MATH221 Mathematics for Computer Science

Tutorial Sheet Week 5

Autumn 2019

Numbers

1. For $a, b \in \mathbb{R}$ with $a \leq b$ we denote by (a, b) the set of all real number strictly between a and b. In set notation we write $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. We denote by $[a, b]$ the set of all real numbers between a and b and including a and b. In set notation we write $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. We let you figure out the sets $(a, b]$ and $[a, b)$. State which of the following sets have least elements and greatest elements, and what these elements are when they exist.

(a) $[0, 1]$

(b) $[0, 1)$

(c) $\left\{1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, \ldots\right\}$

(d) The set of all rational numbers (i.e., whole numbers and fractions) between 0 and 1 but excluding 0 and 1.

Which of the above sets are well-ordered?

2.

(a) Prove by mathematical induction that $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for all $n \in \mathbb{N}$.

(b) Prove by mathematical induction that $1^3 + 2^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$ for all $n \in \mathbb{N}$.

(c) Evaluate

$$\text{(i)} \sum_{i=1}^{5}(2i-5) \qquad \text{(ii)} \sum_{j=-2}^{2} 2^j \qquad \text{(iii)} \sum_{k=0}^{3} \frac{k!}{2} \qquad \text{(iv)} \sum_{\ell=0}^{99} \frac{(-1)^\ell}{3}$$

(d) (i) Express the sum $2 + 6 + 10 + \cdots + (4n - 2)$ using sigma notation.

(ii) Prove by induction that $2 + 6 + 10 + \cdots + (4n - 2) = 2n^2$ for all $n \in \mathbb{N}$.

(e) Consider the statement $2^n \geq n^2$. Test its correctness for a range of values of n, make a conjecture about the range of values for which it is true, and then use a suitable form of mathematical induction to prove it.

(f) Prove by induction that $n! > 2^n$ for $n \geq 4$.

(g) Here is a small example illustrating how mathematical induction may be used in computer science.

Assume that `X` and `Y` have been declared as integer variables, and that initially `X` has value x and `Y` has value y. Consider the following fragment of C or C++ code:

```
while (X != 0)
{
    X = X - 1
    Y = Y + 1
}
```

For $n \in \mathbb{N}$, let CLAIM(n) be the statement "before the nth loop iteration, `X+Y` has the value $x + y$." Prove by induction that CLAIM(n) is true for all n.

Deduce that *if* the loop terminates, then `Y` will have value $x + y$.

(h) Prove by mathematical induction CLAIM(n): "$n + 1 < n$" for all n. However, you find yourself in a bit of a rush, and decide to skip the *Basis step* for an induction proof. Start with the induction step, and observe why the Basis step is actually really, really important.