Homework - 3

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1. $Ax = b$

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

a) Gaussian elimination $Ly = b$ $Ux = y$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & -11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$x_1 = -0.333$
$x_2 = 0.333$
$x_3 = 0.$

Now using L (lower triangular matrix) to find y.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -m_{12} & 1 & 0 \\ -m_{13} & -m_{23} & 1 \end{bmatrix}$$

$m_{12} = -2$, $m_{13} = -3$, $m_{23} = -2$

$$\therefore \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\boxed{y_1 = 1}$,

$2y_1 + y_2 = 1$, $\boxed{y_2 = -1}$

$3y_1 + 2y_2 + y_3 = 1$

$3 \cdot y_1 + 2y_2 + y_3 = 1$

$3 \times -2 + y_3 = 1$

$\boxed{y_3 = 0}$

$$\boxed{\begin{array}{ll} x_1 = -0.333 & y_1 = 1 \\ x_2 = 0.333 & y_2 = -1 \\ x_3 = 0 & y_3 = 0 \end{array}} - \underline{\text{Ans}}$$

1.b LU factorization in matrix A

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}+l_{21} & l_{21}U_{12}+U_{22} & l_{21}U_{13}+U_{23} \\ l_{31}U_{11} & U_{12}l_{31}+U_{22}l_{32} & U_{13}l_{31}+U_{23}l_{32}+U_{33} \end{bmatrix}$$

$\boxed{U_{11} = 1}$, $\boxed{U_{12} = 4}$, $\boxed{U_{13} = 7}$, $7 + U_{23} = 8$

$1 + l_{21} = 2$ $\boxed{l_{21} = 1}$ $\boxed{U_{23} = 1}$, $\boxed{l_{31} = 3}$

$4 \cdot 1 + U_{22} = 5$

$\boxed{U_{22} = 1}$

$12 + l_{32} = 6$, $\boxed{l_{32} = -6}$

$7 \times 3 - 6 + U_{33} = 10$

$\boxed{U_{33} = -5}$

Therefore we get L & U matrices as follows —

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & -6 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 1 & 4 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

2.a LU decomposition using ~~A~~ left-looking method.

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Step 1:
$$\begin{bmatrix} 1 & 0 & 0 \\ l_{12} & x & 0 \\ l_{13} & x & x \end{bmatrix} \begin{bmatrix} U_{11} & x & x \\ 0 & x & x \\ 0 & x & x \end{bmatrix} = \begin{bmatrix} a_{11} & x & x \\ a_{12} & x & x \\ a_{13} & x & x \end{bmatrix}$$

$\boxed{U_{11} = 1}$, $l_{12} \cdot U_{11} = a_{12} = 2$

$\boxed{l_{12} = 2}$

$l_{13} \cdot U_{11} = a_{13}$
$l_{13} \cdot 1 = 3$
$\boxed{l_{13} = 3}$

Step 2:
$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & l_{23} & \cancel{x} \end{bmatrix} \begin{bmatrix} 1 & U_{21} & x \\ 0 & U_{22} & x \\ 0 & 0 & x \end{bmatrix} = \begin{bmatrix} 1 & a_{21} & x \\ 2 & a_{22} & x \\ 4 & a_{23} & x \end{bmatrix}$$

$U_{21} = a_{21} = 4$, $\boxed{U_{21} = 4}$

$2 \times U_{21} + U_{22} = a_{22} = 5$
$2 \times 4 + U_{22} = 5$
$\boxed{U_{22} = -3}$

$$3U_{21} + U_{22} \cdot l_{23} = a_{23} = 6$$

$$3 \times 4 + (-3 \cdot l_{23}) = 6$$

$$-3l_{23} = 6 - 12$$

$$l_{23} = \frac{6}{3}$$

$$\boxed{l_{23} = 2}$$

<u>Step 3</u>

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & U_{31} \\ 0 & -3 & U_{32} \\ 0 & 0 & U_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & a_{31} \\ 2 & 5 & a_{32} \\ 4 & 6 & a_{33} \end{bmatrix}$$

$$U_{31} = a_{31} = 7$$

$$2 \times U_{31} + U_{32} = a_{32} = 8$$

$$2 \times 7 + U_{32} = 8$$

$$U_{32} = 8 - 14$$

$$\boxed{U_{32} = -6}$$

$$3 \times 7 + 2 \times (-6) + U_{33} = a_{33} = 10$$

$$21 - 12 + U_{33} = 10$$

$$\boxed{U_{33} = 1}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 4 & 6 & 10 \end{bmatrix}$$

P. T. O →

2.b $Ly = b \quad Ux = y$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $\boxed{y_1 = 1}$
- $2y_1 + y_2 = 1$

$2 + y_2 = 1$

$\boxed{y_2 = -1}$

- $3y_1 + 2y_2 + y_3 = 1$

$3 - 2 + y_3 = 1$

$\boxed{y_3 = 0}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$Ux = y$

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

- $\boxed{x_3 = 0}$
- $-3x_2 - 6x_3 = -1$

$-3x_2 = -1$

$\boxed{x_2 = \frac{1}{3} = 0.333}$

- $x_1 + 4x_2 + 7x_3 = 1$

$x_1 + \frac{4}{3} = 1$

$\boxed{x_1 = -\frac{1}{3}}$

$$\boxed{\begin{array}{lll} \text{Ans} = & x_1 = -1/3 & y_1 = 1 \\ & x_2 = 0.333 & y_2 = -1 \\ & x_3 = 0 & y_3 = 0 \end{array}}$$

Proved, same solution

P.T.O $\Rightarrow$

2.c

$$U = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{y_1 = 1}$

$2y_1 + y_2 = 0$

$\boxed{y_2 = -2}$

$3y_1 + 2y_2 + y_3 = 0$

$3 - 4 + y_3 = 0$

$\boxed{y_3 = 1}$

$Ux = y$

$$\begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$\boxed{x_3 = 1}$

$-3x_2 - 6x_3 = -2$

$-3x_2 - 6 = -2$

$\boxed{x_2 = -\frac{4}{3} = -1.33}$

$x_1 + 4x_2 + 7x_3 = 1$

$x_1 - \frac{4}{3} \times 4 + 7 = 1$

$x_1 = 1 - \frac{5}{3}$

$\boxed{x_1 = -\frac{2}{3} = -0.667}$

Ans:

$x_1 = -0.667$	$y_1 = 1$
$x_2 = -1.33$	$y_2 = -2$
$x_3 = 1$	$y_3 = 1$

3.

(a)

$$\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & \cdots & N \\
\hline
1 & q_1+q_2 & -q_2 & 0 & 0 & & \\
2 & -q_2 & q_2+q_3 & -q_3 & 0 & & \\
3 & 0 & -q_3 & q_3+q_4 & -q_4 & & \\
4 & 0 & 0 & -q_4 & q_4+q_5 & -q_5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & & \\
N & & & & & &
\end{array}$$

$\therefore$ There are 3 non-zero entries in each row except 1st and last. Therefore it will be $3(n-2)+2+2$

No. of non-zero entries is $= 3n-6+4$
$= (3n-2)$

3.b Inverse of a sparse matrix is a "dense" matrix, so all the elements will be non-zero. Therefore the non-zero entries are (n^2).

3.c L matrix will have
$2(n-1)+1$ as two elements in each row except 1st row.
$\therefore 2n-2+1 = (2n-1)$.
U matrix will also have the same as two elements in each row except last row.
$2(n-1)+1 = 2n-2+1$
$\boxed{= 2n-1}$

P.T.O $\Rightarrow$

3.d When we break down the matrix into L and U matrices, the multiplication factor is reduced and all the time complexity ~~is~~ becomes better. In turn, there is less cost associated with LU factorized matrix. And we should try to factorize a matrix into L & U for simplification purpose

4) Markowitz product → $(r_i - 1)(c_i - 1)$

$$A = \begin{array}{|ccccc|c} x & 0 & x & 0 & x & 3 \\ 0 & x & x & x & 0 & 3 \\ x & 0 & 0 & x & 0 & 2 \\ 0 & 0 & x & 0 & 0 & 1 \\ 0 & x & 0 & 0 & x & 2 \\ \hline 2 & 2 & 3 & 2 & 2 & \end{array}$$

Markowitz Product

$$\begin{bmatrix} 2 & 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 & 2 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 1 & 1 \end{bmatrix}$$

P.T.O →

5)

$$\begin{bmatrix} 2.1 & 3.4 & 0 & 0 & 3.4 \\ 0 & 5.4 & 5.6 & 0 & 0 \\ 0 & 3.4 & 0 & 4.5 & 0 \\ 0 & 0 & 6.7 & 0 & 0 \\ 0 & 4.6 & 0 & 0 & 5.6 \end{bmatrix}$$

Compressed Sparsed Row format

Row Ptr : 1 4 6 8 9

Values : $\boxed{2.1 \quad 3.4 \quad 3.4}$ $\boxed{5.2 \quad 5.6}$ $\boxed{3.4 \quad 4.5}$ $\boxed{6.7}$ $\boxed{4.6 \quad 5.6}$

Col. Index. 1 2 5 2 3 2 4 3 2 5

6)

$$A = \begin{bmatrix} w & x \\ y & z \end{bmatrix} \qquad A^T = \begin{bmatrix} w & y \\ x & z \end{bmatrix}$$

$$\det(A^TA - \lambda I) = 0$$

$$\det\left[\begin{bmatrix} w & y \\ x & z \end{bmatrix}\begin{bmatrix} w & x \\ y & z \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] = 0$$

$$\begin{bmatrix} w^2+y^2 & wx+yz \\ wx+yz & x^2+z^2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\det\begin{bmatrix} w^2+y^2-\lambda & wx+yz \\ wx+yz & x^2+z^2-\lambda \end{bmatrix} = 0$$

$$(w^2+y^2-\lambda)(x^2+z^2-\lambda) - (wx+yz)^2 = 0$$

P.T.O →

$$\Rightarrow \cancel{w^2x^2} + y^2x^2 - \lambda x^2 + w^2z^2 + \cancel{y^2z^2} - z^2\lambda - \lambda w^2 - y^2\lambda + \lambda^2 - \cancel{w^2x^2} - \cancel{y^2z^2} + 2wxyz = 0$$

$$\Rightarrow y^2x^2 + w^2z^2 + 2wxyz - \lambda x^2 - \lambda z^2 - \lambda w^2 - \lambda y^2 + \lambda^2 = 0$$

$$\Rightarrow \lambda^2 - \lambda(w^2 + x^2 + z^2 + y^2) + y^2x^2 + w^2z^2 = 0$$

$$\Rightarrow \lambda = \frac{(w^2+x^2+z^2+y^2) \pm \sqrt{(w^2+x^2+z^2+y^2)^2 - 4(y^2x^2+w^2z^2)}}{2}$$

$\boxed{\text{Singular Values}}$

$$\Rightarrow \sigma = \sqrt{\frac{(w^2+x^2+z^2+y^2) \pm \sqrt{(w^2+x^2+z^2+y^2)^2 - 4(y^2x^2+w^2z^2)}}{2}}$$