COMP9020 20T1

Week 3 Problem Set
Functions and Relations

Foundations of Computer Science

1. (Functions)

   Let $\Sigma = \{0, 1\}$. Consider the functions $f, g : \Sigma^* \longrightarrow \Sigma^*$ and $h : \Sigma^* \times \Sigma^* \longrightarrow \Sigma^*$ given by

   - $f(\omega) = \omega\omega$
   - $g(\omega) = 0\omega 1$
   - $h(\nu, \omega) = \nu\omega\nu$

   Compute the following function values:

   a. $(f \circ g)(10)$

   b. $(g \circ f)(10)$

   c. $h(10, f(01))$

   d. $h(f(1), g(0))$

2. (Properties of functions)

   Which of the three functions f, g and h in Exercise 1 is onto? Which are 1-1?

3. (Matrix functions)

   Prove each of the following statements.

   a. $(\mathbf{A}^T)^T = \mathbf{A}$ for any matrix $\mathbf{A}$.

   b. If two matrices $\mathbf{A}$ and $\mathbf{B}$ are of the same size, then $(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$.

   c. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$ for any matrix $\mathbf{A}$ of size $m \times n$ and matrices $\mathbf{B}$, $\mathbf{C}$ of size $n \times p$.

4. (Boolean functions)

   a. Give all elements of BOOL(2), that is, all functions $\mathbb{B}^2 \longrightarrow \mathbb{B}$ over two Boolean variables.

   b. Show that there are 2^{2^n} elements in BOOL(n) for $n \in \mathbb{P}$.

5. (Properties of binary relations)

   a. Consider the relation $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}$ defined by

   $$(a, b) \in \mathcal{R} \text{ iff } b + 0.5 \geq a \geq b - 0.5$$

   Which of the following standard properties does $\mathcal{R}$ satisfy?

- Reflexivity
- Antireflexivity
- Symmetry
- Antisymmetry
- Transitivity

b. For each of the following statements, give a valid proof if it is true for all relations $\mathcal{R}_1 \subseteq S \times S$ and $\mathcal{R}_2 \subseteq S \times S$ over arbitrary sets S. If the statement is not always true, provide a counterexample.

- If $\mathcal{R}_1$ and $\mathcal{R}_2$ are symmetric, then $\mathcal{R}_1 \cap \mathcal{R}_2$ is symmetric.
- If $\mathcal{R}_1$ and $\mathcal{R}_2$ are antisymmetric, then $\mathcal{R}_1 \cup \mathcal{R}_2$ is antisymmetric.

6. **Challenge Exercise**

Consider a set U and the binary relation $\mathcal{R}$ on $Pow(U)$ defined by $(A, B) \in \mathcal{R}$ iff $|A \cap B| \geq 1$. Prove that $\mathcal{R}$ is transitive if and only if $|U| \leq 1$.

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 3 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 12 March 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult