Week 5 - Practice

Problem 1

Prove by induction the following formula

$$2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + (n+1) \times 2^n = n \times 2^{n+1}$$

Solution

- We write LHS = $2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + (n+1) \times 2^n$ and RHS = $n \times 2^{n+1}$
- Case n=1, we have LHS = 2x2 = 4, and RHS = 1x$2^{1+1} = 2$. Hence LHS = RHS.
- Assume it holds for the case n, i.e.,
$$2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + (n+1) \times 2^n = n \times 2^{n+1}$$
- We need to prove for the case n+1, i.e., we need to show that
$$2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + (n+1) \times 2^n + (n+2) \times 2^{n+1}$$
$$= (n+1) \times 2^{n+2}$$

- We have

LHS = $2 \times 2 + 3 \times 2^2 + 4 \times 2^3 + \cdots + (n+1) \times 2^n + (n+2) \times 2^{n+1}$

= $n \times 2^{n+1} + (n+2) \times 2^{n+1}$ (by hypothesis)

= $(n + n + 2) \times 2^{n+1}$

= $2\mathrm{x}(n+1) \times 2^{n+1} = (n+1) \times 2^{n+2}$ = RHS , as desired

Problem 2

Using the method in Workshop Week 5, find the formula of the following sum:

$$1^3 + 2^3 + \cdots + n^3$$

Solution

Set S(n) = $1^3 + 2^3 + \cdots + n^3$ and P(n) = a + bn + cn^2 + dn^3 + en^4

S(0) = 0 and P(0) = a and so a = 0

S(1) = 1 and P(1) = b + c + d + e. Hence

b + c + d + e = 1 (1)

S(2) = 1 + 2^3= 9, and P(2) = 2b + 4c + 8d + 16e. Hence

2b + 4c + 8d + 16e = 9 (2)

S(3) = $1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$, and P(3) = 3b + 9c + 27d + 81e

Hence 3b + 9c + 27d + 81e = 36 (3)

- S(4) = $1^3 + 2^3 + 3^3 + 4^3$ = 100, and P(4) = 4b + 16c + 64d + 256e.
  Hence 4b + 16c + 64d + 256e = 100 (4)
- It follows from (1),(2),(3),(4) that b = 0, c = ¼, d = ½ and e = ¼
- Hence
- $$1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2 + \frac{1}{2}n^3 + \frac{1}{4}n^4$$
  $$= \frac{1}{4}n^2(1 + 2n + n^2)$$
  $$= \frac{1}{4}n^2(n+1)^2$$