Week 2 Problem Set
Logic, Proofs, Boolean Algebra

Before you start:

Download and read a short essay on Good Mathematical Writing *and write up your solutions to the following exercises with these guidelines in mind.*

1. (Entailment)

    a. Prove that $\neg K$ follows logically from $H \wedge \neg J$ and $(H \wedge K) \Rightarrow J$.

    b. Which of the following formulae are logically entailed by $P \wedge (Q \vee \neg R)$?

        i. $\neg P$

        ii. Q

        iii. $\neg Q \vee R$

        iv. $R \Rightarrow P$

        v. $R \Rightarrow Q$

        vi. $\neg R \Rightarrow \neg Q$

        vii. $\neg P \Rightarrow R$

2. (Logical reasoning)

    a. See pages 21–23 of the lecture slides week 2 and answer the two questions.

    b. The country of Mew is inhabited by two types of people: liars always lie and "truars" always tell the truth. At a cocktail party the newly appointed Australian ambassador to Mew talked to three inhabitants. Peter remarked that Joan and Shane were liars. Shane denied he was a liar, but Joan said that Shane was indeed a liar. Now the ambassador wondered how many of the three were liars.

       Use propositional logic formulae to help the ambassador.

3. (Mathematical proofs)

    a. Prove that $\left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{2} \right\rceil = n$ for all integers n.

       Hint: Give a proof by cases.

    b. Prove that $8 \mid (n^2 - 1)$ for every odd integer n (that is, for every $n \in \mathbb{Z}$ such that $2 \nmid n$).

4. (Boolean algebra)

    Consider a boolean algebra over a set T. For each of the following, either prove that the equation is true for all $x, y \in T$ or give a counterexample.

    a. $x + (y' \cdot x') = x + y'$

    b. $x' + (y' \cdot x) = x + y'$

    c. $y + (x + y') = x + y + (x' \cdot y')$

   d. $y' + (x \cdot y)' = y'$

   e. $x \cdot (y + x') = x \cdot y$

   f. $y \cdot (x' + y) = x' \cdot y$

5. (Disjunctive normal form)

   a. Consider the formulae $\phi_1 = (r \Rightarrow p)$ and $\phi_2 = (p \Rightarrow (q \vee \neg r))$. Transform the formula $\neg q \Rightarrow (\phi_1 \wedge \phi_2)$ into **DNF**. Simplify the result as much as possible.

   b. Consider the following canonical DNF of a Boolean function $f(v, w, x, y)$:

$$v\,w\,x\,y' \;+\; v\,w\,x'\,y' \;+\; v\,w'\,x\,y \;+\; v'\,w'\,x\,y' \;+\; v'\,w'\,x'\,y \;+\; v'\,w\,x\,y' \;+\; v'\,w\,x'\,y' \;+\; v'\,w\,x'\,y$$

What is the minimal number of clauses required in any DNF representation of f? Justify your answer visually by drawing a Karnaugh map.

6. **Challenge Exercise**

Digital circuits are often built only from **nand**-gates with two inputs and one output. The function **nand**: $\mathbb{B} \times \mathbb{B} \longrightarrow \mathbb{B}$ is defined by $(A \textbf{ nand } B) \mapsto (A \cdot B)'$ or, equivalently, $\neg(A \wedge B)$. Show that *any* Boolean function can be encoded with only **nand**-gates.

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 2 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 5 March 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult