COMP9020 17s1 • Problem Set 2 • 10 March 2017

Logic

Before you start.

Download and read a short essay on

> *Good Mathematical Writing*

and write up your solutions to the following exercises with these guidelines in mind.

Hint: If the link above does not work, you can also find the Pdf on the course webpage.

Exercise 1. Prove that $\neg N$ follows logically from $H \wedge \neg R$ and $(H \wedge N) \Rightarrow R$.

Exercise 2. See pages 37–39 of the lecture slides week 2 and answer the two questions.

$*$**Exercise 3.** Prove that $8 \mid (n^2 - 1)$ for every odd number n (that is, for every $n \in \mathbb{N}$ such that $2 \nmid n$).

Exercise 4. The country of Mew is inhabited by two types of people: liars always lie and truars always tell the truth. At a cocktail party the newly appointed Australian ambassador to Mew talked to three inhabitants. Joan remarked that Shane and Peter were liars. Shane denied he was a liar, but Peter said that Shane was indeed a liar. Now the ambassodor wondered how many of the three were liars.

Use propositional logic formulae to help the ambassador.

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Solutions

Exercise 1. We are given

1. $H \wedge \neg R$
2. $(H \wedge N) \Rightarrow R$

From 1 we can conclude H and $\neg R$. If N were true, then from H and N we could conclude R by 2, which contradicts $\neg R$. Hence, N cannot be true, which proves $\neg N$.

Hint: You can also use a truth table to show that $\neg N$ is true in every row in which formulae 1 and 2 are true.

Exercise 2.

- First question: Yes. In fact, the conclusion follows directly from just the first requirement.
- Second question: No. The third requirement states that the alarm should sound whenever there is a fire. On the other hand, the first requirement does not require the alarm to sound at all (it only states a requirement about when the alarm should *not* sound); and the second requirement mentions nothing about fire at all.

Exercise 3. If n is odd, then $n-1$ and $n+1$ are both even and one of them must be divisible by 4. It follows that $n^2 - 1 = (n+1)(n-1) = 2k \cdot 4\ell = 8k\ell$, for some $k, \ell \in \mathbb{N}$. Therefore, $8 \mid (n^2 - 1)$.

Hint: Other proofs are possible.

Exercise 4. Model the "character" of each of the three persons (Joan, Shane and Peter) with a proposition J,S,P. These are true if and only if that person is a truar. Then, we write their statements as follows:

$J \Leftrightarrow \neg S \wedge \neg P$

$S \Leftrightarrow \neg\neg S$

$P \Leftrightarrow \neg S$

Using a truth table we can see that the only assignments consistent with the above are: J=F, S=T, P=F or J=F, S=F, P=T. In both cases there are two liars and one truar.

COMP9020 17s1 • Problem Set 3 • 17 March 2017

Logic — Boolean Expressions

Exercise 1. Consider the formulae $\phi_1 = (r \Rightarrow p)$ and $\phi_2 = (p \Rightarrow (q \vee \neg r))$. Transform the formula $\phi = (\neg q \Rightarrow (\phi_1 \wedge \phi_2))$ into **DNF**. Simplify the result as much as possible.

Exercise 2. Digital circuits are often built only from **nand**-gates with two inputs. A **nand** B is defined as $\overline{A \cdot B}$, that is, $\neg(A \wedge B)$. Prove that **nand**-gates are sufficient to encode any Boolean function.

$*$**Exercise 3.**

(a) Give all elements of BOOL(1), that is, all functions over a single boolean variable.

(b) Prove that there are 2^{2^n} elements in BOOL(n) for $n \in \mathbb{P}$.

Exercise 4. In Mew (cf. Week 2, Exercise 4), the Nelsons, who are truars, were leaving their four children with the new babysitter, Nancy, for the evening. Before they left, they told Nancy that three of their children were consistent liars but that one of them was a truar. While she was preparing dinner, one of the children broke a vase in the next room. When she asked who broke the vase the children's answers were:

Betty: Steve broke the vase,

Steve: John broke it,

Laura: I didn't break it,

John: Steve lied when he said I broke it.

Who did it?

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Solutions

Exercise 1. $\neg q \Rightarrow ((r \Rightarrow p) \wedge (p \Rightarrow (q \vee \neg r))) \;\equiv\; q \vee ((\neg r \vee p) \wedge (\neg p \vee q \vee \neg r))$

$$\begin{array}{lll} & q + ((\overline{r} + p) \cdot (\overline{p} + q + \overline{r})) & \\ = & q + \overline{r}\,\overline{p} + \overline{r}\,q + \overline{r}\,\overline{r} + p\overline{p} + p\,q + p\,\overline{r} & \text{(by \textit{distribution})} \\ = & q + \overline{r}\,\overline{p} + \overline{r}\,q + \overline{r} + p\,q + p\,\overline{r} & \text{(since } \overline{r}\cdot\overline{r} = \overline{r} \text{ and omitting } p \cdot \overline{p} = 0) \\ = & q + \overline{r} & \text{(by \textit{absorption})} \end{array}$$

Exercise 2. It suffices to show that the three basic operations can be expressed via the **nand** operation:

- $\overline{A} = \overline{A \cdot A} = A$ **nand** A
- $A \cdot B = \overline{\overline{A \cdot B}} = \overline{\overline{A \cdot B} \cdot \overline{A \cdot B}} = (A$ **nand** $B)$ **nand** $(A$ **nand** $B)$
- $A + B = \overline{\overline{A} \cdot \overline{B}} = \overline{\overline{A \cdot A} \cdot \overline{B \cdot B}} = (A$ **nand** $A)$ **nand** $(B$ **nand** $B)$

Therefore, any Boolean expression can be expressed via just the **nand** operation.

Exercise 3.

(a) $\textsc{bool}(1)$ contains $2^{2^1} = 4$ elements:

$$\begin{array}{lllll} f_1: & p & \mapsto & 0 & \text{, i.e. } f_1(0) = 0,\; f_1(1) = 0 \\ f_2: & p & \mapsto & p & \text{, i.e. } f_2(0) = 0,\; f_2(1) = 1 \\ f_3: & p & \mapsto & \overline{p} & \text{, i.e. } f_3(0) = 1,\; f_3(1) = 0 \\ f_4: & p & \mapsto & 1 & \text{, i.e. } f_4(0) = 1,\; f_4(1) = 1 \end{array}$$

(b) If $f \in \textsc{bool}(n)$ then f is a function over n boolean variables, that is, $\mathsf{Dom}(f) = \times_{i=1}^{n}\{0,1\}$, hence $|\mathsf{Dom}(f)| = 2^n$. The result of a boolean function is either 0 or 1, that is, $\mathsf{Codom}(f) = \{0,1\}$, hence $|\mathsf{Codom}(f)| = 2$. For each element in the domain, a function can chose any of the elements from the codomain as the function value, hence there are $|\mathsf{Codom}(f)|^{|\mathsf{Dom}(f)|} = 2^{2^n}$ different functions.

Exercise 4. We use B, S, L, J to denote that the corresponding child is a truar, and b, s, l, j to denote that the corresponding child dropped the vase. The information from the parents translates to:

$(B \cdot \overline{S} \cdot \overline{L} \cdot \overline{J}) + (\overline{B} \cdot S \cdot \overline{L} \cdot \overline{J}) + (\overline{B} \cdot \overline{S} \cdot L \cdot \overline{J}) + (\overline{B} \cdot \overline{S} \cdot \overline{L} \cdot J)$

Also, we know that just one child broke the vase, which means:

$(b \cdot \overline{s} \cdot \overline{l} \cdot \overline{j}) + (\overline{b} \cdot s \cdot \overline{l} \cdot \overline{j}) + (\overline{b} \cdot \overline{s} \cdot l \cdot \overline{j}) + (\overline{b} \cdot \overline{s} \cdot \overline{l} \cdot j)$

The children's claims translate to:

$B \Leftrightarrow s$

$S \Leftrightarrow j$

$L \Leftrightarrow \overline{l}$

$J \Leftrightarrow \overline{S}$ (or $J \Leftrightarrow \overline{S} \cdot \overline{j}$)

Now, use a truth table to find the only possible truth assignment. Alternatively, reason as follows: from the last claim, we conclude that either Steve or John must be a truar and the other one must be a liar. This means that the other two children must both be liars. Since Laura is a liar, her claim is false, so Laura broke the vase. (We can now also conclude that the truar is John.)

COMP9020 17s1 • Problem Set 4 • 24 March 2017

Functions and Relations

Exercise 1. Consider the relation $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}$ defined by $a\mathcal{R}b$ if, and only if, $b + 0.5 \geq a \geq b - 0.5$. Is $\mathcal{R}$

(a) reflexive?

(b) antireflexive?

(c) symmetric?

(d) antisymmetric?

(e) transitive?

Exercise 2. Prove each of the following statements.

(a) $(\mathbf{A}^T)^T = \mathbf{A}$ for any matrix $\mathbf{A}$.

(b) If two matrices $\mathbf{A}$ and $\mathbf{B}$ are of the same size, then $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

(c) $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ for any matrix $\mathbf{A}$ of size $m \times n$ and matrices $\mathbf{B}, \mathbf{C}$ of size $n \times p$.

$*$**Exercise 3.** Consider a relation $\mathcal{R}$ on $\mathrm{Pow}(U)$ for some set U defined by $A\mathcal{R}B$ iff $|A \cap B| \geq 1$.
Prove that $\mathcal{R}$ is transitive iff $|U| \leq 1$.

COMP9020 17s1 • Problem Set 4 • 24 March 2017

Solutions

Exercise 1.

(a) Yes, since $a + 0.5 \geq a \geq a - 0.5$ for all $a \in \mathbb{R}$

(b) No; see (a)

(c) Yes, since $(b + 0.5 \geq a) \wedge (a \geq b - 0.5)$ implies $(b \geq a - 0.5) \wedge (a + 0.5 \geq b)$.

(d) No; e.g. $(0, 0.1) \in \mathcal{R}$ and $(0.1, 0) \in \mathcal{R}$.

(e) No; e.g. $(1.1, 1.5) \in \mathcal{R}$ and $(1.5, 1.9) \in \mathcal{R}$ but $(1.1, 1.9) \notin \mathcal{R}$ since $1.9 - 0.5 > 1.1$

Exercise 2.

(a) If $\mathbf{A}$ is of size $m \times n$ then $\mathbf{A}^T$ is of size $n \times m$, hence $(\mathbf{A}^T)^T$ is of size $m \times n$, the same as $\mathbf{A}$. The (i, j)-th entry of $\mathbf{A}^T$ is a_{ji}, hence the (i, j)-th entry of $(\mathbf{A}^T)^T$ is a_{ij}, the same as $\mathbf{A}$.

(b) The (j, i)-th entry of $\mathbf{A} + \mathbf{B}$ is $a_{ji} + b_{ji}$, hence the (i, j)-th entry of $(\mathbf{A} + \mathbf{B})^T$ is $a_{ji} + b_{ji}$. The (i, j)-th entry of $\mathbf{A}^T$ is a_{ji} and the (i, j)-th entry of $\mathbf{B}^T$ is b_{ji}, hence the (i, j)-th entry of $\mathbf{A}^T + \mathbf{B}^T$ is $a_{ji} + b_{ji}$. This proves the claim.

(c) The (j, k)-th entry of $\mathbf{B} + \mathbf{C}$ is $b_{jk} + c_{jk}$, hence the (i, k)-th entry of $\mathbf{A}(\mathbf{B} + \mathbf{C})$ is

$$\sum_{j=1}^{n} a_{ij}(b_{jk} + c_{jk}) \tag{1}$$

The (i, k)-th entry of $\mathbf{AB}$ is $\sum_{j=1}^{n} a_{ij} b_{jk}$ and the (i, k)-th entry of $\mathbf{AC}$ is $\sum_{j=1}^{n} a_{ij} c_{jk}$, hence the (i, k)-th entry of $\mathbf{AB} + \mathbf{AC}$ is

$$\sum_{j=1}^{n} a_{ij} b_{jk} + \sum_{j=1}^{n} a_{ij} c_{jk} \tag{2}$$

This proves the claim since $(1) = \sum_{j=1}^{n}(a_{ij} b_{jk} + a_{ij} c_{jk}) = (2)$

Exercise 3. If $|U| = 0$, there is only one subset (empty) that is not related to itself so there is nothing to violate the transitivity (it holds vacuously). If $|U| = 1$, there is only one case when $\mathcal{R}$ holds, which is $U \mathcal{R} U$ and again nothing violates transitivity. If $|U| \geq 2$, choose any two elements in U (say, a and b) and observe that $\{a\}\mathcal{R}\{a, b\}$ and $\{a, b\}\mathcal{R}\{b\}$, but $\{a\}$ is not in a relation with $\{b\}$, so $\mathcal{R}$ is not transitive.

COMP9020 17s1 • Problem Set 5 • 31 March 2017

Equivalence Relations, Orderings

Exercise 1. Prove or disprove that $\mathcal{R} \subseteq \mathbb{Z} \times \mathbb{Z}$ is an equivalence relation if

(a) $\mathcal{R}$ is defined by $a\mathcal{R}b$ iff $a + b$ is divisible by 3;

(b) $\mathcal{R}$ is defined by $a\mathcal{R}b$ iff $a + 2b$ is divisible by 3.

Exercise 2. Prove that if $m, n \in \mathbb{Z}$ and $m = n \pmod{p}$ then $m^2 = n^2 \pmod{p}$.

Exercise 3. Consider the relation $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}$ defined by $(a, b) \in \mathcal{R}$ iff either $a \leq b - 0.5$ or $a = b$. Show that $\mathcal{R}$ is a partial order, but not a total order.

Exercise 4. Let binary relation $\mathcal{R}$ on $\{1, \ldots, 20\}$ be defined by $a\mathcal{R}b$ iff either $a = b$ or $a - b > 10$.

(a) Show that $\mathcal{R}$ is a partial order.

(b) Is $\langle \{1, \ldots, 20\}, \mathcal{R} \rangle$ a lattice?

$*$**Exercise 5.** Using the set $\{1, \ldots, 10\}$ with the natural total order, define $A = \{1, \ldots, 10\} \times \{1 \ldots, 10\}$ and consider the two orderings over A:

- product $\sqsubseteq_P$
- lexicographic $\sqsubseteq_L$

Find the maximal length of a chain $a_1 \sqsubseteq a_2 \sqsubseteq \ldots \sqsubseteq a_n$ (such that $a_i \neq a_{i+1}$) for each of these orderings.

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Solutions

Exercise 1.

(a) $\mathcal{R}$ is not an equivalence relation since it is neither reflexive nor transitive: $(1,2) \in \mathcal{R}$ and $(2,1) \in \mathcal{R}$, but $(1,1) \notin \mathcal{R}$.

(b) Yes, $\mathcal{R}$ is an equivalence relation: Notice that $a + 2b$ is divisible by 3 whenever $a - b$ is divisible by 3, hence $(a,b) \in \mathcal{R}$ iff $a \bmod 3 = b \bmod 3$, which is reflexive, symmetric and transitive.

Exercise 2. If $m = n \pmod p$ then $m = k \cdot p + r$ and $n = l \cdot p + r$ for some $k, l \in \mathbb{Z}$ and $r \in \{0, \ldots, p-1\}$. Then,

$$\begin{aligned} m^2 &= k^2p^2 + 2kpr + r^2 \\ n^2 &= l^2p^2 + 2lpr + r^2 \end{aligned}$$

Hence, $m^2 \bmod p = r^2 \bmod p = n^2 \bmod p$, so $m^2 = n^2 \pmod p$.

Exercise 3.

- $\mathcal{R}$ is reflexive: for every a, b such that $a = b$, by definition $(a,b) \in \mathcal{R}$.
- $\mathcal{R}$ is antisymmetric: for any $a \neq b$, if $(a,b) \in \mathcal{R}$ then it must be that $a \leq b - 0.5$, therefore $b \geq a + 0.5 > a - 0.5$ so $(b,a) \notin \mathcal{R}$.
- $\mathcal{R}$ is transitive: for any a, b, c, this is trivial if $a = b$ or $b = c$, otherwise if $a \neq b$ and $b \neq c$ then $(a,b) \in \mathcal{R} \wedge (b,c) \in \mathcal{R} \Rightarrow a \leq b - 0.5 \wedge b \leq c - 0.5 \Rightarrow a \leq c - 1 \Rightarrow a \leq c - 0.5 \Rightarrow (a,c) \in \mathcal{R}$.

Therefore $\mathcal{R}$ is a partial order. It is not a total order since any pair a, b where $a < b < a + 0.5$ (for instance, 1.1 and 1.2) are not related in either direction.

Exercise 4.

(a) Check that $\mathcal{R}$ is reflexive, antisymmetric and transitive in the same way as for the relation in Exercise 2.

(b) It is not a lattice; for example, the pair (1,2) do not have a greatest lower bound (or, in fact, any lower bound — both 1 and 2 are minimal elements).

Exercise 5. For the product order the maximum length is 19; for example,

$$(1,1) \sqsubseteq_P (1,2) \sqsubseteq_P \ldots \sqsubseteq_P (1,9) \sqsubseteq_P (1,10) \sqsubseteq_P (2,10) \sqsubseteq_P \ldots \sqsubseteq_P (9,10) \sqsubseteq_P (10,10)$$

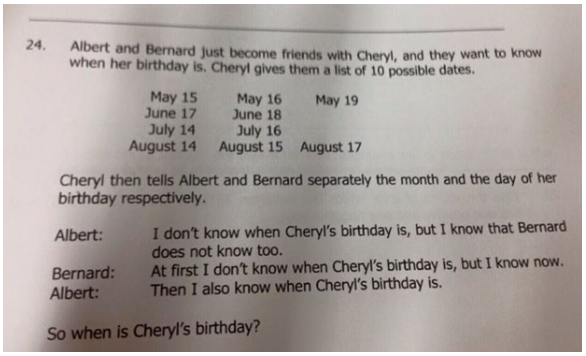
For the lexicographic order, since it is a total order, the longest chain contains all of the 100 elements in the set:

$$(1,1) \sqsubseteq_L (1,2) \sqsubseteq_L \ldots \sqsubseteq_L (1,10) \sqsubseteq_L (2,1) \sqsubseteq_L \ldots \sqsubseteq_L (9,10) \sqsubseteq_L (10,1) \sqsubseteq_L \ldots \sqsubseteq_L (10,10)$$

COMP9020 17s1 • Fun with Logic • 7 April 17

Logic

Exercise 1. The following problem from a School Math Olympiad in Singapore went viral a while ago.



24. Albert and Bernard just become friends with Cheryl, and they want to know when her birthday is. Cheryl gives them a list of 10 possible dates.

May 15	May 16	May 19
June 17	June 18	
July 14	July 16	
August 14	August 15	August 17

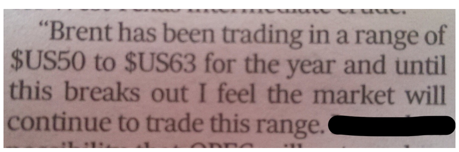
Cheryl then tells Albert and Bernard separately the month and the day of her birthday respectively.

Albert: I don't know when Cheryl's birthday is, but I know that Bernard does not know too.
Bernard: At first I don't know when Cheryl's birthday is, but I know now.
Albert: Then I also know when Cheryl's birthday is.

So when is Cheryl's birthday?

Can you solve it?

Exercise 2. From the *Australian Financial Review*:



"Brent has been trading in a range of $US50 to $US63 for the year and until this breaks out I feel the market will continue to trade this range. [redacted]

An analyst tries to predict the future price of Brent oil. How informative do you think this prediction is?

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Solutions

Exercise 1.

1. Albert knows that Bernard does not know the date. This rules out that Albert has been told May (since otherwise he would have to consider the possibility that Bernard had been told 19 and hence would know Cheryl's birthday) or June (for the same reason, since otherwise Bernard might have known the date because he could have been told 18).

2. From the above, Bernard can conclude that Albert must have been told July or August, but he wouldn't know which of the two if he had been told 14. Hence, Bernard now knowing the date rules out that possibility.

3. Albert now also knows the date, which rules out that he has been told August (since otherwise he would have to consider both August 15 and August 17 as possible dates). This leaves **July 16** as the only possibility.

Exercise 2. The analyst essentially says, "Brent oil will trade in the range [\$US50 .. \$US63] until it no longer trades in that range." In logic, "A is true until $\neg A$ becomes true." Market analysts are paid well for their thoughtful predictions ...

Graphs

Exercise 1. True or false?

(a) The complete bipartite graph $K_{5,5}$ has no cycle of length five.

(b) If T is a tree with at least four edges, then $\chi(T) = 3$.

(c) Let C_n denote a cycle on n vertices. For all $n \geq 5$ it holds $\chi(C_n) \neq \chi(C_{n-1})$.

(d) It is possible to remove two edges from K_6 so that the resulting graph has a clique number of 4.

Exercise 2. Let G be an undirected graph on 20 vertices with exactly two connected components. What is the minimum and the maximum possible number of edges in G?

Exercise 3. For what pairs of integers (i, j), $i \geq j \geq 1$, are the graphs $K_{i,j}$ planar?

Exercise 4. Consider the complete 3-partite graphs $K_{4,1,1}$, $K_{3,2,1}$, $K_{2,2,2}$.

(a) What is the chromatic number of each of these graph?

(b) Which of these graphs are planar?

$*$**Exercise 5.** What is the minimum number of edges that need to be removed from K_5 so that the resulting graph has a chromatic number of

(a) 3?

(b) 2?

(c) 1?

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Solutions

Exercise 1.

(a) True. In general, any bipartite graph can only have cycles of an even length. If V_1, V_2 denote the two groups of nodes in the bipartite graph, then every path starting from, say, a node $v \in V_1$ must alternate its vertices between V_1 and V_2. If it is a cycle (that is, eventually returns to node v), it must have an even number of such alternations, in other words, it contains an even number of edges.

(b) False. For any tree T, $\chi(T) = 2$.

(c) True (in fact, true for all $n \geq 2$), because $\chi(C_n) = 2$ for all even n and $\chi(C_n) = 3$ for all odd n.

(d) True. Remove any two edges which have no common vertex. In the resulting graph there will be no five fully interconnected vertices; in other words it will not contain K_5, and will have a clique number of 4.

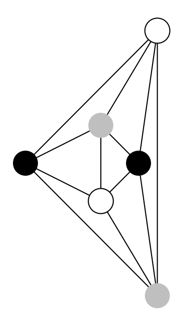
Exercise 2. Let the two connected components have n and m vertices respectively, with $n+m = 20$. The maximum number of edges is achieved by creating two complete graphs K_n and K_m with $\frac{n(n-1)}{2} + \frac{m(m-1)}{2}$ edges overall. This number is maximal for $n = 19$, $m = 1$ which gives 171 edges. The minimum number of edges is obtained when both components are trees with $n-1$ and $m-1$ edges respectively, for a total of $n+m-2 = 18$ edges.

Exercise 3. $K_{i,1}$ and $K_{i,2}$ for all $i \geq 1$. They can be easily presented without self-intersection in a planar drawing. For $i, j \geq 3$ the corresponding bipartite graph would contain $K_{3,3}$, which is known not to be planar.

Exercise 4.

(a) For every 3-partite graph three colours suffice: use a different colour for each of the groups of vertices. Three colours are also necessary: any three vertices selected from three different groups will form a clique.

(b) $K_{4,1,1}$ can be easily drawn without intersections. $K_{3,2,1}$ contains $K_{3,3}$ hence is not planar. A planar drawing of $K_{2,2,2}$ is:



Exercise 5.

(a) 2 edges. To achieve $\chi = 3$, one needs (at least) to avoid having any 4-cliques. Removing one edge leaves a 4-clique (actually two such cliques — including any one but not both of that edge's endpoints, plus the remaining three vertices). Removing two edges suffices — remove any pair of edges which do not share a common vertex; the remaining graph can then be coloured with 3 colours.

(b) 4 edges. A chromatic number of 2 means that the graph is bipartite, with two groups of nodes where each group can be painted with one colour. To minimise the number of *removed* edges, we want to have as many edges as possible in the remaining bipartite graph. We therefore look at *complete bipartite* graphs with a total of 5 vertices. As $K_{1,4}$ has four edges and $K_{2,3}$ has six edges, the latter is the better choice. To reach it we need to remove 4 of the edges in K_5.

(c) 10 edges. A chromatic number of 1 means a fully disconnected graph, with no edges at all. Therefore all 10 edges of the original graph must be removed.

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Induction and Recursion

Exercise 1.

(a) Prove by induction that

$$1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! = (n+1)! - 1 \quad \text{for } n \geq 1$$

(b) Given the recursive definition,

$$\begin{array}{ll} \text{(B)} & s_1 = 1 \\ \text{(R)} & s_{n+1} = \frac{1}{1+s_n} \end{array}$$

prove by induction that

$$s_n = \frac{\text{FIB}(n)}{\text{FIB}(n+1)} \quad \text{for } n \geq 1$$

Exercise 2. Prove that in any rooted tree, the number of leaves is one more than the number of nodes with a right sibling.

Hint: This assumes a given order among the children of every node from left to right; see page 38 of the lecture slides week 7 for an example.

$*$**Exercise 3.** Prove by induction that every *connected* graph $G = (V, E)$ must satisfy $e(G) \geq v(G) - 1$.

Hint: You can use the fact from the previous lecture that $\sum_{v \in V} deg(v) = 2 \cdot e(G)$.

Exercise 4. Let $T(n)$ be defined by the recurrence

$$T(n) = T(n-1) + g(n) \quad \text{for } n > 1$$

Prove by induction on i that if $1 \leq i < n$, then

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j)$$

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Solutions

Exercise 1.

(a) Base case: $1 \cdot 1! = 1 = (1 + 1!) - 1$

Inductive step:

$$\begin{array}{rcll} 1 \cdot 1! + 2 \cdot 2! + \ldots + n \cdot n! + (n+1) \cdot (n+1)! & = & (n+1)! - 1 + (n+1) \cdot (n+1)! & \text{by ind. hyp.} \\ & = & (1 + (n+1)) \cdot (n+1)! - 1 & \\ & = & (n+2) \cdot (n+1)! - 1 & \\ & = & (n+2)! - 1 & \text{by def. of !} \end{array}$$

(b) Base case: $s_1 = 1 = \frac{1}{1} = \frac{\text{FIB}(1)}{\text{FIB}(2)}$

Inductive step:

$$s_{n+1} = \frac{1}{1 + s_n} = \frac{1}{1 + \frac{\text{FIB}(n)}{\text{FIB}(n+1)}} = \frac{1}{\frac{\text{FIB}(n+1)+\text{FIB}(n)}{\text{FIB}(n+1)}} = \frac{1}{\frac{\text{FIB}(n+2)}{\text{FIB}(n+1)}} = \frac{\text{FIB}(n+1)}{\text{FIB}(n+2)}$$

Exercise 2. For a tree T, let $\ell(T)$ and $r(T)$ denote, respectively, the number of leaves and the number of vertices with a right sibling.

Base case: A tree consisting of just a root has 1 leaf and no vertex with a right sibling:

$$T = \langle v; \rangle \;\Rightarrow\; \ell(T) = 1 = 0 + 1 = r(T) + 1$$

Inductive step: The leaves of a tree $T = \langle r; T_1, T_2, \ldots, T_k \rangle$ are all the leaves of all the subtrees T_i. Moreover, in T the roots of all the subtrees $T_1, \ldots, T_{k-1}$ (but not T_k) are vertices with a right sibling, in addition to all the other vertices in all the subtrees that have a right sibling. Hence,

$$T = \langle r; T_1, T_2, \ldots, T_k \rangle \;\Rightarrow\; \ell(T) = \sum_{i=1}^{k} \ell(T_i) \text{ and } r(T) = (k-1) + \sum_{i=1}^{k} r(T_i)$$

From the induction hypothesis, $\ell(T_i) = r(T_i) + 1$ for all $1 \le i \le k$, it follows that

$$\ell(T) = \sum_{i=1}^{k} \ell(T_i) = \sum_{i=1}^{k} (r(T_i) + 1) = k + \sum_{i=1}^{k} r(T_i) = r(T) + 1$$

Exercise 3. Base case: A graph with $v(G) = 1$ node is connected, has $e(G) = 0$ edges and hence satisfies $e(G) \ge v(G) - 1$.

Inductive step (proof by contradiction): Consider graph G with $v(G) \ge 2$ nodes such that $e(G) < v(G) - 1$. We will show that G is not connected. From the lecture we know that $\sum_{v \in V} deg(v) = 2e(G) < 2v(G) - 2$. It follows that there is at least one vertex $v_0 \in V$ with $deg(v_0) \le 1$. If $deg(v_0) = 0$ then G is not connected and we are done. If $deg(v_0) = 1$, consider the graph G' obtained by removing v_0 and its only connecting edge from G. It follows that $e(G') < v(G') - 1$ since $e(G') = e(G) - 1$, $v(G') = v(G) - 1$ and $e(G) < v(G) - 1$. By the induction hypothesis, G' is not connected. But then neither is G: if v and w are vertices with no path between them in G' then adding v_0 doesn't help.

Exercise 4. Base case: For $i = 1$, we have

$$T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j) = T(n-1) + \sum_{j=0}^{0} g(n-j) = T(n-1) + g(n)$$

which is true by the definition of $T(n)$.

Inductive step: Suppose $T(n) = T(n-i) + \sum_{j=0}^{i-1} g(n-j)$ is true for $i = k < n-1$. We show that it holds for $i = k+1$. By the definition of $T(\cdot)$ we have $T(n-k) = T(n-k-1) + g(n-k)$, therefore

$$T(n) = T(n-k-1) + g(n-k) + \sum_{j=0}^{k-1} g(n-j) = T(n-k-1) + \sum_{j=0}^{k} g(n-j)$$

which is exactly what needs to be shown for $i = k+1$.

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Running Time of Programs

Exercise 1. Suppose you have the choice between three algorithms:

(a) Algorithm A solves your problem by dividing it into five subproblems of half the size, recursively solving each subproblem, and then combining the solutions in linear time.

(b) Algorithm B solves problems of size n by recursively solving two subproblems of size $n-1$ and then combining the solutions in constant time.

(c) Algorithm C solves problems of size n by dividing them into nine subproblems of size $\frac{n}{3}$, recursively solving each subproblem, and then combining the solutions in $\mathcal{O}(n^2)$ time.

Estimate the running times of each of these algorithms. Which one would you choose?

Exercise 2. Recall the recurrence for Mergesort: $T(1) = 0$; $T(n) = 2T(\frac{n}{2}) + (n-1)$, for $n > 1$. Prove by induction that

$$T(n) = n \cdot (\log_2 n - 1) + 1 \quad \text{for } n = 2^k,\ k \geq 1$$

Exercise 3. Analyse the complexity of the following recursive algorithm to test whether a number x occurs in an *unordered* list $L = [x_1, x_2, \ldots, x_n]$ of size n. Take the cost to be the number of list element comparison operations.

Search$(x, L = [x_1, x_2, \ldots, x_n])$:

- if $x_1 = x$ then return yes
- else if $n > 1$ then return **Search**$(x, [x_2, \ldots, x_n])$
- else return no

∗**Exercise 4.** Analyse the complexity of the following recursive algorithm to test whether a number x occurs in an *ordered* list $L = [x_1, x_2, \ldots, x_n]$ of size n. Take the cost to be the number of list element comparison operations.

BinarySearch$(x, L = [x_1, x_2, \ldots, x_n])$:

- if $n = 0$ then return no
- else
  - if $x_{\lceil \frac{n}{2} \rceil} > x$ then return **BinarySearch**$(x, [x_1, \ldots, x_{\lceil \frac{n}{2} \rceil - 1}])$
  - else if $x_{\lceil \frac{n}{2} \rceil} < x$ return **BinarySearch**$(x, [x_{\lceil \frac{n}{2} \rceil + 1}, \ldots, x_n])$
  - else return yes

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Solutions

Exercise 1.

(a) $T(n) = 5 \cdot T(\frac{n}{2}) + \mathcal{O}(n)$. The Master Theorem with $d = 2,\ \alpha = \log_2 5,\ \beta = 1$ implies, since $\alpha > \beta$, that $T(n) = \mathcal{O}(n^{\log_2 5}) = \mathcal{O}(n^{2.322})$.

(b) $T(n) = 2 \cdot T(n-1) + O(1)$. The theorem on linear reductions with $c = 2,\ k = 0$ implies, since $c > 1$, that $T(n) = \mathcal{O}(2^n)$.

(c) $T(n) = 9 \cdot T(\frac{n}{3}) + \mathcal{O}(n^2)$. The Master Theorem with $d = 3,\ \alpha = 2,\ \beta = 2$ implies, since $\alpha = \beta$, that $T(n) = \mathcal{O}(n^2 \log n)$.

It follows that algorithm C has the best asymptotic running time.

Exercise 2. Base case: $T(2^1) = 2 \cdot 0 + (2^1 - 1) = 1$, same as $2^1 \cdot (\log_2 2^1 - 1) + 1 = 2 \cdot 0 + 1 = 1$

Inductive step:

$$\begin{array}{rcll}
T(2^{k+1}) & = & 2 \cdot T(\frac{2^{k+1}}{2}) + (2^{k+1} - 1) & \text{by the recurrence} \\
& = & 2 \cdot T(2^k) + (2^{k+1} - 1) & \\
& = & 2 \cdot (2^k \cdot (\log_2 2^k - 1) + 1) + (2^{k+1} - 1) & \text{by ind. hyp.} \\
& = & 2^{k+1} \cdot (\log_2 2^k - 1) + 2 + 2^{k+1} - 1 & \\
& = & 2^{k+1} \cdot ((\log_2 2^k - 1) + 1) + 1 & \\
& = & 2^{k+1} \cdot k + 1 & \\
& = & 2^{k+1} \cdot (\log_2 2^{k+1} - 1) + 1 &
\end{array}$$

Exercise 3. The worst case is when the element occurs last in the list (or not at all). Let $T(n)$ be the total cost of running **Search**$(x, [x_1, \ldots, x_n])$ in this case.

- if $x_1 = x$ then return yes cost $= 1$ (one list element comparison)
- else if $n > 1$ then return **Search**$(x, [x_2, \ldots, x_n])$ cost $= T(n-1)$ (recursive call with list size $n-1$)
- else return no cost $= 0$

This can be described by the recurrence $T(1) = 1;\ T(n) = 1 + T(n-1)$ with the solution $T(n) = \mathcal{O}(n)$.

Exercise 4. Again, the worst case is when the element occurs last in the list (or is larger than the last element). Let $T(n)$ be the total cost of running **BinarySearch**$(x, [x_1, \ldots, x_n])$ in this case.

- if $n = 0$ then return no cost $= 0$
- else if $x_{\lceil \frac{n}{2} \rceil} > x$ then return **BinarySearch**$(x, [x_1, \ldots, x_{\lceil \frac{n}{2} \rceil - 1}])$
  cost $= 1$ (one list element comparison; this condition is not satisfied when x occurs last in the list)
- else if $x_{\lceil \frac{n}{2} \rceil} < x$ return **BinarySearch**$(x, [x_{\lceil \frac{n}{2} \rceil + 1}, \ldots, x_n])$
  cost $= 1 + T(\lfloor \frac{n}{2} \rfloor)$ (one comparison plus cost of recursive call with the second half of the list)
- else return yes cost $= 0$

This can be described by the recurrence $T(0) = 0;\ T(n) = 2 + T\left(\frac{n}{2}\right)$ with the solution $T(n) = \mathcal{O}(\log n)$.

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Counting and Basic Probability

Exercise 1. A management panel at a hospital needs to include at least one member from each of the following three professions: a doctor, a lawyer and an accountant. How many different panels can be formed in each of the following situations?

(a) Each profession offers 5 possible candidates. The panel size is 3.

(b) Each profession offers 4 possible candidates, but A. Brent (doctor) refuses to serve with C. David (lawyer). The panel size is 3.

(c) Each profession offers 5 possible candidates. The panel size is 5.

(d) Each profession offers 4 possible candidates, but A. Brent (doctor) refuses to serve with C. David (lawyer). The panel size is 5.

Exercise 2. Let $S = \{a, b, c, d\}$ and $T = \{e, f, g\}$.

(a) How many different *functions* $f : S \longrightarrow T$ are there?

(b) How many different *relations* on $S \times T$ are there?

∗(c) How many different *onto* functions $f : S \longrightarrow T$ are there?

∗(d) How many different *binary relations* on S are *antireflexive*?

Exercise 3. Three dice are rolled simultaneously.

(a) What is the probability that the sum of the values is a prime number?

(b) What is the probability of a doublet (2 of the 3 values are equal but the third value is different)?

Exercise 4. Let E_1, E_2 be two events. Prove that $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$ implies $P(E_2 \setminus E_1) = 0$.

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Solutions

Exercise 1.

(a) One member from each profession (out of 5) must be selected; therefore $5^3 = 125$ panels.

(b) $4^3 = 64$ panels possible, out of which we need to subtract the 4 panels including Brent and David $\Rightarrow$ 60 panels.

(c) A 5-member panel can either consist of 3 members of one profession and 1 member from each of the other two, so $3 \cdot \binom{5}{1}\binom{5}{1}\binom{5}{3} = 750$ panels; or of 2 members each from two professions and 1 member of the remaining profession, so $3 \cdot \binom{5}{2}\binom{5}{2}\binom{5}{1} = 1500$ panels. The total is 2250 panels.

(d) As above, we find there are a total of $3 \cdot \binom{4}{1}\binom{4}{1}\binom{4}{3} + 3 \cdot \binom{4}{2}\binom{4}{2}\binom{4}{1} = 624$ possible panels. Out of these we need to subtract the panels that include Brent and David; one way to count them is that there are $\binom{10}{3} = 120$ ways to select the remaining 3 members, minus $\binom{6}{3} = 20$ that do not include any accountant, therefore $120 - 20 = 100$ panels that include Brent and David $\Rightarrow$ total of $624 - 100 = 524$ possible panels.

Exercise 2.

(a) For each $x \in S$ there are $|T|$ choices for $f(x)$; hence, the number of functions is $|T|^{|S|} = 3^4 = 81$.

(b) Each pair $(x, y) \in S \times T$ is either related or not; hence, there are $2^{|S| \cdot |T|} = 2^{12} = 4096$ relations.

(c) Among the 81 functions from (a), *not* onto are all the functions $S \longrightarrow \{e, f\}$ ($2^4 = 16$ functions), $S \longrightarrow \{e, g\}$ ($2^4 = 16$ functions), $S \longrightarrow \{f, g\}$ ($2^4 = 16$ functions). This, however, counts twice the functions $S \longrightarrow \{e\}$, $S \longrightarrow \{f\}$ and $S \longrightarrow \{g\}$. Hence, there are $81 - 3 \cdot 16 + 3 = 36$ onto functions $S \longrightarrow T$.

(d) If $\mathcal{R}$ is an antireflexive relation on $S \times S$, then $(x, x) \notin \mathcal{R}$ for all $x \in S$. Each other pair, that is $(x, y) \in S \times S$ with $x \neq y$, is either related or not. There are $|S| \cdot (|S| - 1) = 12$ such pairs; hence, there are $2^{12} = 4096$ such relations.

Exercise 3.

(a) Possible outcomes are $(i, j, k) \in \{1, \ldots, 6\} \times \{1, \ldots, 6\} \times \{1, \ldots, 6\}$. Possible prime sums are 3 (1 outcome), 5 (6 outcomes), 7 (15 outcomes), 11 (27 outcomes), 13 (21 outcomes) and 17 (3 outcomes). Hence, the overall probability is $\frac{1+6+15+27+21+3}{216} = \frac{73}{216} \approx 0.338$.

(b) Select 2 out of 3 dice to have the same value, which can be any of $1 \ldots 6$, while the third number is different, hence one of the 5 remaining values. Thus the overall probability is $\frac{\binom{3}{2} \cdot 6 \cdot 5}{216} = \frac{90}{216} = \frac{5}{12}$.

Exercise 4. By definition,

$$P(E_1 \setminus E_2) = \sum_{\omega \in E_1 \setminus E_2} P(\omega) = \sum_{\omega \in E_1} P(\omega) - \sum_{\omega \in E_1 \cap E_2} P(\omega) = P(E_1) - \sum_{\omega \in E_1 \cap E_2} P(\omega)$$

Hence, if $P(E_1 \setminus E_2) = P(E_1) - P(E_2)$ then $\sum_{\omega \in E_1 \cap E_2} P(\omega) = \sum_{\omega \in E_2} P(\omega)$. Therefore, $P(\omega) = 0$ for all $\omega \in E_2 \setminus E_1$, hence $\sum_{\omega \in E_2 \setminus E_1} P(\omega) = 0$.

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Advanced Counting and Probability

Exercise 1. How many 5-letter words over the alphabet $\Sigma = \{a, c, e, n, s\}$

(a) include the substring ace?

(b) include all letters from Σ with a before e (for example, $canes$)?

(c) have all their letters in alphabetical order (for example, $aceen$)?

Exercise 2. Prove that

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) \;=\; P(A_1) \cdot P(A_2 \,|\, A_1) \cdot P(A_3 \,|\, A_1 \cap A_2) \cdots P(A_n \,|\, A_1 \cap A_2 \cap \ldots \cap A_{n-1})$$

Exercise 3. Consider three urns: Urn 1 contains one red and two black marbles, Urn 2 contains three red and four black marbles, Urn 3 contains two red and two black marbles. One urn is selected at random and then two marbles are randomly drawn from that urn without replacement. Given that these two marbles are red, what is the probability that Urn 2 was chosen?

$*$**Exercise 4.** Alice and Bob repeatedly toss a coin (outcome H – head, or T – tails) until either Alice's or Bob's winning sequence is observed. What is the probability for Alice to win if

(a) Alice's winning sequence is HTH and Bob's is HHH?

(b) Alice's winning sequence is HTH and Bob's is THT?

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Solutions

Exercise 1.

(a) Select $\omega_1, \omega_2 \in \Sigma$ for words $\omega_1\omega_2 ace$, $\omega_1 ace\omega_2$ and $ace\omega_1\omega_2$, hence $5^2 \cdot 3 = 75$ words.

(b) The letters can be arranged in 5! different ways, half of which satisfy the additional requirement, hence $\frac{5!}{2} = 60$ words.

(c) Any word of length n in alphabetical order *that ends in a letter* ω can be obtained by

- a word of length $n-1$ in alphabetical order *that ends in*
  - ω *or*
  - any letter that comes before ω in the lexicographic order.

The following table follows the rule $n_{row,col} = n_{1,col-1} + n_{2,col-1} + \ldots + n_{row,col-1}$

word length	1	2	3	4	5
words ending in a	1	1	1	1	1
words ending in c	1	2	3	4	5
words ending in e	1	3	6	10	15
words ending in n	1	4	10	20	35
words ending in s	1	5	15	35	70
sum					126

Exercise 2. Base case: $P(A_1) = P(A_1)$

Inductive step:

$$\begin{array}{lll} P((A_1 \cap A_2 \cap \ldots \cap A_n) \cap A_{n+1}) & & \\ = P(A_{n+1} \mid A_1 \cap \ldots \cap A_n) \cdot P(A_1 \cap \ldots \cap A_n) & & \text{by def. of cond. probab.} \\ = P(A_{n+1} \mid A_1 \cap \ldots \cap A_n) \cdot P(A_1) \cdot P(A_2 \mid A_1) \cdots P(A_n \mid A_1 \cap \ldots \cap A_{n-1}) & & \text{by ind. hyp.} \end{array}$$

Exercise 3. Let Urn_i denote the event that the i-th urn was selected and TR the event of two red marbles being drawn.

$$\begin{array}{lll} P(TR \mid Urn_1) = 0\ ; & P(TR \mid Urn_2) = \frac{3}{7} \cdot \frac{2}{6}\ ; & P(TR \mid Urn_3) = \frac{2}{4} \cdot \frac{1}{3} \\ P(TR \cap Urn_1) = 0\ ; & P(TR \cap Urn_2) = (\frac{3}{7} \cdot \frac{2}{6}) \cdot \frac{1}{3}\ ; & P(TR \cap Urn_3) = (\frac{2}{4} \cdot \frac{1}{3}) \cdot \frac{1}{3} \end{array}$$

$$P(TR) = P(TR \cap Urn_1) + P(TR \cap Urn_2) + P(TR \cap Urn_3) = \frac{13}{126}$$
$$P(Urn_2 \mid TR) = \frac{P(TR \cap Urn_2)}{P(TR)} = \frac{126}{21 \cdot 13} \approx 0.4615$$

Exercise 4.

(a) Let $p = P(HTH \text{ comes first})$. Consider the (recursive) tree of possible outcomes:

HHH	loss
$HHTH$	win
$HHTT$	p
HTH	win
HTT	p
T	p

Therefore, $p = \frac{1}{16} + \frac{1}{16}p + \frac{1}{8} + \frac{1}{8}p + \frac{1}{2}p$; hence $p = \frac{3}{5}$

(b) This is obviously symmetric, so the probability must be $\frac{1}{2}$.