CSIT113
Problem Solving

Week 11

Complexity

- Is there a sensible way to discuss the complexity of a problem?
- Can we say how hard it is to solve?
- This is not as easy a question as it may appear.

Complexity

- Let us take sorting as an example.
- The first time we looked at this we saw a number of different algorithms:
  - selection sort;
  - insertion sort;
  - bubble sort.
- All of these took around n^2 steps to sort an array of length n.

Sorting

- So, does this mean that it always takes n^2 steps to sort an array?
- No. Consider quicksort.
- It takes $n \log n$ steps which is considerably less.
- Is it possible that there are even faster ways to sort an array?

Theory vs. Practice

- Clearly, there is a difference between:
  1. how complex the problem is;
  2. how complex a particular algorithm to solve it is.
- If we can solve a problem of size n in n log n steps what can we say about the actual complexity of the problem?

Theory vs. Practice

- The problem’s actual complexity is *no worse than n* log *n*.
- This lets us set an *upper limit* on the complexity of the problem.
- Can we find some sensible way to establish a *lower limit*?

Lower Bounds

- In general, this is a much more difficult thing to establish.
- Often the best we can do is to find a relative rather than an absolute answer.
- We can often only say that one problem is *at least as hard* as another problem.
- How can we do this.

Problem Transformation

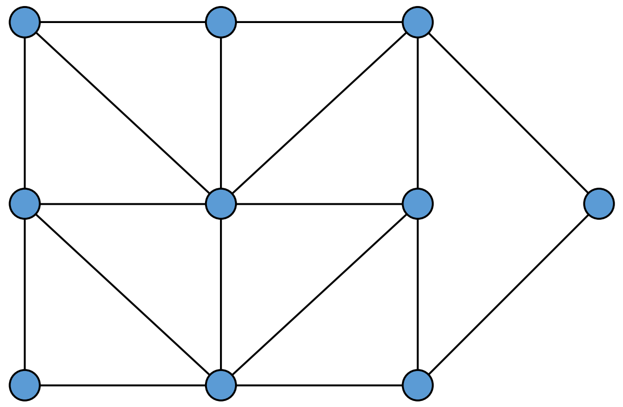
- If we can transform one problem into another problem we can say that the first problem is *no harder* than then the second problem.
- This assumes that the transformation process is relatively simple.
- Let us look at two problems:
  - graph colouring;
  - satisfiability.

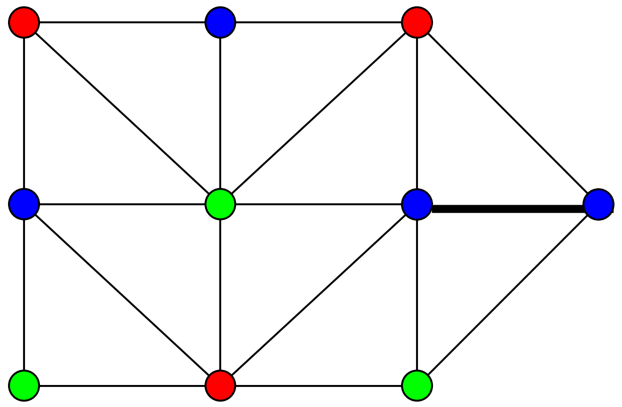
Graph Colouring

- The problem here is to answer the question:
  - Can we colour the vertices of a given graph with three colours so that no two adjacent nodes are of the same colour?
- This is best seen with an example.

Graph Colouring

- Given the graph:-
- We can colour it as follows:-





- But what if we add an extra edge?

Satisfiability

- The problem here is to answer the question:
  - Can we find values for a set of logical variables which satisfies a given logical expression?
- Again, let us look at an example.

Satisfiability

- Given the expression:-
  - $(a \vee b \vee c) \wedge (\sim a \vee b) \wedge (\sim b \vee \sim c) \wedge (a \vee \sim c)$
- Where a, b and c are either true or false, $\vee$ represents “or”, $\wedge$ represents “and” and ~ represents “not”.
- Can we assign values to a, b and c so that the expression has the value true?
  - a = true, b = true, c = false

So What?

- What, if anything do these problems have in common?
- It turns out we can transform the graph colouring problem into the satisfiability problem.
- How?

Transforming the Problem

- Let us label each of the nodes of the graph with an integer value.
- Every node must have a colour
- Let:
  - R_i mean “node *i* is red”;
  - G_i mean “node *i* is green”;
  - B_i mean “node *i* is blue”.
- Then:
  - $(R_i \vee G_i \vee B_i)$

Transforming the Problem

- But every node must have a single colour
- So:
  - $(R_i \wedge \sim G_i \wedge \sim B_i)$ – the node is red;
  - $(\sim R_i \wedge G_i \wedge \sim B_i)$ – the node is green;
  - $(\sim R_i \wedge \sim G_i \wedge B_i)$ – the node is blue.
- We combine these:
  - $(R_i \wedge \sim G_i \wedge \sim B_i) \vee (\sim R_i \wedge G_i \wedge \sim B_i) \vee (\sim R_i \wedge \sim G_i \wedge B_i)$.

Transforming the Problem

- Finally no two connected nodes may have the same colour.
- So, if node *i* is connected to node *j*:
  - $(\sim R_i \vee \sim R_j)$ – the nodes are not both red;
  - $(\sim G_i \vee \sim G_j)$ – the nodes are not both green;
  - $(\sim B_i \vee \sim B_j)$ – the nodes are not both blue.
- Thus:
  - $(\sim R_i \vee \sim R_j) \wedge (\sim G_i \vee \sim G_j) \wedge (\sim B_i \vee \sim B_j)$

Transforming the Problem

- For a graph with *n* nodes and *e* edges we get 4**n* + 3**e* clauses in 3**n* logical variables.
- Thus we can transform any graph colouring problem into an equivalent satisfiability problem.
- So what?

Comparative Complexity

- What this means is that, in the worst case, we can solve a graph colouring problem by transforming it and solving the equivalent satisfiability problem.
- This means that the graph colouring problem is *no harder* than the satisfiability problem.
- This does not mean, of itself, that we cannot find a better way of solving the colouring problem, only that it is never worse than this.

P and NP

- In fact, both the colouring and satisfiability problems are as hard as a problem can be and still be of some interest to us.
- Computer scientists define two sets of problems which are of particular interest.
  - P, the set of problems for which a polynomial complexity solution exists.
  - NP, the set of problems for which a polynomial complexity verification procedure exists.
- Clearly, P is a subset of NP.

Problems and P

- You should note that all problems fall into one of the following categories.
  - Known to be in P.
  - Known not to be in P.
  - Unknown
    - Possibly in P
    - Possibly not in P

Problems and NP

- In a similar way we can categorise problems with respect to NP.
- Of particular interest are the problems which are known to be in NP (we have already found a polynomial verification procedure) but for which we have no current proof of there membership in P (no polynomial solution strategy).

NP-Complete

- Of these problems, there is a subset – known as NP-complete problems – which are as hard as a problem can be and still have a polynomial verification process.
- This set contains, among others, both the satisfiability and colouring problems.

So What?

- So, why is this of interest to computer scientists?
- It has been conjectured that rather than being a subset of NP, P may be equal to NP.
- The consequences of this, if it were shown to be true, are of enormous importance.

If P = NP

- We could find polynomial solutions to all NP problems (including NP-complete problems).
- Any problem in NP would be as easy to solve as it is to verify.

Important Point!

- The identity P = NP is only conjecture.
- Although it would be very nice if it were true, there is currently no basis for assuming that it is.
- It is just that there is equally no basis for concluding that it is not.

More So What!

- Ok, so some problems are basically too hard to solve.
- NP-Complete problems.
- Does that mean we don’t try to solve them?
- No.
- We do one of two things…
    - We use techniques that, hopefully, will bring us to a solution faster than brute force.
    - We use techniques that get us close to a solution within reasonable time.
- Let us look at a new solution technique *branch and bound*.

