COMP9020 20T1

Week 3 Problem Set
Functions and Relations

Foundations of Computer Science

[Show with no answers] [Show with all answers]

1. (Functions)

   Let $\Sigma = \{0, 1\}$. Consider the functions $f, g : \Sigma^* \longrightarrow \Sigma^*$ and $h : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^*$ given by

   - $f(\omega) = \omega\omega$
   - $g(\omega) = 0\omega 1$
   - $h(\nu, \omega) = \nu\omega\nu$

   Compute the following function values:

   a. $(f \circ g)(10)$

   b. $(g \circ f)(10)$

   c. $h(10, f(01))$

   d. $h(f(1), g(0))$

   [hide answer]

   a. $(f \circ g)(10) = f(g(10)) = f(0101) = 01010101$

   b. $(g \circ f)(10) = g(f(10)) = g(1010) = 010101$

   c. $h(10, f(01)) = h(10, 0101) = 10010110$

   d. $h(f(1), g(0)) = h(11, 001) = 1100111$

2. (Properties of functions)

   Which of the three functions f, g and h in Exercise 1 is onto? Which are 1-1?

   [hide answer]

   - f is not onto; e.g. $101 \notin \text{Im}(f)$ because there is no ω such that $f(\omega) = \omega\omega = 101$.

     g is not onto; e.g. $0 \notin \text{Im}(g)$ because there is no ω such that $g(\omega) = 0\omega 1 = 0$.

     h is onto: For every word $\omega \in \Sigma^*$ we have that $h(\lambda, \omega) = \lambda\omega\lambda = \omega$, hence $\text{Im}(h) = \Sigma^*$.

   - f is 1-1: You cannot find two different words $\omega \neq \omega'$ such that $\omega\omega = \omega'\omega'$.

     g is 1-1: You cannot find two different words $\omega \neq \omega'$ such that $0\omega 1 = 0\omega' 1$.

     h is not 1-1; e.g. $h(1, 0) = h(\lambda, 101) = 101$, hence $h(\nu, \omega) = h(\nu', \omega')$ does not imply $(\nu, \omega) = (\nu', \omega')$.

3. (Matrix functions)

   Prove each of the following statements.

a. $(\mathbf{A}^T)^T = \mathbf{A}$ for any matrix **A**.

b. If two matrices **A** and **B** are of the same size, then $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

c. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ for any matrix **A** of size $m \times n$ and matrices **B**, **C** of size $n \times p$.

[hide answer]

a. If **A** is of size $m \times n$ then $\mathbf{A}^\mathrm{T}$ is of size $n \times m$, hence $(\mathbf{A}^\mathrm{T})^\mathrm{T}$ is of size $m \times n$, the same as **A**. The *(i,j)*-th entry of $\mathbf{A}^\mathrm{T}$ is a_{ji}, hence the *(i,j)*-th entry of $(\mathbf{A}^\mathrm{T})^\mathrm{T}$ is a_{ij}, the same as **A**.

b. The *(j,i)*-th entry of **A** + **B** is $a_{ji} + b_{ji}$, hence the *(i,j)*-th entry of $(\mathbf{A} + \mathbf{B})^\mathrm{T}$ is $a_{ji} + b_{ji}$. The *(i,j)*-th entry of $\mathbf{A}^\mathrm{T}$ is a_{ji} and the *(i,j)*-th entry of $\mathbf{B}^\mathrm{T}$ is b_{ji}, hence the *(i,j)*-th entry of $\mathbf{A}^\mathrm{T} + \mathbf{B}^\mathrm{T}$ is $a_{ji} + b_{ji}$. This proves the claim.

c. The *(j,k)*-th entry of **B** + **C** is $b_{jk} + c_{jk}$, hence the *(i,k)*-th entry of **A**(**B** + **C**) is $\sum_{j=1}^{n} a_{ij} \cdot (b_{jk} + c_{jk})$.

The *(i,k)*-th entry of **AB** is $\sum_{j=1}^{n} a_{ij} b_{ij}$ and the *(i,k)*-th entry of **AC** is $\sum_{j=1}^{n} a_{ij} c_{jk}$, hence the *(i,k)*-th entry of **AB** + **AC** is $\sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk}$.

This proves the claim since $\sum_{j=1}^{n} a_{ij} \cdot (b_{jk} + c_{jk}) = \sum_{j=1}^{n} a_{ij} b_{jk} + a_{ij} c_{jk} = \sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk}$.

4. (Boolean functions)

a. Give all elements of BOOL(2), that is, all functions $\mathbb{B}^2 \longrightarrow \mathbb{B}$ over two Boolean variables.

b. Show that there are 2^{2^n} elements in BOOL(*n*) for $n \in \mathbb{P}$.

[hide answer]

a. There are 16 functions of type f: $\mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$. They can be described as follows:

x	**y**	**f_1(x,y)**
0	0	0
0	1	0
1	0	0
1	1	0

x	**y**	**f_2(x,y)**
0	0	0
0	1	0
1	0	0
1	1	1

x	**y**	**f_3(x,y)**
0	0	0
0	1	0
1	0	1
1	1	0

…

x	**y**	**f_{16}(x,y)**
0	0	1
0	1	1
1	0	1
1	1	1

Alternatively, you can specify them by Boolean expressions:

- f_1: $(x,y) \mapsto 0$
- f_2: $(x,y) \mapsto x \cdot y$
- f_3: $(x,y) \mapsto x \cdot y'$
- …
- f_{16}: $(x,y) \mapsto 1$

b. If $f \in \text{BOOL}(n)$ then f is a function over n Boolean variables, that is, $\text{Dom}(f) = \times_{i=1}^{n}\{0,1\}$; hence, $|\text{Dom}(f)| = 2^n$. The result of a Boolean function is either 0 or 1, that is, $\text{Codom}(f) = \{0,1\}$, hence $|\text{Codom}(f)| = 2$.
For each element in the domain, a function can choose any of the elements from the codomain as the function value; hence, there are $|\text{Codom}(f)|^{|\text{Dom}(f)|} = 2^{2^n}$ different functions.

5. (Properties of binary relations)

a. Consider the relation $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}$ defined by

$$(a,b) \in \mathcal{R} \text{ iff } b + 0.5 \geq a \geq b - 0.5$$

Which of the following standard properties does $\mathcal{R}$ satisfy?

- Reflexivity
- Antireflexivity
- Symmetry
- Antisymmetry
- Transitivity

b. For each of the following statements, give a valid proof if it is true for all relations $\mathcal{R}_1 \subseteq S \times S$ and $\mathcal{R}_2 \subseteq S \times S$ over arbitrary sets S. If the statement is not always true, provide a counterexample.

- If $\mathcal{R}_1$ and $\mathcal{R}_2$ are symmetric, then $\mathcal{R}_1 \cap \mathcal{R}_2$ is symmetric.
- If $\mathcal{R}_1$ and $\mathcal{R}_2$ are antisymmetric, then $\mathcal{R}_1 \cup \mathcal{R}_2$ is antisymmetric.

[hide answer]

a. Relation $\mathcal{R}$ is

- reflexive, since $a + 0.5 \geq a \geq a - 0.5$ for all $a \in \mathbb{R}$;
- not irreflexive, see above;
- symmetric, since $(b + 0.5 \geq a) \wedge (a \geq b - 0.5)$ implies $(b \geq a - 0.5) \wedge (a + 0.5 \geq b)$;
- not antisymmetric, e.g. $(0, 0.1) \in \mathcal{R}$ and $(0.1, 0) \in \mathcal{R}$;
- not transitive, e.g. $(1.1, 1.5) \in \mathcal{R}$ and $(1.5, 1.9) \in \mathcal{R}$ but $(1.1, 1.9) \notin \mathcal{R}$ since $1.9 - 0.5 > 1.1$.

b. The first statement is true: If $(x,y) \in \mathcal{R}_1 \cap \mathcal{R}_2$, then $(x,y) \in \mathcal{R}_1$ and $(x,y) \in \mathcal{R}_2$. Since $\mathcal{R}_1$ and $\mathcal{R}_2$ are both symmetric, it follows that $(y,x) \in \mathcal{R}_1$ and $(y,x) \in \mathcal{R}_2$, hence $(y,x) \in \mathcal{R}_1 \cap \mathcal{R}_2$.

The second statement is false. Counterexample: For $S = \{a,b\}$, let $\mathcal{R}_1 = \{(a,b)\}$ and $\mathcal{R}_2 = \{(b,a)\}$ then both $\mathcal{R}_1$ and $\mathcal{R}_2$ are antisymmetric, but $\mathcal{R}_1 \cup \mathcal{R}_2 = \{(a,b),(b,a)\}$ is not.

6. **Challenge Exercise**

Consider a set U and the binary relation $\mathcal{R}$ on $Pow(U)$ defined by $(A,B) \in \mathcal{R}$ iff $|A \cap B| \geq 1$. Prove that $\mathcal{R}$ is transitive if and only if $|U| \leq 1$.

[show answer]

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 3 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 12 March 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult