Francis's (Implicit) QR Algorithm

Single shift or degree one: Let $A_0 = A$.

for $j = 1, 2, \ldots$

(i) Find reflector $Q_{j-1}^{(1)}$ *such that*

$$Q_{j-1}^{(1)}(A_{j-1} - \rho_{j-1} I)e_1 = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and compute $Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)}$.

(i) Find reflectors $\hat{Q}_{j-1}^{(2)}, \ldots, \hat{Q}_{j-1}^{(n-2)}$ *such that*

$$A_j = \hat{Q}_{j-1}^{(n-1)} \cdots \hat{Q}_{j-1}^{(2)} Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)} \hat{Q}_{j-1}^{(2)} \cdots \hat{Q}_{j-1}^{(n-1)}$$

is upper Hessenberg.

Francis's (Implicit) QR Algorithm

Double shift or degree two: Let $A_0 = A$.

for $j = 1, 2, \ldots$

(i) Find reflector $Q_{j-1}^{(1)}$ *such that*

$$Q_{j-1}^{(1)}(A_{j-1} - \rho_{j-1} I)(A_{j-1} - \tau_{j-1} I)e_1 = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and compute $Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)}$.

(i) Find reflectors $\hat{Q}_{j-1}^{(2)}, \ldots, \hat{Q}_{j-1}^{(n-1)}$ *such that*

$$A_j = \hat{Q}_{j-1}^{(n-1)} \cdots \hat{Q}_{j-1}^{(2)} Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)} \hat{Q}_{j-1}^{(2)} \cdots \hat{Q}_{j-1}^{(n-1)}$$

is upper Hessenberg.

Francis's QR Algorithm of degree 2

Let

$$A_0 = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}.$$

Then $(A_0 - \rho I)(A_0 - \tau I)e_1 = \begin{bmatrix} (a_{11} - \rho_1)(a_{11} - \tau) + a_{12}a_{21} \\ a_{21}((a_{11} + a_{22}) - (\rho + \tau)) \\ a_{32}a_{21} \\ 0 \end{bmatrix}.$

Let $Q_0^{(1)} = \begin{bmatrix} \tilde{Q}_0^{(1)} & 0 \\ 0 & 1 \end{bmatrix}$ where

$$\tilde{Q}_0^{(1)} \begin{bmatrix} (a_{11} - \rho)(a_{11} - \tau) + a_{12}a_{21} \\ a_{21}((a_{11} + a_{22}) - (\rho + \tau)) \\ a_{32}a_{21} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}.$$

Francis's QR Algorithm of degree 2

Then $Q_0^{(1)} A_0 = \begin{bmatrix} \tilde{Q}_0^{(1)} & 0 \\ 0 & 1 \end{bmatrix} A_0 = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ b & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}$

and $Q_0^{(1)} A_0 Q_0^{(1)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ b_1 & \times & \times & \times \\ b_2 & b_3 & \times & \times \end{bmatrix}$.

Bulge chasing:

Let $\hat{Q}_0^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q}_0^{(2)} \end{bmatrix}$ where $\tilde{Q}_0^{(2)} \begin{bmatrix} \times \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \times \\ 0 \\ 0 \end{bmatrix}$.

Francis's QR Algorithm of degree 2

Then $\hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & b_4 & \times & \times \end{bmatrix}$, and

$$\hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & b_5 & \times & \times \end{bmatrix}.$$

Let $\hat{Q}_0^{(3)} = \begin{bmatrix} I_2 & 0 \\ 0 & \tilde{Q}_0^{(3)} \end{bmatrix}$ where $\tilde{Q}_0^{(3)} \begin{bmatrix} \times \\ b_5 \end{bmatrix} = \begin{bmatrix} \times \\ 0 \end{bmatrix}$.

Francis's QR Algorithm of degree 2

Then $\hat{Q}_0^{(3)} \hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}$, and

$$\hat{Q}_0^{(3)} \hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} \hat{Q}_0^{(3)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} = A_1 .$$

Francis's QR Algorithm of degree 2

Then $\hat{Q}_0^{(3)} \hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}$, and

$$\hat{Q}_0^{(3)} \hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} \hat{Q}_0^{(3)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} = A_1 .$$

The bulge is gone and A_1 is in upper Hessenberg form!