COMP9020 20T1 | Foundations of Computer Science

Week 9 Problem Set
Probability

[Show with no answers] [Show with all answers]

1. (General probability)

   Consider two events A and B such that $0 < P(B) < 1$. For each of the following statements, decide whether they are always true for any such A and B, always false or could be either.

   a. $P(A \cup B) \geq P(A) + P(B)$
   b. $P(A \mid B) \cdot P(A) = P(A \cap B)$
   c. $P(A \cup B \mid B) \geq P(A \mid B)$
   d. $P(A \cap B \mid B) + P(A \cap B^c \mid B) = 1$
   e. $P(A \mid B) + P(A^c \mid B) < 1$
   f. $A \perp B \Rightarrow P(A \mid B) = P(A \mid B^c)$
   g. $A \perp B \Rightarrow P(A \cup B) = P(A) + P(B)$

   [hide answer]

   a. **Could be either**. For example, if $A = \emptyset$ then $P(A) = 0$, hence $P(A \cup B) = P(B) = P(A) + P(B)$; but if $A = B$ then $P(A) = P(B)$ and $P(A \cup B) = P(B) < P(B) + P(B)$ (since $P(B) > 0$).
   b. **Could be either.** By definition, $P(B \mid A) \cdot P(A) = P(A \cap B)$, but $P(B \mid A)$ and $P(A \mid B)$ are not related in general.
   c. **Always true.** $B \subseteq A \cup B$, hence $P(A \cup B \mid B) = 1 \geq P(A \mid B)$.
   d. **Could be either.** By definition, $P(A \cap B^c \mid B) = \frac{P(A \cap B^c \cap B)}{P(B)} = 0$, since $A \cap B \cap B^c = \emptyset$. Hence, $P(A \cap B \mid B) + P(A \cap B^c \mid B) = 1$ iff $P(A \cap B \mid B) = 1$ iff $P(A \mid B) = 1$, which may or may not be true.
   e. **Always false.** $P(A^c \mid B) = 1 - P(A \mid B)$, hence $P(A^c \mid B) + P(A \mid B) = 1$.
   f. **Always true.** If $A \perp B$ then also $A \perp B^c$ (note that $P(B^c) > 0$ since $P(B) < 1$ by assumption). It follows that $P(A \mid B) = P(A) = P(A \mid B^c)$.
   g. **Could be either.** If $A \perp B$ then $P(A \cap B) = P(A) \cdot P(B)$. From $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ it then follows that $P(A \cup B) = P(A) + P(B)$ iff $P(A) \cdot P(B) = 0$, which is true if $P(A) = 0$ and false if $P(A) > 0$ (since $P(B) > 0$).

2. (Events and probability)

   a. Let E_1, E_2 be two events. Prove that $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$ implies $P(E_2 \setminus E_1) = 0$.
   b. A red, a blue and a green die are rolled simultaneously.
      i. What is the probability that the sum of the 3 values is a prime number?
      ii. What is the probability of a doublet (2 of the 3 values are equal but the third value is different)?
      iii. What is the probability that all 3 values are different and the value of the red die is higher than the other two?

   [hide answer]

   a. By definition,

   $$P(E_1 \setminus E_2) = \sum_{\omega \in E_1 \setminus E_2} P(\omega) = \sum_{\omega \in E_1} P(\omega) - \sum_{\omega \in E_1 \cap E_2} P(\omega) = P(E_1) - \sum_{\omega \in E_1 \cap E_2} P(\omega)$$

   Hence, if $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$ then $\Sigma_{\omega \in E_1 \cap E_2} P(\omega) = \Sigma_{\omega \in E_2} P(\omega)$. Therefore, $P(\omega) = 0$ for all $\omega \in E_2 \setminus E_1$, hence $\Sigma_{\omega \in E_2 \setminus E_1} P(\omega) = 0$.

   b. Possible outcomes are $(r,b,g) \in \Omega = \{1,\ldots,6\} \times \{1,\ldots,6\} \times \{1,\ldots,6\}$. Hence $|\Omega| = 216$.

i. Possible prime sums are: 3 (1 outcome), 5 (6 outcomes), 7 (15 outcomes), 11 (27 outcomes), 13 (21 outcomes) and 17 (3 outcomes). Hence, the overall probability is $\frac{1+6+15+27+21+3}{216} = \frac{73}{216} \approx 0.338$.

ii. Select 2 out of 3 dice to have the same value, which can be any of 1…6, while the third number is different, hence one of the 5 remaining values. Thus the overall probability is:

$$\frac{\binom{3}{2} \cdot 6 \cdot 5}{216} = \frac{90}{216} = \frac{5}{12} \approx 0.417$$

iii. Possible values for the red die are $r \in \{3,\dots,6\}$. For each outcome r there are $\Pi(r\text{-}1,2)$ possible vaues for the other two dice. Hence the overall probability is:

$$\frac{2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4}{216} = \frac{40}{216} = \frac{5}{27} \approx 0.185$$

3. (Counting and probability)

Let $\Sigma = \{a, c, e, n, s\}$. Suppose we choose a 4-letter word at random from Σ^4. What is the probability that the letters of the word are in alphabetical order (e.g. as in $aces$ or $cees$ but not in $sees$)?

[hide answer]

There are $|\Sigma|^4 = 625$ words in Σ^4. Of these, $1 + 4 + 10 + 20 + 35 = 70$ are in alphabetical order (see Exercise 4c in Problem Set Week 8 on how to count them). Hence, the probability is $\frac{70}{625} = 0.112$.

