CSIT113
Problem Solving

Workshop - Week 11

Magic square

A magic square of order 3 is a 3 x 3 table filled with nine distinct integers from 1 to 9 so that the sum of the numbers in each row, column, and two corner-to-corner diagonals is the same.

Find all the magic squares of order 3.

- By hypothesis, the sum of each row, column and two corner-to-corner diagonals will be $(1+2+\ldots+9)/3 = 15$. This is called the magic sum.

- Denoting the numbers in the rows as the following

a	b	c
d	e	f
g	h	i

- We have

$$(d + e + f) + (b + e + h) + (a + e + i) + (g + e + c) =$$

$$= 3e + (a + b + c) + (d + e + f) + (g + h + i) = 3e + 3 \times 15 = 4 \times 15$$

- This implies that $e = 5$ – the central cell.
- It remains to arrange pairs (1,9), (2,8), (3,7) and (4,6) around it

- Now we put (1,9) first.
- How many ways to put them in the table?
- Due to the symmetry of the table, there are only two qualitatively different ways to put 1 and hence 9: in the table's corner and not in the table's corner

1		
	5	
		9

	1	
	5	
	9	

- Consider the first one. How can you put other pairs into the table?

1		
	5	
		9

- If we put a number less than 5 in the upper right corner, we will not be able to have the magic sum of 15 in the first row, and if we put there a number larger than 5, we will have the same problem with the last column.

1		4
	5	
		9

1		6
	5	
		9

- So this first arrangement does not work.

- Consider the second one

	1	
	5	
	9	

- There are also three other ways to put 1 and 9 in the same row or the same column with 5

9	5	1

	9	
	5	
	1	

1	5	9

- How to put the pair (6,8) into the table?

	1	
	5	
	9	

- The line (row or column) containing 1 must then be filled with 6 and 8, which can be done in two ways.

6	1	8
	5	
	9	

8	1	6
	5	
	9	

- The numbers for the remaining cells are now uniquely determined

6	1	8
	5	
	9	4

6	1	8
	5	3
	9	4

6	1	8
	5	3
2	9	4

6	1	8
7	5	3
2	9	4

- Similarly, we also have

8	1	6
3	5	7
4	9	2

- As conclusion, we have totally 8 magic squares of order 3

6	1	8
7	5	3
2	9	4

2	7	6
9	5	1
4	3	8

4	9	2
3	5	7
8	1	6

8	3	4
1	5	9
6	7	2

8	1	6
3	5	7
4	9	2

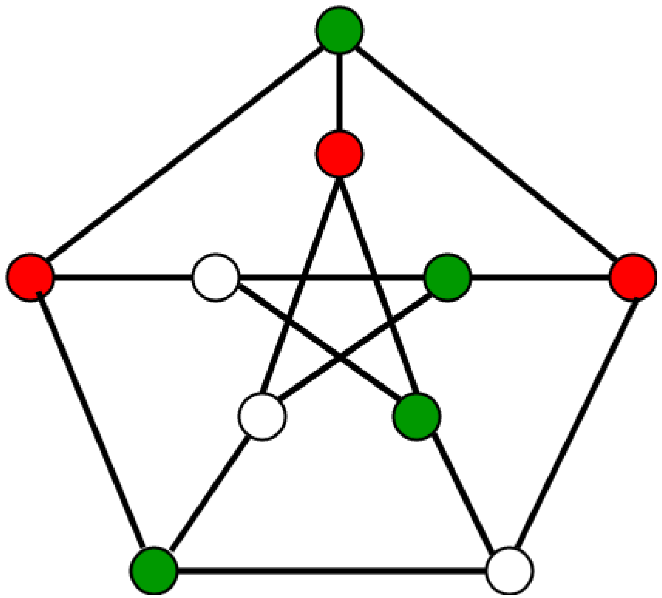
4	3	8
9	5	1
2	7	6

2	9	4
7	5	3
6	1	8

6	7	2
1	5	9
8	3	4

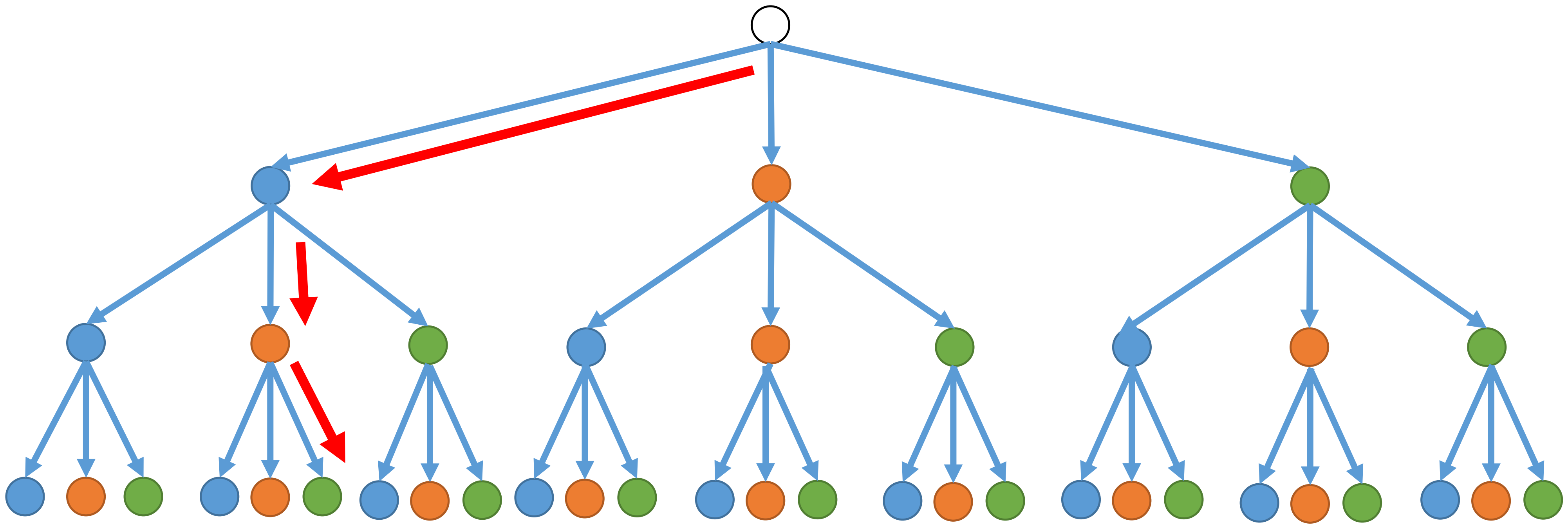
Graph coloring

- Given an undirected graph and a number m, determine if the graph can be coloured with at most m colours such that no two adjacent vertices of the graph are colored with the same color.
- Here is an example of a graph that can be coloured with 3 different colours. (n = 10 is the number of vertices, m = 3)



Case n = 3, m = 3

- We can represent a tree for search space

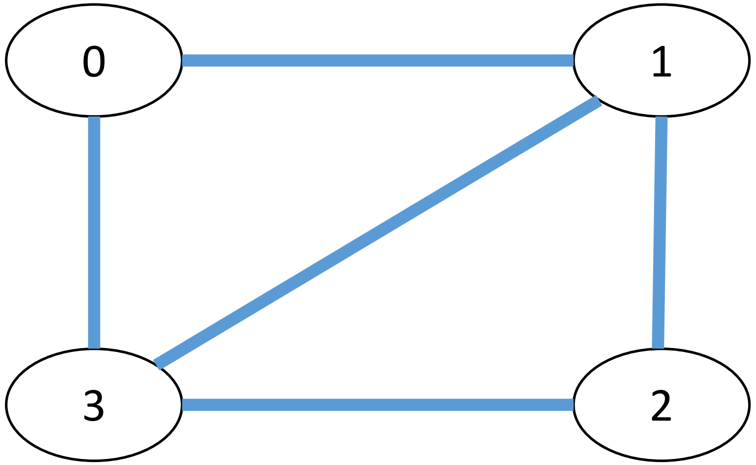


Graph coloring - backtracking

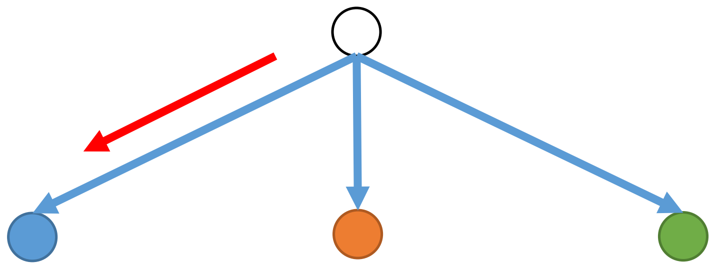
- Assign colors one by one to different vertices, starting from the vertex 0.
- Before assigning a color, check for safety by considering already assigned colors to the adjacent vertices i.e check if the adjacent vertices have the same color or not.
- If there is any color assignment that does not violate the conditions, mark the color assignment as part of the solution.
- If no assignment of color is possible then backtrack

