Lecture 20 Tuesday, 13 Sep.

Note Title 9/14/2022

§ Permutation group: Let $X \neq \phi$.

$S(X)$ = the set of all the bijections from $X \to X$

$= \{ f : X \to X \mid f \text{ is 1-1 and onto} \}$

= the set of all the permutations on X.

- $S(X)$ is a group under composition of functions.
- If X is infinite, then $S(X)$ is also infinite.
- If $X = \{1, 2, \dots, n\}$, then $S(X)$ is denoted by S_n.
- $|S_n| = n!$

Definition (r-cycle): Let $n \geq 1$ be fixed. Let $1 \leq r \leq n$.

$f \in S_n$ is said to be an r-cycle if $\exists\, i_1, \dots, i_r \in \{1, 2, \dots, n\}$ such that $f(i_1) = i_2,\ f(i_2) = i_3, \dots, f(i_{r-1}) = i_r,\ f(i_r) = i_1$ and

$f(x) = x \quad \forall\, x \notin \{i_1, i_2, \dots, i_r\}$.

Notation: $(i_1\ i_2\ \dots\ i_r)$

Theorem 1: Let $\alpha = (i_1\ i_2\ \dots\ i_r) \in S_n$. Then, $o(\alpha) = r$.

Proof: $\alpha(i_1) = i_2,\quad \alpha^2(i_1) = \alpha(\alpha(i_1)) = \alpha(i_2) = i_3, \dots\dots, \alpha^{r-1}(i_1) = i_r$

$\therefore\ o(\alpha) > r-1$.

But, $\alpha^r(i_1) = i_1,\ \alpha^r(i_2) = i_2, \dots, \alpha^r(i_r) = i_r$. Hence, $\alpha^r = id$

$\Rightarrow o(\alpha) = r$.

Number of cycles in S_n:

$r=1$: $(1)=(2)=\cdots=(n)$. Thus, 1-cycle is the identity element.

$\therefore$ Number of 1-cycle is one.

$r \geq 2$: Number of r-cycles $= \dfrac{{}^nP_r}{r}$.

$S_1=\{(1)\}$, $S_2=\{(1),(1\,2)\}$, $S_3=\{(1),(1\,2),(1\,3),(2\,3),(1\,2\,3),(1\,3\,2)\}$.

In S_4, # 1-cycle $=1$

\# 2-cycles $=6$

\# 3-cycles $=8$

\# 4-cycles $=6$

$\therefore$ # cycles in $S_4 = 1+6+8+6 = 21$.

$\therefore$ There are 3 permutations in S_4 which are not cycles.

Definition (disjoint permutation): Let $f, g \in S(X)$. We say that f and g are disjoint if $f(a) \neq a$ for some $a \in X$, then $g(a) = a$.

- Two cycles $(i_1\, i_2 \cdots i_r)$ and $(j_1\, j_2 \cdots j_s)$ are disjoint if $i_k \neq j_l$ $\forall k$ and $\forall l$, that is,

$$\{i_1, i_2, \dots, i_r\} \cap \{j_1, j_2, \dots, j_s\} = \phi.$$

For example, $(1\,2\,3)$ and $(4\,7\,9)$ are two disjoint permutations of S_{10}.

- $(1\,2\,3)$ and $(2\,5\,6)$ are <u>not</u> disjoint permutations in S_{10}.

Theorem 2: If $f, g \in S(X)$ are disjoint, then $f \circ g = g \circ f$.

Proof: We need to prove that $(fg)(x) = (gf)(x)$ $\forall x \in X$.

Let $x \in X$.

- If $f(x) = x = g(x)$, then $(fg)(x) = (gf)(x) = x$.

- Let $f(x) \neq x$. Let $f(x) = y$. Since f and g are disjoint,

  so $g(x) = x$.

  Now, $(fg)(x) = f(g(x)) = f(x) = y$

  $(gf)(x) = g(f(x)) = g(y)$

Enough to prove that $g(y) = y$.

Suppose that $g(y) \neq y$. Then, $f(y) = y = f(x)$

$\Rightarrow y = x$, a contradiction.

$(\because y = f(x) \neq x)$.

$\therefore g(y) = y$.

That is, $(fg)(x) = (gf)(x)$.

This completes the proof.

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