CSIT113
Problem Solving

Workshop – Week 5

Proofs by induction

Prove the following formulae by induction

1) $1 + 2 + .. + n = \frac{1}{2}n(n+1)$

2) $1^2 + 2^2 \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Steps

- Check the initial case (either n=0 or n=1)
- Assume it holds for n
- Prove for the case n+1

1) $1 + 2 + .. + n = \frac{1}{2}n(n+1)$

LHS := $1 + 2 + .. + n$

RHS := $\frac{1}{2}n(n+1)$

- Check the initial case: for n=1, LHS = 1 and RHS = 1, so LHS=RHS.
- Assume that the formula is correct for n, i.e.,

$$1 + 2 + .. + n = \frac{1}{2}n(n+1)$$

- Now we prove for the case n+1, i.e., we need to prove

$$1 + 2 + .. + n + (n+1) = \frac{1}{2}(n+1)(n+2)$$

Now

LHS $= 1 + 2 + \ldots + n + (n + 1)$

$= \frac{1}{2} n(n + 1) + (n+1)$ (by hypothesis for the case n)

$= \frac{1}{2} (n + 1)(n + 2)$

= RHS as required.

2) $1^2 + 2^2 \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$

LHS = $1^2 + 2^2 \ldots + n^2$

RHS = $\frac{1}{6}n(n+1)(2n+1)$

- Check the initial case n=1: LHS = 1 = RHS
- Assume the formula is correct for n, i.e.,

$$1^2 + 2^2 \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$$

- We need to prove it holds for n+1, i.e.,

$$1^2 + 2^2 \ldots + n^2 + (n+1)^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$$

Now

LHS $= 1^2 + 2^2 \ldots + n^2 + (n+1)^2$

$= \frac{1}{6} n(n+1)(2n+1) + (n+1)^2$

$= (n+1)\left(\frac{1}{6} n(2n+1) + n + 1\right)$

$= \frac{1}{6}(n+1)\big(n(2n+1) + 6(n+1)\big)$

$= \frac{1}{6}(n+1)(2n^2 + 7n + 6)$

$= \frac{1}{6}(n+1)(n+2)(2n+3)$

= RHS

Notes

- We can prove the above formulae because we know the formula of the left hand side and the right hand side.
- What if we are given only the left hand side and asked to find the formula of the right hand side?
- i.e., find the formulae of the following

$$1^3 + 2^3 + \cdots + n^3$$

- How do we proceed?

Idea

- Using induction to seek for a pattern
- Formulate the pattern in precise mathematical terms
- Then verify the pattern

Simple pattern

- A simple observation in previous example tells us that the sum of k-th powers of the first n numbers is a polynomial in n of degree k+1:

1) $1 + 2 + \ldots + \mathrm{n} = \frac{1}{2}\mathrm{n}(\mathrm{n}+1)$: polynomial of degree 2

2) $1^2 + 2^2 \ldots + \mathrm{n}^2 = \frac{1}{6}\mathrm{n}(\mathrm{n}+1)(2\mathrm{n}+1)$: polynomial of degree 3

It also holds for the case k=0

$1^0 + 2^0 + \cdots + \mathrm{n}^0 = 1 + 1 + \cdots + 1 = n$: polynomial of degree 1

- But how can we get a formula for the sum?

Worked-out Example

- We work out for the case k = 1 (as previously)
- We now know that S(n):= 1+ 2 + .. + n is a polynomial of n in degree 2
- Let that polynomial be P(n) = a + bn + cn^2 for some a,b,c
- Note that S(n) = P(n) for all n
- We need to find a,b,c.
- Now S(0) = 0, because the sum of an empty set of numbers is zero
- Moreover, P(0) = a. Hence a = 0.

- Next, we test for n=1. We have S(1) = 1
- And P(1) = a + b + c = b + c (since a=0). So b + c = 1 (1)
- Now with n=2, we have S(2) = 1 + 2 = 3
- And P(2) = a + 2b + 4c = 2b + 4c. And so 2b + 4c = 3 (2)
- From equations (1), (2), we obtain that b = ½ and c = ½
- Therefore S(n) = 1 + 2 + … + n = $\frac{1}{2}n + \frac{1}{2}n^2 = \frac{1}{2}$n(n+1) as desired.
- Your turn to work out for the sums $1^2 + 2^2 \ldots + \mathrm{n}^2$

- Let S(n) = $1^2 + 2^2 \ldots + n^2$
- And P(n) = a + bn + cn^2+ dn^3 for some a,b,c,d
- We are looking for a,b,c,d.
- First S(0) = 0 and P(0) = a and hence again a = 0
- S(1) = 1 and P(1) = b + c + d. Hence

  b + c + d = 1 (1)
- S(2) = 1 + 4 = 5, and P(2) = 2b + 4c + 8d. Hence

  2b + 4c + 8d = 5 (2)
- S(3) = 1 + 4 + 9 = 14, and P(3) = 3b + 9c + 27d. Hence

  3b + 9c + 27d = 14 (3)

It follows from (1),(2),(3), we obtain $b = \frac{1}{6}, c = \frac{1}{2}, d = \frac{1}{3}$

Hence we have

$$1^2 + 2^2 \ldots + \mathrm{n}^2 = \frac{1}{6}n + \frac{1}{2}\mathrm{n}^2 + \frac{1}{3}\mathrm{n}^3$$

$$= \frac{1}{6}n(1 + 3n + 2\mathrm{n}^2)$$

$$= \frac{1}{6}n(n + 1)(2n + 1)$$

Problem in Lecture

Consider the sequence produced by adding successive powers of 2.

- 1 + 2 + 4 + 8…
- 1, 3, 7, 15, …
- Prove that this sequence is of the form $2^n - 1$

- Initial case: $n=1$ then it is clearly true.
- Assume it is true for n, i.e., the sum $1 + 2 + 4 + \ldots + 2^{n-1}$ is $2^n - 1$
- We prove for the case n+1, i.e., we want to show that
  $1 + 2 + 4 + \ldots + 2^n$ is $2^{n+1} - 1$
- We have $1 + 2 + 4 + \ldots + 2^n = 1 + 2 + 4 + \ldots + 2^{n-1} + 2^n$

$$= 2^n - 1 + 2^n$$

$$= 2^{n+1} - 1$$

Problem in Lecture

Similarly the sequence produced by adding successive powers of 5…

- 1 + 5 + 25 + 125…
- 1, 6, 31, 156, …

Prove the sequence to be of the form $(5^n - 1)/4$.