Branch and Bound

- A further approach to problem solving is often adopted when the simpler methods (greedy, divide and conquer) fail is that of *Branch and Bound*.
- In this method, we attempt to ‘prune’ the solution tree by eliminating branches where we can safely assert that the (optimal) solution cannot be located.

Branch and Bound

- In general terms the algorithm starts with the following steps:
  - Find an initial possible solution – which, generally, is not the best solution. This establishes a *bound* on the final result.
  - Find the best solution – which, generally, is not possible. This establishes a second bound on the final result.
- We can now say that the actual solution lies between these two bounds.

Branch and Bound

- We now proceed as follows:
  - For each branch of the solution tree establish an estimate of the best expected solution that we can get by following this branch.
  - Eliminate all branches that are outside the bounds we have established.
  - Pick the branch with the best expectation as the next one to expand.
  - If we get a complete solution we use it to update the appropriate bound.

Branch and Bound

- This technique is best seen with an example problem.
- We will use the assignment problem as an example.
- Simply stated the problem is to assign *n* jobs to *n* employees – one job to each employee.
- The problem is that each combination has a different cost and we must find the cheapest solution.

Branch and bound

- The assignment problem
  - A set of *n* agents are assigned *n* tasks.
  - Each agent performs exactly one task.
  - If agent *i* is assigned task *j* then a cost c_{ij} is associated with this combination.
  - The problem is to minimise the total cost *C.*

The assignment problem

- For example suppose agents a, b and c are assigned tasks 1, 2 and 3 with the following cost matrix:

	1	2	3
a	4	7	3
b	2	6	1
c	3	9	4

- If we allocate task 1 to agent c, task 2 to agent b and task 3 to agent a the total cost is 3 + 6 + 3 = 12

The assignment problem

- The assignment problem has many applications:
  - Buildings and sites – the cost of erecting building i on site j.
  - Trucks and destinations – the cost of sending truck i to destination j.
  - Etc.
- In general, with n agents and n jobs there are $n!$ possible assignments.
- This is too many to consider, even for relatively small values of n.

The assignment problem

- Suppose we have to solve the following instance

	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- Note the assignment a $\Rightarrow$ 1, b $\Rightarrow$ 2, c $\Rightarrow$ 3, d $\Rightarrow$ 4 has cost 73.

Upper Bound

- This represents a feasible solution
- This cost of 73, therefore, provides an *upper bound* on the solution.
- The minimal cost must be ≤ 73
- Can we get a *lower bound*?

Lower Bound

- The cost of 73 provides an upper bound on the solution.
- The minimal cost must be $\leq$ 73
- Can we get a lower bound?
  - Task 1 must cost at least 11
  - Task 2 Must cost at least 12
  - Task 3 Must cost at least 13
  - Task 4 Must cost at least 22
- The minimal cost must be ≥ 58
- 58 ≤ C ≤ 73

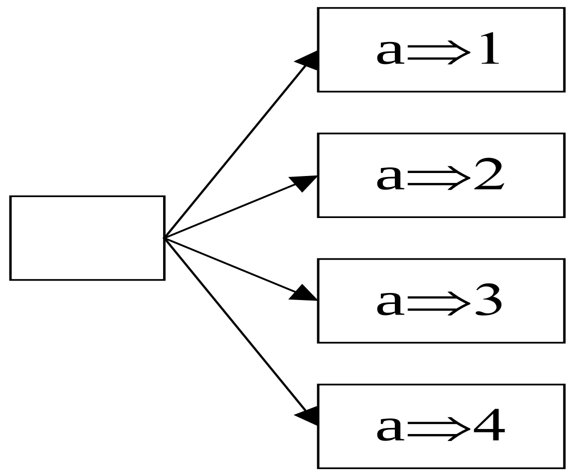
	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

The Solution Strategy

- The solution proceeds as follows:
  - We explore a tree whose nodes correspond to partial assignments.
  - At the root of the tree, no assignments have been made.
  - At each level we fix the assignment of one more agent.
  - At each node we calculate a bound on the costs that can be achieved by completing this sub tree.
  - We prune any sub tree with too high a minimum cost bound.

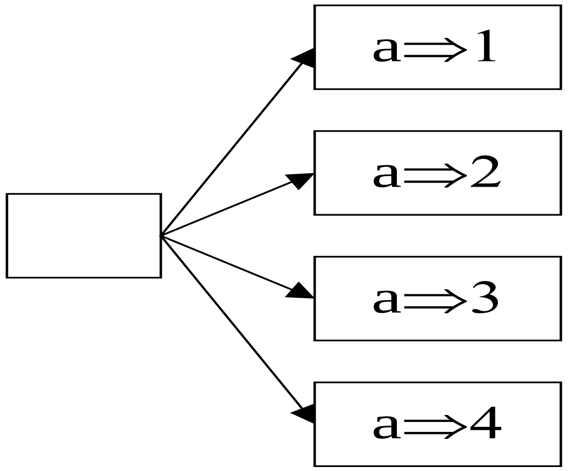
The assignment problem: 58 $\leq$ C $\leq$ 73

- Assign agent a



The assignment problem: $58 \le C \le 73$

- Assign agent a, calculate lower bounds from here

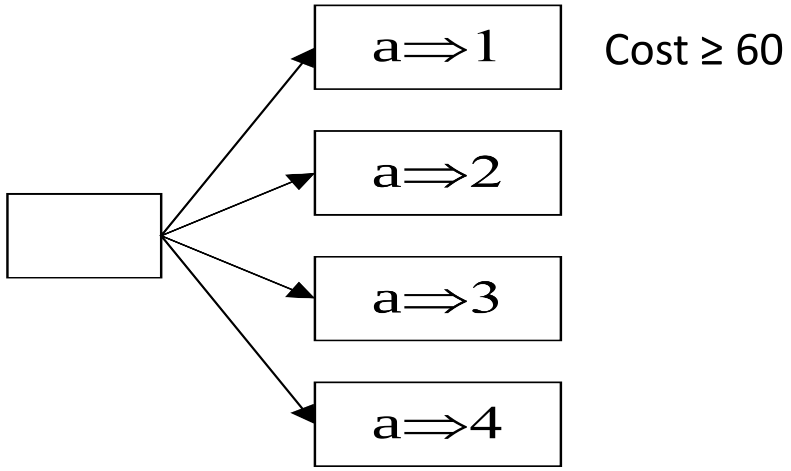


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If a $\Rightarrow$ 1, the lower bound cost is 11 + 14 + 13 + 22 = 60

The assignment problem: 58 $\leq$ C $\leq$ 73

- Assign agent a, calculate lower bounds from here

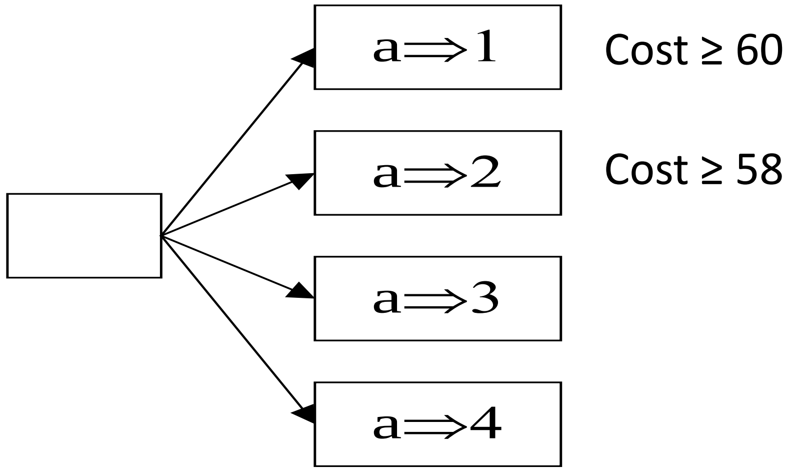


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If a $\Rightarrow$ 2, the lower bound cost is 12 + 11 + 13 + 22 = 58

