COMP9020 20T1 Foundations of Computer Science

Week 10 Problem Set
Expectation

[Show with no answers] [Show with all answers]

Congratulations on reaching the end of this course!

Please note:

- Just 3 (+1 challenge) exercises this week.
- Note the early submission deadline: Tuesday, 28 April, at 10:00:00am.

1. (Expectation)

   a. You randomly draw one card at a time from a deck of 52 Poker cards: {*2, 3, …, 10, J, Q, K, A*} × {♠, ♣, ♥, ♦}. The cards are **not** put back into the deck after each drawing.

      i. Is the event of drawing a specific card independent of the previous draw?

      ii. What is the expected number of drawing attempts until a card **other than** an ace is drawn?

      iii. What is the expected number of drawing attempts until the sum of the cards drawn is ≥5? (Assume that 2–10 are counted as their numeric value; *J, Q, K* are counted as 10; *A* is counted as 11.)

   b. Answer questions (i)–(iii) for the case when the cards **are** put back after each drawing.

[hide answer]

a. Cards are not put back:

   i. No, for example: $P(\blacklozenge A_{t=2} \wedge \blacklozenge A_{t=1}) = 0$ (you can't draw ♦A twice); while $P(\blacklozenge A_{t=2}) > 0$ and $P(\blacklozenge A_{t=1}) > 0$, hence $P(\blacklozenge A_{t=2}) \cdot P(\blacklozenge A_{t=1}) > 0$.

   ii. Expected number of attempts to draw a non-ace:

$$1 \cdot \frac{48}{52} + 2 \cdot \frac{4}{52} \cdot \frac{48}{51} + 3 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{48}{50} + 4 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{48}{49} + 5 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} \cdot \frac{1}{49} \approx 1.0816$$

   iii. The possible outcomes are:

   - 1 drawing attempt: any card of value ≥5
   - 2 drawing attempts: a 2 followed by a card of value ≥3, or a 3 or 4 followed by any card
   - 3 drawing attempts: two 2's followed by any card

   Hence,

$$1 \cdot \frac{40}{52} + 2 \cdot \left(\frac{4}{52} \cdot \frac{48}{51} + \frac{8}{52} \cdot \frac{51}{51} \right) + 3 \cdot \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{50}{50} \approx 1.2353$$

b. Cards are put back:

   i. If each card drawn is put back into the deck, then each draw is independent since the deck from which a card is randomly selected is the same for each drawing attempt.

   ii. Each event of drawing a non-ace has the probability $p = \frac{48}{52} = \frac{12}{13}$. Hence, the expected number of drawing attempts is $\frac{1}{p} = \frac{13}{12} \approx 1.0833$.

   iii. Expected number of draws to obtain a sum ≥5:

$$1 \cdot \frac{40}{52} + 2 \cdot \left(\frac{4}{52} \cdot \frac{48}{52} + \frac{8}{52} \cdot \frac{52}{52} \right) + 3 \cdot \frac{4}{52} \cdot \frac{4}{52} \cdot \frac{52}{52} = \frac{209}{169} \approx 1.2367$$

2. (Variance)

   Consider an urn with four balls: one ball is worth 5, two balls are worth 20 each, and one ball is worth 25.

   Suppose you randomly draw two balls from the urn at the same time. Let random variable *X* denote the sum of the values of these two balls.

   a. What is the variance of *X* ?

b. How does your answer to question a. change if you double the initial value of each ball?

[hide answer]

a. There are $\binom{4}{2} = 6$ possible draws, with 4 possible outcomes: 25 (two draws), 30 (one draw), 40 (one draw) and 45 (two draws). Therefore:

$$E(X) = \frac{2}{6} \cdot 25 + \frac{1}{6} \cdot 30 + \frac{1}{6} \cdot 40 + \frac{2}{6} \cdot 45 = \frac{210}{6} = 35$$

$$E(X^2) = \frac{2}{6} \cdot 25^2 + \frac{1}{6} \cdot 30^2 + \frac{1}{6} \cdot 40^2 + \frac{2}{6} \cdot 45^2 = \frac{7800}{6} = 1300$$

Hence, the variance σ^2 is $1300 - 35^2 = 75$.

b. If all values are multiplied by 2 then the value of the random variable doubles for each possible outcome. Hence, *E(X)* doubles while *E(X^2)* quadruples; therefore, σ^2 increses from 75 to $5200 - 70^2 = 300$.

Generally, multiplying a random variable by a constant always increases the variance by the square of that constant.

3. (Decision making)

An airline is selling tickets for AU$100 each for a plane with 10 seats. Each ticket holder independently has the probability of $\frac{1}{8}$ of not turning up to the flight – in which case the airline keeps the AU$100 for the ticket. Suppose 12 people want tickets. The airline has a choice of three strategies:

- *X*: sell 10 tickets
- *Y*: sell 11 tickets, but if everyone turns up the airline has to pay AU$400 in compensation.
- *Z*: sell 12 tickets, but if 11 people turn up the airline has to pay AU$300 in compensation, and if 12 people turn up the compensation will be AU$600.

Calculate the expected values *E(X)*, *E(Y)* and *E(Z)* to determine the best strategy.

[hide answer]

- *X* has a value of $1000 with probability 1, hence *E(X)* = $1000.
- For strategy *Y*, the probability of all ticket holders turning up is $(7/8)^{11}$. If they all turn up, *Y* has a value of $1100 - $400, otherwise *Y* has a value of $1100. Hence,

  $$E(Y) = \$1100 - (7/8)^{11} \cdot \$400 = \$1007.92$$

- For strategy *Z*, the probability of all ticket holders turning up is $(7/8)^{12}$. The probability of 11 of the 12 ticket holders turning up is $12 \cdot 1/8 \cdot (7/8)^{11}$. If they all turn up, *Y* has a value of $1200 - $600 = $600; if 11 turn up, *Y* has a value of $1200 - $300 = $900; otherwise, *Y* has a value of $1200. Hence,

  $$E(Z) = \$600 \cdot (7/8)^{12} + \$900 \cdot 12 \cdot 1/8 \cdot (7/8)^{11} + \$1200 \cdot [1 - (7/8)^{12} - 12 \cdot 1/8 \cdot (7/8)^{11}] = \$975.56$$

In summary, the second strategy has the highest expected payoff.

4. **Challenge Exercise**

You keep rolling a die until you have observed each of the 6 numbers at least once. Calculate the expected number of times you have to roll the die.

[hide answer]

With probability 1 you obtain a new outcome on your first roll. The likelihood to obtain a different result with the next roll is $\frac{5}{6}$, hence on average it will require $\frac{6}{5} = 1.2$ rolls before you get the second number. The likelihood to then roll a number different from the first two is $\frac{4}{6}$, hence will require $\frac{6}{4} = 1.5$ rolls on average, and so on. Thus the number of rolls you expect to need before you see all 6 outcomes is $1 + \frac{6}{5} + \frac{6}{4} + \ldots + \frac{6}{1}$, or:

$$\sum_{k=1}^{6} \frac{6}{k} = 1 + 1.2 + 1.5 + 2 + 3 + 6 = 14.7$$

NB. This is a variation of the "Coupon Collector's Problem".

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 10 to answer 4 quiz questions on this week's problem set (Exercises 1-3 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Tuesday, 28 April 10:00:00am**.

Please also for this final quiz respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Tuesday

Do not …

- post specific questions about the quiz **before** the Tuesday deadline
- agonise too much about a question that you find too difficult