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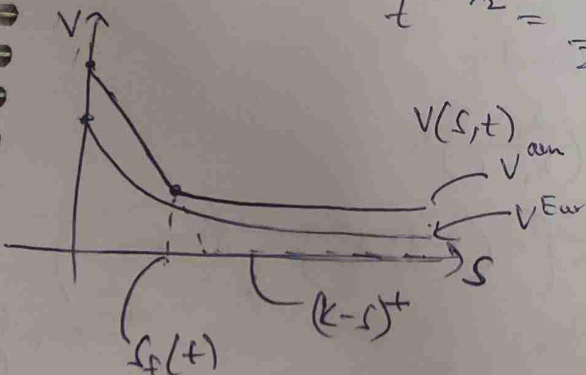
$$B\dot{w} + b = -Aw - a$$

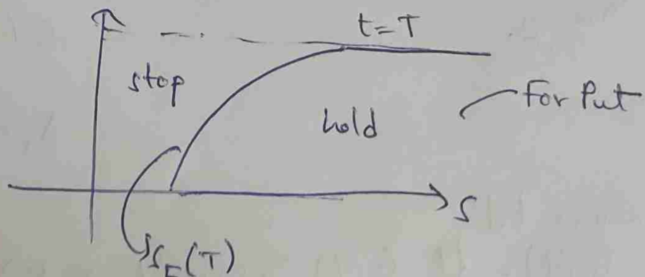
$$w_j(0) = \alpha(x_j) - \phi_0(x_j, 0)$$

Crank-Nicolson:-

$$\left(B + \frac{\delta t A}{2}\right) w^{n+1} = \left(B - \frac{\delta t}{2} A\right) w^{(n)} - \frac{\delta t}{2}\left(a^{(n)} + a^{(n+1)} + b^{(n)} + b^{(n+1)}\right)$$

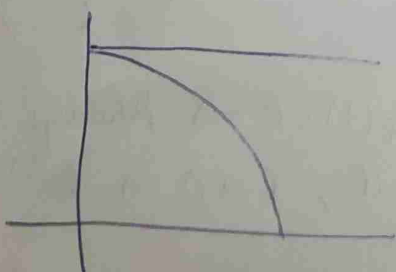
$$t^{n+1/2} = \frac{1}{2}\left(t^{n} + t^{(n+1)}\right)$$





$$V_p^{Am}(S,t) \geq (K-S)^+$$

$$V_c^{Am}(S,t) \geq (S-K)^+$$



$$\cancel{V_p(S,t) = K - S}$$

$$V_p^{Am}(S,t) > (K-S)^+, \quad S > S_f(t)$$

The $S_f(t)$ divides the curve into 2 parts

The Early exercise is monotonic, smooth.

$S_F(t)$ is not known apriori $\therefore$ American opt^n is referred as free bdary problem.

Early - Exercise Curve :-

Put :-

1) S_f is cont diff $0 \le t < T$

2) S_f is non decreasing

3) $S_f(t) > \left(\frac{\lambda_2}{\lambda_2 - 1}\right) K$

$$\lambda_2 = \frac{1}{\sigma^2}\left\{ -\left(r - \delta - \frac{\sigma^2}{2}\right) - \sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r} \right\}$$

4) Upper bdd

$$S_f(t) < \lim_{\substack{t \to T \\ t < T}} S_f(t) = \min\left(K, \frac{rK}{\delta}\right)$$

Call :-

1) Same as put

2) S_f - non increasing

3) Upper bound

$$S_f(t) < \left(\frac{\lambda_1}{\lambda_1 - 1}\right) K$$

$$\lambda_1 = \frac{1}{\sigma^2}\left\{ -\left(r - \delta - \frac{\sigma^2}{2}\right) + \sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2 + 2\sigma^2 r} \right\}$$

4) Lower bd

$$S_f(t) = \lim_{\substack{t \to T \\ t < T}} S_f(t) > \max\left(K, \frac{r}{\delta} K\right)$$

Put :- $r \to 0$, $S_f(t) = 0 \Rightarrow$ Always hold $\Rightarrow$ same as European

For American call, $\delta \to 0 \Rightarrow S_f(t) \to \infty \Rightarrow$ same as " .

$$V_p^{Am}(S_f(t), t) = K - S_f(t) \qquad ; \qquad V_c^{Am}(S_f(t), t) = S_f(t) - K$$

$$\frac{\partial V_p(S_f(t), t)}{\partial S} = -1 \qquad\qquad \frac{\partial V_c(\cdot\,\cdot)}{\partial S} = 1$$

Finance Interpretation?

Black Scholes

$\mathcal{L}_{BS}(V) =$

$\frac{\partial V}{\partial t} +$

$① \Rightarrow \frac{\partial V}{\partial t}$

(From

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We kno

Black Scholes Inequality

$$\mathcal{L}_{BS}(V) = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV \rightarrow (1)$$

$$\frac{\partial V}{\partial t} + \boxed{\frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r-\delta) S \frac{\partial V}{\partial S} - rV} = 0$$

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = 0$$

$$V = K - S$$

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial V}{\partial S} = -1 \qquad \frac{\partial^2 V}{\partial S^2} = 0$$

$$(1) \Rightarrow \frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) = (r-\delta) S(-1) - r\underbrace{V}_{(K-S)} = \delta S - rK$$

$$= \delta S - rK$$

$$\frac{\partial V}{\partial t} + \mathcal{L}_{BS}(V) < 0$$

(From the upper bound of Early Exercise curve)

6/9/24 T.S.T S_f

We know that when $t = T$, $V_p(S, T) = K - S$, $0 S < K$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} \underbrace{\frac{\partial^2 V}{\partial S^2}}_{0} + (r-\delta) S \underbrace{\frac{\partial V}{\partial S}}_{-1} - rV = 0$$

$$\frac{\partial V}{\partial t}(S, T) = rK - \delta S$$

Claim :- $\frac{\partial V}{\partial t}(S, T) \le 0$

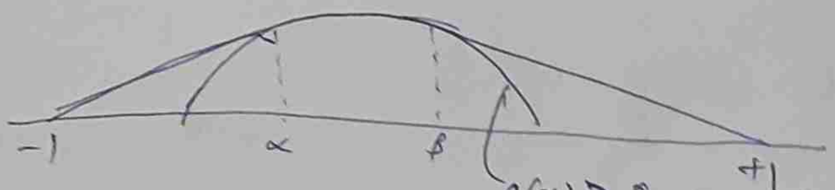
Pf :- Suppose ≥ 0

$$\frac{V(S, T) - V(S, T - \delta t)}{\delta t} \ge 0$$

$$(K - S) \ge V(S, T - \delta t) \Rightarrow \Leftarrow$$

$$S_f(T) = \lim_{\substack{t \to T \\ t < T}} S_f(t)$$

Obstacle Problem :-



$g(x) \geq 0$ for $x \in [\alpha, \beta]$

$g'' < 0$

$g(-1) < 0,\ g(+1) < 0$

Find min length of the elastic band that will keep the $g(x)$ bowl intact

for $-1 < x < \alpha$, $u'' = 0$ $\quad\equiv\quad$ If $u > g$, $u'' = 0$

$\alpha < x < \beta$, $u = g$ $\quad\quad$ $u = g$, ~~$u'' \leq 0$~~ $u'' = g'' \leq 0$

$\beta < x < 1$, $u'' = 0$ $\quad\quad$ (Same like American)

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The above can also be rewritten as :-

Linear Complementarity problem (LCP) :-

$$u''(u-g) = 0,\ -u'' \geq 0\quad u - g \geq 0$$ ← Constraints like a linear programming problem

$$u(\pm 1) = 0$$

$$(U_{i-1} - 2U_i + U_{i+1})(U_i - g_i) = 0,\ -(U_{i-1} - 2U_i + U_{i+1}) \geq 0,$$
$$(U_i - g_i) \geq 0$$

$$(U-g)^T BU = 0 \qquad B = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$
$$BU \geq 0$$
$$U \geq g$$

In Order to solve LCP, we have to use some iterative methods to satisfy $U_i \geq g_i$ $\equiv$ At every iteratⁿ, make $U_i = 0$ when $U_i < g_i$ O.W let it be as it is while solving $BU = 0$

BVP will be evaluated at each time level for American options i.e Black Scholes PDE will be discretized to be equivalent to LCP

↑

Parametrized Successive OverRelaxatⁿ (PSOR)

American Put ($\delta = 0$)

$$\frac{\partial y}{\partial \tau} = \frac{\partial^2 y}{\partial S^2}$$

$$V_p \geq (K-S)^+$$

$$V(S,t) \geq (K - Ke^{x})^+ = K\max\{1-e^{x}, 0\}$$

$$y(x,\tau) \geq \exp\left\{\frac{1}{2}(q-1)x + \frac{1}{4}(q+1)^2\tau\right\} \max\{1-e^{x}, 0\}$$

$$= \underbrace{\exp\left(\frac{1}{4}(q+1)^2\tau\right)\max\left\{e^{\frac{1}{2}(q-1)x} - e^{\frac{1}{2}(q+1)x}, 0\right\}}_{g(x,\tau)}$$

~~$y(x,0) = g(x,\tau)$~~

$$y(x,0) = g(x,0)$$

$$y(x,\tau) \geq g(x,\tau)$$

$$g(x_{min},\tau) = g(x_{min},\tau), \quad y(x_{max},\tau) = g(x_{max},\tau)$$

$$\left.\begin{aligned} &\left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right)(y-g) = 0 \\ &\left(\frac{\partial y}{\partial \tau} - \frac{\partial^2 y}{\partial x^2}\right) \geq 0, \quad y - g \geq 0 \quad y(x,0) = g(x,0) \end{aligned}\right\} \to \mathrm{I}$$

I is an LCP and doesn't require the locatⁿ of x in the holding / stopping region.

We can discretize this LCP by using Crank-Nicolson Central Difference.

$$\frac{U_m^{n+1} - U_m^n}{\Delta t} = \frac{\theta}{h^2}\left(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1}\right) + \frac{(1-\theta)}{h^2}\left(U_{m+1}^{n} - 2U_m^{n} + U_{m-1}^{n}\right)$$

$\theta = 0$ – FTCS

$\theta = 1 \to$ BTCS

$\theta = 1/2 \to$ CN

We need 3 subroutines for Jacobi, Gauss-Seidel, SOR eq also for obstacle problem.

We can reformulate LCP

$x := u - q$

Cryer Problem: $y := Au - b$ (A & b are known)

Find x, y ∋

$$\left.\begin{array}{l} \hat{b} = b - Aq \\ Ax - y = \hat{b}, \; x \geq 0, \; y \geq 0 \\ x^T y = 0 \end{array}\right\} \circledast$$

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~~$x := U - q$~~ Thm: Cryer's problem $\circledast$ is equivalent to minimization for Cryer

$\min_{x \geq 0} G(x)$, $G(x) = \frac{1}{2}(x^T A x) - \hat{b}^T x$ is strictly convex.

Pf:

$G_x = Ax - \hat{b}$

$G_{xx} = A$

where the tridiagonal matrix ~~A~~

$$A = \begin{bmatrix} 1+2\lambda\theta & -\lambda\theta & 0 & & 0 \\ -\lambda\theta & 1+2\lambda\theta & -\lambda\theta & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & \ddots \end{bmatrix}$$

The eigen values of A are $+^{ve}$.

∴ Hessian of $G = G_{xx}$ is symmetric $+^{ve}$ definite.

∴ G is strictly convex and has a unique min on any convex set in $\mathbb{R}^n$, in particular for $x \geq 0$

From KKT,

$\min G(x)$

$H_i(x) \leq 0 \quad i = 1:n$

Here we want constraint to be $x \geq 0$, $H(x) = -x$

$\frac{\partial H}{\partial x} = -1$

which is equivalent to have a vector x_0 ∋ ∃ Lagrangian multiplier $y \geq 0$ ∋

$$\text{grad } G(x_0) + \left(\frac{\partial H(x_0)}{\partial x}\right)^T y = 0$$

$$y^T H(x) = 0$$

$$\Leftrightarrow (Ax - \hat{b}) - y = 0$$

$$-y^T x = 0$$