CSIT113
Problem Solving

Workshop

Week 3

Jugs of water…

- You have two jugs, one capable of holding 3L, the other capable of holding 5L.
- What volumes of liquid can you obtain just using those two jugs?
- What difference does it make if we have a large reservoir that we can store liquid in?
- What strategy is appropriate here?
- What if we change the numbers?

Jugs of water…

- We can obtain any integer amount of water between 1 and 8 litres with just the jugs.
- Use the notation
  - +3 for “fill the 3L jug”
  - +5 for “fill the 5 litre jug”
  - -3 for “empty the 3L jug”
  - -5 for “empty the 5 litre jug”
  - 3>5 for “pour the 3 litre jug into the 5 litre jug”
  - 5>3 for “pour the 5 litre jug into the 3 litre jug”

Jugs of water…

• +3, 3>5, +3, 3>5	1 litre in the 3 litre jug.
• +5, 5>3	2 litres in the 5 litre jug
• +3	3 litres in the 3 litre jug
• +5, 5>3, -3, 5>3, +5, 5>3	4 litres in the 5 litre jug
• +5	5 litres in the 5 litre jug
• +3, 3>5, +3	6 litres across the two jugs
• +5, 5>3, -3, 5>3, +5	7 litres across the two jugs
• +3, +5	8 litres across the two jugs

Jugs of water…

- If we have a storage reservoir we can obtain any integer volume of water as follows:
- Any volume N can be expressed as $5i+j$ where $0 \le j < 5$. Therefore we fill the reservoir i times with the 5 litre jug and then add between 0 and 4 litres as required, obtained using the procedures above.
- Alternately N can be expressed as $3k+l$ where $0 \le l < 3$ and we use the 3 litre jug to fill the reservoir.
- This problem is probably best solved by “informed trial and error” which is essentially brute force.

Jugs of water…

- Why the problem above can be solved?
- What if we have jugs with capacities, say 4 and 6?
- We can only obtain volume which are multiples of their GCD (in this case multiples of 2)
- The jugs do not need capacities which are prime numbers – just that they be relatively prime.
- Thus capacities of 4 and 9 litres will allow all integer volumes to be produced.

Stamps…

- A post office has lots of 3c and 5c stamps.
  - (Okay it’s an old post office!).
- What postage values can the post office handle?
- How is this different from the water?
- What strategy is appropriate here?
- Can you find a pattern?
  - Can you prove something?
- What if we change the numbers here?

Stamps…

- Note that we cannot subtract stamps, the result we got with the jugs is not possible here.
- Clearly, there is no way to get values of 1c and 2c.
- 7c is also unobtainable
- What are possibilities?

Stamps…

- 3c 1 3c stamp
- 5c 1 5c stamp
- 6c 2 3c stamp
- 8c 1 3c + 1 5c stamp
- 9c 3 3c stamp
- 10c 2 5c stamp
- What about stamps with greater than 10c? e.g., 11c,12c,13c….

Stamps…

- Note we now have 3 consecutive values 8, 9, and 10c
- We can add an appropriate number of 3c stamps to one of these to obtain any value > 10c.
- For example: 11c = 1 3c stamp + 1 8c stamp
- Similarly, once we have 5 consecutive values, we could add an appropriate number of 5c stamps.
- The observation about the stamp values needing to be relatively prime holds in this example as well.

Weights…

- You have an old fashioned balance scale.
- I also have 5 weights, each an integer number of kg.
- I need to be able to weigh objects to the nearest kg.
- What are the most useful weights I can have?
- What is the largest object I can weigh?

Weights…

- Where can you put weights on the pan?

Left Pan	1	3	3	3+1	9	9	9+1	9	9	9+1	9+3	9+3
Right Pan	0	1	0	0	3+1	3	3	1	0	0	1	0
L-R	1	2	3	4	5	6	7	8	9	10	11	12

- The answer is: 1, 3, 8, 27 and 81, for a maximum of 121kg
- The values correspond to *ternary* numbers where the columns are powers of 3. For more information, please refer to here https://en.wikipedia.org/wiki/Ternary_numeral_system

Row and Column Exchanges

Can one transform the left table into the right table by exchanging its rows and columns?

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

→

12	10	11	9
16	14	5	13
8	6	7	15
4	2	3	1

Row and Column Exchanges

- The answer is “NO”
- Row exchanges preserve the numbers in rows, column exchanges preserve the numbers in columns.
- This is not the case for the given tables: for example, 5 and 6 are in the same row in the initial table but in the different rows in the target table.

Remaining Number

The first 50 natural numbers—1, 2, . . . , 50—are written on a board. You have to apply the following operation 49 times: select two of the numbers on the board, *a* and *b,* write the absolute value of their difference $|a - b|$ on the board, and then erase both *a* and *b.* Determine all possible values of the remaining number that can be obtained in this manner.

Remaining Number

- Answer: Any positive odd integer less than 50 can be obtained.
- What is an invariant here?
- Hint: think of parity

Remaining Number

- The initial sum of the numbers on the board is equal to $1+2+...+50 = 1275$, which is odd
- Using the formula $1+2+....+n = n(n+1)/2$
- Each operation of replacing a and b by $|a - b|$, where $a \le b$ without loss of generality, decreases the sum by $2a$:
- $S_{new} = S_{old} - a - b + |a - b| = S_{old} - b - a + b - a = S_{old} - 2a$.
- This immediately implies that the new sum must be odd if the old sum was odd.
- Therefore no even number can result from repeated applications of such an operation starting with 1275

Remaining Number

- Further, all the numbers on the board are always nonnegative. They are also less than or equal to 50, since $|a - b|$ is always less than or equal to the maximum of a and b for nonnegative a and b.
- We will show now that any odd integer from 1 to 49, inclusive, can be obtained by applying the puzzle's operation 49 times.
- Let k be such a number. We can obtain it on the first iteration by subtracting 1 from $k + 1$.
- Then we can apply the operation to the pairs of the remaining consecutive integers,

$$(2, 3), (4, 5), \ldots, (k - 1, k), (k + 2, k + 3), \ldots, (49, 50),$$

to get 24 ones on the board while erasing the above pairs.

Remaining Number

- What we have now are k 1 1 1 … 1
- Applying the operation to the 24 pairs of ones yields 12 zeros, which can be reduced to a single zero after applying the operation 11 times.
- So what we have left now are 0 k
- Finally, applying the operation to the two remaining numbers, *k* and 0, yields *k*.