MA 101 (Mathematics-I)

Multivariable Calculus part 2: Hint/solution Practice Problem Sheet 2

1. Consider the transformation $T : [0, 2\pi] \times [0, 1] \to \mathbb{R}^2$ given by $T(u, v) = (2v \cos u, v \sin u)$.

   (a) For a fixed $v_o \in [0, 1]$, describe the set $\{T(u, v_o) : \; u \in [0, 2\pi]\}$.

   (b) Describe the set $\{T(u, v) : \; [0, 2\pi] \times [0, 1]\}$.

   Solution: (a) If $x = 2v_o \cos u$ and $y = v_o \sin u$, then $\frac{x^2}{4} + \frac{y^2}{1} = v_o^2$. The set $\{T(u, v_o) : \; u \in [0, 2\pi]\}$ is an ellipse.

   (b) The set is the region enclosed by $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

2. Let R be the region in $\mathbb{R}^2$ bounded by the straight lines $y = x, \; y = 3x$ and $x + y = 4$. Consider the transformation $T(u, v) = (u - v, u + v)$. Find the set S satisfying $T(S) = R$.

   Solution: If $x = u - v$ and $y = u + v$, then $y = x$ is mapped to $v = 0$ and $y = 3x$ is mapped to $v = \frac{u}{2}$. The line $x + y = 4$ is mapped to $u = 2$. Please see Figure 1.

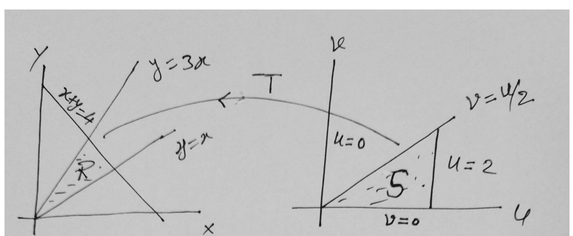


Figure 1

3. Evaluate $\iint\limits_R x\,dxdy$ where R is the region $1 \leq x(1 - y) \leq 2$ and $1 \leq xy \leq 2$.

   Solution: Let $u = x(1 - v)$ and $v = xy$. Since $xy \neq 0$, we can solve as $x = u + v$ and $y = \frac{v}{u+v}$. Here $J(u, v) = \frac{1}{u+v}$. The required integral is $\int_1^2 \int_1^2 (u + v) \frac{1}{|u+v|} dudv = 1$

4. Evaluate

   (a) $\int\limits_0^{\frac{1}{\sqrt{2}}} \int\limits_{x=y}^{\sqrt{1-y^2}} (x + y)dxdy.$

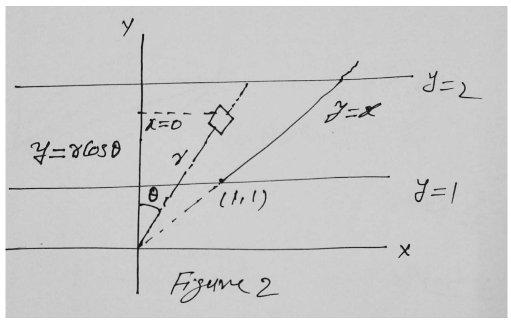
(b) $\int\limits_{1}^{2} \int\limits_{x=0}^{y} \frac{1}{(x^2+y^2)^{\frac{3}{2}}} dxdy.$

(c) $\int\limits_{0}^{2} \int\limits_{y=0}^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}dydx.$

Solution: (a) The given integral is $\iint\limits_{D}(x+y)dxdy$, where D is the region bounded by $y=0,\ y=x$ and the circle $x^2+y^2=1$. By polar coordinates

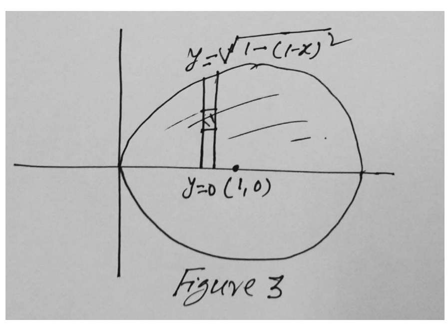
$$\iint\limits_{D}(x+y)dxdy = \int_0^{\frac{\pi}{4}}\int_0^1 r(\cos\theta+\sin\theta)rdrd\theta.$$

(b) Please see Figure 2.



By polar coordinate, the given integral becomes $\int_0^{\frac{\pi}{4}}\int_{\sec\theta}^{2\sec\theta}\frac{1}{r^3}rdrd\theta.$

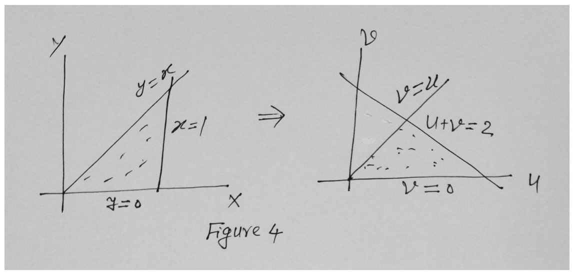
(c) Please see Figure 3.



The given integral becomes $\iint\limits_{D}\sqrt{x+y}\ dxdy$, where D is the region in the first quadrant bounded by the circle $(x-1)^2+y^2=1$ and the x-axis. Using polar coordinate, the circle $(x-1)^2+y^2=1$ can be represented by $r=2\cos\theta$. Hence the required integral is $\int_0^{\frac{\pi}{2}}\int_0^{2\cos\theta} r^2drd\theta.$

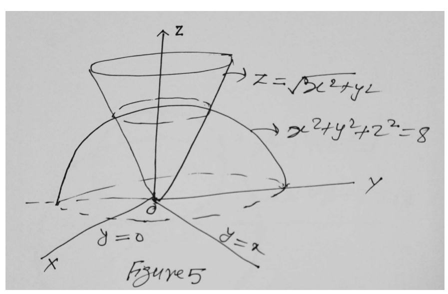
5. Using change of variables $u = x + y$ and $v = x - y$, show that
$$\int_0^1 \int_{y=0}^{y=x} (x - y)dydx = \int_0^1 \int_{u=v}^{2-v} \frac{v}{2} dudv.$$

Solution: We have $u + v = 2x$ and $u - v = 2y$. The line $x = y$ is mapped to $v = 0$ and $x = 1$ to $u + v = 2$. The x-axis is mapped to $v = u$. Here $J(u, v) = \frac{1}{2}$. Please see the Figure 4.



6. Find the volume of the solid in the first octant bounded below by the surface $z = \sqrt{x^2 + y^2}$ above by $x^2 + y^2 + z^2 = 8$ as well as the planes $y = 0$ and $y = x$.

Solution: The given solid lies above the region D, where D is in the first quadrant in $\mathbb{R}^2$ bounded by the circle $x^2 + y^2 = 4$ and the line $y = x$ and $y = 0$. Please see Figure 5.



Therefore the required volume is given by $\iint\limits_D (\sqrt{8 - x^2 - y^2} - \sqrt{x^2 + y^2})dxdy = \int_0^{\frac{\pi}{4}} \int_0^2 (\sqrt{8 - r^2} - r)rdrd\theta$.

7. Find the volume of the solid bounded by the surfaces $z = 3(x^2 + y^2)$ and $z = 4 - (x^2 + y^2)$.

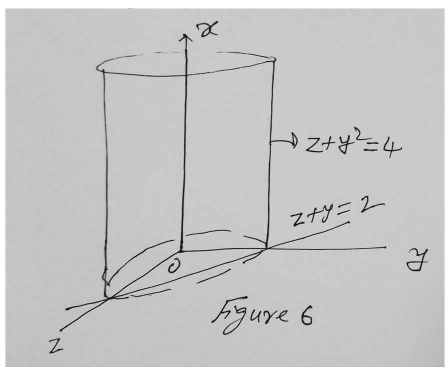
Solution: The intersection of the surfaces is the set $\{(x, y, 3) : x^2 + Y^2 = 1\}$. Therefore the volume is given by $\iint\limits_D (4-x^2-y^2-3(x^2+y^2)dxdy$, where D is the region in $\mathbb{R}^2$ enclosed by the circle $x^2 + y^2 = 1$. By polar coordinate the integral becomes $\int_0^{2\pi} \int_0^1 (4 - 4r^2)rdrd\theta$.