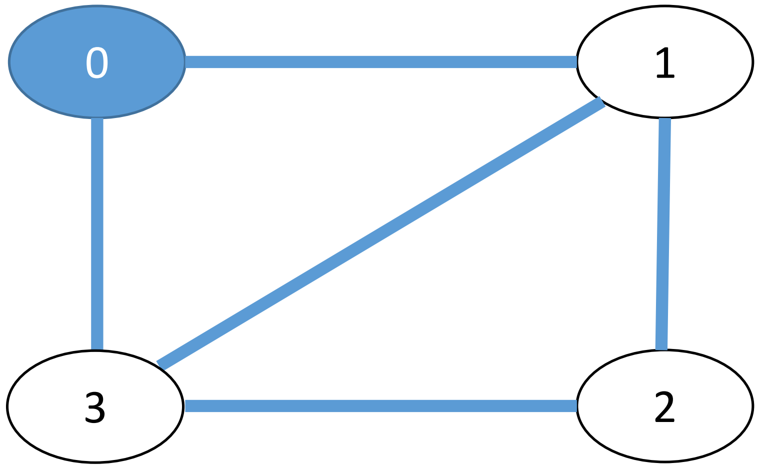
Example n = 4, m = 3

- Coloring the following graph with 3 colors using backtracking method

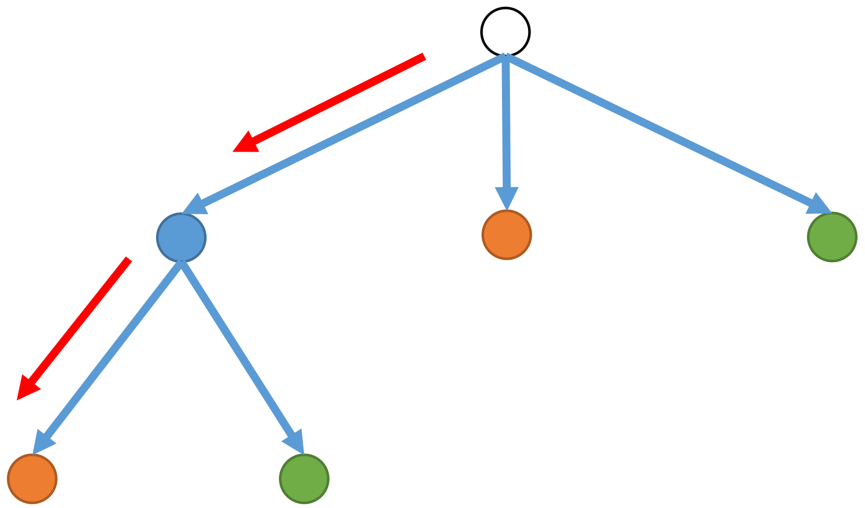


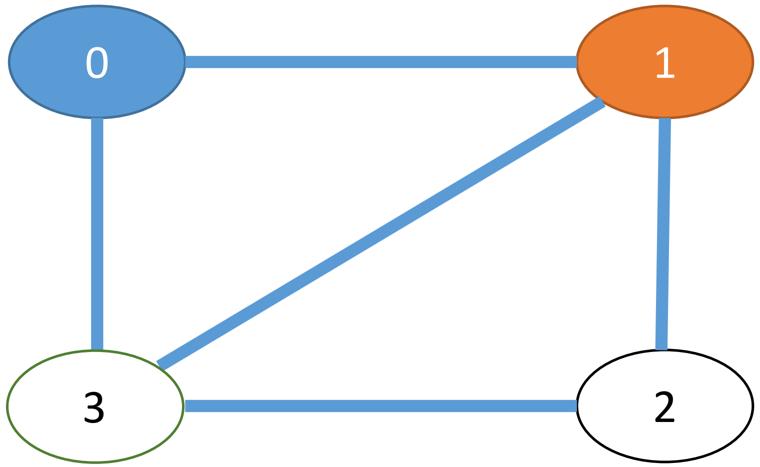
Step 1: Color one vertex with one color, say 0 with blue



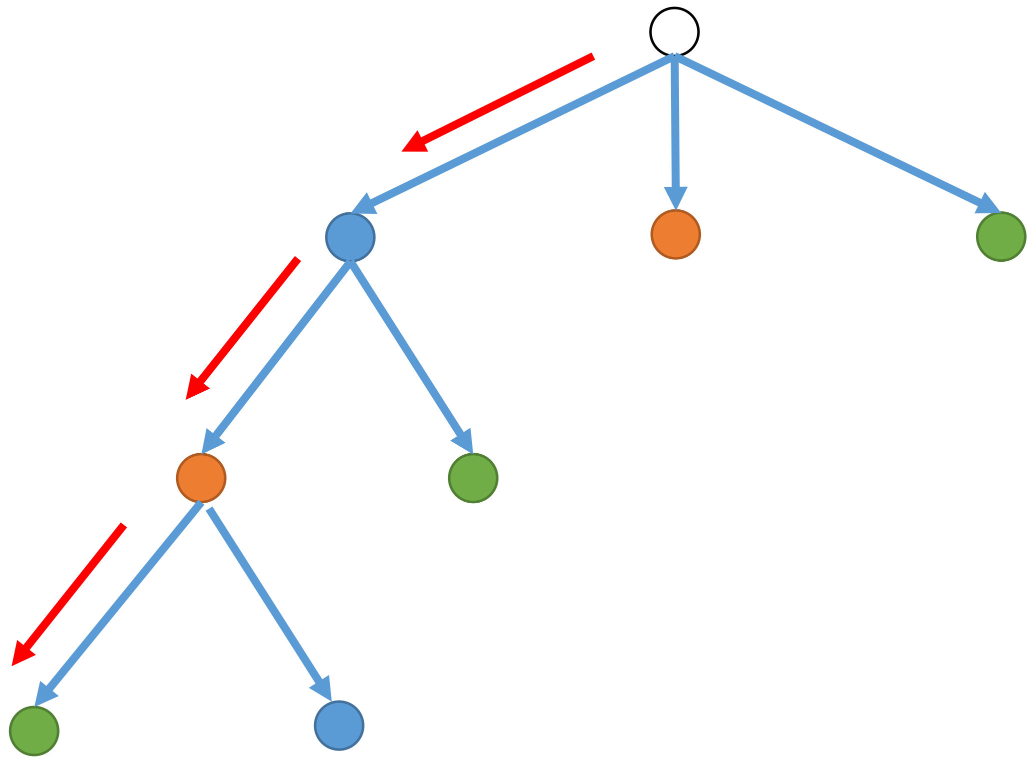


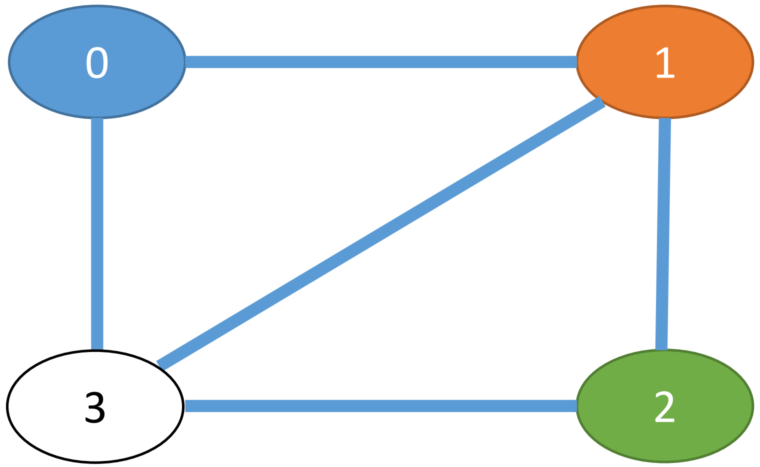
Step 2: Color next adjacent vertex 1 with different color



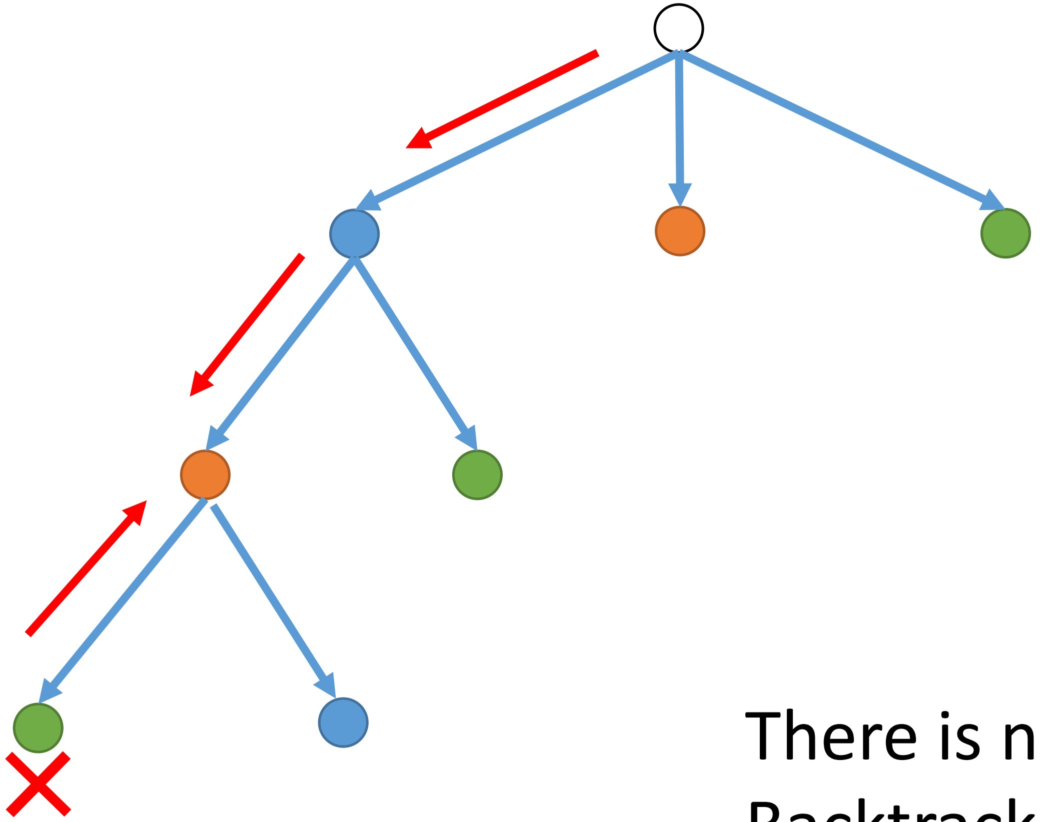


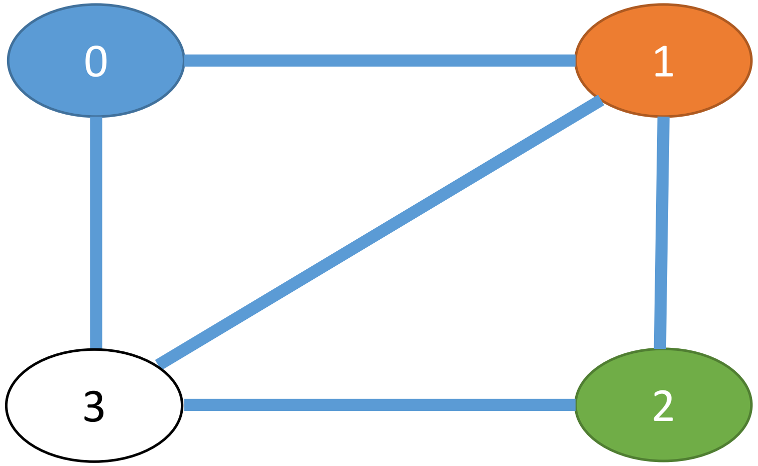
Step 3: Color vertex 2 with different color from adjacent vertices (1, 3)