The assignment problem: $58 \leq C \leq 73$

- Assign agent a, calculate lower bounds from here

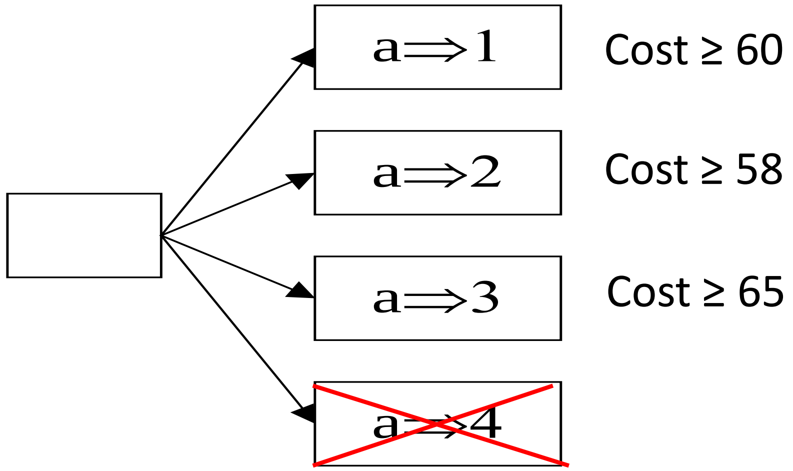


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If a $\Rightarrow$ 3, the lower bound cost is 18 + 11 + 14 + 22 = 65

The assignment problem: 58 $\leq$ C $\leq$ 73

- Assign agent a, calculate lower bounds from here

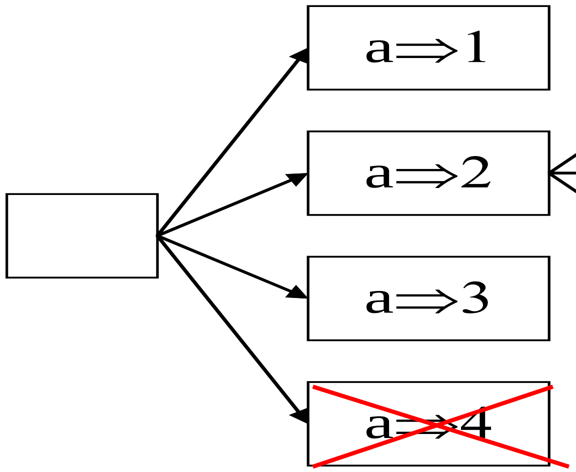


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If a ⇒ 4, the lower bound cost is 40 + 11 + 14 + 13 =78
- The minimum cost for this path is bigger than the upper bound
- We can eliminate this branch

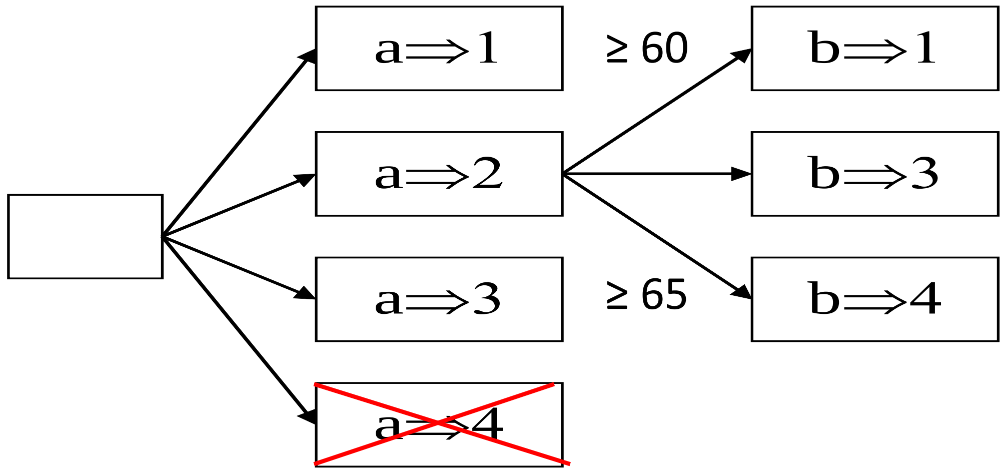
The assignment problem: 58 $\leq$ C $\leq$ 73

- Expand a $\Rightarrow$ 2



The assignment problem: $58 \leq C \leq 73$

- Assign agent b, calculate lower bounds from here

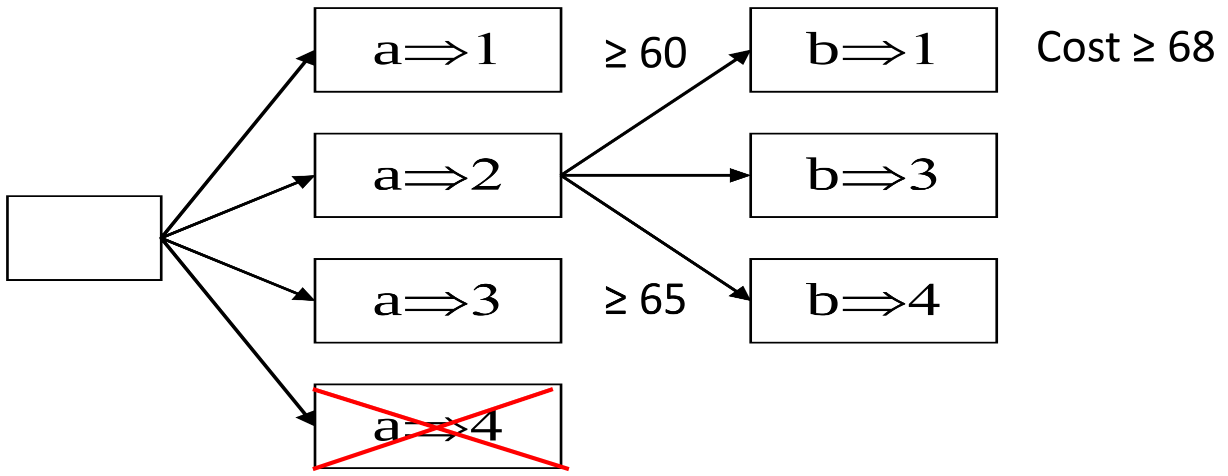


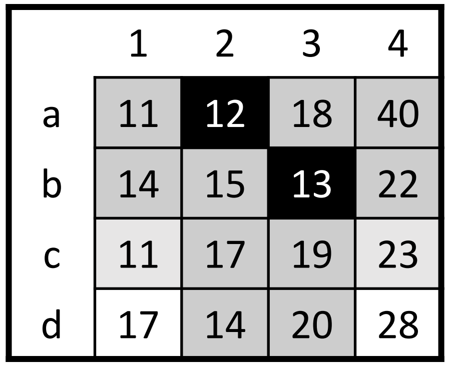
	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If b ⇒ 1, the lower bound cost is 12 + 14 + 19 + 23 = 68

