Singular Value Decomposition(SVD)

Corollary Let $A = U\Sigma V^*$ be an SVD of $A \in \mathbb{F}^{n\times m}$.

(a) If A is square and nonsingular, then $A^{-1} = (VF)(F\Sigma^{-1}F)(UF)^*$ is an SVD of A^{-1} and where F is the $n \times n$ 'flip' matrix and $\|A^{-1}\|_2 = \frac{1}{\sigma_n}$. (Exercise!)

(b) If $p = \min\{m, n\}$, then assuming $\kappa_2(A) = \frac{\text{maxmag}\, A^T}{\text{minmag}\, A^T}$ if $n < m$,
$$\kappa_2(A) = \begin{cases} \frac{\sigma_1}{\sigma_p} & \text{if } \operatorname{rank} A = p, \\ \infty & \text{otherwise} \end{cases}$$
(Exercise!)

(c) Assuming, $\sigma_k = 0$ for $k > \min\{m, n\}$, $A^*Av_i = \sigma_i^2 v_i$, $i = 1, \ldots, m$, and $AA^*u_j = \sigma_j^2 u_j$, $j = 1, \ldots, n$. (Exercise!)

(d) If $n = m$ and A is a singular matrix, then for any $\epsilon > 0$, there exists a nonsingular matrix $B \in \mathbb{F}^{n\times n}$ such that $\|A - B\|_2 < \epsilon$.

Condensed Singular Value Decomposition

Let $A = U\Sigma V^*$ be an SVD of $A \in \mathbb{F}^{n\times m}$ with $\operatorname{rank} A = r$. Let $U = [u_1\; u_2 \cdots u_r] \in \mathbb{F}^{n\times r}$, $V_r = [v_1\; v_2 \cdots v_r] \in \mathbb{F}^{m\times r}$ and $\Sigma_r = \operatorname{diag}(\sigma_1, \ldots \sigma_r) \in \mathbb{F}^{r\times r}$. Then

$$A = U_r \Sigma_r V_r^*$$

is called the Condensed Singular Value Decomposition of A.

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Computing the Condensed SVD for small matrices:

1. Find the nonzero eigenvalues, say λ_i, $i = 1, \dots, r$, of A^*A or AA^*, whichever is smaller in size and corresponding eigenvectors. Here $\operatorname{rank} A = r$.
2. Set $\Sigma_r = \operatorname{diag}(\sigma_1 \ \cdots \ \sigma_r)$ where $\sigma_i = \sqrt{\lambda_i}$, $i = 1, \dots, r$.
3. If the eigenvectors of A^*A were found, call them $v_i, i = 1, \dots, r$. Compute $u_i = \frac{Av_i}{\sigma_i}$, $i = 1, \dots, r$ and set $U_r = \begin{bmatrix} u_1 & \cdots & u_r \end{bmatrix}$. Otherwise if the eigenvectors of AA^* were found, call them u_i, $i = 1, \dots, r$. Compute $v_i = \frac{A^* u_i}{\sigma_i}$, $i = 1, \dots, r$, and set $V_r = \begin{bmatrix} v_1 & \cdots & v_r \end{bmatrix}$.
4. Then $A = U_r \Sigma_r V_r^*$ is a Condensed SVD of A.

Moore-Penrose Pseudoinverse

Let $A = U\Sigma V^*$ be an SVD of $A \in \mathbb{F}^{n\times m}$ with $\operatorname{rank} A = r$. The Moore-Penrose pseudoinverse $A^\dagger$ of A is defined as

$$A^\dagger := V\Sigma^\dagger U^*$$

where $\Sigma^\dagger = \operatorname{diag}(\sigma_1^{-1}, \ldots, \sigma_r^{-1}, 0, \ldots 0) \in \mathbb{R}^{m\times n}$.

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Examples: The SVD of

$$A := \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T.$$

Therefore,

$$A^\dagger = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} -1/3 & 2/3 \\ -2/3 & -1/3 \end{bmatrix} = A^{-1}.$$

Moore-Penrose Pseudoinverse

The SVD of

$$B := \begin{bmatrix} 1 & 2 \end{bmatrix} = [1] \begin{bmatrix} \sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}^T .$$

Therefore,

$$B^\dagger = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \end{bmatrix} [1] = \begin{bmatrix} 1/5 \\ 2/5 \end{bmatrix}$$

Moore-Penrose Pseudoinverse

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Finally the SVD of

$$D := \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} [1]^T .$$

Therefore,

$$D^\dagger = [1] \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Properties of the Moore-Penrose Pseudoinverse

Theorem Let $A \in \mathbb{F}^{n \times m}$. Then,

(a) $A^{-1} = A^\dagger$ if $n = m$ and A is nonsingular. (Exercise!)

(b) $A^\dagger = (A^* A)^{-1} A^*$ if $\operatorname{rank} A = m$ (Exercise!)

(c) $A^\dagger = A^* (A A^*)^{-1}$ if $\operatorname{rank} A = n$. (Exercise!)

(d) $(\underbrace{AA^\dagger}_{=P})^* = AA^\dagger$, $\quad (A^\dagger A)^* = A^\dagger A$, $\quad AA^\dagger A = A$, $\quad A^\dagger A A^\dagger = A^\dagger$.

$P^* = P$

$A^+ A A^+ = A^+ \Rightarrow AA^+AA^+ = AA^+ \Rightarrow P^2 = P$

Also, if $B \in \mathbb{F}^{m \times n}$, such that
$(AB)^* = AB$, $\quad (BA)^* = BA$, $\quad ABA = A$, $\quad BAB = B$, then
$B = A^\dagger$. (Exercise!)

(f) $(A^\dagger)^* = (A^*)^\dagger$. (Exercise!)

(g) $A^\dagger = V_r \Sigma_r^{-1} U_r^*$. (Exercise!)

Moore-Penrose Pseudoinverse and the LSP

Theorem Let $Ax = b$ where $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$ with $n \geq m$. Then $x_0 = A^\dagger b$ is the unique least squares solution of the system $Ax = b$ if $\operatorname{rank} A = m$.

If $\operatorname{rank} A < m$, then x_0 is the least squares solution of the system with the smallest 2-norm.