Convergence of QR Algorithm

Invariant Subspaces

Consider $A : \mathbb{F}^n \mapsto \mathbb{F}^n$. A subspace $V \subset \mathbb{F}^n$ is said to be invariant with respect to A, if $Av \in V$ for all $v \in V$.

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Examples:

1. For $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$, $A\left(\mathrm{span}\{e_1, e_3\}\right) \subseteq \mathrm{span}\{e_1, e_3\}$.
2. For $A = \begin{bmatrix} 4 & 3 & -5 \\ 0 & -3 & 3 \\ 0 & -2 & 3 \end{bmatrix}$, $A(\mathrm{span}\{e_1\}) \subseteq \mathrm{span}\{e_1\}$ and $A\left(\mathrm{span}\{e_1 + e_2 + e_3, e_1 - e_2\}\right) \subseteq \mathrm{span}\{e_1 + e_2 + e_3, e_1 - e_2\}$.

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Facts:

1. The trivial subspaces $\mathbb{F}^n$ and $\{0\}$ are always invariant with respect to every $A \in \mathbb{F}^{n \times n}$.
2. $V \subseteq \mathbb{F}^n$ is a one dimensional subspace of $\mathbb{F}^n$ invariant with respect to $A \in \mathbb{F}^{n \times n}$ if and only if $V = \mathrm{span}\{v\}$ for some eigenvector v of A.
3. Eigenvectors of $A \in \mathbb{F}^{n \times n}$ span invariant subspaces.

Invariant Subspaces

Theorem: Let $A \in \mathbb{F}^{n \times n}$. The first k columns of an invertible $S \in \mathbb{F}^{n \times n}$ span a subspace of $\mathbb{F}^n$ invariant with respect to A if and only if

$$S^{-1}AS = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline & A_{22} \end{array}\right]$$

where $A_{11} \in \mathbb{F}^{k \times k}$, $A_{12} \in \mathbb{F}^{k \times n-k}$ and $A_{22} \in \mathbb{F}^{n-k \times n-k}$.

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Corollary: Let $A \in \mathbb{F}^{n \times n}$ and $S = [s_1 \ \cdots \ s_n] \in \mathbb{F}^{n \times n}$ be a invertible matrix. Then the first k columns of S span subspaces of $\mathbb{F}^n$ that are invariant with respect to A $\underline{\text{for each } k = 1, \ldots, n-1,}$ if and only if $S^{-1}AS$ is an upper triangular matrix.

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Schur's Theorem: Given $A \in \mathbb{C}^{n\times n}$, there exists an orthonormal basis $\{q_1, \ldots, q_n\}$ of $\mathbb{C}^n$ such that

$$A(\mathrm{span}\{q_1, \ldots, q_k\}) \subseteq \mathrm{span}\{q_1, \ldots, q_k\}$$

for each $k = 1, \ldots, n-1$.