HOMEWORK 4 – CS210

Print this PDF and write your solutions in the space given. Your submission should consist of a scan of this PDF with addional scanned pages that you think are necessary to better explain your answer. Remember to fill out your name and student ID. You may write your answer directly into the PDF fields.

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Question	1	2	3	4	5	6	7	8
Points	10	15	10	10	15	10	10	20

1. For each of the following statements, indicate whether the statement is true or false.

   T/F F All the eigenvalues of a real matrix are necessarily real.

   T/F F If $\lambda = 0$ for every eigenvalue λ of a matrix A, then $A = 0$.

   T/F T The condition number of a matrix with respect to solving linear systems also determines the conditioning of its eigenvalues.

   T/F T The eigenvalues of a real symmetric Hermitian matrix are always well-conditioned.

   T/F T If two matrices have the same eigenvalues, then the two matrices are similar.

   T/F T If two matrices are similar, then they have the same eigenvectors.

   T/F F The eigenvalues and singular values of a square matrix are the same.

2. Let

$$A = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$$

The answers to the following questions should be numeric and specific to the matrix A.

a) What is the characteristic polynomial of A?

$\det(A - \lambda I) = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)^2 - 4$

$= \lambda^2 - 2\lambda - 3$ is the characteristic polynomial of A.

b) What are the eigenvalues of A?

For eigen values $\det(A - \lambda I) = 0$

$\therefore \lambda^2 - 2\lambda - 3 = 0$

$\lambda = 3$ & $\lambda = -1$

$\therefore$ Eigen values are $\lambda = 3, -1$

c) What are the eigenvectors of A?

$$v_{\lambda=3} = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}, \quad v_{\lambda=-1} = \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

We find the above Eigenvectors using $Av = \lambda v$

d) If QR iteration were applied to A, to what form would it converge: diagonal or triangular?

- If a matrix is symmetric, then it converts to tridiagonal
- If non-symmetric — upper Hessenberg matrix
- If QR iteration is applied to A, then it would converge to a <u>triangular</u> matrix.

3. Show that the Householder matrix H_v is *involutory*, meaning $H_v^2 = I$.

$$H_v^2 = \left(I - \frac{2vv^T}{v^Tv}\right)\left(I - \frac{2vv^T}{v^Tv}\right)$$

$$= I - \frac{2vv^T}{v^Tv} - \frac{2vv^T}{v^Tv} + \frac{4vv^Tvv^T}{v^Tvv^Tv}$$

$$= I - \frac{4vv^T}{v^Tv} + \frac{4vv^T}{v^Tv}$$

$$= I \quad \text{(Proved)}$$

4. Show that a $n \times n$ Householder matrix $H = I - 2\frac{vv^T}{v^Tv}$ has an eigenvalue of 1 with multiplicity $n-1$ and an eigenvalue of -1 with multiplicity 1. What are the eigenvectors of H?

$H = I - 2uu^T$, $u = v/\sqrt{v^Tv}$ = unit vector.

$|H - \lambda I| = 0$

$\therefore |I - 2uu^T - \lambda I| = 0$

$\Rightarrow |I(1-\lambda) - 2uu^T| = 0$

$\Rightarrow |I(1-\lambda)|\left(1 - 2u^T(I(1-\lambda))^{-1}u\right) = 0$

$\Rightarrow |I(1-\lambda)|\left(1 - 2(1-\lambda)^{-1}u^TIu\right) = 0$

$\Rightarrow |1-\lambda|\left(1 - 2(1-\lambda)^{-1}u^TIu\right) = 0$

$\Rightarrow (1-\lambda)\left(1 - 2(1-\lambda)^{-1}\right) = 0$

$\Rightarrow (1-\lambda)^{n-1}(1-\lambda-2) = 0$

$= (1-\lambda)^{n-1}(-1-\lambda) = 0$

$\therefore$ one root for $\lambda = -1$ and other for $(n-1)$ is 1.

