Shifted QR algorithm

Let $\rho \in \mathbb{F}$. If λ_k be the eigenvalue of A closest to ρ, then let the eigenvalue set of A be reordered if necessary such that

$$|\lambda_1 - \rho| \geq |\lambda_2 - \rho| \geq \cdots \geq |\lambda_{n-1} - \rho| > |\lambda_n - \rho|.$$

If the QR iterations are performed on $B = A - \rho I$ to produce iterates B_j, then $b^{(j)}(n, n-1) \to 0$ linearly at the rate $|\lambda_n - \rho|/|\lambda_{n-1} - \rho|$.

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Shifted QR algorithm: Let $A_0 = A$ and $\rho_j \in \mathbb{F}$ for $j = 0, 1, \ldots.$
for j =1,2, . . .
(i) Find a QR decomposition $\underline{A_{j-1} - \rho_{j-1} I} = Q_{j-1} R_{j-1}$
(ii) Set $A_j = R_{j-1} Q_{j-1} + \rho_{j-1} I$. $= Q_{j-1}^{*} Q_{j-1} R_{j-1} Q_{j-1} + \rho_{j-1} Q_{j-1}^{*} Q_{j-1}$

$= Q_{j-1}^{*} [Q_{j-1} R_{j-1} + \rho_{j-1} I] Q_{j-1}$

$= Q_{j-1}^{*} (A_{j-1}) Q_{j-1}$

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Clearly $A_j = Q_{j-1}^* A_{j-1} Q_{j-1}$ and is again upper Hessenberg !

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Shifted QR algorithm:

for j =1,2, . . .

(i) Find reflectors $Q_{j-1}^{(1)}, Q_{j-1}^{(2)}, \ldots Q_{j-1}^{(n-1)}$ *such that*

$$Q_{j-1}^{(n-1)} \cdots Q_{j-1}^{(2)} Q_{j-1}^{(1)} (A_{j-1} - \rho_{j-1} I) = R_{j-1}$$

(ii) Set $A_j = Q_{j-1}^{(n-1)} \cdots Q_{j-1}^{(2)} Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)} Q_{j-1}^{(2)} \cdots Q_{j-1}^{(n-1)}$.

This costs $O(n^2)$ flops if A is upper Hessenberg and $O(n)$ flops if A is tridiagonal.

Shifted QR algorithms

Where do we get good shifts ?

Raleigh quotient shifts: $\rho_j = a_{nn}^{(j)}$.

Wilkinson shifts: $\rho_j = \lambda$ where λ is the eigenvalue of

$$\begin{bmatrix} a_{n-1,n-1}^{(j)} & a_{n-1,n}^{(j)} \\ a_{n,n-1}^{(j)} & a_{nn}^{(j)} \end{bmatrix}$$

closest to $a_{n,n}^{(j)}$. In case of a tie, the eigenvalue smallest in magnitude is chosen.

For symmetric tridiagonal matrices *QR* iterations with Wilkinson shifts always converge. With Rayleigh Quotient shifts, they almost always converge. Also, whenever convergence occurs, the rate of convergence is usually cubic.

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However, for general matrices both shifting strategies can fail!