \text{ by (2)}} = K(Q^* A Q, e_1)$$

$= Q^* K(A, Qe_1) \longrightarrow$ from property (1)

$= Q^* K(A, \alpha p(A) e_1)$

$= \alpha Q^* K(A, p(A) e_1)$

$= \alpha Q^* p(A) \underbrace{K(A, e_1)}_{\text{upper } \Delta^{lar} \text{ \& non singular by (3)}}$

$\therefore p(A) = \frac{1}{\alpha} Q K(\hat{A}, e_1) [K(A, e_1)]^{-1}$

Let $R = \frac{1}{\alpha} K(\hat{A}, e_1) [K(A, e_1)]^{-1}$

Clearly R is upper Δ^{lar} & $p(A) = QR$

T.S-T $Q_j := \hat{Q}^{(1)}_{j-1} \hat{Q}^{(2)}_{j-1} \cdots \hat{Q}^{(n-1)}_{j-1}$: $Q_j e_1$ is proportional to $p_j(A) e_1$?

Pf:- $Q := \hat{Q}^{(1)} \hat{Q}^{(2)} \cdots \hat{Q}^{(n-1)}$

$\hat{Q}^{(1)} p_j(A) e_1 = \alpha e_1$ As $p_j(A) = (A - \rho I)(A - \tau I)$ or. $(A - \rho I)$ — In either cases, $p_j(A) e_1 \neq 0$

$$p_j(A) e_1 = \begin{bmatrix} * \\ * \\ a_{32} a_{21} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

this entry is non-zero as A is upper Hessenberg

$p_j(A) e_1 \neq 0$

$$\therefore \left(Q^{(1)}\right)^2 p_j(A)e_1 = \alpha Q^{(1)} e_1$$

$$Q^{(1)} e_1 = \frac{1}{\alpha} p_j(A) e_1$$

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Francis Q-R

$[\hat{Q}, H] = \text{hess}(A)$

$A_0 := \hat{Q}^* A \hat{Q}$ $\quad Q_0 = \hat{Q}$ $\quad \hat{Q}$ itself is built column by column

$A_1 := Q_1^* A_0 Q_1$ $\quad Q = \hat{Q} Q_1$ by matlab (as in condensed QR)

$\vdots$

$A_j := Q_j^* A_{j-1} Q_j$ $\quad Q = \underbrace{\hat{Q} Q_1 \dots Q_p}_{\text{apply reflect on } \hat{Q}}$

$= Q_j^* \dots Q_1^* \hat{Q}^* A_0 Q Q_1 \dots Q_j$

While deflating,

$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, U can't throw away A_{12}. U should apply $Q A_{12} Q$ to it.

How to avoid complex no's in finding eigen ~~values~~ vectors of T - ? when T is real & λ is complex

Invariant subspaces :-

$S = [s_1 \;\dots\; s_k, s_{k+1} \dots s_n] \rightarrow$ invertible

Suppose $A[\text{span}\{s_1, \dots, s_k\}] \subseteq \text{span}\{s_1, \dots, s_k\}$

$\forall\ i = 1:k$

$$As_i = \alpha_{1i} s_i + \dots + \alpha_{ki} s_k$$

$\forall\ i = k+1:n$

$$As_i = \alpha_{1i} s_i + \dots + \alpha_{ki} s_k + \alpha_{k+1,i} s_{k+1} + \dots + \alpha_{n,i} s_n$$

$$AS = [As_1 \dots As_k \; As_{k+1} \dots As_n] = [s_1 \dots s_k \; s_{k+1} \dots s_n] \left[\begin{array}{ccc|cc} \alpha_{11} & \dots & \alpha_{1k} & \alpha_{1k+1} & \alpha_{1n} \\ \alpha_{k1} & & \alpha_{kk} & & \\ \hline & 0 & & \alpha_{nk+1} & \alpha_{nn} \end{array}\right]$$

$\|V \Sigma V^*\|_2$

$= \sup \frac{\|\Sigma x\|}{\|x\|}$

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$$\|U_2^T U_1\|_2 = \sigma_1 \le \|U_2^T\|_2 \|U_1\|_2 = \|U_2\|_2 \|U_1\|_2 = 1$$

$\therefore$ All σ_i's are ≤ 1 and

$\theta_i = \cos^{-1}(\sigma_i)$ is defined

$U \Sigma V^T$

$(Q_1 U) \Sigma V^T Q_2$

$(Q_2^T V)^T$

Also, instead of U_1 we have $\hat{U}_1$,

U_2 - - - $\hat{U}_2$,

$\hat{U}_2^T \hat{U}_1$ will also have the same singular values ~~orthonormal bases~~.

S.T -?

1) $d(S_1, S_2) = d(S_2, S_1)$

2) $d(S_1, S_2) = 0 \Leftrightarrow S_1 = S_2$

3) $d(S_1 + S_2, S_3) \le d(S_1, S_3) + d(S_2, S_3)$

Convergence in Subspace iteratⁿs :-

$S \cap \mathcal{V}_k = \{0\}$ means if every vectors in S has some component in $v_1, \dots, v_k$ means if $s = c_1 v_1 + c_2 v_2 + \dots + c_k v_k + \dots + c_n v_n$, then $\prod_{i=1}^{k} c_i \neq 0$

Ex:-

a) $\text{Null}(A^m) \subseteq U_k$ ~~For m = i~~

Null space is the subspace spanned by the '0' eigen value.

A^m has the same eigen vectors with $|\lambda_1|^m \ge |\lambda_2|^m \ge \dots$

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$\{x_1, \ldots, x_k\} \rightarrow$ basis of S

$\{Ax_1, \ldots, Ax_k\} \rightarrow$ basis of AS obv that $\text{span}\{Ax_1, \ldots, Ax_k\} = A(S)$

Suppose $c_1 A x_1 + \cdots + c_k A x_k = 0 \rightarrow$

$$A(c_1 x_1 + \cdots + c_k x_k) = 0$$

$$c_1 x_1 + \cdots + c_k x_k \in \text{Null}(A) \subseteq U_k$$

$$\Rightarrow c_1 x_1 + \cdots + c_k x_k \in S \cap U_k$$

$$\Rightarrow c_1 x_1 + \cdots + c_k x_k = 0 \quad (\because S \cap U_k = \{0\})$$

$$\Rightarrow c_1 = \cdots = c_k = 0 \ [\text{as } \{x_1, \ldots, x_k\} \text{ is l.i}]$$

If A is upper Hessenberg & the starting vectors are $e_1, \ldots, e_n$, then automatically $S_j \cap U_j = \{0\}$

Some-linear algebra

$$[x]_B = B^{-1} x \qquad [T]_B = B^{-1} T B$$

$$\cancel{[T]_{B'} = B'^{-1} [T]_B B'}$$

Thm:- $[T]_{B_2} = S^{-1} [T]_{B_1} S$ where $S = B_1^{-1} B_2$