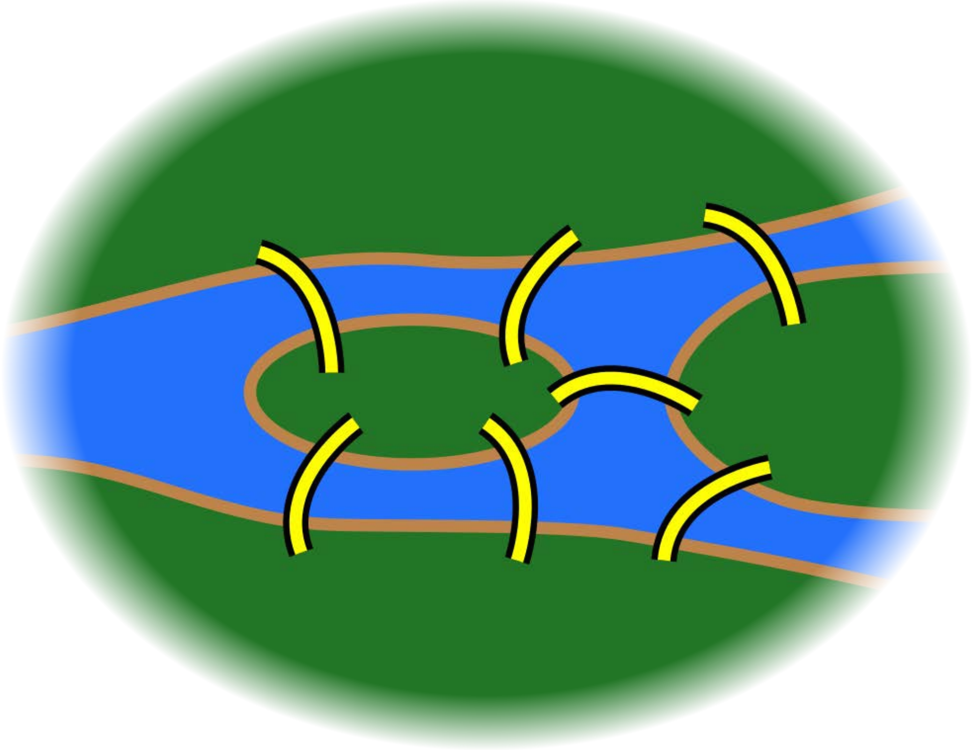
# Week 4 - Practice

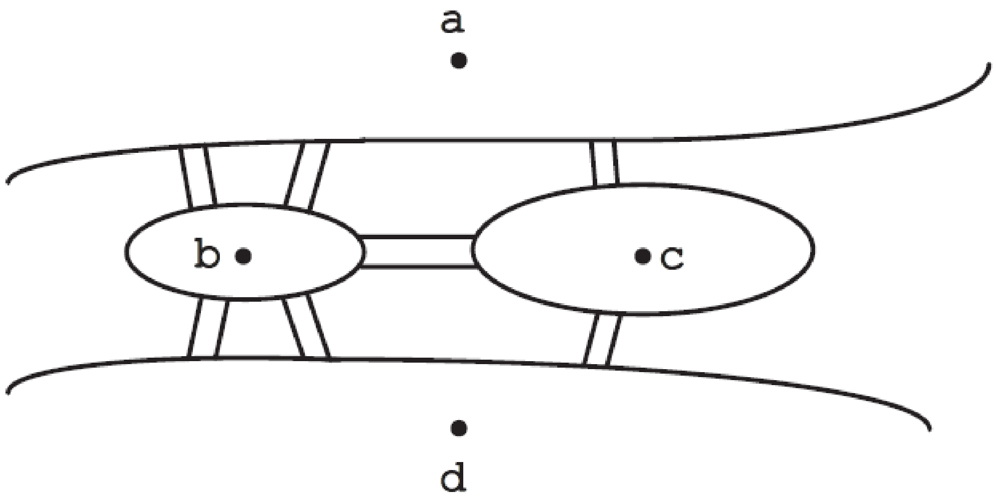
# Problem 1: Königsberg Bridge Problem

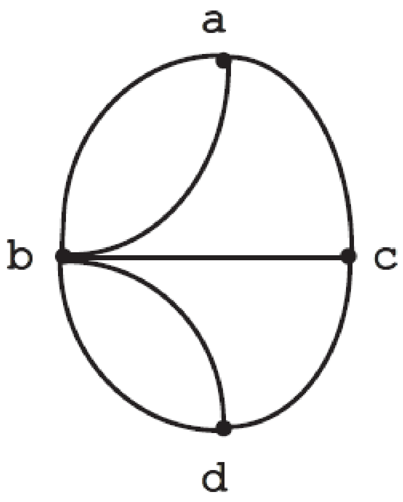
Is it possible, in a single stroll, to cross all seven bridges of Königsberg exactly once and return to the starting point? A sketch of the river with its two islands and seven bridges is shown below



# Solution

- It was solved by Leonhard Euler (1708-1783)
- Euler realized that walking along a land mass – a bank of the river or an island – is irrelevant to the problem.
- Only pertinent information: connections provided by the bridges
- Transform the problem to the question about the graph below



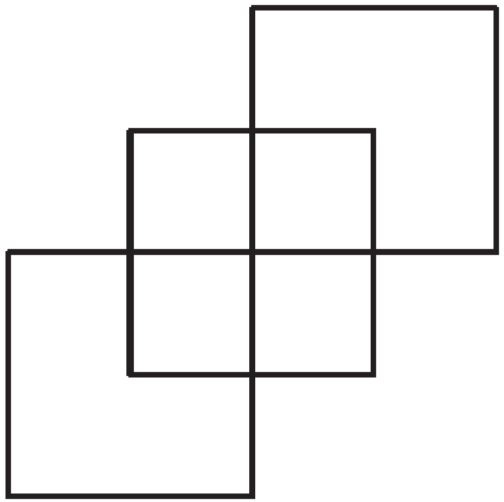



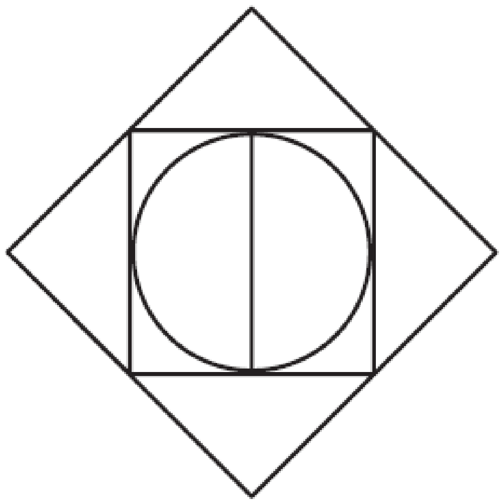
- Question now becomes: whether there exists a sequence of adjacent vertices that traverses all the edges exactly once before returning to the starting vertex (called Eucler circuit)?
- Any such circuit would have to enter a vertex exactly the same number of times it leaves the vertex.
- Hence such a circuit can only exist in a graph in which the number of edges touching a vertex – called its degree – is even for each vertex
- This invariant implies that the Königsberg Bridge Problem has no solution, since every vertex of the graph has an odd degree.

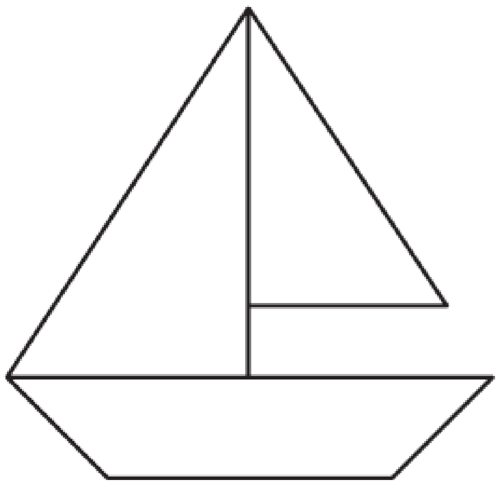
- Same analysis implies that there was no walk crossing all the bridges exactly once even without the requirement of returning to the starting point. Such a walk is called an Euler path.
- An Euler path in a graph exists if all the vertices have an even degree except exactly two vertices in which the walk starts and ends
- These conditions turn out to be necessary and sufficient condition for existence of an Euler circuit and an Euler path in a connected graph
  - A graph is connected if there is a path between every pair of its vertices

# Problem 2: Figure tracing

For each of the three figures below, either trace the figure without lifting your pen off the paper or going back over any line in it, or prove that it is impossible to do so.


(a)


(b)


(c)

# Solution

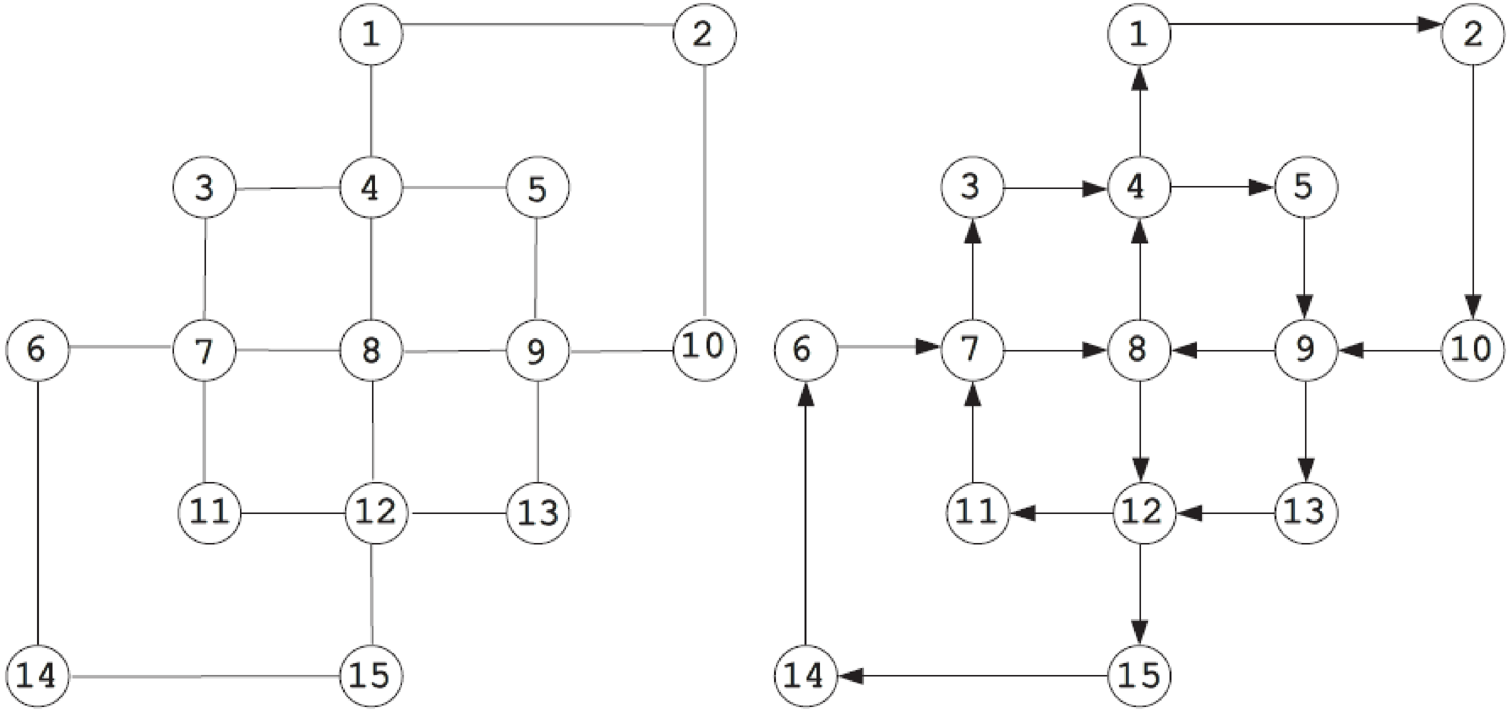
It follows from Problem 1 that a figure can be traced without lifting pen off the paper or going back over any line in it if and only if the graph of the figure is connected and satisfies one of the two conditions:

- All the vertices of the multigraph have even degrees (i.e., an even number of edges for which the vertex is an endpoint)—then a trace can start at any of its vertices where it will end.
- Exactly two of its vertices have odd degrees—then a trace must start at one of these odd vertices and end at the other

a) The first figure can be traced with the restrictions imposed: its graph is connected and all its vertices have even degrees.