The assignment problem: 58 $\leq$ C $\leq$ 73

- Assign agent b, calculate lower bounds from here



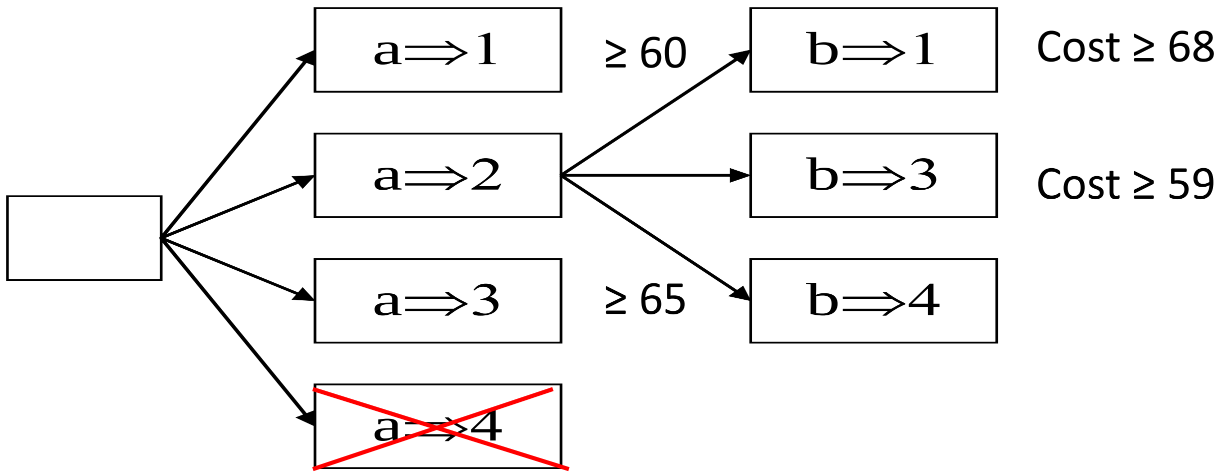


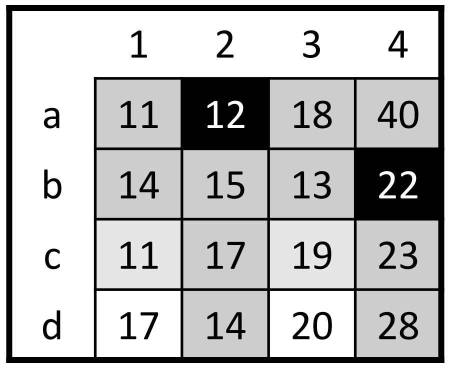
	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If b ⇒ 3, the lower bound cost is 12 + 13 + 11 + 23 = 59

The assignment problem: $58 \leq C \leq 73$

- Assign agent b, calculate lower bounds from here



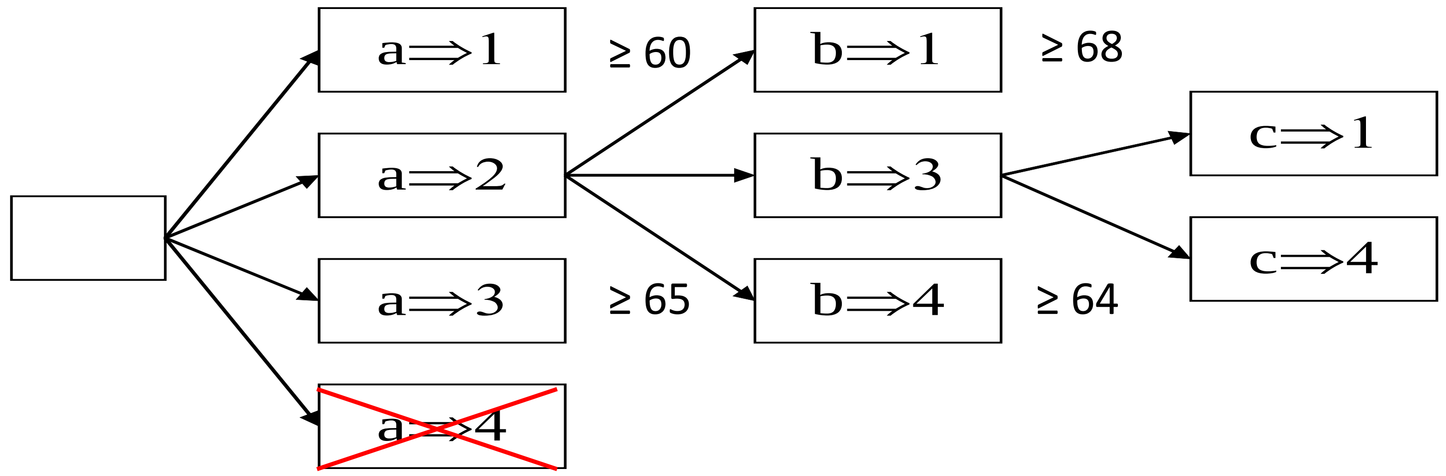


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If b ⇒ 4, the lower bound cost is 12 + 22 + 11 + 19 = 64
- Our best choice seems to be to assign b to task 3.

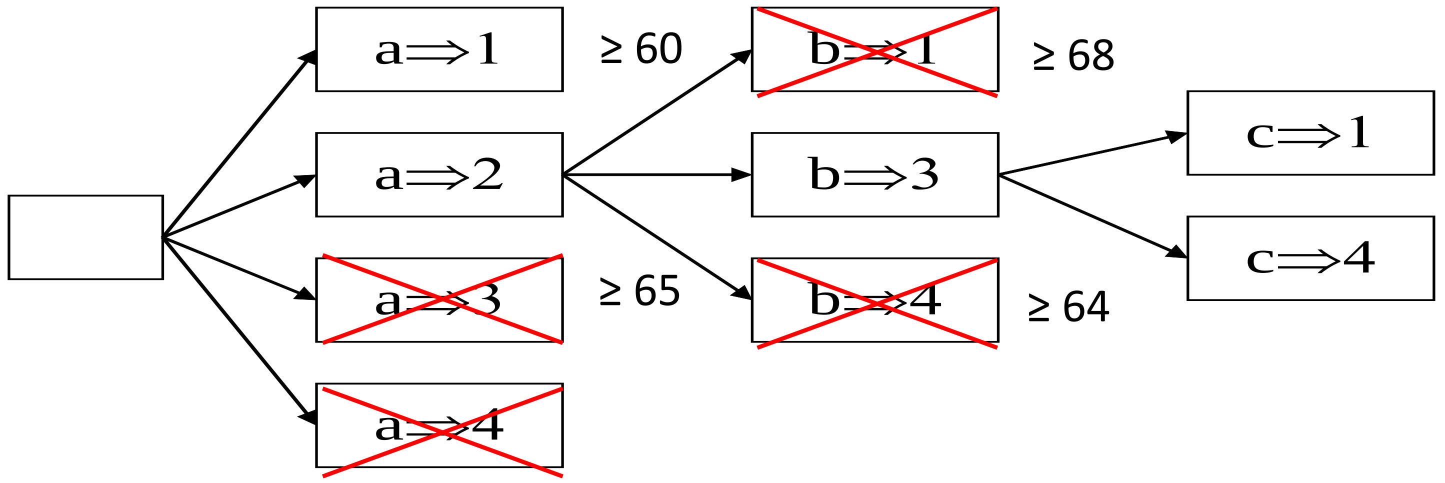
The assignment problem: 58 $\underline{<}$ C $\underline{<}$ 73

