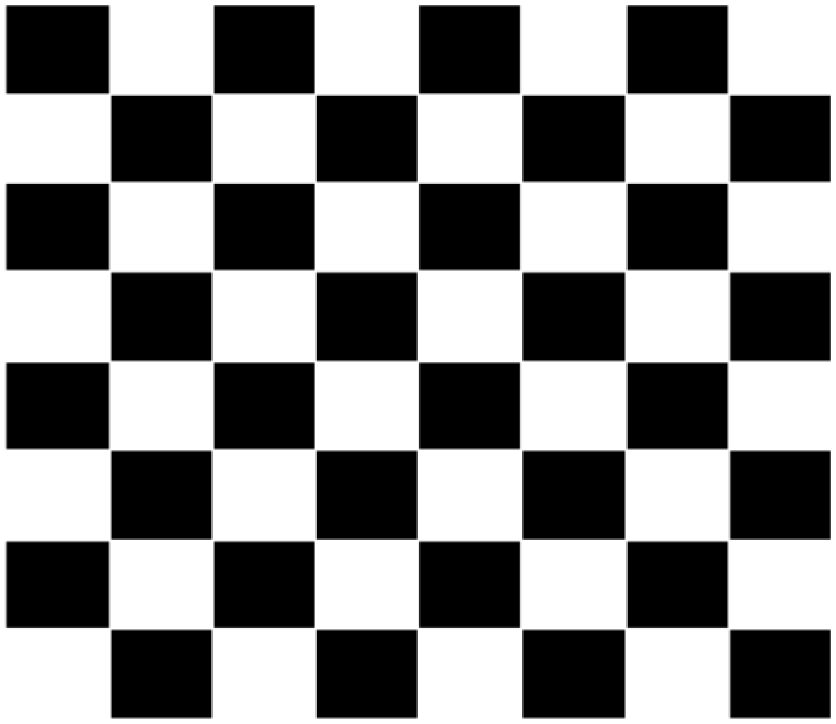
CSIT113
Problem Solving

Workshop – Week 6

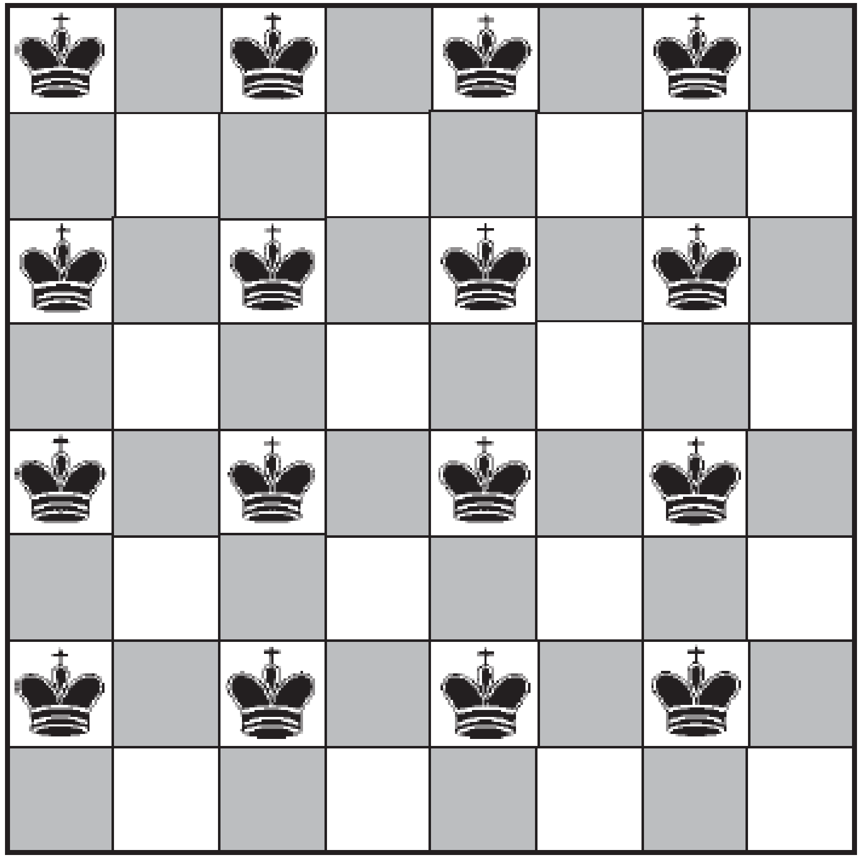
Non-Attacking Kings

Place the greatest possible number of kings on an 8 × 8 chessboard so that no two kings are placed on adjacent—vertically, horizontally, or diagonally—squares.



Greedy strategy

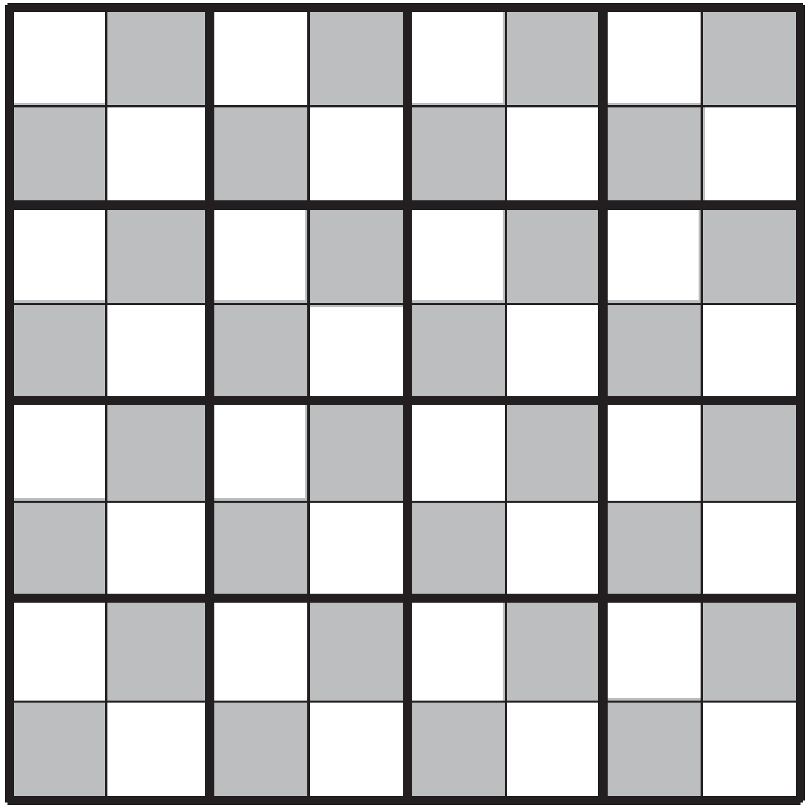
- Place the maximum number of non-adjacent kings in the first column of the board.
- How many? Four
- What next?
- Skip the next column. Why?
- Because each of its squares is adjacent to one of the placed kings in the first column
- Place four kings in the third column.
- Continue to the end. How many kings on the board?
- 16



- Is 16 the maximum number of kings you can place on the board?
- If yes, prove it?

A proof

- Partition the board into 16 four-by-four squares.
- Now it is impossible to place more than one king in each of these squares
- Hence the maximum number of nonadjacent kings on the board cannot exceed 16

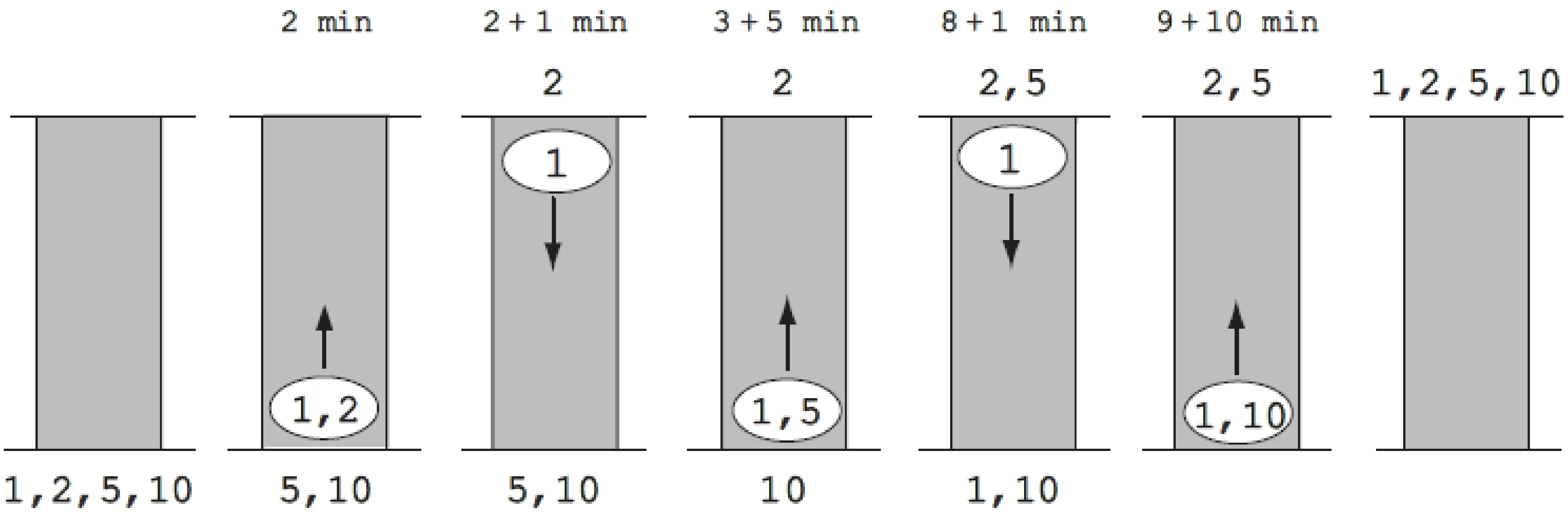


Bridge crossing at night

- A group of four people, who have one flashlight, need to cross a rickety bridge at night.
- A maximum of two people can cross the bridge at one time, and any party that crosses (either one or two people) must have the flashlight with them. The flashlight must be walked back and forth; it cannot be thrown.
- Person A takes 1 minute to cross the bridge, person B takes 2 minutes, person C takes 5 minutes, and person D takes 10 minutes.
- A pair must walk together at the rate of the slower person’s pace.
- Find the fastest way they can accomplish this task.

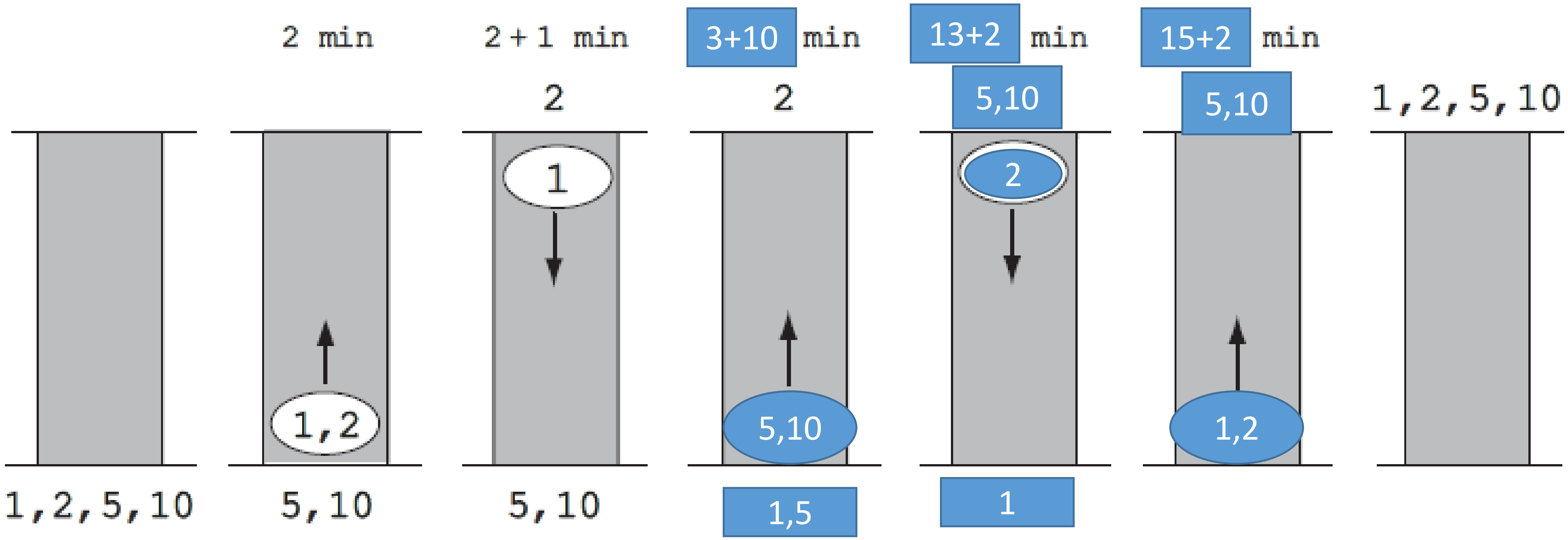
First approach

- Send to the other side the two fastest people, persons A and B. It will take 2 minutes. Then return the flashlight with the fastest of the two, that is, with person A. It takes 1 more minute.
- Send to the other side the two fastest persons available, that is, persons A and C (5 minutes) and return the flashlight with the fastest person A (1 minute).
- Finally, the two persons remaining will cross the river together (10 minutes).
- The total amount of time this greedy-based schedule requires is (2+1)+(5+1)+10 = 19 minutes



- But it is not the fastest possible solution.
- Can you think of the fastest one?

Another solution: 17 minutes



A proof

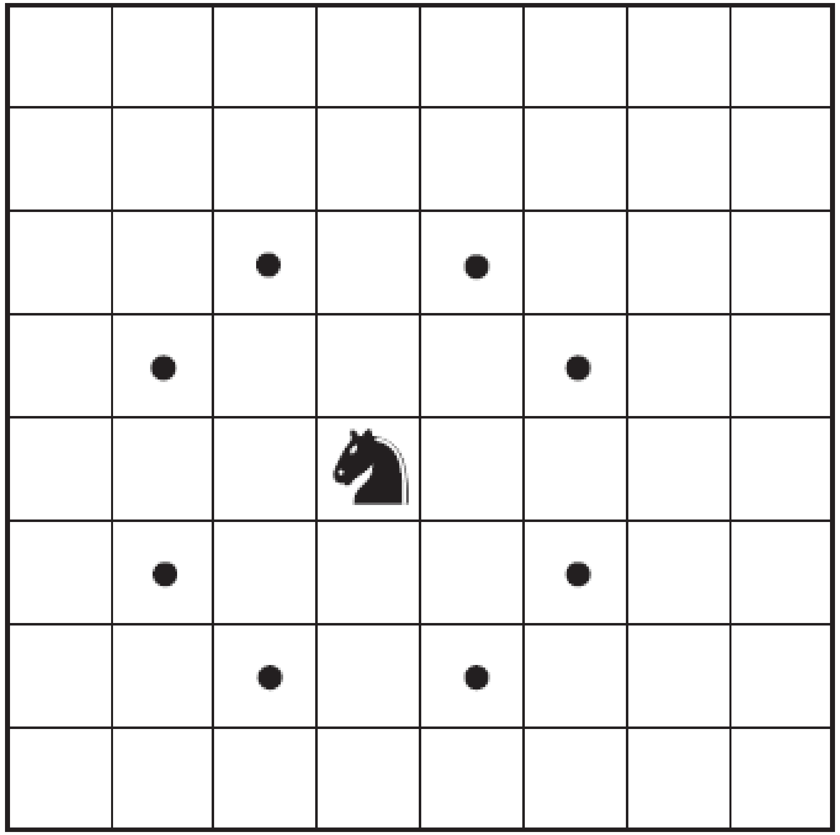
- We prove that 17 minutes is the minimum amount of time needed.
- It is obvious (and can be formally proved) that in an optimal solution two persons have to cross the bridge together and one person has to return the flashlight, if not, all the people are already on the other side.
- Thus three two-person trips and two one-person trips are needed for four people to get to the other side in a minimum amount of time.
- If the flashlight is returned by the fastest person on both back trips, the fastest person will have to participate in each pair going to the other side, for the total time of (10 + 1) + (5 + 1) + 2 = 19 minutes

- If one of the two back trips is not done by the fastest person, then the return crossing times will be at least 2+1 = 3 minutes, and trips to the other side will be at least 10+2 +2 = 14 minutes because at least one pair will have to include the slowest person and hence take 10 minutes to cross the bridge whereas the other two pairs will take at least 2 minutes each.
- Hence, the total crossing time will have to take at least 17 minutes.

Chessboard colourings

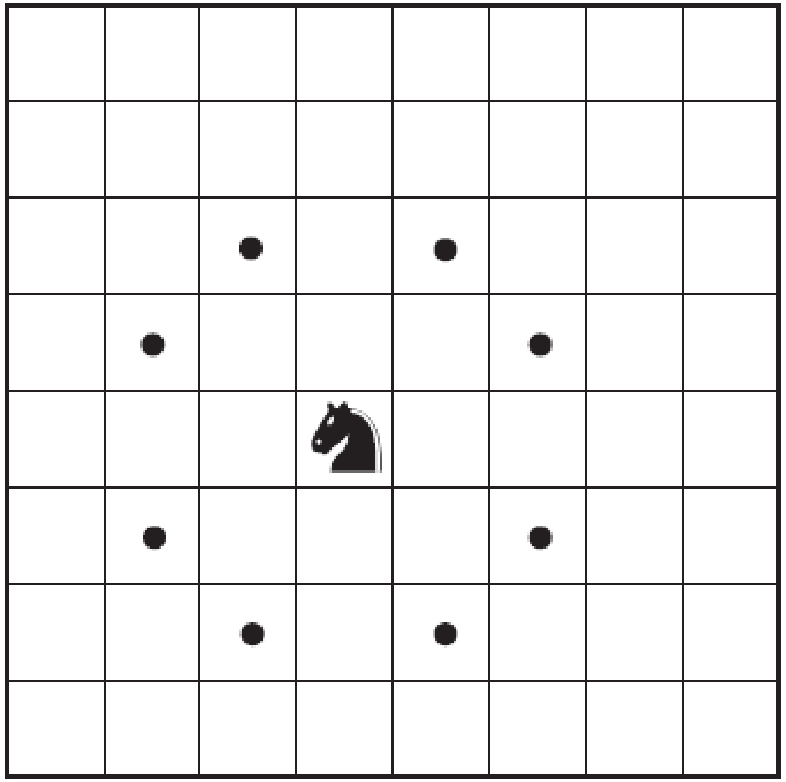
For each of the following chess pieces, find the minimum number of colors needed to color an $n \times n$ chessboard ($n > 1$) so that no two pieces in question placed on two squares of the same color can threaten each other, i.e., after colouring the board, the piece mentioned in the question (wherever it be placed in the board) will have to make the next move to a square of different color.

a) The knight. (The knight threatens any square that is two squares horizontally and one square vertically, or two squares vertically and one square horizontally from the square it occupies.)

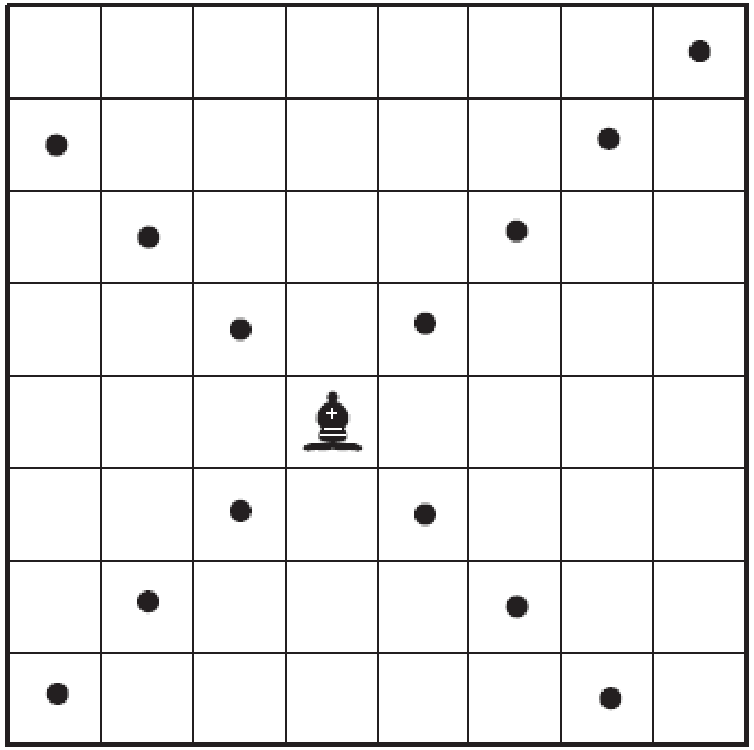


Solution

- The minimum number of colors for $n > 2$ is two:
- more than one color is obviously needed.
- the standard coloring of the board in two opposite colors provides a coloring required by the question. It is one for $n = 2$: no two knights threaten each other on such a small board.



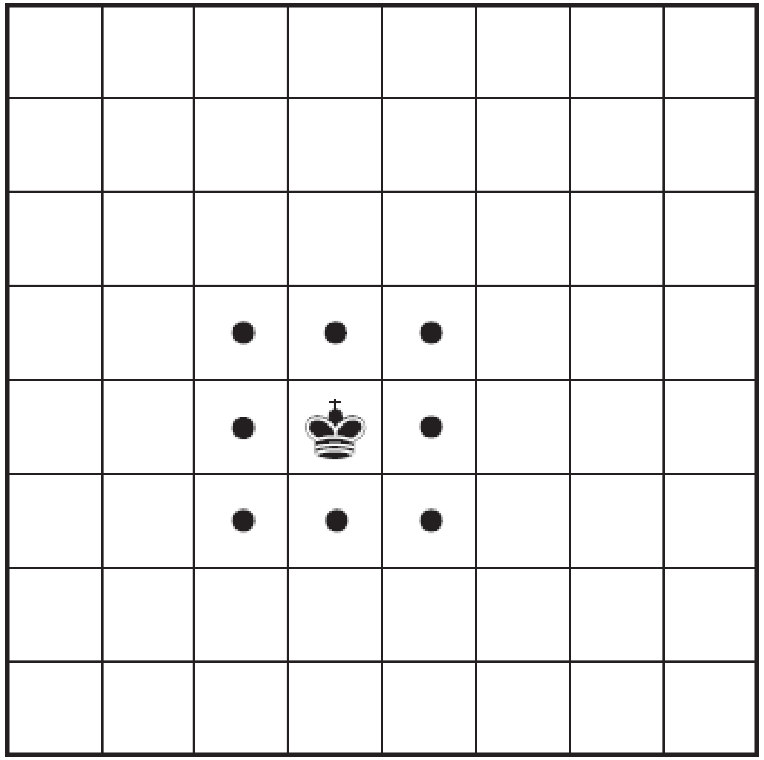
b) The bishop. (The bishop threatens any square that is on the same diagonal.)



