Problem Set 4

MA 221: Discrete Mathematics

November 18, 2022

Problem 1. Let G be a simple graph on 10 vertices and 28 edges. Prove that G contains a cycle of length 4.

Problem 2. Let G be a graph. We say that H is an induced subgraph of G if the vertex set of H is a subset of that of G, and if x and y are two vertices of H, then xy is an edge in H if and only if xy is an edge in G. Let G be a simple graph that has 10 vertices and 38 edges. Prove that G contains K_4 (the complete graph on four vertices) as an induced subgraph.

Problem 3. Let $c(n)$ be the number of connected graphs on the vertex set $[n]$, and let $C(x)$ be the exponential generating function of the sequence $\{c(n)\}$. Find $C(x)$. Do not look for a closed form. Look for a functional equation that enables us to compute the values $c(n)$.

Problem 4. Prove that a tournament is transitive if and only if it has only one Hamiltonian path.

Problem 5. Let G be a simple graph in which all vertices have degree four. Prove that it is possible to color the edges of G orange or blue so that each vertex is adjacent to two orange edges and two blue edges.

Problem 6. Let G be a graph on labeled vertices, let A be its adjacency matrix, and let k be a positive integer. Prove that $A^k_{i,j}$ is equal to the number of walks from i to j that are of length k.

Problem 7. Prove that in any tree T , any two longest paths cross each other.

Problem 8. How many different labeled trees are there on $[n]$ that have no vertices with degree more than 2?

Problem 9. Let A be the graph obtained from K_n by deleting an edge. Find a formula for the number of spanning trees of A.

Problem 10. Let $c_o(G-S)$ be the number of components of $G-S$ that have an odd number of vertices. Prove that a graph G has a perfect matching if and only if, for all subsets S of the vertex set of G, the inequality $c_o(G-S) \leq |S|$ holds. This is called Tutte's theorem.

Problem 11. Let G be any simple graph with labeled vertices, and let $p(n)$ be the number of ways to properly n-color G. Prove that p is a polynomial function of n. What is the degree of that polynomial? Note that $p(n)$ is called the chromatic polynomial of G.

Problem 12. Prove that the faces of planar graph G are 2-colorable if and only if all vertices of G have even degree.

Problem 13. Is it true that if a connected graph satisfies $E \leq 3V - 6$, then that graph is planar?