Statistical Inference and Multivariate Analysis (MA324)

LECTURE SLIDES
Lecture 20

Likelihood Ratio Test

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Likelihood Ratio Tests

- We have seen that the UMP level α test **does not exist** for testing $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$ for all values of $\alpha \in (0, 1)$. Nontheless, these hypotheses are quite meaningful in practice.
- Here, we will discuss an alternative method to find a meaningful test.
- This test is called **likelihood ratio test (LRT)**. Note that **LRT may not be best**, but they are quite intuitive and useful.

Likelihood Ratio Tests

The general procedure to obtain a LRT can be described as follows. Suppose that we want to test $H_0 : \boldsymbol{\theta} \in \Theta_0$ versus $H_1 : \boldsymbol{\theta} \in \Theta_1$ at level $\alpha \in (0, 1)$. Consider the ratio

$$\Lambda(\boldsymbol{x}) = \frac{\sup\limits_{\boldsymbol{\theta} \in \Theta_0} L(\boldsymbol{\theta}, \boldsymbol{x})}{\sup\limits_{\boldsymbol{\theta} \in \Theta_0 \cup \Theta_1} L(\boldsymbol{\theta}, \boldsymbol{x})}.$$

This ratio $\Lambda(\boldsymbol{x})$ is called **likelihood ratio test statistic.** The test function of a LRT is given by

$$\psi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \Lambda(\boldsymbol{x}) < k \\ \gamma & \text{if } \Lambda(\boldsymbol{x}) = k \\ 0 & \text{if } \Lambda(\boldsymbol{x}) > k, \end{cases}$$

where $\gamma \in (0, 1)$ and $k > 0$ are such that $\sup\limits_{\boldsymbol{\theta} \in \Theta_0} E_\theta\left(\psi(\boldsymbol{X})\right) = \alpha$.

Likelihood Ratio Tests: Interpretation

- Note that $\sup_{\boldsymbol{\theta} \in \Theta_0} L(\theta, \boldsymbol{x})$ can be interpreted as the **best evidence** for observing $\boldsymbol{X} = \boldsymbol{x}$, when the parameter $\boldsymbol{\theta}$ is restricted in Θ_0. Similarly, $\sup_{\boldsymbol{\theta} \in \Theta_0 \cup \Theta_1} L(\boldsymbol{\theta}, \boldsymbol{x})$ can be interpreted as the **overall best evidence** for observing $\boldsymbol{X} = \boldsymbol{x}$.
- LRT **rejects the null hypothesis** if the value of $\Lambda(\boldsymbol{x})$ is "**small**". The rationale is that if the best evidence of observing $\boldsymbol{X} = \boldsymbol{x}$ under the null hypothesis is weak compared with overall evidence, then H_0 ought to be rejected.
- It is clear from the definition of LRT statistic $\Lambda(\boldsymbol{x})$ that $0 \leq \Lambda(\boldsymbol{x}) \leq 1$. To see it, notice that the supremum in the **numerator** is over a **smaller** set than that in the **denominator**.
- Therefore, we need to find the cutoff point $k \in (0, 1)$ and $\gamma \in (0, 1)$ such that the test **satisfies level condition**, *i.e.*,

$$\sup_{\boldsymbol{\theta} \in \Theta_0} E_{\boldsymbol{\theta}} \left(\psi \left(\boldsymbol{X} \right) \right) = \alpha.$$

Example 1: Let $X_1,\, X_2,\, \ldots,\, X_n \overset{i.i.d.}{\sim} N(\mu,\, \sigma^2)$, where σ is known. Let μ_0 be a real number. We are interested to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Here, $\Theta_0 = \{\mu_0\}$, $\Theta_1 = \mathbb{R} \setminus \{\mu_0\}$, and $\Theta_0 \cup \Theta_1 = \mathbb{R}$.

Remark: $\dfrac{\sqrt{n}|\bar{x}-\mu_0|}{\sigma}$ can be considered as standardized distance between $\bar{x}$ and μ_0. Hence, this test rejects H_0 if the distance is large, which is quite intuitive.

Example 2: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$. Let μ_0 be a real number. We are interested to test $H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$. Here, $\Theta_0 = \{\mu_0\} \times \mathbb{R}^+$, $\Theta_1 = \mathbb{R} \setminus \{\mu_0\} \times \mathbb{R}^+$, and hence, $\Theta_0 \cup \Theta_1 = \mathbb{R} \times \mathbb{R}^+$. Now, Example 3: Let $X_1, X_2, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$. Let σ_0 be a positive real number. We are interested to test $H_0 : \sigma^2 = \sigma_0^2$ against $H_1 : \sigma^2 \neq \sigma_0^2$. Here, $\Theta_0 = \mathbb{R} \times \left\{\sigma_0^2\right\}$, $\Theta_1 = \mathbb{R} \times \mathbb{R}^+ \setminus \left\{\sigma_0^2\right\}$, and hence $\Theta_0 \cup \Theta_1 = \mathbb{R} \times \mathbb{R}^+$.