- Expand b $\Rightarrow$ 3



The assignment problem: $58 \leq C \leq 73$

- Assign agent c, calculate lower bounds from here

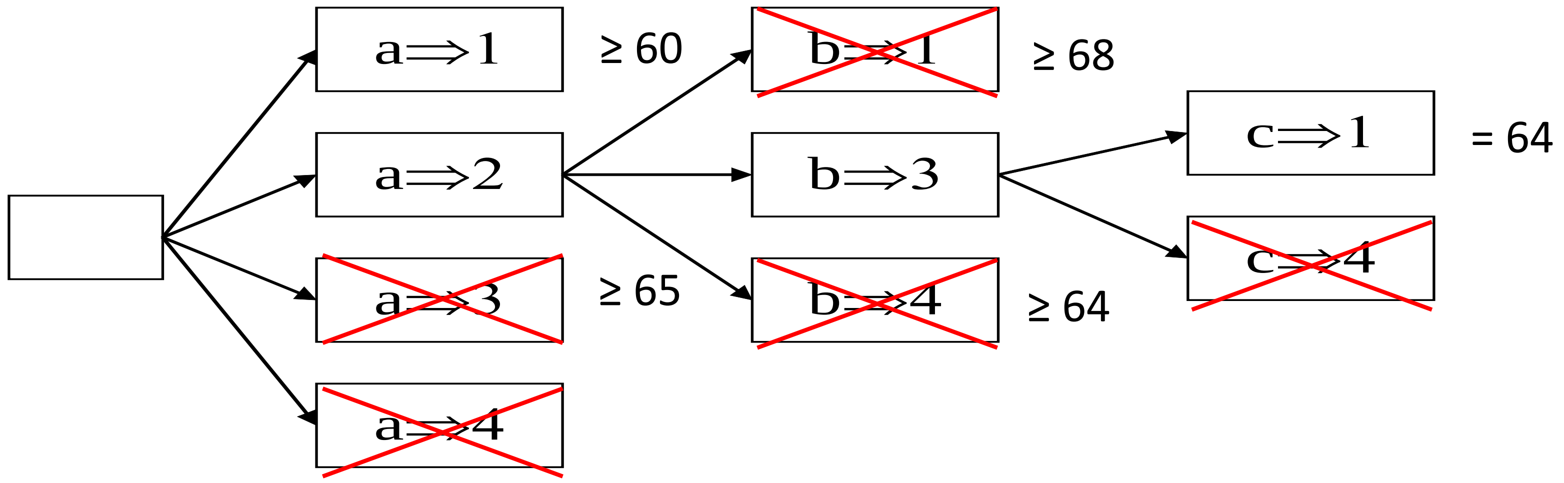


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If c $\Rightarrow$ 1, the lower bound cost is 12 + 13 + 11 + 28 = 64
- This is a feasible solution and has a value < 73
- Thus, 64 is a new upper bound
- This means we can eliminate several sub trees

The assignment problem: 58 ≤ C ≤ 64

- Assign agent c, calculate lower bounds from here

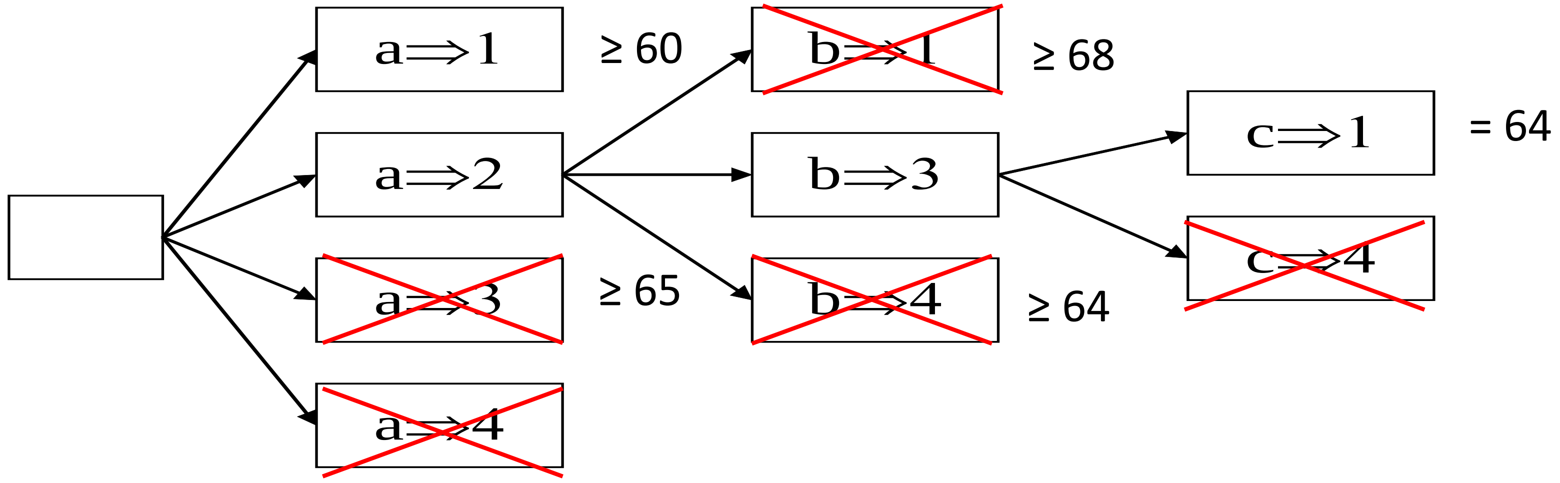


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If c ⇒ 4, the lower bound cost is 12 + 13 + 23 + 17 = 65
- This is larger than the new upper bound so we can eliminate it too.

The assignment problem: $58 \leq C \leq 64$

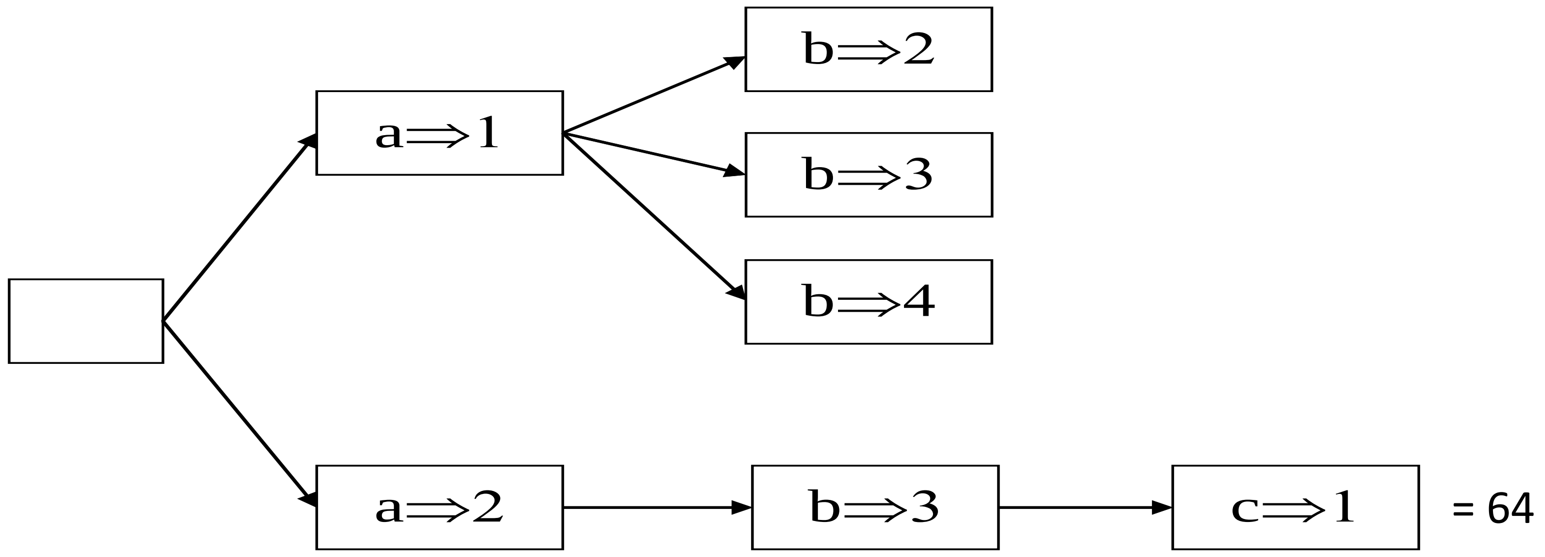
- The only node left to explore is a $\Rightarrow$ 1



