Indian Institute of Technology Guwahati

Mid Semester Examination, September 19, 2022

MA201: Mathematics III

Answers without proper justifications will fetch zero mark.

Time: 2 hours (9.00AM - 11.00 AM) Marks: 30

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1. Prove or disprove the following statements with proper justification.

   (a) If f is an entire function and $f^n(0) \neq 0,\ \forall n = 0, 1, \cdots$, then $f(z) \neq 0$ for all $z \in \mathbb{C}$.
   Answer: Disprove. Take $f(z) = e^z - \alpha$ with $\alpha \neq 1$. Then $f^n(0) \neq 0$ for $n = 0, 1, 2 \cdots$. If we take $z_0 = \log|\alpha| + iArg(\alpha)$ then $f(z_0) = 0$

   (b) There exists an entire function f such that $f'(z) = \dfrac{1}{z}$ for all $z \in \mathbb{C} \setminus \{0\}$.
   Answer: Disprove. If exists an entire function f such that $f'(z) = \dfrac{1}{z}$ for all $z \in \mathbb{C} \setminus \{0\}$ then $\displaystyle\int_{|z|=1} \frac{1}{z} dz = 0$. But $\displaystyle\int_{|z|=1} \frac{1}{z} dz = 2\pi i \neq 0$.

   (c) If f is an entire function satisfying $|f(z) - z| \leq |z|$ on $|z| = \frac{3}{2}$, then $|f'(a)| \leq (1 + |a|)^{-2}\ \forall a \in B(0, \frac{1}{2})$.
   Answer: Disprove. Take $f(z) = z + \frac{1}{2}$. Then for any $a \in B(0, \frac{1}{2}) \setminus \{0\}$ the inequality $|f'(a)| \leq (1 + |a|)^{-2}\ \forall a \in B(0, \frac{1}{2})$ does not hold.

   (d) If f is a non-constant entire function, then there exists $a \in \mathbb{C}$ such that $\lim_{n\to\infty} f(a_n) \neq a$ for any $\{a_n\} \subset \mathbb{C}$.
   Answer: Disprove. Assume that the given statement is true i. e. there exists an $\epsilon > 0$ such that $|f(z) - a| > \epsilon$ for all $z \in \mathbb{C}$. Then define $g(z) = \frac{1}{f(z)-a}$. By Liouville's theorem, g is constant implying f is constant.

$\boxed{2\times4}$

2. (a) Find all entire functions f such that $f'(0) = f(0) = 1$ and $f''(1+\frac{1}{n}) = 7-\frac{3}{n}$ for all $n \in \mathbb{N}$.
   Answer. Define $g(z) = f''(z) + 3z - 10$. Since $g(1 + \frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, by identity theorem $f''(z) = 10 - 3z$ for all $z \in \mathbb{C}$. Using the conditions $f'(0) = f(0) = 1$ we get $f(z) = -\frac{1}{3}z^3 + 5z^2 + z + 1$ for all $z \in \mathbb{C}$.

   (b) Let f and g be non-constant entire functions such that $|f(z)+g(z)| = |f(z)|$ for all $z \in \mathbb{C}$. Then show that for any $R > 0$, f and g have the same number of zeros in $B(0, R)$.
   Answer. Define $h(z) = \frac{f(z)+g(z)}{f(z)}$. Then h is analytic on $B(0, R)$ for every $R > 0$ with $|h(z)| = 1$ (on $|z| = R$). So by maximum and minimum modulus theorem h is constant. So $h(z) = e^{i\theta_0}$ for some θ (independent of z) for all $z \in B(0, R)$, which yields $f(z) + g(z) = e^{i\theta_0} f(z)\ z \in B(0, R)$. Note that $\theta_0 \neq 2k\pi$ for any $k \in \mathbb{Z}$. (If $\theta_0 = 2k\pi$ for some $k \in \mathbb{Z}$, then $g(z) = 0$ for all $z \in \mathbb{C}$ contradicting the assumption that g is non constant).

