Week 5 Problem Set
Graphs and Trees

[Show with no answers] [Show with all answers]

1. (Graph properties)

   True or false?

   a. The complete bipartite graph $K_{5,5}$ has no cycle of length five.

   b. If you add a new edge to a cycle, the resulting graph will always contain a 3-clique.

   c. It is possible to remove two edges from K_6 so that the resulting graph has a clique number of 4.

   d. There are exactly 3 automorphisms of K_3.

   [hide answer]

   a. **True.** In general, any bipartite graph can only have cycles of an even length. If V_1, V_2 denote the two groups of nodes in the bipartite graph, then every path starting from, say, a node $v \in V_1$ must alternate its vertices between V_1 and V_2. If it is a cycle (that is, eventually returns to node v, it must have an even number of such alternations, in other words, it contains an even number of edges.

   b. **False.** For example, a cycle C_6 will still have a clique number of 2 (i.e., no 3-cliques) after adding an edge between two nodes that are 3 edges apart.

   c. **True.** Remove any two edges which have no common vertex. In the resulting graph there will be no five fully interconnected vertices; in other words it will not contain K_5, and will have a clique number of 4.

   d. **False.** There are 6 automorphisms of K_3, one for each permutation of the vertices:

   - $v_1 \mapsto v_1$, $v_2 \mapsto v_2$, $v_3 \mapsto v_3$
   - $v_1 \mapsto v_1$, $v_2 \mapsto v_3$, $v_3 \mapsto v_2$
   - $v_1 \mapsto v_2$, $v_2 \mapsto v_1$, $v_3 \mapsto v_3$
   - $v_1 \mapsto v_2$, $v_2 \mapsto v_3$, $v_3 \mapsto v_1$
   - $v_1 \mapsto v_3$, $v_2 \mapsto v_1$, $v_3 \mapsto v_2$
   - $v_1 \mapsto v_3$, $v_2 \mapsto v_2$, $v_3 \mapsto v_1$

2. (Graph traversal)

   For each of the following graphs, show a Hamiltonian path or argue why no such path exists.

   a. The graph on slide 18 (week 5).

   b. $K_{3,4}$

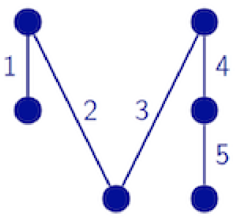
   c. $K_{1,4,1}$

   d. $K_{2,2,4}$

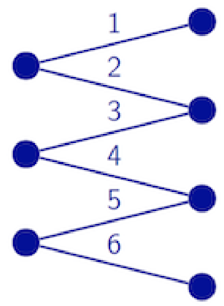
   e. A graph with 5 nodes and degree sequence 0, 0, 5.

   f. A tree with 5 nodes and degree sequence 0, 3, 1, 1.

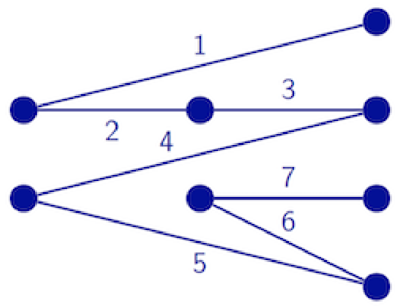
   [hide answer]



a.



b.



d.

   c. There is no Hamiltonion path in $K_{1,4,1}$: Even if you start with one of the 4 vertices in the largest partition, you can visit at most 3 of them before you have to return to a previously visited vertex from one of the other two partitions.

   e. Any graph on 5 nodes and where each node has degree 2 constitutes a cycle C_5, which is a Hamiltonian path by itself.

f. The top right tree on slide 17 (week 5) is the only tree on 5 nodes that has one vertex of degree 3, one vertex of degree 2 plus three vertices of degree 1. This tree has no Hamiltonian path since you have to go through the top node at least twice in order to visit all three nodes of degree 1.

3. (Graph colouring)

For each of the following graphs *G*, determine its chromatic number $\chi(G)$.

a. The graph on slide 18 (week 5).

b. A graph obtained by adding one new edge to C_4.

c. A graph obtained by removing one edge from $K_{2,2}$.

d. A graph obtained by removing one edge from K_4.

[hide answer]

a. The graph has a 3-clique, hence 3 colours are necessary. It is easy to find such a 3-colouring, hence $\chi(G) = 3$.

b. $\chi(C_4) = 2$, but with one edge added the graph will have a 3-clique, hence $\chi(G) = 3$.

c. Removing one edge from $K_{2,2}$ results in a tree, hence $\chi(G) = 2$.

d. $\chi(K_4) = 4$, but if one edge (*v*,*w*) is removed, then *v* and *w* can be coloured the same, hence $\chi(G) = 3$.

4. (Planar graphs)

True or false?

a. A forest is always planar.

b. All graphs with 6 nodes and 8 edges are planar.

c. All graphs whose clique number is 2 are planar.

d. You can obtain a nonplanar graph by adding 3 edges to a cycle.

e. When you remove two edges from K_6, you will never obtain a planar graph.

f. All graphs whose chromatic number is 4 are planar.

[hide answer]

a. **True.** Trees are acyclic, hence cannot contain, as a minor, K_5 or $K_{3,3}$ (both of which have cycles).

b. **True.** K_5 requires 10 edges and $K_{3,3}$ 9 edges, hence there cannot be a nonplanar graph with only 8 edges.

c. **False.** For example, $K_{3,3}$ has a clique number of 2 but is not planar.

d. **True.** You can add 3 edges to C_6 (a graph with 6 vertices and 6 edges) in such a way that you obtain the nonplanar $K_{3,3}$ (a graph with 6 vertices and 9 edges).

e. **True.** The remaining graph will always contain $K_{3,3}$, which is nonplanar.

f. **False.** For example, $K_{3,3}$ has a chromatic number of only 2 but is not planar. You can add more edges to turn $K_{3,3}$ into a graph whose chromatic number is 4 (and which, of course, remains nonplanar).

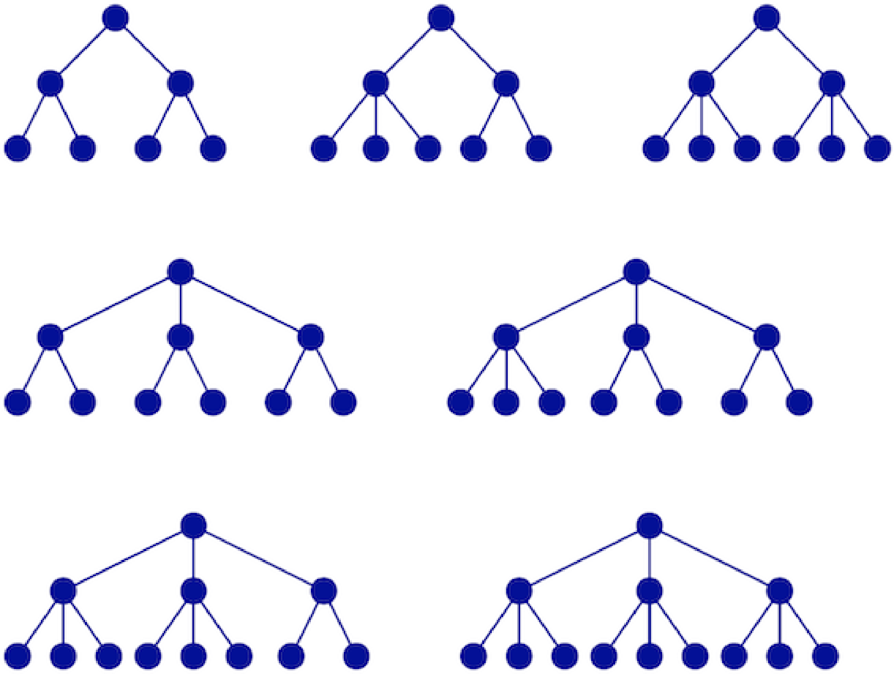
5. (Constructing graphs)

A graph *G* is a 2-3 tree if:

- *G* is a rooted tree.
- Each node has either 2 or 3 children (unless it is a *leaf* node, which has no children).
- All paths from the root to the leaves have the same length.

There are seven different types of 2-3 trees of height 2 (i.e., which are non-isomorphic). Draw one tree of each type.

[hide answer]



6. **Challenge Exercise**

   a. What is the minimum number of edges that need to be removed from K_5 so that the resulting graph has a chromatic number of

      - 3 ?
      - 2 ?
      - 1 ?

   b. Give a planar drawing of $K_{2,2,2}$. Can you find one with only straight lines?

[hide answer]

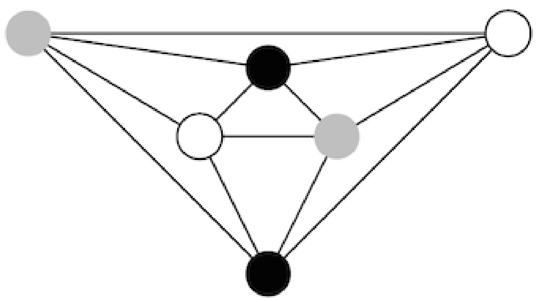
a. $\chi(K_5) = 5$. Minimum number of edges that need to be removed:

   - 2 edges. To achieve $\chi = 3$, one needs (at least) to avoid having any 4-cliques. Removing one edge leaves a 4-clique (actually two such cliques — including any one but not both of that edge's endpoints, plus the remaining three vertices). Removing two edges suffices — remove any pair of edges which do not share a common vertex; the remaining graph can then be coloured with 3 colours.

   - 4 edges. A chromatic number of 2 means that the graph is bipartite, with two groups of nodes where each group can be painted with one colour. To minimise the number of *removed* edges, we want to have as many edges as possible in the remaining bipartite graph. We therefore look at *complete bipartite* graphs with a total of 5 vertices. As $K_{1,4}$ has four edges and $K_{2,3}$ has six edges, the latter is the better choice. To reach it we need to remove 4 of the edges in K_5.

   - 10 edges. A chromatic number of 1 means a fully disconnected graph, with no edges at all. Therefore all 10 edges of the original graph must be removed.

b. A planar drawing of $K_{2,2,2}$ with straight lines:



7. Mid-term practice test

   To prepare for the mid-term online test on **Friday, 27 March, 2:30pm** go to COMP9020 20T1 Practice Test to see 3 sample questions.

   Note:
   - The practice test will automatically close 1 hour (60 minutes) after you have started it, just like the mid-term test will.
   - The practice test will end on Thursday, 26 March, 10:00am. It will no longer be available afterwards.
   - Your answers to the open questions will not be marked, but you will get to see sample solutions after the practice test has ended on Thursday.

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 5 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 26 March 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult