COMP9020 20T1

Week 4 Problem Set Equivalence Relations, Orderings

Foundations of Computer Science

[Show with no answers] [Show with all answers]

1. (Equivalence relations)

For each of the following relations $\mathcal{R}$, prove or disprove that $\mathcal{R}$ is an equivalence relation.

a. Real number r is related to real number s iff $|r-s| \leq 1$.

b. Pair of integers (i,j) is related to pair of integers (m,n) iff $i+j \equiv m-n \pmod 2$.

c. Set A is related to set B iff $A \cap B = A$ and $B \in Pow(A)$.

d. Let $\Sigma = \{a,b\}$. Word $\nu \in \Sigma^*$ is related to word $\omega \in \Sigma^*$ iff $\nu = \omega\chi$ for some word $\chi \in \Sigma^*$.

e. Propositional formula ϕ is related to propositional formula ψ iff $\phi \models \psi$ and $\models \psi \Rightarrow \phi$.

f. Over the standard Boolean algebra, $x, y \in \mathbb{B} = \{0,1\}$ are related iff $(x+y') \cdot (x'+y) = 0$.

g. 1×2 matrix **A** is related to 1×2 matrix **B** iff **A** = **B** or **A**·**B**^{T} = 0.

[hide answer]

a. Not an equivalence since the relation is not transitive. Example: 3.1 and 2.2 are related, 2.2 and 1.3 are related, but 3.1 and 1.3 are not related.

b. Equivalence. Proof: $x+y$ is even iff x and y are of the same *parity*, that is, they are both even or both odd. Likewise, $x-y$ is even iff x and y are of the same parity. Hence, all pairs with numbers of the same parity are related to each other and all pairs with numbers of different parity are related to each other. This divides all pairs of numbers into two equivalence classes, hence the relation is an equivalence.

c. Equivalence. Proof: $A \cap B = A$ is true iff $A \subseteq B$, and $B \in Pow(A)$ is true iff $B \subseteq A$. Hence, A and B are related iff $A = B$, which is obviously reflexive, symmetric and transitive.

d. Not an equivalence since the relation is not symmetric. Example: ab is related to a since $ab = a\chi$ for $\chi = b$, but a is not related to ab since there is no χ such that $a = ab\chi$.

e. Equivalence. Proof: $\phi \models \psi$ is the same as $\models \phi \Rightarrow \psi$. Hence, ϕ is related to ψ iff $\models (\phi \Rightarrow \psi) \wedge (\psi \Rightarrow \phi)$, that is, iff $\models \phi \Leftrightarrow \psi$. The latter means that ϕ and ψ are *logically* equivalent, which can be easily shown to satisfy all properties of an equivalence relation.

f. Not an equivalence since the relation is not reflexive: $x = 0$ is not related to itself since $(0+0') \cdot (0'+0) = 1 \cdot 1 = 1$.

g. Not an equivalence since the relation is not transitive. Example: $[1\ 1]$ and $[-1\ 1]$ are related, $[-1\ 1]$ and $[-1\ \ -1]$ are related, but $[1\ 1] \cdot [-1\ \ -1]^T = -2$, hence $[1\ 1]$ and $[-1\ \ -1]$ are not related.

2. (Modular arithmetic)

a. Prove that if $m, n \in \mathbb{Z}$ and $m \equiv n \pmod{p}$ then $m^2 \equiv n^2 \pmod{p}$.

b. Let $p = 3$. Give all equivalence classes for $m^2 \equiv n^2 \pmod{p}$ where $m, n \in \mathbb{Z}$.

[hide answer]

a. If $m \equiv n \pmod{p}$ then $m = k \cdot p + r$ and $n = l \cdot p + r$ for some $k, l \in \mathbb{Z}$ and $r \in \{0, \ldots, p-1\}$. Then,

$$\begin{aligned} m^2 &= k^2p^2 + 2kpr + r^2 \\ n^2 &= l^2p^2 + 2lpr + r^2 \end{aligned}$$

Hence, $m^2 \bmod p = r^2 \bmod p = n^2 \bmod p$, therefore $m^2 \equiv n^2 \pmod{p}$.

b. From a. we know that $m^2 \equiv n^2 \pmod{3}$ if~~, and only if,~~ $m \equiv n \pmod{3}$. Also, $m^2 \equiv n^2 \pmod{3}$ if $m \equiv -n \pmod{3}$. Hence the ~~three~~ two equivalence classes are:

- $\{\ldots, -6, -3, 0, 3, 6, \ldots\}$
- $\{\ldots, -5, -4, -2, -1, 1, 2, 4, 5 \ldots\}$

3. (Partial versus total orders)

Consider the relation $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}$ defined by $(a, b) \in \mathcal{R}$ iff either $a \leq b - 0.5$ or $a = b$.

Show that $\mathcal{R}$ is a partial order, but not a total order.

[hide answer]

- $\mathcal{R}$ is reflexive: For every a, b such that $a = b$, by definition $(a, b) \in \mathcal{R}$.
- $\mathcal{R}$ is antisymmetric: For any $a \neq b$, if $(a, b) \in \mathcal{R}$ then it must be that $a \leq b - 0.5$, therefore $b \geq a + 0.5 > a - 0.5$ so $(b, a) \notin \mathcal{R}$.
- $\mathcal{R}$ is transitive: For any a, b, c, this is trivial if $a = b$ or $b = c$, otherwise if $a \neq b$ and $b \neq c$ then

$$\begin{aligned} & (a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \\ \Rightarrow\ & a \leq b - 0.5 \wedge b \leq c - 0.5 \\ \Rightarrow\ & a \leq c - 1 \\ \Rightarrow\ & a \leq c - 0.5 \\ \Rightarrow\ & (a, c) \in \mathcal{R} \end{aligned}$$

Therefore $\mathcal{R}$ is a partial order. It is not a total order since any pair a, b that satisfies $a < b < a + 0.5$ (e.g. $a = 0$ and $b = 0.1$) are not related in either direction (e.g. $0 \not\mathcal{R}\, 0.1$ and $0.1 \not\mathcal{R}\, 0$).

