CSIT113
Problem Solving

Week 3

Invariants

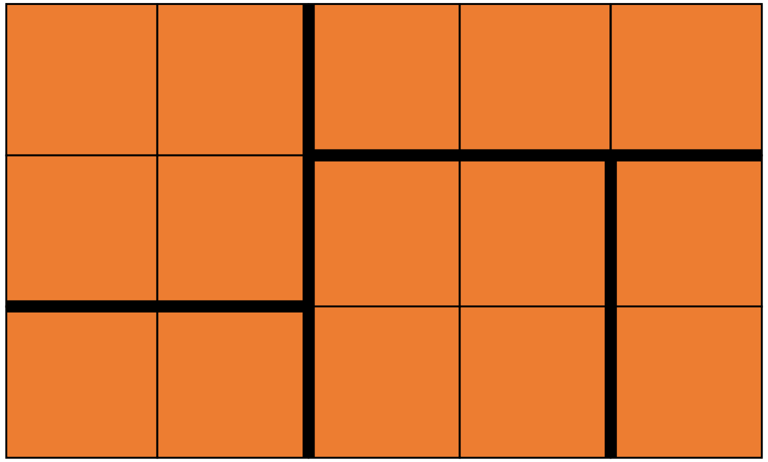
- An invariant is something that does not change when we apply the rules for a problem.
- Often, identifying the invariant will provide a means to solve the problem.
- The following problems will illustrate this idea.

Problem 1 : Chocolate Bars

- A rectangular chocolate bar is divided into squares by horizontal and vertical grooves.
- It is to be broken into individual squares.
- A break is made by taking a single piece and breaking it along one of the grooves.
- How many breaks are needed to completely break the bar into all its individual squares?

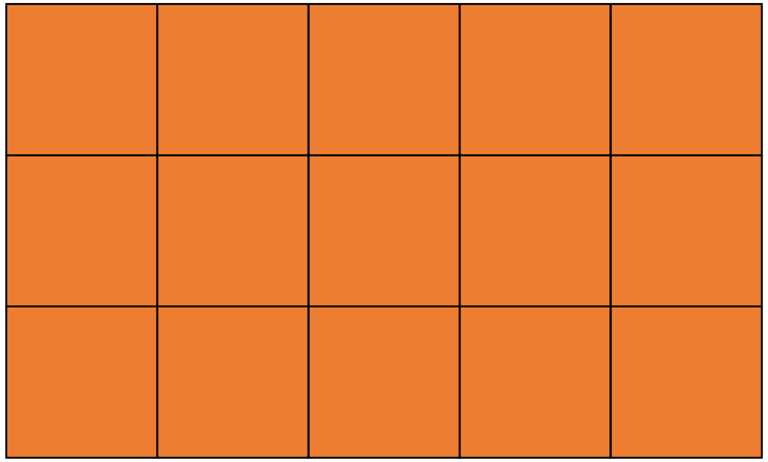
Problem 1

- Here we have a 3x5 bar which has been broken 4 times.



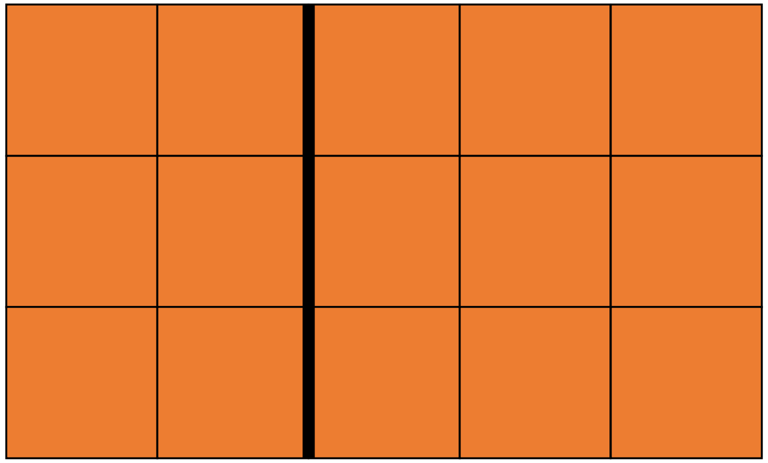
Problem 1

- Whenever a break is made, the number of breaks (C) increases by one, and the number of pieces (P) increases by one.
- At the beginning P=1 and C=0



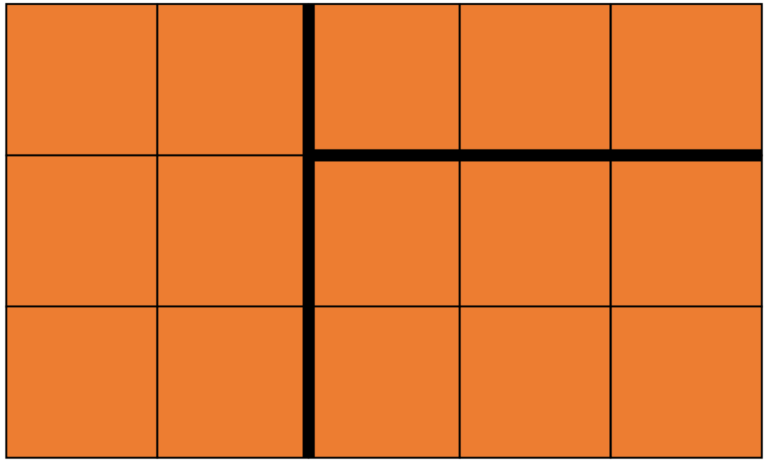
Problem 1

- Making a break as in the picture, we now have P = 2, C = 1



Problem 1

- One more break, we have C = 2, P = 3



Problem 1

- So number of breaks and the number of pieces both change.
- What does not change (invariant)?
- It is the difference between the number of breaks and the number of pieces. That is: $P - C = 1$ is an invariant.
- Hence, at the end of the process of breaking the chocolate bar, P will be the number of all squares in the chocolate bar.
- So the number of cuts will be $C = P - 1$, that is one less than the number of squares

What we have used?

Introduce variables (P and C) to characterise the essential elements of the problem (abstraction)

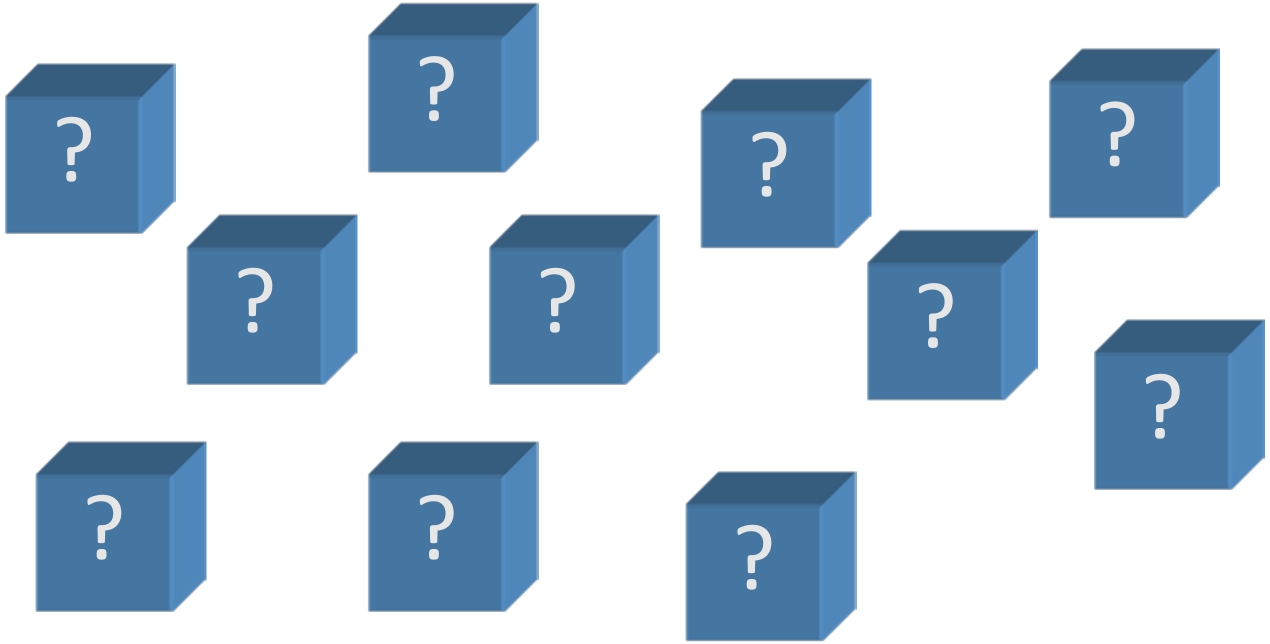
Invariant: find invariance of the function of variables, namely P-C