<u>Eigenvector</u>

For any $v \perp u$, we have $Hv = v$, while $Hu = -u$

So H encodes a reflection with respect to the orthogonal subspace of u. So $H = I + 2uu^T$ maps any $v \perp u$ into itself and u into $3u$. In both cases, a base of eigenvectors is given by u and a base of the orthogonal subspace of u.

5. In this problem, we will derive a technique known as Newton-Raphson division which is often implemented in hardware due to its fast convergence.

(a) Show how the reciprocal $\frac{1}{a}$ of $a \in \mathbb{R}$ can be computed iteratively using Newton's method. Write your iterative formula in a way that requires at most two multiplications, one addition or subtraction, and no divisions.

To compute the reciprocal of a we use the function as,

$f(x) = \frac{1}{x} - a$.

Using Newton's method, we get:-

$f(x) = 1/x - a$

$f'(x) = -1/x^2$ ⊛

⊛ $x_1 = x_0 - \frac{f(x)}{f'(x)}$

$x_1 = x_0 - (1/x_0 - a)/-(1/x_0^2)$

$x_1 = (2x_0 - ax_0^2)$

It is expressed as two multiplication & one substuction.

(b) Take x_k to be the estimate of $\frac{1}{a}$ during the $k-th$ iteration of Newton's method. If we define $\epsilon_k \equiv ax_k - 1$, show that $\epsilon_{k+1} = -\epsilon_k^2$.

$\varepsilon_k \equiv ax_k - 1$, $x_{k+1} = 2x_k - ax_k^2$

$\varepsilon_{k+1} = ax_{k+1} - 1$

$= a(2x_k - ax_k^2) - 1$

$= 2ax_k - a^2x_k^2 - 1$

$= -a^2x_k^2 + 2ax_k - 1 = -(ax_k - 1) = -\varepsilon_k^2$

(c) Approximately how many iterations of Newton's method are needed to compute $\frac{1}{a}$ within d binary decimal points? Write your anwser in terms of ϵ_0 and d, and assume $|\epsilon_0| < 1$.

$\varepsilon_{k+1} = -\varepsilon_k^2$

$\varepsilon_k = -\varepsilon_{k-1}^2$

$\varepsilon_{k-1} = -\varepsilon_{k-2}^2$

$\varepsilon_{k-1}^2 = -\varepsilon_k^4 - 2$

and we compute the range of $\varepsilon_0 \in (0, \frac{2}{a})$. See answer at the last page.

(d) Is this method always convergent regardless of the initial guess of $\frac{1}{a}$? Why?

- This method doesn't converges with the starting guess of zero.
- The root that the process converges to is dependent upon the sign of initial guess.
- If the starting guess is not in the open interval $(0, \frac{2}{a})$ Newton's method will not converge at all.
- So Newton's method does not converge if the derivative is zero for one of the iteration terms, if there is no root ~~for one of the iteration terms, if there is~~ to be found in the first place, or if the iteration enter a cycle and alternates back and forth between diff values

6. Suppose we have a polynomial $p(x) = a_k x^k + ... + a_1 x + a_0$. You can assume $a_k \neq 0$ and $k \geq 1$. Suppose the derivative $p'(x)$ has no roots in the interval (a, b). How many roots can $p(x)$ have in the interval (a, b)?

Let $p(x)$ has two roots and they are a and b. Now by assumptions, we know $f(x)$ is continuous and differentiable everywhere and so in particular it is continuous on $[a,b]$ and differentiable on (a,b). So by Mean Value theorem we get $f(c) = \frac{f(b) - f(a)}{b-a} = 0$. Therefore $p(x)$ has two roots.

7. Consider the nonlinear equation $f(x) = sin(x)$

(a) With $x_0 = \pi/4$ as a starting point, what is the value of x_1 if you use Newton's method for solving this problem?

$f'(x) = \cos x$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = \frac{\pi}{4} - \frac{\sin \pi/4}{\cos \pi/4} = \cancel{0.785} = \frac{\pi}{4} - 1 = -0.21460$

(b) With $x_0 = \pi/4$ and $x_1 = \pi/2$ as a starting points, what is the value of x_2 if you use the secant method for solving this problem?

$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = \frac{\pi}{2} - \frac{\sin \pi/2 \left(\frac{\pi}{2} - \frac{\pi}{4}\right)}{\sin \frac{\pi}{2} - \sin \pi/4} = \frac{\pi}{2} - \left(\frac{\frac{\pi}{2} - \pi/4}{1 - \frac{1}{\sqrt{2}}}\right)$

$= \cancel{3.025} \; -1.110713$

8. **Programming assignment.** Implement the "Brent-Dekker" method to find the roots of the following polynomial

$$p(x) = x^5 - \frac{29x^4}{20} + \frac{29x^3}{36} - \frac{31x^2}{144} + \frac{x}{36} - \frac{1}{720}$$

on the interval $x \in [0, 1]$. You can find the algorithm in the Wikipedia page:

https://en.wikipedia.org/wiki/Brent's_method

Answer the following questions and attach your code in your submission along with this PDF.

(a) How many unique roots are there?

We can find roots for the resp intervals.
0 - 0.1 = 0.1
0.1 - 0.2 = 0.1828
0.2 - 0.3 = 0.3
0.3 - 0.4 = 0.3046
0.4 - 0.5 = 0.5
and 3 are unique

There are total five roots, in which 3 are unique roots.

Roots are ~~94.6156 + 0.0001i, 0.0661 + 0.0736i, 0.0661 - 0.0736i~~

~~-0.0651 + 0.0709i, -0.0651 - 0.0709i~~

(b) What are the root x-values of the polynomial to at least 10 digits of accuracy?

X values are → 0.00832838
→ 0.0533983
→ 0.526699
→ 0.390492
→ 0.416919
→ 0.471809
→ 0.499254
→ 0.500256
→ 0.499997
→ 0.500000

S.C

$$\varepsilon_{k+1} = -\varepsilon_k^2$$

$$\varepsilon_k = -\varepsilon_{k-1}^2$$

$$\varepsilon_{k-1} = -\varepsilon_{k-2}^2$$

$$\varepsilon_{k-1}^2 = -\varepsilon_{k-2}^4$$

$$\therefore \varepsilon_k = (-1)^{\sum_{n=1}^{i}(2^{n-1})} \varepsilon_0^{2^i}$$

$$\therefore S_n = \frac{a(1-r^n)}{1-r}$$

$$x_0 = \left(\frac{1}{x_0} - a\right)$$

$$\varepsilon_i = (-1)^{(2i-1)} \varepsilon_0^{2i}$$

$$\varepsilon_i = (-1)^{2i-1} \left(\frac{1}{x_0} - a\right)^{2i}$$

$$\varepsilon_i = (-1)^{2i-1} \left(\frac{1}{x_0^{2i}} - a^{2i}\right)$$

$$= \frac{(-1)^{2i-1}}{x_0^{2i}} - (-1)^{2i-1} a^{2i}$$

$$= -\frac{(-1)^{2i}}{x_0^{2i}} + (-1)^{2i} a^{2i}$$

$x_0 = \varepsilon_0$
$i = d.$

$$= -\left(\frac{1}{x_0}\right)^{2i} + (a)^{2i}$$

$$= a^{2i} - \frac{1}{x_0^{2i}}$$

$$= \frac{(a x_0)^{2i} - 1}{x_0^{2i}} \qquad \therefore (a x_0)^{2i} - 1 < 1$$

$$\left|(a\varepsilon_0)^{2d} - 1\right| < 1$$

$$(a\varepsilon_0)^{2d} < 2$$

$$a\varepsilon_0 < 2^{1/2d}.$$

$$0 < \varepsilon_0 < \frac{1}{a} 2^{1/2d}$$

$$a\varepsilon_0^{2d} > 0$$

$$\therefore \text{range} = \varepsilon_0 \in \left(0, \frac{2^{1/2d}}{a}\right)$$