Matrix Computation (Q in slides)

Preliminaries

Slide 11:- If $\| \|_F$ is vector induced norm, then

$\|I\|_F = \sup_{x \neq 0} \frac{\|Ix\|_F}{\|x\|_F} = 1$ but $\|I\|_F$ by defn $= \sqrt{n}$

Slide 12:-

$\|x\|_2 \leq \|x\|_1 \leq \sqrt{n}\|x\|_2$

by squaring on both sides

AM $\leq$ RMS

i.e $\frac{a+b+c+\ldots}{n} \leq \sqrt{\frac{a^2+b^2+c^2+\ldots}{n}}$

2.) If $x = \langle x_1, x_2, \ldots, x_n \rangle$ Let $|x_l| \geq |x_i| \ \forall i$ $(\therefore \|x\|_\infty = |x_l|)$

$\|x_2\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} \leq \sqrt{n}\,|x_l| = \sqrt{n}\,\|x\|_\infty$

$\Updownarrow$

$x_1^2 + x_2^2 + \ldots + x_n^2 \leq n x_l^2$

true since $x_i^2 \leq x_l^2 \ \forall i$

$\|x\|_\infty \leq \|x\|_2$

$\Updownarrow$

$|x_l| \leq \sqrt{x_1^2 + \ldots + x_l^2}$

1) Cauchy Schwartz

~~1) $\|A\|_1 = \sup_{x \neq 0} \frac{\|Ax\|_1}{\|x\|_1}$~~

$A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix}$

~~$= \frac{|a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n| + |a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n| + \ldots}{|x_1| + |x_2| + \ldots + |x_n|}$~~

$|a_i^T x| \leq \|a_i\|\|x\|$

From Cauchy Schwartz, for the equality to hold

$|a_i^T x|$ $\lambda a_i = x$

$$\cancel{= \left|a_{11} + a_{12}\frac{x_2}{x_1} + \dots + a_{1n}\cdot\frac{x_n}{x_1}\right| + \left|a_{21} + a_{22}\cdot\frac{x_2}{x_1} + \dots + a_{2n}\cdot\frac{x_n}{x_1}\right| + \dots}$$

$$\cancel{= \frac{\qquad}{1 + \frac{|x_2|}{|x_1|} + \dots +}}$$

Q) $\|Ax\| \le \|A\|\|x\|$ for a ~~matrix~~ vector norm induced matrix norm. S.T equality holds for some x.

Sol) $\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$ so $\exists$ a sequence $(x_1, x_2, \dots, \cdot)$ $\ni$

$\lim_{i\to\infty} \frac{\|Ax_i\|}{\|x_i\|} = \|A\|$. As $\mathbb{R}^n$ is compact, for $\lim_{i\to\infty} x_i$, equality holds

See from Watkins Pg 119

1) $\|A\|_1 = \sup_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1}$ Let m be the column in A with the max column sum

$$= \frac{|a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n| + \dots}{\|x\|}$$

$$\le \frac{|a_{11}x_1| + |a_{12}x_2| + \dots + |a_{1n}x_n| + \dots}{\|x\|}$$

$$= \frac{(|a_{11}| + |a_{21}| + |a_{31}| + \dots)|x_1| + (|a_{12}| + |a_{22}| + \dots)|x_2| + \dots}{\|x\|_1}$$

$$\le \frac{\left(\sum_{i=1}^{n}|a_{im}|\right)|x_1| + \left(\sum_{i=1}^{n}|a_{im}|\right)|x_2| + \dots}{\|x\|_1}$$

$$= \sum_{i=1}^{n}|a_{im}|$$

And the equality satisfies for $x = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \end{bmatrix}$ — in m^{th} posit^n

$$\|A\|_\infty = \sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

$$= \max_i \frac{|a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n|}{\max_i |x_i|}$$

$$\cancel{\leq \max_i |a_{i1} + a_{i2} + \dots + a_{in}| \max_i |x_i|}$$

Let the maximum row sum occur at m

~~Equality holds when $x=$~~

$$\leq \frac{|a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n|}{\max_i |x_i|}$$

$$\leq \frac{|a_{m1}x_1| + |a_{m2}x_2| + \dots + |a_{mn}x_n|}{\max_i |x_i|}$$

$$\leq \frac{(|a_{m1}| + |a_{m2}| + \dots + |a_{mn}|) \max_i |x_i|}{\max_i |x_i|}$$

and equality occurs when $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad x_i = \begin{cases} +1 & \text{if } a_{mi} \geq 0 \\ -1 & \text{if } a_{mi} < 0 \end{cases}$

3) $\cancel{\|Ax\| \leq \|A\| \|x\|}$

2.1.28 b) $\|A\|_2 = \sqrt{\lambda_{max}(A^*A)}$ $\quad \sigma_{max}(A)$

$\|A\|_F = \sqrt{\sum \sigma_i^2(A)}$

2.1.32 ☆ — Proving $\|A\|_1 \leq \sqrt{n}\|A\|_2 \leq n\|A\|_1$

$\|A\|_\infty \leq \sqrt{n}\|A\|_2 \leq n\|A\|_\infty$

GE-1

Slide 15:-

8) P.T $M_k = I_n - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T \qquad M_k^{-1} = I_n + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T$

Pf:-

$$\left(\begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T\right)\left(\begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T\right)$$

$$= \begin{bmatrix} 0 \\ \text{Rest all zeroes} \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \begin{bmatrix} (\text{Rest all zeroes}) \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} = \begin{bmatrix} 0 \ldots 0 \\ = 0 \end{bmatrix}$$

GE-all
Slide-21

LU decompositⁿ is unique

Pf:- $A = \begin{bmatrix} 1 & & & \\ l_{11} & 1 & & 0 \\ l_{12} & l_{22} & \ddots & \\ \vdots & & & \vdots \\ - & - & - & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & \cdots & \cdots \\ & u_{22} & \cdots & \cdots \\ & 0 & \ddots & \vdots \\ & & & \vdots \end{bmatrix}$

~~Find all~~ If u know 1st k columns of L & 1st k rows of U, u can find $(k+1)^{th}$ .. of L & $(k+1)^{th}$ .. of U

Slide-28

Let $P_k = P_{k+1} \ldots P_{n-1}$. Then $P_k = \begin{bmatrix} I_k & \\ & \tilde{P}_{k+1} \ldots \tilde{P}_{n-1} \end{bmatrix}$ where

$\tilde{P}_j$ are transpositⁿs of size $n-k \times n-k$.

For $P_{n-2} = P_{n-1} = \begin{bmatrix} I_{n-2} & \\ & \underbrace{\hat{P}_{n-1}}_{2} \end{bmatrix} \}2$

Remember $P_k = \begin{array}{c} \scriptstyle k-1 \\ \scriptstyle n-k+1 \end{array} \begin{bmatrix} I_{k-1} & \\ & \underbrace{\hat{P}_k}_{n-k+1} \end{bmatrix}$ $\forall k$

$\hat{P}_k$ is a transposit

Suppose true for $k+1$. T.S.T true for k

$$P_{k+1} = P_{k+2} \cdots P_{n-1} = \begin{bmatrix} I_{k+1} & \\ & \hat{P}_{k+2} \cdots \hat{P}_{n-1} \end{bmatrix}$$

$$P_k = P_{k+1} P_{k+1} = \begin{bmatrix} I_k & \\ & \hat{P}_{k+1} \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 \cdots 0 & 1 \\ & & \hat{P}_{k+2} \cdots \hat{P}_{n-1} \end{bmatrix}$$

$$I_{k+1} = \begin{pmatrix} I_k & \begin{matrix} 0 \\ 0 \\ 0 \\ \vdots \end{matrix} \\ 0\,0\,0 \cdots & 1 \end{pmatrix}$$

$$= \begin{bmatrix} I_k & \\ & \hat{P}_{k+1} \end{bmatrix} \begin{bmatrix} I_k & \begin{matrix} 0 & \cdots & \cdots \\ 0 & \cdots & \cdots \\ \vdots & & \end{matrix} \\ 0\,0\,0 \cdots 0 & \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \hat{P}_{k+2} & \cdots & \hat{P}_{n-1} \\ 0 & & & \end{matrix} \end{bmatrix}$$

$$= \begin{array}{c|c} I_k & 0 \\ \hline 0 & \end{array}$$

Something is wrong in this

2. $\tilde{M}_k = P_k^T \left[I_n - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T \right] P_k$

$$= I_n - P_k^T \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix} \left[k^{th} \text{ row of } P_k \right] = e_k^T \text{ (see 1. in slide)}$$

$$= I_n - P_{n-1} \cdots P_{k+1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ m_{k+1} \\ \vdots \\ m_{nk} \end{bmatrix} e_k^T$$

~~Thm:- $PA = LU$~~
~~$A = P^T LU$~~

Pf for slide 29:- i) If A is non-singular:-

~~We need to show that $\exists$ a $P \ni PA$ is a matrix whose principal sub-matrices are non-singular.~~

Let P be as defined by partial pivoting $P_n P_{n-1} \ldots P_1$. We know we can do partial pivoting unless at k^{th} stage, $a_{kj}\ \forall j \ni k \leq j \leq n$ are all zero i.e. it is of the form

$$P_k P_{k-1} \cdots A = \left\{ \begin{array}{c|c} \overbrace{B_{11}}^{k-1} & B_{12} \\ \hline 0 & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\ B_{22} \end{array} \right.$$

B_{11} is a upper triangular matrix

$= |B_{11}||B_{22}|$.

So if 1^{st} column of $B_{22} = 0$, then $|A| = 0$.

$\therefore$ PA has a LU decomposit^n

i) If A is singular:-

then while doing LU decomposit^n, u will encounter the case where $a_{kj}\ \forall j \ni k \leq j \leq n$, $a_{kj} = 0$.

$$\begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 & 7 \end{matrix}$$

(Bcuz if u have not encountered that case $\Rightarrow$ u cud have done LU decomp (even without partial pivoting (theoretically))

$\Rightarrow$ there was a LU decomposit^n where the diagonal entries of U are non-zero

$\Rightarrow |A| = |U| \neq 0$ ~~$\Rightarrow$ A is no~~

$\Rightarrow$ A is non singular

A is non-singular $\Rightarrow$ u have not encountered that case has been shown above

bcuz U is singular $|A| = |U|$ — so a diagonal entry $= 0$

Then, u simply skip for that k & go to next one (Slly in the permutat^n matrix)

GE-all lectures Pg 64/67

1. $U = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ & u_{22} & & \vdots \\ 0 & & \ddots & \vdots \\ & & & u_{nn} \end{pmatrix}$

$A = LU$
$= LDV$

$D = \begin{bmatrix} d_{11} & & 0 \\ & d_{22} & \\ 0 & & \ddots \ d_{nn} \end{bmatrix}$ $V = \begin{bmatrix} 1 & v_{12} & \cdots & v_{1n} \\ & 1 & & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$

$\rightarrow U$ (a unit upper Δ^{lar} matrix) has unique decomp $U = DV$
$\rightarrow V$ is unit upper $\Delta^{lar} \rightarrow$ V^{-1} is also unit upper Δ^{lar}

$DV = \begin{bmatrix} d_{11} & d_{11}v_{12} & \cdots & d_{11}v_{1n} \\ 0 & d_{22} & & d_{22}v_{2n} \\ & & \ddots & \vdots \\ 0 & & & d_{nn} \end{bmatrix}$

Clearly $U = DV \Rightarrow$
~~$d_{ii} = u_{ii}$~~
$v_{ij} = \frac{u_{ij}}{u_{ii}}$ $(i \neq j)$
$\therefore U = DV$ is also unique

Suppose two decompositions exists.
$A = L_1 D_1 V_1 = L_2 D_2 V_2$
Since A has unique LU decomposition, $L_1 = L_2$ and $D_1V_1 = D_2V_2$
$\Rightarrow D_1^{-1} D_2 = V_1 V_2^{-1} \Rightarrow$ (V_2^{-1} is also unit upper Δ^{lar}, product of 2 unit upper Δ^{lar} also unit upper Δ^{lar}) $\Rightarrow V_1 V_2^{-1}$ also unit upper $\Delta^{lar} \Rightarrow$ $D_1^{-1} D_2$ also unit upper Δ^{lar}
$\Rightarrow D_1^{-1} D_2 = I$ $(D_1 = D_2) \Rightarrow V_1 = V_2$

Lemma:-
V_2 is unit upper $\Delta^{lar} \Rightarrow V_2^{-1}$ is unit upper Δ^{lar}

$\begin{bmatrix} 1 & v_{12} & \cdots & v_{1n} \\ & 1 & & \vdots \\ 0 & & \ddots & \vdots \\ & & & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} = I$

$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 & v_{12} & \cdots & v_{1n} \\ & 1 & & \vdots \\ 0 & & \ddots & \vdots \\ & & & 1 \end{bmatrix} = I$

~~$a_{11} \cdot 1 = 1$~~
$a_{11} \cdot v_{12} + a_{12} = 1$
$a_{21} \cdot 1 = 0 \Rightarrow a_{21} = 0$

Multiply i^{th} row, j^{th} col then $(i+1)^{th}$ row, j^{th} col (finish all rows) and then other cols. This gives $A = V_2^{-1}$ is upper Δ^{lar}.

2. $A = LDV$

$A^T = \cancel{L}V^T D L^T = LDV$

Since LDV decomposit^n is unique, $V = L^T$

3. D has positive diagonal entries $\Leftarrow x^T A x > 0 \ \forall x$
$(A = LDL^T)$

$\Rightarrow$: $x^T A x > 0 \ \forall x$

$x^T L D L^T x > 0$

Take $x \ni L^T x = e_i \quad x = (L^T)^{-1} e_i$

$\therefore e_i^T D e_i > 0 \Rightarrow d_{ii} > 0$

$\Leftarrow$

$x^T L D L^T x = y^T D y$ (where $y = L^T x$)

$$\begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \begin{bmatrix} d_1 y_1 & d_2 y_2 & \cdots & d_n y_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = d_1 y_1^2 + \cdots > 0 \ \forall y$$

$\therefore d_i > 0$

4. Cholesky

Slide 9:

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$

$$\Leftarrow \begin{bmatrix} x_1 a_{11} + x_2 a_{21} & x_1 a_{12} + x_2 a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} > 0$$

$$a_{11} x_1^2 + x_2 x_1 (a_{21} + a_{12}) + a_{22} x_2^2 > 0$$

~~$\therefore A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$ there is no 2x2 PD matrix~~

$\therefore A = \begin{bmatrix} 3 & -1 \\ +1 & 5 \end{bmatrix}$ is a PD matrix

$x^* A x = (A^* x)^* x = x^*(A^* x)^*$ — since $x^* A x \in \mathbb{R}$

$= ((A^* x)^* x)^*$

$= x^* A^* x$

$\therefore x^*(A - A^*)x = 0 \quad \forall x \rightarrow ①$

~~$(x+y)^*(A-A^*)(x+y) = 0$ ← Put $x+y$ in place of x in ①~~

~~$(x+iy)$ ← Put $x+iy$ in place of x in ①~~

Claim: If $x^* B x = 0 \Rightarrow B = 0$

Pf: Put $x+ky$ in place of x in $x^* B x$

$(x+ky)^* B(x+ky) = \cancel{x^* B x} + \bar{k} y^* B x + x^* B k y + \cancel{\bar{k} y^* B k y}$ → it is of the form x^*

Put $k=1$ & $k=i$

$\Rightarrow y^* B x = 0 \quad \forall y \in \mathbb{C}$

$\Rightarrow Bx = 0 \quad \forall x$

Slide 5

If A is PD, its leading principal submatrices are PD.

Pf:-

$$A = \begin{array}{c} k\{ \\ n-k\{ \end{array} \left[\begin{array}{c|c} \hat{A}_{11} & \hat{A}_{12} \\ \hline \hat{A}_{21} & \hat{A}_{22} \end{array} \right]$$

If $x = [x_1, \dots, x_k, 0, 0, 0, \dots]$

$x^* A x = \hat{x}^* \hat{A}_{11} \hat{x}$, where $\hat{x} = [x_1 \dots x$

6 - Finite Precision systems

35 $(\beta, p, E_{min}, E_{max})$
$(2, 3, -1, 1)$

b) $\epsilon = (1.01)_2 - 1$ $\quad N_{min} = (1.00)_2 \times 2^{-1}$

$u = \epsilon/2$ $\quad N_{max} = (1.11)_2 \times 2^{1}$

0.11

c) $x = 1/3 \quad y = 1/2 \quad z = 1/3$ $\quad fl\left(\frac{5}{6}\right) = 0.11$ $\quad x = 0.10$, $x_+ = 0.11$

$\left|\frac{1}{2} - \frac{5}{6}\right| = \frac{1}{3}$ $\quad \left|\frac{1}{2} + \frac{1}{4} - \frac{5}{6}\right| = \frac{1}{12}$

$\frac{3}{4} + \frac{1}{3} = \frac{13}{12}$ $\quad 1 + \frac{1}{12}$ $\quad \frac{1}{2} + \frac{1}{4} + \frac{1}{3}$

~~$fl(x+z)$~~

$fl(x+y+z) = fl(fl(x+y)+z) = fl\left((0.11)_2 + \frac{1}{3}\right) = fl\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{3}\right)$ $\quad \frac{3}{4} - \frac{5}{6}$

$fl(fl(z+x)+y) = fl\left((0.11)_2 + \frac{1}{2}\right) = (1.01)_2$ $\quad$ ~~(1.00)~~ $= (1.00)_2$

$fl\left(\frac{2}{3}\right) = (0.11)$ $\quad \frac{1}{12}$

Slide 10

1) $S^\perp$ is always a subspace of $\mathbb{F}^n$

Pf:- $x \in S^\perp$, $y \in S^\perp$ then $\langle x, s\rangle = 0$ $\langle y, s\rangle = 0$ $\forall s \in S$

$\langle ax+by, s\rangle = a\langle x, s\rangle + b\langle y, s\rangle = 0$

$\therefore ax+by \in S^\perp$

~~2) S~~

~~Ex:- $N(A)^\perp = \langle y, x\rangle = 0$~~

Some properties:-

1) If $x \in N(A^TA)$ then Ax is in both $R(A)$ & $N(A^T)$

2) $N(A^TA) = N(A)$ ($N(A) \subseteq N(A^TA) \to$ obv, $x \in N(A^TA) \Rightarrow A^TAx = 0$

$\Downarrow$

$Ax = 0 \Leftarrow x^TA^TAx = 0$

$\Leftrightarrow (Ax)^TAx = 0$)

3) A & A^TA have same rank

Use rank-Nullity Thm,

$A_{m\times n}$ $\text{rank}(A) = \dim(R(A))$ (definitn)

$\text{rank}(A) + \text{nullity}(A) = \underbrace{\dim(A)}_{=n}$ (rank-nullity thm)

$\text{rank}(A^TA) + \text{nullity}(A^TA) = \dim(A^TA) = n$

From 2), as null spaces are same, nullities are same

$\text{rank}(A) = \text{rank}(A^TA)$

~~Ex:- $(Ax)^T y = 0$ $\Rightarrow$ I~~

4) If A has l.i columns, A^TA is non-singular

As $\text{rank}(A) = n$, $\text{rank}(A^TA) = n$ & hence non-singular

$\to$ $N(A) = R(A^T)^\perp$

Pf:- $Ax = 0 \Rightarrow x^TA^T = 0 \Rightarrow x$ is orthogonal to the columns of A^T

$\Downarrow$

$S^\perp = \text{Span}(S^\perp) = R(A^T)^\perp \Leftarrow N(A) = S^\perp$ where S is set of rows of A (columns of A^T)

yA^Tx

Thm:- Let U & W be 2 subspaces of $\mathbb{F}^n$ $\ni$ $F^n = U+W$. Then

① $\dim U + \dim W - \dim(U \cap W) = n$ &

$\text{span}(U \cup W) = U + W \to$ ②

Pf:- ② - Obv

Let $B' = \{u_1, \dots, u_r\}$ be a basis of $U \cap W$

Extend this to get

$B_1 = \{u_1, \dots, u_r, v_1, \dots, v_s\} \to$ basis of U

$B_2 = \{u_1, \dots, u_r, w_1, \dots, w_t\} \to$ basis of W

T.S.T $B = \{u_1, \dots, u_r, v_1, \dots, v_s, w_1, \dots, w_t\}$ is a basis of $U+W$

$$\sum_{i=1}^{r} a_i u_i + \sum_{i=1}^{s} b_i v_i + \sum_{i=1}^{t} c_i w_i = 0 \to ④$$

$$\underbrace{\sum_{i=1}^{r} a_i u_i + \sum_{i=1}^{s} b_i v_i}_{\in U} = \underbrace{-\sum_{i=1}^{t} c_i w_i}_{\in W}$$

