Problem Set 3

MA 221: Discrete Mathematics

October 29, 2022

Problem 1. Prove that there exists a positive integer n so that $44^n - 1$ is divisible by 7.

Problem 2. Prove that the sequence 1967, 19671967, 196719671967,... , contains an element that is divisible by 1969.

Problem 3. Let r be any irrational real number. Prove that there exists a positive integer n so that the distance of nr from the closest integer is less than 10^{-10}.

Problem 4. Prove that for any positive integer n, it is possible to partition any triangle T into $3n + 1$ similar triangles.

Problem 5. In how many different ways can we place eight identical rooks on a chess board so that no two of them attack each other?

Problem 6. A magic square is a square matrix with nonnegative integer entries in which all row sums and column sums are equal. Let $H_3(r)$ be the number of magic squares of size 3×3 in which each row and column have sum r. Prove that

$$H_3(r) = \binom{r+4}{4} + \binom{r+3}{4} + \binom{r+2}{4},$$

where $H_3(r)$ is the number of 3×3 magic squares of line sum r.

Problem 7. How many n-element subsets $S \subseteq [2n]$ are there so that there are no two elements x and y in S satisfying $x + y = 2n + 1$?

Problem 8. Prove, by a combinatorial argument, that for all positive integers n, the number $\binom{3n}{n,n,n}$ is divisible by six.

Problem 9. Prove that if $n \geq 2$,then $n! < S(2n, n) < (2n)!$.

Problem 10. Prove that for all integers $n \geq 2$, the number $p(n) - p(n-1)$ is equal to the number of partitions of n in which the two largest parts are equal.

Problem 11. How many n-permutations contain entries 1, 2 and 3 in the same cycle?

Problem 12. How many positive integers $k \leq 210$ are relatively prime to 210?

Problem 13. Let m be a positive integer. Denote by $\phi(m)$ the number of integers in $[m]$ that are relatively prime to m. Let p, q, and r be distinct prime numbers. Compute $\phi(pqr)$.

Problem 14. Let a_n be the number of ways to pay n dollars using ten-dollar bills, five-dollar bills, and one-dollar bills only. Find the ordinary generating $A(x) = \sum_{n\geq 0} a_n x^n$.

Problem 15. Let $f(n)$ be the number of subsets of $[n]$ in which the distance of any two elements is at least three. Find the generating function of $f(n)$.