ME101: Engineering Mechanics (3 1 0 8)

2021-2022 (II Semester)

KINEMATICS OF A PARTICLE

LECTURE: 22

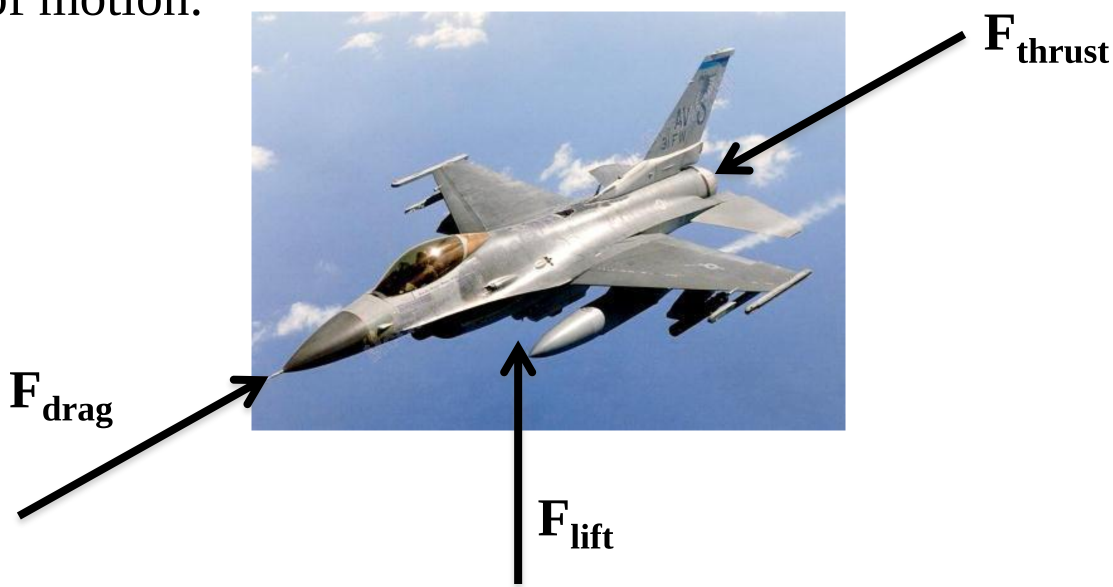
Contents

Kinematic relationships are used to help us determine the trajectory of a golf ball, the orbital speed of a satellite, and the accelerations during acrobatic flying.

Introduction

- **Dynamics includes:**

Kinematics: study of the geometry of motion.
Relates displacement, velocity, acceleration, and time ***without reference*** to the cause of motion.



Kinetics: study of the relations existing between the forces acting on a body, the mass of the body, and the motion of the body. Kinetics is used to predict the motion caused by given forces or to determine the forces required to produce a given motion.

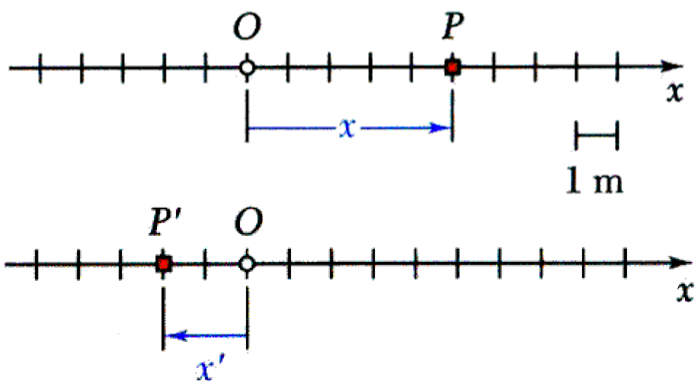
Introduction

- **Particle kinetics includes:**

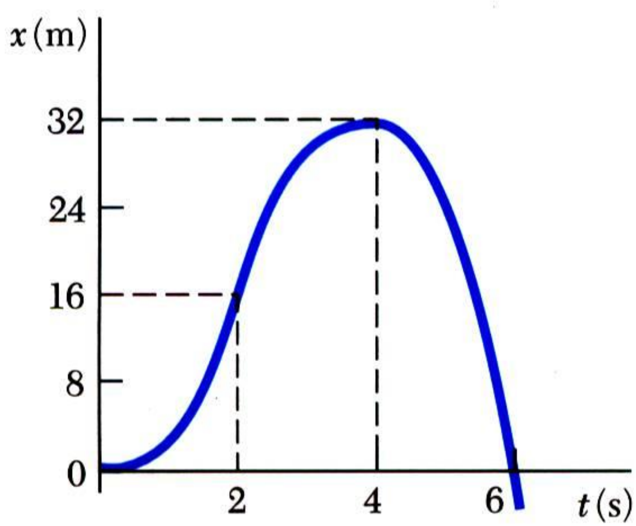
  - ***<u>Rectilinear motion</u>***: position, velocity, and acceleration of a particle as it moves along a straight line.

- ***<u>Curvilinear motion</u>***: position, velocity, and acceleration of a particle as it moves along a curved line in two or three dimensions.

Rectilinear Motion: Position, Velocity & Acceleration



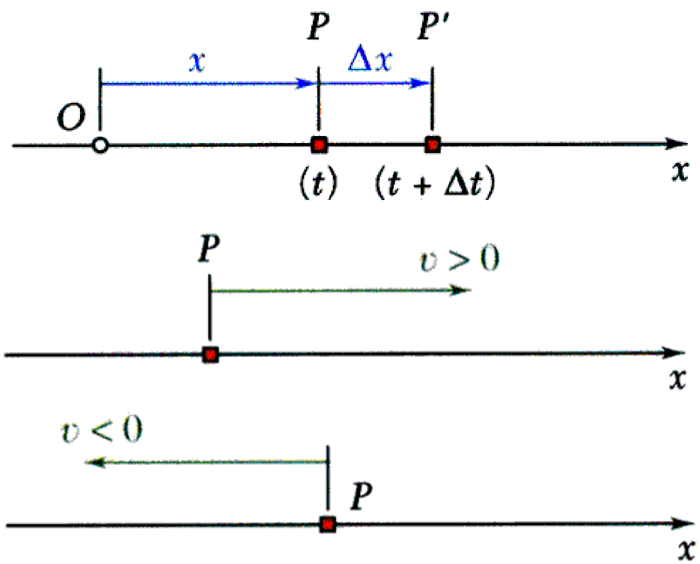
- ***Rectilinear motion:*** particle moving along a straight line
- ***Position coordinate:*** defined by positive or negative distance from a fixed origin on the line.

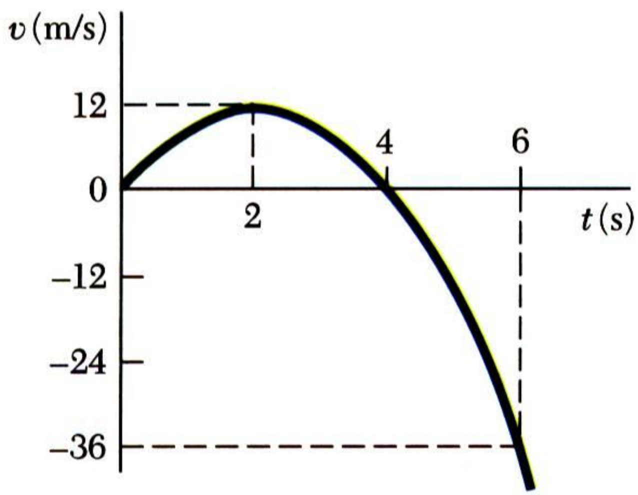


- The ***motion*** of a particle is known if the position coordinate for particle is known for every value of time *t*.
- May be expressed in the form of a function, e.g., $x = 6t^2 - t^3$

or in the form of a graph *x* vs. *t*.

Rectilinear Motion: Position, Velocity & Acceleration





- Consider particle which occupies position P at time t and P' at $t+\Delta t$,

$$\textit{Average velocity} = \frac{\Delta x}{\Delta t}$$

$$\textit{Instantaneous velocity} = v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

- Instantaneous velocity may be positive or negative. Magnitude of velocity is referred to as *particle speed*.

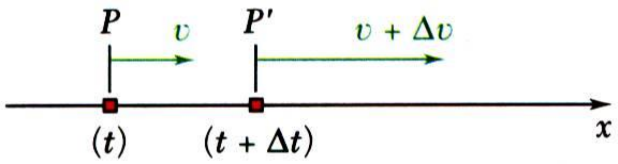
- From the definition of a derivative,

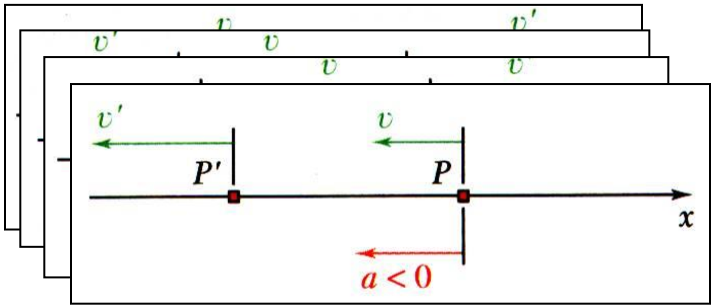
$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

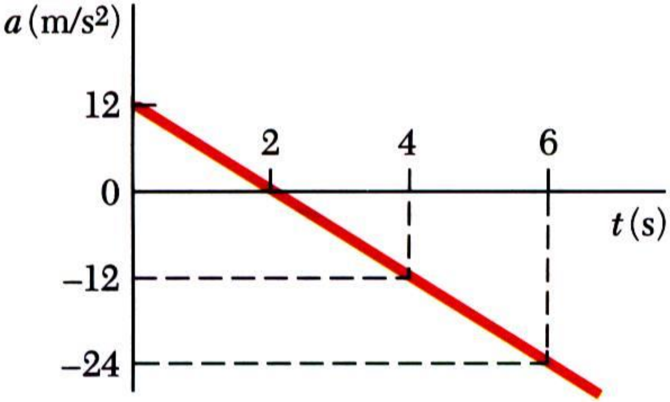
e.g., $x = 6t^2 - t^3$

$$v = \frac{dx}{dt} = 12t - 3t^2$$

Rectilinear Motion: Position, Velocity & Acceleration







- Consider particle with velocity v at time t and v' at $t+\Delta t$,

$$\textit{Instantaneous acceleration} = a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- Instantaneous acceleration may be:
  - positive: increasing positive velocity or decreasing negative velocity
  - negative: decreasing positive velocity or increasing negative velocity.

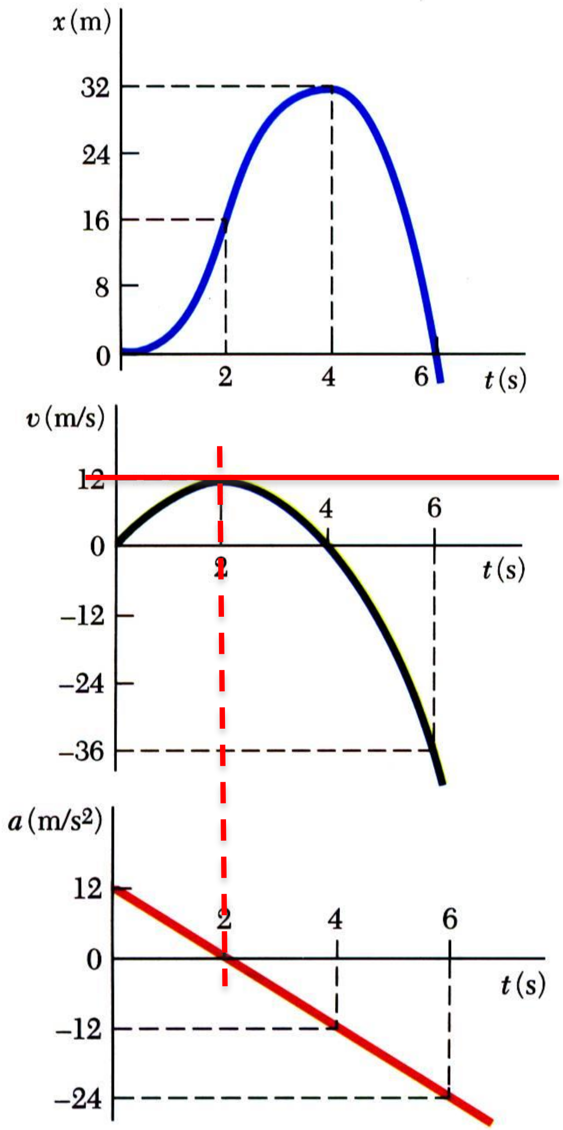
- From the definition of a derivative,

$$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt} = \frac{d^2 x}{dt^2}$$

e.g. $v = 12t - 3t^2$

$$a = \frac{dv}{dt} = 12 - 6t$$

Rectilinear Motion: Position, Velocity & Acceleration



- From our example,

$$x = 6t^2 - t^3$$

$$v = \frac{dx}{dt} = 12t - 3t^2$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = 12 - 6t$$

- What are *x*, *v*, and *a* at *t* = 2 s ?

- at $t = 2$ s, $x = 16$ m, $v = v_{max} = 12$ m/s, $a = 0$

- Note that v_{max} occurs when a=0, and that the slope of the velocity curve is zero at this point.
- What are *x*, *v*, and *a* at *t* = 4 s ?

- at $t = 4$ s, $x = x_{max} = 32$ m, $v = 0$, $a = -12$ m/s^2

Determination of the Motion of a Particle

- **We often determine accelerations from the forces applied (kinetics will be covered later)**
- **Generally have three classes of motion**
  - acceleration given as a function of *time*, $a = f(t)$
  - acceleration given as a function of *position*, $a = f(x)$
  - acceleration given as a function of *velocity*, $a = f(v)$

- **Can you think of a physical example of when force is a function of position? When force is a function of velocity?**

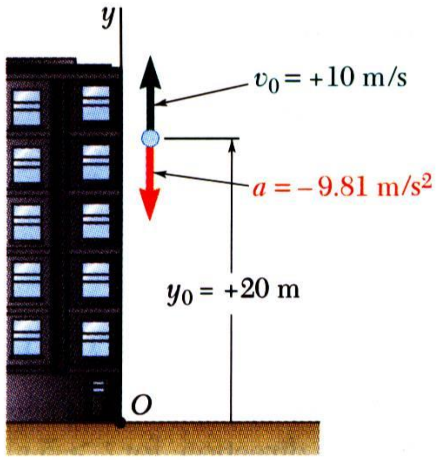
a spring

drag

Acceleration as a function of time, position, or velocity

If….	Kinematic relationship	Integrate
$a = a(t)$	$\dfrac{dv}{dt} = a(t)$	$\displaystyle\int_{v_0}^{v} dv = \int_{0}^{t} a(t)\,dt$
$a = a(x)$	$dt = \dfrac{dx}{v}$ and $a = \dfrac{dv}{dt}$ $\downarrow$ $v\,dv = a(x)\,dx$	$\displaystyle\int_{v_0}^{v} v\,dv = \int_{x_0}^{x} a(x)\,dx$
$a = a(v)$	$\dfrac{dv}{dt} = a(v)$	$\displaystyle\int_{v_0}^{v} \frac{dv}{a(v)} = \int_{0}^{t} dt$
	$v\dfrac{dv}{dx} = a(v)$	$\displaystyle\int_{x_0}^{x} dx = \int_{v_0}^{v} \frac{v\,dv}{a(v)}$

Sample Problem



Ball tossed with 10 m/s vertical velocity from window 20 m above ground.

