COMP9020 20T1

Week 8 Problem Set
Counting

Foundations of Computer Science

[Show with no answers] [Show with all answers]

1. (Inclusion-Exclusion)

   Let $S = \{a, b, c, d\}$ and $T = \{e, f, g\}$.

   a. How many different relations on $S \times S$ are there?

   b. How many different **antireflexive** relations on $S \times S$ are there?

   c. How many different functions $f : S \longrightarrow T$ are there?

   d. How many different **onto** functions $f : S \longrightarrow T$ are there?

   [hide answer]

   a. Each pair $(x, y) \in S \times S$ is either related or not. There are $|S|^2$ = 16 such pairs; hence, there are 2^{16} = 65,536 relations on $S \times S$.

   b. If $\mathcal{R}$ is an antireflexive relation on $S \times S$, then $(x, x) \notin \mathcal{R}$ for all $x \in S$. Each other pair, that is $(x, y) \in S \times S$ with $x \neq y$, is either related or not. There are $|S| \cdot (|S| - 1)$ = 12 such pairs; hence, there are 2^{12} = 4,096 such relations.

   c. There are $|T|^{|S|}$ = 3^4 = 81 functions from S to T.

   d. Among the 81 functions from S to T, *not* onto are all the functions $S \longrightarrow \{e, f\}$ (2^4 = 16 functions), $S \longrightarrow \{e, g\}$ (2^4 = 16 functions), $S \longrightarrow \{f, g\}$ (2^4 = 16 functions). This, however, counts twice the functions $S \longrightarrow \{e\}$, $S \longrightarrow \{f\}$ and $S \longrightarrow \{g\}$. Hence, there are 81 - 3·16 + 3 = 36 onto functions $S \longrightarrow T$.

2. (Combinations)

   A management panel at a hospital needs to include at least one member from each of the following three professions: a doctor, a lawyer and an accountant. How many different panels can be formed in each of the following situations?

   a. Each profession offers 5 possible candidates. The panel size is 3.

   b. Each profession offers 4 possible candidates, but A. Brent (doctor) refuses to serve with C. David (lawyer). The panel size is 3.

   c. Each profession offers 5 possible candidates. The panel size is 5.

   d. Each profession offers 4 possible candidates, but A. Brent (doctor) refuses to serve with C. David (lawyer). The panel size is 5.

   [hide answer]

   a. One member from each profession (out of 5) must be selected; therefore 5^3 = 125 panels.

   b. 4^3 = 64 panels possible, out of which we need to subtract the 4 panels including Brent and David; therefore 60 panels.

   c. A 5-member panel can either consist of 3 members of one profession and 1 member from each of the other two, so $3 \cdot \binom{5}{1}\binom{5}{1}\binom{5}{3}$ = 750 panels; or of 2 members each from two professions and 1 member of the remaining profession, so $3 \cdot \binom{5}{2}\binom{5}{2}\binom{5}{1}$ = 1,500 panels. The total is 2,250 panels.

   d. As above, we find there are a total of $3 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{3} + 3 \cdot \binom{4}{2}\binom{4}{2}\binom{4}{1}$ = 624 possible panels. Out of these we need to subtract the panels that include Brent and David; one way to count them is that there are $\binom{10}{3}$ = 120 ways to select the remaining 3 members, minus $\binom{6}{3}$ = 20 that do not include any accountant, therefore 120 - 20 = 100 panels that include Brent and David. Hence there are 624 - 100 = 524 possible panels without Brent or David.

3. (Sequences)

How many sequences of 10 coin flips have at most 3 heads?

[hide answer]

In order to determine the number of seuqences with exactly *k=0,…3* heads, we need to choose *k* of the 10 coin flips to be heads. The remaining flips will be tails. Hence there are

$$\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\binom{10}{3}=1+10+\frac{10!}{2!\cdot 8!}+\frac{10!}{3!\cdot 7!}=1+10+45+120=176$$

possible sequences.

4. (Advanced Counting)

How many 5-letter words over the alphabet Σ = {*a, c, e, n, s*}

a. include the substring *ace*?

b. include all letters from Σ with *a* before *e* (for example, *canes*)?

c. have all their letters in alphabetical order (for example, *aceen*)?

[hide answer]

a. Select $\omega_1, \omega_2 \in \Sigma$ for words $\omega_1\omega_2 ace$, $\omega_1 ace \omega_2$ and $ace\omega_1\omega_2$, hence $5^2 \cdot 3$ = 75 words.

b. The letters can be arranged in 5! different ways, half of which satisfy the additional requirement, hence $\frac{5!}{2}$ = 60 words.

c. Any word of length *n* in alphabetical order *that ends in a letter ω* can be obtained by
   - a word of length *n* - 1 in alphabetical order *that ends in*
     - ω *or*
     - any letter that comes before ω in the lexicographic order.

The table below follows the rule
- $n_{row,1}$ = 1 (the number of 1-letter words ending in a given letter)
- $n_{row,col} = n_{1,col-1} + n_{2,col-1} + \ldots + n_{row,col-1}$ (the number of *col*-letter words ending in a given letter, for *col* ≥ 2)

word length	1	2	3	4	5
# words ending in *a*	1	1	1	1	1
# words ending in *c*	1	2	3	4	5
# words ending in *e*	1	3	6	10	15
# words ending in *n*	1	4	10	20	35
# words ending in *s*	1	5	15	35	70
total					126

Hence, there are 126 words in Σ^5 that have all letters in alphabetical order.

5. **Challenge Exercise**

In a movie theatre, 4 couples are sitting in one row with men and women alternating. If no couple is sitting together, how many arrangements are possible?

[hide answer]

If you place a man in seat 1, his partner can be in seat 4, 6 or 8. If you place a man in seat 3, his partner must be in seat 6 or 8. If you place a man in seat 5, his partner must be in seat 2 or 8. If you place a man in seat 7, his partner must be in seat 2 or 4.

So, once you determine where all the men are seated when placed in odd-numbered seats: `M1-?-M2-?-M3-?-M4-?`, there are three possible seating arrangments for the women:

- `M1-W3-M2-W4-M3-W1-M4-W2`
- `M1-W3-M2-W4-M3-W2-M4-W1`
- `M1-W4-M2-W1-M3-W2-M4-W3`

There are $4! = 24$ possible ways to assign the seats 1, 3, 5 and 7 to the four men, so there are $24 \cdot 3 = 72$ such arrangements. And for each of these arrangements you can swap the seats for men and women without changing anything, hence there is a total of $72 \cdot 2 = 144$ possible arrangements.

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 8 to answer 4 quiz questions on this week's problem set (Exercises 1-4 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 16 April 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult