COMP9020 20T1

Week 1 Problem Set
Numbers, Sets, Words, Logic

Foundations of Computer Science

[Show with no answers] [Show with all answers]

1. (Numbers)

   How many numbers in the interval [1431,9758] are

   a. divisible by 3?

   b. divisible by 5?

   c. divisible by 3 and 5?

   d. divisible by 3 or 5?

   [hide answer]

   Using the formula $\left\lfloor \frac{m}{k} \right\rfloor - \left\lfloor \frac{n-1}{k} \right\rfloor$:

   a. $\left\lfloor \frac{9758}{3} \right\rfloor - \left\lfloor \frac{1430}{3} \right\rfloor = 2776$ numbers divisible by 3 $(1431, 1434, \ldots, 9756)$.

   b. $\left\lfloor \frac{9758}{5} \right\rfloor - \left\lfloor \frac{1430}{5} \right\rfloor = 1665$ numbers divisible by 5 $(1430, 1435, \ldots, 9755)$.

   c. $\left\lfloor \frac{9758}{15} \right\rfloor - \left\lfloor \frac{1430}{15} \right\rfloor = 555$ numbers divisible by 3 and 5 $(1440, 1455, \ldots, 9750)$.

   d. From the numbers counted in a. and b. we need to subtract those that were counted twice, i.e. that are divisible by 15 (answer c.); hence $2776 + 1665 - 555 = 3886$ numbers are divisible by 3 or 5.

2. (Sets)

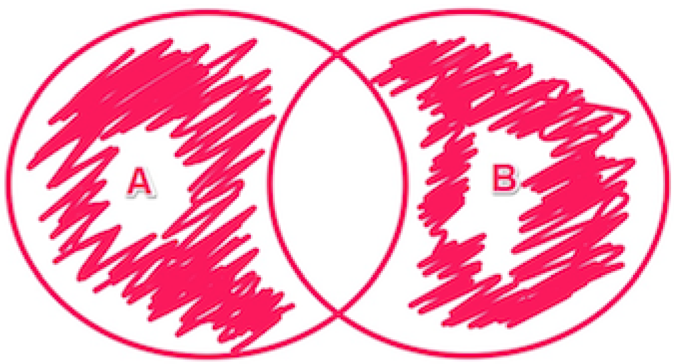
   Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$

   a. using Venn diagrams,

   b. without Venn diagrams.

   [hide answer]

   a. Drawing both $(A \setminus B) \cup (B \setminus A)$ and $(A \cup B) \setminus (A \cap B)$ result in the same diagram:

   b. We show in both directions that if an element belongs to $(A \setminus B) \cup (B \setminus A)$ then it also belongs to $(A \cup B) \setminus (A \cap B)$ and vice versa:

- Consider an element $x \in (A \setminus B) \cup (B \setminus A)$. Then either $x \in A \setminus B$ or $x \in B \setminus A$. In both cases it follows that $x \in A \cup B$ and (by the definition of set difference) $x \notin A \cap B$. Therefore, $x \in (A \cup B) \setminus (A \cap B)$.

- Suppose that $x \in (A \cup B) \setminus (A \cap B)$. This means that $x \in A \cup B$ (and, therefore, either $x \in A$ or $x \in B$), but $x \notin A \cap B$. If $x \in A$ and $x \notin A \cap B$, then $x \in A \setminus B$; alternatively, if $x \in B$ and $x \notin A \cap B$, then $x \in B \setminus A$. In either case, we conclude that $x \in (A \setminus B) \cup (B \setminus A)$.

3. (Alphabets and Words)

   Let $\Sigma = \{a, b, c\}$ and $\Psi = \{a, c, e\}$.

   a. How many words are in the set Σ^2 ?

   b. What are the elements of $\Sigma^2 \setminus \Psi^*$?

   c. Is it true that $\Sigma^* \setminus \Psi^* = (\Sigma \setminus \Psi)^*$? Why or why not?

   [hide answer]

   a. $\Sigma^2 = \{aa, ab, ac, ba, \ldots, cc\}$, hence $|\Sigma^2| = 3 \cdot 3 = 9$.

   b. $\Sigma^2 \setminus \Psi^* = \{ab, ba, bb, bc, cb\}$, that is, all words in Σ^2 with the letter b.

   c. This is not true: For example, $ab \in \Sigma^*$ and $ab \notin \Psi^*$, hence $ab \in \Sigma^* \setminus \Psi^*$; but $\Sigma \setminus \Psi = \{b\}$, hence $ab \notin (\Sigma \setminus \Psi)^*$.

