MATH221: Mathematics for Computer Science
Tutorial Sheet Week 3 - Autumn 2021

1. Use this exercise to practice the "quick method"; do not use truth tables. Determine which of the following statements are tautologies.
   (i) $(p \Rightarrow q) \vee (p \Rightarrow \sim q)$
   (ii) $\sim (p \Rightarrow q) \vee (q \Rightarrow p)$
   (iii) $(p \wedge q) \Rightarrow [\sim r \vee (p \Rightarrow q)]$

2. Using the laws seen in lecture, write the following expressions using only $\vee$, $\wedge$ and $\sim$. Further, write the expression in the simplest form.
   (i) $(p \wedge q) \Rightarrow r$
   (ii) $p \Rightarrow (p \vee q)$

3. Let p, q and r be statements. Using the laws seen in lecture, prove the following.
   (i) $\sim (p \Rightarrow q) \equiv (p \wedge \sim q)$.
   (ii) $\big((p \wedge \sim q) \Rightarrow r\big) \equiv \big(p \Rightarrow (q \vee r)\big)$

4. Let p, q and r be statements.Using the laws seen in lecture, prove that the following compound statements are tautologies. (i) $p \Rightarrow (q \vee p)$ (ii) $(p \wedge q) \Rightarrow \big(\sim r \vee (p \Rightarrow q)\big)$

5. In each case, decide whether the proposition is True or False. Give reasons.
   (i) If x is a positive integer and $x^2 \leq 3$ then $x = 1$.
   (ii) $\big(\sim (x > 1) \vee \sim (y \leq 0)\big) \iff \sim \big((x \leq 1) \wedge (y > 0)\big)$

6. (i) Write the following logical expressions using $\vee$ and $\wedge$ only (even without $\sim$).

   (a) $\sim (x > 1) \Rightarrow \sim (y \leq 0)$

   (b) $(y \leq 0) \Rightarrow (x > 1)$

   (ii) Simplify the expression $\sim (\sim (p \vee q) \wedge \sim q)$.

7. Write each of the following statements in words. Write down whether you think the statement is true or false.
   (i) $\forall x \in \mathbb{R},\ (x \neq 0 \Rightarrow (x > 0 \vee x < 0))$
   (ii) $\forall x \in \mathbb{N},\ \sqrt{x} \in \mathbb{N}$
   (iii) $\forall$ student s in MATH221, $\exists$ assigned problem x s.t. s can correctly solve x.

8. Write each of the following statements using logical quantifiers and variables. Write down whether you think the statement is true or false.
   (i) If the product of two real numbers is 0, then both of the numbers are 0.
   (ii) Each real number is less than or equal to some integer.
   (iii) There is a student in MATH221 who has never laughed at any lecturer's jokes.

9. Translate each of the following statements into the notation of predicate logic and simplify the negation of each statement. Which statements do you think are true?
   (i) Someone loves everybody.
   (ii) Everybody loves everybody.
   (iii) Somebody loves somebody.
   (iv) Everybody loves somebody.
   (v) All rational numbers are integers.
   (vi) Not all natural numbers are even.
   (vii) There exists a natural number that is not prime.
   (viii) Every triangle is a right triangle.

10. Are the following statements true or false? Give brief reasons why.
(i) $\forall x \in \mathbb{R},\ (x > 1 \ \Rightarrow\ x > 0)$
(ii) $\forall x \in \mathbb{R},\ (x > 1 \ \Rightarrow\ x > 2)$
(iii) $\exists x \in \mathbb{R}$ s.t. $(x > 1 \ \Rightarrow\ x^2 > x)$
(iv)$\exists x \in \mathbb{R}$ s.t. $\big(x > 1 \ \Rightarrow\ \frac{x}{x^2+1} < \frac{1}{3}\big)$
(v) $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x^2 + y^2 = 9$
(vi) $\forall x \in \mathbb{R},\ \exists y \in \mathbb{R}$ s.t. $\ x^2 < y + 1$
(vii) $\exists x \in \mathbb{R}$ s.t. $\ \forall y \in \mathbb{R},\ x^2 + y^2 \geq 0$
(viii) $\exists x \in \mathbb{R},\ \exists y \in \mathbb{R}$ s.t. $\ (x < y \ \Rightarrow\ x^2 < y^2)$

11. For each of the following statements, (a) write down the negation of the statement, (b) write down whether the statement or its negation is false, and (c) think about how you would disprove it.
(i) $\forall \varepsilon > 0,\ \exists x \neq 0$ s.t. $|x| < \varepsilon$.
(ii) $\exists y \in \mathbb{R}$ s.t. $\forall x \in \mathbb{R},\ y < x^2$.
(iii) $\forall y \in \mathbb{R},\ \forall x \in \mathbb{R},\ \big(x < y \ \ \Rightarrow\ \ x < \frac{x+y}{2} < y\big)$.