- Let’s clean up the tree a bit first

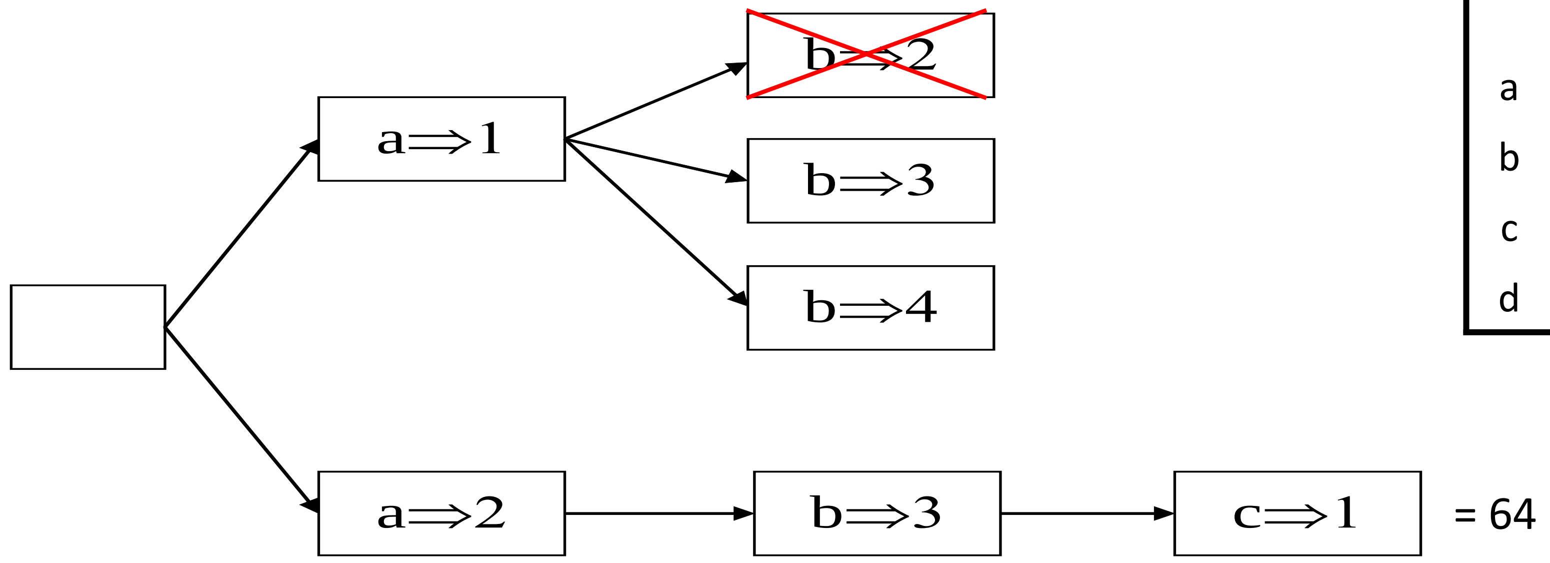
The assignment problem: 58 $\leq$ C $\leq$ 64

- Expand node a $\Rightarrow$ 1



The assignment problem: $58 \leq C \leq 64$

- Assign agent b, calculate lower bounds from here

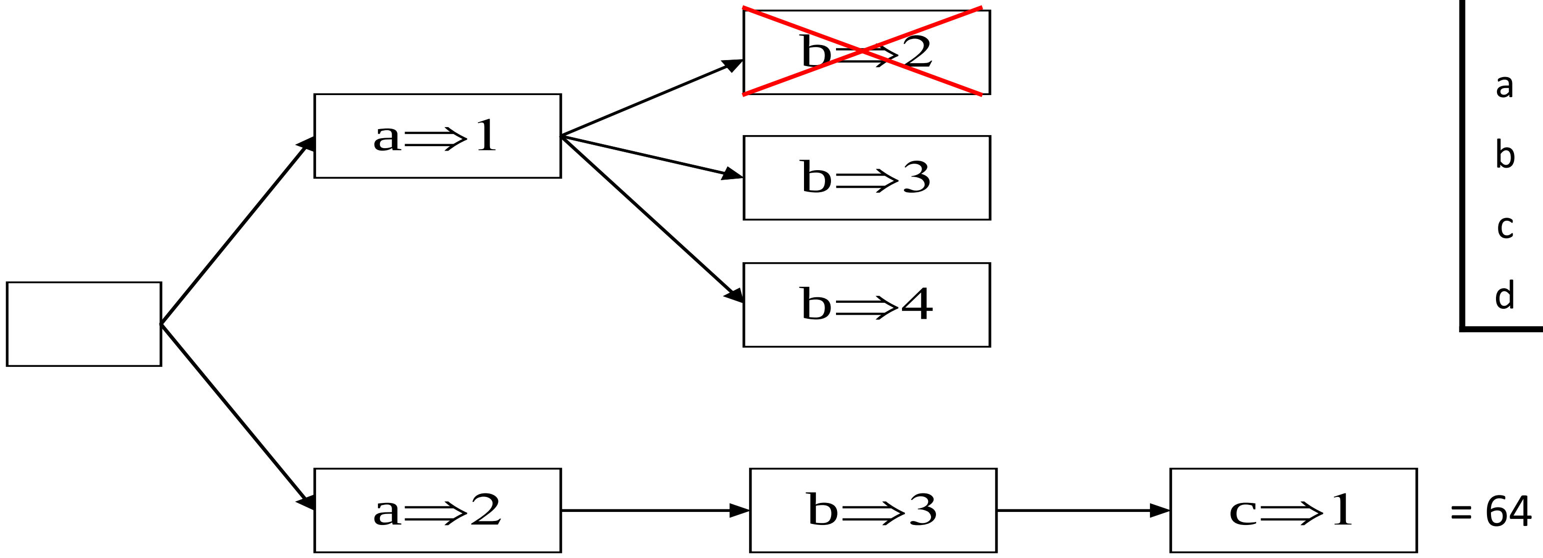


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If b $\Rightarrow$ 2, the lower bound cost is 11 + 15 + 19 + 23 = 68
- We can eliminate this branch

The assignment problem: 58 $\leq$ C $\leq$ 64

- Assign agent b, calculate lower bounds from here

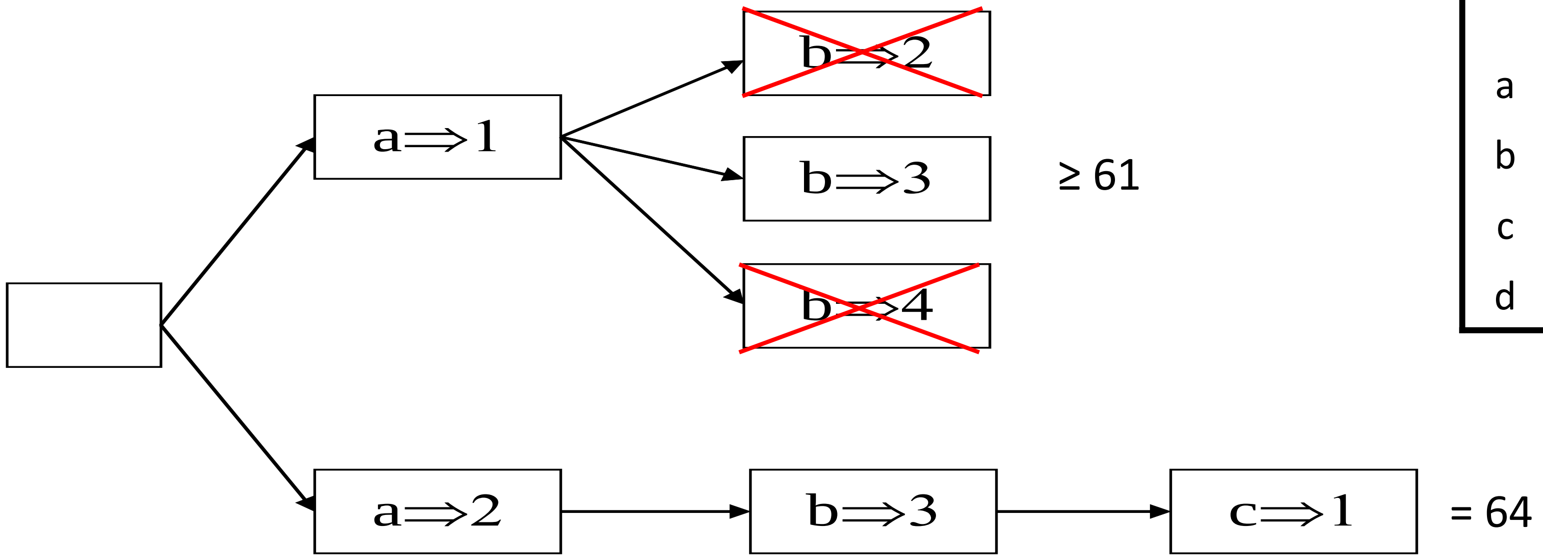


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If b ⇒ 3, the lower bound cost is 11 + 13 + 14 + 23 = 61

The assignment problem: 58 ≤ C ≤ 64

- Assign agent b, calculate lower bounds from here

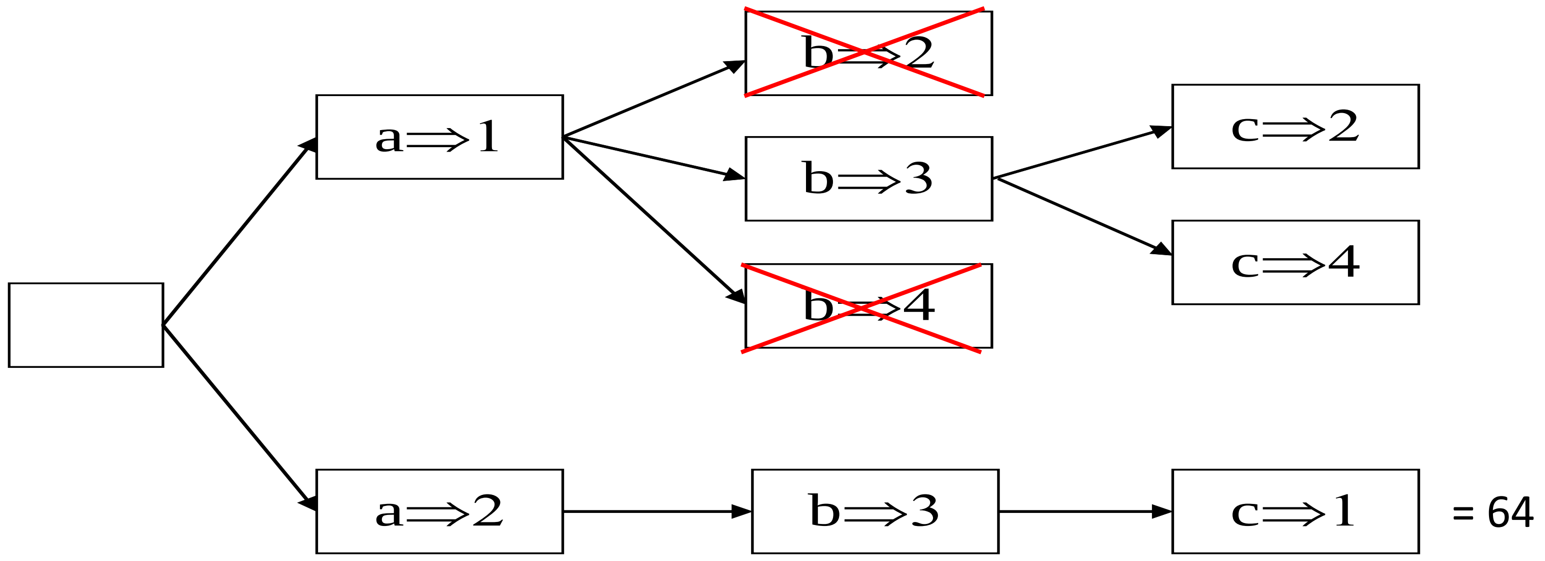


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If b ⇒ 4, the lower bound cost is 11 + 22 + 14 + 19 = 66
- We can eliminate this branch

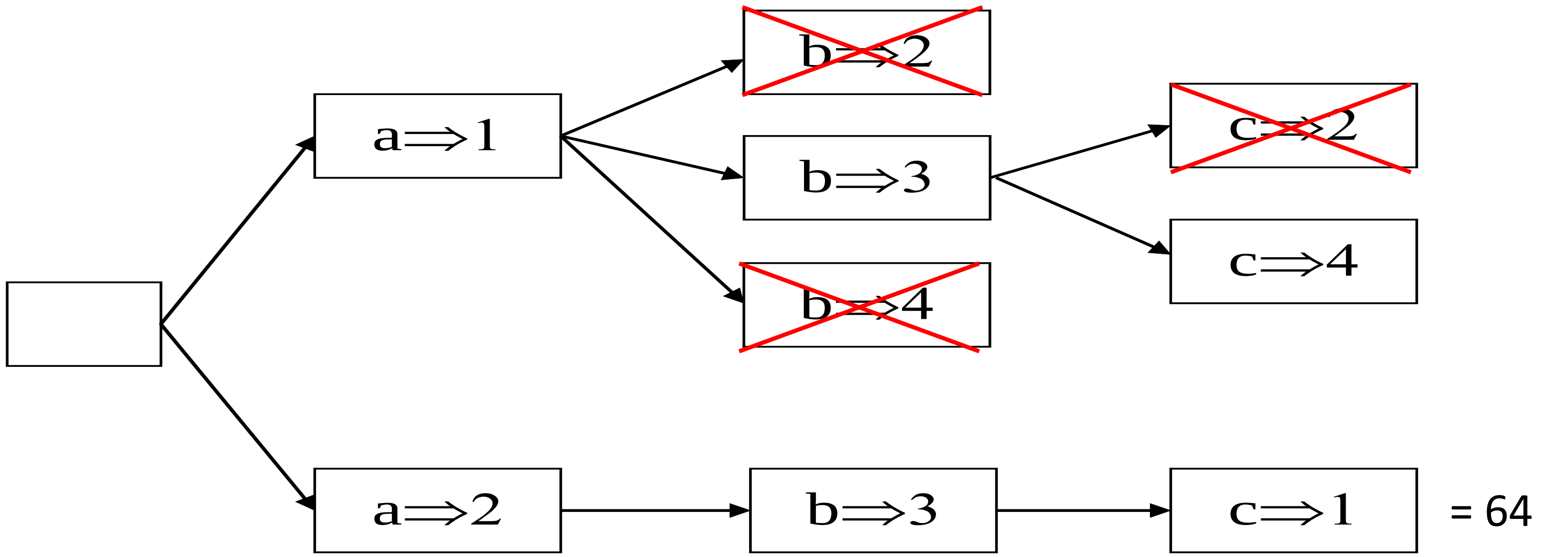
The assignment problem: $58 \leq C \leq 64$

