KAMALIKA PODDAR — EE 213

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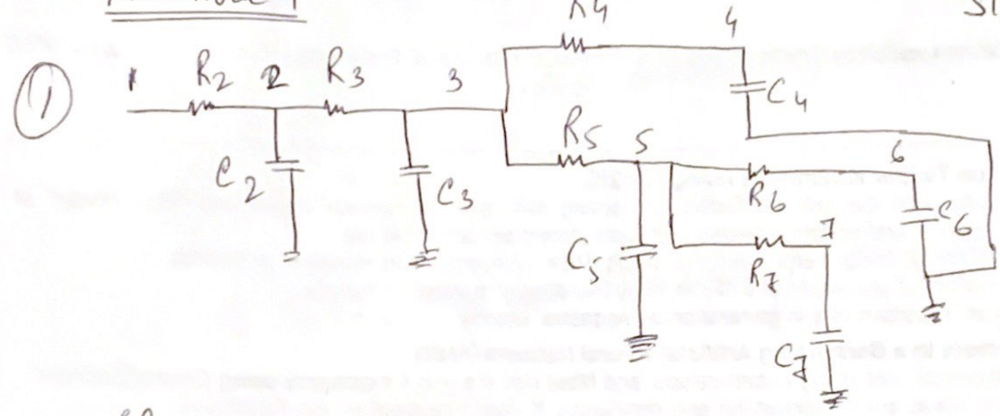
Homework 4

(1)



Elmore Delay

Resistance

$$T_{D15} = R_2(C_2+C_3+C_4+C_5+C_6+C_7) + R_3(C_3+C_4+C_5+C_6+C_7)$$

$$R_5(C_5 + \cancel{\ } + C_6 + C_7)$$

$$T_{D16} = R_2(C_2+C_3+C_4+C_5+C_6+C_7) + R_3(C_3+C_4+C_5+C_6+C_7)$$

$$R_5(C_5+C_6+C_7) + R_6(C_6)$$

Capacitance

$$T_{D15} = C_2R_2 + C_3(R_2+R_3) + C_5(R_2+R_3+R_5)$$

$$C_4(R_2+R_3) + C_6(R_2+R_3+R_5 \cancel{\ })$$

$$+C_7(R_2+R_3+R_5 + \cancel{\ })$$

$$T_{D16} = C_2R_2 + C_3(R_2+R_3) + C_5(R_2+R_3) + C_4(R_2+R_3)$$

$$C_6(R_2+R_3+R_5+R_6) + C_7(R_2+R_3+R_5).$$

(2) $\dot{x} = Ax + Bu(t)$

where $u(t)$ is a step ~~transform~~ function, so Laplace transform of $u(t) = \frac{1}{s}$

$\therefore$ taking Laplace transform of the above equation, we get

$$sX(s) - x(0) = AX(s) + \frac{B}{s}$$

given $x(0) = 0$

$$sX(s) = AX(s) + \frac{B}{s}$$

$$X(s)(s - A) = \frac{B}{s}$$

$$X(s) = (s-A)^{-1}\frac{B}{s}$$

$$= \left[(-A)(1 - sA^{-1})\right]^{-1}\frac{B}{s}$$

$$= -A^{-1}(1 - sA^{-1})^{-1}\frac{B}{s}$$

Expanding $(1-sA^{-1})^{-1}$ about $s=0$:

$$X(s) = -A^{-1}\left(1 + sA^{-1} + s^2A^{-2} + s^3A^{-3} + \dots\right)\frac{B}{s}$$

$$X(s) = -A^{-1}\left(\frac{1}{s} + A^{-1} + sA^{-2} + s^2A^{-3}\dots\right)B$$

The co-efficient powers of s are directly related as.

$m_0 = -A^{-1}B$

$m_1 = -A^{-2}B = A^{-1}m_0$

$m_2 = -A^{-3}B = A^{-1}m_1$

$\vdots$

$m_q = -A^{-(q+1)}B = A^{-1}m_{q-1}$ $\rightarrow$ (Recursive moment matching formula)

(3) $A = \cancel{0}\,4 \times 10^{-4}\ m^2$

$l = 8 \times 10^{-2}\ m.$

$g = 2 \times 10^4\ W/m^3$

$k = 20\ W/mC$

$\rho = 80\ Kg/m^3$

$C_p = 60\ J/KgC$

$\Delta t = 0.1\ s$

a) Finite element differential equation.

$$[C]\{\dot{T}\} + [K]\{T\} = \{F\}$$

Two nodes, so length is $\frac{8}{2} = 4\ cm$

Two element

$$[K_1] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{4 \times 10^{-4} \times 20}{4 \times 10^{-2}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + 600 \times 4 \times 10^{-4}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.24 \end{bmatrix}$$

$$[K_1] = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.44 \end{bmatrix}$$

$$[K_2] = \frac{Ak}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + hA\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} + 300 \times 4 \times 10^{-4}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.12 \end{bmatrix}$$

$$[K_2] = \begin{bmatrix} 0.2 & -0.2 \\ -0.2 & 0.32 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.2 & -0.2 & 0 \\ -0.2 & 0.64 & -0.2 \\ 0 & -0.2 & 0.32 \end{bmatrix}$$

To find C

$$C_1 = \frac{\rho C_p A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{80 \times 60 \times 4 \times 10^{-4} \times \cancel{8}^{4} \times 10^{-2}}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \cancel{153600}\ \cancel{25600}^{12800} \times 10^{-6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \cancel{0.0256}^{0.0128} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[C_1] = \begin{bmatrix} 0.0256 & 0.0128 \\ 0.0128 & \cancel{0.0512}\ 0.0256 \end{bmatrix} = [C_2]$$

$$[C] = \begin{bmatrix} 0.0512 & 0.0256 & 0 \\ 0.0256 & 0.1024 & 0.0256 \\ 0 & 0.0256 & 0.0512 \end{bmatrix}$$

$$[C] = \begin{bmatrix} 0.0256 & 0.0128 & 0 \\ 0.0128 & 0.0512 & 0.0128 \\ 0 & 0.0128 & 0.0256 \end{bmatrix}$$

$$[F_1] = \frac{h_1 P l T_{a_1}}{2}\begin{bmatrix}1\\1\end{bmatrix} = \cancel{\frac{600 \times 80 \times 8\times10^{-2} \times 30}{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$4\times10^{-2}$$

$$= \frac{600 \times 4\times10^{-4} \times \cancel{4}\ 4\times10^{-2} \times 30}{2}\begin{bmatrix}1\\1\end{bmatrix} \qquad \cancel{PA} \quad \cancel{Pl} = A$$

$$= 0.\cancel{144}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \cancel{\begin{bmatrix}0.288\\0.288\end{bmatrix}} = \begin{bmatrix}0.144\\0.144\end{bmatrix}$$

$$[F_2] = \frac{300 \times 4\times10^{-4} \times \cancel{2\times10^{-2}}\ 4\times10^{-2} \times 100}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= 0.24\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \cancel{\begin{bmatrix}0.48\\0.48\end{bmatrix}} = \begin{bmatrix}0.24\\0.24\end{bmatrix}$$

$$F = \cancel{\begin{bmatrix}0.288\\0.768\\0.48\end{bmatrix}}\begin{bmatrix}0.144\\0.384\\0.24\end{bmatrix}$$

Therefore the differential equation is:- $[C]\{\dot{T}\}+[K]\{T\}=\{F\}$

$$\begin{bmatrix}0.0256 & 0.0128 & 0\\ 0.0128 & 0.0512 & 0.0128\\ 0 & 0.0128 & 0.0256\end{bmatrix}\begin{bmatrix}\dot{T}_1\\ \dot{T}_2\\ \dot{T}_3\end{bmatrix} + \begin{bmatrix}0.2 & -0.2 & 0\\ -0.2 & 0.64 & -0.2\\ 0 & -0.2 & 0.32\end{bmatrix}\begin{bmatrix}T_1\\T_2\\T_3\end{bmatrix} = \begin{bmatrix}0.144\\0.384\\0.24\end{bmatrix}$$

$$[C]\{\dot{T}\}+[K]\{T\}=[F]$$

b) finite element equation using Back Euler Method

Back Euler:

$$T_{n+1} = T_n + \Delta t \, \dot{T}_{n+1}$$

$$C T_{n+1} = C T_n + \Delta t \, C \dot{T}_{n+1}$$

$$= C T_n + \Delta t \left[F - K T_{n+1} \right]$$

$$(C + \Delta t K) T_{n+1} = C T_n + \Delta t F$$

$$T^0 = \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix}$$

$$T_1 = (C + \Delta t K)^{-1} \left[C \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix} + \Delta t F \right]$$

