COMP9020 20T1

Week 4 Problem Set
Equivalence Relations, Orderings

Foundations of Computer Science

1. (Equivalence relations)

   For each of the following relations $\mathcal{R}$, prove or disprove that $\mathcal{R}$ is an equivalence relation.

   a. Real number r is related to real number s iff $|r - s| \leq 1$.

   b. Pair of integers (i, j) is related to pair of integers (m, n) iff $i + j \equiv m - n \pmod 2$.

   c. Set A is related to set B iff $A \cap B = A$ and $B \in Pow(A)$.

   d. Let $\Sigma = \{a, b\}$. Word $\nu \in \Sigma^*$ is related to word $\omega \in \Sigma^*$ iff $\nu = \omega\chi$ for some word $\chi \in \Sigma^*$.

   e. Propositional formula ϕ is related to propositional formula ψ iff $\phi \models \psi$ and $\models \psi \Rightarrow \phi$.

   f. Over the standard Boolean algebra, $x, y \in \mathbb{B} = \{0, 1\}$ are related iff $(x + y') \cdot (x' + y) = 0$.

   g. 1×2 matrix **A** is related to 1×2 matrix **B** iff **A** = **B** or $\mathbf{A} \cdot \mathbf{B}^T = 0$.

2. (Modular arithmetic)

   a. Prove that if $m, n \in \mathbb{Z}$ and $m \equiv n \pmod p$ then $m^2 \equiv n^2 \pmod p$.

   b. Let $p = 3$. Give all equivalence classes for $m^2 \equiv n^2 \pmod p$ where $m, n \in \mathbb{Z}$.

3. (Partial versus total orders)

   Consider the relation $\mathcal{R} \subseteq \mathbb{R} \times \mathbb{R}$ defined by $(a, b) \in \mathcal{R}$ iff either $a \leq b - 0.5$ or $a = b$.

   Show that $\mathcal{R}$ is a partial order, but not a total order.

4. (Partial orders)

   For each of the following relations, prove or disprove that it is a partial order.

   a. Natural number m is related to natural number n iff $n^3 > m^2 + 1$.

   b. Natural number m is related to natural number n iff $\lceil m + 0.5 \rceil > n$.

   c. Positive integer m is related to positive integer n iff $3m > 2n$.

   d. Positive integer m is related to positive integer n iff $\gcd(m, n) = n$.

   e. Positive integer m is related to positive integer n iff m is a prime divisor of n.

   f. Integer m is related to integer n iff $m^3 \leq n^3$.

5. (Lattices)

   a. Draw a Hasse diagram for the following partially ordered set:

$$S = \{1, 2, 3, 5, 6, 10, 15, 30\}$$
$$x \preceq y \text{ iff } x \mid y$$

Is $(S, \preceq)$ a lattice? Why or why not?

b. Let binary relation $\mathcal{R}$ on $\{1, \ldots, 20\}$ be defined by $a\mathcal{R}b$ iff either $a = b$ or $a - b > 10$. Show that $(\{1, \ldots, 20\}, \mathcal{R})$ is a partial order. Is $(\{1, \ldots, 20\}, \mathcal{R})$ a lattice? Why or why not?

6. **Challenge Exercise**

Using the set $\{1, \ldots, 10\}$ with the natural total order, define $A = \{1, \ldots, 10\} \times \{1, \ldots, 10\}$ and consider these two orderings over A:

a. product $\sqsubseteq_P$

b. lexicographic $\sqsubseteq_L$

Find the maximal length of a chain $a_1 \sqsubseteq a_2 \sqsubseteq \ldots \sqsubseteq a_n$ (such that $a_i \neq a_{i+1}$) for each of the two orderings.

Assessment

After you have solved the exercises, go to COMP9020 20T1 Quiz Week 4 to answer 4 quiz questions on this week's problem set (Exercises 1-5 only) and lecture.

The quiz is worth 2.5 marks.

There is no time limit on the quiz once you have started it, but the deadline for submitting your quiz answers is **Thursday, 19 March 10:00:00am**.

Please continue to respect the **quiz rules**:

Do …

- use your own best judgement to understand & solve a question
- discuss quizzes on the forum only **after** the deadline on Thursday

Do not …

- post specific questions about the quiz **before** the Thursday deadline
- agonise too much about a question that you find too difficult