CSIT113
Problem Solving

Workshop – Week 7

$2n$-Counters Problem

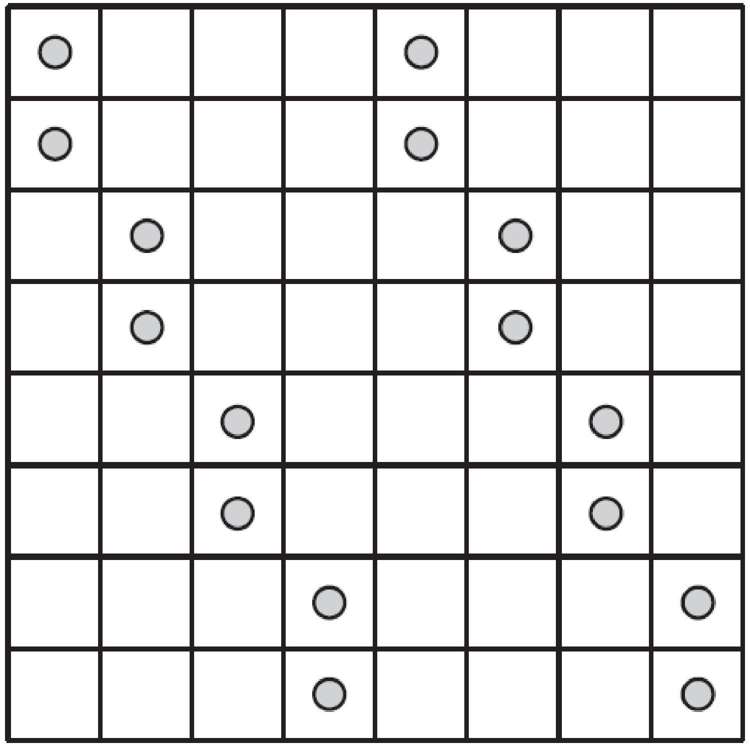
For any $n > 1$, place $2n$ counters on an $n \times n$ board so that no more than two counters are in the same row, column, or diagonal.

Solution

- Since $2n$ counters need to be placed into n rows and n columns of the board with at most two counters in the same row or in the same column, exactly two counters have to be placed in each row and column.
- We consider two cases: n is even and n is odd.

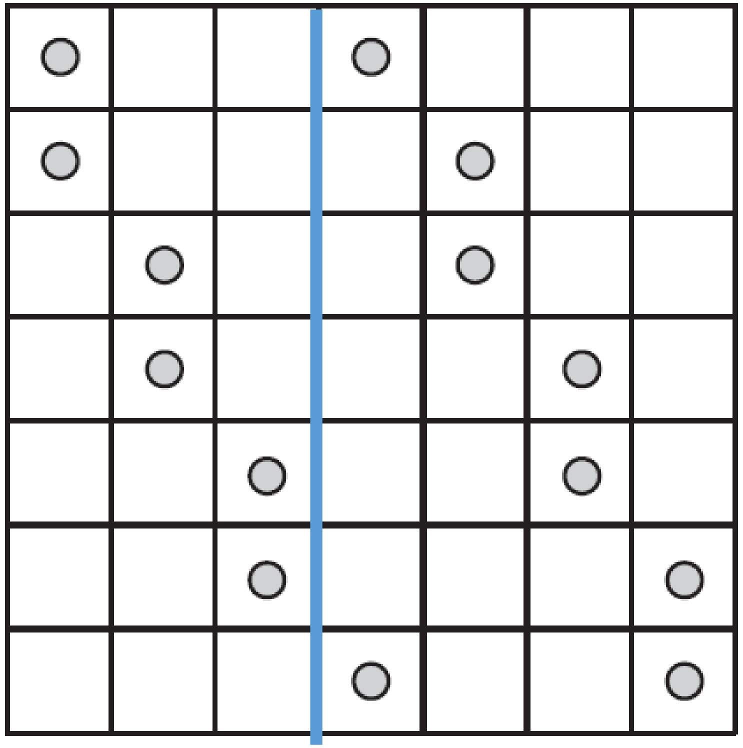
n is even, i.e., n= 2k for some k

- We assume that rows and columns of the board are numbered top to bottom and left to right, respectively
- Place two counters in the first two rows of columns 1 and $k + 1$
- Place two counters in rows 3 and 4 of columns 2 and $k + 2$,
- and so on until finally counters are placed in rows $n - 1$ and n of columns k and $2k$
- Figure on the right shows example for n = 8



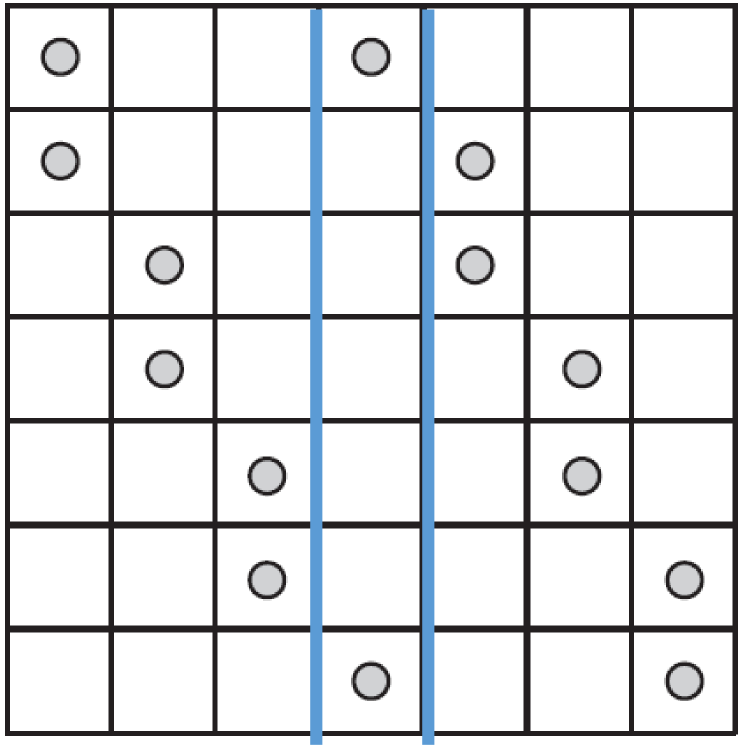
n is odd, i.e., n = 2k+1 for some k

- Place two counters in rows 1 and 2 of column 1
- Place two counters in rows 3 and 4 of column 2
- Continue until counters are placed in rows $n - 2$ and $n - 1$ of column k
- Figure on the right shows example for the case of n=7 (look at the left side of the board)
- Then two counters are placed in the first and last rows of column k +1



n is odd, i.e., n = 2k+1 for some k

- After that, k counters are placed in the right part of the board symmetrically with respect to the board's central square to those in the left part
- Two counters are placed in rows 2 and 3 of column $k + 2$
- Place two counters in rows 4 and 5 of column $k + 3$
- Continue until rows $n - 1$ and n of the last column
- Figure on the right shows example for the case of n=7



Straight Tromino Tiling

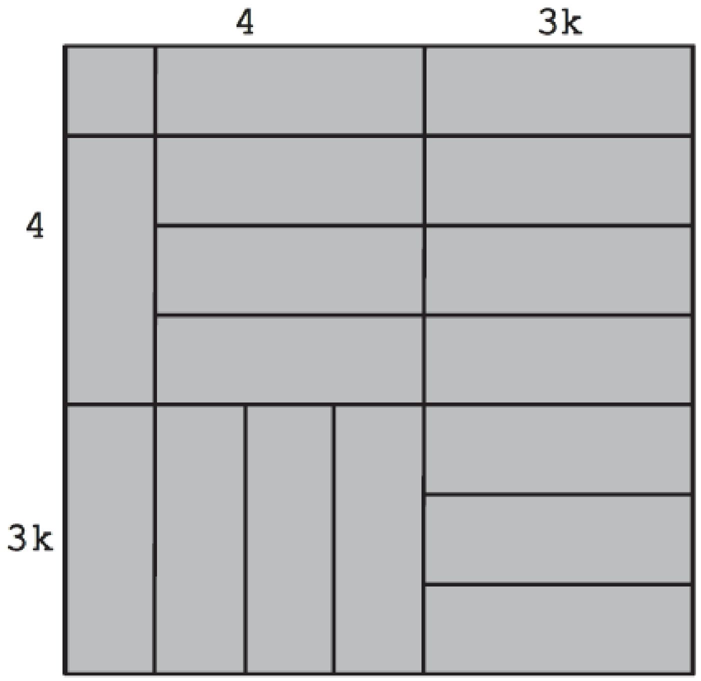
A straight tromino is a 3×1 tile. Obviously, one can tile any $n \times n$ square with straight trominoes if n is divisible by 3. Is it true that for every $n > 3$ that is not divisible by 3, one can tile an $n \times n$ square with straight trominoes and a single 1×1 tile called a monomino? If it is possible, explain how; if it is not, explain why.

Solution : a tiling is always possible

- We divide into 2 cases : $n = 1 \bmod 3$ and $n = 2 \bmod 3$
- Clearly $n > 3$ since otherwise we cannot fit the straight tromino into the board

$n = 1 \bmod 3$

- We can write n of the form $n = 4 + 3k$ where $k \geq 0$
- We can divide the square into three subregions : the 4×4 square in the upper left corner, the $4 \times 3k$ rectangle, and $3k \times (4 + 3k)$ rectangle.
- The 4×4 square requires a monomino placed in one of its corners to make it possible to tile the rest of it.
- Tiling the other two rectangles (if $k > 0$) is trivial since both of them have a side equal to $3k$



n = 2 mod 3

- We write n = 5 + 3k where k ≥ 0
- We use the same method as in previous case
- Look at the figure for the solution