$$T_2 = (C + \Delta t K)^{-1} \left[C T_1 + \Delta t F \right]$$

c) at $t = 0.1$

$$(C + \Delta t K)^{-1} = \left(\begin{bmatrix} 0.0256 & 0.0128 & 0 \\ 0.0128 & 0.0512 & 0.0128 \\ 0 & 0.0128 & 0.0256 \end{bmatrix} + \begin{bmatrix} 0.02 & -0.02 & 0 \\ -0.02 & 0.094 & -0.02 \\ 0 & -0.02 & 0.032 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} 0.0456 & -0.0072 & 0 \\ -0.0072 & 0.1452 & -0.0072 \\ 0 & -0.0072 & 0.0576 \end{bmatrix}^{-1} = \begin{bmatrix} 16.807 & -1.83252 & 0.3905 \\ -1.83252 & 7.18915 & -1.53211 \\ 0.390538 & -1.53211 & 13.9877 \end{bmatrix}$$

$$= \begin{bmatrix} 22.1501 & 1.395 & 0.1744 \\ 1.395 & 8.836 & 1.1046 \\ 1.7441 & 1.1046 & 17.499 \end{bmatrix}$$

$$CT_0 = \begin{matrix} 0.0456 & -0.0072 & 0 \\ -0.0072 & 0.1152 & -0.00 \end{matrix}$$

$$CT_0 = \begin{bmatrix} 0.0256 & 0.0128 & 0 \\ 0.0128 & 0.0512 & 0.0128 \\ 0 & 0.0128 & 0.0256 \end{bmatrix} \begin{bmatrix} 50 \\ 50 \\ 50 \end{bmatrix}$$

$$= \begin{bmatrix} 1.92 \\ 3.84 \\ 1.92 \end{bmatrix}$$

$$CT_0 + \Delta t F = \begin{bmatrix} 1.92 \\ 3.84 \\ 1.92 \end{bmatrix} + \begin{bmatrix} 0.014 \\ 0.038 \\ 0.024 \end{bmatrix}$$

$$= \begin{bmatrix} 1.934 \\ 3.878 \\ 1.944 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 22.1501 & 1.395 & 0.1744 \\ 1.395 & 8.836 & 1.1049 \\ 1.7441 & 1.1046 & 17.499 \end{bmatrix} \begin{bmatrix} 1.934 \\ 3.878 \\ 1.944 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 48.587 \\ 39.1118 \\ 41.684 \end{bmatrix}$$

c) at $t = 0.2$

$$(C + \Delta t K)^{-1} = \begin{bmatrix} 0.0256 & 0.0128 & 0 \\ 0.0128 & 0.0512 & 0.0128 \\ 0 & 0.0128 & 0.0256 \end{bmatrix} + \begin{bmatrix} 0.04 & -0.04 & 0 \\ -0.04 & 0.128 & -0.04 \\ 0 & -0.04 & 0.064 \end{bmatrix}$$

$$= \begin{bmatrix} 0.0656 & -0.0272 & 0 \\ -0.0272 & 0.1792 & -0.0272 \\ 0 & -0.0272 & 0.0896 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 16.3219 & 2.5997 & 0.8072 \\ 2.599 & 6.2705 & 1.9470 \\ 0.8072 & 1.9470 & 12.0201 \end{bmatrix}$$

$$CT_0 = \begin{bmatrix} 1.92 \\ 3.84 \\ 1.92 \end{bmatrix}$$

$$CT_0 + \Delta t F = \begin{bmatrix} 1.92 \\ 3.84 \\ 1.92 \end{bmatrix} + \begin{bmatrix} 0.0288 \\ 0.0768 \\ 0.048 \end{bmatrix}$$

$$= \begin{bmatrix} 1.9488 \\ 3.9168 \\ 1.968 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 16.3219 & 2.5997 & 0.8072 \\ 2.599 & 6.2705 & 1.9470 \\ 0.8072 & 1.9470 & 12.0201 \end{bmatrix} \begin{bmatrix} 1.9488 \\ 3.9168 \\ 1.968 \end{bmatrix}$$

$$= \begin{bmatrix} 47.9837 \\ 33.456 \\ 32.854 \end{bmatrix}$$