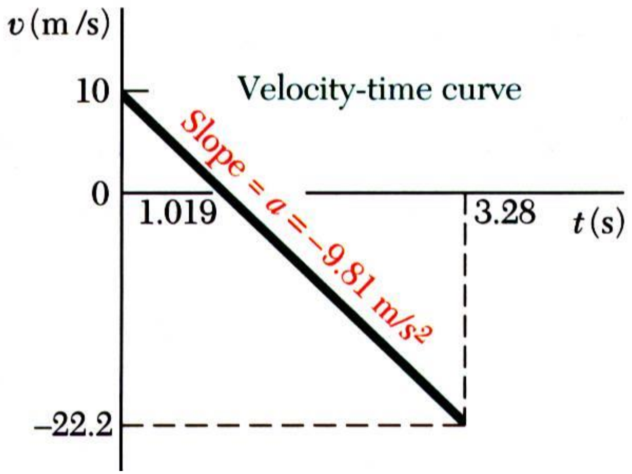
Determine:

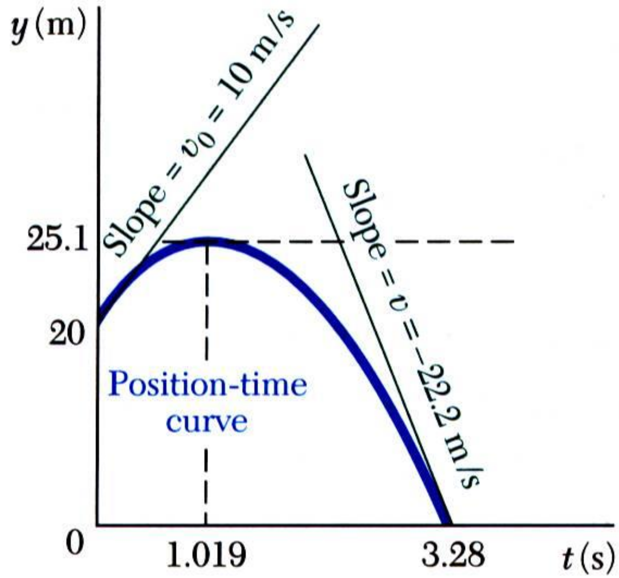
- velocity and elevation above ground at time *t*,
- highest elevation reached by ball and corresponding time, and
- time when ball will hit the ground and corresponding velocity.

SOLUTION:

- Integrate twice to find $v(t)$ and $y(t)$.
- Solve for *t* when velocity equals zero (time for maximum elevation) and evaluate corresponding altitude.
- Solve for *t* when altitude equals zero (time for ground impact) and evaluate corresponding velocity.

Sample Problem





SOLUTION:

- Integrate twice to find $v(t)$ and $y(t)$.

$$\frac{dv}{dt} = a = -9.81\,\mathrm{m/s^2}$$

$$\int_{v_0}^{v(t)} dv = -\int_0^t 9.81\,dt \qquad v(t) - v_0 = -9.81t$$

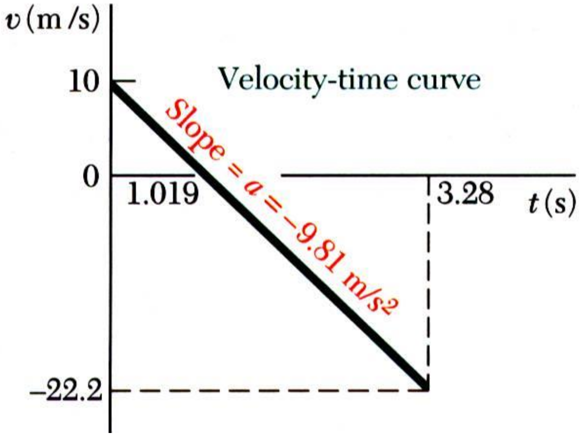
$$\boxed{v(t) = 10\,\frac{\mathrm{m}}{\mathrm{s}} - \left(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t}$$

$$\frac{dy}{dt} = v = 10 - 9.81t$$

$$\int_{y_0}^{y(t)} dy = \int_0^t (10 - 9.81t)\,dt \qquad y(t) - y_0 = 10t - \tfrac{1}{2}9.81t^2$$

$$\boxed{y(t) = 20\,\mathrm{m} + \left(10\,\frac{\mathrm{m}}{\mathrm{s}}\right)t - \left(4.905\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2}$$

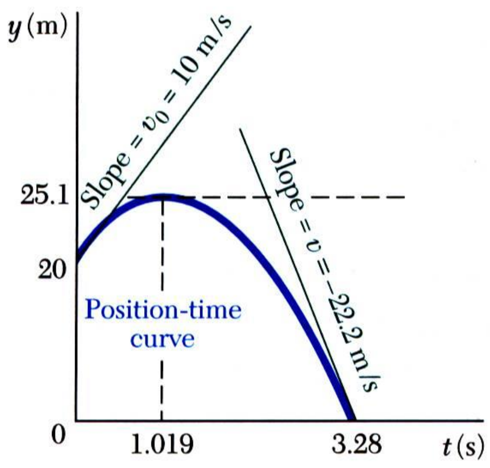
Sample Problem



- Solve for t when velocity equals zero and evaluate corresponding altitude.

$$v(t) = 10\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t = 0$$

$$\boxed{t = 1.019\,\text{s}}$$



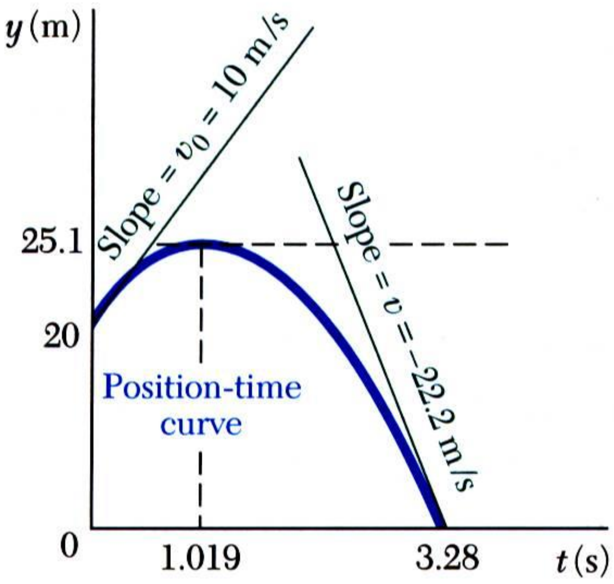
- Solve for t when altitude equals zero and evaluate corresponding velocity.

$$y(t) = 20\,\text{m} + \left(10\frac{\text{m}}{\text{s}}\right)t - \left(4.905\frac{\text{m}}{\text{s}^2}\right)t^2$$

$$y = 20\,\text{m} + \left(10\frac{\text{m}}{\text{s}}\right)(1.019\,\text{s}) - \left(4.905\frac{\text{m}}{\text{s}^2}\right)(1.019\,\text{s})^2$$

$$\boxed{y = 25.1\,\text{m}}$$

Sample Problem



- Solve for *t* when altitude equals zero and evaluate corresponding velocity.

$$y(t) = 20\,\mathrm{m} + \left(10\,\frac{\mathrm{m}}{\mathrm{s}}\right)t - \left(4.905\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2 = 0$$

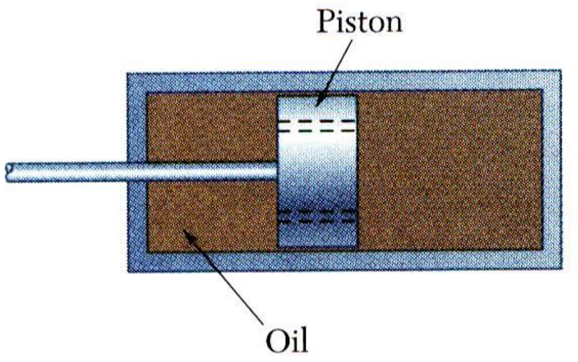
$$\boxed{\begin{aligned} t &= -1.243\,\mathrm{s} \ \ (\mathrm{meaningless}) \\ t &= 3.28\,\mathrm{s} \end{aligned}}$$

$$v(t) = 10\,\frac{\mathrm{m}}{\mathrm{s}} - \left(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)t$$

$$v(3.28\,\mathrm{s}) = 10\,\frac{\mathrm{m}}{\mathrm{s}} - \left(9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}\right)(3.28\,\mathrm{s})$$

$$\boxed{v = -22.2\,\frac{\mathrm{m}}{\mathrm{s}}}$$

Sample Problem



$a = -kv$

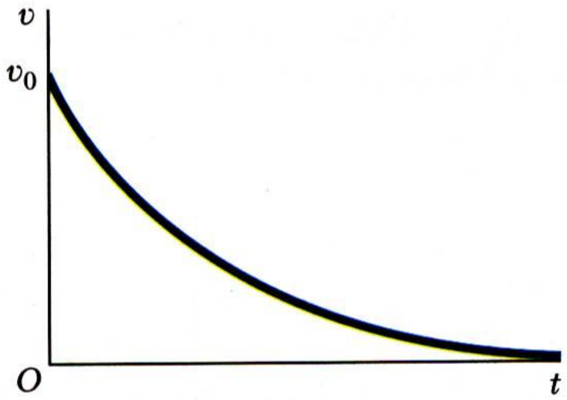
Brake mechanism used to reduce gun recoil consists of piston attached to barrel moving in fixed cylinder filled with oil. As barrel recoils with initial velocity v_0, piston moves and oil is forced through orifices in piston, causing piston and cylinder to decelerate at rate proportional to their velocity.

Determine $v(t)$, $x(t)$, and $v(x)$.

SOLUTION:

- Integrate $a = dv/dt = -kv$ to find $v(t)$.
- Integrate $v(t) = dx/dt$ to find $x(t)$.
- Integrate $a = v\, dv/dx = -kv$ to find $v(x)$.

Sample Problem

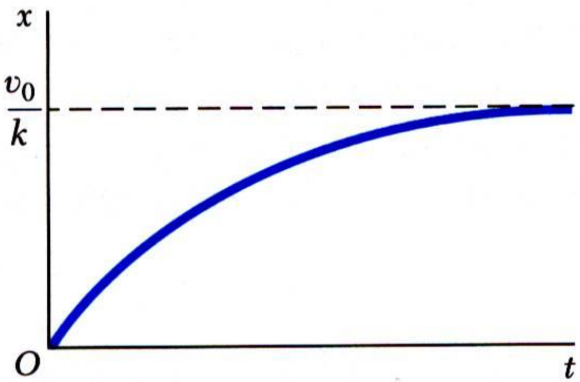


SOLUTION:

- Integrate $a = dv/dt = -kv$ to find $v(t)$.

$$a = \frac{dv}{dt} = -kv \qquad \int_{v_0}^{v} \frac{dv}{v} = -k\int_0^t dt \qquad \ln\frac{v(t)}{v_0} = -kt$$

$$\boxed{v(t) = v_0 e^{-kt}}$$



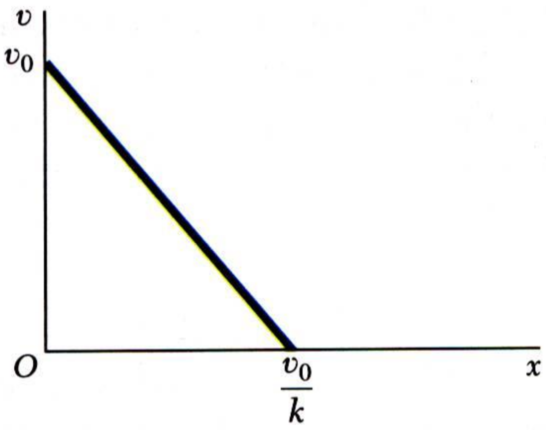
- Integrate $v(t) = dx/dt$ to find $x(t)$.

$$v(t) = \frac{dx}{dt} = v_0 e^{-kt}$$

$$\int_0^x dx = v_0 \int_0^t e^{-kt} dt \qquad x(t) = v_0\left[-\frac{1}{k}e^{-kt}\right]_0^t$$

$$\boxed{x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)}$$

Sample Problem



- Integrate $a = v\,dv/dx = -kv$ to find $v(x)$.

$$a = v\frac{dv}{dx} = -kv \qquad dv = -k\,dx \qquad \int_{v_0}^{v} dv = -k\int_{0}^{x} dx$$

$$v - v_0 = -kx$$

$$\boxed{v = v_0 - kx}$$

- Alternatively,

with $\quad x(t) = \dfrac{v_0}{k}\left(1 - e^{-kt}\right)$

and $\quad v(t) = v_0 e^{-kt}$ or $e^{-kt} = \dfrac{v(t)}{v_0}$

then $\quad x(t) = \dfrac{v_0}{k}\left(1 - \dfrac{v(t)}{v_0}\right)$

$$\boxed{v = v_0 - kx}$$

Uniform Rectilinear Motion

During free-fall, a parachutist reaches terminal velocity when her weight equals the drag force. If motion is in a straight line, this is uniform rectilinear motion.

For a particle in uniform rectilinear motion, the acceleration is zero and the velocity is constant.

$$\frac{dx}{dt} = v = \text{constant}$$

$$\int_{x_0}^{x} dx = v \int_{0}^{t} dt$$

$$x - x_0 = vt$$

$$x = x_0 + vt$$

Careful – these only apply to uniform rectilinear motion!

Uniformly Accelerated Rectilinear Motion

If forces applied to a body are constant (and in a constant direction), then you have uniformly accelerated rectilinear motion.

