Week 3 - Practice

Exercise 1: Tromino tilings

- Tight trominoes are L-shaped tiles formed by three adjacent squares. In a tiling, trominoes can be oriented in different ways, but they should cover all the squares of the board exactly with no overlaps
- For each of the three cases, prove or disprove that for every n>0, all the boards of the following dimensions can be tiled by right trominoes:

$a)$ $3^n \times 3^n$

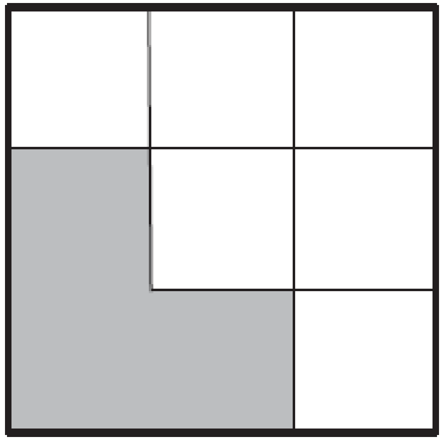
$b)$ $5^n \times 5^n$

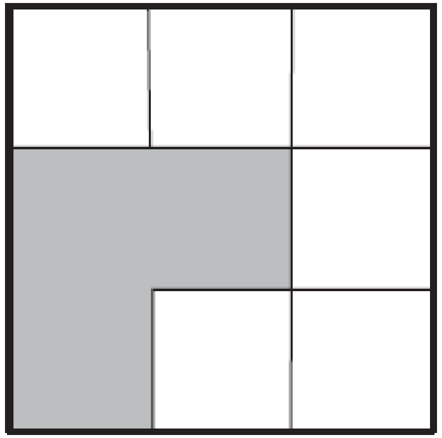
$c)$ $6^n \times 6^n$

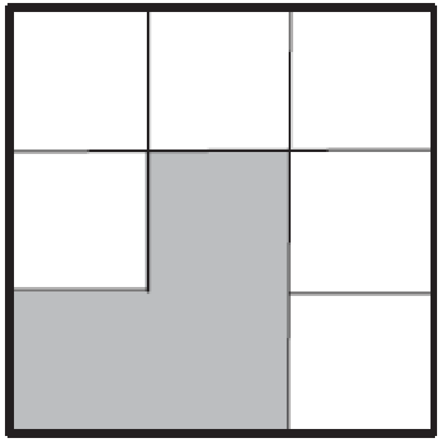
Exercise 1: Tromino tilings

- The answer is “no” for parts (a) and (b) and “yes” for part (c)

a) The answer is “no” because a 3x3 board cannot be tiled with right trominoes. Indeed, a corner of the board, say, the lower left one, can be tiled in three different ways, and each of them leaves a room only for one more right tromino. See the figures below:



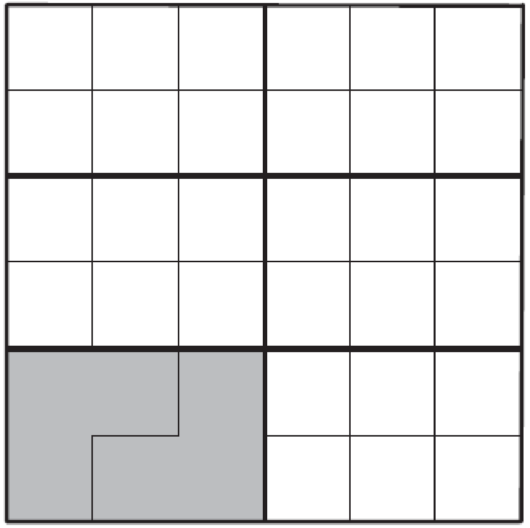




Exercise 1: Tromino tilings

b) The answer is “no” because the total number of squares in any $5^n \times 5^n$ board is not divisible by 3

c) A desired tiling can be easily obtained by dividing the board into 2x3 rectangles to tile each of them with two trominoes



Exercise 2: Tetromino tiling (again)

- We continue with tetromino tiling as in lecture
- Is It possible to tile an 8x8 chessboard with the following?

a) 16 I tetrominoes

b) 16 square tetrominoes

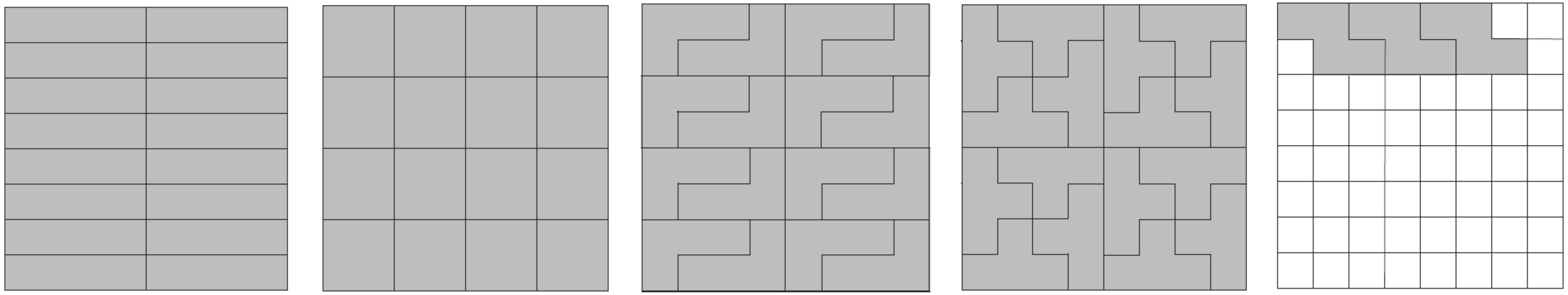
c) 16 L-tetrominoes

d) 16 T-tetrominoes

e) 16 Z-tetrominoes

Exercise 2: Tetromino tiling (again)

- The tilings are shown in the photos below



- First fours cases are possible: the tiling of one quarter of the board is repeated for the three other quarters
- The last case is not possible: putting such a tile to cover a corner of the board makes it necessary to continue putting two more tiles along the boarder with no possibility to cover the two remaining squares in the first row

Exercise 3: Plus and Minus

- The n consecutive integers from 1 to n are written in row. Design an algorithm that puts signs “+” and “-” in front of them so that the expression obtained is equal to 0 or, if the task is impossible to do, returns the message “no solution”. Your algorithm should be much more efficient than an examination of all possible ways to place the signs. Write a pseudocode for your algorithm.

Solution

- It has a solution if and only if either n or n+1 is divisible by 4
- The problem is equivalent to partitioning n integers from 1 to n into two disjoint (i.e., without common elements) subsets with the same sum: a subset of numbers with a plus before them and a subset of numbers with a minus before them.
- Since $S = 1+2+\cdot\cdot\cdot+n = n(n+1)/2$, the sum of the numbers in each of the subsets must be equal to exactly one half of S.
- This implies that $n(n + 1)/2$ must be even as a necessary condition for the puzzle to have a solution. We will show that this condition is also sufficient for a solution’s existence.

- Note that $n(n + 1)/2$ is even if and only if either n is a multiple of 4 or $n + 1$ is a multiple of 4.
- Indeed, if $n(n + 1)/2 = 2k$, then $n(n + 1) = 4k$, and since either n or $n + 1$ is odd, the other must be divisible by 4.
- Conversely, if either n or $n + 1$ is a multiple of 4, $n(n + 1)/2$ is obviously even.

- If n is divisible by 4, we can, for example, partition the sequence of integers from 1 to n into $n/4$ groups of four consecutive integers and then put pluses before the first and fourth number and minuses before the second and third number in each of these groups:

$$(1 - 2 - 3 + 4) + \cdots + ((n - 3) - (n - 2) - (n - 1) + n) = 0. \qquad (1)$$

- If $n + 1$ is divisible by 4, then $n = 4k - 1 = 3 + 4(k - 1)$, and we can exploit the same idea by first taking care of the first three numbers as follows

$$(1+2-3)+(4-5-6+7)+\cdots+((n-3)-(n-2)-(n-1)+n) = 0. \qquad (2)$$

Algorithm

The problem can be solved by the following algorithm.

- Compute n mod 4 (the remainder of the division of n by 4).
- If the remainder is equal to 0, insert the “+” and “−” signs as indicated by formula (1);
- If the remainder is equal to 3, insert the “+” and “−” signs as indicated by formula (2);
- Otherwise, return the “no solution” message.