8. Let D denote the solid bounded by surfaces $y = x,\ y = x^2,\ z = x$ and $z = 0$. Evaluate $\iiint\limits_D ydxdydz$.

   Solution: The projection of the solid D on the xy-plane is give by $R = \{(x, y) :\ 0 \leq x \leq 1,\ x^2 \leq y \leq x\}$. The solid D lies above the surface $z = f_1(x, y) = 0$ and below $z = f_2(x, y) = x$. Therefore, $\iiint\limits_D ydxdydz = \int_{x=0}^1 \left(\int_{y=x^2}^x \left(\int_{z=0}^x ydz\right) dy\right) dx$.

9. Let D denote the solid bounded below by the plane $z + y = 2$, above by the cylinder $z + y^2 = 4$ and on the sides $x = 0$ and $x = 2$. Evaluate $\iiint\limits_D xdxdydz$.

   Solution: Please see Figure 6.



Solving $4 - y^2 = 2 - y$ implies $y = -1, 2$. The projection of the solid D on the xy-plane is given by $R = [0, 2] \times [-1, 2]$. The solid lies above the surface $z = f_1(x, y) = 2 - y$ and below $z = f_2(x, y) = 4 - y^2$. Therefore

$$\iiint\limits_D xdxdydz = \iint\limits_R \left(\int_{z=2-y}^{4-y^2} xdz\right) dxdy = \int_{x=0}^2 \int_{y=-1}^2 \int_{z=2-y}^{4-y^2} xdzdydx.$$

10. Let $D = \{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{4} + \frac{y^2}{16} + \frac{z^2}{9} \leq 1\}$ and $E = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 \leq 1\}$. Show that $\iiint\limits_D dxdydz = \iiint\limits_E 24dudvdw$.

    Solution: Note that the transformation $T(u, v, w) = (2u, 3v, 4w) = (x, y, z)$ maps E onto D and $J(u, v, w) = 24$.

11. Let D be the solid that lies inside the cylinder $x^2+y^2=1$, below the cone $z=\sqrt{4(x^2+y^2)}$ and above the plane $z=0$. Evaluate $\iiint\limits_D x^2dxdydz$.

Solution: The projection of the solid D on the xy-plane is $\{(x,y)\in\mathbb{R}^2 :\ x^2+y^2\le 1\}$. By changing to the cylindrical coordinates, the solid D is bounded by $z=0$ and $z=2r$. Therefore

$$\iiint\limits_D x^2dxdydz=\int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta r dzdrd\theta.$$

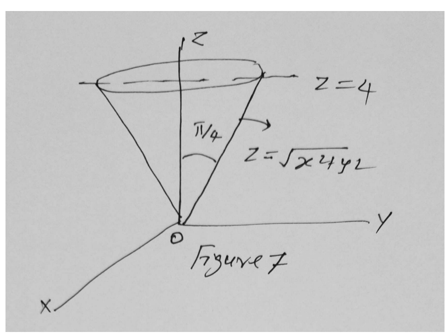
12. Evaluate $\int\limits_{-2}^{2}\int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int\limits_{x^2+y^2}^{4} xdzdydx$.

Solution: Note that $\int\limits_{-2}^{2}\int\limits_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int\limits_{x^2+y^2}^{4} xdzdydx=\iiint\limits_D xdxdydz$, where where D is the solid bounded below by $z=x^2+y^2$ and above by $z=4$. The projection of the solid D on the xy-plane is given by $\{(x,y)\in\mathbb{R}^2 :\ x^2+y^2\le 4\}$. By the cylindrical coordinates

$$\iint\limits_D xdxdydz=\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\cos\theta r dzdrd\theta.$$

13. Let D denote the solid bounded above by the plane $z=4$ and below by the cone $z=\sqrt{x^2+y^2}$. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2}dxdydz$.

Solution: Please see Figure 7.



We use the spherical coordinates. The equation $z=\sqrt{x^2+y^2}$ changes to $\rho\cos\phi=\rho\sin\phi$. This implies that $\phi=\frac{\pi}{4}$. The equation $z=4$ is written as $\rho\cos\phi=4$. That is, $\rho=\frac{4}{\sec\phi}$. Therefore,

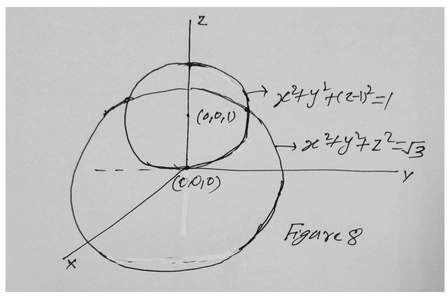
$$\iiint\limits_D \sqrt{x^2+y^2+z^2}dxdydz=\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{4\sec\phi}\rho\rho^2\sin\phi d\rho d\phi d\theta=2\pi=4^3\int_0^{\frac{\pi}{4}}\frac{\sin\phi}{\cos^4\phi}.$$

14. Parametrize the part of the sphere $x^2 + y^2 + z^2 = 16$, $-2 \leq z \leq 2$ using the spherical co-ordinates.

    Solution: By the spherical coordinates we can write the required surface as $S := r(\theta, \phi) = (4 \sin \phi \cos \theta,\ 4 \sin \phi \sin \theta,\ 4 \cos \phi)$, where $0 \leq \theta \leq 2\pi$, $\frac{\pi}{3} \leq \phi \leq \frac{2\pi}{3}$.

15. Let D denote the solid enclosed by the spheres $x^2 + y^2 + (z-1)^2 = 1$ and $x^2 + y^2 + z^2 = 3$. Using the spherical coordinates, set up iterated integral that gives the volume of D.

    Solution: Please Figure 8.



    By solving $x^2 + y^2 + (z-1)^2 = 1$ and $x^2 + y^2 + z^2 = 3$ we get $z = \frac{3}{2}$. That is, $\rho \cos \phi = \frac{3}{2}$. the equation $x^2 + y^2 + (z-1)^2 = 1$ becomes $\rho = 2 \cos \phi$ in the spherical coordinates.

    The required volumes is the sum of the volume of the portion of the region $x^2+y^2+z^2 \leq 3$ that lies inside the cone $\rho = \frac{\pi}{6}$ and the volume of the portion of the region $x^2+y^2+(z-1)^2 \leq 1$ that lies inside the sphere $x^2 + y^2 + z^2 = 3$. Therefore the required volume is given by
$$\int_0^{2\pi} \int_0^{\frac{\pi}{6}} \int_0^{\sqrt{3}} \rho^2 \sin \phi d\rho d\phi d\theta + \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^{2 \cos \phi} \rho^2 \sin \phi d\rho d\phi d\theta.$$

16. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$. Parametrize S by considering it as a graph and again by using the spherical coordinates.

    Solution: The surface S is bounded below by $z = \sqrt{2}$ and above by $z = 2$. By spherical coordinates, we get $\sqrt{2} \leq 2 \cos \phi \leq 2$. This implies that $0 \leq \phi \leq \frac{\pi}{4}$. Hence $S := r(\theta, \phi) = (2 \sin \phi \cos \theta,\ 2 \sin \phi \sin \theta,\ 2 \cos \phi)$, where $0 \leq \theta \leq 2\pi$, $0 \leq \phi \leq \frac{\pi}{4}$.

17. Let S denote the part of the plane $2x+5y+z=10$ that lies inside the cylinder $x^2+y^2=9$. Find the area of S.

   (a) By considering S as a part of the graph $z=f(x,y)$, where $f(x,y)=10-2x-5y$.

   (b) By considering S as a parametric surface $r(u,v)=(u\cos v, u\sin v, 10-u(2\cos v+5\sin v))$, $0\le u\le 3$ and $0\le v\le 2\pi$.

   Solution: (a) The projection D of the surface on the xy-plane is $\{(x,y): x^2+y^2=9\}$. The required area is $\iint\limits_D \sqrt{1+f_x^2 f_y^2}\, dxdy = \iint\limits_D \sqrt{30}\, dxdy = 9\sqrt{30}\pi$.
   (b) The area is $\int_0^3\int_0^{2\pi} |r_u\times r_v| dudv = \int_0^3\int_0^{2\pi} u\sqrt{30}\, dudv$.

18. Find the area of the surface $x=uv, y=u+v, z=u-v$, where $(u,v)\in D=\{(s,t)\in \mathbb{R}^2 : s^2+t^2\le 1\}$.

   Solution: The surface is given by $r(u,v)=(uv,u+v,u-v)$ and hence $|r_u\times r_v| = \sqrt{4+2(u^2+v^2)}$. Therefore the required area is
$$\iint\limits_D \sqrt{4+2(u^2+v^2)}dudv = \int_0^{2\pi}\int_0^1 \sqrt{4+2r^2}rdrd\theta.$$

19. Find the area of the part of the surface $z=x^2+y^2$ that lies between the cylinders $x^2+y^2=4$ and $x^2+y^2=16$.

   Solution: The given surface $z=x^2+y^2$ can be parameterized as $R(r,\theta)=(r\cos\theta, r\sin\theta, r^2)$, $r\ge 0$, and $0\le\theta\le 2\pi$. Hence $|R_r\times R_\theta| = r\sqrt{4r^2+1}$. Since the projection of the part of the surface on the xy-plane is the region between $x^2+y^2=4$ and $x^2+y^2=4$, we get $2\le r\le 4$. Therefore the required area is $\int_0^{2\pi}\int_2^4 r\sqrt{4r^2+1}drd\theta$.

20. Let S be the part of the cylinder $y^2+z^2=1$ that lies between the planes $x=0$ and $x=3$ in the first octant. Evaluate $\iint\limits_S (z+2xy)d\sigma$.

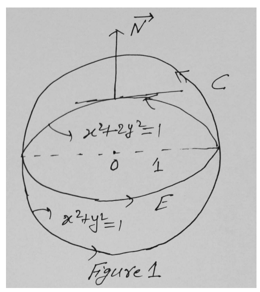
   Solution: The surface is $r(x,\theta)=(x,\cos\theta,\sin\theta)$, $0\le x\le 3$ and $0\le\theta\le\frac{\pi}{3}$. This implies $|r_x\times r_\theta|=1$. Hence
$$\iint\limits_S (z+2xy)d\sigma = \int_0^{\frac{\pi}{2}}\int_0^3 (\sin\theta+2x\cos\theta)(1)dxd\theta = \int_0^{\frac{\pi}{2}} (3\sin\theta+9\cos\theta)d\theta.$$

MA 101 (Mathematics-I)

Multivariable Calculus part 2: Hint/solution Practice Problem Sheet 3

1. Let $\vec{N}$ be the unit outward normal vector on the ellipse $x^2 + 2y^2 = 1$. Evaluate the line integral $\int\limits_C \vec{N}.d\vec{R}$ along the unit circle $C = \{(x, y) : \ x^2 + y^2 = 1\}$.

   Solution: The ellipse $x^2 + 2y^2 = 1$ can be represented by $E(t) = \left(\cos t, \frac{\sin t}{\sqrt{2}}\right)$ with $0 \le t < 2\pi$. This implies that normal vector to E will be $(y'(t), -x'(t)) = \left(\frac{\cos t}{\sqrt{2}}, \cos t\right)$. Hence the unit normal vector $\vec{N}(t) = \sqrt{\frac{2}{3}}\left(\frac{\cos t}{\sqrt{2}}, \cos t\right)$. Let C be represented by $R(t) = (\cos t, \sin t),\ 0 \le t < 2\pi$. Please refer to Figure 1.



   Thus,
$$\int\limits_C \vec{N}.d\vec{R} = \sqrt{\frac{2}{3}}\left(\frac{\cos t}{\sqrt{2}}, \cos t\right).(-\sin t, \cos t)dt.$$

2. Use second fundamental theorem of calculus for the line integral to show that $\int_C ydx + (x + z)dy + ydz$ is independent of any path C joining the points $(2, 1, 4)$ and $(8, 3, -1)$.

   Solution: Let $F(x, y, z) = (y, x+z, y)$. Consider $f(x, y, z) = xy+yz+c$. Then $\nabla f(x, y, z) = F(x, y, z)$. Hence, by second FTC for line integral
$$\int_C \nabla f \cdot dR = f(2, 1, 4) - f(8, 3, -1).$$
   That is, the given line integral is path independent. **Note that** one can $\nabla f = F$ for f by doing indefinite integral.

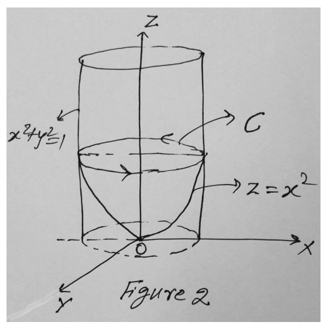
3. Consider the curve C which is the intersection of the surfaces $x^2 + y^2 = 1$ and $z = x^2$. Assume that C is oriented counterclockwise as seen from the positive z-axis. Evaluate $\int\limits_C zdx - xydy - xdz$.

Solution:

Let $F(x, y, z) = (z, -xy, -x)$. The cylinder $x^2 + y^2 = 1$ can be parameterized as

$$\{(\cos\theta, \sin\theta, z) : \ 0 \le \theta < 2\pi, z \in \mathbb{R}\}.$$

Since C also lies in $z = x^2$, this implies $C = \{(\cos\theta, \sin\theta, \cos^2\theta) : \ 0 \le \theta < 2\pi\}$. Please see the Figure 2.



The required line integral is

$$\int\limits_C f \cdot dR = \int_0^{2\pi} f(R(\theta)) \cdot R'(\theta) d\theta = 0.$$

4. Let $f(x, y, z) = (x^2, xy, 1)$. Show that that there is no ϕ such that $\nabla\phi = f$.

Solution: If there exists ϕ such that $\nabla\phi = f$, then $0 = \text{curl } \nabla\phi = \text{curl } f$ should be satisfied. But that is not the case here.

5. Let C be a curve represented by two parametric representations such that $C = \{R_1(s) : \ s \in [a, b]\} = \{R_2(t) : \ t \in [c, d]\}$, where $R_1 : [a, b] \to \mathbb{R}^3$ and $R_2 : [c, d] \to \mathbb{R}^3$ be two distinct differentiable one-one maps.

   (a) Show that there exists a function $h : [c, d] \to [a, b]$ such that $R_2(t) = R_1(h(t))$.

   (b) If R_1 and R_2 trace out C in the same direction, then $\int\limits_C f \cdot dR_1 = \int\limits_C f \cdot dR_2$.

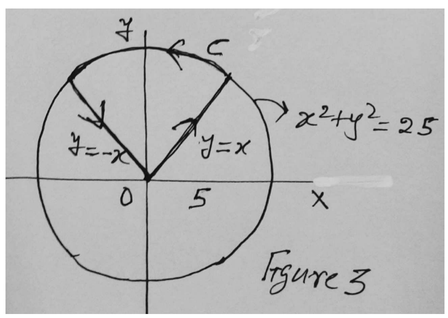
(c) If R_1 and R_2 trace out C in the opposite direction, then $\int\limits_C f \cdot dR_1 = -\int\limits_C f \cdot dR_2$.

Solution: (a) Consider $h(t) = R_1^{-1}(R_2(t))$.

(b & c) By chain rule $R_2'(t) = R_1'(t)h'(t)$. Therefore, the required line integral
$$\int_C f.dR_2 = \int_c^d f(R_2(t)) \cdot R_2'(t)dt = \int_c^d f(R_1(h(t)))) \cdot R_1'(h(t))h'(t)dt.$$
Let $u = h(t)$. Then
$$\int_C f.dR_2 = \int_{h(c)}^{h(d)} f(R_1(u)) \cdot R_1'(u)du = \pm \int_a^b f(R_1(u)) \cdot R_1'(u)du = \pm \int_C f \cdot dR_1.$$

6. Evaluate the line integral $\oint\limits_C (x^2 \sin^2 x - y^3)dx + (y^2 \cos^2 y - y)dy$, where C is the closed curve consisting $x + y = 0$, $x^2 + y^2 = 25$ and $y = x$ and lying in the first and fourth quadrants.

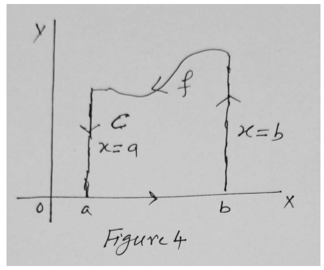
Solution: Let D be the domain enclosed by C as shown in Figure 3.



By Green's Theorem,
$$\oint\limits_C (x^2 \sin^2 x - y^3)dx + (y^2 \cos^2 y - y)dy = \iint\limits_D 3y^2 dxdy = \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_0^5 3r^2 \sin^2 \theta dr d\theta.$$

7. Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuously differentiable function. Suppose C is the boundary of the region bounded above by the graph of f, below by the x-axis and on the sides by the lines $x = a$ and $x = b$. Show that $\int\limits_a^b f(x)dx = -\oint\limits_C ydx$.