- We still need to expand b $\Rightarrow$ 3



The assignment problem: 58 $\leq$ C $\leq$ 64

- Assign agent c, calculate lower bounds from here

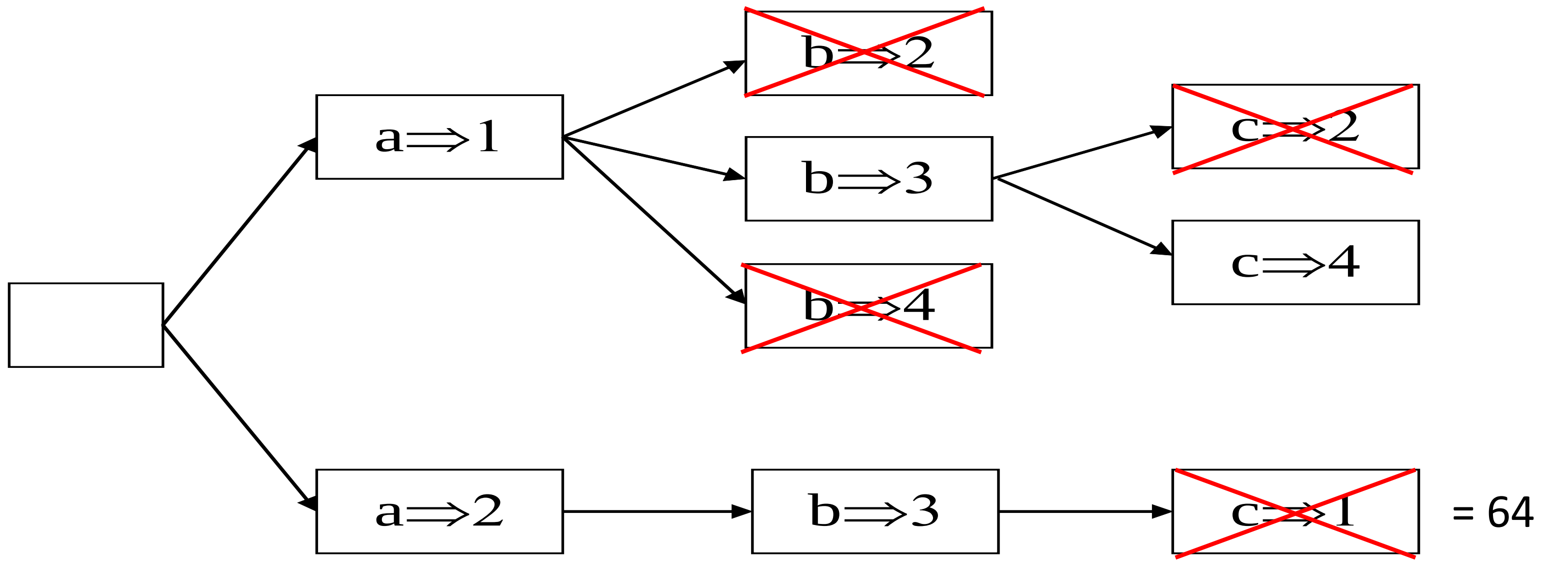


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If c $\Rightarrow$ 2, the lower bound cost is 11 + 13 + 17 + 28 = 69
- We can eliminate this branch

The assignment problem: 58 $\leq$ C $\leq$ 64

- Assign agent c, calculate lower bounds from here

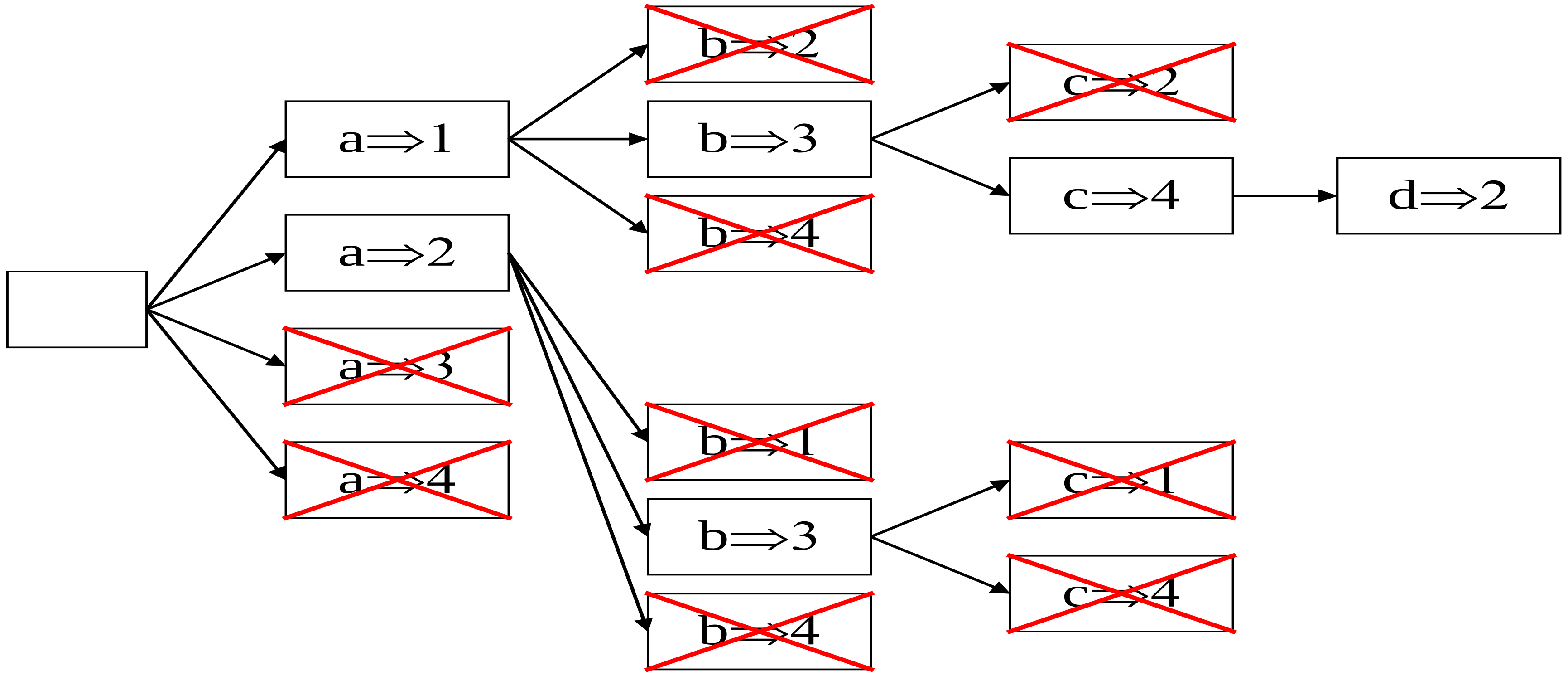


	1	2	3	4
a	11	12	18	40
b	14	15	13	22
c	11	17	19	23
d	17	14	20	28

- If c ⇒ 4, the lower bound cost is 11 + 13 + 23 + 14 = 61
- This gives us a new upper bound and we can eliminate another branch

The assignment problem: 58 ≤ C ≤ 61

- We have now eliminated all branches except one



- Our final solution is a ⇒ 1, b ⇒ 3, c ⇒ 4 and d ⇒ 2; cost = 61

Branch and bound

- The basic idea behind branch and bound algorithms is shown in the preceding example:
  - Explore the sample space of possible solutions.
  - At each stage, prune off any part of the sample space that cannot lead to a solution better than the best one found so far.
  - If we have not found any solution so far we can still prune off any part of the sample space for which the best possible solution is worse than the worst possible solution.
  - That is any branch with a lower bound greater in value than the current upper bound.

Depth-First vs. Breadth-First

- In branch and bound we look at all of the branches and select the most promising at each step, jumping from branch to branch as needed.
- This is called a *breadth-first* strategy.
- Alternatively we could follow the most promising branch down to its end before looking at other branches.
- This is called a *depth-first* strategy.
- (In the example problem it doesn’t matter much which strategy we choose.)

Depth-First vs. Breadth-First

- Several strategies that we have already seen are depth first.
- Greedy:
  - Follow a single branch from root to leaf and then stop.
  - Note that we are also pruning all other branches at each level.
- Backtracking:
  - Follow a single branch as far as possible.
  - Go back up the tree only until you find an unexplored branch.
- Generally, the breadth-first strategy will find a solution more quickly (it will require less of the solution space to be explored) but will involve more housekeeping for the program.

Branch and Bound

- Many difficult problems can be attacked by using a branch and bound technique.
- All that is required are ways to estimate the bounds.
- The closer the estimate is to the real value, the better the strategy will work.
- It should be noted that, although branch and bound usually dramatically reduces the part of the state tree that needs to be examined, the strategy does not guarantee that a given problem will be any easier to solve than pure brute force.
- Generally, however, branch and bound will massively reduce the work to be done.