Solution

- Since the bishop threatens all the squares on the same diagonal and no other squares, at least n colors are needed to color the main diagonal of the board from its upper left corner to the lower right corner.
- The easiest way to extend this coloring to the entire board is to color all the squares in the same column the same color as the square on the main diagonal.
- Thus, the answer for the bishop is n.

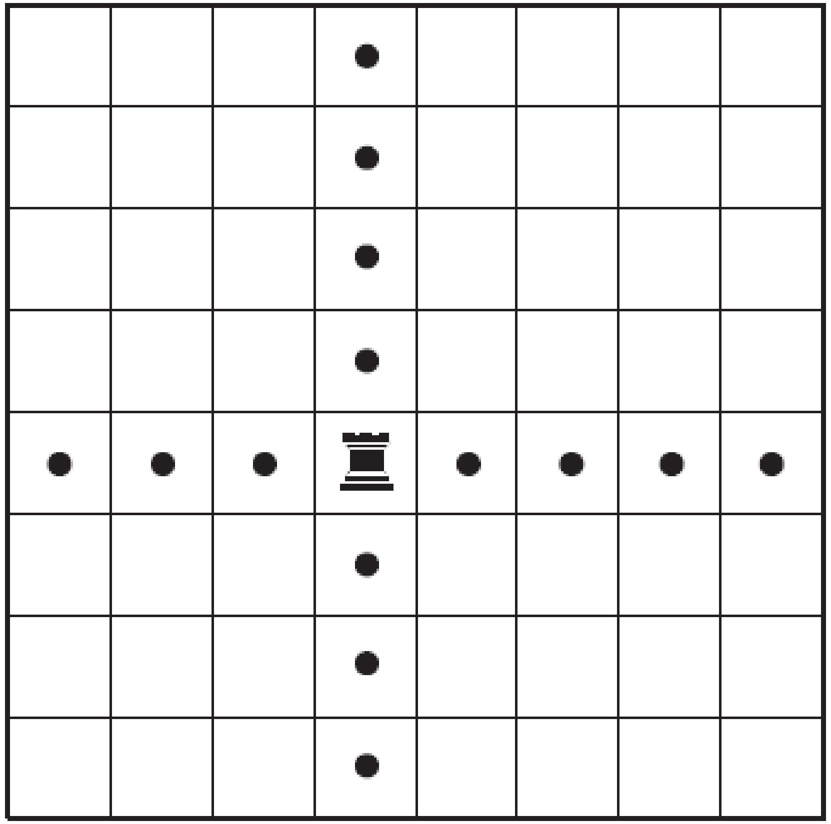
c) The king. (The king threatens any square adjacent to it horizontally, vertically, or diagonally.)



Solution

- Since the king threatens only every square adjacent to it horizontally, vertically, or diagonally, at least four colors are needed to color each 2×2 region of the board.
- Dividing the board into such disjoint regions (some of which may degenerate to smaller rectangles that can be thought of as parts 4×4 regions with some squares outside the board)
- and coloring each 2×2 region with four colors using the same coloring scheme implies that the answer for the king is four.

d) The rook. (The rook threatens any square in the same row or in the same column.)



Solution

- Since the rook only threatens all the squares in the same row or column, at
- least *n* colors are needed to color every line (row or column). This number is not only necessary but also sufficient.
- The easiest way to get a coloring in *n* colors so that no two squares in the same line are colored the same, is to color, say,
- the first row in *n* colors and then shift the coloring scheme one column to the right with wrapping the colors that fall outside the board to color the squares in the leftmost columns.

An example for $n = 5$

1	2	3	4	5
5	1	2	3	4
4	5	1	2	3
3	4	5	1	2
2	3	4	5	1

Coloring a 5×5 board in five colors with no two squares in the same row or column colored the same color.