MA 322: Scientific Computing

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Chapter 5: Numerical Differentiations and Initial Value Problems for ODEs

Error in second order numerical differentiation

$$f(x) = -\cos x$$

h	$f''(0) - D_h^{(2)} f(0)$	Ratio	$f''(0) - \tilde{D}_h^{(2)} f(0)$
.5	2.07E − 2		2.07E − 2
.25	5.20E − 3	3.98	5.20E − 3
.125	1.30E − 3	3.99	1.30E − 3
.0625	3.25E − 4	4.00	3.25E − 4
.03125	8.14E − 5	4.00	8.45E − 5
.015625	2.03E − 5	4.00	2.56E − 6
.0078125	5.09E − 6	4.00	−7.94E − 5
.00390625	1.27E − 6	4.00	−7.94E − 5
.001953125	3.18E − 7	4.00	−1.39E − 3

Euler method

$$y' = y \qquad y(0) = 1.$$

	x	$y_h(x)$	$Y(x)$	$Y(x) - y_h(x)$
$h = .2$	.40	1.44000	1.49182	.05182
	.80	2.07360	2.22554	.15194
	1.20	2.98598	3.32012	.33413
	1.60	4.29982	4.95303	.65321
	2.00	6.19174	7.38906	1.19732
$h = .1$	.40	1.46410	1.49182	.02772
	.80	2.14359	2.22554	.08195
	1.20	3.13843	3.32012	.18169
	1.60	4.59497	4.95303	.35806
	2.00	6.72750	7.38906	.66156
$h = .05$	.40	1.47746	1.49182	.01437
	.80	2.18287	2.22554	.04267
	1.20	3.22510	3.32012	.09502
	1.60	4.76494	4.95303	.18809
	2.00	7.03999	7.38906	.34907

Euler method

$$y' = 1/(1+x^2) - 2y^2 \qquad y(0) = 0.$$

	x	$y_h(x)$	$Y(x)$	$Y(x) - y_h(x)$
$h = .2$	0.00	0.0	0.0	0.0
	.40	.37631	.34483	−.03148
	.80	.54228	.48780	−.05448
	1.20	.52709	.49180	−.03529
	1.60	.46632	.44944	−.01689
	2.00	.40682	.40000	−.00682
$h = .1$	.40	.36085	.34483	−.01603
	.80	.51371	.48780	−.02590
	1.20	.50961	.49180	−.01781
	1.60	.45872	.44944	−.00928
	2.00	.40419	.40000	−.00419
$h = .05$	.40	.35287	.34483	−.00804
	.80	.50049	.48780	−.01268
	1.20	.50073	.49180	−.00892
	1.60	.45425	.44944	−.00481
	2.00	.40227	.40000	−.00227

Rounding effects in Euler's method

h	x	Chopped Decimal	Rounded Decimal	Exact Arithmetic
.04	1	−1.00E − 2	−1.70E − 2	−1.70E − 2
	2	−1.17E − 2	1.83E − 2	−1.83E − 2
	3	−1.20E − 3	−2.80E − 3	−2.78E − 3
	4	1.00E − 2	1.60E − 2	1.53E − 2
	5	1.13E − 2	1.96E − 2	1.94E − 2
.02	1	7.00E − 3	−9.00E − 3	−8.46E − 3
	2	4.00E − 3	−9.10E − 3	−9.13E − 3
	3	2.30E − 3	−1.40E − 3	−1.40E − 3
	4	−6.00E − 3	8.00E − 3	7.62E − 3
	5	−6.00E − 3	8.50E − 3	9.63E − 3
.01	1	2.80E − 2	−3.00E − 3	−4.22E − 3
	2	2.28E − 2	−4.30E − 3	−4.56E − 3
	3	7.40E − 3	−4.00E − 4	−7.03E − 4
	4	−2.30E − 2	3.00E − 3	3.80E − 3
	5	−2.41E − 2	4.60E − 3	4.81E − 3

Theorem

Assume that the solution $Y(x)$ of IVP has a bounded second derivative on $[x_0, b]$. Then the solution $\{y_h(x_n) : x_0 \leq x \leq b\}$ obtained by Euler's method satisfies

$$\max_{x_0 \leq x \leq b} |Y(x_n) - y_h(x_n)| \leq e^{(b-x_0)K} |e_0 + \left[\frac{2^{(b-x_0)K} - 1}{K}\right] \tau(h),$$

where $\tau(h) = \frac{h}{2}\|Y''\|_\infty$ and $e_0 = Y_0 - y_h(x_0)$. If in addition to the conditions to the above theorem,

$$|Y_0 - y_h(x_0)| \leq c_1 h \qquad as \;\; h \to 0$$

for some $c_0 \geq 0$, then there is a constant $B \geq 0$ for which

$$\max_{x_0 \leq x \leq b} |Y(x_n) - y_h(x_n)| \leq Bh.$$