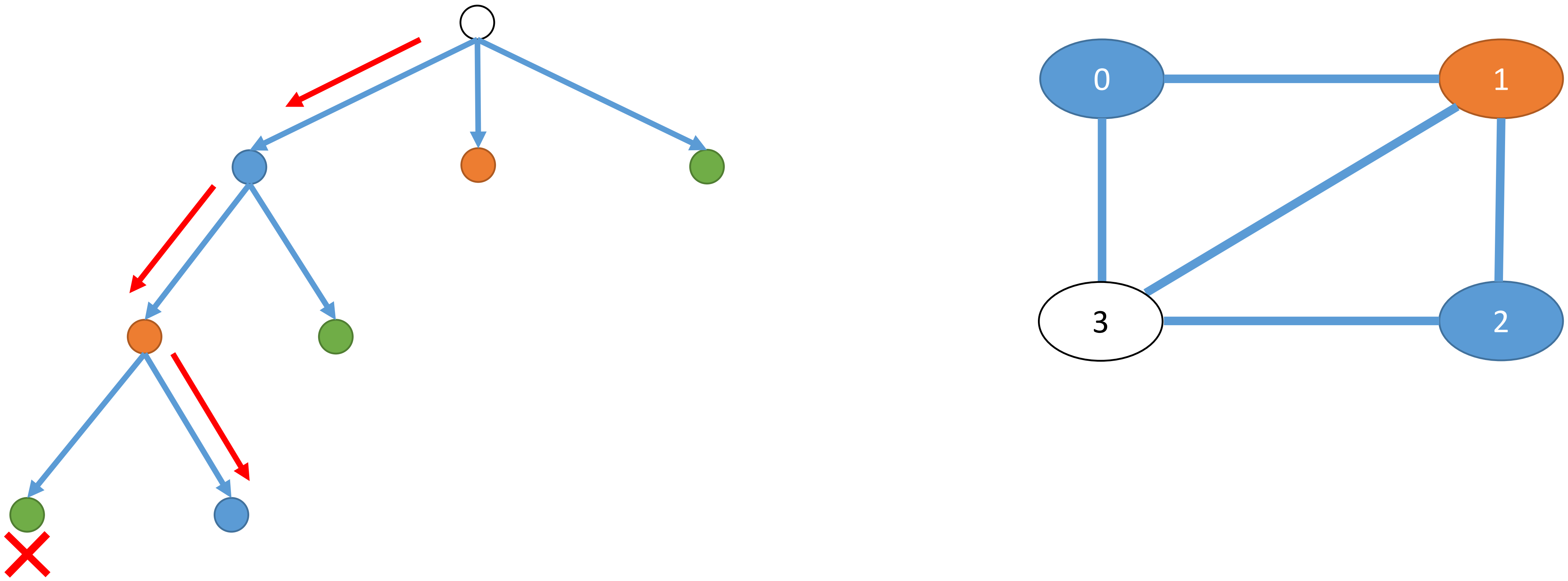
Step 4: backtrack



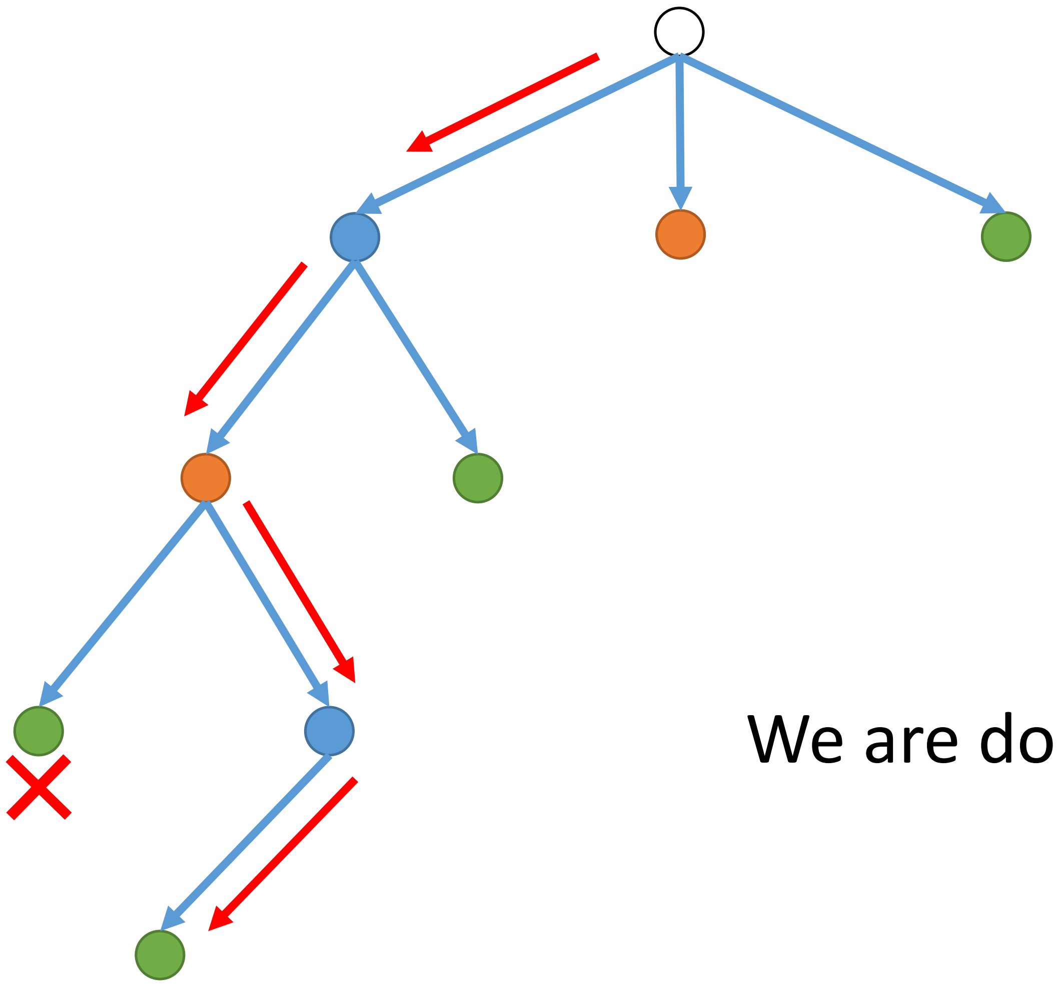


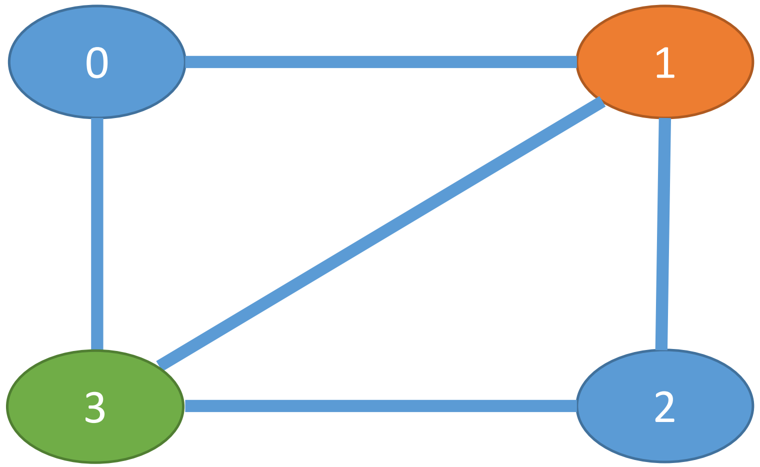
There is no possibility for node 3.
Backtrack

Step 5: color vertex 2 with a different safe color



Step 6: color vertex 3 with a safe color, which is different from vertices 0, 1, 2





We are done

Subset sum problem

- Subset Sum Problem: Give a set T = $\{t_1, \ldots, t_n\}$ of positive integers and an integer M. Find a subset S of T such that $\sum_{x \in S} x = M$
- Problem of today: Let T = {4,7,6,3,1} and M = 10. Find a subset S of T such that $\sum_{x \in S} x = 10$

Rephase