4. (Propositional Logic)

   For each of the following formulae, determine all the truth assignments to A, B and C under which the formula is true.

   a. $A \wedge (\neg C \Rightarrow (B \vee \neg A))$

   b. $(A \wedge \neg C) \Rightarrow (B \vee \neg A)$

   c. $(\neg C \Rightarrow \neg A) \wedge (B \Rightarrow (A \wedge \neg C))$

   d. $\neg(C \Rightarrow A) \wedge (A \vee (B \wedge \neg C))$

   [hide answer]

   a. There are 3 truth assignments under which the formula is true:

      - $A = \mathsf{T},\ B = \mathsf{F},\ C = \mathsf{T}$
      - $A = \mathsf{T},\ B = \mathsf{T},\ C = \mathsf{F}$
      - $A = \mathsf{T},\ B = \mathsf{T},\ C = \mathsf{T}$

   b. The formula is true under all truth assignments except for $A = \mathsf{T},\ B = \mathsf{F},\ C = \mathsf{F}$.

   c. There are 3 truth assignments under which the formula is true:

      - $A = \mathsf{F},\ B = \mathsf{F},\ C = \mathsf{F}$
      - $A = \mathsf{F},\ B = \mathsf{F},\ C = \mathsf{T}$
      - $A = \mathsf{T},\ B = \mathsf{F},\ C = \mathsf{T}$

   d. There is no truth assignment under which the formula is true: $\neg(C \Rightarrow A)$ requires $C = \mathsf{T}$ and $A = \mathsf{F}$, but then $A \vee (B \wedge \neg C)$ can't be T.

5. (Proving properties of algorithms)

   Recall the algorithm for computing the greatest common divisor (gcd) of two positive numbers:

$$gcd(m,n) = \begin{cases} m & \text{if } m = n \\ gcd(m-n,n) & \text{if } m > n \\ gcd(m,n-m) & \text{if } m < n \end{cases}$$

   Recall the correctness proof given in class. What needs to be changed to adapt it to the faster version below?

$$gcd(m,n) = \begin{cases} m & \text{if } n = 0 \\ gcd(n, m \,\mathsf{mod}\, n) & \text{if } n > 0 \end{cases}$$

   [hide answer]

   The original Euclid's algorithm for the gcd of two positive numbers works because if *d,a,b* are integers and *d* is a common divisor of *m* and *n*, then *d* is a common divisor of *m – n* and *n* (or of *n – m* and *m*) and vice versa.

   For the second, faster version, *gcd(m,0) = m* by definition; and if *n > 0* and *r = m* **mod** *n*, then there must be some integer *a* such that *r = m – a·n*. Therefore, again, if *d* is a common divisor of *m* and *n*, then *m = k·d* and *n = l·d* for some *k,l* $\in \mathbb{Z}$, hence *r = k·d – a· l·d = d(k – a·l)*, which implies that *d* is a divisor of *r = m* **mod** *n* and vice versa.

   The second version terminates for any *m,n* $\in \mathbb{N}$ because if *n > 0* then

   - if *m* ≥ *n* then *m* **mod** *n* < *n*, so the next recursive call operates on a smaller pair of numbers;
   - if *m* < *n* then *m* **mod** *n* = *m* and the next recursive call operates on *n,m*, to which the case above applies since *n* ≥ *m*.

6. **Challenge Exercise**

   A multiplication magic square has the product of the numbers in each row, column and diagonal the same. If the diagram below is filled with positive integers to form a multiplicative magic square, then give the value of *y*.

5		*y*
4		
	1	

   [hide answer]

   y = 2.

   You can reason as follows:

5	D	*y*
4	C	

A	1	B

Let p be the product of the numbers in each row, column or diagonal. Then A must be $p/20$. It follows that B = 20. This in turn implies that C must be $p/100$. Therefore D = 100. Hence, y is $p/500$. Since $y \cdot C \cdot A$ must also result in p, it follows that $(p/500)\cdot(p/100)\cdot(p/20) = p^3/1{,}000{,}000 = p$. Hence $p = 1000$, which implies $y = p/500 = 2$.

The complete magic square is

5	100	2
4	10	25
50	1	20

Assessment

This first problem set is meant to give you your first practice and will not count towards your mark for the weekly homework.

However, in order to familiarise yourself with the environment and structure of the weekly assessments, go to COMP9020 20T1 Quiz Week 1 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 27 February 10:00:00am**.

Please always be mindful of the following **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult