The Analemma for Latitudinally-Challenged People

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An academic exercise presented in partial fulfillment for the degree of
Bachelor of Science with Honours in Mathematics

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2001/2002

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Acknowledgements

Firstly, I would like to take this opportunity to express my sincere appreciation and gratitude to my supervisor, Associate Professor Helmer Aslaksen, for his guidance throughout the preparation of this project. To work independently on a project is a new experience for me and it would never be possible without his help. I am especially thankful to him for his continual patience in helping me to improve my thesis.

Next, I would like to thank my family for their encouragements whenever I was stressed up by the heavy workload. Last but not least, I would like to thank Huihui and May for their support throughout my Honours year.

Summary

In the northern hemisphere the shortest day falls on December 21 and the longest day falls on June 21. These two dates correspond to the winter solstice and summer solstice, respectively. Most people assume that the earliest sunrise and the latest sunset will fall on the summer solstice; and the latest sunrise and the earliest sunset will fall on the winter solstice. But this is not true. For example in Florida, latitude 30°N, the earliest sunrise and the latest sunset fall on June 10 and June 30, respectively; the latest sunrise and the earliest sunset fall on January 9 and December 2, respectively. In Singapore, about latitude 1° N, the earliest sunrise and the latest sunset fall on November 1 and February 13, respectively; the latest sunrise and the earliest sunset fall on February 9 and November 4, respectively. Although the earliest sunrise and the latest sunset do not fall on the summer solstice, they lie close to it at high latitudes. However, as latitude decreases, the earliest sunrise and the latest sunset move further away from the summer solstice. Likewise, the latest sunrise and the earliest sunset fall near the winter solstice at high latitudes, but move further away from the winter solstice at low latitudes.

At the equator, the length of daylight is the same every day, thus it seems reasonable to suppose that the time of sunrise and the time of sunset will be the same every day. However, the fact is the difference in time between the earliest and the latest sunrise is around 30 minutes.

The main objective of this project is to explain the above phenomena and to discuss how the equation of time, together with the latitude of the observer, affects the dates on which the extrema of the sunrise and sunset occur. The first chapter gives a more detailed introduction to the topic, using specific latitudes for examples. In the second chapter, I introduce some basic terminology of astronomy required to facilitate our discussion in the later chapters. In the third chapter, I first explain the two factors that result in the equation of time, leading to an approximated formula for it. In the fourth chapter, I discuss the characteristics of the analemma curve and its position above our horizon at different times of the day. Finally in the fifth chapter, I discuss how the analemma curve can be used to explain some of the peculiarities in the time of sunrise and sunset in the tropics.

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Statement of Author's Contributions

The idea of "analemma-rise" was introduced by Roger W. Sinnott in ([12]). I have expanded on this idea and presented the theory from a "latitudinally-correct" point of view. I have focused on how the phenomenon changes with latitude and the "tug of war" between the latitude of the observer and the equation of time. Many of the pictures were made using the Mathematica version of the code from the book by Nachum Dershowitz and Edward M. Reingold ([2]). The conversion from Lisp to the Mathematica package Calendrica was done by Robert C. McNally.

Chapter 1

Introduction

In the northern hemisphere, the shortest day is December 21 and the longest day is June 21. In the southern hemisphere, the reverse is true. It seems reasonable to suppose that the earliest sunrise and the latest sunset would correspond to the longest day of the year. Likewise, most people assume that the latest sunrise and the earliest sunset would fall on the shortest day of the year. But surprisingly this is not the case. For example, for an observer at latitude 44°N, the latest sunrise actually falls on January 4 and the earliest sunset falls on December 9. Similarly the earliest sunrise and the latest sunset at latitude 44°N are around 2 weeks before and after the summer solstice, respectively. Nevertheless, the earliest sunrise and the latest sunset still fall near the summer solstice; the latest sunrise and the earliest sunset still fall near the winter solstice. However as we move towards the equator, the extrema of the sunrise/sunset move further away from the solstices. For example in Singapore, the earliest sunrise is around November 1 and the latest sunrise is around February 13, which are not near the solstices. Regardless of latitude, the earliest sunrise and the latest sunset are about equal number of days away from the summer solstice; the latest sunrise and the earliest sunset are about equal number of days away from the winter solstice.

Places near the equator have no seasons and have about equal amount of daylight every day. Thus it seems logical to assume that the time of the earliest sunrise and the time of the latest sunrise would not differ much. However, this is not the case. The difference is around 30 minutes, even at the equator.

The equation of time is responsible for the above phenomena. However, it is not the only factor involved. In fact there is a constant “tug-of-war” between the latitude of the observer and the equation of time. The equation of time always attempts to move the earliest sunrise and the latest sunset away from the summer solstice; however as latitude increases, the earliest sunrise and the latest sunset move towards the summer solstice.

This project aims to explain the above, using a graphical representation of the equation of time, called the analemma. In fact the analemma makes it easier for us to understand the “tug-of-war”, as it pictorially illustrates the effects of these two co-existing factors.

Chapter 2

Basic Astronomy

2.1 Celestial Sphere

In order to understand the origin of the equation of time, we need to first understand some of the basic terminology and facts of astronomy. For the purpose of reference, I will go into more detail than is strictly necessary, so the reader may skip this chapter if he already knows some basic astronomy. I rely on the book by Kaler ([7]) and the online astronomy notes of Strobel ([13]).

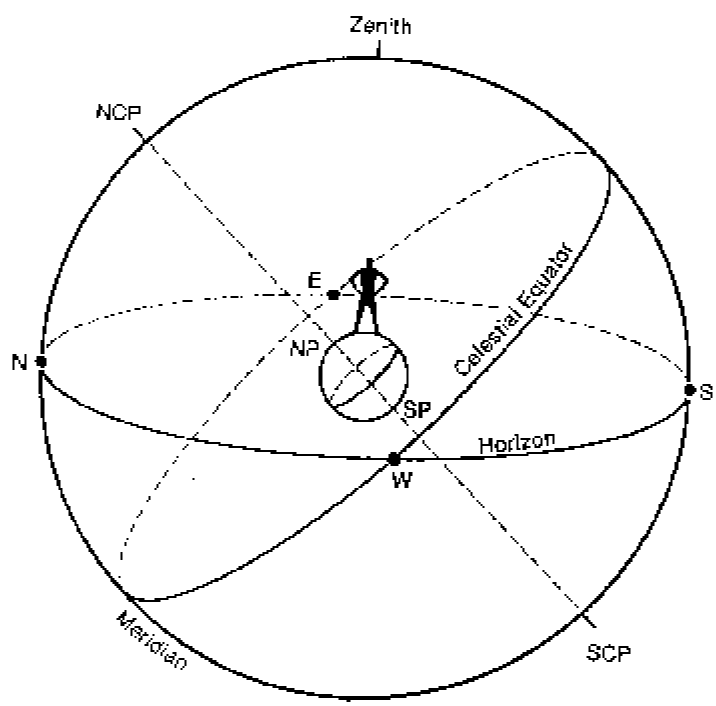


Figure 1. Reference markers on the celestial sphere

Imagine the stars as fixed on the surface of the hemisphere with an observer at the center. The whole sphere of which this hemisphere is part is called the **celestial sphere**.

The point directly over the observer's head is called the **zenith**. The observer's **horizon** is a great circle around him, whose plane is perpendicular to the line joining the observer and his zenith. (A **great circle** is any circle on a sphere that divides it into equal halves.)

If we extend the Earth's axis into space in both directions, the line will intersect the celestial sphere at two points, called the North Celestial Pole (NCP) and the South Celestial Pole (SCP). The NCP is directly above the Earth's North Pole, and the SCP is directly above the South Pole. The celestial poles are to the sky what the terrestrial poles are to the Earth. The Earth makes one anti-clockwise rotation about its own axis in one day. The sky appears to rotate clockwise about its axis that passes through the celestial poles. The great circle that goes through the NCP and the observer's zenith is called the **Meridian**. For any individual there is only one Meridian. The **celestial equator** is an extension of the Earth's own equator onto the celestial sphere. The Meridian and the horizon intersect at two opposite points. The intersection point nearest to the NCP is called "north" (N). The one nearest to the SCP is called "south" (S). Halfway between north and south are the "east" and "west" points, which are 90° clockwise from north and south respectively. Figure 1 shows all the above reference markers.

The Earth makes one anti-clockwise revolution around the Sun in one year. The Earth is tilted on its axis 23.5° in relation to the plane of its orbit around the Sun. This orbit is not a circle, instead it is an ellipse. From our point of view from the Earth, the Sun appears to move in a great circle about us. The path of the Sun throughout the year on the celestial sphere is called the **ecliptic**. The angle that the ecliptic makes with the celestial equator is the same as the tilt of the Earth's axis.

The ecliptic and celestial equator intersect at two points: the **vernal (spring) equinox** and **autumnal (fall) equinox**. The Sun crosses the celestial equator moving northward at the vernal equinox around March 21 and crosses the celestial equator moving southward at the autumnal equinox around September 22. When the Sun is on the celestial equator at the equinoxes, everybody on the Earth experiences 12 hours of daylight and 12 hours of night for those 2 days (hence, the name "equinox" for "equal night"). On those two days of the year, the Sun will rise in the exact east direction, follow an arc right along the celestial equator and set in the exact west direction.

Since the ecliptic is tilted 23.5° with respect to the celestial equator, the Sun's maximum angular distance from the celestial equator is 23.5°. This happens at the **solstices**. For an observer in the northern hemisphere, the farthest northern point above the celestial equator is the **summer solstice** and the farthest southern point is the **winter solstice**.

The word "solstice" means "the Sun standing still" because the Sun stops moving northward or southward at those points on the ecliptic. In the northern hemisphere, the Sun reaches winter solstice at around December 21. On that day, you see its shortest diurnal path of the year. This is the day of the least amount of daylight. The Sun reaches the summer solstice at around June 21. On that day, you see its longest diurnal path of the year. This is the day of the most amount of daylight. The season in the northern hemisphere is always opposite to that in the southern hemisphere. For example, it is summer in the northern hemisphere when it is winter in the southern hemisphere. Figure 2 shows the position of the solstices and the equinoxes on the ecliptic.

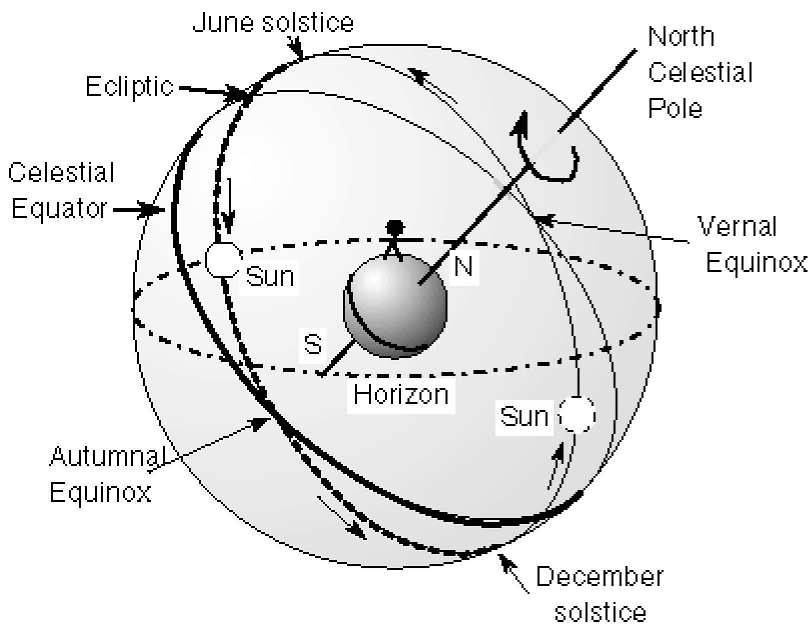


Figure 2. The solstices and the equinoxes

2.2 Equatorial Coordinate System

There are a number of celestial coordinate systems we can use to indicate the position of celestial bodies on the celestial sphere. One of them is the **equatorial coordinate system**. This system is very similar to the longitude-latitude system used to specify positions on the Earth's surface. The lines on a map of the Earth that run north-south are lines of longitude and when projected onto the sky, they become lines of **right ascension**. Because the stars were used to measure time, right ascension (RA) is measured in terms of hours, minutes, and seconds instead of degrees and increases in an easterly direction. For two stars one hour of RA apart, you will see one star cross your Meridian one hour of time before the other. If they were 30 minutes of RA apart, you would see one rise half an hour before the other and cross your Meridian half an hour before the other. Zero RA is where the Sun crosses the celestial equator at the vernal equinox. The full 360 degrees of the Earth's rotation is broken up into 24 hours, so one hour of RA = 15 degrees of rotation. The lines of RA all converge at the celestial poles so two stars one hour of RA apart will not necessarily be 15 degrees in angular separation on the sky. In fact, only when both stars are on the celestial equator will they be 15 degrees apart.

The lines on a map of the Earth that run east-west parallel to the equator are lines of latitude and when projected onto the sky, they become lines of **declination**. Declination works on the surface of the celestial sphere much like latitude does on the surface of the Earth. It measures the angular distance of a celestial object north or south of the celestial equator as shown in the figure 3. To obtain the declination of a celestial body, measure the angle between the celestial equator and the position of the body, along the body's line of right ascension. An object lying on the celestial equator has a declination of 0°. The declination increases as you move away from the celestial equator to the celestial poles. Therefore, at the north celestial pole, the declination is 90°. Declinations in the northern celestial hemisphere are positive. Declinations in the southern hemisphere are negative. Since the Sun moves along the ecliptic on the celestial sphere, its declination changes throughout the year. Specifically, its declination is 0° at the vernal and autumnal equinoxes and +23.5° and –23.5° at the summer solstice and the winter solstice respectively. Declination is calculated with the following formula ([5]), where N equals day number starting from January 1:

$$\text{Declination} = 23.45 \sin [(360 / 365)\times(284 + N)]$$

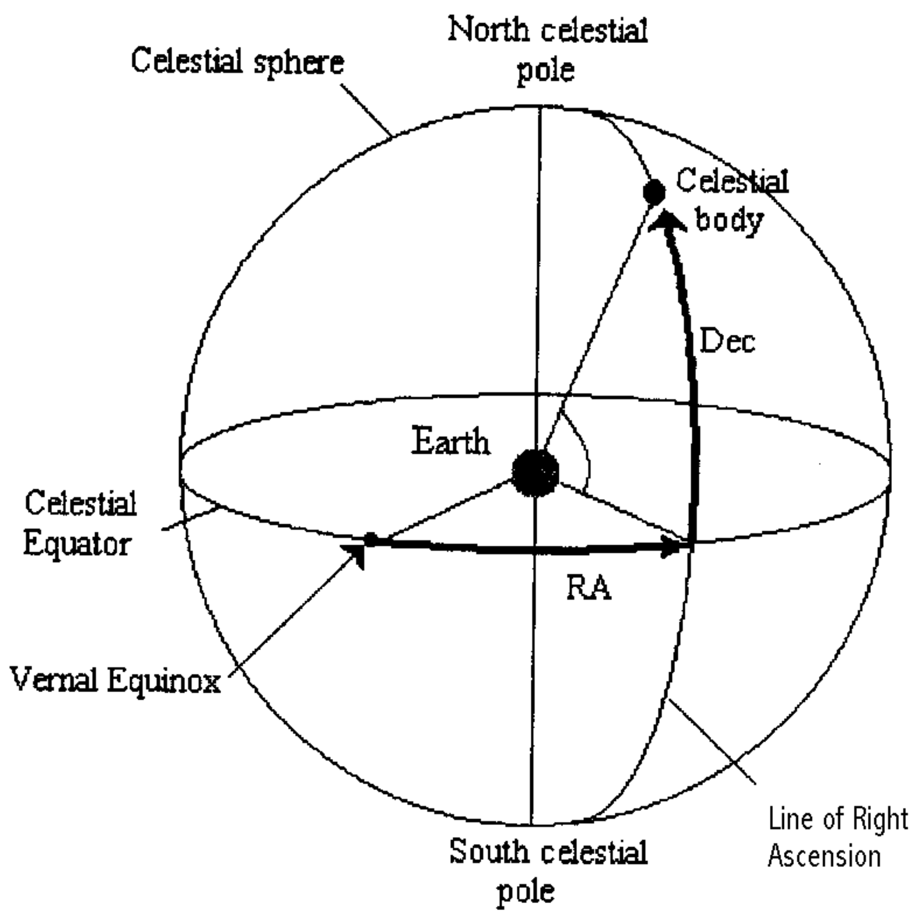


Figure 3. Right ascension and declination

2.3 Declination and the Daily Motion of the Sun

An observer at any place on Earth will always see ½ of the celestial equator's arc ([13]). Since the sky appears to rotate around us in 24 hours, anything on the celestial equator takes 12 hours to go from exact east to exact west. Every celestial object's diurnal motion is parallel to the celestial equator. So for northern observers, anything south of the celestial equator (negative declination) takes less than 12 hours between rise and set, because most of its rotation arc around the observer is hidden below the horizon. Anything north of the celestial equator (positive declination), takes more than 12 hours between rising and setting because most of its rotation arc is above the horizon. For observers in the southern hemisphere, the situation is reversed.

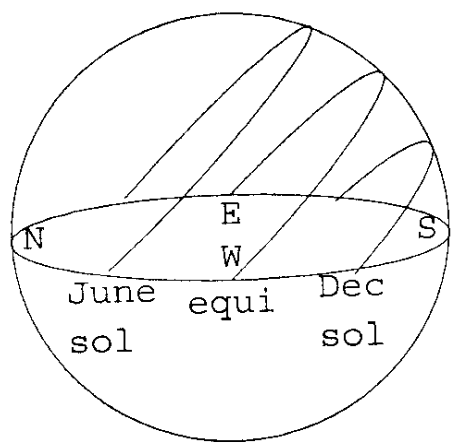


Figure 4. The Sun's diurnal path at high latitudes in the northern hemisphere

As a result, when the Sun is at the equinoxes (the Sun is on the celestial equator), everyplace on earth experiences 12 hours of daylight. When the Sun's declination is positive (i.e. the Sun is above the celestial equator) during the seasons of spring and summer, an observer in the northern hemisphere will have more than 12 hours of daylight. The Sun will rise in the northeast, follow a long, high arc north of the celestial equator, and set in the northwest. When the Sun's declination is negative (i.e. the Sun is below the celestial equator) during the seasons of autumn and winter, an observer in the northern hemisphere will have less than 12 hours of daylight. The Sun will rise in the southeast, follow a short, low arc south of the celestial equator, and set in the southwest. Figure 4 shows the diurnal paths of the Sun at the equinoxes and the solstices for an observer in the northern hemisphere.

When the observer is at the equator, the celestial equator is perpendicular to his horizon. Since a celestial body's motion is parallel to the celestial equator, its diurnal path will also be perpendicular to the celestial equator. As a result, half of its 24-hour path will be above the horizon. Therefore, for the case of the Sun, it will be above the horizon for exactly 12 hours for every day of the year, regardless of its declination. This explains why places near to the equator have approximately equal amount of daylight throughout the year.

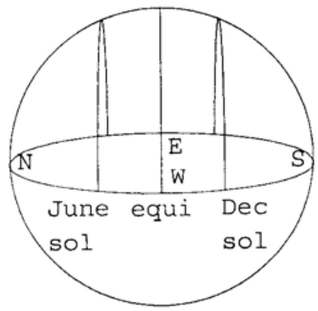


Figure 5. Diurnal path of the Sun for an observer at the equator

We mention earlier that the Sun's diurnal motion is parallel to the celestial equator. However, this is not exactly true. Since there is a continuous motion of the Sun along the ecliptic, the Sun's declination during the course of a day changes. Nevertheless the change is minute, since the Sun's declination only varies very slightly between two consecutive days. As a result, if we were to join up all the daily paths of the Sun (inclusive of the Sun's path below the horizon), we would obtain a spiral of circles moving from maximum declination of 23.5° to minimum declination of –23.5° and then back to maximum declination again.

Recall that we introduce the lines of RA in section 2.2. They are assigned values based on their angular distance from the vernal equinox. Thus the values assigned to the lines of RA are independent of the location of the observer. However there is a way to assign values to these lines of longitudes such that their values are dependent on the location of the observer. The line of longitude that cuts the East point of the observer's horizon is called the 6 a.m. meridian. The line of longitude that cuts the West point of the observer's horizon is called the 6 p.m. meridian. The other lines of longitude are assigned values in between, based on their angular distance from the 6 a.m. and 6 p.m. meridian. Note that the plane of a meridian cuts the plane of the celestial equator at right angles. (To avoid confusion, the Meridian defined in section 2.1 will always take a capital "M".)

Most people will assume that if we were to observe the Sun's position at a same time everyday on a place in the northern hemisphere, it will move along the meridian of that specific time, reaching maximum height above the horizon at the summer solstice and minimum height above the horizon at the winter solstice. Note that since the Sun is very far from the observer, as viewed from Earth the movement on the meridian will not appear to be an arc, instead it would appear to be a line in the sky. Figure 6 shows where we would expect the Sun to be positioned at 12 noon and 3 p.m., at different days of the year. However, if we actually observe the Sun's position at the same time throughout the year, we would not see the Sun move along a line, instead we would see a figure-of-eight curve in the sky. We would explain the origin of this figure-of-eight curve in the next chapter.

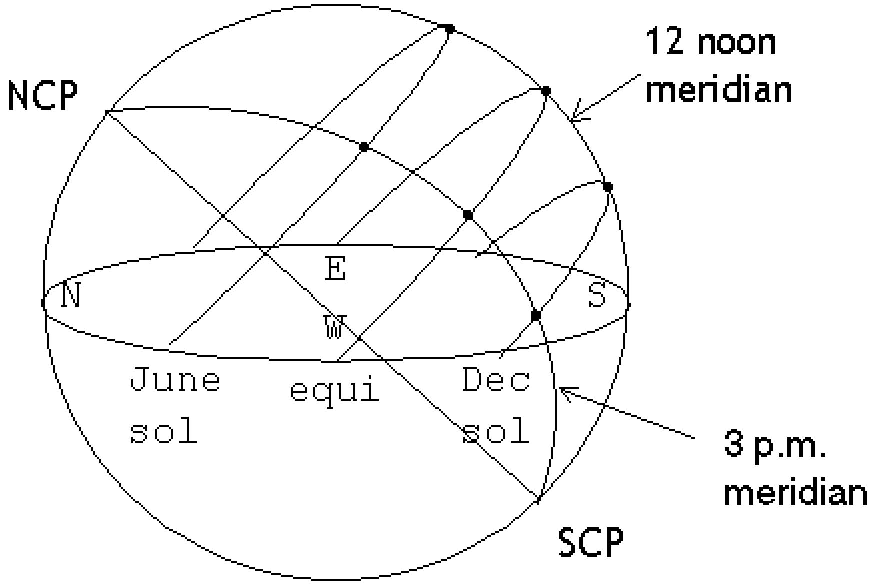


Figure 6. The meridians on the celestial sphere

Chapter 3

Equation of Time

3.1 Factors of the Equation of Time

In section 2.3, we mention why we expect the Sun's position to change along a line if we observe its position at the same time every day. In this chapter, we would explain why this is not true. The difference in the Sun's position within 2 consecutive days might be minute, but the overall effect throughout a year results in a figure-8 path of the Sun, called the **analemma** ([15]). The north-south movement of the Sun is due to the change in the Sun's declination. The east-west movement of the Sun is due to the **equation of time**. Figure 7 is a photograph taken by Dennis di Cicco ([11]). It shows how the position of the 8.30 a.m. Sun differs throughout the year.

Figure 7. The analemma in the sky

There are two reasons why the Sun takes this strange path. These two factors are completely independent but their sum causes the analemma ([10]).

1. The Earth is tilted on its axis 23.5° in relation to the plane of its orbit around the Sun.

2. The Earth does not orbit the Sun in a circle, but in an ellipse.

3.2 Elliptical Orbit Effect

The time taken by the Earth to make one full rotation is 23 hours 56 minutes. This time period is called the **sidereal day**. Suppose there is a fictitious Sun moving along the celestial equator at a constant speed and making one full circuit of 360 degrees in a year of 365.24 days, very close to 1 degree per day. This is equivalent to the Earth moving at a constant speed along a circular untilted orbit around the Sun. We call this Sun the **mean Sun** ([9]).

Between two consecutive Meridian crossings of this mean Sun, the Earth has to turn nearly 361 degrees, not 360 degrees. One way of looking at this additional one-degree rotation is to interpret it as the amount of spinning the Earth does while making up the shortfall caused by the fact that, from the point of view of an observer on Earth, the mean Sun has moved by one degree. Figure 8 illustrates how this extra rotation results.

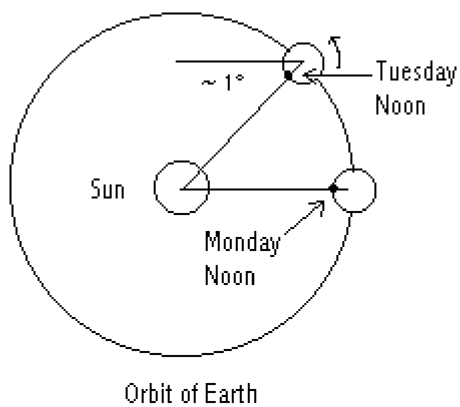


Figure 8. Two consecutive Meridian passings of the mean Sun

Since the rotation rate of the Earth is $\frac{360}{(23\times60)+56}$ degrees per minute, it takes 24 hours for the Sun to rotate 361 degrees. As a result, the time between two consecutive Meridian crossings of the mean Sun is 24 hours. However, the Earth does not travel around the true Sun in a circle, instead it takes the path of an ellipse ([13]). As a result the speed of the Earth along its elliptical orbit varies throughout the year. The speed of the Earth is fastest when it is closest to the Sun (at perihelion), which is in January. The speed of the Earth is slowest when it is farthest from the Sun (at aphelion), which is in July. In other words, in January, the true Sun will be moving faster than average, and in July it will be moving slower than average. Figure 9 shows the position of the Sun at perihelion and aphelion.

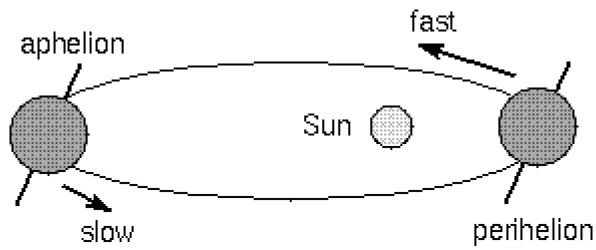


Figure 9. Elliptical orbit of the Earth

In this section, let us assume that the Earth is <u>not</u> tilted 23.5 degrees on the axis. Figure 10 shows the difference in revolution speed of the Earth when its orbit is circular and when it is elliptical. Let us denote the Earth in a circular orbit as Earth A and that in an elliptical orbit as Earth B.

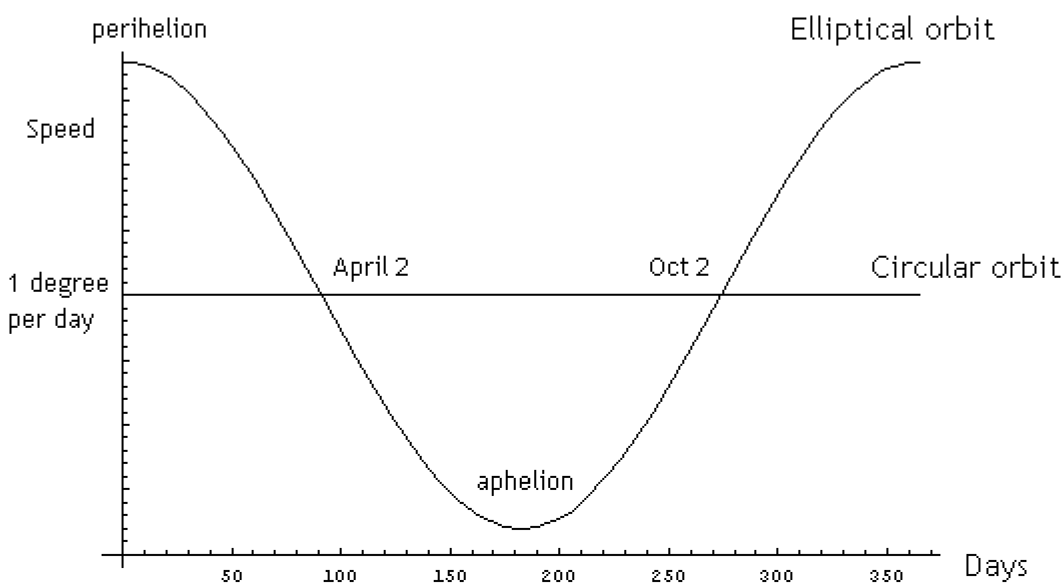


Figure 10. Revolution speed of Earth A and Earth B

The revolution speed of Earth A is constant at nearly one degree per day. From January 2 (perihelion) to April 2, Earth B's revolution speed decreases from a maximum. However during this period, Earth B's speed is faster than that of Earth A. On April 2, Earth A and Earth B have the same revolution speed. From April 2 to July 3 (aphelion), Earth B's revolution speed becomes slower than that of Earth A and it reaches a minimum on July 3. From July 3 to October 2, Earth B picks up speed and eventually catches up with Earth A on October 2. From October 2 to January 2, Earth B revolves faster than Earth A and reaches a maximum speed on January 2.

Figure 11 illustrates the difference in position of Earth A and Earth B one day after the perihelion ([15]). On January 2 (perihelion), Earth A and Earth B start from the same point. At noon, the Sun is directly overhead for both. After 24 hours, both of them have rotated 361 degrees. However due to the difference in revolution speed, Earth A has revolved 1 degree around the Sun, while Earth B has revolved more than 1 degree. As a result after 24 hours, the Sun would still appear to be directly overhead Earth A, but slightly to the east for Earth B. After another 24 hours, Earth B is still continuing to move faster than average. This error in time will accumulate and the Sun will continue for a time to appear to move farther and farther east in the sky, in comparison to what the watch reads at noon. The difference accumulates each day and continues to accumulate until around April 2 when the speed of Earth A and Earth B are the same. At that time, the position of the Sun in the sky will have reached its maximum "offset" to the east. The time difference between the Sun and the watch will be almost 8 minutes. From April 2 until around July 3 the Sun will drift back towards the west because the speed of Earth B is getting slower (moving towards aphelion). Then from July 3 to October 2 (after passing the aphelion), Earth B picks up speed, but its speed is still slower than Earth A's. The Sun continues to drift to the west but at a slower rate. On October 2, the Sun reaches its maximum "offset" in the west, since the speed of Earth A and Earth B are the same again. Then from October 2, Earth B begins to move faster than Earth A and the Sun drifts back towards the east until it reaches its starting position on January 2.

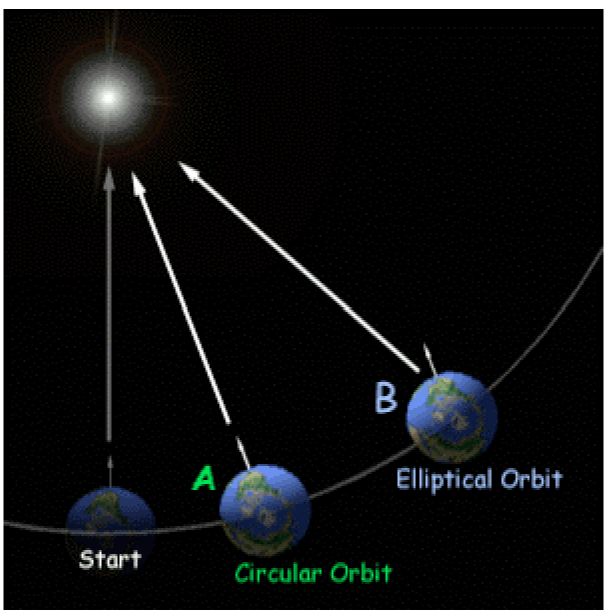


Figure 11. Elliptical orbit effect

In calculating the position of the Sun, we need to know the eccentricity of the elliptical orbit of the Earth. The motions of the planets around the Sun are controlled by the action of gravity, that is the mutual force of attraction between the bodies. This force of attraction is dependent on the distance between the planet and the Sun. When the distance is shorter, the force of attraction is stronger and as a result, the planet revolves around the Sun at a faster speed. An ellipse can be imagined as a squashed circle; in fact, a circle is a special case of an ellipse, where the two foci F_1 and F_2 have coincided. Figure 12 shows the features of an ellipse.

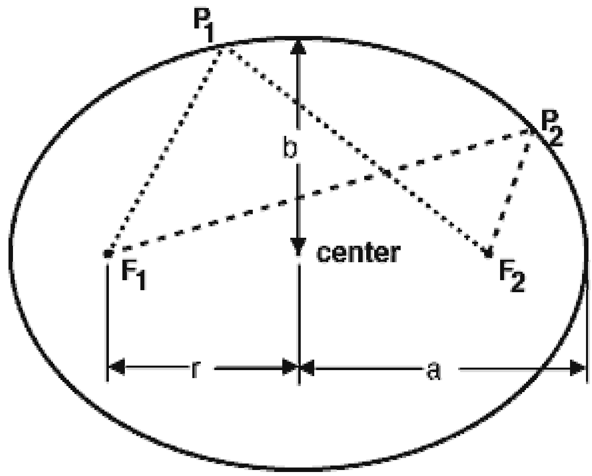


Figure 12. Features of an ellipse

The amount of squashing is measured by the **eccentricity**, **e**; for a circle, e = 0. Most planetary orbits have eccentricities less than 0.1 so that their deviations from circular orbits are small. The eccentricity of the Earth's orbit around the Sun is 0.0167. For the Earth-Sun system, F_1 is the position of the Sun; F_2 is an imaginary point in space, while the Earth follows the path of the ellipse. The distance *a* is the semimajor axis, while the distance *b* is the semiminor axis. The eccentricity e can be calculated as follows:

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{r}{a}$$

We will assume that the perihelion occurs at around January 2. We need to find the angle *v* the Earth makes in relation with the Sun after perihelion and compare it to the angle *x* the Earth would make with the Sun if the orbit were circular. Since the speed of the Earth in a circular orbit is constant, the average angle travelled is $\frac{360}{365.24}$ = 0.986 ° per day. For the Earth travelling in an elliptical path, its angle with the Sun can be calculated as follows:

N days after perihelion,
$x = 0.986N$
e = eccentricity of elliptical orbit = 0.0167
$v = x + (360/\pi)e \sin x = x + 1.915 \sin x$
$x - v$ = angular difference between the Earth in a circular path and the Earth in an ellipse.

Since it takes 3.989 minutes for the Earth to make 1-degree rotation, we can convert the angular difference to time difference by multiplying it with a factor of 3.989.
For example, to calculate the time difference for the Sun's position in relation to our watch on April 2:

N = 91
$x = 0.986 \times 91 = 89.726°$
$v = 89.726° + 1.915 \sin 89.726° \approx (89.726 + 1.915)°$
$x - v = -1.915°$
Equation of time = $-1.915° \times 3.989 = -7.64$ minutes

Note that the negative sign indicates that the true Sun's position is 7.64 minutes behind the position that it should be as indicated by the watch. Recall that April 2 is the day when the Sun is at maximum offset to the East, therefore the maximum magnitude of the equation of time, only taking into account of the elliptical orbit component, is 7.64. Using the formula above, we can obtain figure 13, which is actually a sine curve. The Earth revolves around the Sun the fastest when it is at perihelion (January 2). On that day, the rate of the Sun's drift to the East is the fastest. Thus the graph is steepest on January 2, which corresponds to the point of inflection of the curve. Similarly, there is a point of inflection on July 3 (aphelion).

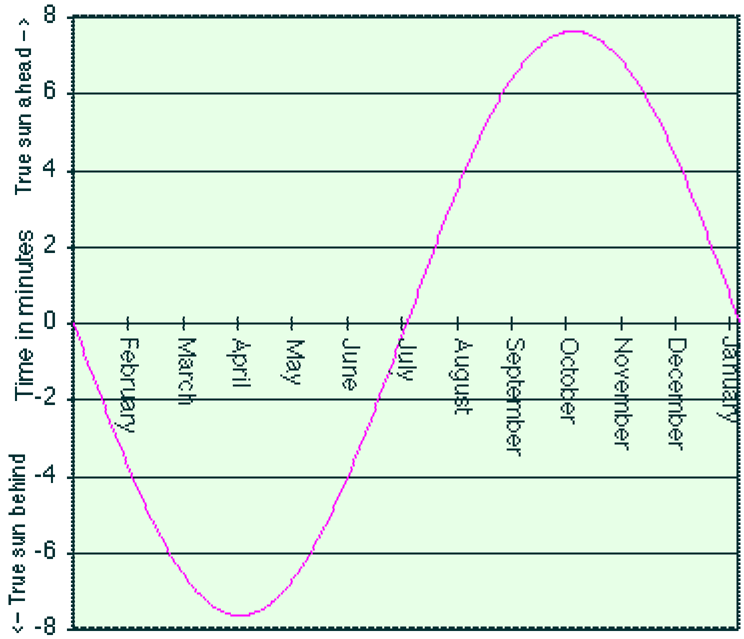


Figure 13. Equation of time graph for one year -- elliptical orbit

3.3 Obliquity of the Ecliptic

We have seen how the ellipticity of the Earth's orbit around the Sun causes irregularity in the Sun's time keeping. A separate irregularity is caused by the Sun apparent movement on the ecliptic. The ecliptic is inclined to the equator at an angle of 23.5 degrees. The equator is actually the basis of our time measurements. Our daily time system is based on the Earth's revolution round the polar axis, and the equator is the plane at right angles to this axis ([10]).

In this section, let us assume that the Earth's orbit is circular (i.e. constant speed of revolution). As a result of the tilt of the ecliptic, the actual Sun's drift against the stars is not uniform. The non-uniformity is due to the fact that on top of the general eastward drift among the stars, the Sun is moving along the ecliptic northward or southward with respect to the celestial equator. Figure 14 illustrates how the east-west component of the Sun's velocity varies as it moves along the ecliptic ([13]).

Remember that we are assuming the Earth's orbit around the sun is circular. The velocity of the mean Sun and the true Sun are constant, each one taking one year to make a complete trip around the celestial sphere. Though the true Sun is moving at a constant speed along the ecliptic, its eastward motion will still be faster during some periods as compared to others. For example, its eastward motion is greatest when it is at the solstices and smallest when it is at the equinoxes. Apparent solar time is based on the component of the true Sun's motion parallel to the celestial equator. This effect alone would account for as much as 9 minutes difference between the true Sun and a fictional mean Sun moving uniformly along the celestial equator.

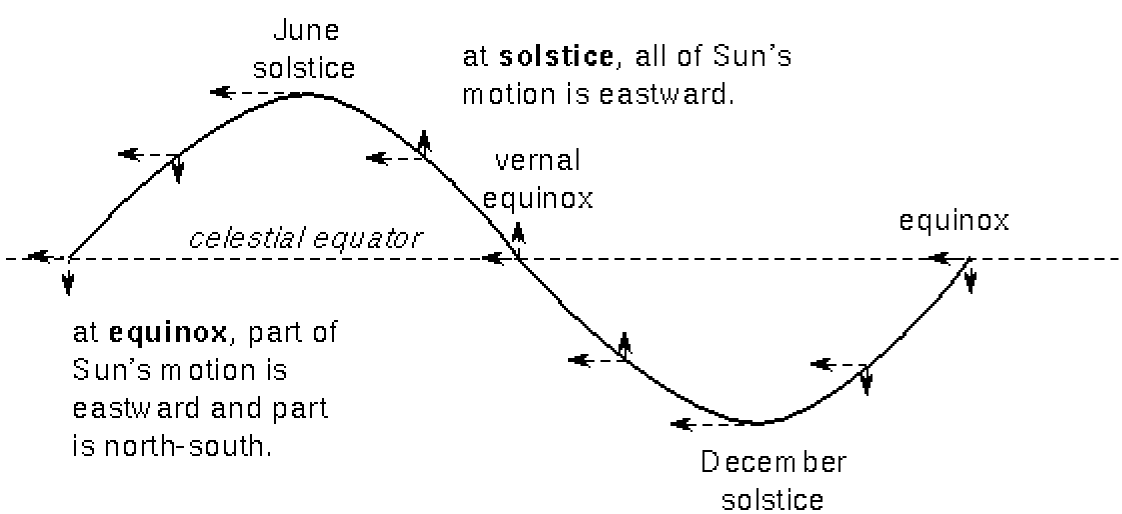


Figure 14. Eastward and north-south resolution of the Sun's velocity along the ecliptic

It is important to make a clear distinction between the slow drift of the Sun along the ecliptic during the year and the fast motion of the rising and setting Sun during a day. We would not be able to see the Sun's movement on the ecliptic, but we could feel its effect as it influences the daily motion of the rising and setting Sun.

At the vernal equinox and the autumnal equinox, the true Sun and the mean Sun are in the same position. Let's look at a close-up of the positions of the mean Sun and the true Sun a day after the vernal equinox ([15]). (We are observing from a point on the equator outside the celestial sphere).

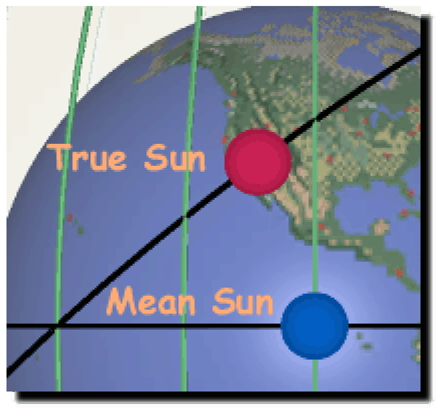


Figure 15. One day after vernal equinox

The true Sun and the mean Sun will each travel one degree on the ecliptic and the celestial equator respectively. However along the celestial equator, the true Sun's progress will only be $1°(\cos 23.5°) \approx 0.917°$. Therefore the true Sun's right ascension (RA) will be smaller than that of the mean Sun. Recall that in section 2.2, we mention that a celestial body with a smaller RA will rise first. Therefore one day after the vernal equinox, the true Sun will rise before the mean Sun and crosses the Meridian before 12 noon. At 12 noon, the true Sun would be in the western sky. In other words, the true Sun will be ahead of the clock time, which is measured with respect to the mean Sun. Thus we can see that whenever the RA of the true Sun is smaller than that of the mean Sun, the true Sun will be ahead of the clock time and the equation of time will be positive. In fact, **the equation of time takes the same value as the difference in RA of the mean Sun and the true Sun.**

The rate of increase of RA for the mean Sun will be faster than that of the true Sun from the vernal equinox to May 4 (halfway between the vernal equinox and the summer solstice). The difference in RA for the mean Sun and the true Sun will increase during this period, as a result the true Sun will rise earlier and earlier during this period. After May 4, the rate of increase of RA will be faster for the true Sun. This is because as the true Sun moves towards the summer solstice, the eastward component of its motion increases (figure 14). Moreover, as the true Sun moves to higher declination towards the summer solstice, the lines of RA get closer together. Therefore after May 4, one-degree separation on the ecliptic will result in more than one-degree separation in RA. As a result, the difference in RA between the mean Sun and the true Sun will start to decrease after May 4. Eventually at the summer solstice, the mean Sun and the true Sun will have the same RA, which means the difference in RA is zero. This means that the equation of time will increase from zero at the vernal equinox, reaches a maximum value halfway to the summer solstice, and decreases to zero at the summer solstice.

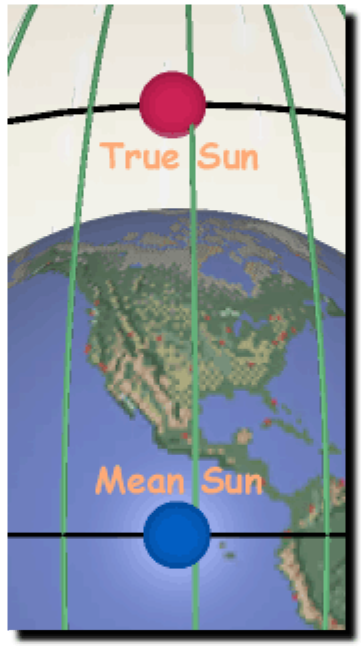


Figure 16. At the summer solstice

As the true Sun moves towards the autumnal equinox, the reverse takes place. Initially, the RA of the true Sun will be greater than that of the mean Sun, and thus the true Sun will rise later than the mean Sun. The difference in RA of the mean Sun and the true Sun will be negative, and thus the equation of time is also negative. But after August 5 (half way to the autumnal equinox), the rate of increase of RA for the mean Sun becomes faster than that of the true Sun. Thus the difference in RA of the mean Sun and the true Sun reaches its most negative value on August 5. From then on, the difference in RA of the mean Sun and the true Sun will become less and less negative and reaches zero on the autumnal equinox. For the case in which the true Sun moves from the autumnal equinox to the winter solstice and back to the vernal equinox, we can apply the same theory to deduce the equation of time for that period. From the above, we can deduce that the equation of time will be zero four times a year, at the equinoxes and at the solstices.

We can derive the formulas for the effect of the Earth's tilt on the equation of time as follows:

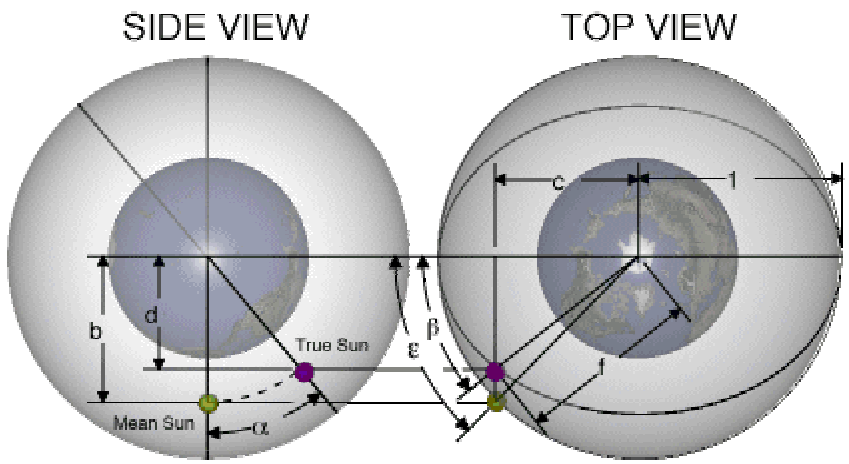


Figure 17. The positions of the true Sun and the mean Sun

All angles are expressed in radians.

Looking at the side view:
α = tilt of the Earth's axis = 0.408983
$d = b \cos \alpha$

Looking at the side and top view:
$b = \sin \varepsilon$
$c = \cos \varepsilon$
$d = (\sin \varepsilon)(\cos \alpha) = 0.9175 \sin \varepsilon$
N = number of days since vernal equinox
ε = the angle of the mean Sun after N days = $(2\pi/365.24)$ N = 0.172 N

We are interested in finding the angle β. This is the angle of the true Sun on N days after the vernal equinox. From this, we can easily calculate the time difference between the true Sun and the mean Sun. Again from the top view, we can see:
$\tan \beta = d/c$

To find the equation of time on May 4, 44 days after the vernal equinox and half way from the vernal equinox to the summer solstice:
N = 44
$\varepsilon = 0.17203\text{N} = 0.756932$
$\beta = \arctan (0.9175 \sin \varepsilon / \cos \varepsilon) = 0.714712$
$\varepsilon - \beta = 0.0422 = 2.419$ degrees = 9.65 minutes

Since the answer is positive, this is the amount that the true Sun will be ahead of the mean Sun. (i.e. the true Sun will appear to be west of the position that it should be according to the watch). Remember that this does not take into consideration the effect of the Earth's elliptical orbit around the Sun. Recall that we have deduced that the equation of time will be zero at the equinoxes and the solstices; and has maximum magnitude at May 4. And together with the above formulas, we can obtain the following graph:

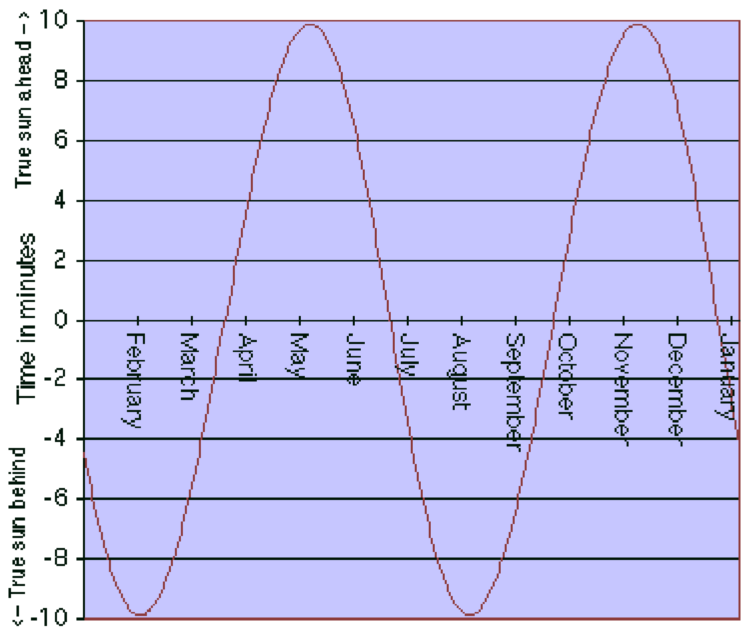


Figure 18. Equation of time graph for one year -- tilt = 23.4°

3.4 The Equation of Time

The **equation of time** is the amount by which the true Sun is ahead of the mean Sun, in minutes and seconds. It is the result of the sum of the Earth's elliptical orbit around the Sun and the tilt of the Earth's axis in relation to the plane of its orbit around the Sun. Figure 19 shows the effect of this summation ([15]). The equation of time is zero four times a year and reaches a local maximum or local minimum value four times a year.

From figure 19, we notice that the equation of time graph preserves the general shape of the graph corresponding to the tilt of the Earth's axis. This is expected since the summation of the Earth's tilt graph with period 2 and the elliptical orbit graph with period 1 should give rise to a resultant graph of period 2. In addition, we see that the maximum magnitude of the elliptical orbit graph is smaller than that of the Earth's tilt graph. These two features imply that the more important component of the equation of time is that which is due to the obliquity of the Earth's axis. Consequently, even if the Earth's orbit were exactly circular, the equation of time would still exist: There would still be four zeros and four local extrema in a year.

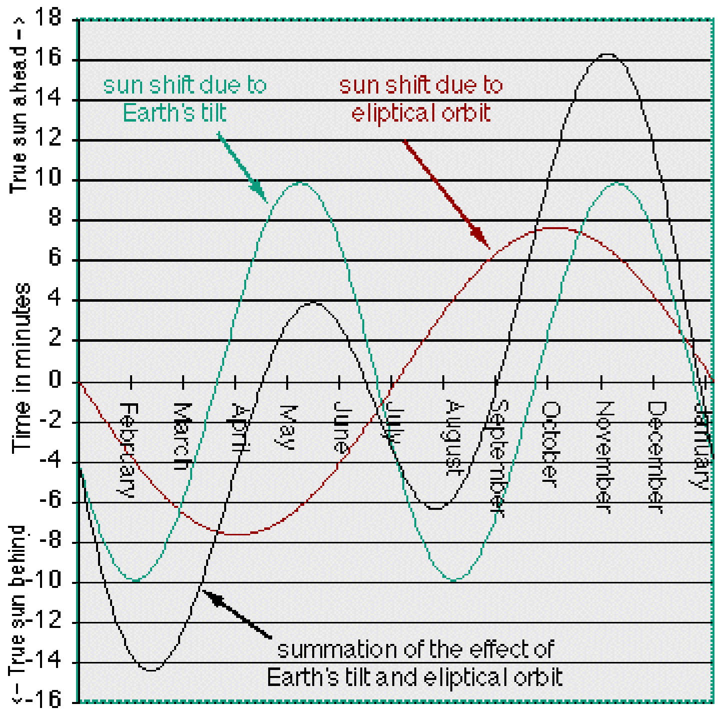


Figure 19. The equation of time

From figure 19, we can see that the graph of the Earth's tilt effect and the graph of the elliptical orbit effect are actually sine curves. Let N represents the number of days after perihelion.

Let the graph of the elliptical orbit effect be approximated by:

$\mathbf{E_1 = -7.64 \sin (360N/365.24)}$

This approximation is made by noting that:

1. The graph is a reflected sine curve about the x-axis.
2. Its maximum magnitude is 7.64.
3. It begins its cycle on the perihelion and completes one cycle after about 365 days.

Let the graph of the Earth's tilt effect be approximated by:

$\mathbf{E_2 = 9.65 \sin \left(\frac{(N-78)180}{92}\right)}$

This approximation is made by noting that:

1. The graph is a sine graph.
2. Its maximum magnitude is 9.65.
3. It begins its cycle at the vernal equinox (78 days after the perihelion) and completes one cycle after about 183 days, at the autumnal equinox. From the autumnal equinox to the next vernal equinox, it repeats one cycle.

Therefore we now have a workable approximate formula for finding the equation of time, using the number of days that have passed after perihelion:

$$\mathbf{E = E_1 + E_2 = -7.64 \sin (360N/365.24) + 9.65 \sin \left(\frac{(N-78)180}{92}\right)}$$

The exact formula for the equation of time can be found from the book "Astronomical Algorithms" by Jean Meeus ([8]).

At this point of time, it might be interesting to let the readers know that the definition of the equation of time might be different in different references. For example, in reference ([8]) it is defined as $E = T - M$; in French almanacs and in older textbooks ([4]), it is defined as $E = M - T$. (E denotes equation of time, M denotes mean Sun time and T denotes true Sun time.) The difference in the definition might be related to the purpose of the equation. In olden times when clocks have not been invented, people depend on sundial for time. In other words, the time employed is the true Sun time. Thus to obtain the mean Sun time, they need to add E as defined by the second equation to the true Sun time. However after clocks are invented, people depend on mean Sun time. For them to obtain true Sun time from the mean Sun time (clock time), they need to add E as defined by the first equation to the mean Sun time. In this thesis, we employ the first equation, which is $E = T - M$.

3.5 Time of Sunrise and Sunset for Latitude 44°N

In this section, I would like to discuss a case study by Stan Wagon ([14]). All the data in this article are for the year 1988 at latitude 44°N. The equation of time graph that we have obtained in figure 19 can be manipulated to show how the time of solar noon changes throughout the year. **Solar noon** is the time of Meridian passage of the Sun. Figure 20 is the plot of the time difference of solar noon from 12:00 noon according to the watch against the days of the year. The graph in figure 20 is actually a reflection of the equation of time graph with respect to the x-axis. This is because when the true Sun is behind the mean Sun by t minutes (as shown by negative values in the equation of time graph), it means that solar noon will occur t minutes after 12:00 noon (i.e. represented by positive values in figure 20). If we know the amount of half daylight for a particular day, we can obtain its sunrise and sunset time by subtracting and adding half daylight time respectively to the time of solar noon. Figure 21 shows the amount of half daylight over the year. As expected, the amount of half daylight is least at the winter solstice and greatest at the summer solstice. Note that the magnitude of the peaks in figure 21 will change as the observer's latitude changes.

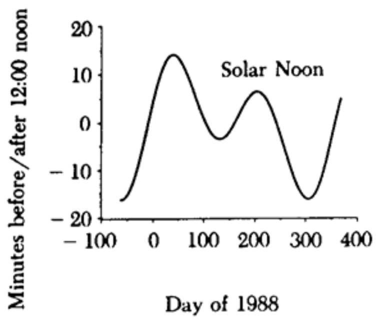


Figure 20. Solar Noon

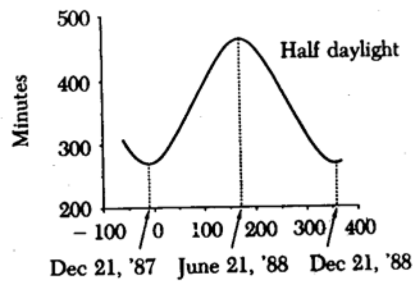


Figure 21. Half daylight

By subtracting figure 21 from figure 20, we can obtain the variation of time of sunrise throughout the year, figure 22. Similarly, by adding figure 21 to figure 20, we can obtain the variation of time of sunset throughout the year, figure 23. By observing the extrema for each resultant graph, we would be able to deduce the earliest or latest sunrise and sunset. For example, the latest sunrise is on January 4 and the earliest sunset is on December 9.

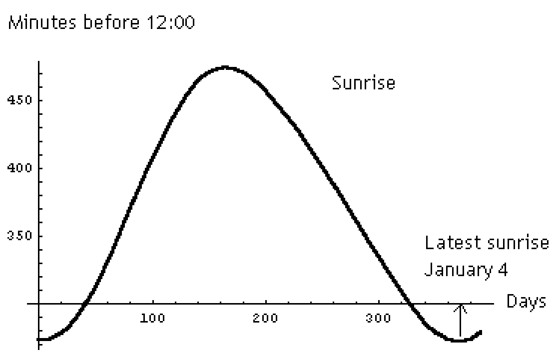


Figure 22. Time of sunrise

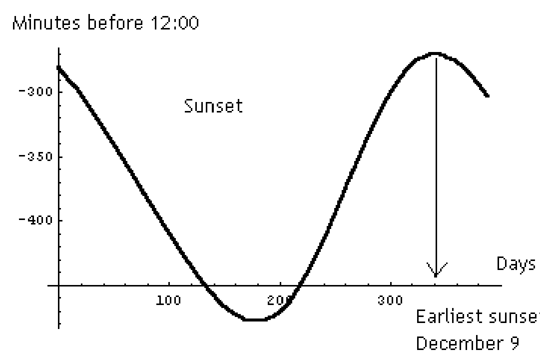


Figure 23. Time of sunset

Figure 24 plots both the sunrise and sunset curves on the same graph. From figure 24, we can locate the extrema of the sunrise and sunset and the longest and shortest day of the year. The length of daylight is obtained by finding the difference in time between the sunrise and the sunset. The vertical length between the two graphs on any day is representative of the length of daylight on that day. If the extrema A and B fall on the same day, we can conclude easily that the shortest day of the year fall on that particular day. However from figure 24, we see that this is not the case. The latest sunrise and the earliest sunset do not fall on the shortest day.

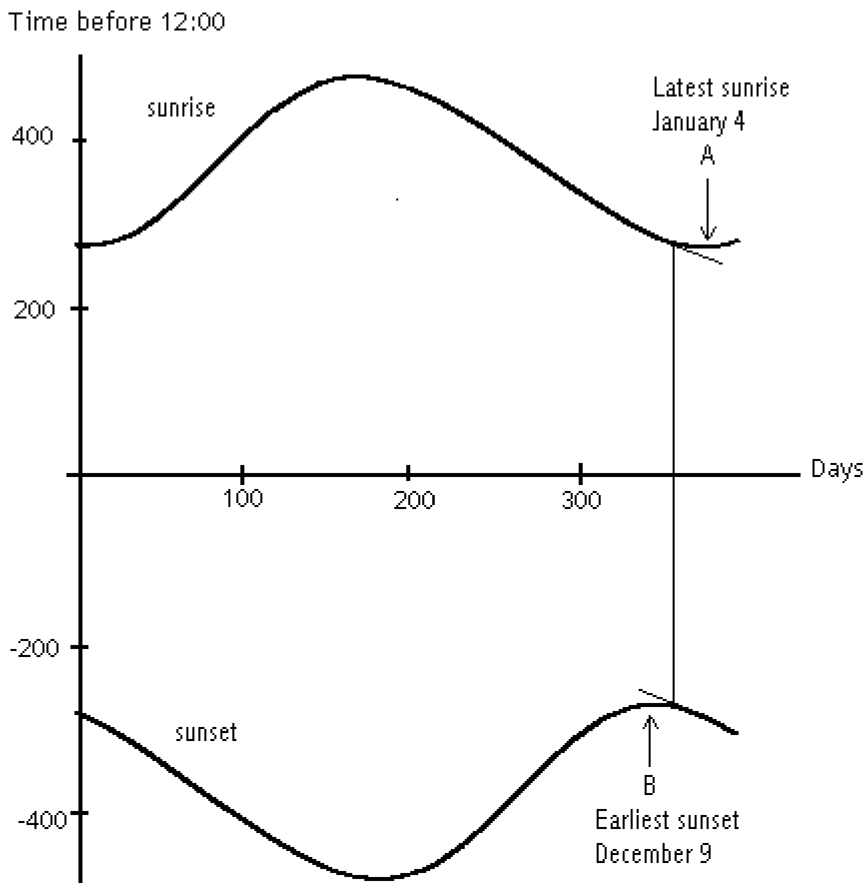


Figure 24. Variation in sunrise and sunset times over a year

From December 9, the Sun starts to set later, thus it seems doubtful that the shortest day will fall after December 9. However this is possible because during this period, the Sun is rising later each day and reaches the latest sunrise on January 4. We know that for points near the turning point, the rate of change is small; and for points away from the turning point, the rate of change is big. A few days after December 9, the rate at which the Sun is setting later is smaller than the rate at which the Sun is rising later, thus the length of daylight gets shorter. But as we move further away from December 9 towards December 21, the rate at which the Sun sets later increases and the rate at which the Sun rises later decreases. Eventually on December 21, the two rates are equal. On the other hand, after passing December 21 the rate at which the Sun rises later is smaller than the rate at which the Sun sets later, thus the day starts to get longer.

From the above we can understand why the earliest sunset and the latest sunrise need not fall on the shortest day. To see it graphically, let function f equals the length of daylight:

$$f = \text{Time of sunset} - \text{Time of sunrise}$$

Minimum and maximum daytime occur when the derivative of f is zero. This happens when the derivative of the sunset curve is equal to the derivative of the sunrise curve. Therefore when the tangents of the two curves are the same, that point is either the shortest day or the longest day. Figure 24 shows the point for the shortest day (winter solstice) where the gradients of the two graphs are equal. We can do the same for the longest day.

The article by Stan Wagon also introduces the term **solar day**, which is the period from one solar noon to the next solar noon. Note that the average duration of a solar day is 24 hours, which is different from our usual 12-hour definition of "day" to be the period from sunrise to sunset. One surprising feature of the solar noon graph (figure 20) is that the longest solar day is the day from December 22-23: that is when the graph has the most positive slope. This is surprising because the longest solar day actually falls very close to the day commonly considered to be the shortest day of the year (December 21). Figure 25 shows us how the solar day length varies throughout the year. We simply need to compute the derivatives of the solar noon graph to obtain figure 25.

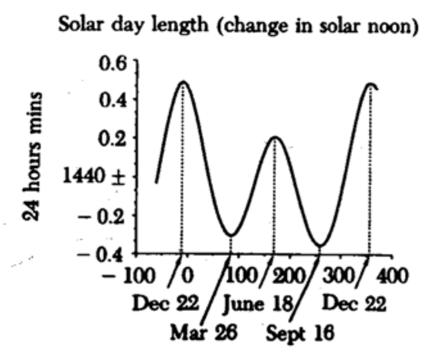


Figure 25. Solar day length

From figure 25, we see that if we define day to be from one solar noon to the next solar noon, the longest day of the year will be on Dec 22-23 and the shortest day of the year will be on Sept 16-17. We have already mentioned how the solar noon graph is related to the equation of time. If there were no equation of time, the solar noon graph will be a horizontal line intersecting the y-axis at 12:00 noon. In other words, the time from one solar noon to the other solar noon will be exactly 24 hours every day.

From the above, we see that the definition of the word "day" is critical to our discussion of the dates of the "longest" or "shortest" day. Note that for the other sections in this thesis, the word "day" is of the conventional definition (12-hour period from sunrise to sunset).

Chapter 4

Analemma

4.1 Analemma Curve

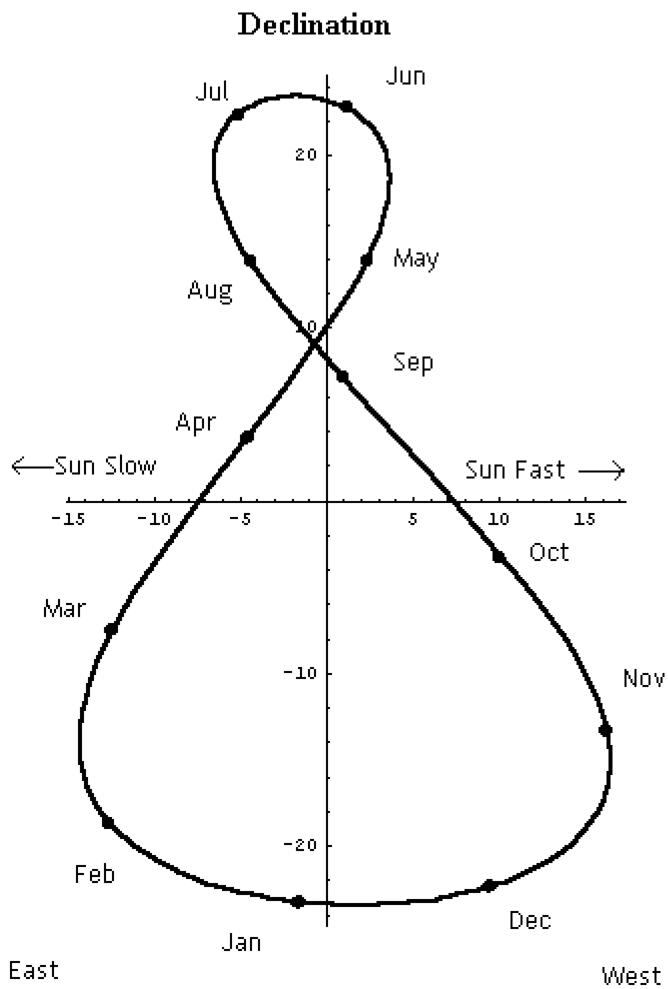


Figure 26. Analemma curve

Figure 26 is an **analemma curve**. The word " analemma" originated from the Latin name of a sort of sundial. The analemma curve shows the positions of the true Sun in the sky, at the same time (which can be arbitrarily chosen) throughout the year. The y-axis in the graph represents the declination of the Sun in the sky for one year, going from -23.45° in the winter to +23.45° in the summer. The x-axis represents the difference in time from what your watch reads to the actual position of the Sun in the sky. In other words, the x-axis represents the equation of time. Recall that when the equation of time is negative, it means that the Sun is to the east of where it should have been according to the watch (i.e. the true Sun is behind the mean Sun). When the equation of time is positive, the opposite applies.

If the Sun's annual path around the sky is along the celestial equator and that its motion is at a uniform rate, we would not be able to obtain this curve. Instead, we would just get a point at the zero coordinate of the declination-time graph.

As mentioned earlier, if you could record the position of the Sun in the sky at the same time everyday, you would be able to obtain the analemma curve. There is an easier way to see this effect. First of all, we need to find a place where the Sun shines on the ground at noon all year long. Then, place a rod of length about 3 feet long into the ground. On the first day of every month, at the same time every day, another short rod is placed onto the spot where the Sun makes a shadow with the tip of the longer original rod. At the end of 12 months, we would see that the 12 short rods form a slightly distorted figure-of-eight pattern.

Notice that the analemma crosses itself at one point. This intersection point represents two dates of the year. The Sun's declination and the equation of time for one are equal to those of the other. Since this point does not correspond to zero equation of time, the cross is located slightly away from the vertical axis. In addition, the equation of time is also not zero at the solstices or equinoxes. As a result, the analemma is skewed somewhat and does not line up precisely with the vertical axis.

Figure 27 shows how the analemma would look like if the Earth's orbit were tilted but circular. The resulting analemma is a perfect figure-of-eight with the points representing the solstices and equinoxes lying on the vertical axis. The reason why we can obtain this perfect figure-of-eight is because when the Sun's declination changes from +23.5° to –23.5° from summer solstice to winter solstice, the tilt effect graph coincidentally complete one cycle; and when the Sun's declination changes from –23.5° to +23.5° from winter solstice to the next summer solstice, the tilt effect graph repeats another cycle. Figure 28 shows how the analemma would look like if the Earth's orbit were elliptical but untilted. Since the Sun's declination does not affect the dates of the perihelion and aphelion, the solstices do not have any "special" values for the equation of time. In particular, the solstice points of this analemma do not lie on the vertical axis. When the tilt effect and the eccentricity effect are summed, we obtained figure 26 and this accounts for the resulting analemma being skewed.

As mentioned earlier, the points representing the solstices have zero equation of time if we only consider the tilt effect. However for the eccentricity effect, the point representing the summer solstice has approximately –1.4 minute equation of time; and the point representing the winter solstice has approximately +1.4 minute equation of time. Summing up both effects, we see that the points representing the summer solstice and the winter solstice will have –1.4 minute equation of time and +1.4 minute equation of time respectively. Therefore the resultant analemma will have its upper tip tilted to the left and the lower tip tilted to the right.

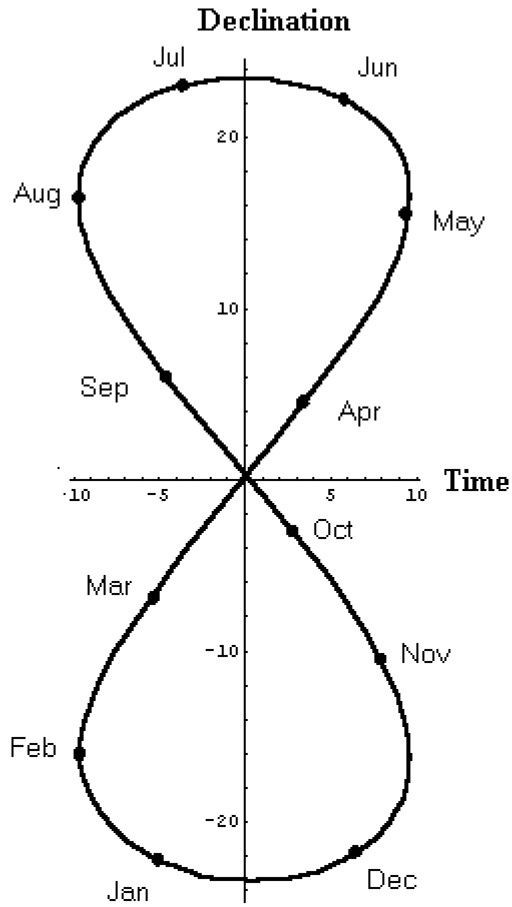


Figure 27. Tilt effect

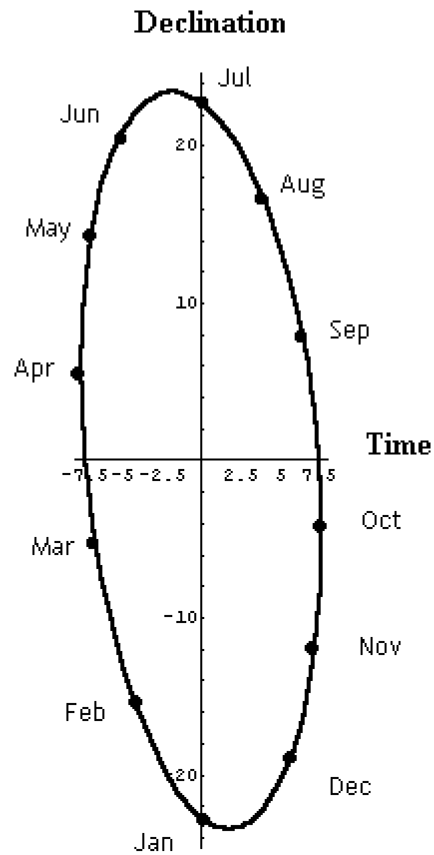


Figure 28. Eccentricity effect

It is necessary to note that from the vernal equinox to the autumnal equinox, the points representing this period on figure 27 and figure 28 have opposite signs. For example, the point representing April 1 on figure 27 has positive equation of time; but on figure 28, it has negative equation of time. This is due to the tilt effect following a figure-of-eight graph and changes sign on passing the equinoxes and the solstices. Nevertheless, there is a small discrepancy in the above comment. From the summer solstice to July 3 (aphelion), the points representing this period have negative equation of time for both figure 27 and figure 28. On the other hand, most of the points representing the period from the autumnal equinox to the vernal equinox have the same sign on both graphs. As a result, from the vernal equinox to the autumnal equinox, the tilt effect and the eccentricity effect counteract each other; from the autumnal equinox to the vernal equinox, the tilt effect and the eccentricity effect reinforce each other. Therefore, the resultant analemma curve on figure 26 is a distorted figure-of-eight with a "small head" and a "heavy bottom".

It is important for the readers to note that the assignment of the months to the analemma in figure 26 is based on how the noon Sun will vary in position in the sky. However in some references such as ([10]) and ([14]), the analemma curve is obtained using sundial, and as a result the assignment of the months to the analemma is based on how the shadow of the noon Sun will vary in position. As a result, the analemma obtained by the first method will be a reflection of that obtained by the second method, along the y-axis. Thus it might be confusing to see that, for example, sometimes the month February is located to the left of the y-axis (figure 26) and sometimes it is located to the right of the y-axis (figure 29). The point is we must be clear of how the analemma curve is obtained, via observation of the Sun's position in the sky or via the sundial.

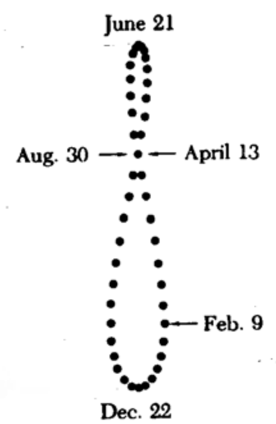


Figure 29. Analemma obtained via sundial

4.2 Azimuth and Altitude

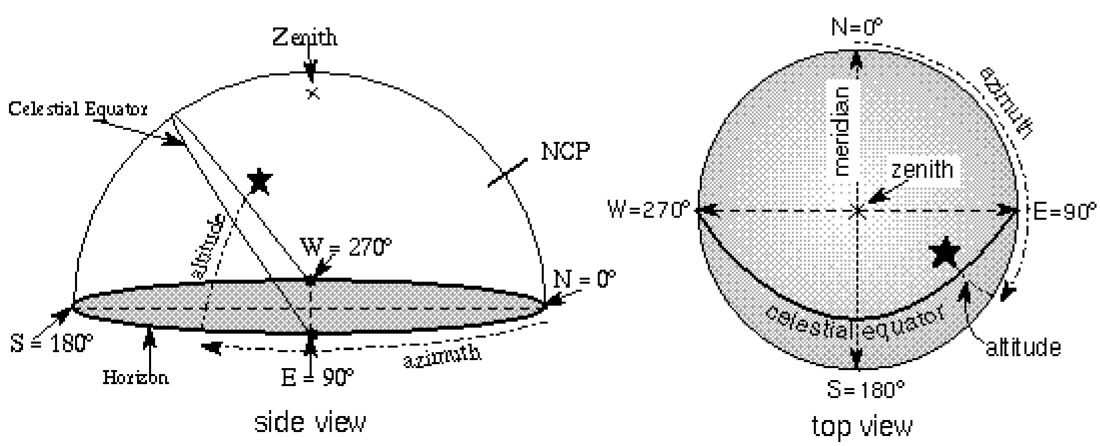


Figure 30. Azimuth-altitude coordinate system

In order to determine the position of the analemma at our horizon at different time, we need to first know the altitude-azimuth coordinate system. The **altitude** of a celestial body is the number of degrees by which it is above the horizon ([13]). Its values range from 0° to 90°. The **azimuth** of a celestial body is the number of degrees by which it is along the horizon and corresponds to the compass direction. Azimuth starts from exactly North = 0 degrees azimuth and increases clockwise: exactly East = 90 degrees, exactly South = 180 degrees, exactly West = 270 degrees. The above is the conventional definition for azimuth. However it must be noted that in some books, azimuth is defined to be zero at South and increases as we move clockwise ([8]). The azimuth-altitude coordinate system depends on the position of the observer's horizon. Therefore two observers at different latitudes, looking at the same star will have different azimuth-altitude coordinates for the star.

The altitude of the Sun is dependent on its declination. Figure 31 shows how altitude of the noon Sun differs on the solstices and equinoxes [6]. For an observer in the northern hemisphere, when the declination of the Sun is x and the latitude of the observer is l, altitude of the noon Sun = $90° - l + x$

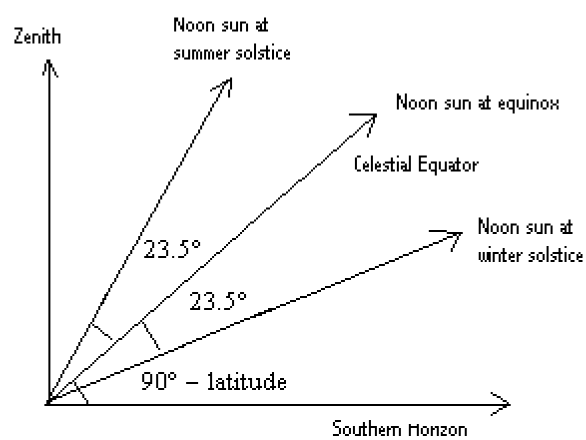


Figure 31. Altitude of the Sun

If we know the latitude Ψ of the observer, declination δ of the Sun and its hour angle H, we would be able to calculate its altitude h and azimuth A using the following equations ([8]):

$$\tan A = \frac{\sin H}{\cos H \sin \Psi - \tan\delta \cos \Psi}$$

$$\sin h = \sin\Psi \sin\delta + \cos\Psi \cos\delta \cos H$$

4.3 Rising Analemma

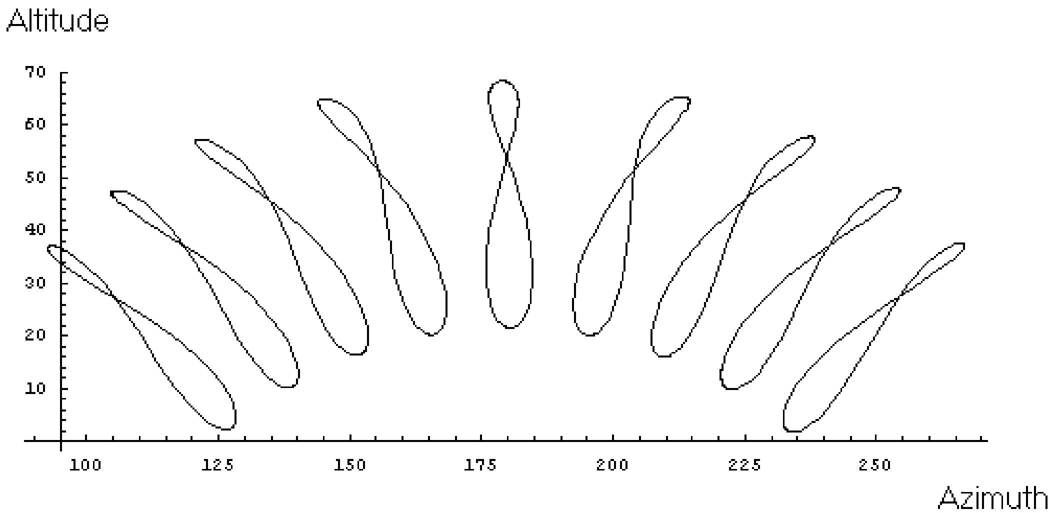


Figure 32. Analemma across the sky

By imagining the existence of the analemma in the sky, rising in the morning and setting in the evening, the problem of earliest or latest sunrise and sunset can be easily solved. Figure 32 shows the analemma's path across the sky starting from the east to the west for an observer in the northern hemisphere. Note that some of the analemmas look distorted. These distortions result from the flattening of the azimuth-altitude hemisphere of the observer to a 2-dimensional view in figure 32. In actual fact, when the analemma is in the sky, we will not see these distortions. (Refer to appendix for the 3-dimensional view.) The analemma reaches maximum height when it is at noon, since the altitude of the Sun is maximum when the Sun passes the Meridian. Figure 32 is generated using the Mathematica package Calendrica and the altitude and azimuth equations in section 4.2.

Let us denote the axis of the analemma to be the line passing through the points representing zero equation of time on the analemma. Recall that the meridians on the celestial sphere are a measure of mean time and their planes are perpendicular to the celestial equator. Thus if the time is say 9 a.m., the axis of the analemma should lie on the 9 a.m. meridian, since the 9 a.m. meridian represents zero equation of time at that moment.

Since the meridians lie on the surface of the celestial sphere, each meridian curves in such a way that it is only parallel to the North-South axis of the celestial sphere at the halfway mark of the meridian. This halfway mark is actually the intersection point of the celestial equator and the meridian. The angle between the horizon and the North-South axis is equal to the latitude of the observer. Therefore at the point where the horizon intersects with the halfway mark of a meridian, we can take the angle between the horizon and the meridian to be the latitude of the observer. Note that this particular point, if it exists, is actually the intersection point of three great circles: the celestial equator, a meridian, and the horizon. There are 2 meridians that satisfy the above condition: the 6 a.m. meridian and the 6 p.m. meridian.

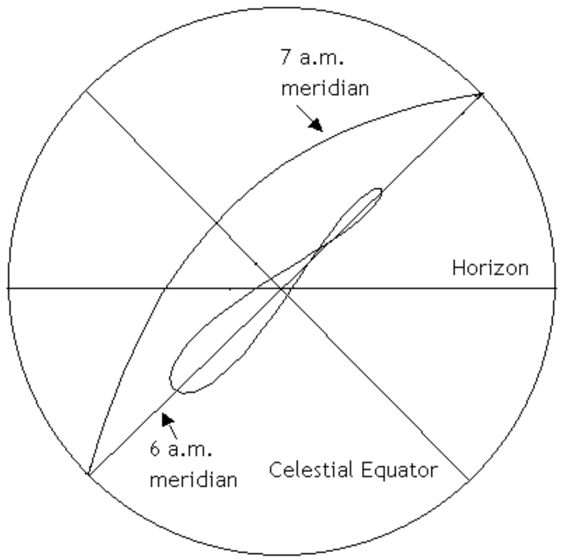


Figure 33. Intersection point of horizon, celestial equator and meridian

Thus if we define the tilt of the analemma to be the angle between the analemma axis and the horizon, the tilt of the 6 a.m. or 6 p.m. analemma gives the latitude of the observer. However as can be seen from figure 33, at 6 a.m. or 6 p.m. part of the analemma would be hidden under the horizon, thus it would be difficult to locate the axis of the analemma. As a result, to approximate the latitude of the observer, we use the first complete analemma that is visible above the horizon or the last complete analemma that is visible above the horizon (before it sets).

Let the "rising analemma" refer to the analemma that is rising and having is last point on the horizon. Let the "setting analemma" refer to the analemma that is setting and having its first point on the horizon. From the tilt of the rising analemma or setting analemma, we can approximate the latitude of the observer. Note that this angle is not exactly equal to the latitude of the observer. For an observer at low latitudes, the rising analemma can be seen less than one hour after 6 a.m. and the setting analemma can be seen less than one hour before 6 p.m. As a result, the deviation is very slight at low latitudes. However as the latitude of the observer increases, the deviation increases too. Nevertheless, the deviation is not too much to be noticeable.

If we ignore the deviation, for an observer at the North Pole the analemma always stands straight up and down, since the 'rising analemma' will be tilted 90 degrees from the horizon. For an observer at the equator, the analemma rises and sets lying on its side. In other words we will see a horizontal rising and setting analemma.

Figure 34 illustrates how the position of the rising analemma changes with the change in latitude of the observer. Let the latitude of the observer be *l*. Turn the above figure anti-clockwise by (90° - l). The resulting analemma would be the orientation of the rising analemma at latitude l .

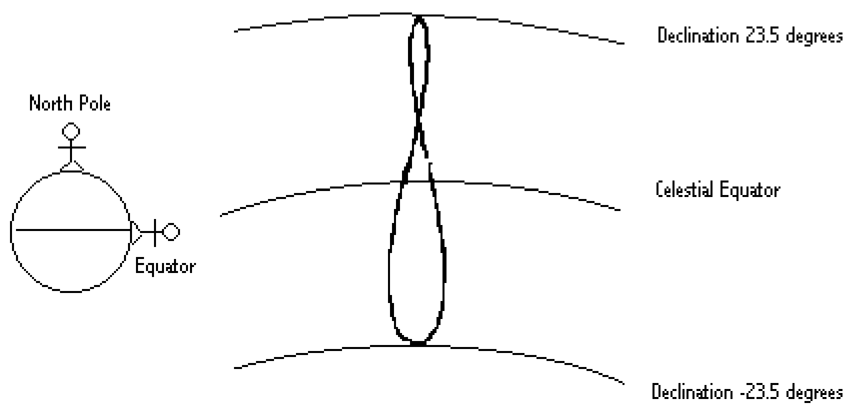


Figure 34. Rising position of analemma

Chapter 5

Tropical Issues

5.1 Analemma Rise and Set

We can use the rising analemma to help us deduce the dates when the earliest and the latest sunrise occur ([12]). Figure 35 shows us the "analemma-rise" for the northern hemisphere, which is the time when the lowest part of the figure–8 leaves the southeastern horizon. This happens at the same clock time every morning. The Sun's position at this particular clock time will change throughout the year, along this analemma. When the Sun reaches the particular spot, which is the lowest part of this tilted analemma, the latest sunrise of the year occurs. Similarly, the earliest sunrise occurs when the Sun reaches the spot on the analemma that is the first point to leave the horizon every morning. It is clear from the diagram that the earliest sunrise does not occur at the summer solstice and the latest sunrise does not occur at the winter solstice. Instead the earliest sunrise occurs before the summer solstice, at around early June; and the latest sunrise occurs after the winter solstice, at around early January.

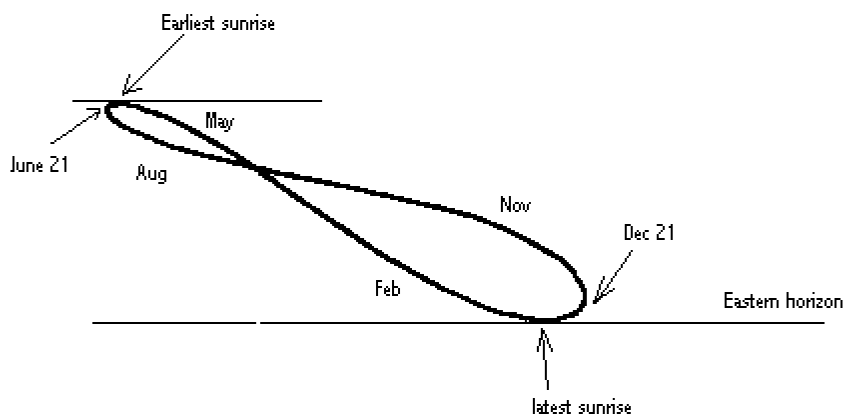


Figure 35. Analemma-rise in the northern hemisphere

Now visualize the analemma-set in the western sky at a place in the northern hemisphere. At the same time each afternoon, the lowest part of the analemma dips below the horizon. On the calendar date when the Sun reaches this spot on the curve, the earliest sunset of the year occurs. Similarly, when the Sun reaches the spot on the curve that is the last to dip below the horizon, the latest sunset occurs. Figure 36 shows the analemma-set. Again, we see that the earliest sunset does not fall on the winter solstice, instead it occurs before the winter solstice, at around early December; the latest sunset falls after the summer solstice at around late June.

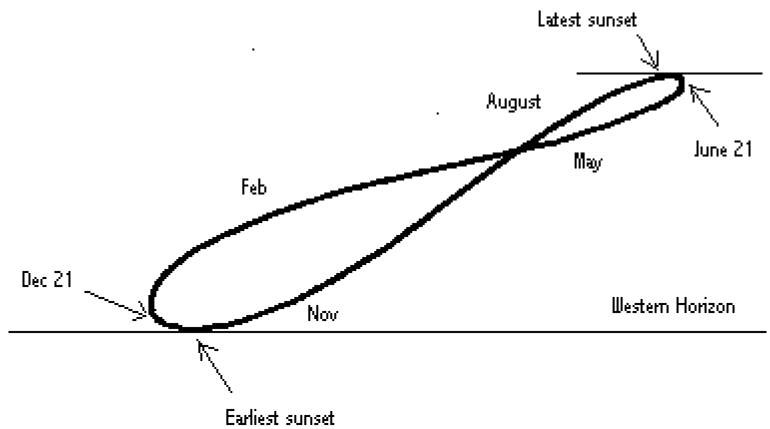


Figure 36. Analemma-set in the northern hemisphere

The shortest day lies about midway between the dates of latest sunrise and earliest sunset. Similarly, the longest day lies about midway between the dates of the earliest sunrise and the latest sunset. All of these events rely on the fact that the analemma has a curve at either end. If it did not (if there were no equation of time), then the analemma would be a north-south line instead of a figure-8. The earliest sunrise, longest day, and latest sunset would coincide on the same day, the summer solstice. The latest sunrise, the shortest day and the earliest sunset would coincide on the winter solstice.

Recall that in section 4.3 we mentioned that the angle between the axis of the rising analemma and the horizon is equal to the latitude of observer. Similarly the angle between the axis of the setting analemma and the horizon is equal to the latitude of the observer. Thus in figure 37, the angle between the eastern horizon and the axis should be equal to the angle between the western horizon and the axis. Note that the points A, B, C and D represent the earliest sunrise, latest sunset, earliest sunset and latest sunrise, respectively. If the analemma were a perfect figure-of-eight, we would expect the number of days between A and the summer solstice to be equal to the number of days between B and the summer solstice. Similarly, the number of days between C and the winter solstice should be equal to the number of days between D and the winter solstice. However since the analemma is slightly distorted, there is no exact symmetry. Nevertheless, the deviation is very slight.

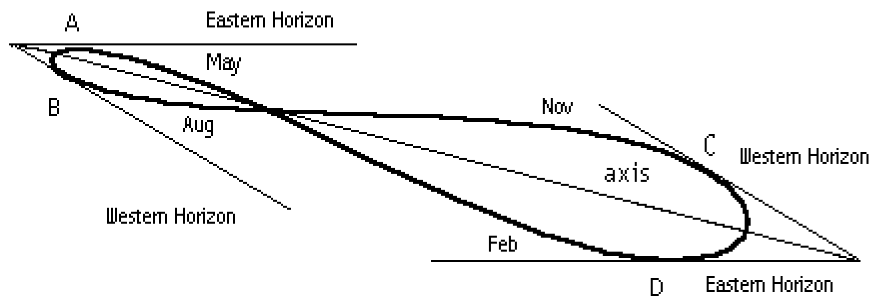


Figure 37. Relationship between the axis and the horizon

The curve on the top of the analemma is smaller than the curve at the bottom of the analemma. This is expected as we can see that the 2 maximas of the equation of time graph are not of the same magnitude. As a result, the number of days in which the earliest sunset and the latest sunrise are away from the winter solstice may not be equal to the number of days in which the latest sunset and the earliest sunrise are away from the summer solstice.

Since the angle between the analemma and the horizon changes with latitude, we would expect that the day on which the earliest sunrise occurs varies with latitude too. In section 5.2, we would discuss more on the relationship of the latitude with the date of the earliest and latest sunrise.

It is important to note that the rising analemma for the southern hemisphere is orientated differently from that of the northern hemisphere. The rising analemma in the northern hemisphere has its bottom tipped down and the top tipped up. However, the rising analemma of the southern hemisphere has the bottom tipped up and the top tipped down. The dates of the summer solstice and winter solstice in the southern hemisphere are reversed from those of the northern hemisphere. Thus we would expect the earliest sunrise and the latest sunset to occur somewhere in December; the latest sunrise and the earliest sunset to occur somewhere in June. Figure 38 illustrates the case for the southern hemisphere.

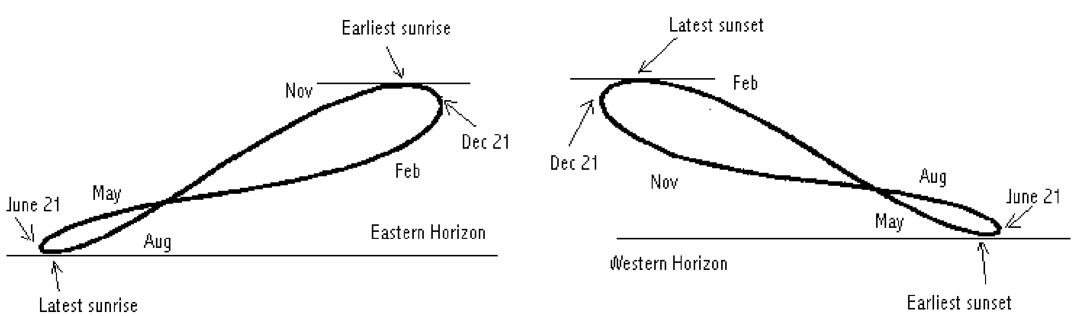


Figure 38. Rising and setting analemmas in the southern hemisphere

Now, let us compare the case for the equator with those of higher latitudes. For higher latitudes, the extrema of the sunrise and sunset are found near the tips of the analemmas. But at the equator, due to the rising and setting analemmas being horizontal, these extrema are located further away from the tips. As a result the earliest and the latest sunrise are located more than one month away from the winter solstice. The earliest sunrise occurs on November 3 and the latest sunrise occurs on February 10. Figure 39 illustrates why this is so. The earliest sunset and the latest sunset also fall on November 3 and February 10 respectively. This is because the setting analemma is actually a 180 degrees rotation of figure 39. Therefore the point that rises first will also set first.

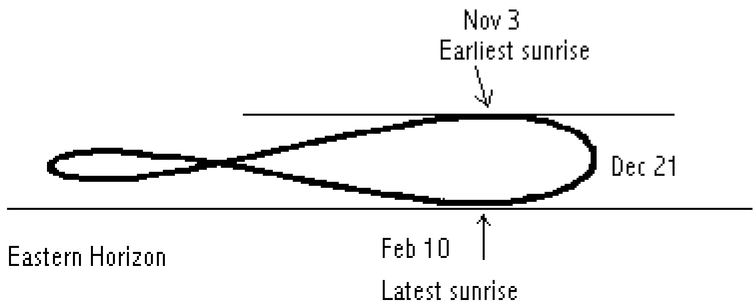


Figure 39. Rising analemma at the equator

Notice that the two points that represent November 3 and February 10 on the analemma are actually the points farthest away from the line representing zero equation of time. Recall that the maximum magnitude of the equation of time is around 15 minutes. Therefore these two points are around 30 minutes away from each other in terms of equation of time. Thus this explains why the clock time of the earliest sunrise and the clock time of the latest sunrise are around 30 minutes apart, even though the equator experiences equal amount of daylight every day.

It may seem that the earliest sunrise only occurs once in the course of the year. But this is not true for all latitudes. In particular for the latitude of 5° N, we notice that the analemma has two humps as it rises. As a result there are two earliest points that leave the horizon together. When the Sun is at these two points, May 23 and October 24, a place at latitude 5° N experiences its two earliest sunrises of the year.

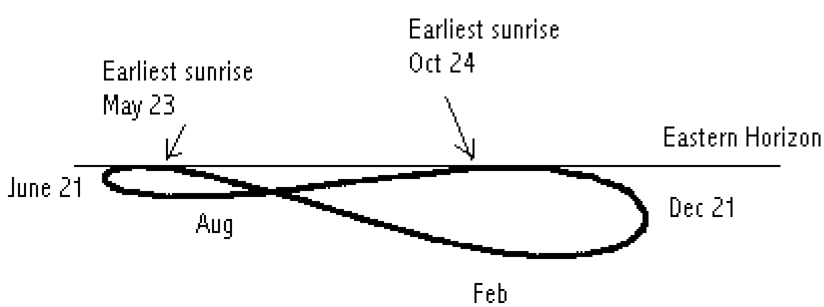


Figure 40. Morning analemma at latitude 5° N

Next, observe that although the analemma curves sharply at the top and bottom, it is almost straight in the middle. Thus, if there exists a particular latitude that will result in this straight part of the analemma to be parallel to the horizon, we would expect a period of dates that will have the same clock time for the sunrise. In particular, a location at latitude 14°N satisfies this condition.

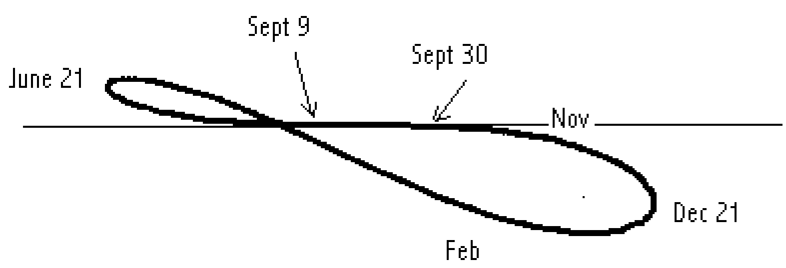


Figure 41. Morning analemma at latitude 14°N

In fact, from about September 9 to September 30, while the Sun travels along this portion of the analemma, sunrise occurs at virtually the same time every morning. Each day the Sun appears on the horizon at a point progressively farther south in azimuth.

5.2 "Tug of War"

From section 5.1, we see that the tilt of the rising analemma and the width of the analemma determine which points leave the horizon first or last. The width of the analemma tends to move the earliest sunrise and the latest sunset away from the summer solstice towards the points on the analemma furthest away from the analemma axis, November 3 and February 10, respectively. However as the tilt of the analemma increases, the width effect is undermined by the tilt effect and the earliest sunrise and the latest sunset shift towards the summer solstice.

The tilt of the rising analemma is dependent on the latitude of the observer and the width of the analemma is dependent on the equation of time. Thus this means that there is a constant tug-of-war between the latitude of the observer and the equation of time in determining the dates of the earliest sunrise and latest sunset.

According to the equation of time, the Sun is fastest on November 3 and slowest on February 10. Thus if we only consider the equation of time, the earliest sunrise should be on November 3 and the latest sunset should be on February 10. However as latitude increases, the earliest sunrise and the latest sunset shift towards the summer solstice. As a result, if the equation of time were the dominating factor, the earliest sunrise would be near to November 3 and the latest sunset would be near to February 10. If the latitude of the observer were the dominating factor, the earliest sunrise and the latest sunset would be near to June 21. Specifically due to the orientation of the months on the analemma, the earliest sunrise would lie in late May to June 21 and the latest sunset would lie in mid July to June 21.

So when does the effect of one dominate that of the other? Let us look at the following table.

Latitude	**Earliest Sunrise**	**Latest Sunset**	**Latest Sunrise**	**Earliest Sunset**
0° N	Nov 3	Feb 10	Feb10	Nov 3
1° N	Nov 1	Feb 12	Feb 9	Nov 4
2° N	Oct 30	Feb 14	Feb 7	Nov 5
3° N	Oct 28	Feb 16, Jul 20	Feb 5	Nov 7
4° N	Oct 26	Jul 19	Feb 4	Nov 8
5° N	Oct 24, May 23	Jul 17	Feb 2	Nov 10
6° N	May 24	Jul 16	Jan 31	Nov 11
7° N	May 25	Jul 15	Jan 30	Nov 12
15° N	June 2	Jul 8	Jan 21	Nov 21

Figure 42. Table for the extrema of sunrise and sunset at different latitudes

From the table, we see that with respect to the earliest sunrise, from latitude 0° N to 5° N, the equation of time component dominates. At latitude 5° N, both effects are on par, thus two earliest sunrises occur, one near to November 3 and one near to June 21. After latitude 5° N, the latitude component dominates and from then on, the earliest sunrise jumps to late May and progresses towards June 21.

With respect to the latest sunset, from latitude 0° N to 3° N, the equation of time component dominates. At latitude 3° N, both effects are on par, thus two latest sunsets occur, one near to February 10 and one near to June 21. After latitude 3° N, the latitude component dominates and from then on, the latest sunset jumps to mid July and progresses towards June 21.

Similarly we would expect the earliest sunset to be located near November 3 at low latitudes and progresses towards December 21 as latitude increases; and the latest sunrise to be located near February 10 at low latitudes and progresses towards December 21 as latitude increases. However the earliest sunset and the latest sunrise progress towards December 21 smoothly, unlike the earliest sunrise and the latest sunset, which progress towards June 21 with a jump in between. To explain this, note that for the earliest sunset, to progress from November 3 to December 21 only take approximately 1½ month. This is similar for the case of the latest sunrise. However for the earliest sunrise to progress from November 3 to June 21 requires approximately 4½ months. Thus a smooth progression is not possible and a jump resulted. The same applies to the latest sunset.

5.3 Daylight Saving Time in Kuching

In Europe and the United States, the clocks are switched an hour forward in the summer. This converted time is known as **Daylight Saving Time** ([3]). The main purpose of Daylight Saving Time is to make better use of daylight and save energy. Energy use and the demand for electricity for lighting in our homes are directly connected to when we go to bed and when we get up. During the summer months, sunrise is very early in the morning. Without Daylight Saving Time, most people will still be asleep many hours after sunrise. By moving the clock ahead one hour, we can make use of this one-hour daylight to do work. In addition, Daylight Saving Time "makes" the sun "set" one hour later and therefore reduces the period between sunset and bedtime by one hour. This means that less electricity would be used for lighting and appliances late in the day.

In the United States, Daylight Saving Time begins on the first Sunday of April and ends on the last Sunday of October. In Europe, Daylight Saving Time begins on the last Sunday in March and ends on the last Sunday in October. Although there is a small irregularity in the actual date when Daylight Saving Time starts, it occurs during the summer. Equatorial and tropical countries (lower latitudes) do not observe Daylight Saving Time since the daylight hours are similar during every season, so there is no advantage to moving clocks forward during the summer.

In Kuching (latitude 1.55°N) from 1935 to 1941, the clocks are switched forward by twenty minutes from September 14 to December 14 ([1]). This Daylight Saving Time in Kuching appears unusual. The existence itself is unexpected since we mentioned earlier that places in the tropics usually do not observer Daylight Saving Time. In addition the time period September 14 to December 14 do not correspond to any season markers. The time adjustment of twenty minutes is different from the usual one hour. However, this Daylight Saving Time is not chosen without reason.

Figure 43 is obtained from the Mathematica package Calendrica. The solid curve in figure 43 represents the time of sunrise for Kuching if there is no Daylight Saving Time. The local extrema A and C fall on February 8 and July 31 respectively. The local extrema B and D fall on May 16 and October 31 respectively. In fact February 8 and October 31 mark the latest and earliest sunrise in Kuching respectively. The time difference between the earliest sunrise and the latest sunrise is around 30 minutes.

Suppose we want to reduce the time between the earliest and latest sunrise, by forcing the earliest sunrise to fall on May 16 instead of October 31. Let us draw a horizontal line tangential to point B (May 16). This tangent line will cut the curve at two points, which we will name as E and F. If we shift the section of the curve from point E to F upward so that point E and F are in line with point A, we would be able to achieve our aim. In fact this is the idea behind the Daylight Saving Time in Kuching. Figure 44 shows how the time of sunrise will change after incorporating the Daylight Saving Time. Point E and F correspond to September 14 and December 14 respectively. It is necessary to note that the time of sunrise at these two points, after the upward shift, are not exactly the same as that of point A (February 8). I suppose it is more convenient to incorporate Daylight Saving Time for a full 3-month period, rather than choosing two dates that exactly correspond to the time of sunrise of point A but result in an awkward duration. To explain why the time adjustment is twenty minutes, we need to look at figure 43 again. This time interval is simply the amount by which point E and F need to be shifted upward to be in line with A. In other words, this is the time difference between the time of sunrise on February 8 and the time of sunrise on May 16 (figure 44). As a result after incorporating Daylight Saving Time, the time difference between the earliest sunrise and the latest sunrise is reduced to around 20 minutes.

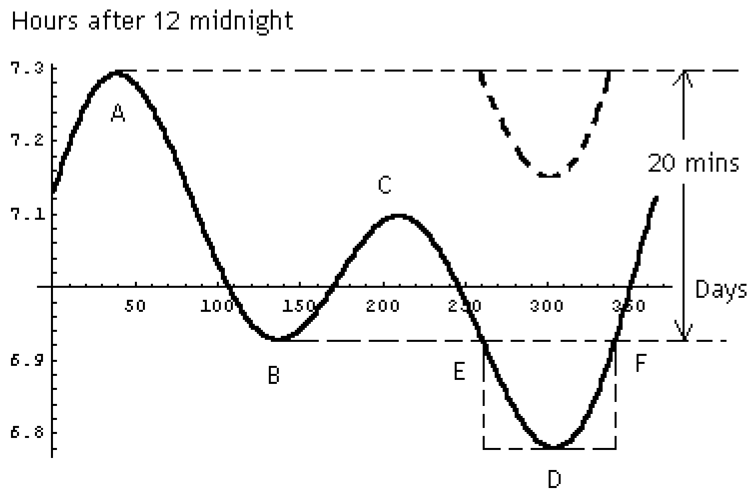


Figure 43. Time of sunrise in Kuching

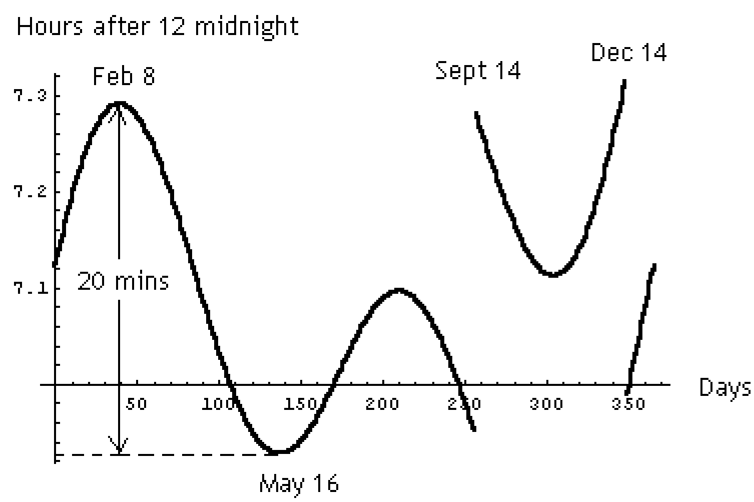
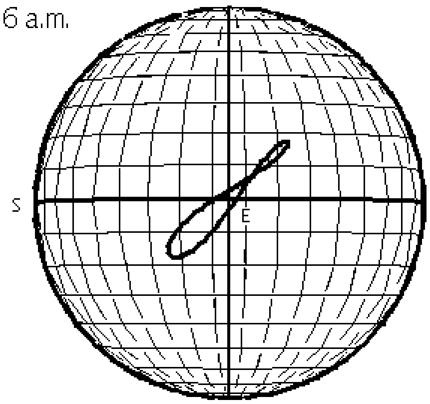


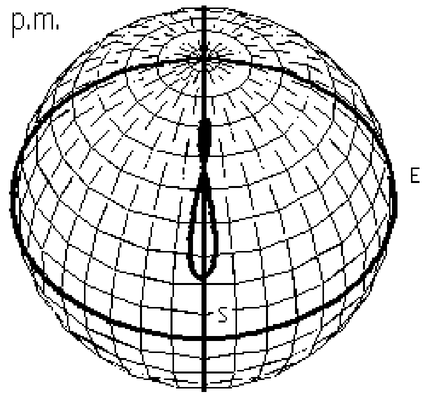
Figure 44. Time of sunrise in Kuching, with daylight saving time

Appendix

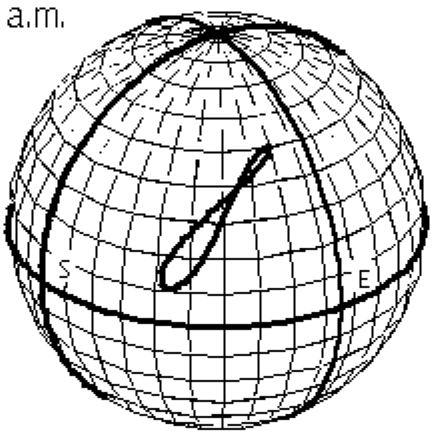
6 a.m.



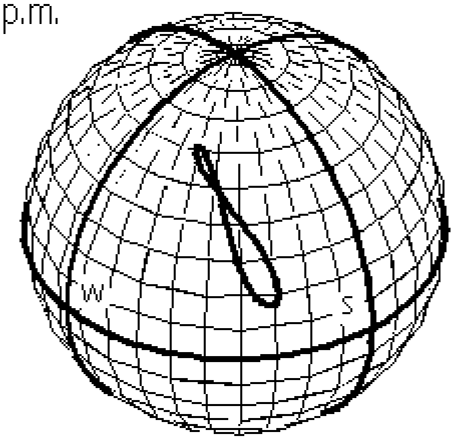
12 p.m.



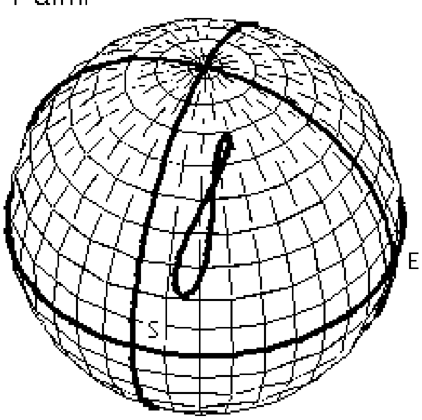
9 a.m.



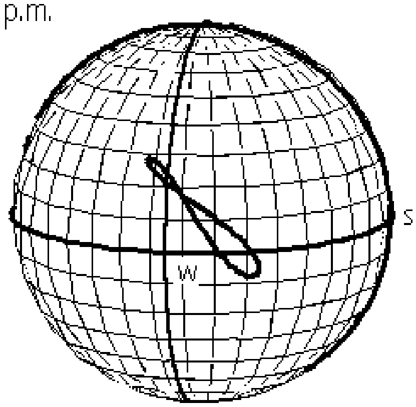
2 p.m.



11 a.m.



5 p.m.



Appendix

These 6 pictures show the position of the analemma at latitude 45° N at various time of the day. Included in the diagrams are the horizon, the Meridian and the line that runs from East to zenith to West. Note that we are viewing the analemma from outside the celestial sphere. These diagrams are created using the Mathematica package Calendrica.

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