- Let $(x_1,\ldots, x_n)$ be a representation of a potential solution such that $x_i = 1$ if $t_i \in S$ and $x_i = 0$ if t_i is not in S.
- Consider T = {4,7,6,3,1} and M = 10.
- (1,0,1,0,0) gives 4 +6 = 10, so it is a valid solution
- (0,0,1,1,1) gives 6 + 3 + 1 = 10, so it is a valid solution
- (1,0,0,1,1) gives 4 + 3 + 1 = 8, so it is not a valid solution

Backtracking strategy

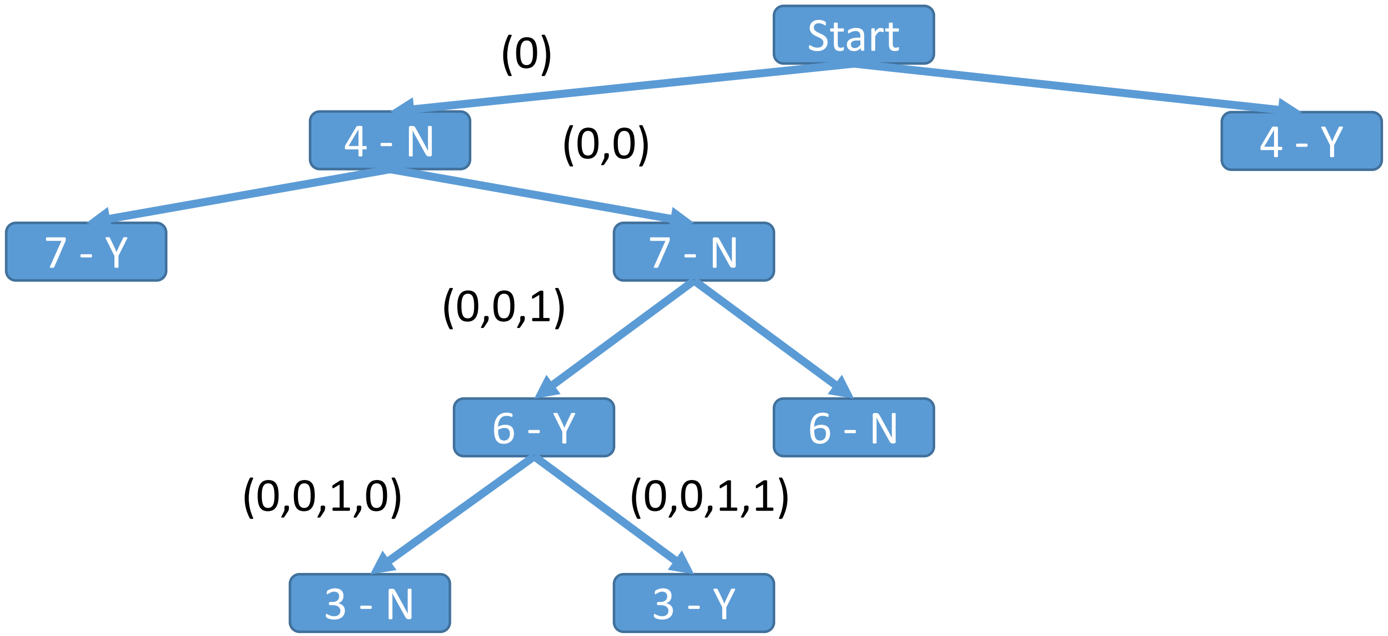
- Start with S = {}
- Given a partial list (of length i-1) of elements used, check if element i can be used
- Backtrack or prune as needed
- Consider a partial solution X = $(x_1,\ldots, x_k)$. We will prune X if

$$\sum_{i=1}^{k} x_i t_i > M$$

- X can be pruned if

$$\sum_{i=1}^{k} x_i t_i + \sum_{i=k+1}^{n} t_i < M$$

Implementation T = {4,7,6,3,1}, M = 10

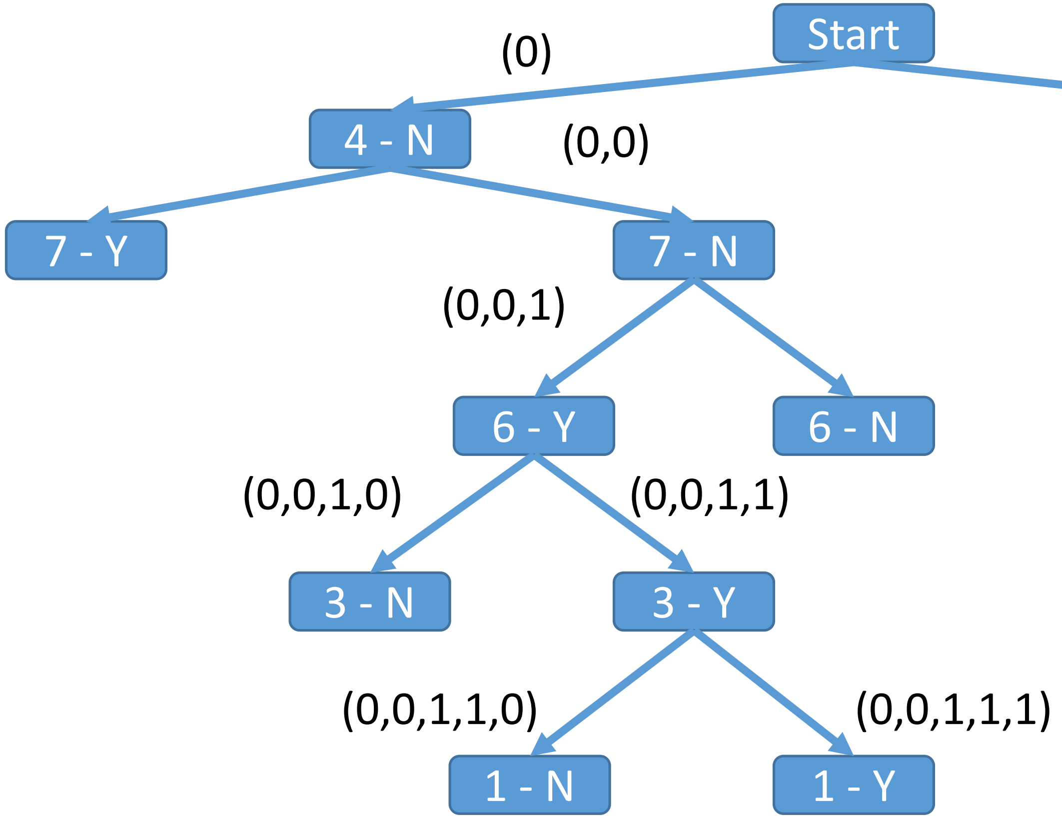


If we do not choose 3 then we get a partial solution (0,0,1,0). We now check:

- 1 x 6 = 6 < 10
- 1x 6 + 1 = 7 < 10

So we prune this. We choose 3, then we have partial solution (0,0,1,1)

Implementation T = {4,7,6,3,1}, M = 10



- If we do not choose 1, then we get solution (0,0,1,1,0).
  We get 6 + 3 = 9 < 10, this does not lead to a solution.
  So we prune.

- If we choose 1, then we get a solution (0,0,1,1,1)
  We get 1x6 + 1x3 + 1x1 = 10, which actually a solution.

- You can then continue.