$\therefore$ LHS = RHS $\in U \cap W$

$$-\sum_{i=1}^{t} c_i w_i = \sum_{i=1}^{r} d_i u_i \quad (\because \text{LHS} \in U \cap W)$$

$$\sum_{i=1}^{t} c_i w_i + \sum_{i=1}^{r} d_i u_i = 0 \Rightarrow d_i = c_i = 0 \ \forall i \to ③$$

$(\because B_2$ is l.i$)$

(Slly $b_i = 0$) $\to$ ②

After ② & ③ in ④, as B' is basis, $a_i = 0$

Obv. that $\text{span}(B) = W_1 + W_2$.

$\therefore \dim(U+W) = r+s+t$

Thm:- Suppose U, W are subspaces of F^n $\ni$ $F^n = U + W$. Then $F^n = U \oplus W \iff u + w = 0$ for $u \in U, w \in W$ implies $u = w = 0$

Pf:- ~~$w_1 + w_2 = v_1 + v_2$~~

$\Rightarrow$ obv

$\Leftarrow$ T.S.T $\quad u_1 + w_1 = u_2 + w_2 \Rightarrow u_1 = u_2$ & $w_1 = w_2$

$$\underbrace{(u_1 - u_2)}_{\in U} + \underbrace{(w_1 - w_2)}_{\in W} = 0$$

$\therefore u_1 - u_2 = 0 \quad w_1 - w_2 = 0$

~~$(N(P))^{\perp} = R(P^T)$ $\quad R(P) = N(P^T)^{\perp}$~~

$N(P) = N(P^T)$

$Px = 0$

When $P^2_{n \times n} = P$

1) c) $N(P) = R(I - P)$ As $I - P$ is also a projexn,

$N(I - P) = R(I - (I - P)) = R(P)$.

b) $(I - P)^2 = I + P^2 - 2P = I - P$

~~$R(I-P) \oplus R(P)$~~

a) ~~$x = (I - P)k_1$~~

~~$y = Pk_2$~~

I have to Prove

$N(P) + R(P) = F^n$ first

Let $x \in F^n$

$x = (Px) + (x - Px)$

$Px \in R(P) \quad x - Px \in N(P) \quad (\because P(x - Px) = 0)$

$x \in N(P) \Rightarrow Px = 0$

T.S.T $\exists y \ni (I - P)y = x$

Let $y = (I - P)x$

$(I - P)y = (I + P^2 - 2P)x \quad (\because Px = 0)$

$= x$

c) $x \in R(I - P) \Rightarrow (I - P)y = x$

T.S.T $\quad Px = 0$

$\therefore Px = P(I - P)y = 0$

T.S.T $F^n = N(P) \oplus R(P)$ (given $F^n = N(P) + R(P)$)

If $u \in N(P)$, $v \in R(P)$ $u + w = 0$

$= R(I-P)$

$u = (I-P)k_1$

$v = \cancel{(I-P)k_2}\, Pk_2$

$\Rightarrow (I-P)k_1 + Pk_2 = 0 \rightarrow (1)$

$P(I-P)k_1 + P^2 k_2 = 0$

$Pk_2 = 0$

$v = 0 \rightarrow (2)$

$(1) \times (I-P) \Rightarrow$

$(I-P)k_1 = 0$

$u = 0 \rightarrow (3)$

From (2) & (3) & Thm in prev slide, (2) & (3) mean sum is direct sum

2) $R(P) \subseteq U$ (obv)

T.S.T ~~N(I-P)~~ $R(P) \supseteq U$

Let $u \in U$

~~$u = (u - Pu) + (Pu)$~~

~~$Pk = (I-P)u = (I-P)u$~~

~~Let $v \in V$. As $F^n = U \oplus V$,~~

~~$Pv = 0$ ($\because$ there is only one way of writing $v \in F^n$ in terms of U & V~~