Another example is free-fall when drag is negligible

Uniformly Accelerated Rectilinear Motion

For a particle in uniformly accelerated rectilinear motion, the acceleration of the particle is constant. You may recognize these constant acceleration equations from your physics courses.

$$\frac{dv}{dt} = a = \text{constant} \qquad \int_{v_0}^{v} dv = a\int_0^t dt \qquad v = v_0 + at$$

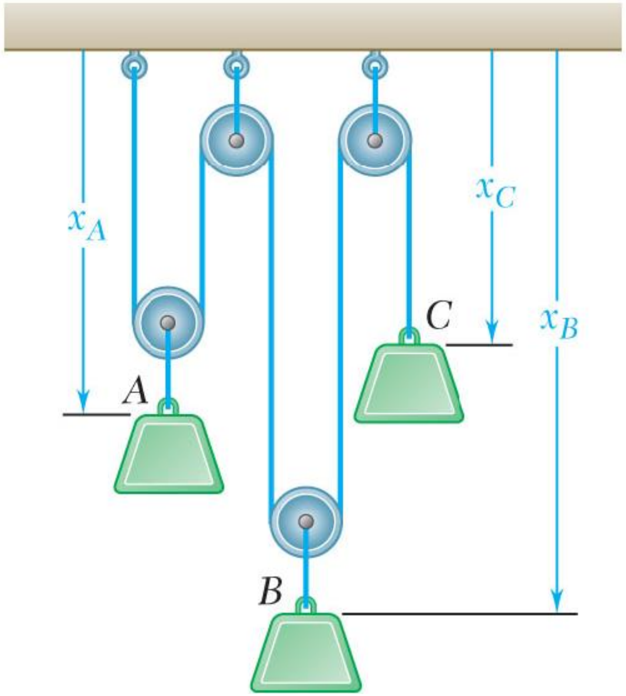
$$\frac{dx}{dt} = v_0 + at \qquad \int_{x_0}^{x} dx = \int_0^t \left(v_0 + at\right) dt \qquad x = x_0 + v_0 t + \tfrac{1}{2} a t^2$$

$$v\frac{dv}{dx} = a = \text{constant} \qquad \int_{v_0}^{v} v\, dv = a\int_{x_0}^{x} dx \qquad v^2 = v_0^2 + 2a\left(x - x_0\right)$$

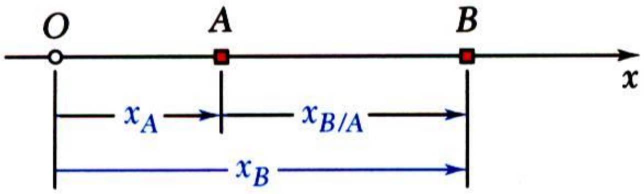
Careful – these only apply to uniformly accelerated rectilinear motion!

Motion of Several Particles

We may be interested in the motion of several different particles, whose motion may be independent or linked together.



Motion of Several Particles: Relative Motion



- For particles moving along the same line, time should be recorded from the same starting instant and displacements should be measured from the same origin in the same direction.

$$x_{B/A} = x_B - x_A = \text{ relative position of } B \text{ with respect to } A$$

$$x_B = x_A + x_{B/A}$$

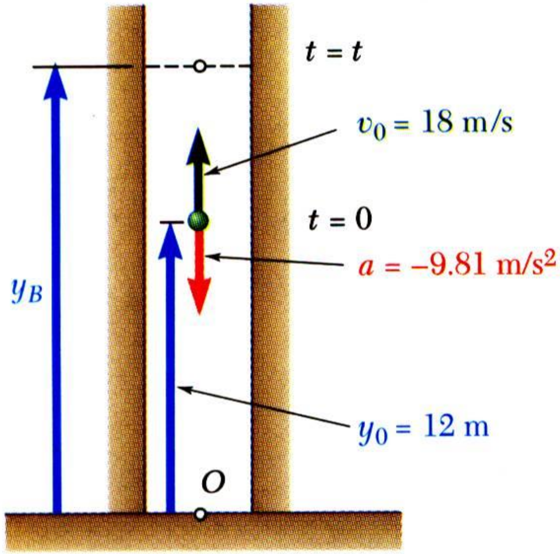
$$v_{B/A} = v_B - v_A = \text{ relative velocity of } B \text{ with respect to } A$$

$$v_B = v_A + v_{B/A}$$

$$a_{B/A} = a_B - a_A = \text{ relative acceleration of } B \text{ with respect to } A$$

$$a_B = a_A + a_{B/A}$$

Sample Problem



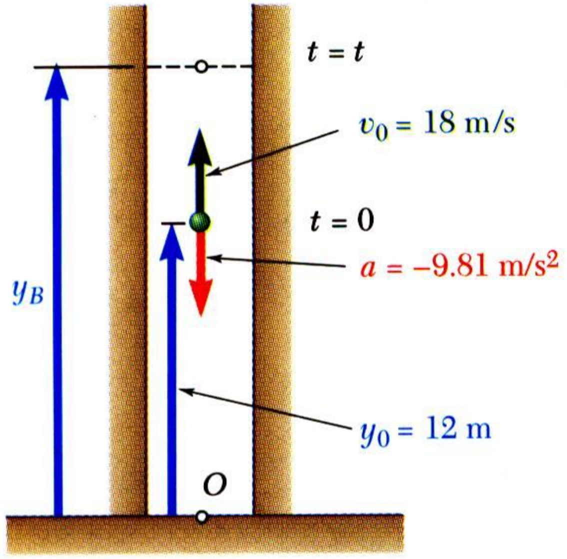
Ball thrown vertically from 12 m level in elevator shaft with initial velocity of 18 m/s. At same instant, open-platform elevator passes 5 m level moving upward at 2 m/s.

Determine *(a)* when and where ball hits elevator and *(b)* relative velocity of ball and elevator at contact.

SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.
- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.
- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.
- Substitute impact time into equation for position of elevator and relative velocity of ball with respect to elevator.

Sample Problem

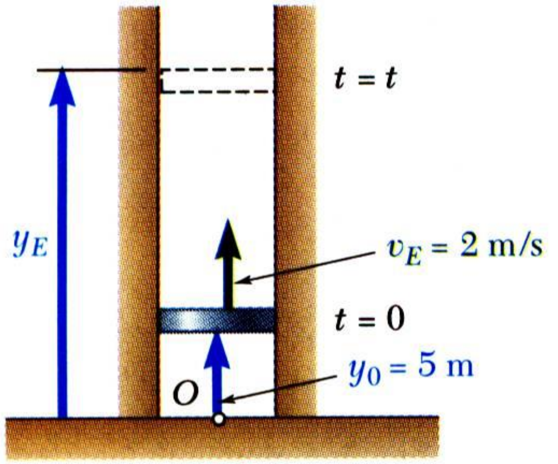


SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

$$v_B = v_0 + at = 18\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t$$

$$y_B = y_0 + v_0 t + \tfrac{1}{2}at^2 = 12\,\text{m} + \left(18\frac{\text{m}}{\text{s}}\right)t - \left(4.905\frac{\text{m}}{\text{s}^2}\right)t^2$$

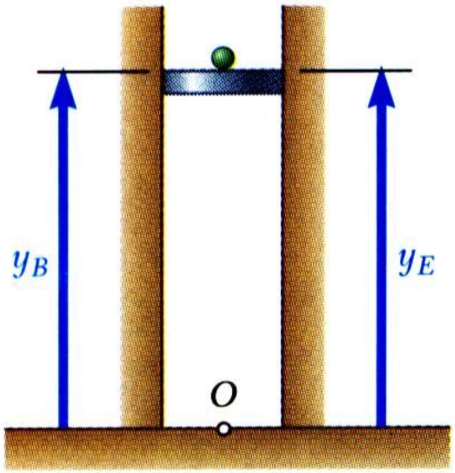


- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

$$v_E = 2\frac{\text{m}}{\text{s}}$$

$$y_E = y_0 + v_E t = 5\,\text{m} + \left(2\frac{\text{m}}{\text{s}}\right)t$$

Sample Problem



- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

$$y_{B/E} = \left(12 + 18t - 4.905t^2\right) - (5 + 2t) = 0$$

$$\boxed{\begin{aligned} t &= -0.39\,\text{s} \ \ (\text{meaningless}) \\ t &= 3.65\,\text{s} \end{aligned}}$$

- Substitute impact time into equations for position of elevator and relative velocity of ball with respect to elevator.

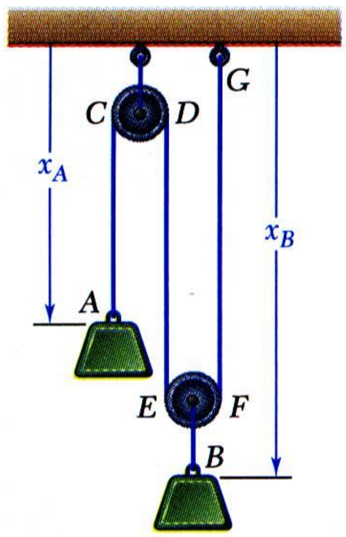
$$y_E = 5 + 2(3.65)$$

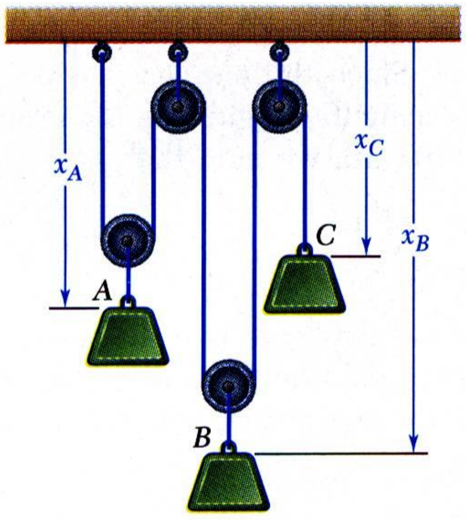
$$\boxed{y_E = 12.3\,\text{m}}$$

$$\begin{aligned} v_{B/E} &= (18 - 9.81t) - 2 \\ &= 16 - 9.81(3.65) \end{aligned}$$

$$\boxed{v_{B/E} = -19.81\,\frac{\text{m}}{\text{s}}}$$

Motion of Several Particles: Dependent Motion





- Position of a particle may *depend* on position of one or more other particles.

- Position of block *B* depends on position of block *A*. Since rope is of constant length, it follows that sum of lengths of segments must be constant.

$$x_A + 2x_B = \text{ constant (one degree of freedom)}$$

- Positions of three blocks are dependent.

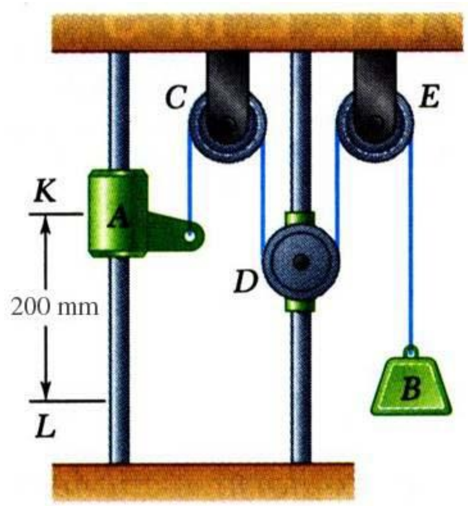
$$2x_A + 2x_B + x_C = \text{constant (two degrees of freedom)}$$

- For linearly related positions, similar relations hold between velocities and accelerations.

$$2\frac{dx_A}{dt} + 2\frac{dx_B}{dt} + \frac{dx_C}{dt} = 0 \quad \text{or} \quad 2v_A + 2v_B + v_C = 0$$

$$2\frac{dv_A}{dt} + 2\frac{dv_B}{dt} + \frac{dv_C}{dt} = 0 \quad \text{or} \quad 2a_A + 2a_B + a_C = 0$$

Sample Problem

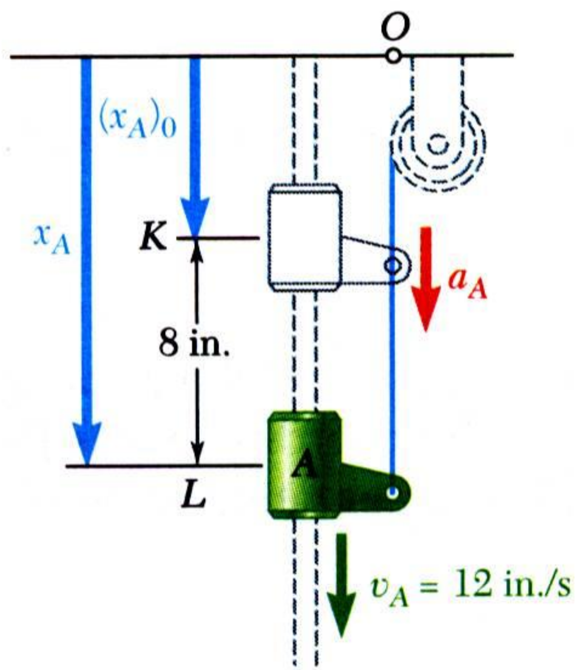


Pulley *D* is attached to a collar which is pulled down at 75 mm/s. At $t = 0$, collar *A* starts moving down from *K* with constant acceleration and zero initial velocity. Knowing that velocity of collar *A* is 300 mm/s as it passes *L*, determine the change in elevation, velocity, and acceleration of block *B* when block *A* is at *L*.

SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.
- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.
- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.
- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.
- Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.

Sample Problem



SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.

- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.

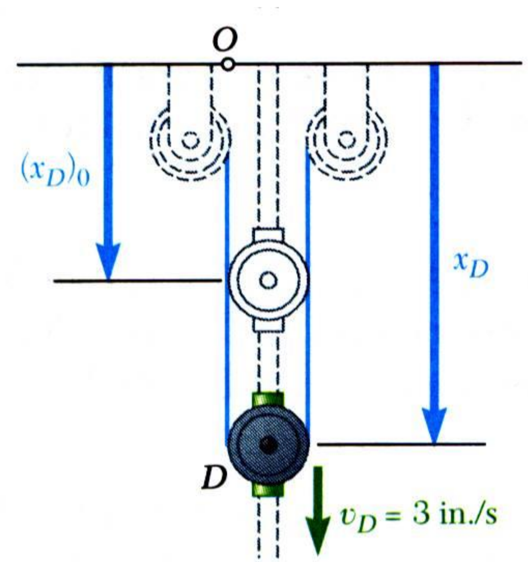
$$v_A^2 = (v_A)_0^2 + 2a_A\left[x_A - (x_A)_0\right]$$

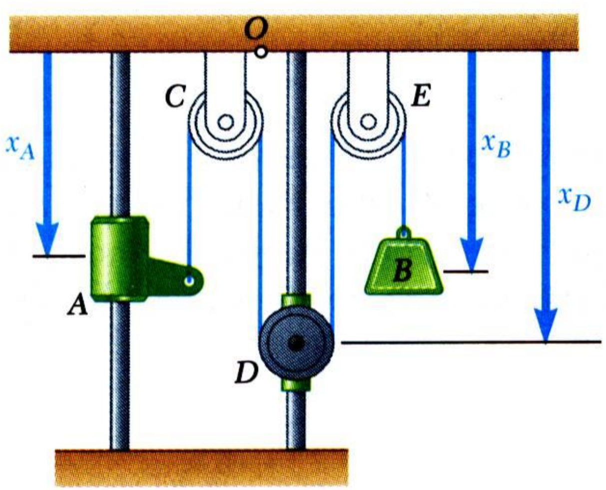
$$\left(300\frac{\text{mm}}{\text{s}}\right)^2 = 2a_A(200\,\text{mm}) \qquad a_A = 225\frac{\text{mm}}{\text{s}^2}$$

$$v_A = v_{A\ 0} + a_A t$$

$$300\frac{\text{mm}}{\text{s}} = 225\frac{\text{mm}}{\text{s}^2}t \qquad t = 1.333\ \text{s}$$

Sample Problem





- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.

$$x_D = (x_D)_0 + v_D t$$

$$x_D - (x_D)_0 = \left(75\,\frac{\text{mm}}{\text{s}}\right)(1.333\text{s}) = 100 \text{ mm}$$

- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.

Total length of cable remains constant,

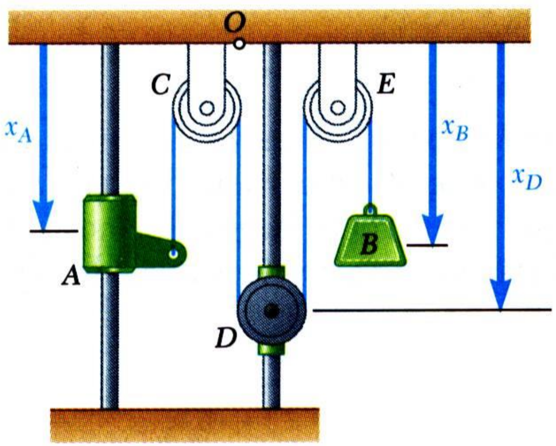
$$x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0$$

$$\left[x_A - (x_A)_0\right] + 2\left[x_D - (x_D)_0\right] + \left[x_B - (x_B)_0\right] = 0$$

$$(200\,\text{mm}) + 2(100\,\text{mm}) + \left[x_B - (x_B)_0\right] = 0$$

$$\boxed{x_B - (x_B)_0 = -400\,\text{mm}}$$

Sample Problem



- Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.

$$x_A + 2x_D + x_B = \text{constant}$$

$$v_A + 2v_D + v_B = 0$$

$$\left(300\frac{\text{mm}}{\text{s}}\right) + 2\left(75\frac{\text{mm}}{\text{s}}\right) + v_B = 0$$

$$\boxed{v_B = -450\frac{\text{mm}}{\text{s}} = 450\frac{\text{mm}}{\text{s}}}$$

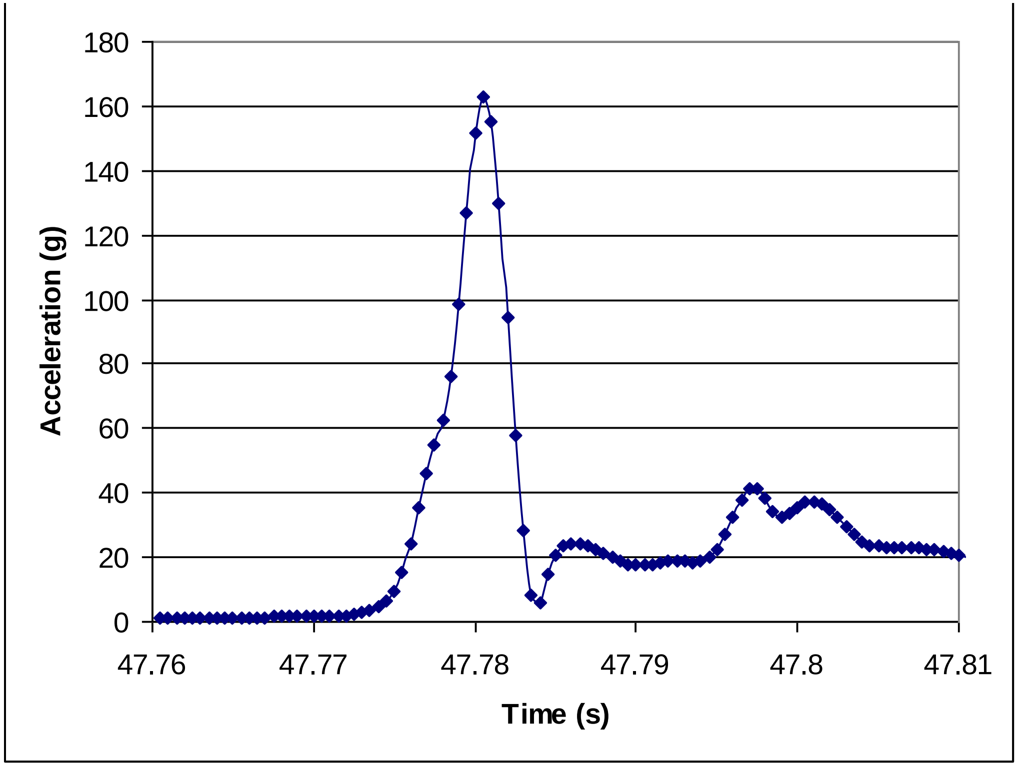
$$a_A + 2a_D + a_B = 0$$

$$\left(225\frac{\text{mm}}{\text{s}^2}\right) + 2(0) + a_B = 0$$

$$\boxed{a_B = -225\frac{\text{mm}}{\text{s}^2} = 225\frac{\text{mm}}{\text{s}^2}}$$

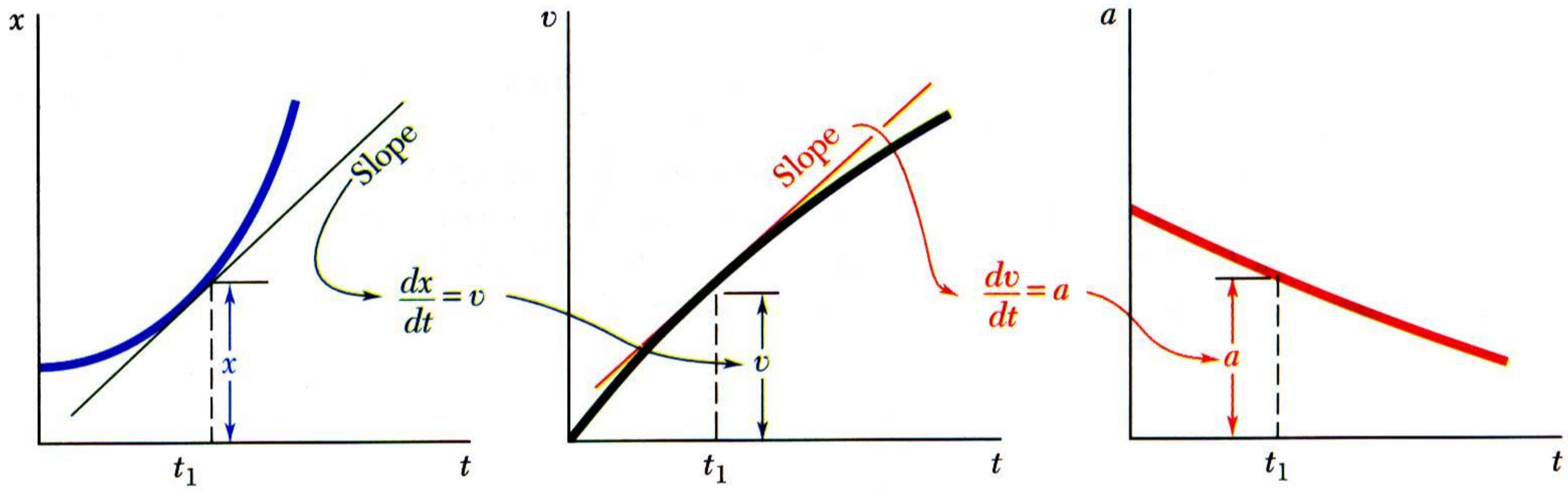
Graphical Solution of Rectilinear-Motion Problems

Engineers often collect position, velocity, and acceleration data. Graphical solutions are often useful in analyzing these data.



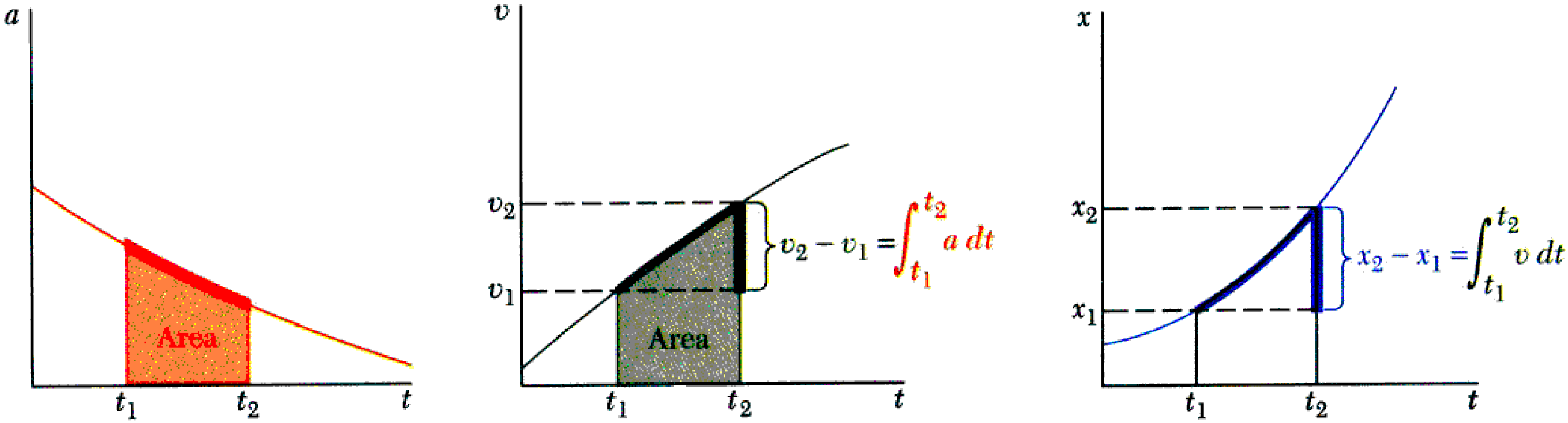
Acceleration data from a head impact during a round of boxing.

Graphical Solution of Rectilinear-Motion Problems



- **Given the *x-t* curve, the *v-t* curve is equal to the *x-t* curve slope.**
- **Given the *v-t* curve, the *a-t* curve is equal to the *v-t* curve slope.**

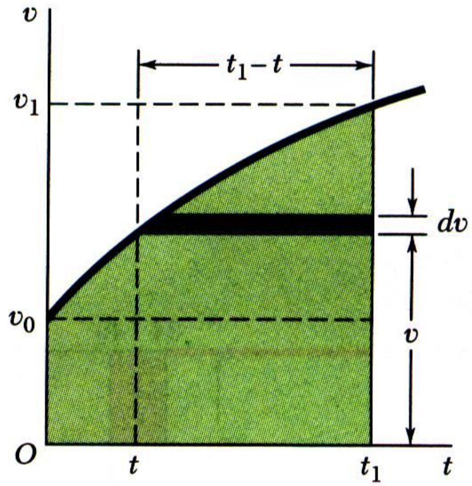
Graphical Solution of Rectilinear-Motion Problems

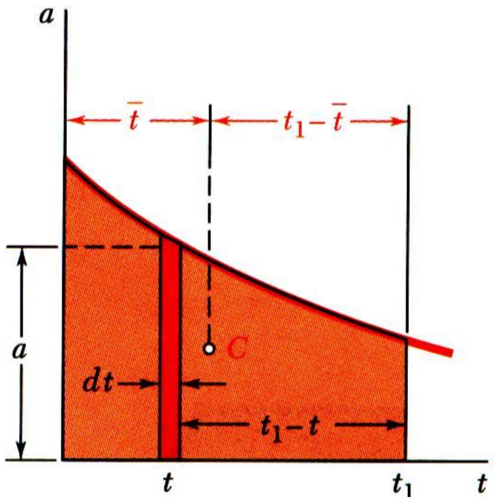


- **Given the *a*-*t* curve, the change in velocity between t_1 and t_2 is equal to the area under the *a*-*t* curve between t_1 and t_2.**

- **Given the *v*-*t* curve, the change in position between t_1 and t_2 is equal to the area under the *v*-*t* curve between t_1 and t_2.**

Other Graphical Methods





- *Moment-area method* to determine particle position at time t directly from the a-t curve:

$$x_1 - x_0 = \text{area under } v - t \text{ curve}$$

$$= v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t) dv$$

using $dv = a\,dt$,

$$x_1 - x_0 = v_0 t_1 + \int_{v_0}^{v_1} (t_1 - t) a\,dt$$

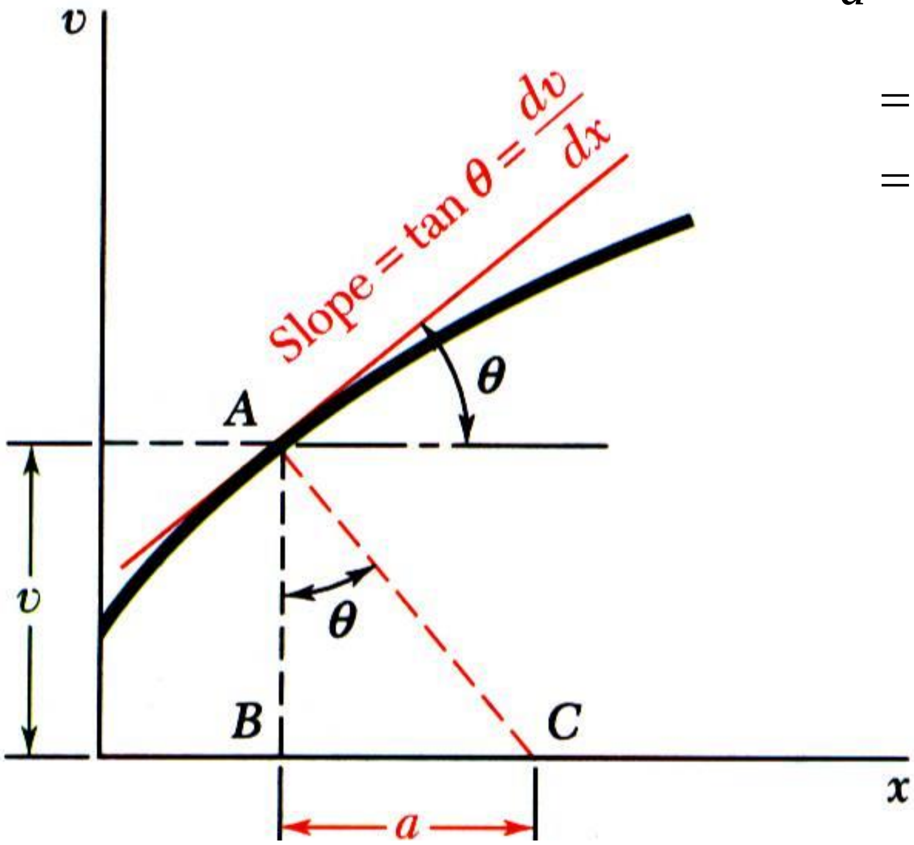
$$\int_{v_0}^{v_1} (t_1 - t) a\,dt = \text{ first moment of area under } a\text{-}t \text{ curve}$$
with respect to $t = t_1$ line.

$$x_1 = x_0 + v_0 t_1 + (\text{area under } a\text{-}t \text{ curve})(t_1 - \bar{t})$$

$$\bar{t} = \text{abscissa of centroid } C$$

Other Graphical Methods

- **Method to determine particle acceleration from *v-x* curve:**



$$a = v\frac{dv}{dx}$$
$$= AB\tan\theta$$
$$= BC = \textit{subnormal} \text{ to } v\text{-}x \text{ curve}$$

Curvilinear Motion: Position, Velocity & Acceleration

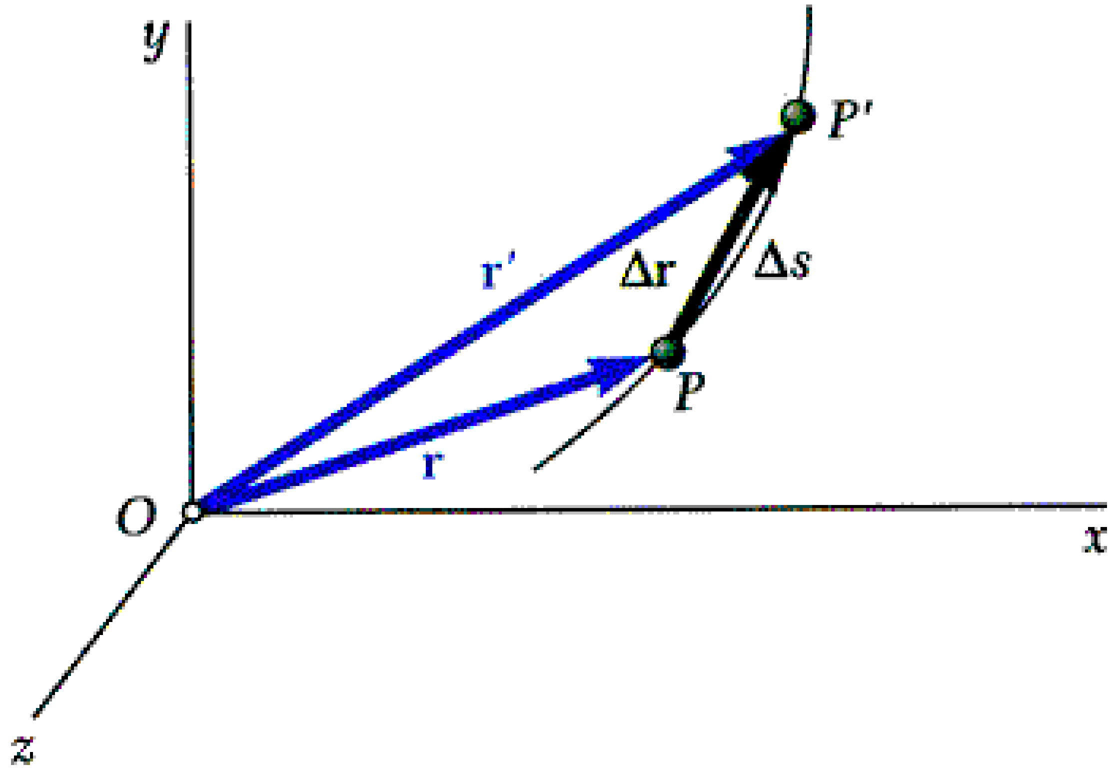
The softball and the car both undergo curvilinear motion.

- A particle moving along a curve other than a straight line is in *curvilinear motion*.

Curvilinear Motion: Position, Velocity & Acceleration

- The *position vector* of a particle at time t is defined by a vector between origin O of a fixed reference frame and the position occupied by particle.

- Consider a particle which occupies position P defined by $\vec{r}$ at time t and P' defined by $\vec{r}\,'$ at $t + \Delta t$,



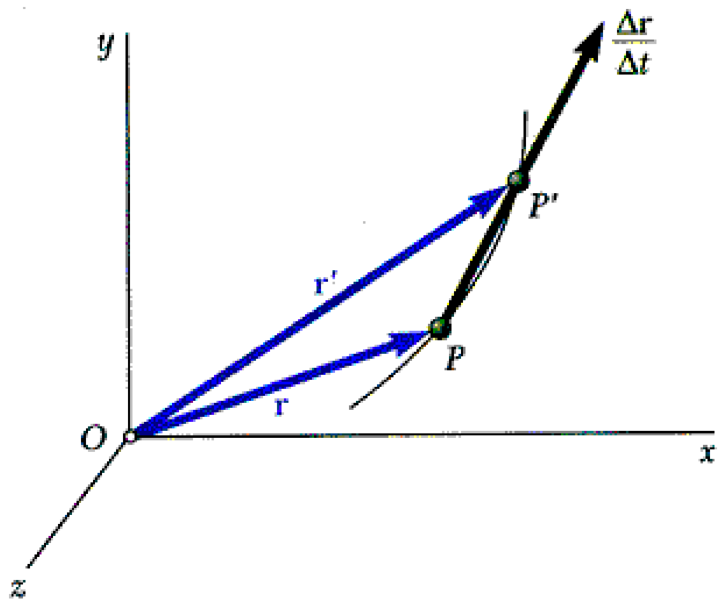
Curvilinear Motion: Position, Velocity & Acceleration

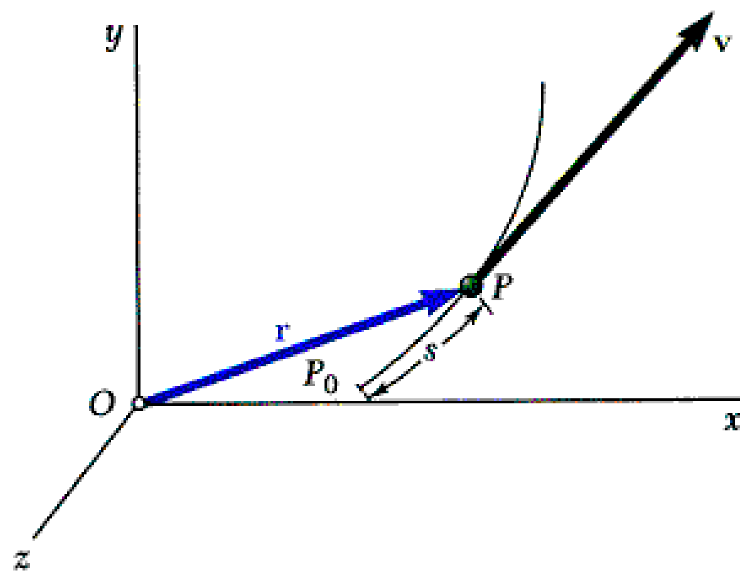
Instantaneous velocity
(vector)

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

Instantaneous speed
(scalar)

$$v = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

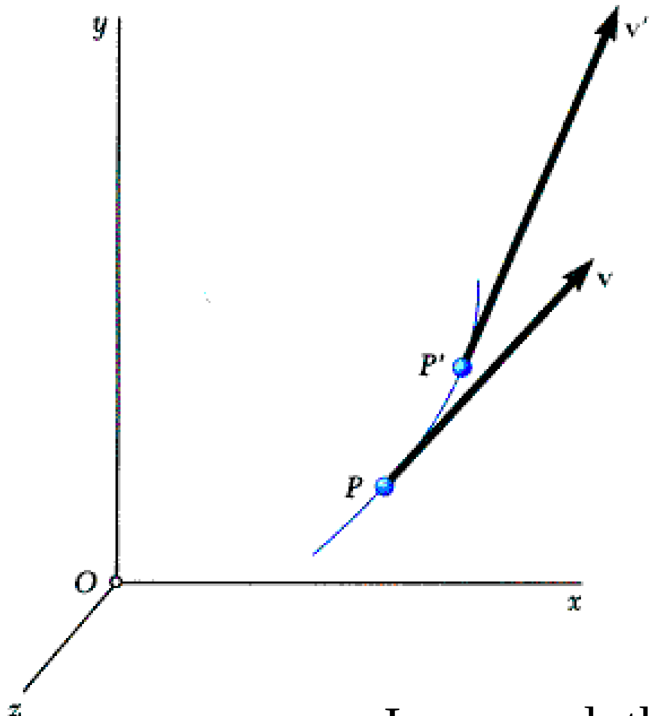


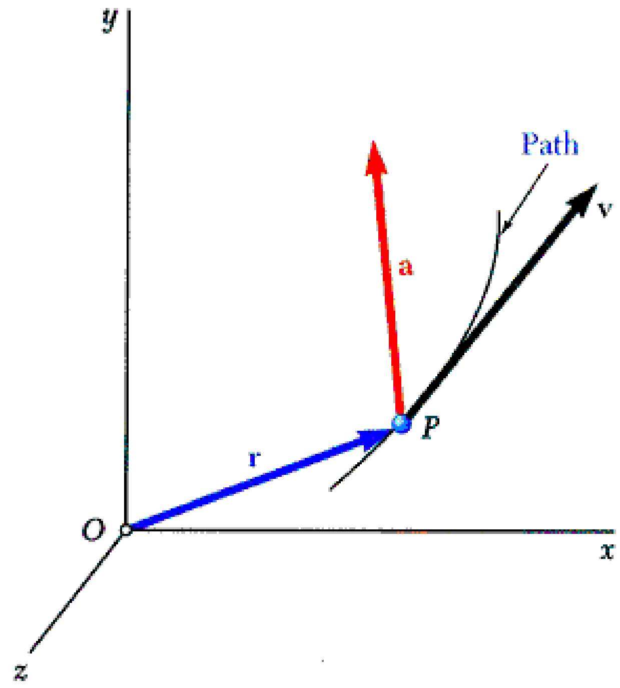


Curvilinear Motion: Position, Velocity & Acceleration

- Consider velocity $\vec{v}$ of a particle at time t and velocity $\vec{v}'$ at $t + \Delta t$,

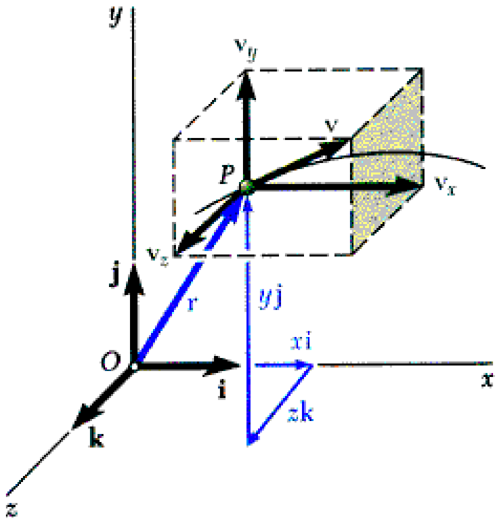
$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \text{ instantaneous acceleration (vector)}$$

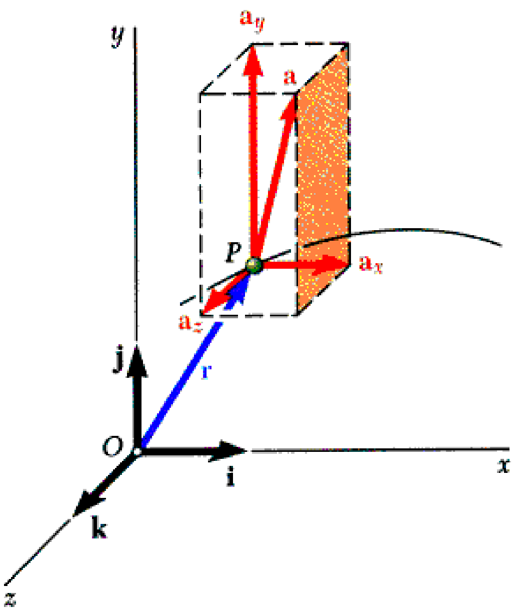




- In general, the acceleration vector is not tangent to the particle path and velocity vector.

Rectangular Components of Velocity & Acceleration





- When position vector of particle *P* is given by its rectangular components,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

- Velocity vector,

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

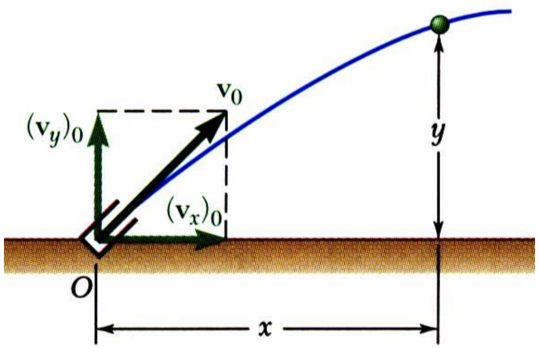
$$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

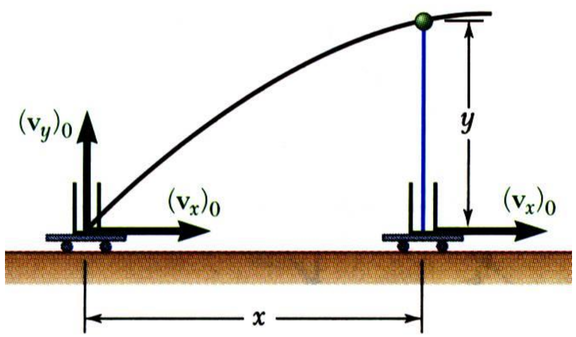
- Acceleration vector,

$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

Rectangular Components of Velocity & Acceleration





- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$

with initial conditions,

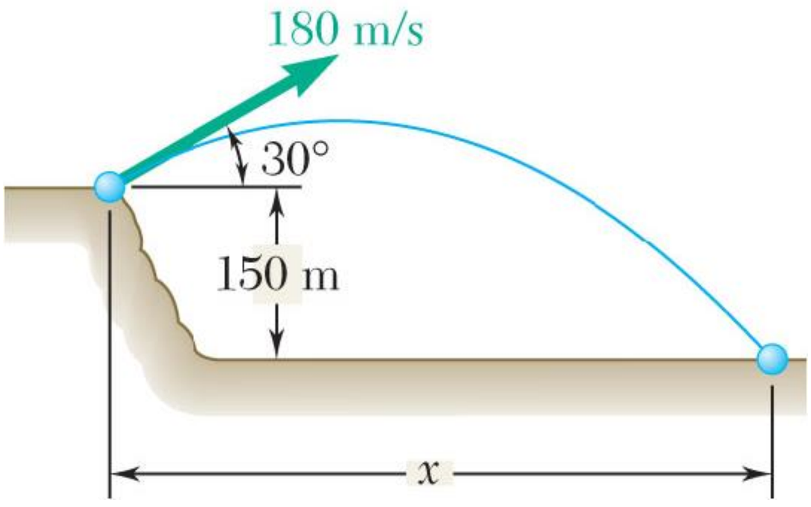
$$x_0 = y_0 = z_0 = 0 \qquad (v_x)_0, (v_y)_0, (v_z)_0 = 0$$

Integrating twice yields

$$v_x = (v_x)_0 \quad v_y = (v_y)_0 - gt \qquad v_z = 0$$
$$x = (v_x)_0 t \quad y = (v_y)_0 y - \tfrac{1}{2} g t^2 \quad z = 0$$

- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.

Sample Problem

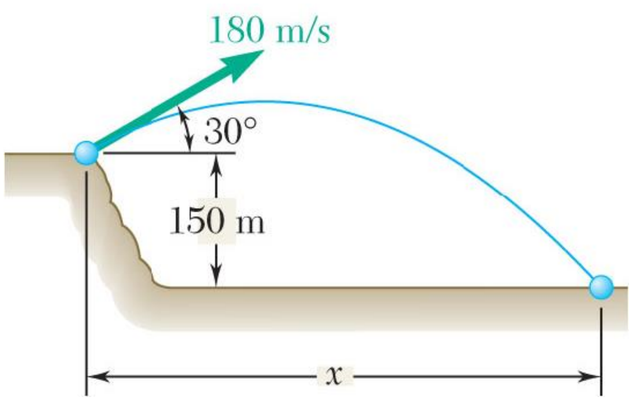


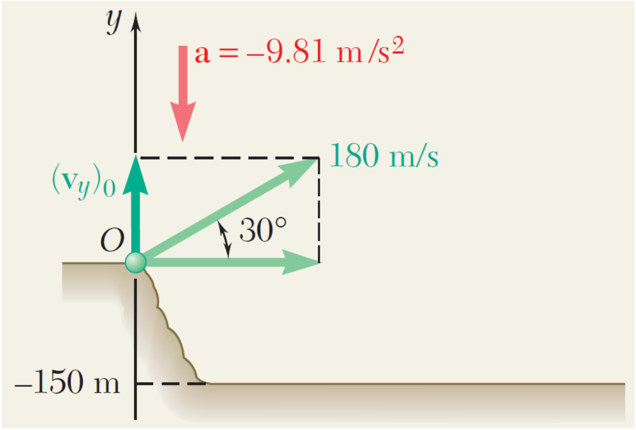
A projectile is fired from the edge of a 150-m cliff with an initial velocity of 180 m/s at an angle of 30° with the horizontal. Neglecting air resistance, find (*a*) the horizontal distance from the gun to the point where the projectile strikes the ground, (*b*) the greatest elevation above the ground reached by the projectile.

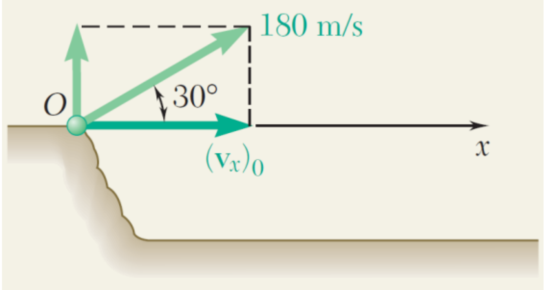
SOLUTION:

- Consider the vertical and horizontal motion separately (they are independent)
- Apply equations of motion in y-direction
- Apply equations of motion in x-direction
- Determine time *t* for projectile to hit the ground, use this to find the horizontal distance
- Maximum elevation occurs when $v_y=0$

Sample Problem







SOLUTION:

Given: $(v)_o = 180$ m/s $(y)_o = 150$ m

$(a)_y = -9.81$ m/s^2 $(a)_x = 0$ m/s^2

Vertical motion – uniformly accelerated:

$$(v_y)_0 = (180 \text{ m/s}) \sin 30° = +90 \text{ m/s}$$

$$v_y = (v_y)_0 + at \qquad v_y = 90 - 9.81t \qquad (1)$$

$$y = (v_y)_0 t + \tfrac{1}{2}at^2 \qquad y = 90t - 4.90t^2 \qquad (2)$$

$$v_y^2 = (v_y)_0^2 + 2ay \qquad v_y^2 = 8100 - 19.62y \qquad (3)$$

Horizontal motion – uniformly accelerated:

Choose positive x to the right as shown

$$(v_x)_0 = (180 \text{ m/s}) \cos 30° = +155.9 \text{ m/s}$$

$$x = (v_x)_0 t \qquad x = 155.9t$$

Sample Problem

SOLUTION:

Horizontal distance

Projectile strikes the ground at: $y = -150 \text{ m}$

Substitute into equation (1) above

$$-150 = 90t - 4.90t^2$$

Solving for t, we take the positive root

$$t^2 - 18.37t - 30.6 = 0 \qquad t = 19.91 \text{ s}$$

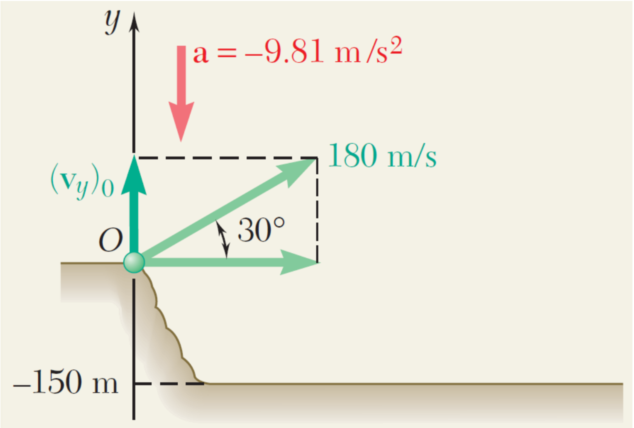
Substitute t into equation (4)

$$x = 155.9(19.91) \qquad x = 3100 \text{ m}$$

Maximum elevation occurs when $v_y=0$

$$0 = 8100 - 19.62y \qquad y = 413 \text{ m}$$

Maximum elevation above the ground = $150 \text{ m} + 413 \text{ m} = 563 \text{ m}$

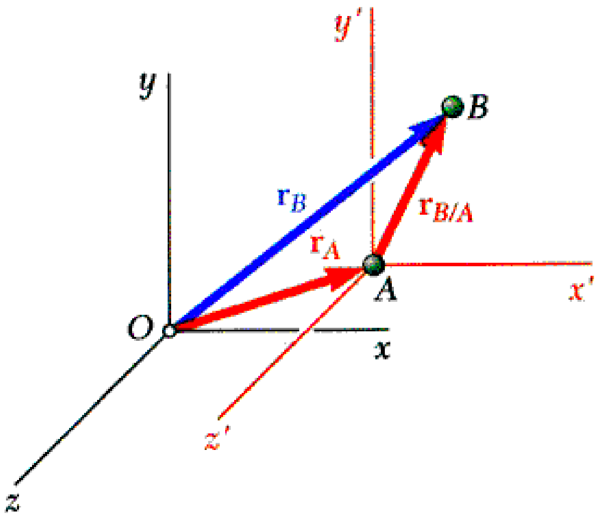


Motion Relative to a Frame in Translation

A soccer player must consider the relative motion of the ball and her teammates when making a pass.

It is critical for a pilot to know the relative motion of his aircraft with respect to the aircraft carrier to make a safe landing.

Motion Relative to a Frame in Translation



- Designate one frame as the *fixed frame of reference*. All other frames not rigidly attached to the fixed reference frame are *moving frames of reference*.
- Position vectors for particles *A* and *B* with respect to the fixed frame of reference *Oxyz* are $\vec{r}_A$ and $\vec{r}_B$.
- Vector $\vec{r}_{B/A}$ joining *A* and *B* defines the position of *B* with respect to the moving frame *Ax'y'z'* and

  $$\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}$$

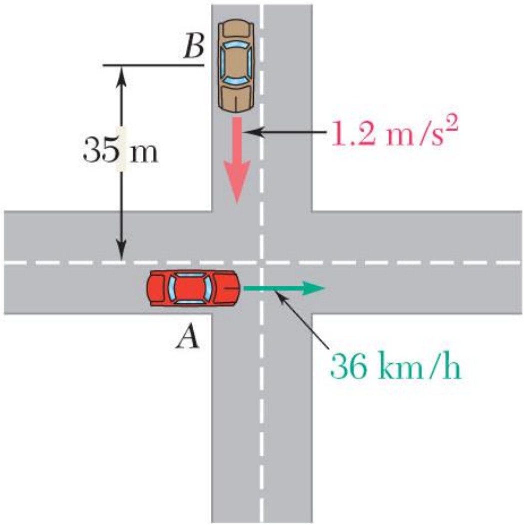
- Differentiating twice,

  $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ $\quad$ $\vec{v}_{B/A} =$ velocity of *B* relative to *A*.

  $\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ $\quad$ $\vec{a}_{B/A} =$ acceleration of *B* relative to *A*.

- Absolute motion of *B* can be obtained by combining motion of *A* with relative motion of *B* with respect to moving reference frame attached to *A*.

Sample Problem

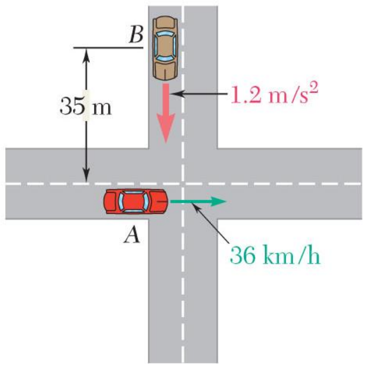


Automobile *A* is traveling east at the constant speed of 36 km/h. As automobile *A* crosses the intersection shown, automobile *B* starts from rest 35 m north of the intersection and moves south with a constant acceleration of 1.2 m/s^2. Determine the position, velocity, and acceleration of *B* relative to *A* 5 *s* after *A* crosses the intersection.

SOLUTION:

- Define inertial axes for the system
- Determine the position, speed, and acceleration of car A at $t = 5$ s
- Determine the position, speed, and acceleration of car B at $t = 5$ s
- Using vectors (Eqs 11.31, 11.33, and 11.34) or a graphical approach, determine the relative position, velocity, and acceleration

Sample Problem

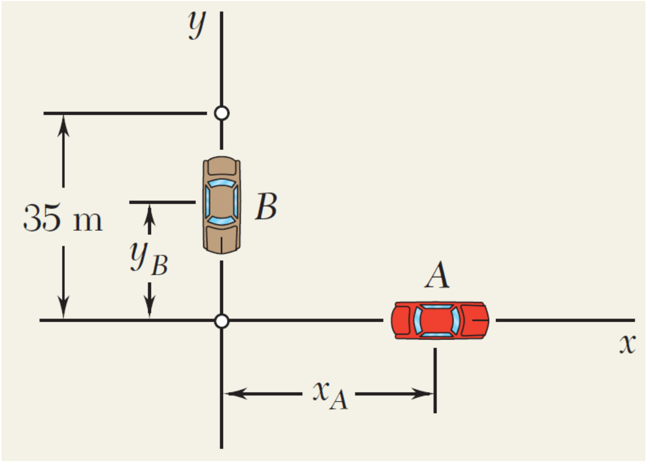


SOLUTION:

- Define axes along the road

Given: v_A=36 km/h, a_A= 0, $(x_A)_0$ = 0

$(v_B)_0$= 0, a_B= - 1.2 m/s^2, $(y_A)_0$ = 35 m



Determine motion of Automobile A:

$$v_A = \left(36\frac{\text{km}}{\text{h}}\right)\left(\frac{1000\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 10\text{ m/s}$$

We have uniform motion for A so:

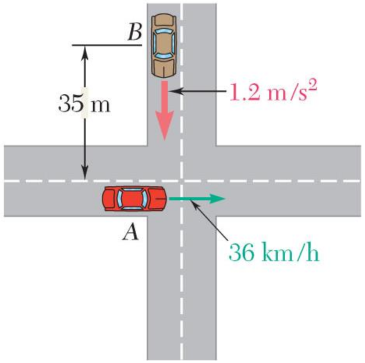
$$\begin{aligned} a_A &= 0 \\ v_A &= +10\text{ m/s} \\ x_A &= (x_A)_0 + v_A t = 0 + 10t \end{aligned}$$

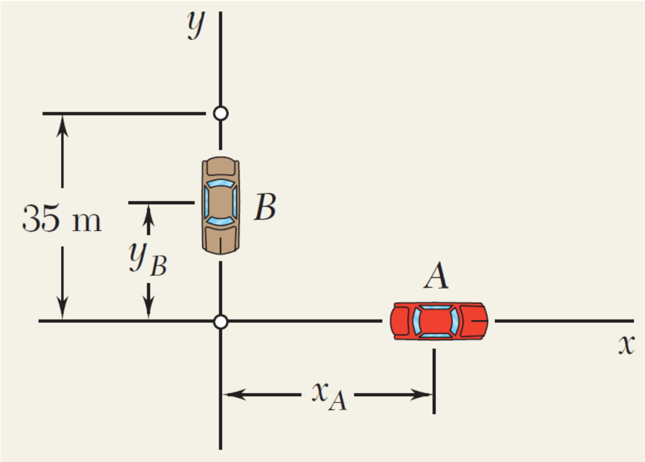
At t = 5 s

$$\begin{aligned} a_A &= 0 \\ v_A &= +10\text{ m/s} \\ x_A &= +(10\text{ m/s})(5\text{ s}) = +50\text{ m} \end{aligned}$$

$$\begin{aligned} \mathbf{a}_A &= 0 \\ \mathbf{v}_A &= 10\text{ m/s} \rightarrow \\ \mathbf{r}_A &= 50\text{ m} \rightarrow \end{aligned}$$

Sample Problem





SOLUTION:

Determine motion of Automobile B:

We have uniform acceleration for B so:

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = (v_B)_0 + at = 0 - 1.2\,t$$
$$y_B = (y_B)_0 + (v_B)_0 t + \tfrac{1}{2} a_B t^2 = 35 + 0 - \tfrac{1}{2}(1.2)t^2$$

At $t = 5$ s

$$a_B = -1.2 \text{ m/s}^2$$
$$v_B = -(1.2 \text{ m/s}^2)(5 \text{ s}) = -6 \text{ m/s}$$
$$y_B = 35 - \tfrac{1}{2}(1.2 \text{ m/s}^2)(5 \text{ s})^2 = +20 \text{ m}$$

$$\mathbf{a}_B = 1.2 \text{ m/s}^2 \downarrow$$
$$\mathbf{v}_B = 6 \text{ m/s} \downarrow$$
$$\mathbf{r}_B = 20 \text{ m} \uparrow$$

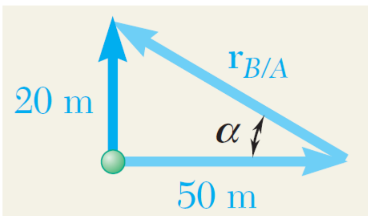
Sample Problem

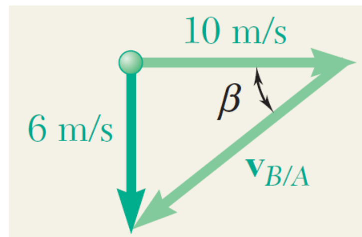
SOLUTION:

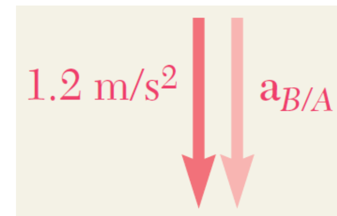
$$\mathbf{a}_A = 0 \qquad \mathbf{v}_A = 10 \text{ m/s} \rightarrow \qquad \mathbf{r}_A = 50 \text{ m} \rightarrow$$

$$\mathbf{a}_B = 1.2 \text{ m/s}^2 \downarrow \qquad \mathbf{v}_B = 6 \text{ m/s} \downarrow \qquad \mathbf{r}_B = 20 \text{ m} \uparrow$$

We can solve the problems geometrically, and apply the arctangent relationship:







$r_{B/A} = 53.9 \text{ m} \qquad \alpha = 21.8°$

$\mathbf{r}_{B/A} = 53.9 \text{ m} \nwarrow 21.8°$

$v_{B/A} = 11.66 \text{ m/s} \qquad \beta = 31.0°$

$\mathbf{v}_{B/A} = 11.66 \text{ m/s} \swarrow 31.0°$

$\mathbf{a}_{B/A} = 1.2 \text{ m/s}^2 \downarrow$

Or we can solve the problems using vectors to obtain equivalent results:

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A}$$
$$20\mathbf{j} = 50\mathbf{i} + \mathbf{r}_{B/A}$$
$$\mathbf{r}_{B/A} = 20\mathbf{j} - 50\mathbf{i} \text{ (m)}$$

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A}$$
$$-6\mathbf{j} = 10\mathbf{i} + \mathbf{v}_{B/A}$$
$$\mathbf{v}_{B/A} = -6\mathbf{j} - 10\mathbf{i} \text{ (m/s)}$$

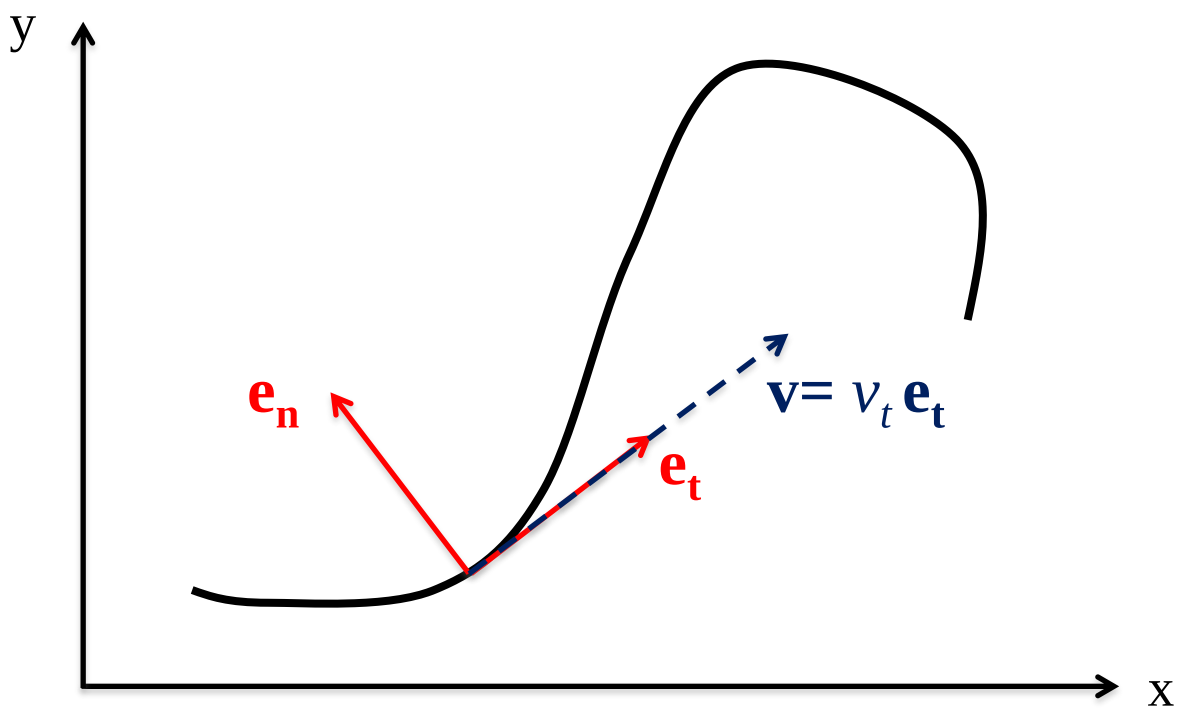
$$\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$
$$-1.2\mathbf{j} = 0\mathbf{i} + \mathbf{a}_{B/A}$$
$$\mathbf{a}_{B/A} = -1.2\mathbf{j} \text{ (m/s}^2\text{)}$$

Physically, a rider in car A would "see" car B travelling south and west.

Tangential and Normal Components

If we have an idea of the path of a vehicle, it is often convenient to analyze the motion using tangential and normal components (sometimes called *path* coordinates).

Tangential and Normal Components



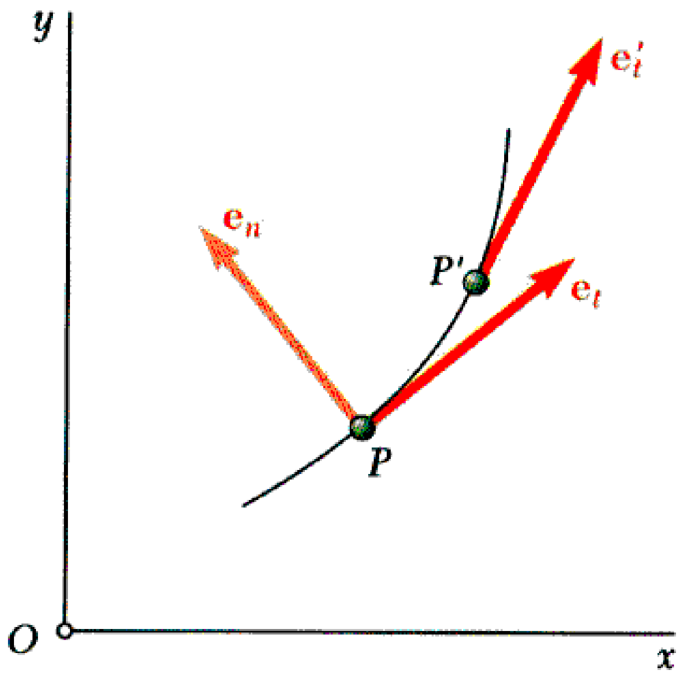
ρ= the instantaneous radius of curvature

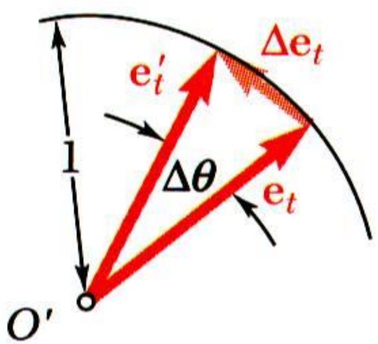
$$\mathbf{v} = v\,\mathbf{e_t}$$

$$\mathbf{a} = \frac{dv}{dt}\mathbf{e_t} + \frac{v^2}{\rho}\mathbf{e_n}$$

- The tangential direction ($\mathbf{e_t}$) is tangent to the path of the particle. This velocity vector of a particle is in this direction
- The normal direction ($\mathbf{e_n}$) is perpendicular to $\mathbf{e_t}$ and points towards the inside of the curve.
- The acceleration can have components in both the $\mathbf{e_n}$ and $\mathbf{e_t}$ directions

Tangential and Normal Components





- To derive the acceleration vector in tangential and normal components, define the motion of a particle as shown in the figure.

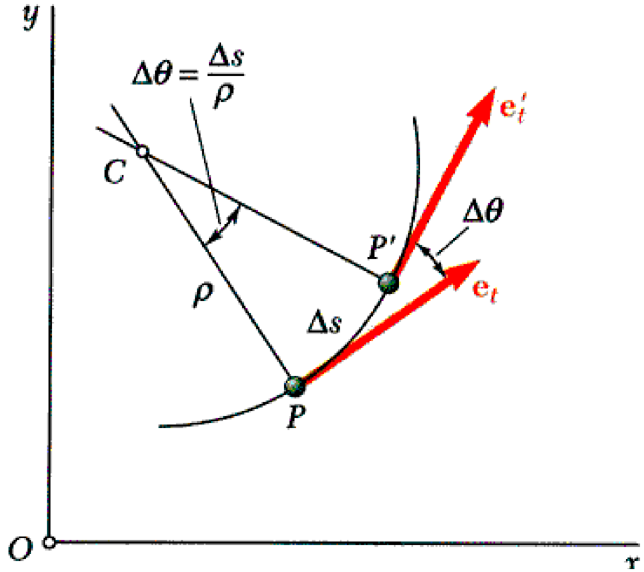
- $\vec{e}_t$ and $\vec{e}_t'$ are tangential unit vectors for the particle path at P and P'. When drawn with respect to the same origin, $\Delta\vec{e}_t = \vec{e}_t' - \vec{e}_t$ and $\Delta\theta$ is the angle between them.

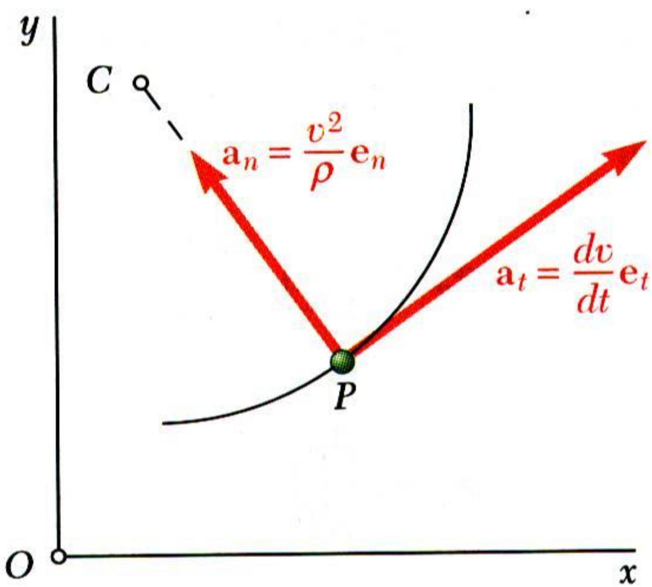
$$\Delta e_t = 2\sin(\Delta\theta/2)$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\vec{e}_t}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\vec{e}_n = \vec{e}_n$$

$$\vec{e}_n = \frac{d\vec{e}_t}{d\theta}$$

Tangential and Normal Components





- With the velocity vector expressed as $\vec{v} = v\vec{e}_t$ the particle acceleration may be written as

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}_t}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}_t}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

but

$$\frac{d\vec{e}_t}{d\theta} = \vec{e}_n \qquad \rho\, d\theta = ds \qquad \frac{ds}{dt} = v$$

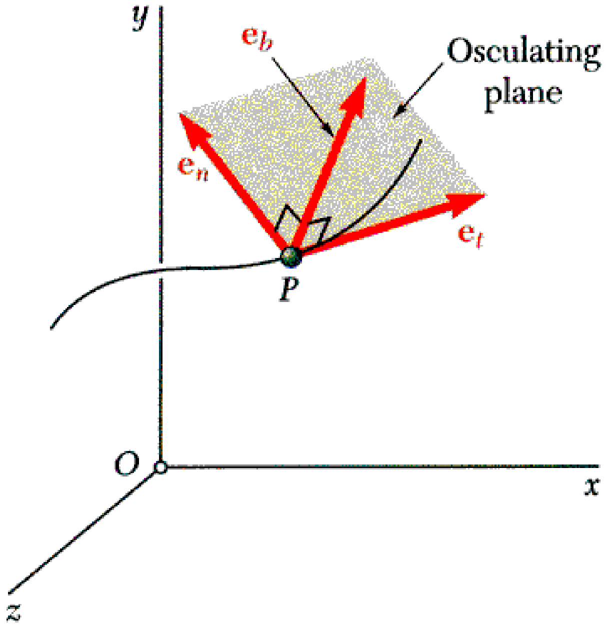
After substituting,

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$

- The tangential component of acceleration reflects change of speed and the normal component reflects change of direction.

- The tangential component may be positive or negative. Normal component always points toward center of path curvature.

Tangential and Normal Components



- Relations for tangential and normal acceleration also apply for particle moving along a space curve.

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \quad a_n = \frac{v^2}{\rho}$$

- The plane containing tangential and normal unit vectors is called the *osculating plane*.

- The normal to the osculating plane is found from

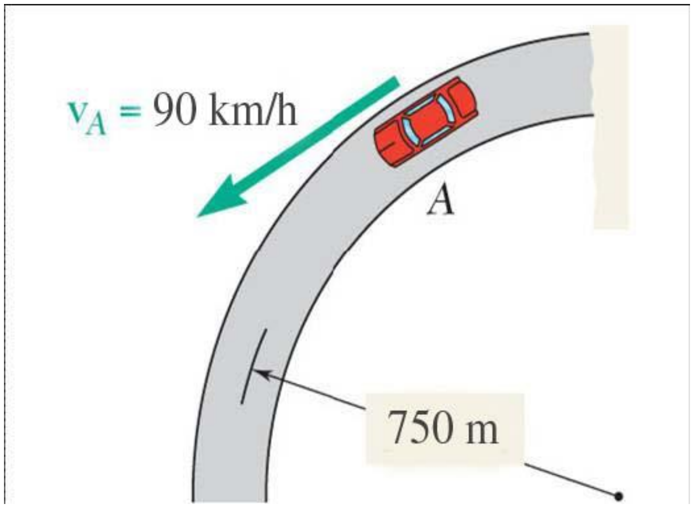
$$\vec{e}_b = \vec{e}_t \times \vec{e}_n$$

$$\vec{e}_n = principal \;\; normal$$

$$\vec{e}_b = binormal$$

- Acceleration has no component along the binormal.

Sample Problem

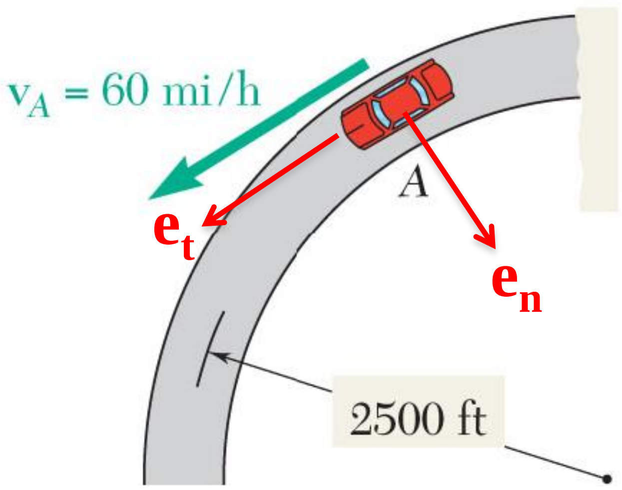


A motorist is traveling on a curved section of highway of radius 750 m at the speed of 90 km/h. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to 72 km/h, determine the acceleration of the automobile immediately after the brakes have been applied.

SOLUTION:

- Define your coordinate system
- Calculate the tangential velocity and tangential acceleration
- Calculate the normal acceleration
- Determine overall acceleration magnitude after the brakes have been applied

Sample Problem



SOLUTION:

- Define your coordinate system
- Determine velocity and acceleration in the tangential direction

$$90 \text{ km/h} = \left(90\frac{\text{km}}{\text{h}}\right)\left(\frac{1000 \text{ m}}{1 \text{ km}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 25 \text{ m/s}$$

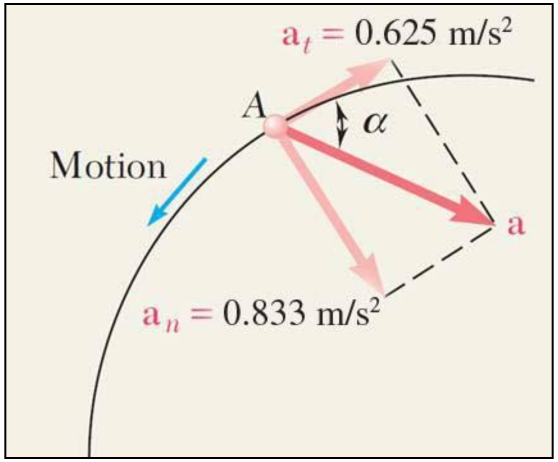
$$72 \text{ km/h} = 20 \text{ m/s}$$

- The deceleration constant, therefore

$$a_t = \text{average } a_t = \frac{\Delta v}{\Delta t} = \frac{20 \text{ m/s} - 25 \text{ m/s}}{8 \text{ s}} = -0.625 \text{ m/s}^2$$

- Immediately after the brakes are applied, the speed is still 25 m/s

$$a_n = \frac{v^2}{\rho} = \frac{(25 \text{ m/s})^2}{750 \text{ m}} = 0.833 \text{ m/s}^2$$



$$a = \sqrt{a_n^2 + a_t^2} = \sqrt{(-0.625)^2 + (0.833)^2}$$

$$\mathbf{a} = 1.041 \text{ m/s}^2$$

$$\tan\alpha = \frac{a_n}{a_t} = \frac{0.833 \text{ m/s}^2}{0.625 \text{ m/s}^2}$$

$$\alpha = 53.1^\circ$$

Tangential and Normal Components

In 2001, a race scheduled at the Texas Motor Speedway was cancelled because the normal accelerations were too high and caused some drivers to experience excessive g-loads (similar to fighter pilots) and possibly pass out. What are some things that could be done to solve this problem?

Some possibilities:

Reduce the allowed speed
Increase the turn radius (difficult and costly)
Have the racers wear g-suits

Radial and Transverse Components

By knowing the distance to the aircraft and the angle of the radar, air traffic controllers can track aircraft.

Fire truck ladders can rotate as well as extend; the motion of the end of the ladder can be analyzed using radial and transverse components.

Radial and Transverse Components

- The position of a particle *P* is expressed as a distance *r* from the origin *O* to *P* – this defines the radial direction $\mathbf{e_r}$. The transverse direction $\mathbf{e_\theta}$ is perpendicular to $\mathbf{e_r}$

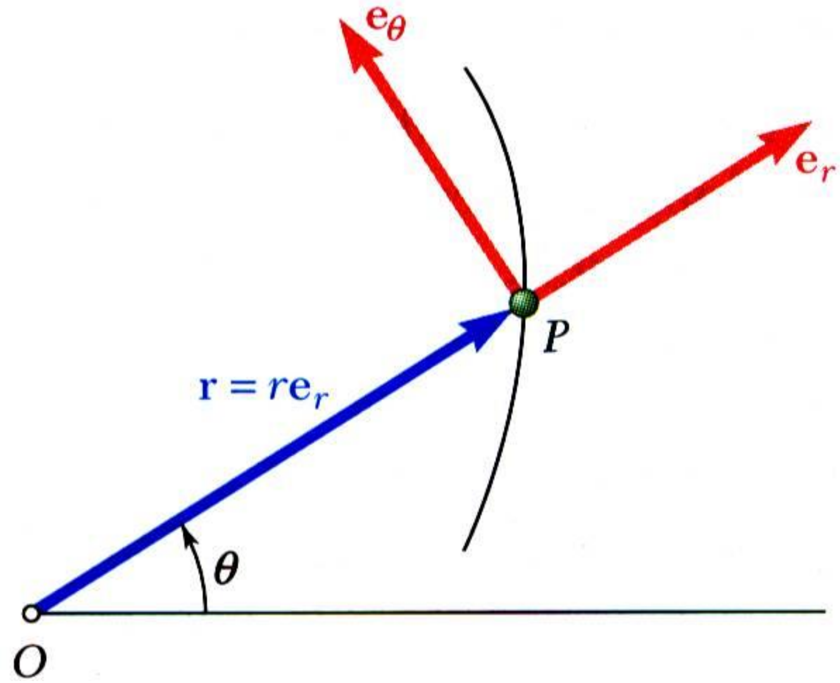
$$\vec{r} = r\vec{e}_r$$

- The particle velocity vector is

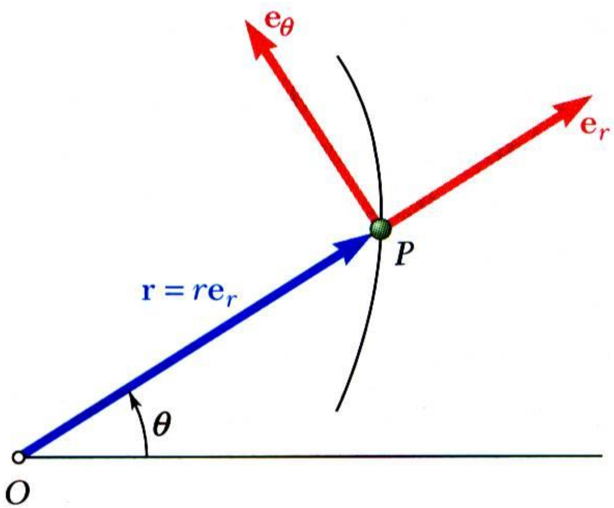
$$\vec{v} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta$$

- The particle acceleration vector is

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$



Radial and Transverse Components



$$\vec{r} = r\vec{e}_r$$

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad \frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$$

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \vec{e}_\theta \frac{d\theta}{dt}$$

$$\frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\vec{e}_r \frac{d\theta}{dt}$$

- We can derive the velocity and acceleration relationships by recognizing that the unit vectors change direction.

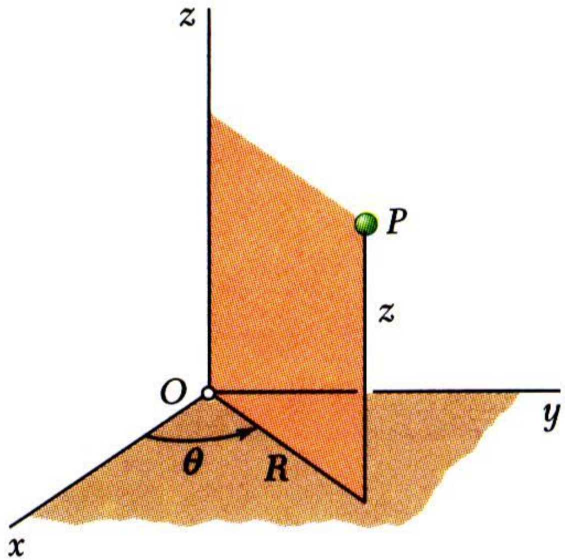
- The particle velocity vector is

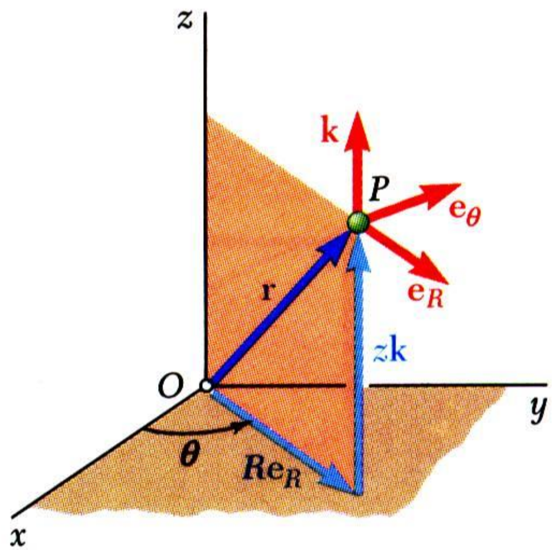
$$\vec{v} = \frac{d}{dt}\left(r\vec{e}_r\right) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$
$$= \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

- Similarly, the particle acceleration vector is

$$\vec{a} = \frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta\right)$$
$$= \frac{d^2 r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}$$
$$= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

Radial and Transverse Components





- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors $\vec{e}_R, \vec{e}_\theta$, and $\vec{k}$.

- Position vector,

$$\vec{r} = R\vec{e}_R + z\vec{k}$$

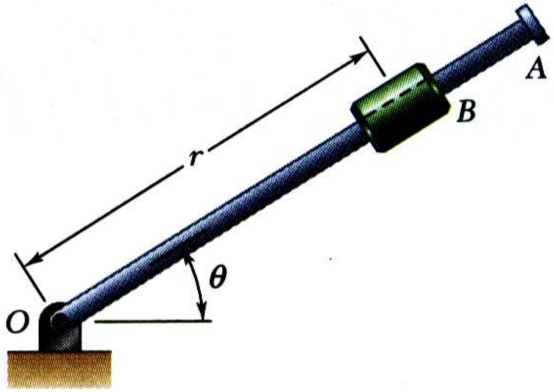
- Velocity vector,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{R}\vec{e}_R + R\dot{\theta}\vec{e}_\theta + \dot{z}\vec{k}$$

- Acceleration vector,

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{R} - R\dot{\theta}^2\right)\vec{e}_R + \left(R\ddot{\theta} + 2\dot{R}\dot{\theta}\right)\vec{e}_\theta + \ddot{z}\vec{k}$$

Sample Problem



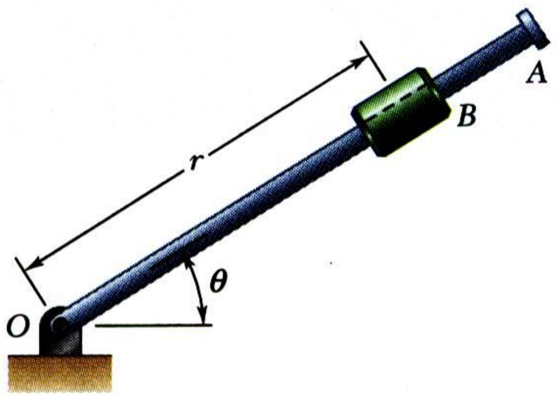
Rotation of the arm about O is defined by $\theta = 0.15t^2$ where θ is in radians and t in seconds. Collar B slides along the arm such that $r = 0.9 - 0.12t^2$ where r is in meters.

After the arm has rotated through 30°, determine *(a)* the total velocity of the collar, *(b)* the total acceleration of the collar, and *(c)* the relative acceleration of the collar with respect to the arm.

SOLUTION:

- Evaluate time t for $\theta = 30^o$.
- Evaluate radial and angular positions, and first and second derivatives at time t.
- Calculate velocity and acceleration in cylindrical coordinates.
- Evaluate acceleration with respect to arm.

Sample Problem



SOLUTION:

- Evaluate time t for $\theta = 30^o$.

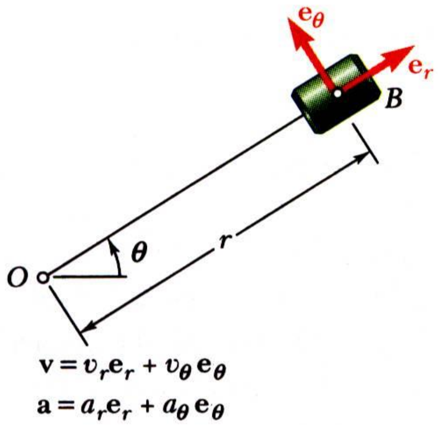
$$\theta = 0.15t^2$$
$$= 30° = 0.524\,\text{rad} \qquad t = 1.869\,\text{s}$$

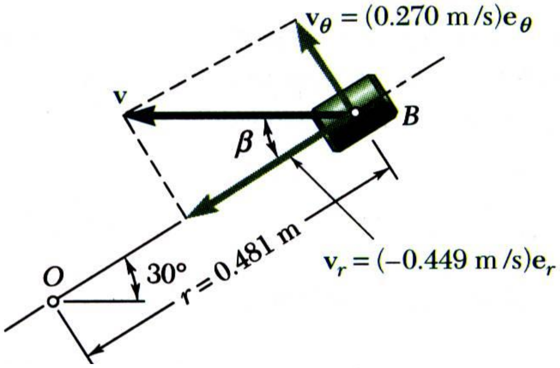
- Evaluate radial and angular positions, and first and second derivatives at time t.

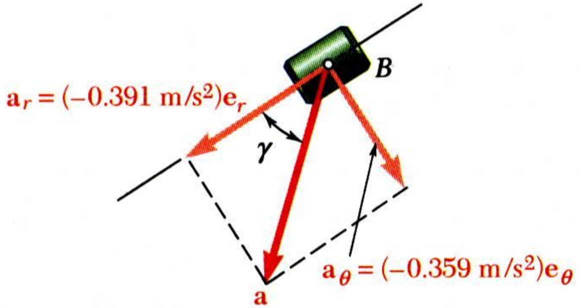
$$r = 0.9 - 0.12t^2 = 0.481\,\text{m}$$
$$\dot{r} = -0.24t = -0.449\,\text{m/s}$$
$$\ddot{r} = -0.24\,\text{m/s}^2$$

$$\theta = 0.15t^2 = 0.524\,\text{rad}$$
$$\dot{\theta} = 0.30t = 0.561\,\text{rad/s}$$
$$\ddot{\theta} = 0.30\,\text{rad/s}^2$$

Sample Problem







- Calculate velocity and acceleration.

$$v_r = \dot{r} = -0.449\,\mathrm{m}/s$$

$$v_\theta = r\dot{\theta} = (0.481\mathrm{m})(0.561\mathrm{rad/s}) = 0.270\,\mathrm{m/s}$$

$$v = \sqrt{v_r^2 + v_\theta^2} \qquad \beta = \tan^{-1}\frac{v_\theta}{v_r}$$

$$\boxed{v = 0.524\,\mathrm{m/s} \qquad \beta = 31.0°}$$

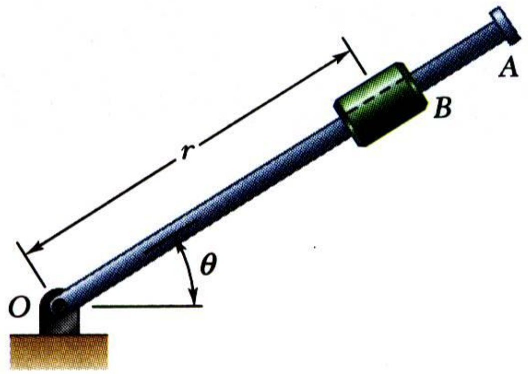
$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$= -0.240\,\mathrm{m/s^2} - (0.481\mathrm{m})(0.561\mathrm{rad/s})^2$$
$$= -0.391\mathrm{m/s^2}$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= (0.481\mathrm{m})\left(0.3\mathrm{rad/s^2}\right) + 2(-0.449\,\mathrm{m/s})(0.561\mathrm{rad/s})$$
$$= -0.359\,\mathrm{m/s^2}$$

$$a = \sqrt{a_r^2 + a_\theta^2} \qquad \gamma = \tan^{-1}\frac{a_\theta}{a_r}$$

$$\boxed{a = 0.531\mathrm{m/s} \qquad \gamma = 42.6°}$$

Sample Problem



- Evaluate acceleration with respect to arm.

  Motion of collar with respect to arm is rectilinear and defined by coordinate *r*.

$$\boxed{a_{B/OA} = \ddot{r} = -0.240\,\mathrm{m/s^2}}$$