Solution: Let D be the domain enclosed by C as shown in Figure 4.

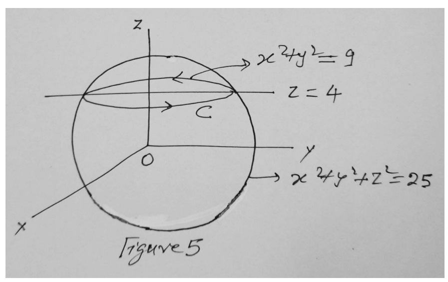


It follows from Greens theorem that

$$\int_a^b f(x)dx = \text{Area}(D) = \iint_D 1dxdy = -\int_C (ydx + 0dy).$$

8. Let $F(x,y,z) = (y, -x, 2z^2 + x^2)$ and S be the part of the sphere $x^2 + y^2 + z^2 = 25$ that lies below the plane $z = 4$. Evaluate $\iint_S \text{curl}\, F \cdot \hat{n} d\sigma$, where $\hat{n}$ is the unit outward normal of S.

   Solution: Let C be the boundary of the surface S as shown in Figure 5.



   Then $C = \{(3\sin\theta, 3\cos\theta, 4) : 0 \le \theta < 2\pi\}$. Note that C is oriented clockwise when viewed from above. By Stoke's Theorem

$$\iint_S \text{curl}\, F \cdot \hat{n} d\sigma = \oint_C F \cdot dR = \int_0^{2\pi} F(R(\theta)) \cdot R'(\theta) d\theta = 18\pi.$$

9. Let C be the boundary of the cone $z = x^2 + y^2$ and $0 \le z \le 1$. Use Stoke's theorem to evaluate the line integral $\int_C \vec{F}.d\vec{R}$ where $\vec{F} = (y, xz, 1)$.

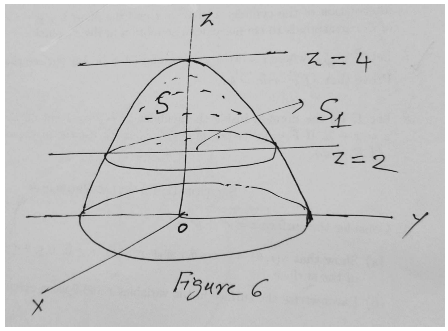
   Solution: Let $f(x,y,x) = x^2 + y^2 - z$. Then $S = \{(x,y,z) : \ f(x,y,x) = 0\}$ will be the surface of the domain. The unit normal to S will be $\hat{n} = \frac{\nabla f}{\|\nabla f\|} = \frac{(2x,2y,-1)}{\sqrt{4(x^2+y^2)+1}}$. Let $z = g(x,y) = x^2 + y^2$. Then $d\sigma = \sqrt{g_x^2 + g_y^2 + 1}dxdy$. By Stoke's Theorem,

$$\int_C \vec{F}.d\vec{R} = \iint_S \text{curl}\, \vec{F} \cdot \hat{n}\ d\sigma = \iint_R \text{curl}\, \vec{F} \cdot \hat{n}\sqrt{1 + g_x^2 + g_y^2} dxdy,$$

where $R = \{(x,y) : \; x^2 + y^2 \leq 1\}$.

10. Let $\vec{F} = (xy, yz, zx)$ and S be the surface $z = 4 - x^2 - y^2$ with $2 \leq z \leq 4$. Use divergence theorem to find the surface integral $\iint\limits_S \vec{F}.\vec{n}\, d\sigma$.

Solution: Let $S_1 = \{(x,y,2) : x^2 + y^2 \leq 2\}$. Please refer to Figure 6.



By divergence theorem,

$$\iint\limits_S \vec{F}.\vec{n}\, d\sigma + \iint\limits_{S_1} \vec{F}.\vec{n_1}\, d\sigma_1 = \iiint\limits_D \text{div } \vec{F}\, dxdydz.$$

Here

$$\iint\limits_{S_1} \vec{F}.\vec{n}\, d\sigma_1 = \iint\limits_{x^2+y^2 \leq 2} (xy, yz, zx).(-k)dxdy$$

11. Let S be the sphere $x^2 + y^2 + z^2 = 1$. If some $\alpha \in \mathbb{R}$ satisfies $\iint\limits_S (zx + \alpha y^2 + xz)d\sigma = \frac{4\pi}{3}$, then find α.

Solution:

Let D denote the solid enclosed by the surface S. By divergence theorem

$$\iint\limits_S (z, \alpha y, x).(x,y,z)d\sigma = \iiint\limits_D \alpha dxdydz = \alpha\frac{4\pi}{3}.$$

Hence $\alpha = 1$.

MA 101 (Mathematics-I)

Multivariable Calculus part 2: Hint/solution Tutorial Problem Sheet 1

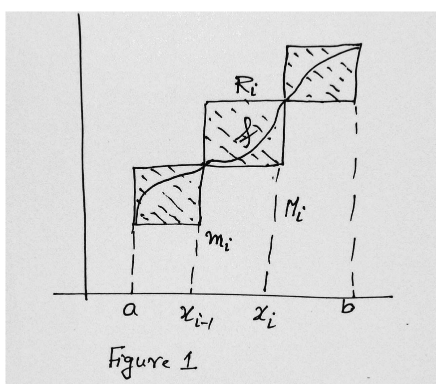
1. If $f : D = [a, b] \times [c, d] \to \mathbb{R}$ is continuous, then f is uniformly continuous.

   Solution: Suppose f is not uniformly continuous on D. Then, there is an $\epsilon > 0$ such that for each $\delta = \frac{1}{n}, n \in \mathbb{N}$, there exist sequences X_n, and Y_n in D such that $\|X_n - Y_n\| < \frac{1}{n}$ but $|f(X_n) - f(Y_n)| \geq \epsilon$. Since D is closed and bounded, by Bolzano-Weierstrass Theorem, there will be subsequence X_{n_k} such that $X_{n_k} \to X \in D$. Similarly, $Y_{n_{kl}}$ has subsequence $Y_{n_{kl}}$ such that $Y_{n_{kl}} \to Y \in D$. Hence, without lose of generality, we can assume that $X_{n_k} \to X$ and $Y_{n_k} \to Y$. Thus, we have $\|X_{n_k} - Y_{n_k}\| < \frac{1}{n_k}$ and $|f(X_{n_k}) - f(Y_{n_k})| \geq \epsilon$. It follows that $X = Y$. By continuity of f at X and Y, we get $|f(X) - f(Y)| \geq \epsilon$, which is a contradiction.

2. Let f be real valued continuous function on $[a, b]$. Show that the graph of f is a set of content zero.

   Solution: Let $G_f = \{(x, f(x)) : \ x \in [a, b]\}$. Note that the function f is uniformly continuous on $[a, b]$. For given $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - y| < \delta$ implies

$$|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}. \tag{1}$$



Let $P = \{x_0, \ldots, x_{i-1}, x_i, \ldots, x_n\}$ be a partition of $[a, b]$ such that $\Delta x_i < \delta$. Then (1) will be satisfied by every pair of points $x, y \in [x_{i-1}, x_i]$. That is,

$$-\frac{\epsilon}{2(b-a)} < f(x) - f(y) < \frac{\epsilon}{2(b-a)}.$$

By taking supremum w.r.t. $x \in [x_{i-1}, x_i]$ keeping y fixed and then supremum w.r.t. y, we get $M_i - m_i < \frac{\epsilon}{2(b-a)}$. Note that $(M_i - m_i)\Delta x_i$ is the area of the rectangle $R_i = [m_i, M_i] \times [x_{i-1}, x_i]$ along the graph of f as shown in Figure 1. This shows that $\sum_{i=1}^{n}(M_i - m_i)\Delta x_i < \epsilon$. Thus, $G_f \subset \bigcup_{i=1}^{n} R_i$ and Area$(\bigcup_{i=1}^{n} R_i) < \epsilon$. Hence G_f is of content zero.

3. Let $D = \{(x, y) : a \leq x \leq b \text{ and } \varphi(x) \leq y \leq \psi(x)\}$, where φ and ψ are continuous functions on $[a, b]$. If f is a bounded continuous functions on D, then

$$\iint_D f(x, y)dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x, y)dy \right) dx.$$

Solution: Since φ and ψ are continuous on $[a, b]$, they are bounded and hence D is a bounded domain in $\mathbb{R}^2$.

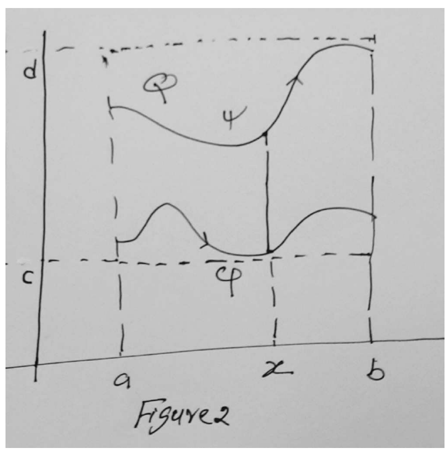


Figure2

Let $Q = [a, b] \times [c, d]$ be a rectangle containing D as shown in Figure 2. Extend f on Q as $\tilde{f} : Q \to \mathbb{R}$, where

$$\tilde{f}(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \in Q \setminus D. \end{cases}$$

By definition of $\tilde{f}$, it is clear that $\tilde{f}$ is continuous on the interior of D. It is clear from Figure 2, the domain D is bounded by the graph of φ, ψ and two vertical line segments, each of content zero. Hence $\tilde{f}$ has discontinuities in Q of content zero. Thus, $\tilde{f}$ is integrable. Now, it only remain to show that

$$\iint\limits_{Q} \tilde{f}(x,y)dxy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} f(x,y)dy \right) dx.$$

Note that for each fixed $x \in [a,b]$, the integral $\int\limits_c^d \tilde{f}(x,y)dy$ exists, since the set of discontinuities of $\tilde{f}(x,\cdot)$ contains at most two points, one each on the graph of φ and ψ. Moreover, $G(x) = \int\limits_c^d \tilde{f}(x,y)dy$ is continuous expect possibly at a and b. Hence G is integrable on $[a,b]$. By applying Fubini's Theorem to $\tilde{f}$ on Q, we get

$$\iint\limits_{D} f(x,y)dxdy = \int_a^b \left(\int_{\varphi(x)}^{\psi(x)} \tilde{f}(x,y)dy \right) dx.$$

But, this follows from the fact that

$$\int_c^d \tilde{f}(x,y)dy = \int_{\varphi(x)}^{\psi(x)} f(x,y)dy.$$

Hence the result followed.

4. Evaluate the following integral applying Fubini's Theorem

   (a) $\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \sqrt{1-y^2}dydx$

   (b) $\int\limits_0^{\pi} \int\limits_x^{\pi} \frac{\sin y}{y}dydx$

   (c) $\int\limits_0^1 \int\limits_y^1 x^2 e^{xy}dxdy$

Solution: (a) The domain of integration is as shown Figure 3.

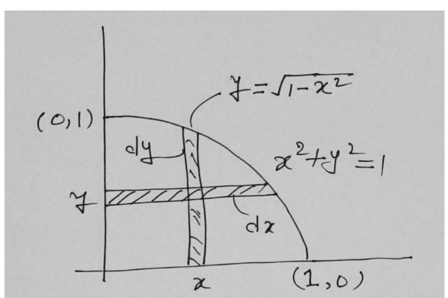


Figure 3

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dydx = \int_{y=0}^1 \left(\int_{x=0}^{\sqrt{1-y^2}} \sqrt{1-y^2}\,dx \right) dy = \int_{y=0}^1 (1-y^2)dy.$$

(b) The domain of integration is as shown Figure 4.

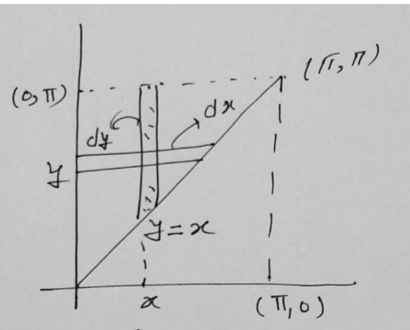


Figure 4

$$\int_0^\pi \left(\int_x^\pi \frac{\sin y}{y} dy \right) dx = \int_{y=0}^\pi \left(\int_{x=0}^y \frac{\sin y}{y} dx \right) dy = \int_{y=0}^\pi \sin y\; dy.$$

(c) The domain of integration is as shown Figure 5.

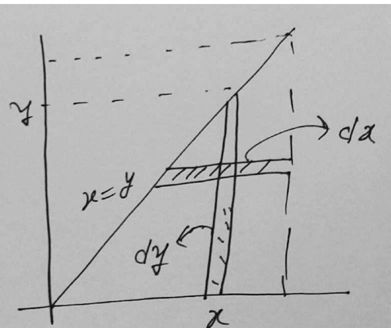


Figure 5

$$\int_0^1 \int_y^1 x^2 e^{xy} dxdy = \int_{x=0}^1 \left(\int_{y=0}^x x^2 e^{xy} dy \right) dx.$$

5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $\int\limits_{y=0}^{x} \int\limits_{t=0}^{y} f(t)dtdy = \int\limits_{t=0}^{x} (x-t)f(t)dt$.

Solution: The domain of integration is as shown Figure 4.

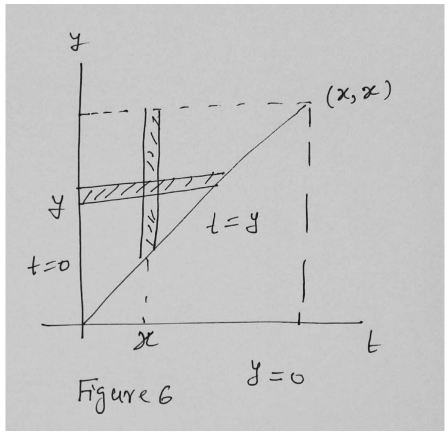


Figure 6

$$\int\limits_{y=0}^{x} \left(\int\limits_{t=0}^{y} f(t)dt \right) dy = \int\limits_{t=0}^{x} \left(\int\limits_{y=t}^{x} f(t)dy \right) dt = \int\limits_{t=0}^{x} (x-t)f(t)dt.$$

6. Let f be a continuous function on the bounded domain D. If $\iint\limits_R f(x,y)dxdy = 0$ for all rectangle R in D, then $f = 0$ on D.

Solution: Suppose there exists $X_o \in D$ such that $f(X_o) \neq 0$. Then without loss of generality we can assume that $f(X_o) > 0$. Since f is continuous at X_o, for $\epsilon = \frac{f(X_o)}{2} > 0$, there exists an open ball $B_\delta(X_o)$ such that $|f(X) - f(X_o)| < \frac{f(X_o)}{2}$. This implies $f(X) > \frac{3f(X_o)}{2}$ for each $X \in B_\delta(X_o)$. Thus,

$$\iint\limits_R f(x,y)dxdy = 0$$

for each rectangle $R \in B_\delta(X_o)$. Since f is continuous on R, it follows that f must be zero on R. If not, then suppose, $f(y_o) > 0$ for some $Y_o \in R$. Then there exists a ball $B_r(Y_o)$ such that $f(X) > \frac{3f(Y_o)}{2}$ for each $X \in B_r(Y_o)$. But, then

$$0 = \iint\limits_R f(x,y)dxdy > \iint\limits_{B_r(Y_o)} f(x,y)dxdy \geq \frac{3f(Y_o)}{2} \iint\limits_{B_r(Y_o)} dxdy = \frac{3f(Y_o)}{2}\pi r^2 > 0.$$

which is a contradiction.

7. Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a continuous function. If f_x, f_y, f_{xy} and f_{yx} are continuous then, by using Fubini's theorem, show that $f_{xy} = f_{yx}$.

Solution: Since f_{xy} is continuous on D, by Fubini's Theorem, we get

$$\begin{aligned}
\int_a^x \int_c^y \frac{\partial^2 f}{\partial x \partial y}(u,v)dvdu &= \int_c^y \int_a^x \frac{\partial^2 f}{\partial x \partial y}(u,v)dudv \\
&= \int_c^y \left[\frac{\partial f}{\partial y}(x,v) - \frac{\partial f}{\partial y}(a,v)\right] dv \\
&= f(x,y) - f(x,c) - f(a,y) + f(a,c).
\end{aligned}$$

Also,

$$\int_a^x \int_c^y \frac{\partial^2 f}{\partial y \partial x}(u,v)dvdu = f(x,y) - f(x,c) - f(a,y) + f(a,c).$$

Hence

$$\int_a^x \int_c^y \frac{\partial^2 f}{\partial x \partial y}(u,v)dvdu = \int_a^x \int_c^y \frac{\partial^2 f}{\partial y \partial x}(u,v)dvdu.$$

Since the above equation holds for every choice of $x, y \in D$, we obtain

$$\iint\limits_R \frac{\partial^2 f}{\partial x \partial y}(u,v)dvdu = \iint\limits_R \frac{\partial^2 f}{\partial y \partial x}(u,v)dvdu$$

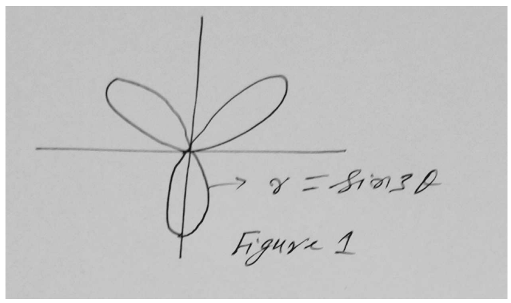
for every rectangle $R \subseteq D$. Thus, $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

MA 101 (Mathematics-I)

Multivariable Calculus part 2: Hint/solution Tutorial Problem Sheet 2

1. Using double integral, find the area enclosed by the curve $r = \sin 3\theta$ given in polar coordinates.

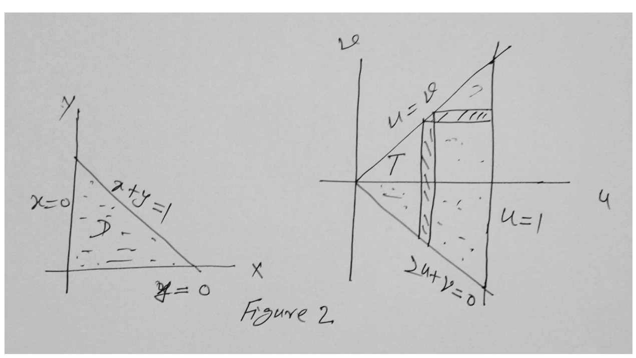
   Solution: Please see Figure 1.



   The curve is given by $r = \sin 3\theta$, where $\theta \in [0, \pi)$. Area $= 3 \int\limits_{0}^{\frac{\pi}{3}} \int\limits_{r=0}^{\sin 3\theta} r dr d\theta$.

2. Evaluate the double integral $\iint\limits_{D} \sqrt{x+y}\,(y-2x)^2 dydx$ over the domain D bounded by the lines $x = 0,\ \ y = 0$ and $x + y = 1$.

   Solution: Let $u = x + y$ and $v = y - 2x$. Then $x = \frac{u-v}{3}$ and $y = \frac{2u+v}{3}$.



   Here $J(u, v) = \frac{1}{3}$. Note that the line $y = 0$ is mapped to $u = x$ and $v = -2x$. Similarly, the line $x = 0$ is mapped to $u = y$ and $v = y$. That is, $x = 0$ is mapped to $u = v$. Also,

$x + y = 1$ is mapped to $u = 1$. Interior of D mapped to the interior of the triangle T as shown in the Figure 2. Hence

$$\iint_D \sqrt{x+y}\,(y-2x)^2 dydx = \frac{1}{3}\iint_T \sqrt{u}\, v^2 dvdu = \int_{u=0}^{1}\left(\int_{v=-2u}^{u} \sqrt{u}\, v^2 dv\right) du.$$

3. Evaluate the integral $\iint_D e^{(x-2y)} dxdy$ over the domain D bounded by the lines $x - 2y = 0$, $2x - y = 0$ and $x + y = 1$ as shown in Figure 3.

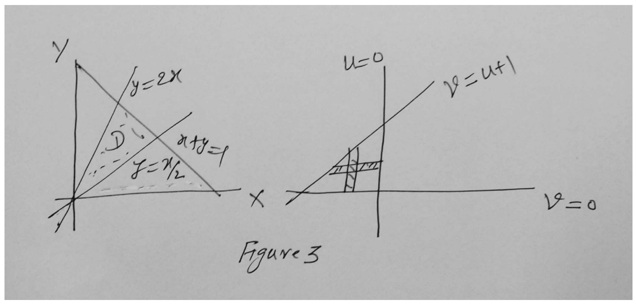


Figure 3

Solution: Put $u = x - 2y$ and $v = 2x - y$. Then $x = \frac{2v-u}{3}$ and $y = \frac{v-2u}{3}$. It is clear that $x - 2y = 0$ is mapped to $u = 0$ and $2x - y = 0$ is mapped to $v = 0$. Also, $x + y = 1$ is mapped to $v - u = 1$. Here $J(u, v) = \frac{1}{3}$. Hence

$$\iint_D e^{(x-2y)} dxdy = 3\int_{u=-1}^{0}\left(\int_{v=0}^{u+1} e^u dv\right) du = 3\int_{u=-1}^{0} e^u (u+1) du.$$

4. Compute $\lim_{a\to\infty} \iint_{D(a)} e^{-(x^2+y^2)} dxdy$, where

$(a)\;\; D(a) = \{(x, y) : x^2 + y^2 \leq a^2\}$ and $(b)\;\; D(a) = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq a\}$

Hence prove that $(c)\;\; \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ $\qquad (d)\;\; \int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}$

Solution: (a) Let $D_1(a) = \{(x, y) : x^2 + y^2 \leq a^2\}$. Then by using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$I_1(a) = \iint_{D_1(a)} e^{-(x^2+y^2)} dxdy = \int_0^{2\pi}\int_{r=0}^{a} e^{-r^2} r dr d\theta = \pi(1 - e^{-a^2}) \to \pi.$$

(b) Write $D_2(a) = \{(x, y) : 0 \leq x \leq a, 0 \leq y \leq a\}$. It is clear from Figure 4

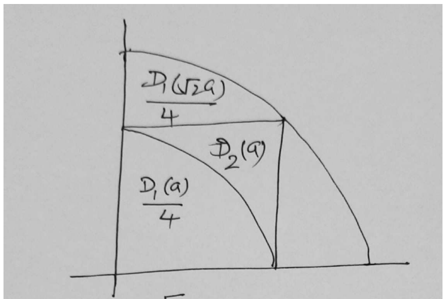


Figure 4

that $\frac{D_1(a)}{4} < D_2(a) < \frac{D_1(\sqrt{2}a)}{4}$. Let $I_2(a) = \iint\limits_{D_2(a)} e^{-(x^2+y^2)}dxdy$. Then the corresponding integrals satisfy $\frac{I_1(a)}{4} < I_2(a) < \frac{I_1(\sqrt{2}a)}{4}$. By sandwich theorem, we get $\lim\limits_{a\to\infty} I_2(a) = \frac{\pi}{4}$.

(c) Let $I(a) = \int_0^a e^{-x^2}dx$. Then by Fubini's theorem,

$$I^2(a) = \left(\int_0^a e^{-x^2}dx\right)\left(\int_0^a e^{-y^2}dy\right) = \int_{x=0}^a \int_{y=0}^a e^{-(x^2+y^2)}dxdy = I_2(a) \to \frac{\pi}{4}.$$

(d) Let $J(a) = \int\limits_0^a x^2e^{-x^2}dx$. Then by Fubini's theorem,

$$J^2(a) = \left(\int_0^a x^2e^{-x^2}dx\right)\left(\int_0^a y^2e^{-y^2}dy\right) = \int_{x=0}^a \int_{y=0}^a x^2y^2e^{-(x^2+y^2)}dxdy$$
$$= \iint_{D_2(a)} x^2y^2e^{-(x^2+y^2)}dxdy.$$

By using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, we can write

$$\iint_{D_1(a)} x^2y^2e^{-(x^2+y^2)}dxdy = \frac{1}{4}\int_0^{2\pi}\int_{r=0}^a r^4(\sin 2\theta)^2e^{-r^2}rdrd\theta.$$

Use similar argument as in solution of (b) to get answer in this case.

5. Let D denote the solid bounded by the surfaces $y = x,\ y = x^2,\ z = x$ and $z = 0$. Evaluate $\iiint\limits_D ydxdydz$.

Solution: Here $y = x,\ y = x^2,\ z = x$ and $z = 0$, implies $y = 0, 1$. Please see Figure 5.

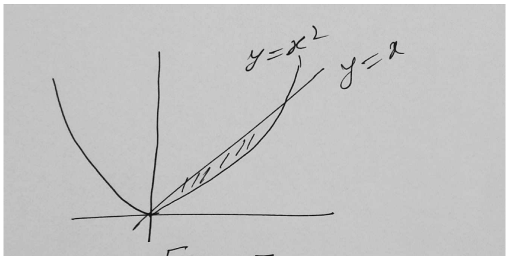


Figure 5

By Fubini's theorem, we get

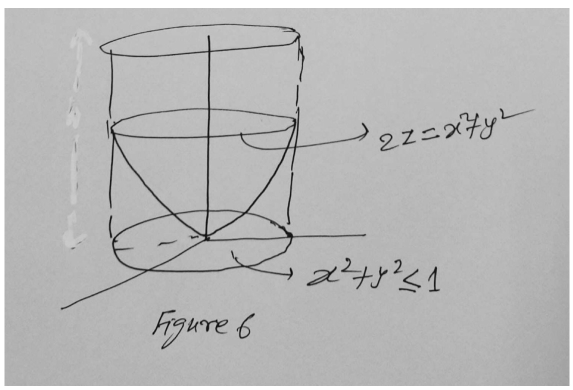
$$\iiint\limits_D y\,dxdydz = \int_{x=0}^{1}\left(\int_{z=0}^{x}\left(\int_{y=x^2}^{x} ydy\right)dz\right)dx.$$

6. Let D denote the solid bounded above by the plane $z = 4$ and below by the cone $z = \sqrt{x^2+y^2}$. Evaluate $\iiint\limits_D \sqrt{x^2+y^2+z^2}\,dxdydz$.

   Solution: Use spherical polar coordinate $x = r\sin\phi\cos\theta,\ y = r\sin\phi\sin\theta,\ z = r\cos\phi$, where $0 \le \theta < 2\pi$ and $0 \le \phi < \frac{\pi}{4}$.

7. Find the surface integral $\iint\limits_S z d\sigma$, where S it the part of the paraboloid $2z = x^2+y^2$ which lies in the cylinder $x^2+y^2 = 1$.

   Solution: Please see Figure 6.



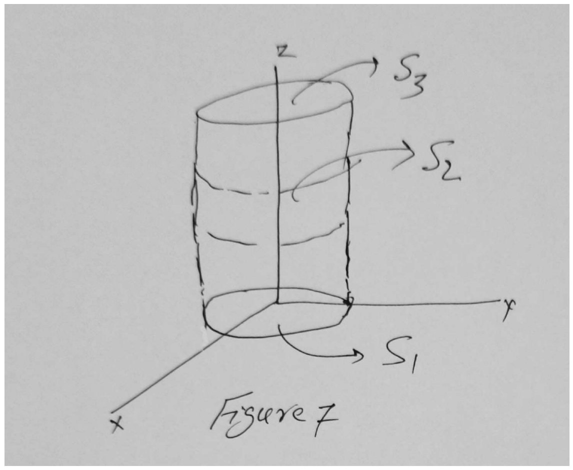
Let $z = f(x,y) = \frac{x^2+y^2}{2}$ and $D = x^2+y^2 \le 1$.

$$\iint\limits_S z d\sigma = \iint\limits_D z\sqrt{1+f_x^2+f_y^2}\,dxdy.$$

Use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ to evaluate the integral on D.

8. What is the integral of the function x^2z taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2+y^2 = a^2$.

   Solution: We divide the surface of the cylinder into three parts $S_i; i = 1,2,3$ as shown in the Figure 7.



$$\iint\limits_{S} x^2 z \, d\sigma = \left(\iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3} \right) x^2 z \, d\sigma.$$

Note that S_1 is the bottom of the cylinder given by $x^2 + y^2 \leq a^2$ and $z = 0$. Hence $\iint\limits_{S_1} x^2 z \, d\sigma = 0$. Here S_2 is the vertical surface given by $r(\alpha, \beta) = (a\cos\alpha, a\sin\alpha, \beta)$, where $0 \leq \alpha < 2\pi$ and $0 \leq \beta \leq h$. Hence

$$\iint\limits_{S_2} x^2 z \, d\sigma = \int_{\beta=0}^{h} \int_{\alpha=0}^{2\pi} (a\cos\alpha)^2 \beta \|r_\alpha \times r_\beta\| d\alpha d\beta = \frac{\pi a^3 h^2}{2}.$$

Here S_3 is the top of the cylinder given by $x^2 + y^2 \leq a^2$ and $z = h$. This can be parametrized by $r(u, v) = (u\cos v, u\sin v, h)$, where $0 \leq u \leq a$ and $0 \leq v < 2\pi$. Thus,

$$\iint\limits_{S_3} x^2 z \, d\sigma = \int_{u=0}^{a} \int_{v=0}^{2\pi} (u\cos v)^2 h \|r_u \times r_v\| du dv = \frac{\pi a^4 h}{4}.$$

MA 101 (Mathematics-I)

Multivariable Calculus part 2: Hint/solution Tutorial Problem Sheet 3

1. Find the line integral of the vector field $F(x, y, z) = y\vec{i} - x\vec{j} + \vec{k}$ along the path $c(t) = (\cos t, \sin t, \frac{t}{2\pi})$, $0 \leq t \leq 2\pi$ joining $(1, 0, 0)$ to $(1, 0, 1)$.

   Solution:
$$\int F \cdot dc = \int_0^{2\pi} F((c(t)) \cdot c'(t)dt = 1 - 2\pi.$$

2. Evaluate $\int\limits_C T \cdot dR$, where C is the circle $x^2 + y^2 = 1$ and T is the unit tangent vector.

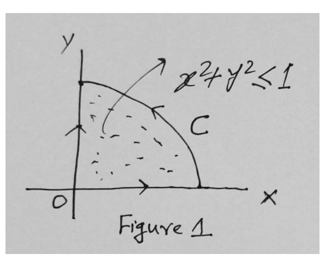
   Solution: The unit circle can be represented by $C = \{R(t) : \ 0 \leq t < 2\pi\}$. The unit tangent vector T to C is given $T(t) = \frac{R'(t)}{\|R'(t)\|}$. Hence
$$\int_C T \cdot dR = \int_0^{2\pi} \frac{R'(t)}{\|R'(t)\|} \cdot R'(t)dt = 2\pi.$$

3. Show that the integral $\int\limits_C yzdx + (xz+1)dy + xydz$ is independent of the path C joining $(1, 0, 0)$ and $(2, 1, 4)$.

   Solution: Let $F(x, y, z) = (xy, xz + 1, xy)$. Consider $f(x, y, z) = xyz + y + c$. Then $\nabla f(x, y, z) = (xy, xz + 1, xy) = F(x, y, z)$. Hence, by second FTC for line integral
$$\int_C \nabla f \cdot dR = f(2, 1, 4) - f(1, 0, 0).$$
   That is, the given line integral is path independent. **Note that** one can $\nabla f = F$ for f by doing indefinite integral.

4. Use Green's Theorem to compute $\int\limits_C (2x^2 - y^2)dx + (x^2 + y^2)dy$ where C is the boundary of the region $\{(x, y) : x, y \geq 0 \text{ and } x^2 + y^2 \leq 1\}$.

   Solution: Let $M(x, y) = 2x^2 - y^2$ and $N = x^2 + y^2$. Then $N_x - M_y = 2(x + y)$. Let $D = \{(x, y) : x, y \geq 0 \text{ and } x^2 + y^2 \leq 1\}$.



Note that C is a simple closed and piece wise smooth curve, as shown in Figure 1. Then by Green's theorem

$$\int_C Mdx + Ndy = \iint\limits_D 2(x+y)dxdy.$$

Use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, where $0 < r \leq 1$ and $0 \leq \theta \leq \frac{\pi}{2}$.

5. If C is any simple closed and smooth curve in $\mathbb{R}^2$ which is not passing through the point $(1,0)$, then evaluate the integral $\int\limits_C \frac{-ydx+(x-1)dy}{(x-1)^2+y^2}$.

Solution: Let $M(x,y) = -\frac{y}{(x-1)^2+y^2}$ and $N(x,y) = \frac{x-1}{(x-1)^2+y^2}$. Note that M and N are not continuous at $(0,0)$. Let C_1 be a circle of radius r centered at $(1,0)$ which is in the interior of domain D enclosed by C. Please see Figure 2.

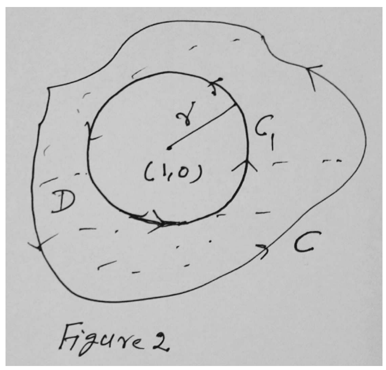


Figure 2

A simple calculation shows that $N_x - M_y = 0$ on D. By Green'n theorem for multiply connected domain,

$$\int_C Mdx + Ndy - \int_{C_1} Mdx + Ndy = \iint\limits_D (N_x - N_y)dxdy = 0.$$

Use the parametrization $x - 1 = r\cos t$ and $y = r\sin t,\ 0 \leq t < 2\pi$. Then

$$\int_C Mdx + Ndy = \int_{C_1} Mdx + Ndy = 2\pi.$$

6. Let $D = \{(x,y):\ x^2 + y^2 < 1\}$. If $f : D \to R^2$ is a continuously differentiable function such that $\int\limits_\Gamma f \cdot dR = 0$ for every curve Γ in D, then f constant.

Solution: Let $P = (x,y)$ be an arbitrary point in D. Then there exists a smooth curve Γ connecting origin and P as shown in Figure 3.

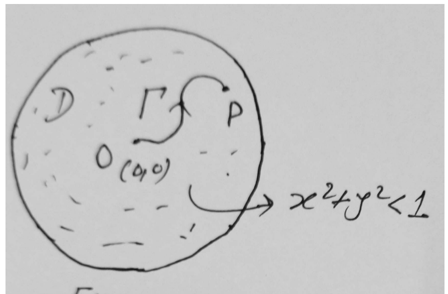


Figure 3

Let $\Gamma = \{R(t) : \ a \le t \le s\}$ and $f = (f_1, f_2)$. Then by the given condition

$$\int_a^s f(R(t)) \cdot R'(t)dt = 0.$$

Since point P is arbitrary, the above condition holds for every choice of $s \le a$. Hence, $f(R(t)) \cdot R'(t) = 0$ for all $t \in [a, s]$. Choose $R(t) = (t - x, (t - y)^2)$ then $R'(s) = (1, 0)$. This implies $f_1(P) = f_1(R(s)) = 0$. Similarly, we can select $R(t)$ such that $R'(s) = (0, 1)$. Hence, $f_2(P) = 0$. Thus, $f = 0$ on D.

7. Use Stokes' Theorem to evaluate the line integral $\int_C -y^3dx + x^3dy - z^3dz$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

   Solution: Let $S = S_1 \cup S_2$, as shown in Figure 4. Note that S_1 is the surface of part of the cylinder whereas S_2 is the base of the cylinder.

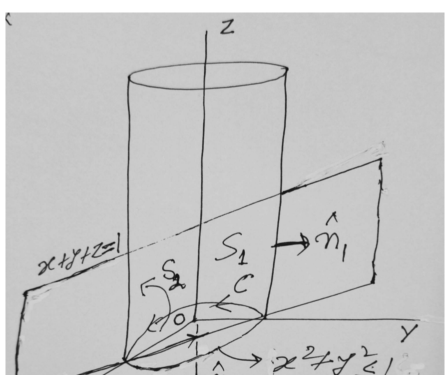


Figure 4

Let $F(x,y,z) = (-y^3, x^3, -z^3)$. Then by stoke's theorem,

$$\oint_C F \cdot dR = \iint_S \text{curl } F \cdot \hat{n}\, d\sigma = \left(\iint_{S_1} + \iint_{S_2} \right) \text{curl } F \cdot \hat{n}\, d\sigma.$$

Note that curl $F(x,y,z) = 3(x^2+y^2)k$. Unit vector $\hat{n}_1$ on S_1 is given by $\hat{n}_1 = \frac{r_\alpha \times r_\beta}{\|r_\alpha \times r_\beta\|}$, and $d\sigma_1(\alpha,\beta) = \|r_\alpha \times r_\beta\|\, d\alpha\, d\beta$, where $r(\alpha,\beta) = (\cos\alpha, \sin\beta, \beta)$ with $0 \le \alpha < 2\pi$, and $0 \le \beta \le 1 - \cos\alpha - \sin\alpha$. Hence

$$\iint_{S_1} \text{curl } F \cdot \hat{n}_1\, d\sigma = \int_{\alpha=0}^{2\pi} \int_{\beta=0}^{1-\cos\alpha-\sin\alpha} (3k) \cdot (r_\alpha \times r_\beta) d\alpha d\beta.$$

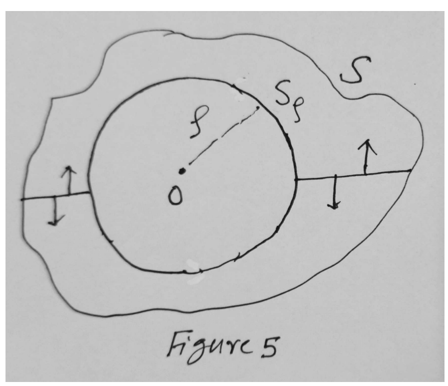
Further, unit vector $\hat{n}_2$ on S_2 is given by $\hat{n}_2(\alpha,\beta) = -k$ and $d\sigma_2(x,y) = dxdy$. Thus,

$$\iint_{S_2} \text{curl } F \cdot \hat{n}_2\, d\sigma = \iint_R -3(x^2+y^2) dxdy,$$

where R is the region $\{(x,y) : x^2+y^2 \le 1 \text{ and } x+y \le 1\}$.

8. Let $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and let S be any surface that surrounds the origin. Prove that $\iint_S \vec{F} \cdot \hat{n} d\sigma = 4\pi$.

   Solution: Note that $\vec{F}$ is not continuous at the origin O. Let S_ρ be the sphere of radius ρ centered at O so that S_ρ is in the interior of the domain D enclosed by S, as shown in Figure 5. A simple calculation show that div $F = 0$ on D.



By divergence theorem, we get

$$\iint_S \vec{F} \cdot \hat{n} d\sigma - \iint_{S_\rho} \vec{F} \cdot \hat{n} d\sigma = \iint_D \text{div } \vec{F} dV = 0.$$

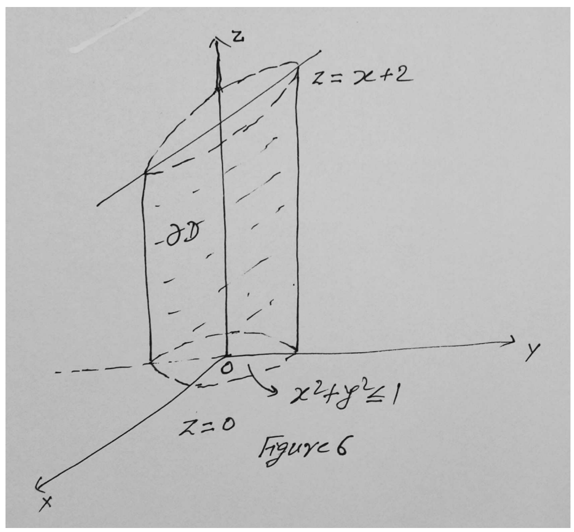
The surface of S_ρ can be represented by $F(x,y,z) = x^2+y^2+z^2-\rho^2$. Hence the unit vector on S_ρ is given by $\hat{n} = \frac{\nabla F}{\|\nabla F\|} = \frac{\vec{r}}{\rho}$. Note that the surface of S_ρ can be represented by

$v(\theta,\phi) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$. Hence surface element on S_ρ will be given by $d\sigma(\theta,\phi) = \|v_\theta \times v_\phi\| d\theta d\phi = \rho^2 \sin\phi d\theta d\phi$, where $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta, z = \rho\cos\phi$, $0 \le \theta < 2\pi$ and $0 \le \phi < \pi$. Thuis,

$$\iint_S \vec{F}\cdot\hat{n} d\sigma = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} \frac{\vec{r}}{|\vec{r}|^3}\cdot\frac{\vec{r}}{\rho}\rho^2\sin\phi d\theta d\phi = 4\pi.$$

9. Let D be the domain inside the cylinder $x^2+y^2=1$ cut off by the planes $z=0$ and $z=x+2$. If $\vec{F} = (x^2+ye^z, y^2+ze^x, z+xe^y)$, use divergence theorem to evaluate $\iint_{\partial D} F\cdot\hat{n}\, d\sigma$.

Solution: Please refer to Figure 6.



By divergence theorem, we get

$$\iint_{\partial D} F\cdot\hat{n}\, d\sigma = \iint_D \operatorname{div}\vec{F} dV = \iint_{x^2+y^2\le 1}\left(\int_{z=0}^{x+2}(2x+2y+1)dz\right)dxdy.$$

Use polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$.