4. (Partial orders)

For each of the following relations, prove or disprove that it is a partial order.

a. Natural number m is related to natural number n iff $n^3 > m^2 + 1$.

b. Natural number m is related to natural number n iff $\lceil m + 0.5 \rceil > n$.

c. Positive integer m is related to positive integer n iff $3m > 2n$.

d. Positive integer m is related to positive integer n iff $\gcd(m,n) = n$.

e. Positive integer m is related to positive integer n iff m is a prime divisor of n.

f. Integer m is related to integer n iff $m^3 \leq n^3$.

[hide answer]

a. Not a partial order. Counterexample: $1^3 \not> 1^2 + 1$, hence the relation is not reflexive.

b. Partial order. $\lceil m + 0.5 \rceil > n$ if, and only if, $m + 1 > n$, which is true iff $m \geq n$. The latter is obviously reflexive, antisymmetric and transitive.

c. Not a partial order. Counterexample: 3 and 4 are related since 9>8, 4 and 3 are related since 12>6, but 3≠4, hence the relation is not antisymmetric.

d. Partial order. $\gcd(m,n) = n$ if, and only if, n is a divsor of m, which is reflexive, antisymmetric and transitive.

e. Not a partial order. Counterexample: 4 is not a prime divisor of 4, hence the relation is not reflexive.

f. Partial order. $m^3 \leq n^3$ if, and only if, $m \leq n$, which is reflexive, antisymmetric and transitive.

5. (Lattices)

a. Draw a Hasse diagram for the following partially ordered set:

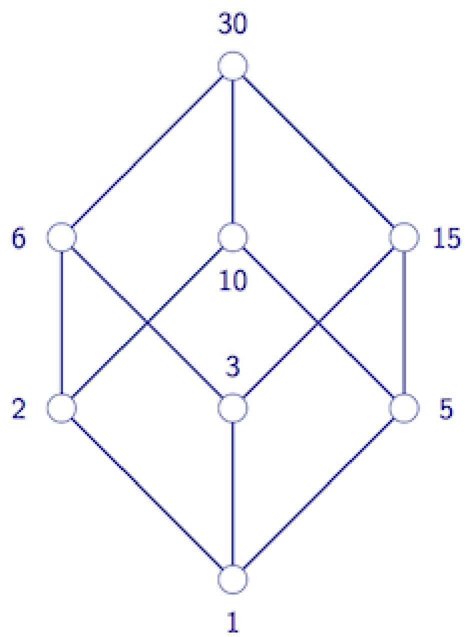
$$S = \{1, 2, 3, 5, 6, 10, 15, 30\}$$
$$x \preceq y \text{ iff } x \mid y$$

Is $(S, \preceq)$ a lattice? Why or why not?

b. Let binary relation $\mathcal{R}$ on $\{1, \ldots, 20\}$ be defined by $a\mathcal{R}b$ iff either $a = b$ or $a - b > 10$. Show that $(\{1, \ldots, 20\}, \mathcal{R})$ is a partial order. Is $(\{1, \ldots, 20\}, \mathcal{R})$ a lattice? Why or why not?

[hide answer]

a. A Hasse diagram:



It is a lattice: S contains all divisors of 30, hence $\text{lub}(x, y) = \text{lcm}(x, y)$ and $\text{glb}(x, y) = \text{gcd}(x, y)$ always exist in S for every $x, y \in S$.

b. It is easy to verify that $\mathcal{R}$ is reflexive, antisymmetric and transitive in the same way as for Exercise 3:

- $\mathcal{R}$ is reflexive: For every a, b such that $a = b$, by definition $(a, b) \in \mathcal{R}$.
- $\mathcal{R}$ is antisymmetric: For any $a \neq b$, if $(a, b) \in \mathcal{R}$ then it must be that $a - b > 10$, therefore $b - a < -10$ so $(b, a) \notin \mathcal{R}$.
- $\mathcal{R}$ is transitive: For any a, b, c, this is trivial if $a = b$ or $b = c$, otherwise if $a \neq b$ and $b \neq c$ then

$$\begin{aligned} & (a, b) \in \mathcal{R} \wedge (b, c) \in \mathcal{R} \\ \Rightarrow\ & a - b > 10\ \wedge\ b - c > 10 \\ \Rightarrow\ & (a - b) + (b - c) > 20 \\ \Rightarrow\ & a - c > 20 \\ \Rightarrow\ & a - c > 10 \\ \Rightarrow\ & (a, c) \in \mathcal{R} \end{aligned}$$

It is not a lattice; for example, {1,2} does not have a greatest lower bound (or, in fact, any lower bound — both 1 and 2 are minimal elements).

6. **Challenge Exercise**

Using the set $\{1, \ldots, 10\}$ with the natural total order, define $A = \{1, \ldots, 10\} \times \{1, \ldots, 10\}$ and consider these two orderings over A:

a. product $\sqsubseteq_P$

b. lexicographic $\sqsubseteq_L$

Find the maximal length of a chain $a_1 \sqsubseteq a_2 \sqsubseteq \ldots \sqsubseteq a_n$ (such that $a_i \neq a_{i+1}$) for each of the two orderings.

[show answer]

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 4 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 19 March 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline

- agonise too much about a question that you find too difficult