MA 322: Scientific Computing

Department of Mathematics
Indian Institute of Technology Guwahati

April 06, 2023

Chapter 5: Numerical Differentiations and Initial Value Problems for ODEs

Errors

In numerically solving a differential equation, several types of arise. These are conveniently classified as follows:

- Local truncation error
- Local round-off error
- Global truncation error
- Global round-off error
- Total error

Errors

- Recall the general form of the multistep methods

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}), \qquad n \geq p.$$

- For any differentiable function $Y(x)$, define the truncation error for integrating $Y'(x)$ by

$$T_n(Y) = Y(x_{n+1}) - \left[\sum_{j=0}^{p} a_j Y(x_{n-j}) + h \sum_{j=-1}^{p} b_j Y'(x_{n-j}) \right] \quad n \geq p.$$

- Define the function $\tau_n(Y)$ by

$$\tau_n(Y) = \frac{1}{h} T_n(Y).$$

Errors

- Convergence of approximate solution $\{y_n : a \leq x_n \leq b\}$ using

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j}), \qquad n \geq p$$

  to the solution $Y(x)$ of the IVP $y' = f(x, y) \qquad y(a) = Y_0$, it is necessary to have

$$\tau(h) = \max_{a \leq x \leq b} |\tau_n(Y)| \to 0 \qquad \text{as} \qquad h \to 0.$$

- This is called the *consistency condition* for the method.

Errors

- The speed of the convergence of the solution $\{y_n\}$ to the true solution $Y(x)$ is related to the speed of convergence in the above limit. Thus, we need to know the conditions under which
$$\tau(h) = \mathbf{O}(h^m)$$
for some desired choice of $m \geq 1$.
- The largest value of m for which the order relation holds is called the *order* or *order of convergence* of the method.

Chapter 6: Boundary Value Problems for ODEs

Shooting method

Consider the BVP

$$y'' = f(x, y, y') \quad a < x < b, \qquad y(a) = y_a, \;\; y(b) = y_b.$$

- Consider an IVP of the form

$$y'' = f(x, y, y') \quad a < x < b, \qquad y(a) = y_a, \;\; y'(a) = \alpha.$$

- Rewrite it as a system of first order equations as follows

$$y' = z, \;\; z' = f(x, y, z) \quad a < x < b, \qquad y(a) = y_a, \;\; z(a) = \alpha_0.$$

- Solve the above IVP to obtain $y(b; \alpha)$.
- We require $\lim_{k \to \infty} y(b; \alpha_k) - y_b = 0$. If $y(b; \alpha_0) \neq y_b$, solve the IVP using $z(a) = \alpha_1$.

Shooting method

- Approximate $\alpha_k = \alpha_{k-1} - \dfrac{y(b;\alpha_{k-1})(\alpha_{k-1} - \alpha_{k-2})}{y(b;\alpha_{k-1}) - y(b;\alpha_{k-1})}$ using Secant method; Or,
- Approximate $\alpha_k = \alpha_{k-1} - \dfrac{y(b;\alpha_{k-1}}{\partial y(b;\alpha_{k-1})/\partial\alpha}$ using Newton's method.

Finite Difference Method

Consider the BVP

$$y'' + a(x)y' + b(x)y = f(x) \quad a < x < b, \qquad y(a) = y_a, \;\; y'(b) = y_b.$$

- Discretize $a \leq x \leq b$ as $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ such that $h = x_{j+1} - x_j,\; 0 \leq j \leq n-1$.
- Approximate the ODE at the points where y is unknown, i.e., $x_1, \; x_2, \; \cdots, \; x_n$ as (using m-th order method)

$$\frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} + a_j \frac{y_{j+1} - y_{j-1}}{2h} + b_j y_j = f_j, \quad 1 \leq j \leq n.$$

$$\Rightarrow \left(\frac{1}{h^2} - \frac{a_j}{2h}\right) y_{j-1} + \left(-\frac{2}{h^2} + b_j\right) y_j + \left(\frac{1}{h^2} + \frac{a_j}{2h}\right) y_{j+1} = f_j, \quad 1 \leq j \leq n.$$

Finite Difference Method

- This can be re-written as as a system of algebraic equations,

$$\mathbf{A}\mathbf{y} = \mathbf{f},$$

where

$$A_{i,i-1} = \frac{1}{h^2} - \frac{a_i}{2h}, \; A_{i,i+1} = \frac{1}{h^2} + \frac{a_i}{2h}, \; A_{i,i} = -\frac{2}{h^2} + b_i.$$

$$\mathbf{y} = [y_1, y_2, \ldots, y_n]^{\mathrm{T}}, \quad \mathbf{f} = [f_1, f_2, \ldots, f_n]^{\mathrm{T}}.$$

- Solve using Gauss-Seidel, Gauss-Jordan, Thomas Algorithm, etc.