Exploit invariance of P-C to find the solution

Problem 2: Empty Boxes

- Eleven large empty boxes are placed on a table
- An unknown number of the large boxes are selected and 8 medium sized boxes are placed in each.
- An unknown number of the medium boxes are selected and 8 small boxes are placed in each.
- At the end of this there are 102 empty boxes.
- How many boxes are there in total?

Problem 2



Similarities to Problem 1

- Given the initial state
- Given some incomplete information of final state
- We are required to completely characterise the final state

Strategy

- **Step 1:** identify what is unknown about the final state and what is known
- **Step 2:** introduce variables to represent the state
- **Step 3:** Model the process of filling boxes
- **Step 4:** Identify an invariant
- **Step 5:** Combine the previous steps to deduce the final state

Step 1

- Unknown: the number of (filled) boxes in the final state
- Known: the number of empty boxes (102)

Step 2: Introduce variables

- F:= the number of filled boxes
- E: = the number of empty boxes
- It is guided by the goal: determine E+F at the final state
- Common mistakes: try to count the number of medium or small boxes.
- Initially, F = 0, E = 11

Step 3: The process of filling boxes

- When a box is filled, the number of filled boxes increases by 1
- Hence F: = F+1 = 1
- The number of empty boxes increases by 8-1=7, since 8 empty boxes are added and 1 is filled.
- Hence E: = E+7 = 18

Step 4: find an invariant

- After n times in succession, we will have
- F: = 0 + nx1 = n
- E := E + nx7 = 11 + 7n
- So E – 7F = 11 is an invariant
- It is consistent with initial state: F =0, E=11

Step 5: find solution

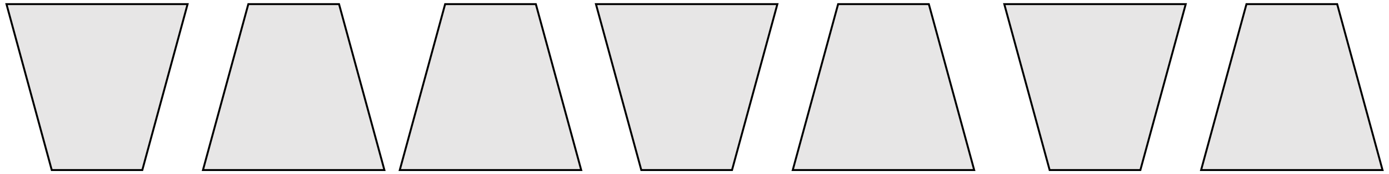
- At the end E = 102
- From the invariant E – 7F = 11 we have F = (E-11)/7 = 13
- Hence at the end, we have E = 102 empty boxes and F=13 filled boxes
- Totally, we have E + F = 115 boxes.

Problem 3: Tumblers

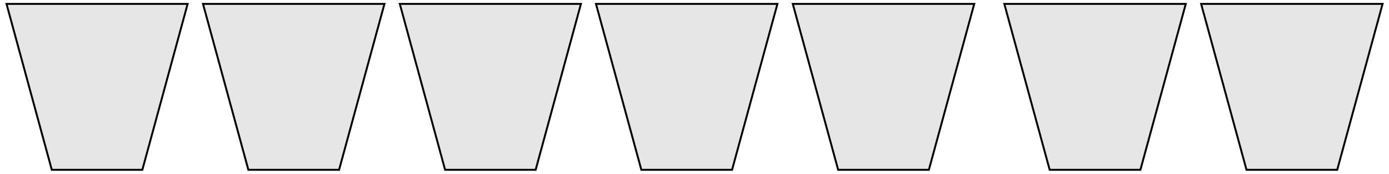
- Several tumblers are placed in a row on a table.
- Some Tumblers are upside down, some are right way up.
- We can only turn tumblers two at a time.
- Under what circumstances can we get all the tumblers right way up?

Tumblers

- Sample initial state.



- Desired final state.



Choose variables

- How many variables do you choose? What are they?
- What we want is to turn all upside down tumblers into upside up
- So the number of upside down tumblers will decrease to zero
- It suggests that we choose only one variable for the number of upside down tumblers
- Call this variable U

Model the process

- There are three possible effects of turning two of the tumblers.
- Two tumblers that are both the right way up are turned upside down.

- This is modelled by the assignment u := u+2

Model the process

- Turning two tumblers that are both upside down has the opposite effect | u decreases by two.

- This is modelled by the assignment u := u-2

Model the process

- Finally, turning two tumblers that are the opposite way up has no effect on u

- This is modelled by the assignment u := u
- The choice of which of these three statements is executed is unspecified.

Find an invariant

- An invariant of the turning process must therefore be an invariant of each of the three.
- We want an invariant of the two assignments u := u+2 and u := u−2.
- What does not change if we add or subtract two from u?
- The answer is: the so-called parity of u.
- No matter how many times we turn two tumblers over, the parity of the number of upside-down tumblers will not change.
- If there is an even number at the outset, there will always be an even number; if there is an odd number at the outset, there will always be an odd number.

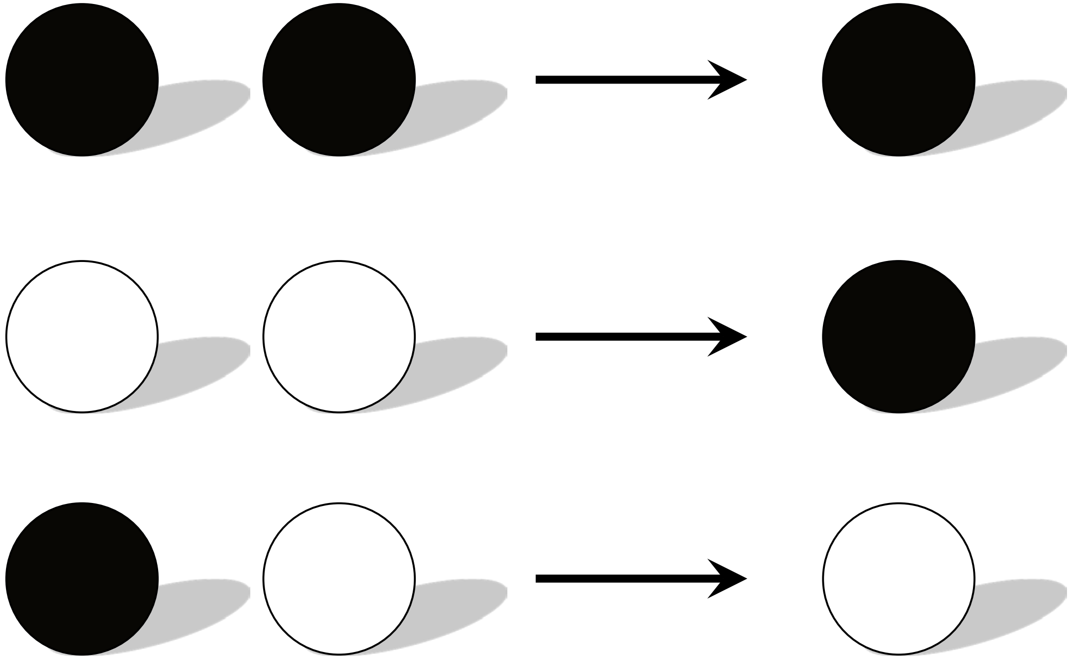
Find the solution

- The goal is to repeat the turning process until there are zero upside-down tumblers, i.e., u =0 at the end
- Zero is an even number
- So what should be u at the beginning?
- The answer is that u is even, i.e., there must be an even number of upside-down tumblers at the outset.

Problem 4: Black and White Balls

- We have an urn containing a number of black and white balls.
- We take two of the balls out at a time.
- If they are both the same colour we replace them with a single black ball.
- If they are of different colours we replace them with a white ball.
- What, if anything, is the relationship between the initial state and the colour of the final ball?

Problem 4



Choosing variables

- How many variables will you choose and what are they?
- Since at the end, we have only one ball left
- It suggests to use one variable only, for either white or black colour
- Choose C to be the number of white balls

Model the process

There are three possibilities:

- If you take out two white balls, then you replace with 1 black ball.
- Hence the number of white balls decreases by 2, i.e., C := C – 2
- If you take out two black balls, then you replace with 1 black ball.
- Hence the number of white balls is unchanged, i.e., C: = C
- If you take out one white and one black ball, then again C: = C

Find an invariant

- What is an invariant here?
- We have seen that, in every step, either C is unchanged or C is decreased by 2.
- So what is unchanged here is, again, the parity of C
- The invariant is hence the parity of C

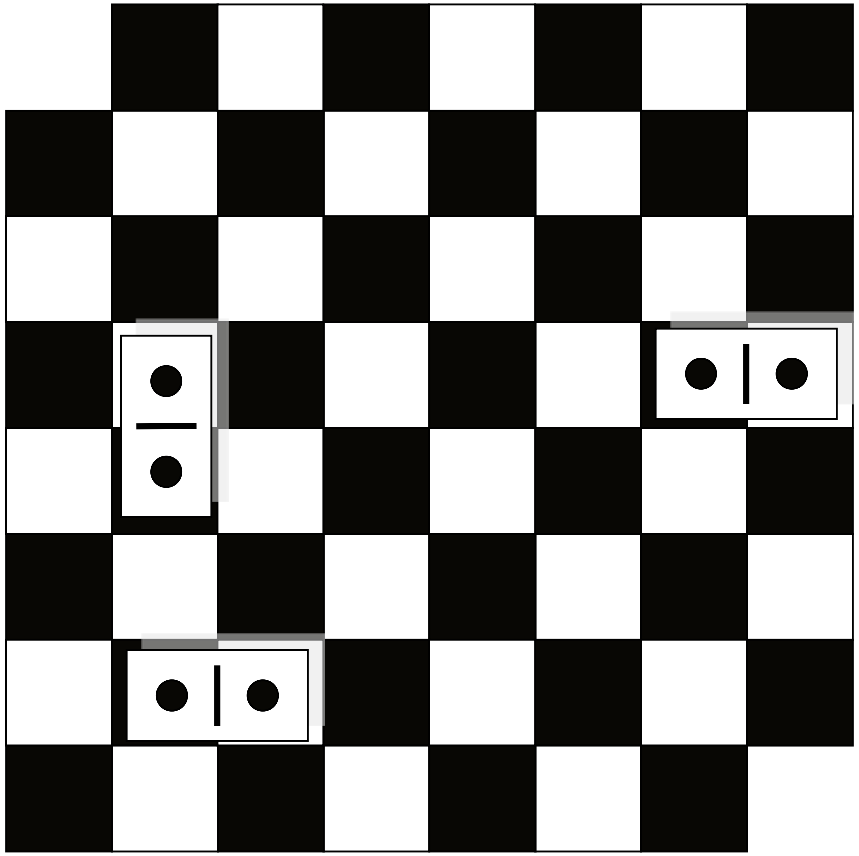
Find the solution

- So, if we start with an even number of white balls. At the end, $C = 0$, and hence the last ball will be black;
- if we start with an odd number of white balls, the last ball will be white.

Problem 5: Chess and Dominoes

- A chess board has its top left and bottom right corner squares cut out leaving 62 squares.
- We have a supply of dominoes, each of which will cover 2 adjacent squares.
- Is there a way to exactly cover the entire board?

Problem 5

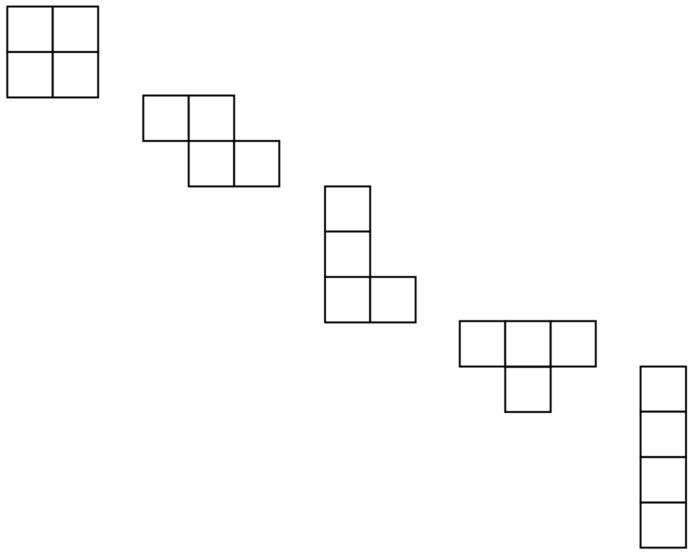


Hint

- Hint: use the colouring of the squares on the chessboard
- How many Black squares are on the chess board?
- How many White squares are on the chess square?
- How many Black squares are on your supply of 31 domino pieces?
- How many White squares are on your supply of 31 domino pieces?
- Is it possible to cover this 62-square chessboard with exactly 31 dominoes?
- Answer: No.

Problem 5: Tetrominoes

- A tetromino is a figure made up of four connected squares.
- There are five different tetrominoes.
- O
- Z
- L
- T
- I



Problem 6

a) If a rectangular board is completely covered with tetrominoes show that at least one side must be of even length.

b) If a rectangular board can be completely covered by T tetrominoes show that the number of squares on the board must be a multiple of 8.

c) If a rectangular board can be completely covered by L tetrominoes show that the number of squares on the board must be a multiple of 8.

d) An 8x8 board cannot be covered by one O and 15 L tetrominoes. Why not?

Choosing variables

- What are variables?
- Since we place a tetromino on the board, and at the end we cover all the squares in the board.
- It suggests to use only one variable, namely the number of covered squares.
- Call this variable C

Find an invariant

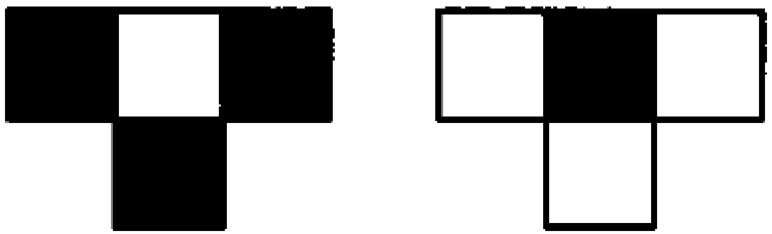
- What is an invariant?
- Every tetromino has 4 squares. Every time you place a tetromino, the number of covered squares increases 4
- Initially C = 0, divisible by 4
- Hence C is always divisible by 4 => an invariant
- We write C mod 4 = 0

Solve (a): cover the board by tetrominos

- The Tetrominoes cover an m×n board. (the number of squares along one side is m and the number along the other side is n.)
- Then, c=m×n and, so, m×n is a multiple of 4
- It must be the case that either m or n (or both) is a multiple of 2
- Hence If a rectangular board is covered by tetrominoes, at least one of the sides of the rectangle must have even length.

Solve (b): cover the board by T tetrominos

- Color the rectangle with black and white squares, as on a chess-board.
- The T-tetrominoes should be colored. This gives us two types
  - one with three black squares and one white square
  - one with three white squares and one black square



Solve (b): cover the board by T tetrominos

- From (a), we have that the board must have equal numbers of black and white squares
- The only way we can do this is to have equal numbers of 3bw and b3w T-tetrominoes.
- This requires that there be a multiple of 8 squares in the rectangular board.

Solve (c): cover the board by L tetrominos

- Similar to (b)
- This time we need to color the board in alternating black and white stripes.
- With this colouring it is clear that we have 3 black 1 white and 3 white 1 black versions.
- Once again, this implies that we need pairs of Tetrominoes to maintain white/black parity

Solve (d): cover 8x8 board by one O and 15L tetrominos

- Use (c)
- O tetromino has even black/white parity
- After we use one O tetromino, the leftover part in the board still requires the same black/white parity
- Odd number of L tetromino (15) cannot have the required black/white parity
- The colouring is not an intrinsic part of the problem. It is merely a solution aid.