The Matrix Eigenvalue Problem

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Let $A \in \mathbb{C}^{n\times n}$. The eigenvalue problem for A consists of finding all $x \in \mathbb{C}^n \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that

$$Ax = \lambda x.$$

The scalars $\lambda \in \mathbb{C}$ are called the eigenvalues of A and the corresponding non zero vector x is called an eigenvector of A.

- The eigenvalues of A are the roots of $\det(A - sI)$.
- $A \in \mathbb{C}^{n\times n}$ can have at most n distinct eigenvalues.
- Any set of eigenvectors of A corresponding to distinct eigenvalues is a linearly independent set.
- Matrices A and B are said to be similar if there exists a nonsingular matrix S such that $B = S^{-1}AS$. If S is a unitary matrix, then A and B are said to be unitarily similar.
- Similar matrices have the same eigenvalues.

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- If $Av = \lambda v$ for some $v \neq 0$, and $B = S^{-1}AS$, then $BS^{-1}v = \lambda S^{-1}v$.
- A matrix A is said to be (unitarily) diagonalizable is it is (unitarily) similar to a diagonal matrix.
- (Unitary) diagonalizability of an $n \times n$ matrix A is equivalent to the existence of a (orthonormal) basis of $\mathbb{C}^n$ consisting of eigenvectors of A.
- Not every matrix is diagonalizable.
- Given any matrix $A \in \mathbb{C}^{n\times n}$, there exists an invertible matrix X and a block diagonal matrix $J = \mathrm{diag}(J_{\lambda_1}, \ldots, J_{\lambda_n})$, with

$$J_{\lambda_i} = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}, \quad i = 1, \ldots, p,$$

such that $A = XJX^{-1}$. J is the called the *Jordan Canonical form of A*.

Schur's Theorem

Issai Schur
(1875-1941)

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$$A^* = A$$
$$T^* = (Q^*AQ)^*$$
$$= Q^*A^*(Q^*)^*$$
$$= Q^*AQ$$
$$= T$$
$$AA^* = A^*A \iff TT^* = T^*T$$

Schur's Theorem: Given any matrix $A \in \mathbb{C}^{n\times n}$, there exists a unitary matrix Q and an upper triangular matrix T such that $Q^*AQ = T$.
T is called a Schur form of A.

Spectral Theorems

Spectral Theorem for Hermitian Matrices $A \in \mathbb{C}^{n \times n}$ is Hermitian if and only if there exists a unitary matrix Q and a real diagonal matrix D such that $Q^* A Q = D$.

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Spectral Theorem for Normal Matrices $A \in \mathbb{C}^{n\times n}$ is normal if and only if there exists a unitary matrix Q and a diagonal matrix D such that $Q^*AQ = D$.

Prove the following:

1. If $AA^* = A^*A$ and $\exists$ a unitary Q such that $Q^*AQ = T$ where T is upper triangular, then, $T^*T = TT^*$.
2. If T is upper triangular & $T^*T = TT^*$, then T is diagonal.
3. Hence prove Spectral Theorem for Normal Matrices.

Spectral Theorems

Spectral Theorem for Hermitian Matrices $A \in \mathbb{C}^{n\times n}$ is Hermitian if and only if there exists a unitary matrix Q and a real diagonal matrix D such that $Q^*AQ = D$.

Spectral Theorem for Normal Matrices $A \in \mathbb{C}^{n\times n}$ is normal if and only if there exists a unitary matrix Q and a diagonal matrix D such that $Q^*AQ = D$.

Spectral Theorem for Symmetric Matrices $A \in \mathbb{R}^{n\times n}$ is symmetric if and only if there exists a real orthogonal matrix Q and a real diagonal matrix D such that $Q^TAQ = D$.

$A \in \mathbb{R}^{n\times n}$ & $A^T = A \Rightarrow A^* = A$. Let $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^n, v \neq 0$ such that $Av = \lambda v \Rightarrow v^*Av = \lambda v^*v \Rightarrow v^*Av = \lambda \|v\|_2^2$ — (1)

$\Rightarrow \overline{v^*Av} = \bar{\lambda}\|v\|_2^2 \Rightarrow v^*A^*v = \bar{\lambda}\|v\|_2^2 \Rightarrow v^*Av = \bar{\lambda}\|v\|_2^2$ (as $A^* = A$) — (2)

So from (1) & (2) $0 = v^*Av - v^*Av = (\lambda - \bar{\lambda})\|v\|_2^2$
$\Rightarrow \lambda = \bar{\lambda}$ (as $v \neq 0$)

Also $Av = \lambda v \Rightarrow A(\mathrm{Re}\, v) + i(A\, \mathrm{Im}\, v) = \lambda\, \mathrm{Re}\, v + i\lambda\, \mathrm{Im}\, v$

$\Rightarrow A\, \mathrm{Re}\, v = \lambda\, \mathrm{Re}\, v$ & $A(\mathrm{Im}\, v) = \lambda(\mathrm{Im}\, v)$ [As A & λ are real]

So A has only real eigenvalues and corresponding eigenvectors can be chosen to be real!