~~$v = 0 + v$)~~

There is only one way of writing $u \in F^n$ in terms of U & V

i.e $u = u + 0$

$\therefore Pu = u \Rightarrow u \in R(P)$

CGS $\equiv$ condensed QR (QR decomposit^n 1 - slide 18/28)

$$\underbrace{\begin{bmatrix} v_1 & \cdots & v_m \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} q_1 & \cdots & q_m \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1m} \\ & r_{22} & \cdots & r_{2m} \\ & & \ddots & \vdots \\ & & & r_{mm} \end{bmatrix}}_{R}$$

$$= \begin{bmatrix} r_{11}q_1 & r_{12}q_1 + r_{22}q_2 & r_{13}q_1 + r_{23}q_2 + r_{33}q_3 & \cdots \end{bmatrix}$$

$$v_j = \sum_{k=1}^{j} r_{kj} q_k$$

Take dot product with q_i on both sides

$$v_j^T q_i = r_{ij} q_i^T q_i + 0 + 0 + \cdots$$

$$= r_{ij}$$

For $i=j$, $r_{ii} q_i = v_i - (r_{1i} q_1 + r_{2i} q_2 + \cdots + r_{i-1\,i} q_{i-1})$

Take 2 norm on both sides

$$r_{ii} = \|\tilde{q}_i\|_2$$

Thm 3.2.46 :- $A \in \mathbb{R}^{n\times n}$ be non-singular. Then $\exists\, Q, R \ni Q$ is orthogonal, R is upper Δ^{lar} with positive main diagonal entries & $A = QR$.

Pf:- Uniqueness :- Pf 1) $A = Q_1 R_1 = Q_2 R_2$

$$A^T A = R_1^T R_1 = R_2^T R_2$$

$\therefore R_1$ & R_2 are cholesky decompositions of $A^T A$.

$\therefore R_1 = R_2$

Pf 2):- $B^T B = I$ (If B is orthogonal & upper Δ^{lar}, B must be a diagonal matrix whose entries are ± 1)

$$\begin{bmatrix} b_{11} & - & - & - & b_{1n} \\ 0 & b_{22} & & & \vdots \\ 0 & & \ddots & & \vdots \\ 0 & & & \ddots & \vdots \\ \vdots & & & & b_{nn} \end{bmatrix}$$

$b_{11} = \pm 1$

$b_{11} b_{12} + 0 = 0$

$b_{12} = 0$ ($\text{IIlly}\ \forall\ b_{i2}$)

Let $Q_1 R_1 = Q_2 R_2$

$Q_2 = Q_1 R_1 R_2^{-1}$ (2)

$Q_1^{-1} Q_2 = (R_1 R_2^{-1}) = K$ (1)

K is orthogonal as well as upper Δ^{lar}.

$\therefore K$ is a diagonal matrix with ± 1 entries

$K = D$

~~$R_2 = DR_1$~~ $R_1 = DR_2$

$Q_2 = Q_1 D$

$Q_1 = Q_2 D^{-1} = Q_2 D$

Ex:- 3.2.68) $Qx = y$

$$(I - \gamma u u^T) x = y$$

$$x - y = \gamma u u^T x$$

$$= (\gamma (u^T x)) u$$

$\therefore$ u is a multiple of $x-y$

3.2.69)

~~a)~~ b) $Gx = 0 \Rightarrow x$ is a multiple of u

$$(I - uv^T) x = 0$$

$$x = (v^T x) u \quad x \text{ is a multiple of } u$$

x is a multiple of $u \Rightarrow Gx = 0$

$$G(\lambda u) = \lambda (u - (v^T u) u)$$

Lemma:- G is singular $\Leftrightarrow v^T u = 1$

G is singular $\Rightarrow Ga = 0$ for $a \neq 0$

$$(I - uv^T) a = 0 \rightarrow 1$$

MOBS to the left with v^T

$$v^T a - (v^T u)(v^T a) = 0$$

$$(v^T a)(1 - v^T u) = 0 \qquad v^T a = 0 \text{ or } v^T u = 1$$

But if $v^T a = 0$, then 1 becomes

$a \cancel{I} - 0 = 0 \Rightarrow\Leftarrow$

$\therefore v^T u = 1$

$v^T u = 1 \Rightarrow G$ is singular

~~$Gv = v - uv^T v$~~

$Gu = u - (v^T u) u = 0$ as $u \neq 0$

As $v^T u = 1$

$$G(\lambda u) = 0$$

~~c) $(I - uv^T) G^{-1} = I$~~

~~$G^{-1} = I + G^{-1} u v^T$~~

~~$G^{-1} u = I + G^{-1} u (v^T u)$~~

~~$G^{-1} u (1 - v^T u) = I$~~

~~$G^{-1}(I - uv^T) = I$~~

~~$G^{-1} = I + G^{-1} u v^T$~~

~~$(I - uv^T)(I - \beta u v^T) = I$~~

~~$\cancel{I} - uv^T(\beta + 1) + \beta (uv^T)^2 = 0$~~

$uv^T u v^T u$

$u - ku(\beta + 1) + \beta k^2 u = 0$

$1 - k(\beta + 1) + \beta k^2 = 0$

c)

$G = I - uv^T$ is non-singular $\Rightarrow$
$G^{-1} = I - \beta uv^T$

Suppose $G^{-1} = I - \beta uv^T$

$G^{-1}G = I$

$(I - \beta uv^T)(I - uv^T) = I$

$-\beta uv^T - uv^T + \beta uv^Tuv^T = 0$ Let $v^Tu = k = u^Tv$

MOBS with u on right

$-\beta ku - ku + \beta k^2 u = 0$

$-\beta - 1 + \beta k = 0$

$\beta(k-1) = 1$

$\beta = \dfrac{1}{k-1} = \dfrac{1}{v^Tu - 1}$

As we know G^{-1} ~~know~~ now, G^{-1} has the form $I - \beta uv^T$

d) $G^T = G^{-1}$

$I - vu^T = I - \beta uv^T$

$vu^T = \beta uv^T$ $\leftarrow$ MOBS with u on the right

$(u^Tu)v = \beta(v^Tu)u \Rightarrow \rho v = \beta u$

Now $u^Tu = 1$

$v = \left(\dfrac{v^Tu}{v^Tu - 1}\right)u = \left(\dfrac{\rho u^Tu}{\rho u^Tu - 1}\right)u = \left(\dfrac{\rho}{\rho - 1}\right)u = \rho u$

$\boxed{\therefore \rho = 2}$

Pg 236 -
3.4.35

$$[\tilde{V}\ v_k] = [\tilde{Q} \mid q_k] \left[\begin{array}{c|c} \tilde{R} & s \\ \hline 0 & r_{kk} \end{array}\right]$$

$$= [\tilde{Q}\tilde{R} \quad \tilde{Q}s + q_k r_{kk}]$$

$$v_k = \tilde{Q}s + q_k r_{kk} \qquad \cancel{q_k^T v_k = r_{kk}}$$

$$\cancel{v_k^T = s^T \tilde{Q}^T + r_{kk} q_k^T}$$

$\tilde{Q}^T \tilde{Q} = I$

(not $\tilde{Q}\tilde{Q}^T = I$)

$$\cancel{v_k^T q_k = r_{kk}} \qquad \cancel{q_k^T v_k = r_{kk}}$$

$$v_k - q_k r_{kk} = \tilde{Q}s$$

$$\tilde{Q}^T v_k - \tilde{Q}^T q_k r_{kk} = s$$

3.4.36)

$$V = [v_1\ \tilde{V}] \qquad Q = [q_1\ \tilde{Q}]$$

$$\cancel{[v_1\ \tilde{V}]} \quad [v_1 \mid \tilde{V}] = [q_1 \mid \tilde{Q}] \left[\begin{array}{c|c} r_{11} & r^T \\ \hline 0 & \tilde{R} \end{array}\right]$$

$$= [q_1 r_{11} \quad q_1 r^T + \tilde{Q}\tilde{R}]$$

$$v_1 = q_1 r_{11}$$

$$\tilde{V} = q_1 r^T + \tilde{Q}\tilde{R}$$

$$\boxed{q_1^T \tilde{V} = r^T}$$