# Proof that $2^{p-1}(2^p - 1)$ is a Perfect Number, when $2^p - 1$ is Prime.

## Theorem

If $p$ is a prime number and $2^p - 1$ is a prime number, then the number $N = 2^{p-1}(2^p - 1)$ is a perfect number.

## Proof

A perfect number is a positive integer that is equal to the sum of its proper divisors. This is equivalent to stating that the sum of all its positive divisors, denoted by the function $\sigma(N)$, is equal to $2N$. Our goal is to prove that $\sigma(N) = 2N$.

Let $N = 2^{p-1}(2^p - 1)$. The two factors, $2^{p-1}$ and $(2^p - 1)$, are coprime.

The sum of divisors function $\sigma(n)$ is multiplicative, meaning that if $a$ and $b$ are coprime integers, then $\sigma(ab) = \sigma(a)\sigma(b)$. Applying this property to $N$:

$$\sigma(N) = \sigma(2^{p-1}) \cdot \sigma(2^p - 1)$$

First, let's find the sum of the divisors of the factor $2^{p-1}$. The divisors of $2^{p-1}$ are $1, 2, 2^2, \ldots, 2^{p-1}$. This is a geometric series.

$$\sigma(2^{p-1}) = 1 + 2 + 2^2 + \cdots + 2^{p-1} = \frac{2^p - 1}{2 - 1} = 2^p - 1$$

Next, let's find the sum of the divisors of the factor $(2^p - 1)$. By the initial condition, $(2^p - 1)$ is a prime number. The only divisors of a prime number are 1 and itself.

$$\sigma(2^p - 1) = 1 + (2^p - 1) = 2^p$$

Now, substitute these results back into the equation for $\sigma(N)$:

$$\sigma(N) = (2^p - 1) \cdot 2^p$$

Rearranging the terms, we get:

$$\sigma(N) = 2 \cdot (2^{p-1}) \cdot (2^p - 1)$$

Since $N = 2^{p-1}(2^p - 1)$, we can see that:

$$\sigma(N) = 2N$$

This proves that $N = 2^{p-1}(2^p - 1)$ is a perfect number.