4. (Probability of sequences)

Alice and Bob repeatedly toss a coin (outcome H – head, or T – tails) until either Alice's or Bob's winning sequence is observed. What is the probability for Alice to win if

a. Alice's winning sequence is HTH and Bob's is HHH?

b. Alice's winning sequence is HTH and Bob's is THT?

[hide answer]

a. Let p = P(HTH comes first). Consider the (recursive) tree of possible outcomes:

HHH	loss (⇒ not counted)
HHTH	win (with probab. 1/16)
HHTT	p
HTH	win (with probab. 1/8)
HTT	p
T	p

Therefore, $p = \frac{1}{16} + \frac{1}{16}p + \frac{1}{8} + \frac{1}{8}p + \frac{1}{2}p \;\Rightarrow p = \frac{3}{5}$.

b. This is obviously symmetric, so the probability must be $\frac{1}{2}$.

5. (Conditional Probability)

a. Prove that

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot \ldots \cdot P(A_n|A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

b. Consider three urns: Urn 1 contains one red and two black marbles, Urn 2 contains three red and four black marbles, Urn 3 contains two red and two black marbles. One urn is selected at random and then two marbles are randomly drawn from that urn without replacement. Given that these two marbles are red, what is the probability that Urn 2 was chosen?

[hide answer]

a. Base case: $P(A_1) = P(A_1)$.

Inductive step:

$$\begin{aligned}
&P((A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}) \\
&\quad = P(A_{n+1} \mid A_1 \cap \ldots \cap A_n) \cdot P(A_1 \cap \ldots \cap A_n) && \text{by def. of cond. probab.} \\
&\quad = P(A_{n+1} \mid A_1 \cap \ldots \cap A_n) \cdot P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \ldots \cap A_{n-1}) && \text{by ind. hyp.}
\end{aligned}$$

b. Let Urn_i denote the event that the i-th urn was selected and TR the event of two red marbles being drawn.

$$P(TR \mid Urn_1) = 0\,; \quad P(TR \mid Urn_2) = \tfrac{3}{7} \cdot \tfrac{2}{6}\,; \qquad P(TR \mid Urn_3) = \tfrac{2}{4} \cdot \tfrac{1}{3}$$
$$P(TR \cap Urn_1) = 0\,; \quad P(TR \cap Urn_2) = (\tfrac{3}{7} \cdot \tfrac{2}{6}) \cdot \tfrac{1}{3}\,; \quad P(TR \cap Urn_3) = (\tfrac{2}{4} \cdot \tfrac{1}{3}) \cdot \tfrac{1}{3}$$

$$P(TR) = P(TR \cap Urn_1) + P(TR \cap Urn_2) + P(TR \cap Urn_3) = \tfrac{13}{126}$$
$$P(Urn_2 \mid TR) = \tfrac{P(TR \cap Urn_2)}{P(TR)} = \tfrac{126}{21 \cdot 13} = \tfrac{6}{13}$$

6. **Challenge Exercise**

a. Jamie and Charlie have two kids, one of which is a girl. Assume that the probability of each gender is ½. What is the probability that the other kid is also a girl?

b. There are three cards in a hat. One card is red on both sides, one card is blue on both sides, and one card is red on one side and blue on the other. You draw one card at random, put it down on the table and then observe that the side of the card facing up is red. What is the probability that the other side is red too?

[hide answer]

a. For a two-child family, there are four equally probable events: girl, girl (probability ¼); girl, boy (probability ¼); boy, girl (probability ¼); boy, boy (probability ¼). The last event does not meet the condition that at least one of the kids is a girl. Of the three remaining possibilities, only one satisfies the condition that the other kid is also a girl. Hence the answer is ⅓.

Note that this answer assumes the following randomisation process: Consider all families consisting of parents named Jamie and Charlie and two children, at least one of which is a girl. Choose one such family at random. The probability, then, that the second child is also a girl is ⅓.

NB. This is a variation of the famous "Boy or Girl paradox".

b. Each of the three cards is drawn with probability ⅓. Given that the side of the card facing up is red, only the red-red or the red-blue card could have been drawn. In case of the red-red card there are two possible sides that may be facing up, whereas in case of the red-blue card there is only one possible side that is facing up. It is therefore twice as likely that you drew the red-red card. Hence the answer is ⅔.

Alternatively, you can work this out directly from the conditional probability:

$$P(\text{Red}_{\textit{underside}} \mid \text{Red}_{\textit{upside}}) = \frac{P(\text{Red}_{\textit{underside}} \cap \text{Red}_{\textit{upside}})}{P(\text{Red}_{\textit{upside}})} = \frac{⅓}{½} = \frac{2}{3}$$

Explanation: There is exactly one card with two red sides, hence the ⅓ in the nominator. And of the six sides that you could see when the card is on the table, exactly three are red. Hence the ½ in the denominator.

(Wait, you may ask: Wouldn't a similar equation show that the other side is blue with probability ⅔ too:

$$P(\text{Blue}_{\textit{underside}} \mid \text{Red}_{\textit{upside}}) = \frac{P(\text{Blue}_{\textit{underside}} \cap \text{Red}_{\textit{upside}})}{P(\text{Red}_{\textit{upside}})} = \frac{⅓}{½} = \frac{2}{3}$$

Why is this **not** correct?)

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 9 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 23 April 10:00:00am**.

As always please respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult