HOMEWORK 6 – Extra credit

Print this PDF and write your solutions in the space given. Your submission should consist of a scan of this PDF with addional scanned pages that you think are necessary to better explain your answer. Remember to fill out your name and student ID. You may write your answer directly into the PDF fields.

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Question	1	2	3	4	5	6
Points	10	10	10	20	20	30

1. For each of the following statements, indicate whether the statement is true or false.

   T/F F The Gauss-Seidel iterative method for solving a system of linear equations $Ax = b$ always converge.

   T/F T The Gauss-Seidel method is a special case of SOR (successive over-relaxation).

   T/F T Given $Ax = b$, where A is a $n \times n$ matrix and x and b are n-vectors, and ignoring rounding errors, the conjugate gradient converges after at most n steps.

   T/F F Preconditioning can be used with conjugate gradients to avoid ill-conditioned problems.

2. Let A be a nonsingular matrix. Denote the strict lower triangular portion of A by L, the diagonal of A by D, and the strict upper triangle by U.

   (a) Express the Jacobi iteration scheme for solving the linear system $Ax = b$ in terms of L, D, and U.

   $D(x_{k+1}) = -(L+U)x_k + b.$

   Detailed explanation is attached. Pg ①

   (b) Express the Gauss-Seidel iteration scheme for solving the linear system $Ax = b$ in terms of L, D, and U.

   $(D+L)(x_{k+1}) = -Ux_k + b$

   Detailed explanation is attached. Pg ①

3. For the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$$

what is the splitting for the Jacobi and Gauss-Seidel methods?

Jacobi:

$D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ $L = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}$ $U = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$

Split is $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ & $\begin{pmatrix} 0 & -3 \\ -5 & 0 \end{pmatrix}$ Pg - 1

Gauss-Seidel:

Split is $\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}$ & $\begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}$ Pg - 2

4. Prove that the SOR method diverges if ω does not lie in the interval $(0, 2)$.

Explaination is attached

Pg - 3

5. Suppose we decompose $A = M - N$, where M is invertible. Show that the iterative scheme $\mathbf{x}_{k+1} = M^{-1}(N\mathbf{x}_k + \mathbf{b})$ converges to $A^{-1}b$ when the maximum eigenvalue of $M^{-1}N$ is less than 1. *Hint:* Define $\mathbf{x}^* = A^{-1}b$ and take $\mathbf{e}_k = \mathbf{x}_k - \mathbf{x}^*$. Show that $\mathbf{e}_k = G^k\mathbf{e}_0$, where $G = M^{-1}N$. For this problem, you can assume that the eigenvectors of G span $\mathbb{R}^n$.

Proof is explained at the end.

Pg - 4

6. For all parts of this problem, assume line search finds a restricted minimizer in each step of optimization.

   (a) Suppose $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ is smooth and bounded. We can run gradient descent on f twice, starting from different points $\mathbf{x}_0$ and $\mathbf{x}_1$. Will the two runs necessarily converge to the same point? Why?

~~Yes~~ No, the two points will not necessary converge to the same point.

Say, we have $y = f(x_1, x_2 \ldots)$ for $f(x_1, x_2 \ldots)$ at least two dimensions. We can construct smooth f such that $f(x, 0, 0) = ax^2$, $f(0, x, 0 \ldots) = ax^2$ & $f(x, x, 0) = bx^2$. In other words a smooth pinch. If we take a chord between $(x, 0, \ldots)$ and $(0, x \ldots)$ then $f = ax^2$ at the ends. This mid point is ~~$(x, x \ldots)$/sqrt(2)~~ $\frac{x, x \ldots}{\sqrt{2}}$ and ~~$f = bx^2/\sqrt{2}$~~ $f = \frac{bx^2}{\sqrt{2}}$. So f is not convex if $\frac{bx^2}{\sqrt{2}} > ax^2$ & $b > \sqrt{2}a$. So we can say a smooth minimum does not have to be convex.

(b) Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite and $\mathbf{b} \in \mathbb{R}^n$. If $f(\mathbf{x})$ from (a) satisfies $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x} - 2\mathbf{x}^T\mathbf{b}$, does your answer to part (a) change? Why?

Explaination Attached Pg - 6

As A is symmetric and positive definite, it is strictly convex.

(c) Suppose we run run an error-prone system from part (c). Conjugate gradients converges within numerical precision to a point $x_0 \in \mathbb{R}^n$, but $||A\mathbf{x}_0 - \mathbf{b}||_2$ is relatively large. Hypothesize what went wrong and propose a method for fixing the problem.

Explaination attached Pg - 8

5, continued $\therefore$ we can write it in the form.

1

2a In $Ax = b$, A is given as $n \times n$ non-singular & b is a column vector with n components, we split A into $A = B - C$.

where B and C are $n \times n$ matrices,

$A = B - C$ is a regular splitting of A if $B^{-1} \geq 0$ and $C \geq 0$.

If the diagonal entries of the matrix A are all non-zero, then we express A as

$$A = D - U - L.$$

(a) For Jacobi method we have the splitting form as

$$x_{(m+1)} = D^{-1}(U+L)\,x_m + D^{-1}k$$

(b) For Gauss Seidel method can be represented in matrix form as a splitting of-

$$x_{(m+1)} = (D-L)^{-1}\,U x^{n}_{(k)} + (D-L)^{-1}k.$$

3. a $$A = \begin{bmatrix} 2 & 3 \\ 5 & 4 \end{bmatrix}$$

Jacobi $A = M - N$

$$M = D$$

$$N = -(L+U)$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad L = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} \qquad U = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$$

$$\therefore N = -(L+U) = -\left[\begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & -3 \\ -5 & 0 \end{pmatrix}$$

$$= \cancel{\begin{pmatrix} 0 & 3 \\ 8 & 0 \end{pmatrix}}$$

Therefore the splitting is done as

$$M - N$$
$$= \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} - \begin{pmatrix} 0 & -3 \\ -5 & 0 \end{pmatrix}$$
$$A = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$$

Therefore the splitting satisfies,

Split is $\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$ & $\begin{pmatrix} 0 & -3 \\ -5 & 0 \end{pmatrix}$

<u>Gauss - Seidel</u>

Therefore here we have

$$M = (D + L)$$
$$N = -U.$$

$$\therefore \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}$$

$$N = \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}$$

$$\therefore M - N = \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} - \begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}$$

Split is $\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}$ & $\begin{pmatrix} 0 & -3 \\ 0 & 0 \end{pmatrix}$

4 The choice of relaxation factor ω in successive Over-relaxation is not necessarily easy and depends on the co-efficient matrix.
According to Ostrowski proves that A is Symmetric and positive definite then $\rho(L_\omega) < 1$ for $0 < \omega < 2$.
Thus, convergence of the iteration process follows.

To determine the convergence rate, we need to assume —

- the relaxation parameter is appropriate: $\omega \in (0,2)$
- Jacobi's iteration matrix $C_{Jac} := I - D^{-1}A$ has only eigenvalues.
- Jacobi's method is convergent $\mu := \rho(C_{Jac} < 1)$
- a unique solution exists: $\det A \neq 0$

Then the Convergence Rate is expressed as

$$\rho(C_\omega) = \begin{cases} \frac{1}{4}\left(\omega\mu \pm \sqrt{\omega^2\mu^2 - 4(\omega-1)}\right)^2, & 0 < \omega \leq \omega_{opt} \\ \omega - 1, & \omega_{opt} < \omega < 2 \end{cases}$$

where the optimal relaxation parameter is given by

$$\omega_{opt} := 1 + \left(\frac{\mu}{1+\sqrt{1+\mu^2}}\right)^2$$

Therefore successive Over Relaxation converges only if it is within the interval of $(0,2)$ and the rest it diverges.
Proved.

5 ~~Simplest way to iterate~~

3 Iterative method for solving a linear system $Ax=b$ has the form

$$x_{k+1} = Gx_k + c,$$

where G is the matrix & vector c as the fixed point of the equation $x = Gx + c$.

The splitting criteria of the matrix is $A = M - N$, with M nonsingular. We then take $G = M^{-1}N$ and $c = M^{-1}b$ so the iteration is

$$x_{k+1} = M^{-1}(\cancel{Gx_k}\, Nx_k + b) \quad —(i)$$

After solving the equation we get,

$$g(x) = M^{-1}Nx + M^{-1}b$$

~~and~~ and the Jacobian is

$$G = M^{-1}N$$

$\therefore$ the iteration scheme converges to

$$\rho(G) = \rho(M^{-1}N) < 1$$

$$x^* = \cancel{A^{-1}b}\, A^{-1}b \qquad e_k = x_k - x^*$$

$$e_k = x_k - A^{-1}b$$

we can write

(1) $$M^{-1}Nx_k + M^{-1}b = M^{-1}(M-A)x_k + M^{-1}b$$

$$= x_k + M^{-1}(b - Ax_k)$$

5, continue

$\therefore$ we can write it in the form

5

$$x_{k+1} = x_k + M^{-1} r_k$$

Where $r_k = (b - Ax_k)$

The matrix M is called a preconditioner, if $x_k \to x_\infty$ then the system has

$$M^{-1}Ax = M^{-1}b$$

Subtracting the iteration from the split system,

$$\left.\begin{aligned} Mx &= Nx + b \\ Mx_{k+1} &= Nx_k + b \end{aligned}\right\} M(x - x_{k+1}) = N(x - x_k)$$

& introducing the error, we get.

$$Me_{k+1} = Ne_k \Rightarrow e_{k+1} = M^{-1}Ne_k$$

$$\therefore\ b - Ax_{k+1} = b - Ax_k - AM^{-1}r_k$$

Therefore from the error relation we conclude

$$\|e_{k+1}\| \leq \|M^{-1}N\| \|e_k\| \leq \cdots \leq \|M^{-1}N\|^{k+1} \|e_0\|$$

$\therefore$ Therefore we have convergence if $\|M^{-1}N\| < 1$.

By induction, we get

$$e_k = (M^{-1}N)^k e_0$$

We know $M^{-1}N = G$

$$\therefore\ e_k = G^k e_0.$$

Therefore Proved

1. False May or may not converge (6)

~~False~~ True — when ω of SOR is equal to 1 ($\omega = 1$)

True — convergence depends on the size of matrix.

False —

6.b $f(x) = x^TAx - 2x^Tb$

$$\nabla f(x) = 2Ax - 2b$$

We can $Ax = b$ setting $\nabla f(x) \to 0$.

Let the generic function be $f(x_{k-1} + \alpha_k d_k)$

$$\therefore g(\alpha) = f(x + \alpha d)$$

$$= (x+\alpha d)^T A (x+\alpha d) - 2(x+\alpha d)^T b$$

$$= (x^TA + \alpha d^TA)(x + \alpha d)$$

$$= x^TAx + \alpha d^TAx + x^T\alpha dA + \alpha d^TA\alpha d - 2x^Tb - 2\alpha d^Tb$$

$$= x^TAx + 2\alpha x^TAd + \alpha^2 d^TAd - 2x^Tb - 2\alpha d^Tb$$

$$= \alpha^2 d^TAd + \alpha(x^TAd - b^Td^T)$$

We minimize g w.r.t α we solve $dg/d\alpha = 0$ to find,

$$\alpha = \frac{d^T(b - Ax)}{d^TAd}$$

For gradient descent, we choose $d_k = b - Ax_k$ so α_k

$$\alpha_k = \frac{\|d_k\|_2^2}{d_k^TAd_k}$$

Since A is positive definite, $\alpha_k > 0$. This leads to iterative gradient descent algorithm. Unlike generic line search, for this problem the choice of α in each iteration is optimal.

Q2) The convexity of a function is given by

$$f(hx_1 + (1-h)x_2) < hf(x_1) + (1-h)f(x_2) \quad 0<h<1$$

when f has two local minimum at x_1 & x_2 with the condition

as $f(x_1) \leq f(x_2) \quad x_1 \neq x_2$

as h is positive $f(x_1) \leq f(x_2) \Rightarrow hf(x_1) \leq hf(x_2)$

which justifies below condition

$$hf(x_1) + (1-h)f(x_2) \leq hf(x_2) + (1-h)f(x_2)$$

$$\Rightarrow hf(x_1) + (1-h)f(x_2) \leq f(x_2)$$

Replacing this condition to the definition of convexity

$$f(hx_1 + (1-h)x_2) < f(x_2)$$

If x_2 is a local minimum the neighborhood must be defined such as $f(x) > f(x_2)$ which is a contradiction with the above condition. To satisfy both condition it must be that $x_1 = x_2$ which shows that f has at most one local minimum.

6.c the given problem has $\|Ax_0 - b\|_2$ ~~is relatively large~~ (8)

with $x_0 \in R^n$.

In iterative methods, this approximately solves this problem, with the cost of each iteration dominated by a matrix vector multiplication.

If we had available an eigenvector basis, the problem would be trivial. Such a basis is impractical to compute and store for large problems.

Throughout $x^* := A^{-1}b$ and $f^* := f(x^*)$.

A induces the inner products $\langle v, w\rangle_A := \langle Av, w\rangle$

and the norm $\|v\|_A := \sqrt{\langle Av, v\rangle}$.

Suppose we have an A-orthogonal basis $\{v_1, v_2 \dots v_n\}$

where n is the dimension of R^n.

$\therefore$ the iterative scheme is define as.

$$\begin{cases} t_k = \operatorname{argmin}_t f(x_k + t v_k) \\ x_{k+1} = x_k + t_k v_k \end{cases}$$

This procedure is called a <u>conjugate direction method</u>.

t_k can be determined from optimality conditions.

The residuals are defined as $r_k := b - Ax_k$.

~~Notice~~ $r_k = -\nabla f(x_k) \rightarrow$ the residuals are simply the negative gradients

References

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