CSIT113
Problem Solving

Week 5

Induction

- Solving a simpler problem
  - Induction is the name given to an approach to problem solving in which we use the solution to small problems to solve larger problems.
- But how do we define the “size” of a problem?

Problem Size

- Normally we can find a property of the problem that can be used as a measure of its size.
- For example, if we have an array of numbers to sort, the number of elements in the array is a good measure of size.
- Or, with the match games, the number of matches in the pile.

The Inductive Process

- The process consists of two steps:
  - First solve the problem for the smallest possible problem (usually size n=0).
  - Show how, given the solution for a problem of size n we can solve a problem of size (n + 1).
- Now we can build up a solution scheme:
  - Use solution0 to find solution1
  - Use solution1 to find solution2
  - Etc.

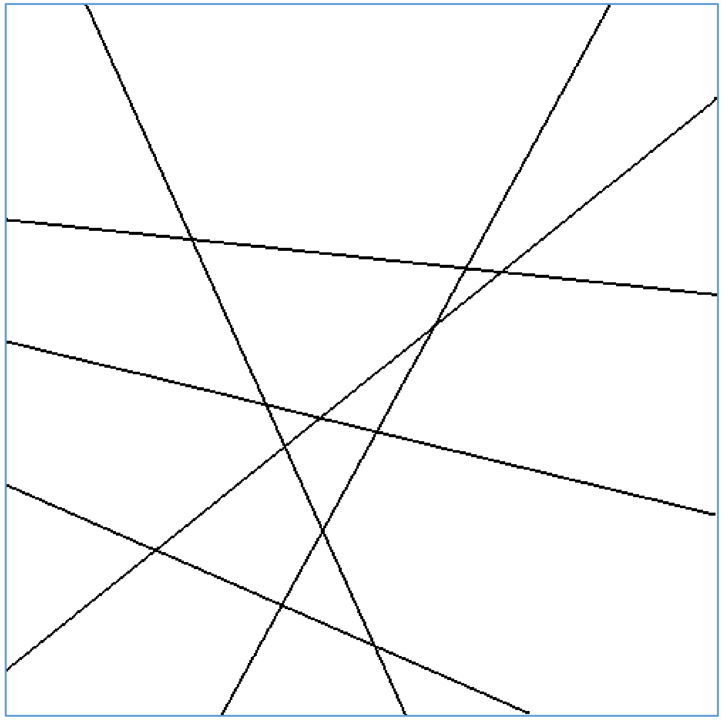
Some Illustrative Problems

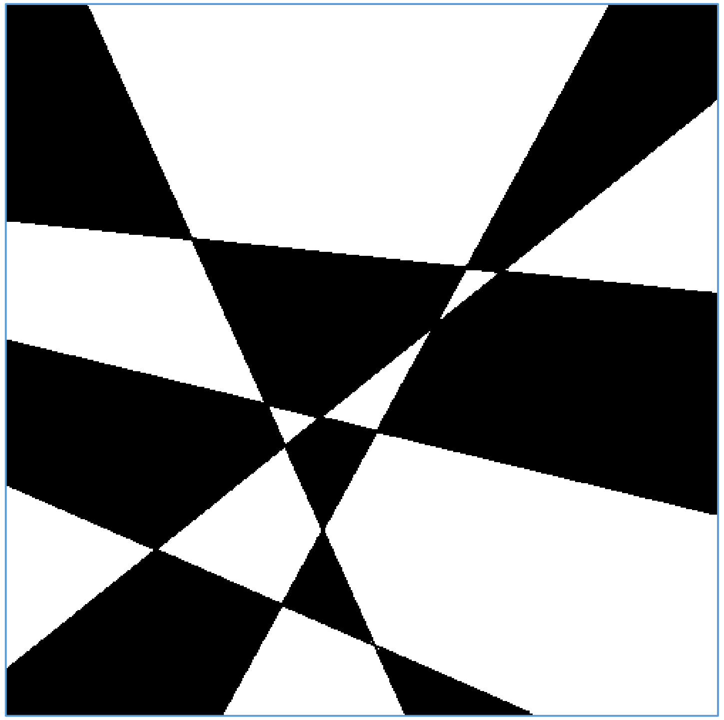
- Each of the problems which follow can be approached by using induction to produce a solution.
- In each problem it should be clear how the size is determined – even if it is not clear how to solve the problem.

Problem 1

- A number of straight lines are drawn across a sheet of paper, each line extending from one edge to another.
- In this way the paper is broken up into a number of regions.
- Show that we can colour the regions using just black and white in such a way that no two adjacent regions have the same colour.

Problem 1

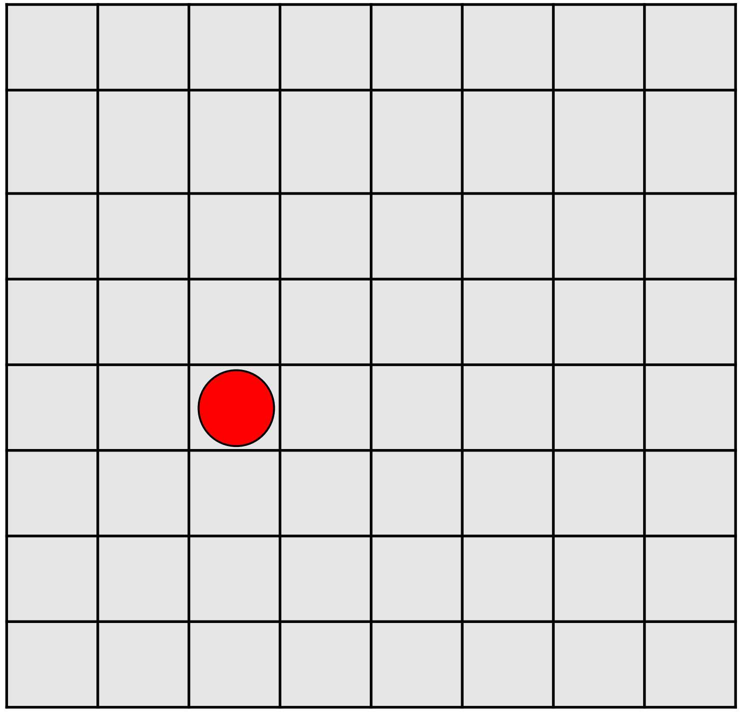


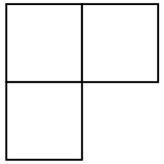


Problem 2

- A square board is divided into a 2^n x 2^n grid.
- One grid square is covered by a coin.
- An L triomino is a shape consisting of three squares arranged in an “L” shape.
- Show that the remaining squares can be covered with L triominoes without any overlap.

Problem 2

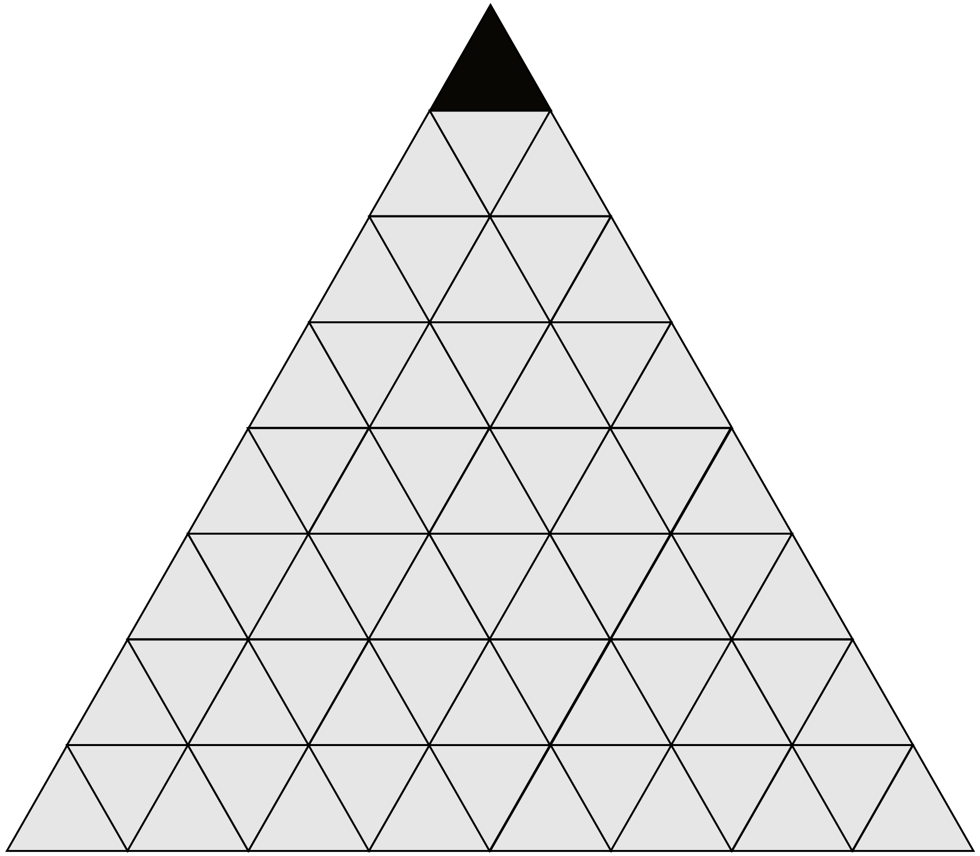


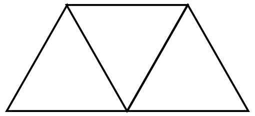


Problem 3

- An equilateral triangle with sides of length 2^n is made up of smaller triangles.
- The topmost triangle is covered.
- Show that it is possible to tile the remainder of the triangle with non-overlapping 3-triangle trapezoids.

Problem 3





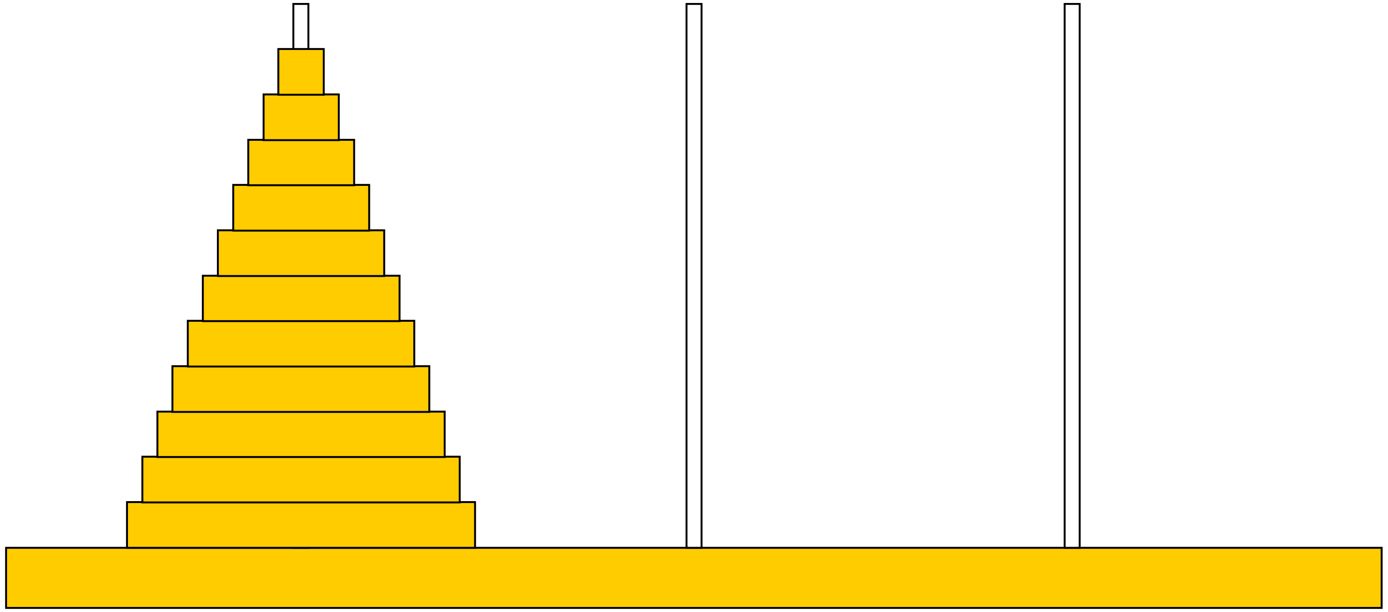
Problem 4

- According to legend, at the time of creation god created three diamond needles set in a slab of pure gold and on one of them he placed 64 discs of pure gold, each smaller than the one below it.
- He tasked a group of monks with moving the disks from their starting needle to another needle.
- But the monks had to obey certain rules:

Problem 4

1. The disks may only be placed on the needles.
2. Only one disc may be moved at a time
3. A disc must never be placed on a disc that is smaller than itself.

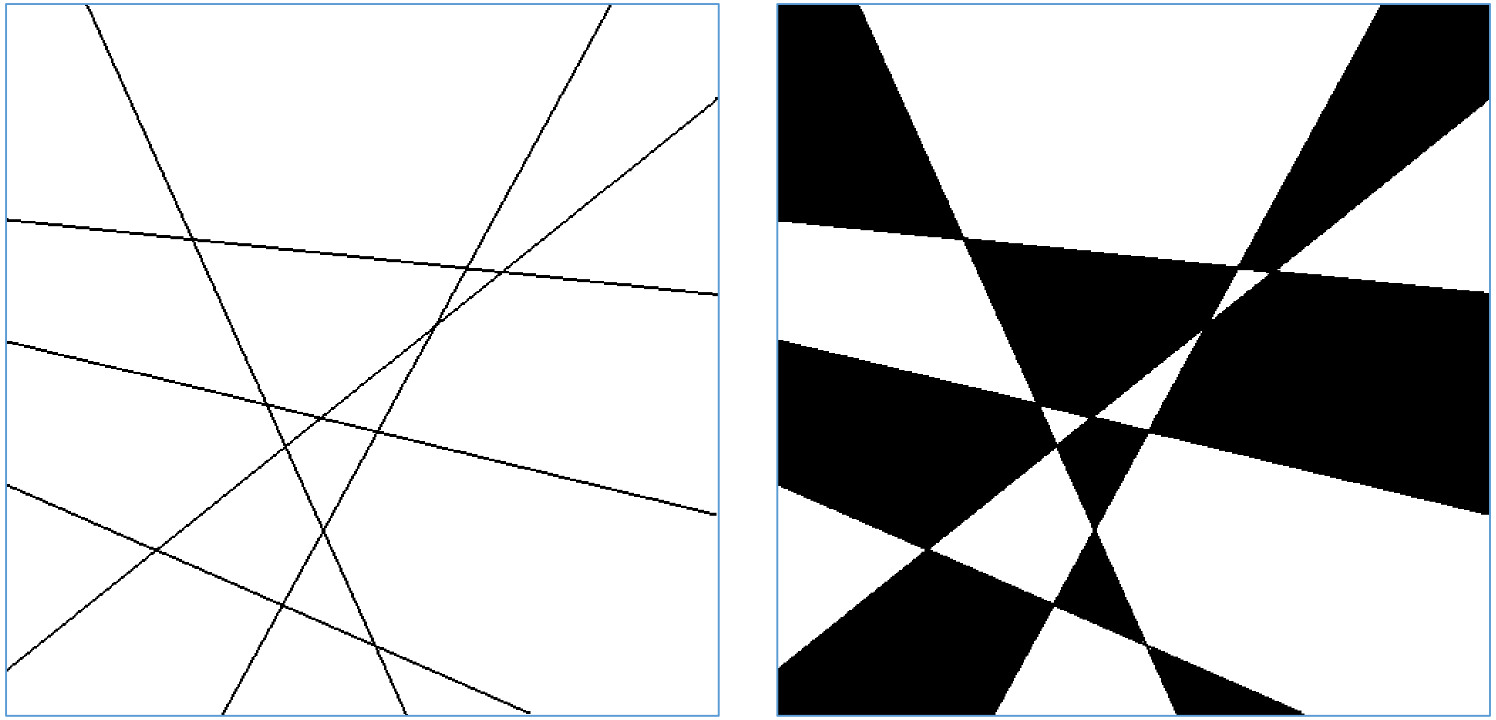
Problem 4



Problem 1

- A number of straight lines are drawn across a sheet of paper, each line extending from one edge to another.
- In this way the paper is broken up into a number of regions.
- Show that we can colour the regions using just black and white in such a way that no two adjacent regions have the same colour.

Problem 1

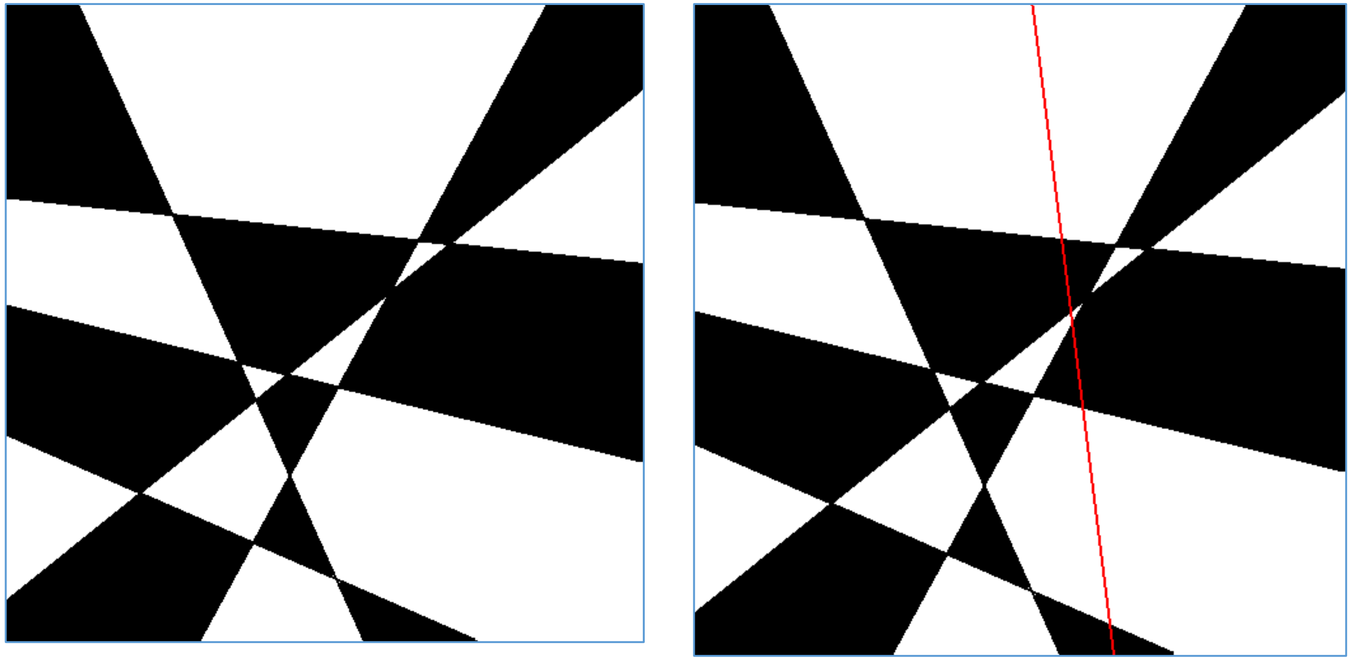


Problem 1

- Here the size of the problem, n, is the number of lines that have been drawn.
- The solution for $n = 0$ is trivial:
  - Colour the paper all white;
  - Colour the paper all black.
- For the induction step we assume that we have a satisfactory colouring for a pattern of n crossing lines.

Problem 1

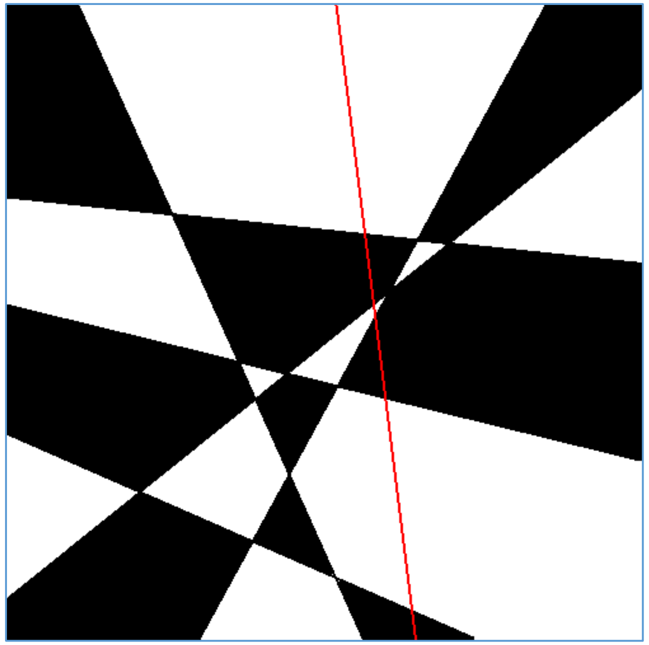
- We now add another line.

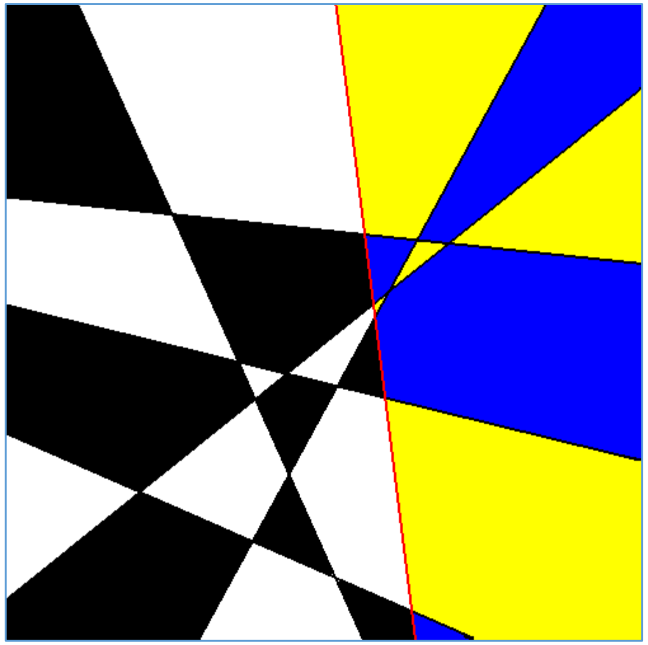


- We see that the regions on each side of this line are correctly coloured.

Problem 1

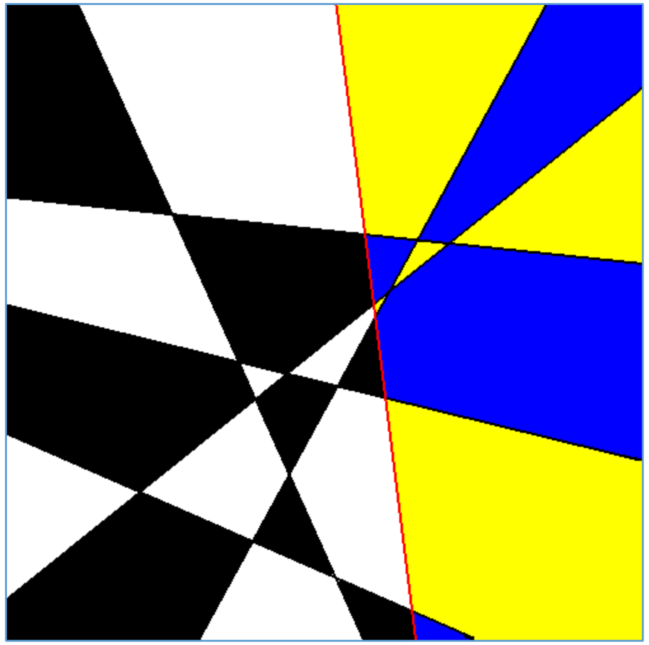
- But the regions across the line are the **same** colour.

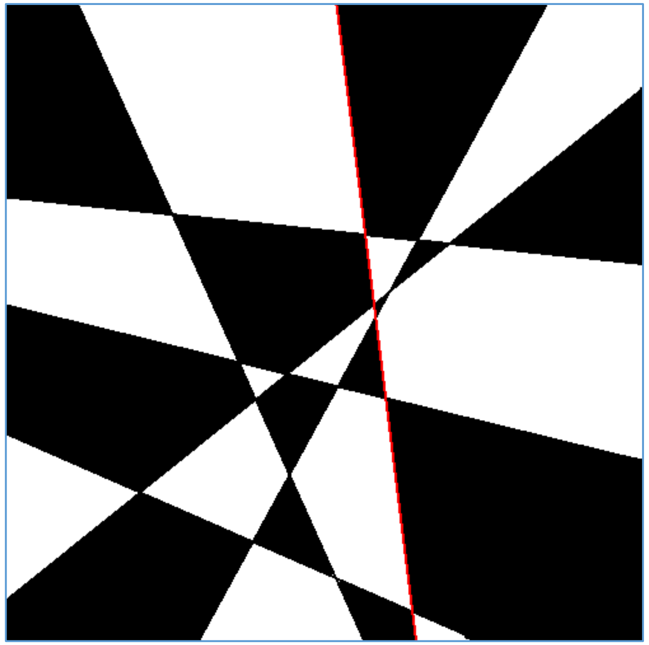




Problem 1

- So we invert the colour of these regions.





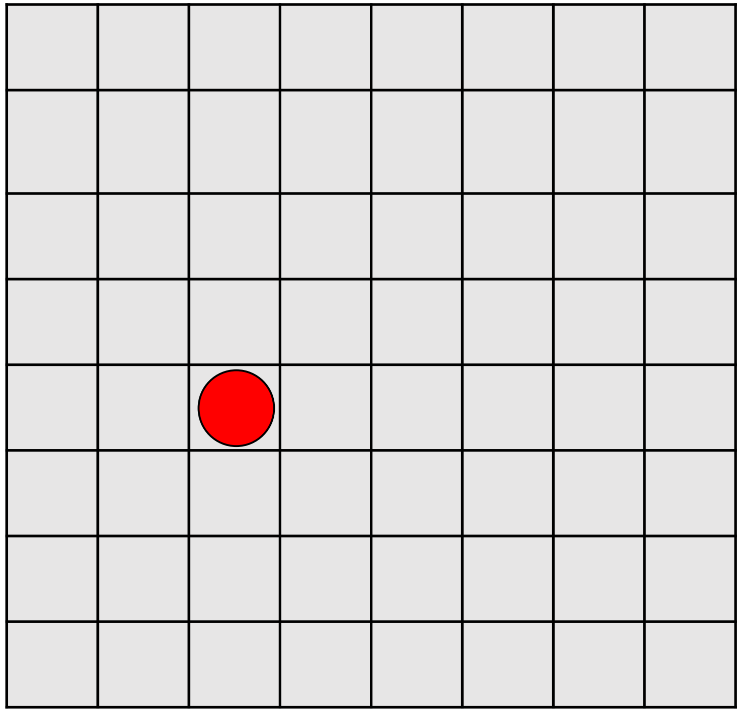
Problem 1

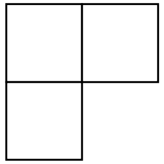
- We now have a solution for any number of lines:
  - Start with no lines.
  - Add lines one at a time.
  - Each time we add a line flip the colours of all the regions on one side of the new line.
- Note that there are two possible solutions for any specific problem.

Problem 2

- A square board is divided into a 2^n x 2^n grid.
- One grid square is covered by a coin.
- A triomino is a shape consisting of three squares arranged in an "L" shape.
- Show that the remaining squares can be covered with triominoes without any overlap.

Problem 2





Problem 2

- The size here is related to the number of squares on the board.
- As each board has sides of length 2^n, n is the obvious choice for measuring the problem size.
- For $n = 0$ we have a 1x1 square which is covered by the coin – problem solved!

Problem 2

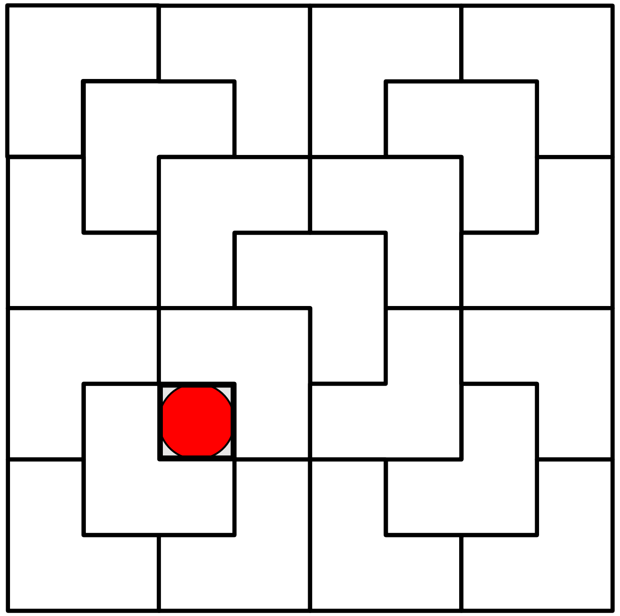
- Let us now consider a problem of size $n + 1$
- We can divide a 2^{n+1} square into 4 2^n sub-squares.
- One of these 4 squares will contain the coin and we assume that this can be solved.
- We can always place a single triomino so that one of its squares lies in each of the 3 empty squares.

Problem 2

- These three squares now each have one grid cell covered and are now soluble in the same way as the square with the coin.
- This gives us a general strategy:
  - Divide the square into 4 equal sized squares;
  - place a single triomino so that it covers one cell in each empty square;
  - Repeat with each of the 4 squares until the whole board is covered.

Problem 2

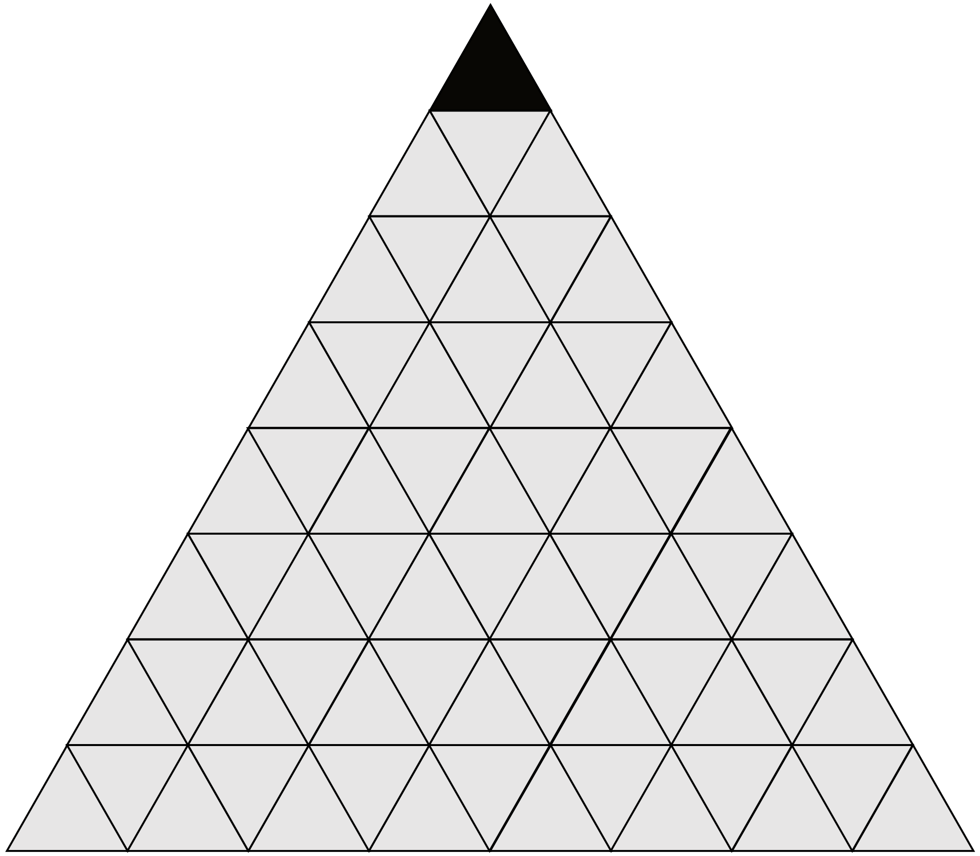
- Here is an example
  - Divide the board
  - Place a triomino
  - Divide each square
  - Place triominoes
  - Divide again
  - Place triominoes
  - Done!

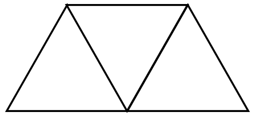


Problem 3

- An equilateral triangle with sides of length 2^n is made up of smaller triangles.
- The topmost triangle is covered.
- Show that it is possible to tile the remainder of the triangle with non-overlapping 3-triangle trapezoids.

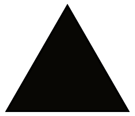
Problem 3





Problem 3

- Once again, we can use the fact that the triangle has sides of length 2^n to derive the problem size as n.
- For $n = 0$ the solution is trivial as we have a single triangle…
- Which will be coloured black.



Problem 3

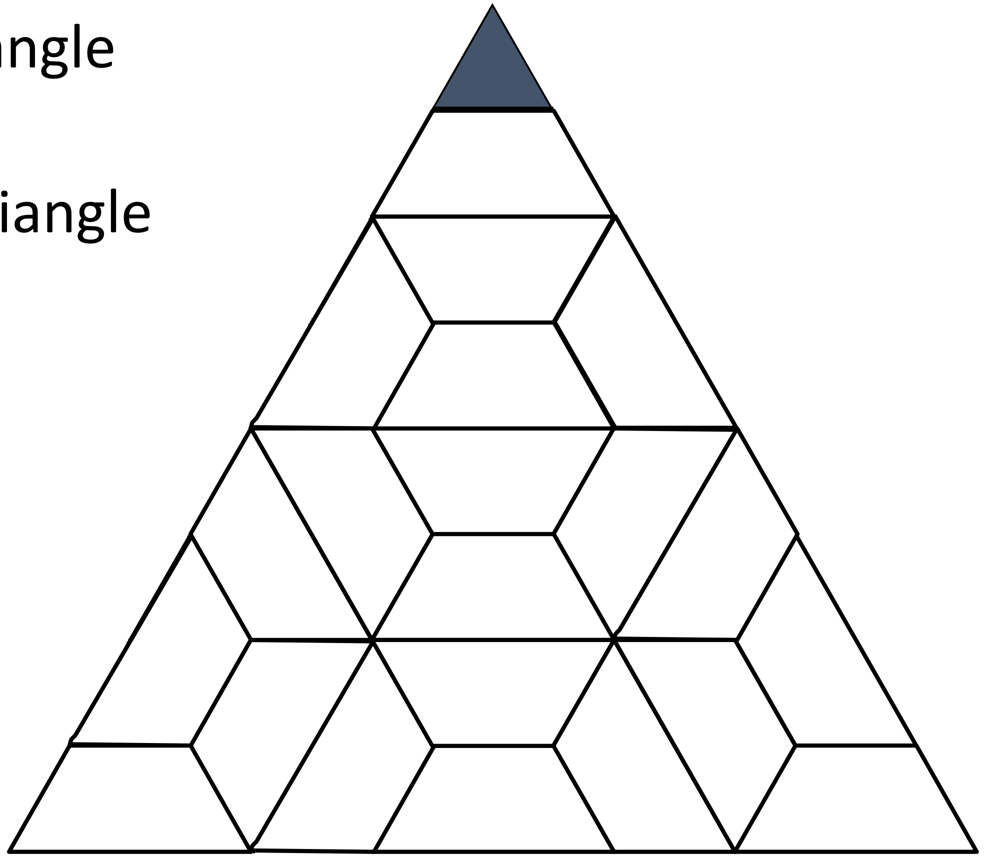
- Let us now consider a problem of size $n + 1$
- We can divide a 2^{n+1} triangle into 4 2^n sub-triangles.
- One of these 4 triangles will have a black vertex and we assume that this can be solved.
- We can always place a single trapezoid so that one of its triangles is at a vertex of each of the 3 empty triangles.

Problem 3

- These three triangles now each have one grid cell covered and are now soluble in the same way as the triangle with the black vertex.
- This gives us a general strategy:
  - Divide the triangle into 4 equal sized triangle s;
  - place a single trapezoid so that it covers one vertex of each empty triangle;
  - Repeat with each of the 4 triangles until the whole board is covered.

Problem 3

- Here is an example
  - Subdivide the triangle
  - Place a trapezoid
  - Subdivide each triangle
  - Place trapezoids
  - Divide again
  - Place
  - Done!

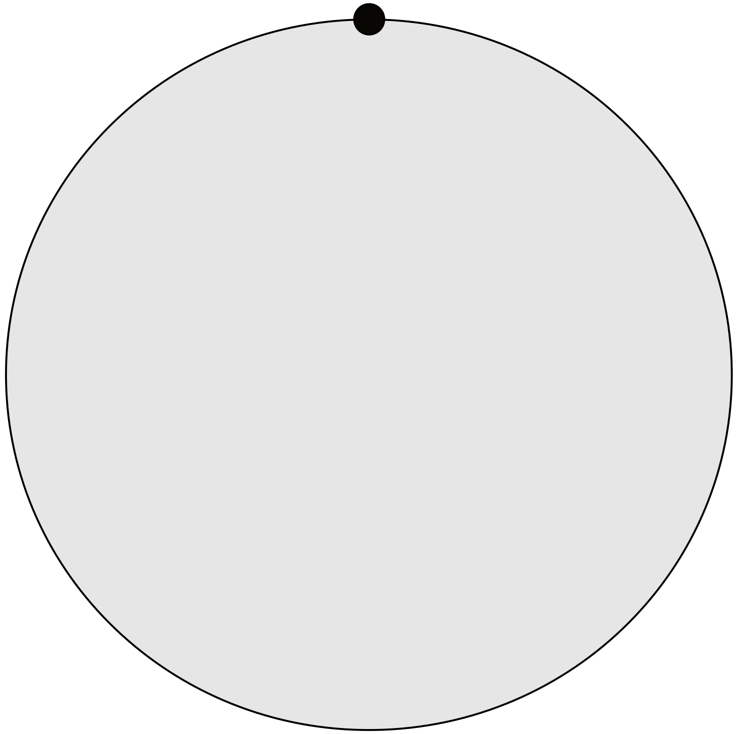


The Need for Proof

- Just because things look like they form a pattern doesn’t mean that they really do.
- Consider the following question.
- I mark *n* evenly spaced points around a circle and connect up all the points.
- How many regions do I produce?
- If we try this experiment we get a surprising result.

The Need for Proof

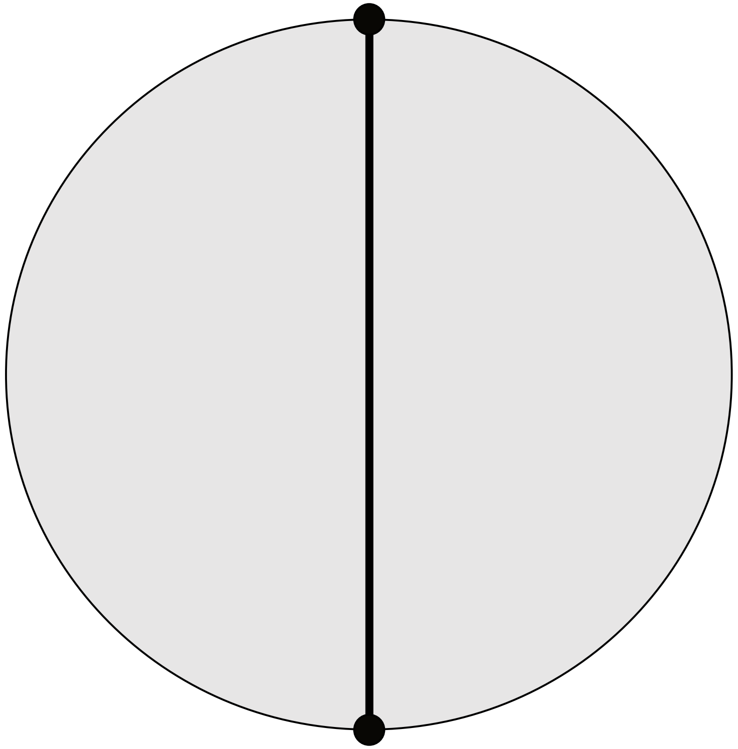
- n = 1
- r = 1



n	r
1	1

The Need for Proof

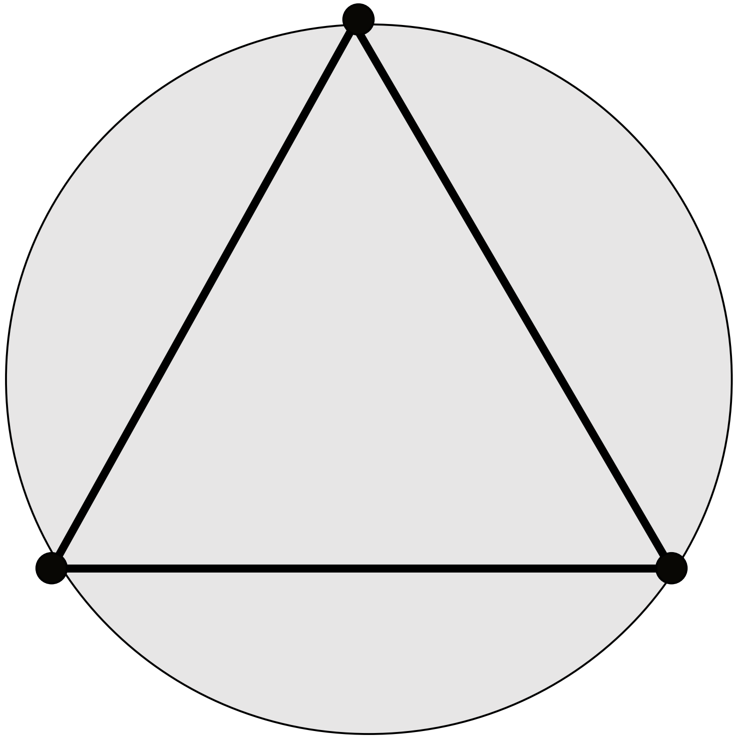
- $n = 2$
- $r = 2$



n	r
1	1
2	2

The Need for Proof

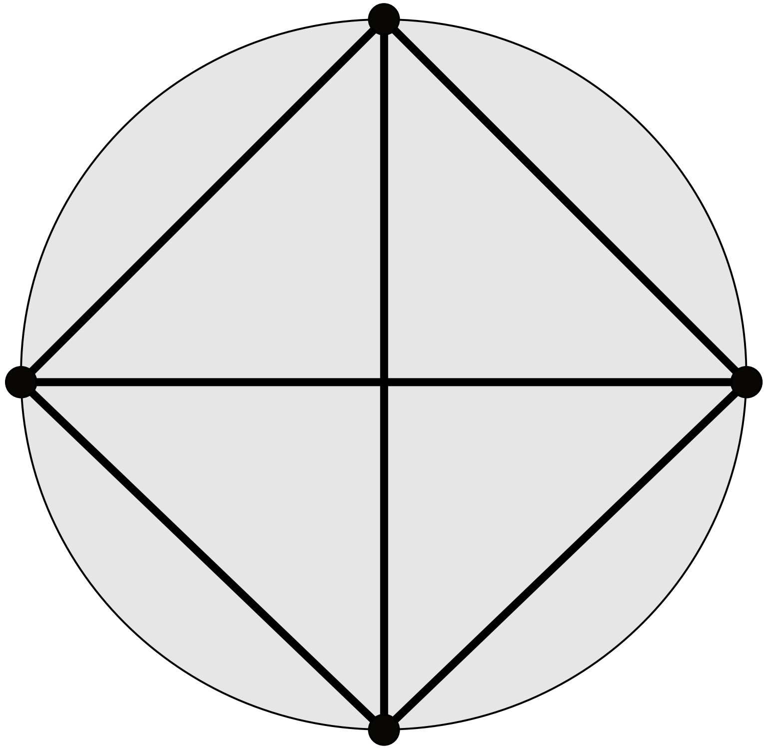
- $n = 3$
- $r = 4$



n	r
1	1
2	2
3	4

The Need for Proof

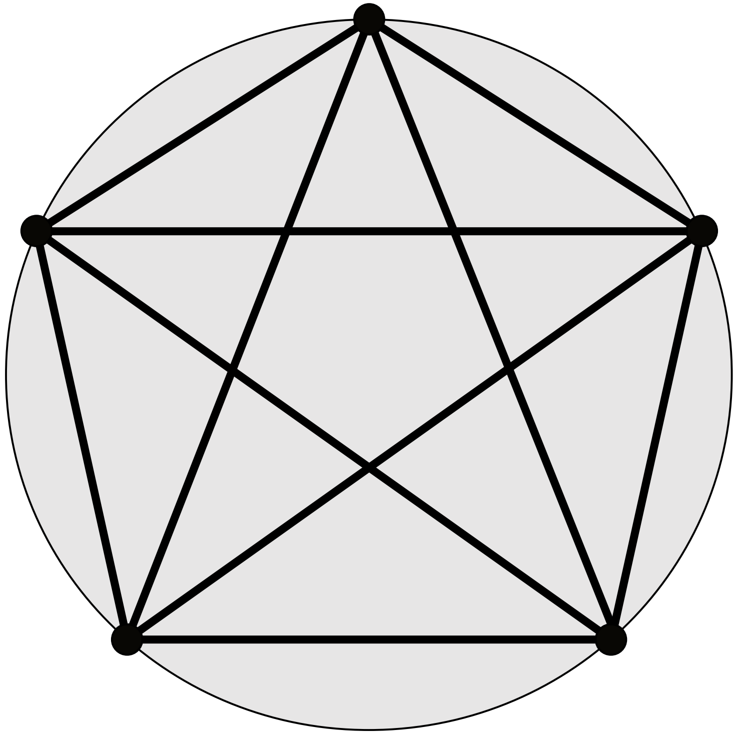
- $n = 4$
- $r = 8$



n	r
1	1
2	2
3	4
4	8

The Need for Proof

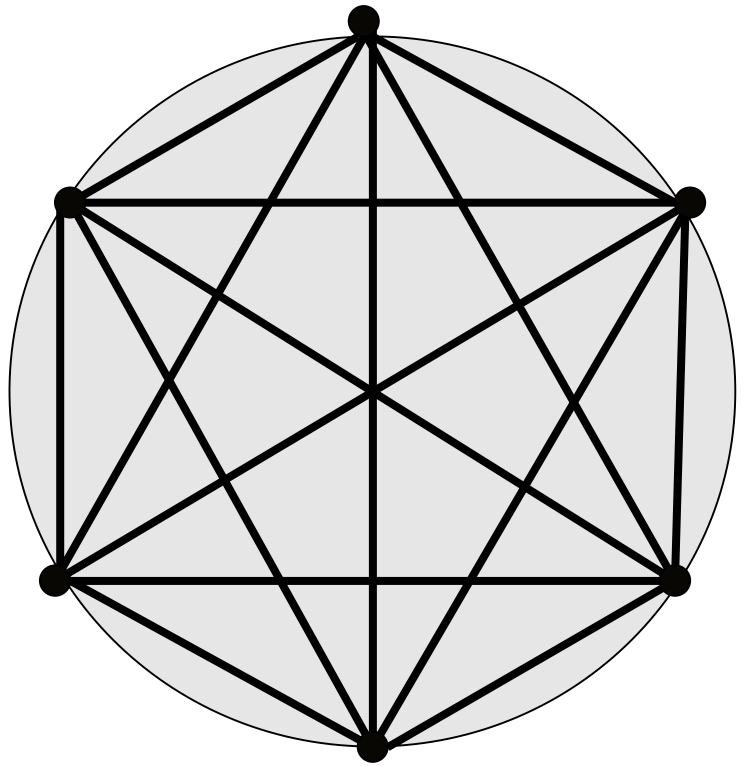
- $n = 5$
- $r = 16$



n	r
1	1
2	2
3	4
4	8
5	16

The Need for Proof

- $n = 6$
- $r = 30$



n	r
1	1
2	2
3	4
4	8
5	16
6	30

The Problem of Induction

- Consider the sequence produced by adding successive powers of 2.
- 1 + 2 + 4 + 8…
- 1, 3, 7, 15, …
- This clearly looks like $2^n – 1$ and we can use induction to prove that this is the case.

The Problem of Induction

- Similarly the sequence produced by adding successive powers of 5…
- 1 + 5 + 25 + 125…
- 1, 6, 31, 156, …
- can be inductively shown to be of the form $(5^n – 1)/4$.

- **What induction will not do is show us what we should be testing**

Problem 4

- According to legend, at the time of creation god created three diamond needles set in a slab of pure gold and on one of them he placed 64 discs of pure gold, each smaller than the one below it.
- He tasked a group of monks with moving the disks from their starting needle to another needle.
- But the monks had to obey certain rules:

Problem 4

1. The disks may only be placed on the needles.
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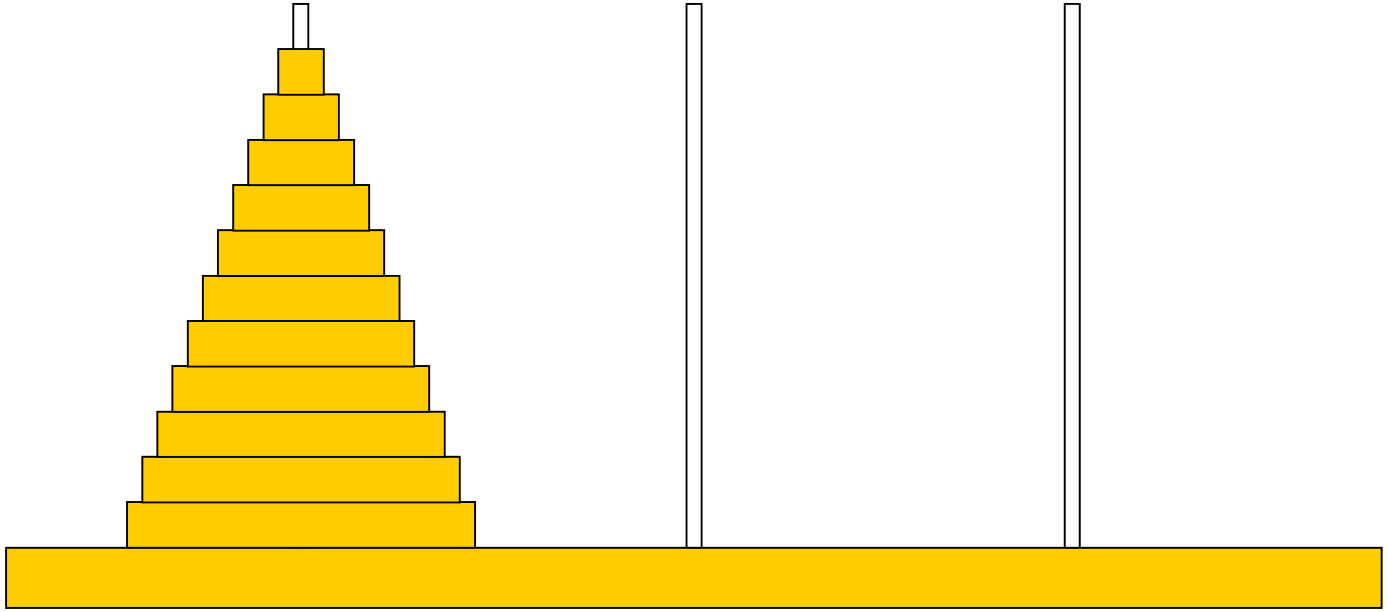
3. A disc must never be placed on a disc that is smaller than itself.

Problem 4

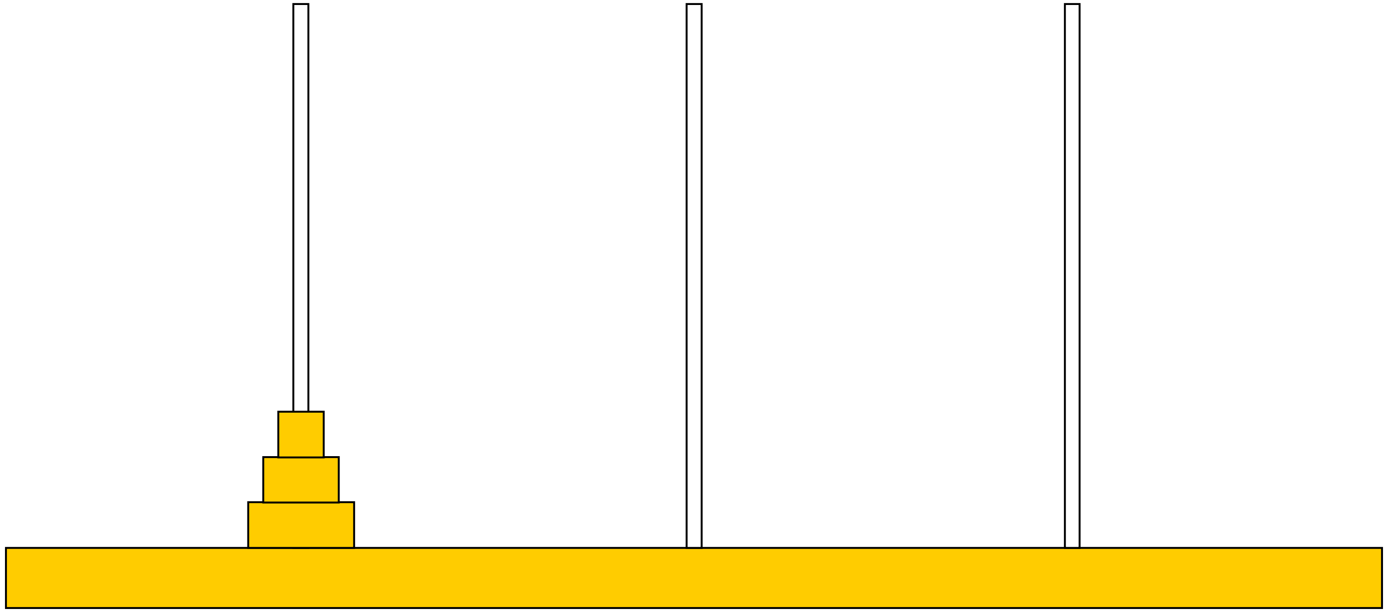


A simple solution

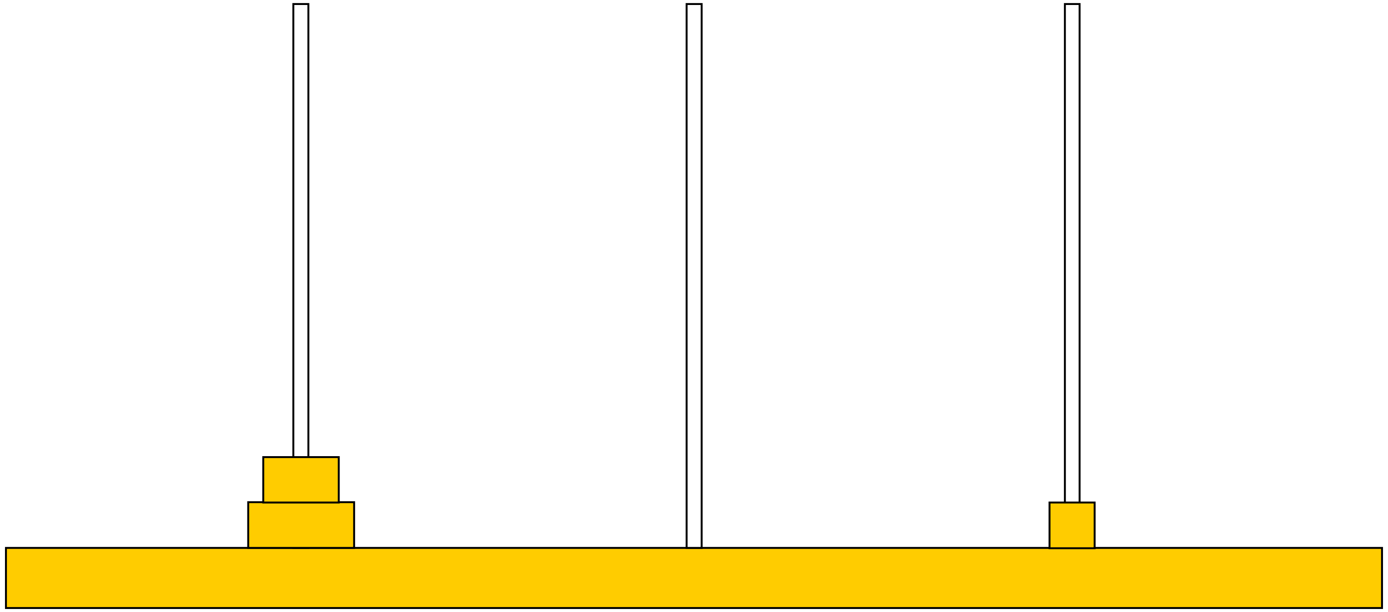
- Let us start with a solution that is simple to use.
  1. move the smallest disc left
  2. If possible, make another move
  3. If we have not got a single pile go to step 1
  4. We are done!
- The problem with this solution is that it provides no insight into the problem.
  - It’s like pulling a rabbit out of a hat.
  - Magic not technology

A small example

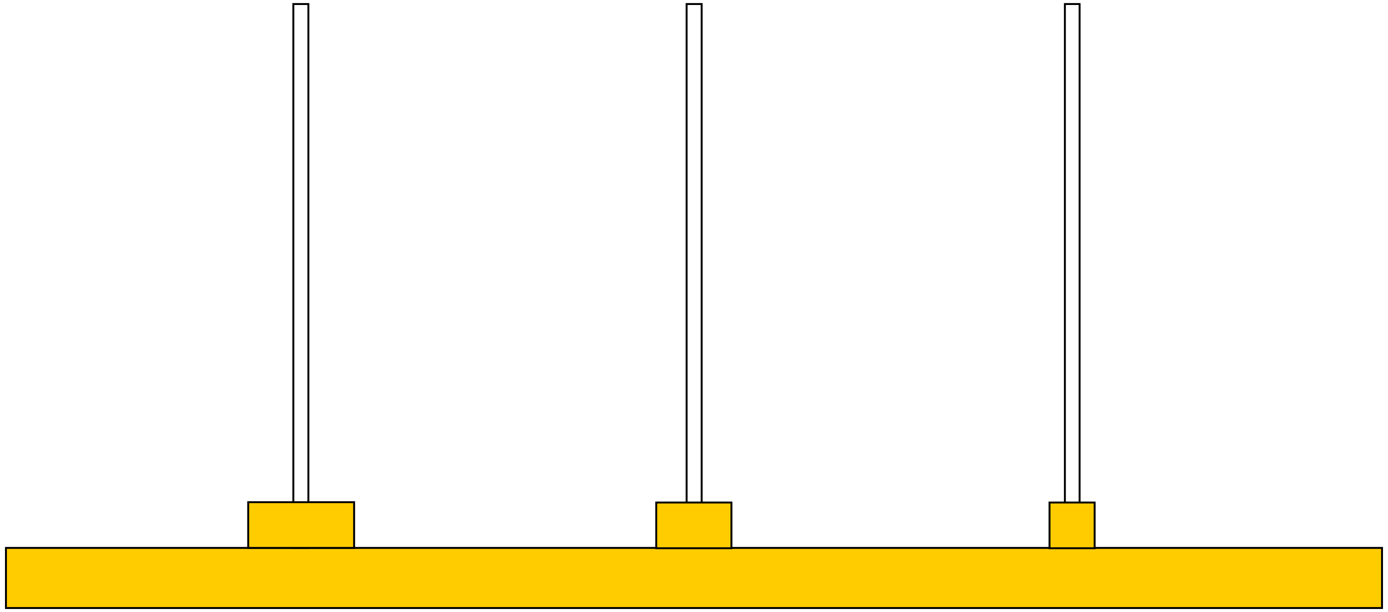
- Let's try a small example.
- We move disc 1 left.



- Step 1



- Step 2



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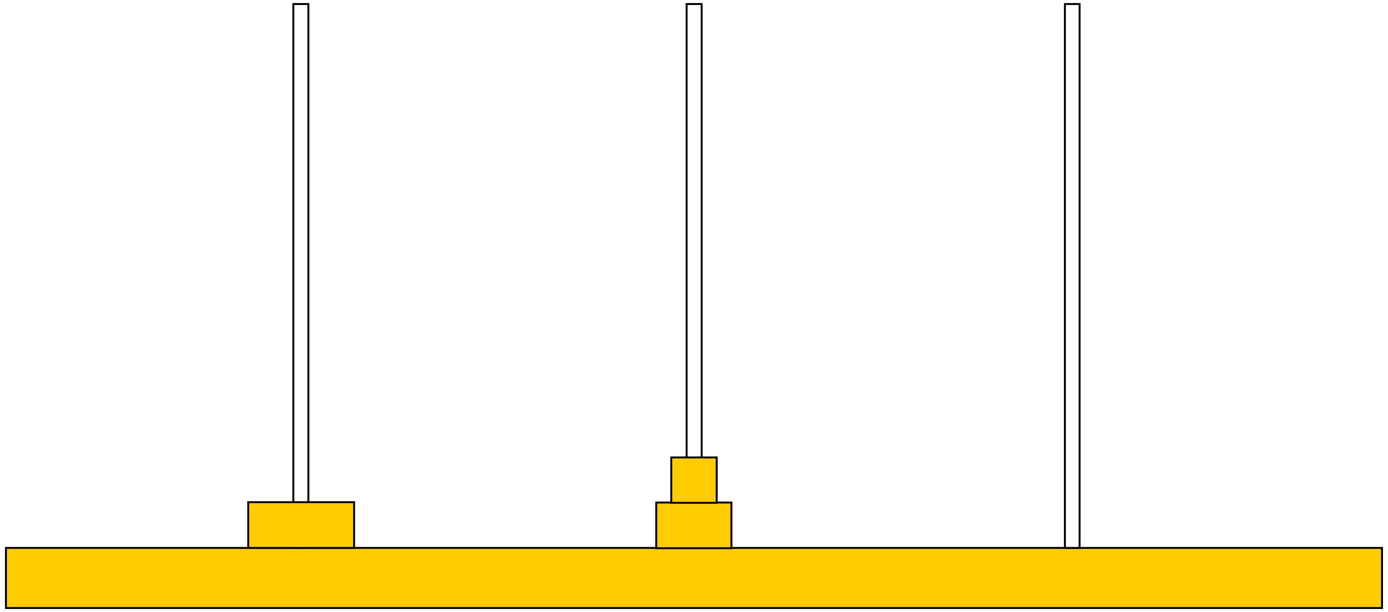
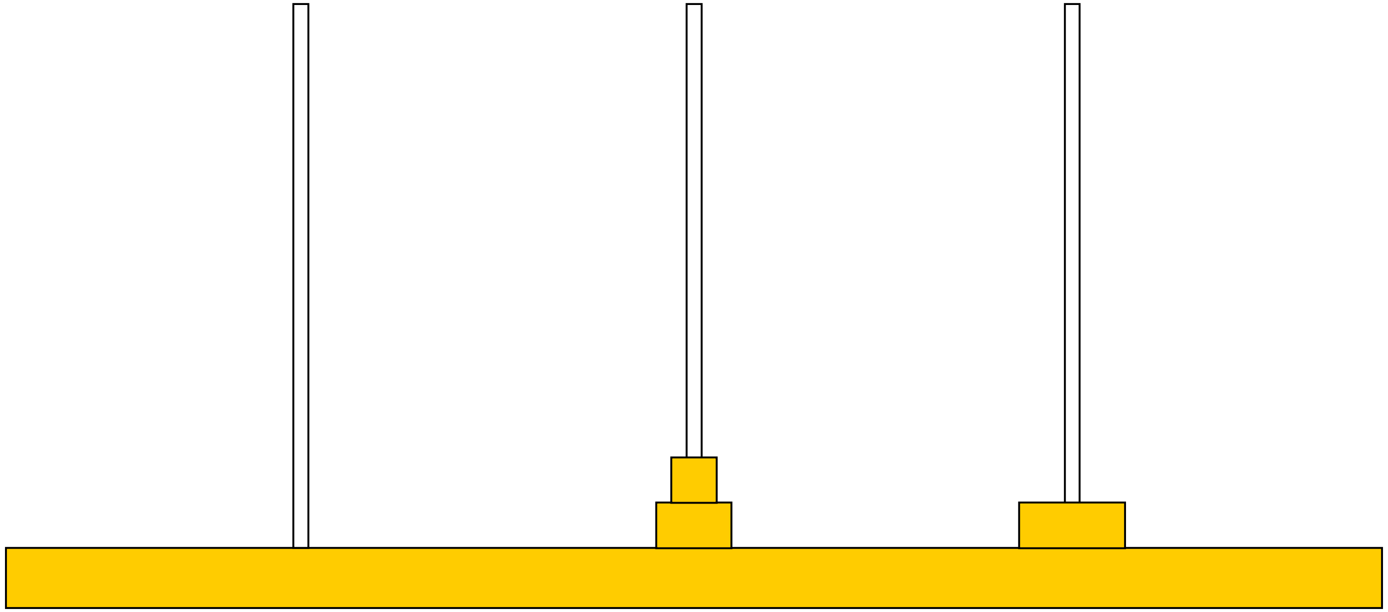


- Step 2



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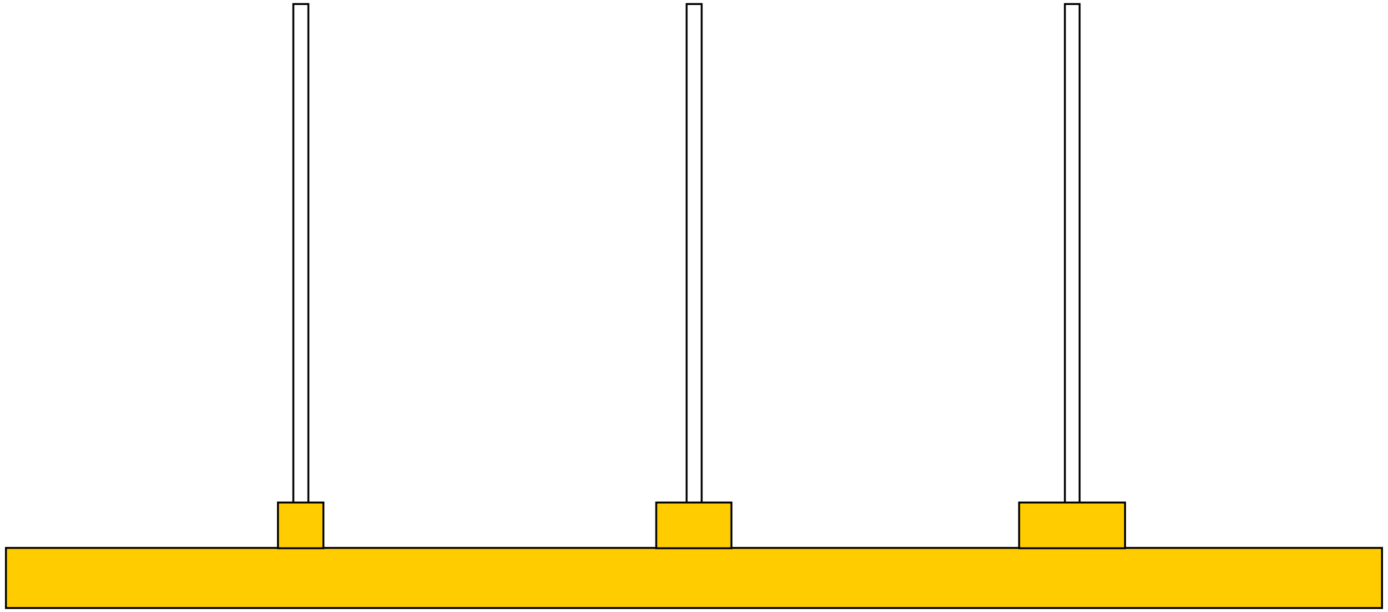
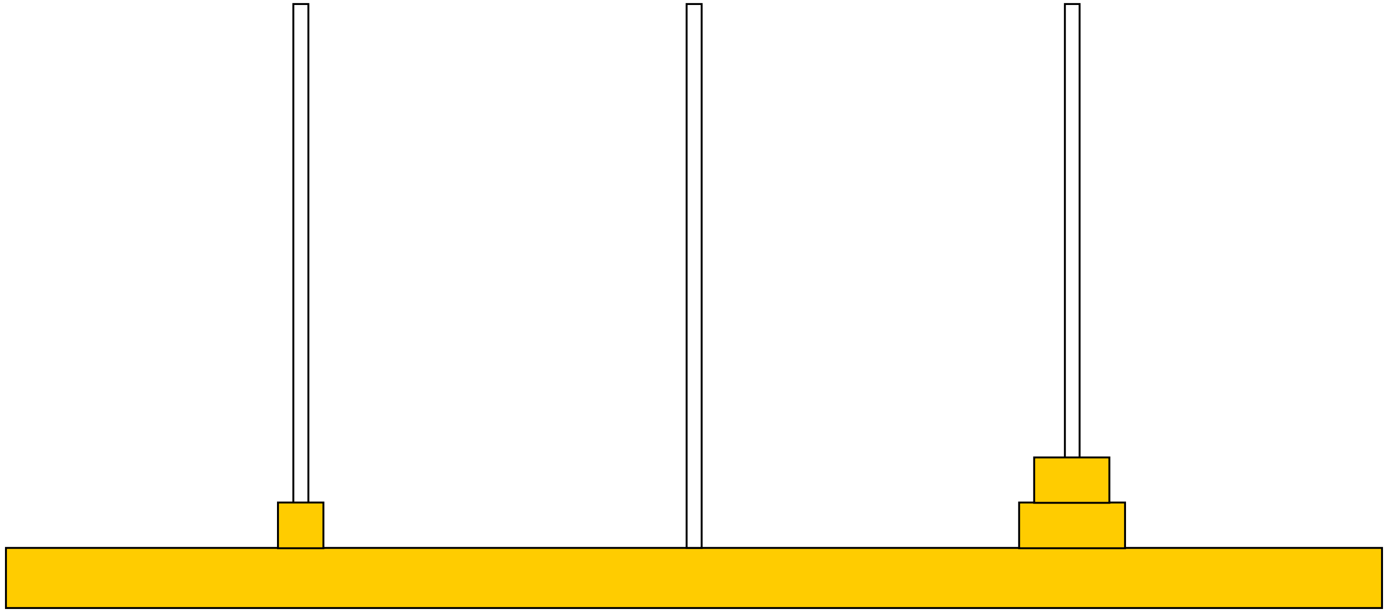


- Step 2



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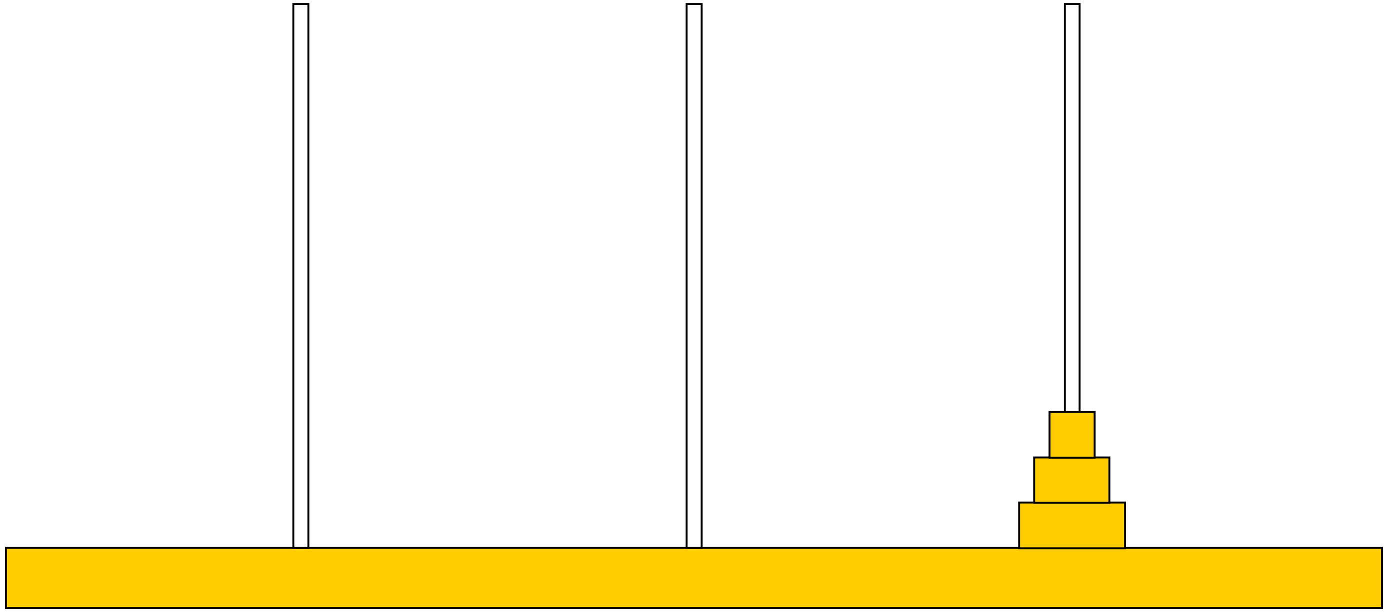
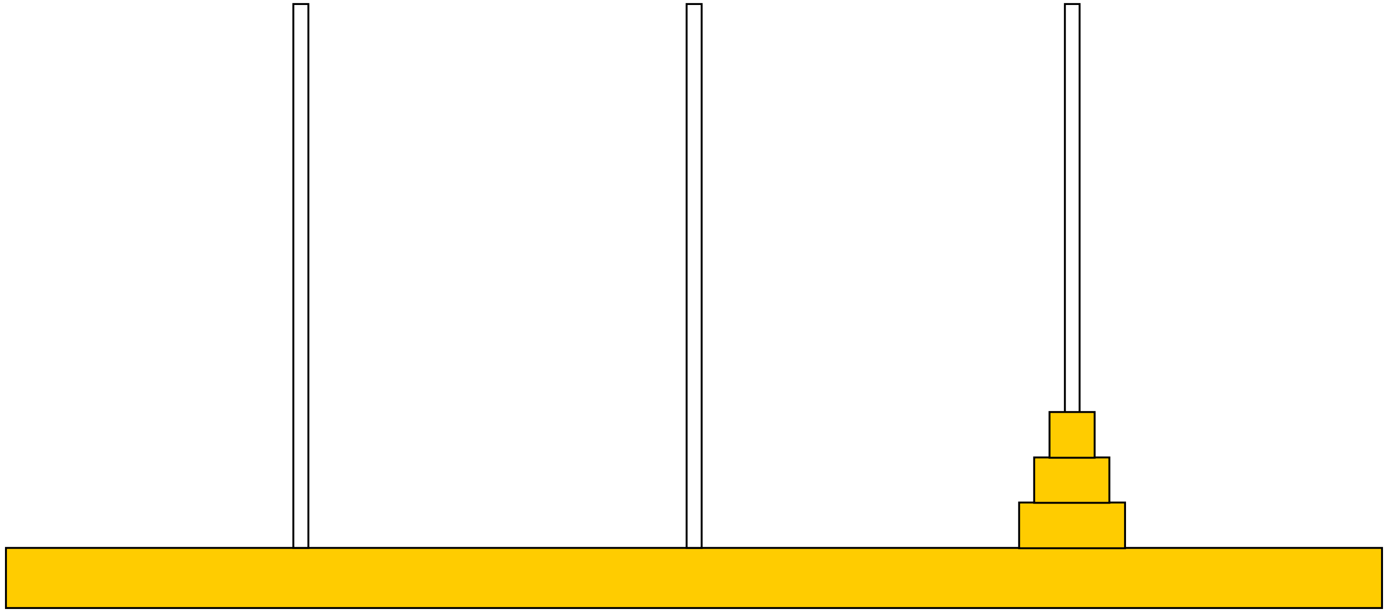