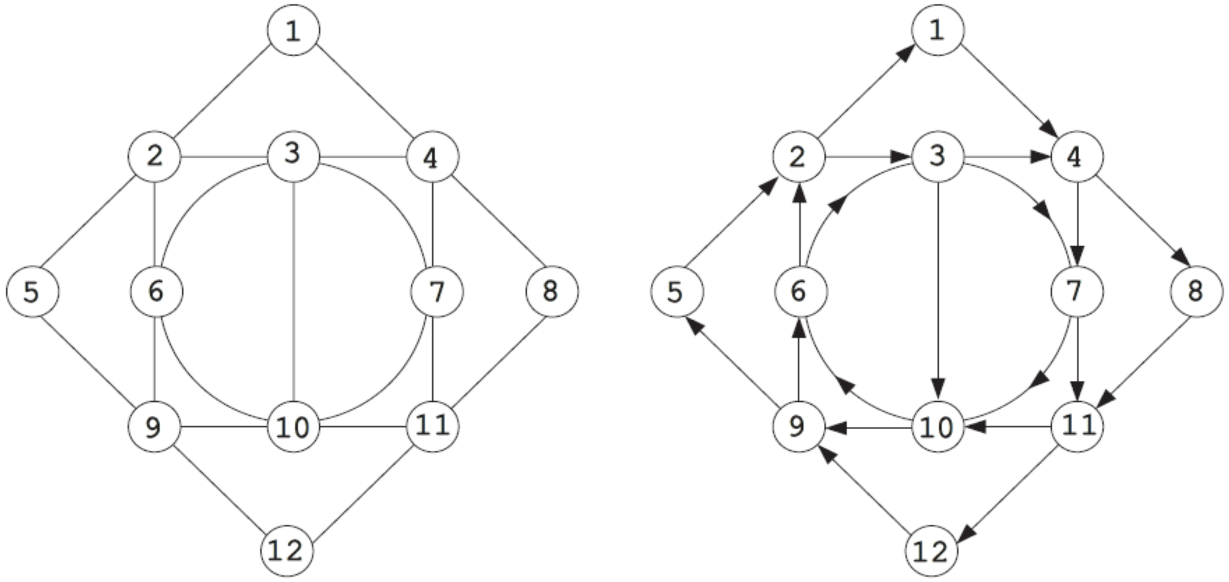
- It starts at an arbitrary chosen vertex and proceeds along previously untraversed edges until either all of them are traversed or the path returns to the starting vertex with no untraversed edge out of it available while some of the graph’s edges remain untraversed.
- In the latter case, the obtained circuit is removed from the graph and the same operation is repeated recursively starting at a vertex that is in both the remaining part of the graph and the removed circuit
- The existence of such a vertex follows from the graph’s connectivity and the fact that all its vertices have even degrees
- Once an Euler circuit is constructed for the remaining part of the graph, it is “spliced” into the first circuit to yield an Euler circuit for the entire graph.

- For example, starting at vertex 1 of the graph and following its “outside” edges, we get the circuit 1 − 2 − 10 − 9 − 13 − 12 − 15 − 14 − 6 − 7 − 3 − 4 − 1.
- Picking, say, vertex 4 as a common vertex with the remaining part of the graph, we will get the following Euler circuit for the remaining part of the graph: 4 − 5 − 9 − 8 − 12 − 11 − 7 − 8 − 4.
- “Splicing” the latter circuit into the former yields the following Euler circuit for the entire graph 1 − 2 − 10 − 9 − 13 − 12 − 15 − 14 − 6 − 7 − 3 − 4 − 5 − 9 − 8 − 12 − 11 − 7 − 8 − 4 − 1.



b) The second figure can be traced with the restrictions imposed: considered as a graph, it is connected and all its vertices have even degrees except two: vertices 3 and 8. Starting at vertex 3 and using essentially the same algorithm, we can get the following path: 3 − 4 − 7 − 11 − 10 − 9 − 6 − 2 − 3 − 7 − 10 − 6 − 3 − 10.

- Then picking, say, vertex 2 as a common vertex with the remaining part of the graph, we get the following Euler circuit for the remaining part of the graph: 2 − 1 − 4 − 8 − 11 − 12 − 9 − 5 − 2.
- “Splicing” the latter circuit into the former path yields the following Euler path for the entire graph: 3 − 4 − 7 − 11 − 10 − 9 − 6 − 2 − 1 − 4 − 8 − 11 − 12− 9 − 5 − 2 − 3 − 7 − 10 − 6 − 3 − 10.



c) It is impossible to trace the third figure because its graph has more than two vertices of an odd degree.