Part A

On Knights and Knaves Island, a native can only be a knight or a knave. Both knights and knaves know everything. Knights always speak truth, knaves always lie. You meet two islanders, Alice and Bob, who make the following statements:

Alice: "One of us is a knight and one of us is a knave."

Bob: "That's right."

Answer:

a) Explain how you can identify Alice and Bob as a knight or a knave, using brute force with a truth table for Alice and Bob.

- To solve by brute force we need to investigate all possible cases for Alice and Bob.

Let A: Alice is a knight

Let B: Bob is a knight

The two statements in the question imply:

$A \Leftrightarrow \neg A \lor \neg B$

$B \Leftrightarrow A$

Combining the two equations above we get, $B \Leftrightarrow (A \Leftrightarrow \neg A \lor \neg B)$

A truth table can be drawn for the equations above to demonstrate the brute force method.

A	B	$\neg A$	$\neg B$	$(\neg A \lor \neg B)$	$(A \Leftrightarrow \neg A \lor \neg B)$	$B \Leftrightarrow (A \Leftrightarrow \neg A \lor \neg B)$
T	T	F	F	F	F	F
T	F	F	T	T	T	F
F	T	T	F	T	F	F
F	F	T	T	T	F	T

From the above truth table, it can be implied that A is false, which means that Alice is a knave and B is false which means Bob is a knave.

b) Explain how you can identify Alice and Bob, with a sequence of logical conclusions from the statements (calculation logic). Do not use brute force by taking different assumptions and then eliminating them. (You may / may not use symbols)

- Using the same equations as questions above, we know.

$B \Leftrightarrow (A \Leftrightarrow \neg A \lor \neg B)$

We also know that.

$X \Leftrightarrow Y \equiv (X \land Y) \lor (\neg X \land \neg Y)$

Given... $A \Leftrightarrow \neg A \vee \neg B$

$(A \wedge (\neg A \vee \neg B)) \vee (\neg A \wedge \neg (\neg A \vee \neg B))$

$[(A \wedge \neg A) \vee (A \wedge \neg B)] \vee [\neg A \wedge (A \wedge B)]$

$(A \wedge \neg B) \vee [(A \wedge \neg A) \wedge (\neg A \wedge B)]$

$(A \wedge \neg B) \vee [0 \wedge (\neg A \wedge B)]$

$(A \wedge \neg B)$

Given... $B \Leftrightarrow (A \Leftrightarrow \neg A \vee \neg B)$

$B \Leftrightarrow (A \wedge \neg B)$

$(B \wedge (A \wedge \neg B)) \vee (\neg B \wedge \neg(A \vee \neg B))$

$[(B \wedge A) \wedge (B \wedge \neg B)] \vee [(\neg B \wedge \neg A) \vee (\neg B \wedge B)]$

$[(B \wedge A) \wedge 0] \vee [(\neg B \wedge \neg A) \vee 0]$

$\neg B \wedge \neg A$

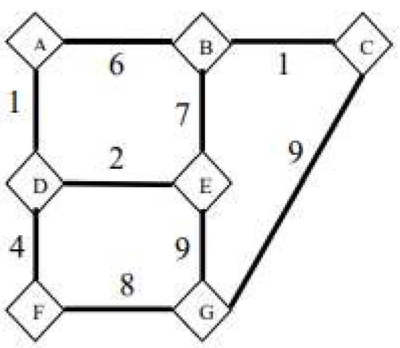
From the above conclusion it can be implied that A is false, which means that Alice is a knave and B is false which means Bob is a knave.

c) Carol, another person you meet on the Island, says “Bob will agree that I am a knight”. What is Carol? Explain you reasoning.

- According to Carol’s statement “Bob will agree that I am a knight”. From our assumption that Alice is a knave and Bob is a knave, we know that Bob always tells a lie. Therefore, if Bob says that Carol is a knight, he is lying, and Carol is instead a knave.

<u>Part B</u>

Questions a to f all refer to the following graph. Yes/No answers without an explanation carries no mark.



<u>Answer:</u>

a) Is this graph weighted? Explain your answer.