- All on the final needle – problem solved.



- How did this solution arise?
- Why does it work?
- Let us approach the problem from a different perspective.
- Let’s try induction.

The Towers of Hanoi

- If we number the towers (needles) in increasing order from left to right the problem becomes:

"Move a pile of *n* discs from tower 1 to tower 3."

- The base case of $n = 0$ is easy to solve.

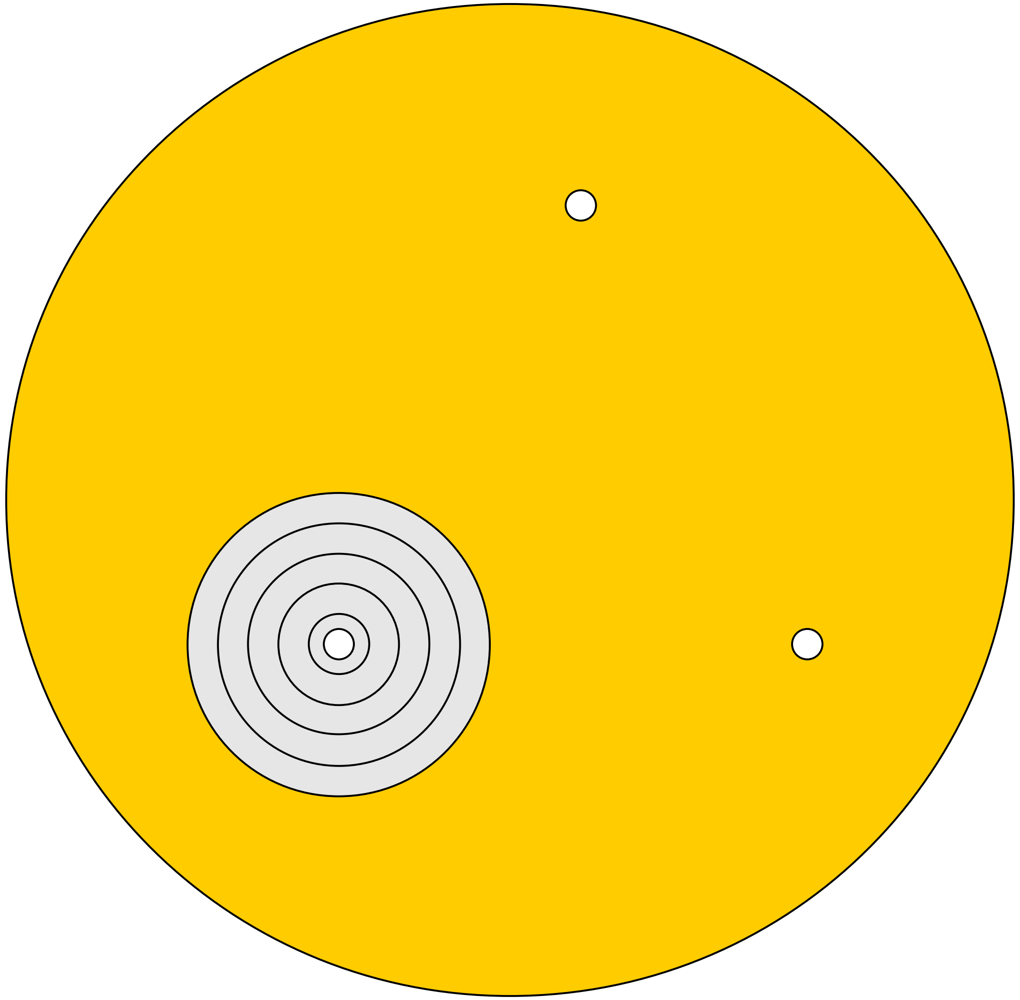
Do nothing!

- Now for the inductive step.

- We assume that we have a way of moving k discs from tower 1 to tower 3.
- Now, can we find a way to use this to move $k + 1$ discs?
- Sadly the answer seems to be no.
- There are only two ways we can start and neither way seems useful.

- Move the top *k* discs from tower 1 to tower 3. After doing this we are stuck because we have no hypothesis that involves moving discs *off* of needle 3! ☹
- Move the smallest disc to tower 2 and then move the remaining discs to tower 3. The problem now is that our hypothesis assumes that all towers are available and tower 2 is blocked by the smallest disk. ☹

- Maybe, numbering the towers was a bad idea.
- Let us make the problem more general.
- Imagine the towers are arranged in a circle.



- Now we can move discs in two directions:
  - clockwise;
  - anticlockwise.

Now the problem becomes:

Move *n* discs to the needle in direction *d*,
where *d* is either clockwise or anticlockwise.

- Now we return to the induction process.
- We have to be more careful in how we state it.
- As we saw earlier, if we have small disks on the towers they block the movement of the larger discs.
- Our induction hypothesis must be that it is possible to move the *n* ***smallest*** disks one step in an arbitrary direction *d* starting from ***any valid position***.
- By ***valid*** we mean any position in which no disc is on top of a smaller disc and the *n* smallest discs are in a single pile.

- For $n = 0$ the solution is still obvious and trivial.
- If we can move k discs in direction d, we can also move k discs in direction ~d, simply by “reversing” the direction of the pattern.
- Now it should be fairly how we can move $k + 1$ discs in direction d.

1. Move k discs in direction **~**d.
2. Move disc $k + 1$ in direction d.
3. Move k discs in direction **~**d.

- Step 1 moves the top k discs out of the way.
- Step 2 moves the next disc to its destination.
- Step 3 moves the top k discs back on top of disc $k + 1$.

Some notation

- We will now introduce some useful notation.
- Let $H_{n.d}$
  (Hanoi solution for n discs in direction d)
  be the sequence of moves required to move the *n* smallest discs in direction *d*.
- Let $\langle k,d \rangle$ represent a single move of disk *k* in direction *d*.

- $H_{0.d} = [\ ]$
- $H_{n+1.d} = H_{n.\neg d}\ ;\ \langle n+1, d\rangle\ ;\ H_{n.\neg d}$
- Note that H_{n+1} involves H_n.
  - This is what is known as a ***recursive*** formulation.
- For the direction we can use
  - c for clockwise and
  - a for anticlockwise.

- Now we can use the definitions to solve a sample problem:
- What is $H_{2.c}$?
  - $H_{2.c} = H_{1.a} ; \langle 2,c \rangle ; H_{1.a}$
  - $H_{1.a} = H_{0.c} ; \langle 1,a \rangle ; H_{0.c}$
  - $H_{0.c} = [\ ]$
- We can substitute upwards to give the following result.

- $H_{0.c} = [\ ]$
- $H_{1.a} = H_{0.c}\ ;\ \langle 1,a \rangle\ ;\ H_{0.c}$
- $H_{1.a} = [\ ]\ ;\ \langle 1,a \rangle\ ;\ [\ ]$ $= \langle 1,a \rangle$
- $H_{2.c} = H_{1.a}\ ;\ \langle 2,c \rangle\ ;\ H_{1.a}$
- $H_{2.c} = \langle 1,a \rangle\ ;\ \langle 2,c \rangle\ ;\ \langle 1,a \rangle$

- We can use this process to determine how to solve a problem of any size.
- However, this is not the easy, iterative solution we saw at the start. ☹

- Remember that the iterative solution involves two parts:
  1. The smallest disc moves consistently in a clockwise or anticlockwise direction;
  2. This move alternates with whatever other move is possible.
- How do we get from the recursive solution to this?

- If we examine the recursive solution, $H_{n.d}$, we note that …
  - The smallest disc always moves in the same direction (d for an odd number of discs and $\neg$d for an even number of discs)
- All that remains is to show that the moves alternate between disc 1 and some other disc (which is the only possible disc to move).
- Examining the problem should show that this must be the case.

- After we move disc 1 we have three possible situations (**cases**):
    1. There are discs on each of the other two towers.
    2. There are discs on only one of the other two towers.
    3. There are no disks on either of the other two towers.
- Let us examine each of these in turn.

- **Case 1:** There are discs on each of the other two towers.
- Clearly one of the two towers must have the smaller of these two disks.
- The only move that makes any sense is to move this disk on top of the larger disk.
- Any other legal move requires us to move disc 1 again.

- **Case 2**: There are discs on only one of the other two towers.
- Now, the only possible move that does not involve disk one is to move the other disc to the empty tower.

- **Case 3**: There are no disks on either of the other two towers.
- Now, no move is possible that does not involve moving disc 1.
- But this is OK because …
- …we have finished!

- Towers of Hanoi: A summary…
  - There is a simple, elegant and uninformative solution.
  - There is a complicated, ugly but informative solution based on induction.
  - We can transform the second solution into the first solution by careful examination.