So $g(z) = (1 - e^{i\theta_0})f(z)$ for all $z \in \mathbb{C}$ with $\theta_0 \neq 2k\pi$ for any $k \in \mathbb{Z}$. Thus for any $R > 0$, f and g have the same number of zeros in $B(0, R)$. $\boxed{3+4}$

3. (a) Let f be an entire function such that $f(z) \neq 0$ for all $z \in \mathbb{C}$. Then show that there exists an entire function g such that $f(z) = e^{g(z)}$ for all $z \in \mathbb{C}$.
Answer. Since f never vanishes, $\frac{f'}{f}$ is an entire functon. Therefore there exist an entire function g such that $g'(z) = \frac{f'(z)}{f(z)}$ for all $z \in \mathbb{C}$. Define $h(z) = e^{g(z)}$ for all $z \in \mathbb{C}$. Then $\frac{f}{h}$ is entire with $\frac{d}{dz}\left(\frac{f(z)}{h(z)}\right) = 0$. So $f(z) = Ce^{g(z)} = e^{g(z)+c'}$ for all $z \in \mathbb{C}$ and for some c'.

(b) Let f be an entire function such that $f(z) \in \mathbb{R}$ on $|z| = 1$. Then show that f is constant.
Answer. Define $g(z) = e^{if(z)}$ on $B(0, 1)$. Then By maximum and minimum modulus theorem $|g(z)| = 1$ for all $z \in B(0, 1)$. By cauchy-Riemann equation g is constant in $B(0, 1)$. Againg by identity theorem g is constant in the whole complex plane. So f is constant. $\boxed{3+4}$

4. (a) Evaluate $\lim\limits_{r\to\infty} \displaystyle\int_{\gamma_r} \frac{e^{i(z-1)}}{z-1}dz$, where $\gamma_r(t) = 1 + re^{it}$, $t \in [\frac{\pi}{2}, \frac{3\pi}{4}]$.

Answer. Note that $I_r = \displaystyle\int_\gamma \frac{e^{i(z-1)}}{z-1}dz = \int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} ie^{ir\cos t}e^{-r\sin t}dt$ and $|I_r| \leq \displaystyle\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} e^{-r\sin t}dt$. Since $\min\limits_{t\in[\frac{\pi}{2},\frac{3\pi}{4}]} \sin t > 0$ therefore $\lim\limits_{r\to\infty} |I_r| = 0$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be a continuous function such that f is analytic on $\mathbb{C} \setminus \{0\}$. If there exists a $M > 0$ such that $|f(z)| \leq M|z|^{\frac{3}{2}}$ whenever $|z| > 1$, then show that $f(z) = az + b$ for all $z \in \mathbb{C}$ and some constants $a, b \in \mathbb{C}$.
Answer. Let $f(0) = b$. Given that f is continuous at 0. In order to apply cauchy integral formula we need to ensure the analyticity at the point $z = 0$. We make use of Morera's theorem to conclude the analyticity. let γ be a simple closed curve lying in $\mathbb{C}$.
Case I: If the point 0 lies in the outer region of γ then $\int_\gamma f(z) = 0$. If the point 0 lies in the interer region of γ then $\int_\gamma f(z)dz = \int_{|z|=r} f(z)dz$ for any $r > 0$. By ML - inequality $|\int_\gamma f(z)dz| \leq M.2\pi r \to 0$ as $r \to 0$.
Case II: If γ passes through the point 0, then by deformation theorem $\int_\gamma f(z)dz = \int_{C_r} f(z)dz$, where C_r is a circle passing through 0 lying in the interier region of γ with any radius $r > 0$. By ML - inequality $|\int_\gamma f(z)dz| \leq M.2\pi r \to 0$ as $r \to 0$.

By Morera's theorem, f is an entire function. Since f is entire f has a power series expansion around 0. i.e. $f(z) = \sum_{k=0} a_k z^k$ where $a_k = \frac{f^k(0)}{k!}$.

Applying Cauchy's integral formula on $\overline{B(0,R)}$, $R > 1$ we have

$$\begin{aligned} |f^k(0)| &= \Big|\frac{k!}{2\pi i}\int_{|z|=R}\frac{f(z)}{z^{k+1}}dz\Big| \\ &\leq \frac{k!}{2\pi}MR^{\frac{3}{2}}\frac{1}{R^{k+1}}2\pi R \\ &\leq \frac{k!M}{R^{(k-\frac{3}{2})}} \to 0. \end{aligned}$$

as $R \to \infty$ for each $k > \frac{3}{2}$.
So $a_2 = a_3 = \cdots = 0$ i.e f is a polynomial of degree at most 1. So $f(z) = az + b$ for all $z \in \mathbb{C}$ and some constants $a, b \in \mathbb{C}$. $\boxed{3+5}$

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