- Yes, the graph is weighted as the all the edges connected to any two nodes has a numeric value associated with it.

b) Is this graph complete? Explain your answer.

- No, the graph is not complete. A complete graph has each pair of vertices connected by at least one unique edge. All the vertices in the graph above are not connect by an edge.

c) Is this graph acyclic? Explain your answer.

- No, it is not acyclic as there are many cycles formed in the graph, such as F->G->E->D->F. An acyclic graph is one which has no cycles formed in it.

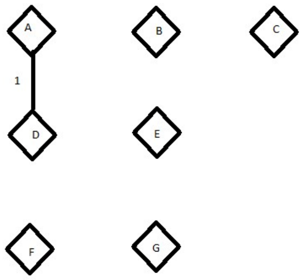
d) Is this graph directed? Explain your answer.

- No, it is not directed graph, as in a directed graph the edges have a direction attached to them. The graph above does not have any direction attached to them.

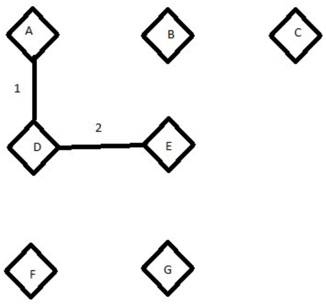
e) Construct a minimal spanning tree using the node-at-a-time algorithm, starting at node A.

- Total weight of minimum spanning tree = 22, spanning tree shown in Figure 1 below.

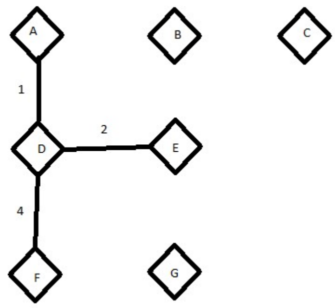
- 1, (A-D)



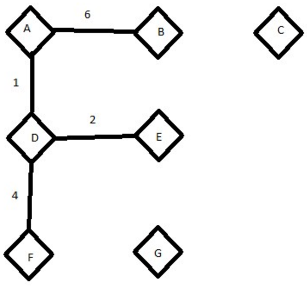
2 , (D-E)



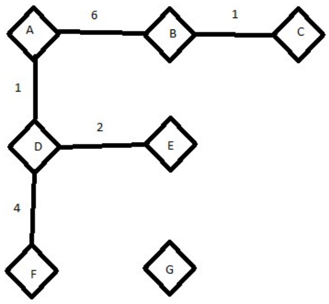
3, (D-F)



4, (A-B)



5, (B-C)



6, (F-G)

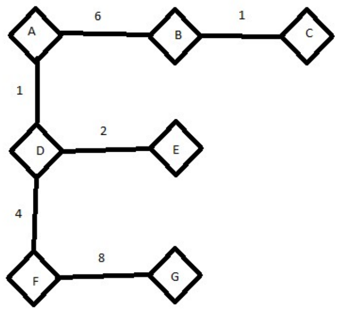
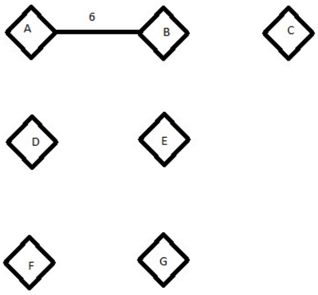


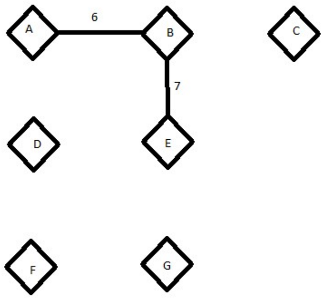
Figure 1: Minimum Spanning Tree

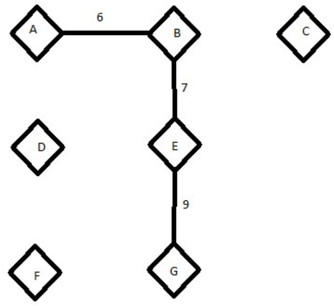
f) Construct a maximal spanning tree (the tree with maximum weight) using the edge at-a-time algorithm, starting at node A.

- Total weight of maximal spanning tree = 43, spanning tree shown in Figure 2 below.

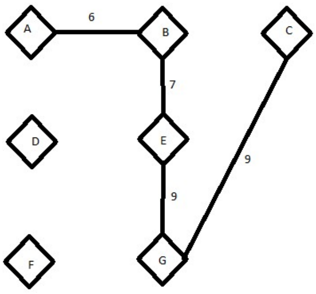
- 1, (A-B) 2, (B-E) 3, (E-G)

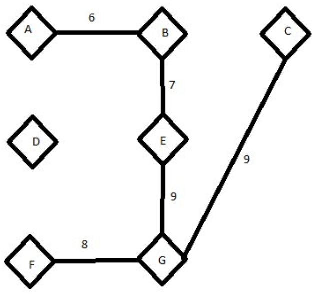






4, (G-C) 5, (G-F) 6, (F-D)





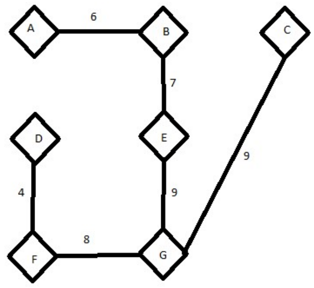


Figure 2: Maximal Spanning Tree

<u>Part C</u>

A factory produces widgets. It has got several orders; each order is about a single widget. For each order, we know the following information:

i) The profit that this widget will generate.

ii) The deadline – the last day, until the end of which, the customer will accept delivery.

The factory can make exactly one widget per day. The following table shows all the orders for a 5-day period.

Order#	1	2	3	4	5	6	7	8	9	10
Profit	42	59	54	95	84	66	33	80	64	75
Deadline	2	5	3	4	5	3	2	4	1	2

Answer:

a) Explain a strategy that can be used, for all similar problems with more or less orders, so as to maximize profit? Provide a clear and concrete explanation of your strategy.

- We would ideally use the greedy approach to solve similar problems to this one, for a problem with n orders, the steps for an algorithm would be:

1) Sort all orders in decreasing order of profit.

2) Initialize the result sequence with the first order of the new sorted list.

3) For the remaining n-1 orders, if the current orders fit in the current result sequence without missing the deadline, add the current job to the result sequence, else move on to the next order.

b) By applying the strategy above, show the sequence in which orders should be chosen each day, for the company, to maximize the profit? What is the maximum profit the company can make with the given orders?

- Using the above-mentioned strategy, we obtained the following results.

For Day 1, Result Sequence = 4(95)

Order#	4	5	8	10	6	9	2	3	1	7
Profit	95	84	80	75	66	64	59	54	42	33
Deadline	4	5	4	2	3	1	5	3	2	2

For Day 2, Result Sequence = 4(95) + 5(84)

Order#	4	5	8	10	6	9	2	3	1	7
Profit	95	84	80	75	66	64	59	54	42	33
Deadline	4	5	4	2	3	1	5	3	2	2

For Day 3, Result Sequence = 4(95) + 5(84) + 8(80)

Order#	4	5	8	10	6	9	2	3	1	7
Profit	95	84	80	75	66	64	59	54	42	33
Deadline	4	5	4	2	3	1	5	3	2	2

For Day 4, Result Sequence = 4(95) + 5(84) + 8(80) + 2(59)

Order#	4	5	8	10	6	9	2	3	1	7
Profit	95	84	80	75	66	64	59	54	42	33
Deadline	4	5	4	2	3	1	5	3	2	2

For Day 5, No other order fits, so no order processed.
For Day 5, Result Sequence = 4(95) + 5(84) + 8(80) + 2(59) + 0

The Maximum Profit the company can make by utilizing the Greedy Algorithm is: 318

Part D

A factory has five workers, Anne, Bob, Carol, Dave and Ethan. It also has 5 tasks which must be done. Each worker does the tasks with a different efficiency. Apply the branch and bound strategy to allocate the tasks with minimum overall cost, when each task must be completed by a different worker.

The cost for each worker to do the tasks is given in the following table:

	Task 1	Task 2	Task 3	Task 4	Task 5
Anne	\$4.00	\$8.00	\$8.00	\$3.00	\$4.00
Bob	\$9.00	\$5.00	\$5.00	\$2.00	\$7.00
Carol	\$4.00	\$2.00	\$4.00	\$1.00	\$3.00
Dave	\$7.00	\$9.00	\$6.00	\$5.00	\$8.00
Ethan	\$3.00	\$6.00	\$4.00	\$4.00	\$5.00

Answer:

a) Describe a suitable method to calculate the lower bound value for all similar problems, where the total number of workers and the total number of tasks are equal?

- To calculate the lower bound value for all similar problems with the same number of workers and tasks, we take each worker, and choose a task with the minimum cost from the list of unassigned tasks and then add them together.

b) Describe a method to calculate a suitable upper bound for similar problems?

- Similar to the lower bound value for all similar problems with the same number of workers and tasks, the upper bound value can be calculated by taking each worker, and choosing a task with the maximum cost from the list of unassigned tasks and then adding them together.

c) Use the branch and bound strategy to find the most cost-efficient allocation of tasks to workers. Show the details of the expansion and pruning (branch and bound) of the decision tree along with reasons.

- Using the brand and bound strategy for finding the lower bound value mentioned above, a decision tree was developed as shown in Figure 3 below. The most cost-efficient allocation of tasks that was derived from the tree is shown below.

- Anne is assigned to Task 5 would cost \$4
- Bob is assigned to Task 4 would cost \$2
- Carol is assigned to Task 2 would cost \$2
- Dave is assigned to Task 3 would cost \$6
- Ethan is assigned to Task 1 would cost \$3

-In the tree below the branch with the lowest LBV at every level was expanded and marked as yellow and the other branches were pruned.

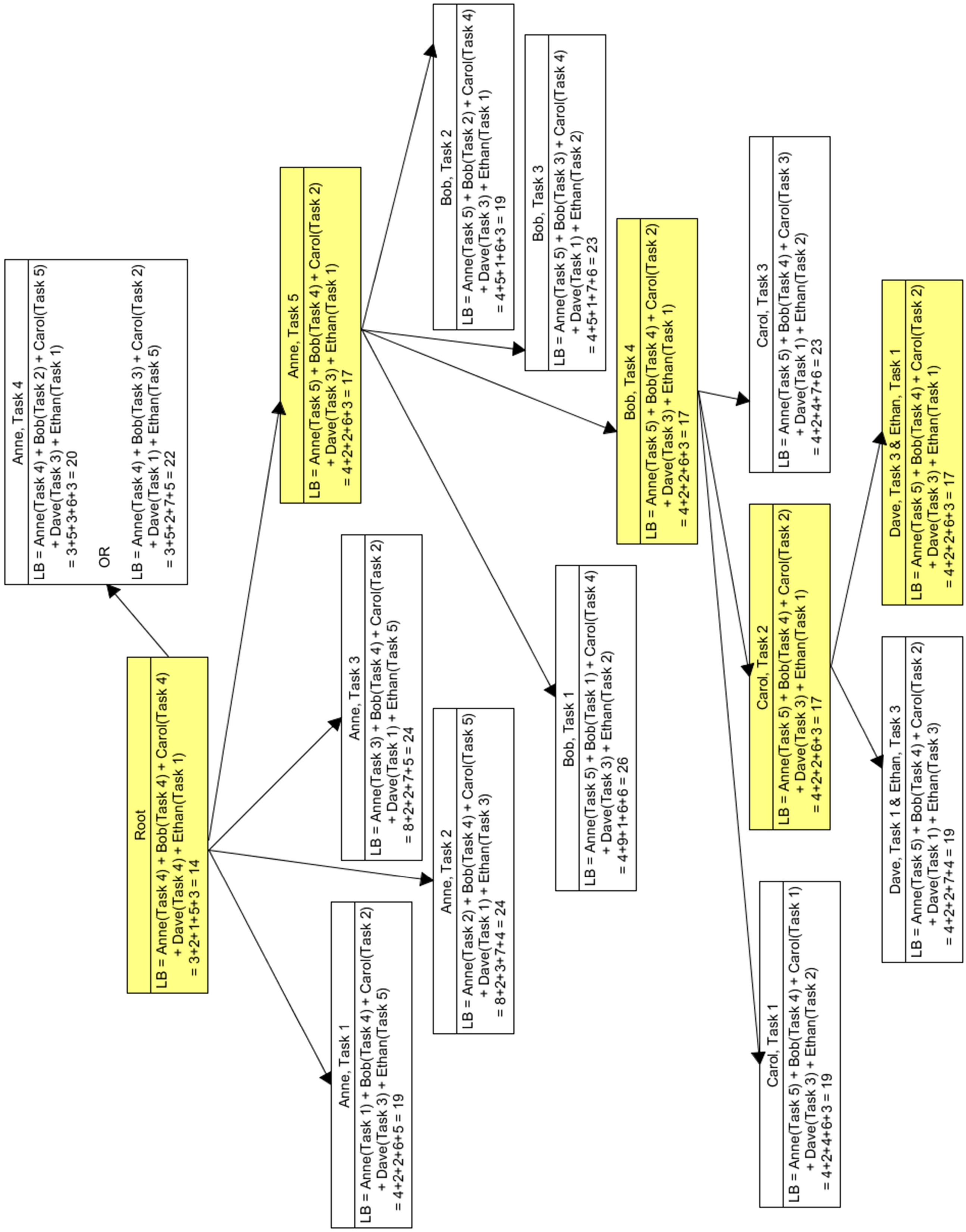
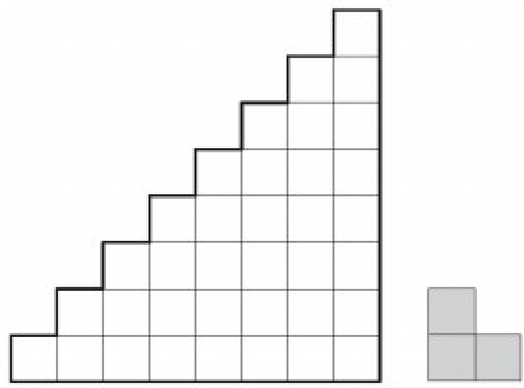


Figure 3: Branch and Bound Decision Tree

Part E

The Figure below shows a staircase region. The breadth and height of the region can be calculated using the small squares which fill it. For the below region, the breadth and height can be calculated as the sum of a side of 8 small squares. Let us call this staircase region, S8 (i.e., n = 8).



Now, let us consider the general case. Find all values of n, where n>1, where a staircase region Sn can be completely tiled with the given triomino. The triomino is shown to the right (you may rotate it).

Prove your finding using any appropriate strategy. Use clear, concrete arguments applicable to general cases.

Answer:

Let the area of a small square be A

If there are n steps, then the number of squares present for n steps will be

$\sum n = 1+2+3+4+\ldots +n$

The sum can be written as

$\sum n = n(n+1)/2$

The total area of the staircase region is

$\sum n \times A = (n(n+1)/2) \times A$

Given that the area of a triomino is 3A

The staircase region with n steps can be completely covered by triominos if $\sum n \times A$ is a multiple of 3A.

This means that $\sum n$ must be a multiple of 3.

Therefore, n(n+1)/2 must be a multiple of 3.

Therefore, either n or n+1 must be a multiple of 3.

The possible values of n in that case are 2,3,5,6,8,9,11,12…….

Furthermore, the triominos cannot be separated into its separate squares and there are two squares on any side of the triomino. There are n squares at the sides of the staircase region, so to fill the staircase completely at the sides, n should be a multiple of 2.

The possible values of n in that case are 2,4,6,8,10,12,14,16……..

Given the two conditions that have been derived:

1) n or n+1 must be a multiple of 3.
2) n must be a multiple of 2.

We can conclude values of n, where a staircase region Sn can be completely tiled with the given triomino is n = 2,6,8,12,14,….

Therefore, starting from 2 all the possible values for n can be found by adding 4 and 2 for successive terms respectively.