MA 201: PDE
Method of Separation of Variables
Finite Vibrating String Problem
Lecture - 11

IBVP for a vibrating string with no external forces

- We consider the problem of vibration of a thin finite string of length L vibrating in a perpendicular plane. Only the transverse vibration is considered.
- The string is assumed to be homogeneous and flexible with no external force acting. The tension along the string is also assumed to be uniform.
- The string is fastened at its two ends so as to have no displacements at these two points.
  (This will give boundary conditions.)
- The string is given a displacement $\phi(x)$ by pulling it from its equilibrium position (say, along the x-axis) and released with a velocity $\psi(x)$.
  (This will give initial conditions.)
- Such problems are known as Initial Boundary Value Problems (IBVP) since both boundary and initial conditions are prescribed. (On the other hand, D'Alembert's solution for the vibration of an infinite string is an Initial Value Problem (IVP) since no boundary condition was associated.) - Refer to earlier lectures.

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- Subsequently, the problem can be considered in a computational domain
$$(x,t) \in [0,L] \times [0,\infty).$$

- The IBVP for the unknown $u(x,t)$ under consideration consists of the following: The governing equation:
$$u_{tt} = c^2 u_{xx}, \quad (x,t) \in (0,L) \times (0,\infty). \tag{1}$$
By classification, this is a hyperbolic equation.
  - The boundary conditions for all $t \geq 0$:
  $$u(0,t) = 0, \quad u(L,t) = 0. \tag{2}$$
  - The initial conditions for $0 \leq x \leq L$ are
  $$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \tag{3}$$

- What does u signify here?
$u = u(x,t)$ gives the displacement of the string at any point x for any $t > 0$.

Separation of variables method

- The main idea of the separation of variables method is to convert the given partial differential equation into several ordinary differential equations which are usually in familiar forms.
- The solution of the equation is assumed to consist of the product of two or more functions.
- The number of such functions depends on the number of independent variables.
- For one-dimensional wave equation, the solution is $u = u(x, t)$. We assume the solution to be of the form $u(x, t) = X(x)T(t)$, where X is a function of x only and T is a function of t only.
- Utilizing this expression, finding the derivatives and inserting them into the given equation will result in a pair of ODEs.
- Note that this method can be used only for bounded domains so that the boundary conditions can be appropriately prescribed.
- This method can be used directly provided the given equation in homogeneous.

Separation of variables method

- This method will also hold for higher dimensional problems when $u = u(x, y, t)$, $u = u(r, \theta, t)$, $u = u(r, \theta, z)$, $u = u(x, y, z)$, $u = u(r, \theta, z, t)$ etc. by considering the appropriate product function.

In order to apply separation of variables method in an appropriate manner, the following are to be noted:

- We are always looking for a non-trivial solution.
- For an IBVP, the boundary conditions must be zero conditions whereas for a BVP, there must one non-zero boundary condition.
- At least one of the given conditions must be non-zero if the equation is homogeneous.
- Finding the solution is not possible if the above two conditions are not met.
- All conditions (including initial conditions) may be zero if the governing equation is non-homogeneous, i.e., if it contains a source.
- Though BVPs or IBVPs cannot be solved directly by this method if BCs are non-zero or the equation is not homogeneous, there are modified methods for finding solution for such problems based on separation of variables method.

IBVP for a vibrating string with no external forces (Contd.)

- Now let us proceed to solve the one-dimensional wave equation:

$$u_{tt} - c^2 u_{xx} = 0, \;\; 0 \le x \le L, \;\; t > 0 \tag{4}$$

  subject to the given BCs and ICs.

- Assume a solution of the form

$$u(x,t) = X(x)T(t). \tag{5}$$

  Here, $X(x)$ is function of x alone and $T(t)$ is a function of t alone.

- Substituting (5) in equation (4)

$$XT'' = c^2 X'' T. \tag{6}$$

IBVP for a vibrating string with no external forces (Contd.)

- Separating the variables

$$\frac{X''}{X} = \frac{T''}{c^2 T}.$$

- Here the left side is a function of x and the right side is a function of t.
- The equality will hold only if both are equal to a constant, say, k. Then,

$$\frac{X''}{X} = \frac{T''}{c^2 T} = k.$$

- It gives us two ordinary differential equations as follows:

$$X'' - kX = 0, \tag{7a}$$

$$T'' - c^2 kT = 0. \tag{7b}$$

IBVP for a vibrating string with no external forces (Contd.)

- Since k is any constant,
  - it can be zero, or
  - it can be positive, or
  - it can be negative.
- Consider all the possibilities and
  examine what value(s) of k lead to a non-trivial solution upon satisfying the given conditions.

IBVP for a vibrating string with no external forces (Contd.)

Case I: $k = 0$

- In this case, equations (7) reduce to

$$X'' = 0, \quad \text{and} \quad T'' = 0,$$

giving rise to solutions

$$X(x) = Ax + B, \quad T(t) = Ct + D.$$

- Boundary conditions

$$u(0,t) = u(L,t) = 0$$

lead to $X(x) = 0$. Hence $u = X(x)T(t) = 0$.

- This case of $k = 0$ is rejected since it gives rise to trivial solution only.

IBVP for a vibrating string with no external forces (Contd.)

Case II: $k > 0$, **let** $k = \lambda^2$ **for some** $\lambda \neq 0$.

- In this case, equations (7) reduce to the equations

$$X'' - \lambda^2 X = 0, \quad \text{and} \quad T'' - c^2\lambda^2 T = 0,$$

giving rise to solutions

$$\begin{aligned} X(x) &= Ae^{\lambda x} + Be^{-\lambda x}, \\ T(t) &= Ce^{c\lambda t} + De^{-c\lambda t}. \end{aligned}$$

- Therefore,

$$u(x,t) = (Ae^{\lambda x} + Be^{-\lambda x})(Ce^{c\lambda t} + De^{-c\lambda t}).$$

IBVP for a vibrating string with no external forces (Contd.)

- Using boundary condition $u(0,t)=0$, we get

$$A+B=0,\ \ B=-A.$$

- Using boundary condition $u(L,t)=0$, we get

$$(Ae^{\lambda L}+Be^{-\lambda L})(Ce^{c\lambda t}+De^{-c\lambda t})=0.$$

- The t part of the solution cannot be zero as it will lead to $T(t)=0$ and then case $k>0$ will be rejected straight way.

- Then, we must have

$$A(e^{\lambda L}-e^{-\lambda L})=0,$$

which leads to $A=0$ as $\lambda\neq 0$. This also implies $B=0$. In other words, $X(x)=0$.

- Thus, $k>0$ also gives rise to trivial solution only:
Therefore, $k>0$ is also rejected.

IBVP for a vibrating string with no external forces (Contd.)

Case III: $k < 0$**, let** $k = -\lambda^2$ **for some** $\lambda \neq 0$**.**

- In this case, equations (7) reduce to equations

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T'' + c^2\lambda^2 T = 0,$$

giving rise to solutions

$$\begin{aligned} X(x) &= A\cos(\lambda x) + B\sin(\lambda x), \\ T(t) &= C\cos(c\lambda t) + D\cos(c\lambda t). \end{aligned}$$

- Therefore,

$$u(x,t) = (A\cos(\lambda x) + B\sin(\lambda x))(C\cos(c\lambda t) + D\sin(c\lambda t)).$$

- Using boundary condition $u(0,t) = 0$, we get $A = 0$.

IBVP for a vibrating string with no external forces (Contd.)

- Using boundary condition $u(L, t) = 0$, we get

$$B \sin(\lambda L) = 0.$$

- $B \neq 0$ since that will lead to a trivial solution as already $A = 0$..

- Hence, we must have

$$\sin(\lambda L) = 0,$$

which gives us

$$\lambda = \frac{n\pi}{L} = \lambda_n, \; n = 1, 2, 3, \ldots.$$

- These λ_n's are called eigenvalues and note that

corresponding to each n, there will be an eigenvalue.

IBVP for a vibrating string with no external forces (Contd.)

- Accordingly, the solution can be written as

$$\begin{aligned} u(x,t) &= (A\cos(\lambda x) + B\sin(\lambda x))(C\cos(c\lambda t) + D\sin(c\lambda t)) \\ &= \sin(\lambda_n x)(BC\cos(c\lambda_n t) + BD\sin(c\lambda_n t)) \\ &= \sin\left(\frac{n\pi x}{L}\right)\left[A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right] \\ &= u_n(x,t). \end{aligned} \tag{8}$$

  This means that there is a solution corresponding to each n.

- The solution corresponding to each eigenvalue is called an eigenfunction.

- Thus, $u_n(x,t)$ is the eigenfunction corresponding to the eigenvalue λ_n.

IBVP for a vibrating string with no external forces (Contd.)

- Since the wave equation is linear and homogeneous, any linear combination will also be a solution.

- Therefore, we can expect the (general) solution to be of the following form:

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) \\
&= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right], \qquad (9)
\end{aligned}
$$

provided
i. A_n and B_n are determined uniquely and
ii. each of the resulting series for those coefficients converges, and
iii. the limit of the series is twice continuously differentiable with respect to x and t both so that it satisfies the equation $u_{tt} - c^2 u_{xx} = 0$.

IBVP for a vibrating string with no external forces (Contd.)

- Using the initial condition $u(x,0) = \phi(x)$, we get

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right). \tag{10}$$

- This series can be recognized as the half-range sine expansion of a function $\phi(x)$ defined in the range $(0, L)$.

- Now A_n can be obtained by multiplying equation (10) by $\sin\left(\frac{n\pi x}{L}\right)$ and integrating with respect to x from 0 to L or by directly writing the form of Fourier coefficient.

- Therefore,

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\, dx,\ \ n = 1, 2, 3, \ldots \tag{11}$$

- Here, we have used the fact that

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx = L.$$

IBVP for a vibrating string with no external forces (Contd.)

In order to utilize the other initial condition $u_t(x,0) = \psi(x)$, we need to differentiate (9) w.r.t. t to get

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{n\pi c}{L}\right)\left[-A_n \sin\left(\frac{n\pi ct}{L}\right) + B_n \cos\left(\frac{n\pi ct}{L}\right)\right].$$

Then

$$\psi(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi x}{L}\right).$$

Similarly, the second coefficient can be found as

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x) \sin\left(\frac{n\pi x}{L}\right)\, dx, \; n = 1, 2, 3, \ldots \tag{12}$$

IBVP for Vibrating string with no external forces (Contd.)

Therefore,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right)\right], \quad (13)$$

with

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\, dx,\ \ n = 1, 2, 3, \ldots \quad (14)$$

and

$$B_n = \frac{2}{n\pi c}\int_0^L \psi(x)\sin\left(\frac{n\pi x}{L}\right)\, dx,\ \ n = 1, 2, 3, \ldots. \quad (15)$$

gives us the solution of our initial value problem for vibration of a finite string under the assumptions and given conditions.

One-dimensional wave equation, as discussed above, is the most familiar hyperbolic equation. Solutions for other hyperbolic equations under the similar conditions may be obtained in a similar manner.

IBVP for Vibrating string with no external forces (Contd.)

The displacement (8) is referred to as the n-th eigenfunction or n-th normal mode of the vibrating string.

The n-th normal mode vibrates with a period of $(2L/nc)$ seconds which corresponds to a frequency of $(nc/2L)$ cycles per second. since $c^2 = (gT/w)$, where g is the acceleration due to gravity, T is the tension and w is the weight of the string per unit length, the frequency is

$$\frac{n}{2L}\left(\frac{gT}{w}\right)^{1/2}.$$

Hence

If a string on a musical instrument vibrates in a normal mode, its pitch may be sharpened (frequency increased) by either decreasing the length L of the string or increasing the tension T in the string.

IBVP for Vibrating string with no external forces (Contd.)

The first normal mode $n = 1$ vibrates with the lowest frequency

$$\frac{1}{2L}\left(\frac{gT}{w}\right)^{1/2}.$$

This is known as the fundamental frequency of the string. If the string can be made to vibrate in a higher normal mode, the frequency is increased by an integer multiple which corresponds to the production of a musical harmonic or overtone.

The solution presented by (9) is regarded as the standing wave solution.