COMP9020 20T1 Foundations of Computer Science

Week 2 Problem Set
Logic, Proofs, Boolean Algebra

[Show with no answers] [Show with all answers]

Before you start:

Download and read a short essay on Good Mathematical Writing and write up your solutions to the following exercises with these guidelines in mind.

1. (Entailment)

   a. Prove that $\neg K$ follows logically from $H \wedge \neg J$ and $(H \wedge K) \Rightarrow J$.

   b. Which of the following formulae are logically entailed by $P \wedge (Q \vee \neg R)$?

      i. $\neg P$

      ii. Q

      iii. $\neg Q \vee R$

      iv. $R \Rightarrow P$

      v. $R \Rightarrow Q$

      vi. $\neg R \Rightarrow \neg Q$

      vii. $\neg P \Rightarrow R$

   [hide answer]

   a. We are given:

      1. $H \wedge \neg J$
      2. $(H \wedge K) \Rightarrow J$

      From 1 we can conclude H and $\neg J$. If K were true, then from H and K we could conclude J by 2, which contradicts $\neg J$. Hence, K cannot be true, which proves $\neg K$.

      Hint: You can also use a truth table to show that $\neg K$ is true in every row in which formulae 1 and 2 are true.

   b. There are 3 truth assignments under which $P \wedge (Q \vee \neg R)$ is true:

      (1) $P =$ T, $Q =$ F, $R =$ F
      (2) $P =$ T, $Q =$ T, $R =$ F
      (3) $P =$ T, $Q =$ T, $R =$ T

      i. Not entailed. $\neg P$ is false in any of the three truth assignments from above.

      ii. Not entailed. Q is false in truth assignment (1).

      iii. Not entailed. Neither $\neg Q$ nor R is true in assignment (2).

      iv. Entailed. In all of the assignments from above in which R is true, P is true as well.

      v. Entailed. In all of the assignments from above in which R is true, Q is true as well.

      vi. Not entailed. There is a truth assignment, (2), in which R is false but Q is true.

      vii. Entailed. In none of the assignments from above, P is false. Hence, the implication is true in all of them.

2. (Logical reasoning)

   a. See pages 21–23 of the lecture slides week 2 and answer the two questions.

b. The country of Mew is inhabited by two types of people: liars always lie and "truars" always tell the truth. At a cocktail party the newly appointed Australian ambassador to Mew talked to three inhabitants. Peter remarked that Joan and Shane were liars. Shane denied he was a liar, but Joan said that Shane was indeed a liar. Now the ambassador wondered how many of the three were liars.

Use propositional logic formulae to help the ambassador.

[hide answer]

a. First question: Yes. In fact, the conclusion follows directly from just the first requirement: S⇒(A∨F) ⊨ (S∧¬F)⇒A. This is so because the two formulae are equivalent (which you can see from the fact that they are both equivalent to ¬S∨A∨F).

Second question: No. The third requirement states that the alarm should sound whenever there is a fire. On the other hand, the first requirement does not require the alarm to sound at all (it only states a requirement about when the alarm should not sound); and the second requirement mentions nothing about fire at all.

More formally, there is a *model* (i.e. a satisfying assignment) for the first two requirements, S⇒(A∨F) and (A∧D)⇒S, which is *not* a model for F⇒S (for example, A=D=S=false, F=true).

b. Model the "character" of each of the three persons (Joan, Shane and Peter) with a proposition J, S, P. These are true if and only if that person is a truar. Then, we write their statements as follows:

- $P \Leftrightarrow \neg J \wedge \neg S$
- $S \Leftrightarrow \neg\neg S$
- $J \Leftrightarrow \neg S$

Using a truth table we can see that there are only two assignments consistent with the above: $J = \mathsf{F}$, $P = \mathsf{F}$, $S = \mathsf{T}$ and $J = \mathsf{T}$, $P = \mathsf{F}$, $S = \mathsf{F}$. In both cases there are two liars and one truar.

3. (Mathematical proofs)

a. Prove that $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = n$ for all integers n.

Hint: Give a proof by cases.

b. Prove that $8 \mid (n^2 - 1)$ for every odd integer n (that is, for every $n \in \mathbb{Z}$ such that $2 \nmid n$).

[hide answer]

a. Proof by cases:

- If n is even, then $n = 2k$ for some $k \in \mathbb{Z}$, hence $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = k + k = n$.
- If n is odd, then $n = 2k + 1$ for some $k \in \mathbb{Z}$, hence $\lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil = \lfloor k + 0.5 \rfloor + \lceil k + 0.5 \rceil = k + \lfloor 0.5 \rfloor + k + \lceil 0.5 \rceil = k + k + 1 = n$.

b. If n is odd, then $n - 1$ and $n + 1$ are both even and one of them must be divisible by 4. It follows that $n^2 - 1 = (n + 1)(n - 1) = 2k \cdot 4l = 8kl$, for some $k, l \in \mathbb{Z}$. Therefore, $8 \mid (n^2 - 1)$.

Hint: Other proofs are possible.

4. (Boolean algebra)

Consider a boolean algebra over a set T. For each of the following, either prove that the equation is true for all $x, y \in T$ or give a counterexample.

a. $x + (y' \cdot x') = x + y'$

b. $x' + (y' \cdot x) = x + y'$

c. $y + (x + y') = x + y + (x' \cdot y')$

d. $y' + (x \cdot y)' = y'$

e. $x \cdot (y + x') = x \cdot y$

f. $y \cdot (x' + y) = x' \cdot y$

[hide answer]

a. True. $x + (y' \cdot x') = (x + y') \cdot (x + x') = (x + y') \cdot 1 = x + y'$.

b. Not always true. Counterexample: Take the standard Boolean algebra over $\mathbb{B} = \{0, 1\}$ with $x = y = 1$. Then $1' + (1' \cdot 1) = 0 + 0 = 0$ but $1 + 1' = 1$.

c. True. $y + (x + y') = x + (y + y') = x + 1 = 1$. Likewise, $x + y + (x' \cdot y') = (x + y + x') \cdot (x + y + y') = 1 \cdot 1 = 1$.

d. Not always true. Counterexample: Take the standard Boolean algebra over $\mathbb{B} = \{0, 1\}$ with $x = 0$ and $y = 1$. Then $1' + (0 \cdot 1)' = 0 + 1 = 1$ but $1' = 0$.

e. True. $x \cdot (y + x') = x \cdot y + x \cdot x' = x \cdot y + 0 = x \cdot y$

f. Not always true. Counterexample: Take the standard Boolean algebra over $\mathbb{B} = \{0, 1\}$ with $x = y = 1$. Then $1 \cdot (1' + 1) = 1 \cdot 1 = 1$ but $1' \cdot 1 = 0$.

5. (Disjunctive normal form)

a. Consider the formulae $\phi_1 = (r \Rightarrow p)$ and $\phi_2 = (p \Rightarrow (q \vee \neg r))$. Transform the formula $\neg q \Rightarrow (\phi_1 \wedge \phi_2)$ into **DNF**. Simplify the result as much as possible.

b. Consider the following canonical DNF of a Boolean function $f(v, w, x, y)$:

$$v\,w\,x\,y' \;+\; v\,w\,x'\,y' \;+\; v\,w'\,x\,y \;+\; v'\,w'\,x\,y' \;+\; v'\,w'\,x'\,y \;+\; v'\,w\,x\,y' \;+\; v'\,w\,x'\,y' \;+\; v'\,w\,x'\,y$$

What is the minimal number of clauses required in any DNF representation of f? Justify your answer visually by drawing a Karnaugh map.

[hide answer]

a. Eliminating "$\Rightarrow$":

$\phi_1 \,:\, r \Rightarrow p \;\equiv\; \neg r \vee p$

$\phi_2 \,:\, p \Rightarrow (q \vee \neg r) \;\equiv\; \neg p \vee q \vee \neg r$

$\neg q \Rightarrow (\phi_1 \wedge \phi_2) \;\equiv\; q \,\vee\, ((\neg r \vee p) \,\wedge\, (\neg p \vee q \vee \neg r))$

Transforming to DNF:

$$\begin{array}{lll} & q \,+\, ((r' + p) \,\cdot\, (p' + q + r')) & \\ = & q + r'\,p' \,+\, r'\,q + r'\,r' \,+\, p\,p' \,+\, p\,q + p\,r' & \text{(by distribution)} \\ = & q + r'\,p' \,+\, r'\,q \,+\, r' \,+\, p\,q \,+\, p\,r' & \text{(since } r' \cdot r' = r' \text{ and omitting } p \cdot p' = 0) \\ = & q + r' & \text{(by absorption)} \end{array}$$

b. A Karnaugh map for the given function:

	vw	vw'	$v'w'$	$v'w$
xy		++		
xy'	++		++	++
$x'y'$	++			++
$x'y$			++	++

It can be seen that at least 4 rectangles (colour-coded above) are required to cover the map: one 2×2 square (that wraps around the edges), two 2×1 rectangles and 1 single square. The resulting minimal DNF is

$$w\,y' + v'\,x\,y' + v'\,x'\,y + v\,w'\,x\,y$$

6. **Challenge Exercise**

   Digital circuits are often built only from **nand**-gates with two inputs and one output. The function **nand**: $\mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ is defined by $(A\ \textbf{nand}\ B)\ \mapsto\ (A \cdot B)'$ or, equivalently, $\neg(A \wedge B)$. Show that *any* Boolean function can be encoded with only **nand**-gates.

   [show answer]

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 2 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 5 March 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult