The Matrix Eigenvalue Problem

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Let $A \in \mathbb{C}^{n \times n}$. The eigenvalue problem for A consists of finding all $x \in \mathbb{C}^n \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that

$$Ax = \lambda x$$
.

The scalars $\lambda \in \mathbb{C}$ are called the eigenvalues of A and the corresponding non zero vector x is called an eigenvector of A.

- ▶ The eigenvalues of *A* are the roots of det(A sI).
- ▶ $A \in \mathbb{C}^{n \times n}$ can have at most n distinct eigenvalues.
- ▶ Any set of eigenvectors of *A* corresponding to distinct eigenvalues is a linearly independent set.
- ▶ Matrices A and B are said to be similar if there exists a nonsingular matrix S such that $B = S^{-1}AS$. If S is a unitary matrix, then A and B are said to be unitarily similar.
- Similar matrices have the same eigenvalues.



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- ▶ If $Av = \lambda v$ for some $v \neq 0$, and $B = S^{-1}AS$, then $BS^{-1}v = \lambda S^{-1}v$.
- ► A matrix A is said to be (unitarily) diagonalizable is it is (unitarily) similar to a diagonal matrix.
- ▶ (Unitary) diagonalizability of an $n \times n$ matrix A is equivalent to the existence of a (orthonormal) basis of \mathbb{C}^n consisting of eigenvectors of A.
- Not every matrix is diagonalizable.
- ▶ Given any matrix $A \in \mathbb{C}^{n \times n}$, there exists an invertible matrix X and a block diagonal matrix $J = \text{diag}(J_{\lambda_1}, \dots, J_{\lambda_n})$, with

$$J_{\lambda_i} = \left[egin{array}{cccc} \lambda_i & 1 & & & & \\ & \lambda_i & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \lambda_i & 1 & \\ & & & & \lambda_i \end{array}
ight], \ i=1,\ldots,p,$$

such that $A = XJX^{-1}$. J is the called the *Jordan Canonical form* of A.

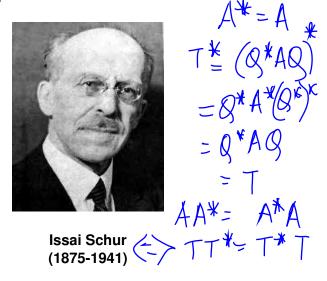


Schur's Theorem



Issai Schur (1875-1941)

Schur's Theorem



Schur's Theorem: Given any matrix $A \in \mathbb{C}^{n \times n}$, there exists a unitary matrix Q and an upper triangular matrix T such that $Q^*AQ = T$. T is called a Schur form of A.

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Spectral Theorems

Spectral Theorem for Hermitian Matrices $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if there exists a unitary matrix Q and a real diagonal matrix D such that $Q^*AQ = D$.

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Spectral Theorem for Normal Matrices $A \in \mathbb{C}^{n \times n}$ is normal if and only if there exists a unitary matrix Q and a diagonal matrix D such that $Q^*AQ = D$.

Prove the following:

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8*AB = T where To upper tonorigular, then, TT = TT,

If I is upper trangular & TXT=TTo then I'm

Henre prove Spectoal Theorem for Normal Matrices.

Spectral Theorems

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Spectral Theorem for Normal Matrices $A \in \mathbb{C}^{n \times n}$ is normal if and only if there exists a unitary matrix Q and a diagonal matrix D such that $Q^*AQ = D$.

Spectral Theorem for Symmetric Matrices $A \in \mathbb{R}^{n \times n}$ is symmetric if

and only if there exists a real orthogonal matrix Q and a real diagonal A $\in \mathbb{R}^{n \times n}$ & $AT = A \Rightarrow A^* = A$. Let $\lambda \in \mathcal{L}$ and $b \in \mathcal{L}^n, v \neq 0$, and that $A \circ = \lambda \circ = 0$ $\Rightarrow A \circ = \lambda \circ = 0$ $\Rightarrow A \circ = \lambda \circ = 0$ 8. From () & (0) 0 = (0 × A v - (0 × A v = (λ - λ) || v ||₂) = =) λ = λ (an (v + 0)) = Also Au= Au => A (Res)+i(AJme)=> Reso (=) A Rev = & Rev & A (Imv) = & (Imv)[Ro AR & we red] (as A*=A)-(2)
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