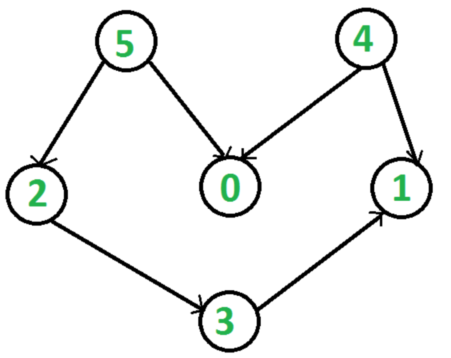
Kahn’s algorithm for Topological Sorting

Topological sorting for **D**irected **A**cyclic **G**raph (DAG) is a linear ordering of vertices such that for every directed edge uv, vertex u comes before v in the ordering. Topological Sorting for a graph is not possible if the graph is not a DAG.

For example, a topological sorting of the following graph is “5 4 2 3 1 0?. There can be more than one topological sorting for a graph. For example, another topological sorting of the following graph is “4 5 2 0 3 1". The first vertex in topological sorting is always a vertex with in-degree as 0 (a vertex with no in-coming edges).



A DFS based solution to find a topological sort has already been discussed.

In this article we will see another way to find the linear ordering of vertices in a directed acyclic graph (DAG). The approach is based on the below fact :

A DAG G has at least one vertex with in-degree 0 and one vertex with out-degree 0.
Proof: There’s a simple proof to the above fact is that a DAG does not contain a cycle which means that all paths will be of finite length. Now let S be the longest path from u(source) to v(destination). Since S is the longest path there can be no incoming edge to u and no outgoing edge from v, if this situation had occurred then S would not have been the longest path
=> indegree(u) = 0 and outdegree(v) = 0

Algorithm:
Steps involved in finding the topological ordering of a DAG:

Step-1: Compute in-degree (number of incoming edges) for each of the vertex present in the DAG and initialize the count of visited nodes as 0.

Step-2: Pick all the vertices with in-degree as 0 and add them into a queue (Enqueue operation)

Step-3: Remove a vertex from the queue (Dequeue operation) and then.

1. Increment count of visited nodes by 1.
2. Decrease in-degree by 1 for all its neighboring nodes.
3. If in-degree of a neighboring nodes is reduced to zero, then add it to the queue.

Step 5: Repeat Step 3 until the queue is empty.

Step 5: If count of visited nodes is **not** equal to the number of nodes in the graph then the topological sort is not possible for the given graph.

How to find in-degree of each node?
There are 2 ways to calculate in-degree of every vertex:
Take an in-degree array which will keep track of
1) Traverse the array of edges and simply increase the counter of the destination node by 1.

```
for each node in Nodes
 indegree[node] = 0;
for each edge(src,dest) in Edges
 indegree[dest]++
```

Time Complexity: O(V+E)

2) Traverse the list for every node and then increment the in-degree of all the nodes connected to it by 1.

```
 for each node in Nodes
  If (list[node].size()!=0) then
  for each dest in list
   indegree[dest]++;
```

Time Complexity: The outer for loop will be executed V number of times and the inner for loop will be executed E number of times, Thus overall time complexity is O(V+E).

The overall time complexity of the algorithm is O(V+E)

Below is C++ implementation of above algorithm. The implementation uses method 2 discussed above for finding indegrees.

C++

```
// A C++ program to print topological sorting of a graph
// using indegrees.
#include<bits/stdc++.h>
using namespace std;

// Class to represent a graph
class Graph
{
    int V;    // No. of vertices'

    // Pointer to an array containing adjacency listsList
    list<int> *adj;

public:
    Graph(int V);   // Constructor

    // function to add an edge to graph
    void addEdge(int u, int v);

    // prints a Topological Sort of the complete graph
    void topologicalSort();
};

Graph::Graph(int V)
{
    this->V = V;
    adj = new list<int>[V];
}

void Graph::addEdge(int u, int v)
{
    adj[u].push_back(v);
}

// The function to do Topological Sort.
void Graph::topologicalSort()
{
    // Create a vector to store indegrees of all
    // vertices. Initialize all indegrees as 0.
    vector<int> in_degree(V, 0);

    // Traverse adjacency lists to fill indegrees of
    // vertices.  This step takes O(V+E) time
    for (int u=0; u<V; u++)
    {
        list<int>::iterator itr;
        for (itr = adj[u].begin(); itr != adj[u].end(); itr++)
             in_degree[*itr]++;
    }

    // Create an queue and enqueue all vertices with
    // indegree 0
    queue<int> q;
```

```
    queue<int> q;
    for (int i = 0; i < V; i++)
        if (in_degree[i] == 0)
            q.push(i);

    // Initialize count of visited vertices
    int cnt = 0;

    // Create a vector to store result (A topological
    // ordering of the vertices)
    vector <int> top_order;

    // One by one dequeue vertices from queue and enqueue
    // adjacents if indegree of adjacent becomes 0
    while (!q.empty())
    {
        // Extract front of queue (or perform dequeue)
        // and add it to topological order
        int u = q.front();
        q.pop();
        top_order.push_back(u);

        // Iterate through all its neighbouring nodes
        // of dequeued node u and decrease their in-degree
        // by 1
        list<int>::iterator itr;
        for (itr = adj[u].begin(); itr != adj[u].end(); itr++)

            // If in-degree becomes zero, add it to queue
            if (--in_degree[*itr] == 0)
                q.push(*itr);

        cnt++;
    }

    // Check if there was a cycle
    if (cnt != V)
    {
        cout << "There exists a cycle in the graph\n";
        return;
    }

    // Print topological order
    for (int i=0; i<top_order.size(); i++)
        cout << top_order[i] << " ";
    cout << endl;
}

// Driver program to test above functions
int main()
{
    // Create a graph given in the above diagram
    Graph g(6);
    g.addEdge(5, 2);
    g.addEdge(5, 0);
    g.addEdge(4, 0);
    g.addEdge(4, 1);
    g.addEdge(2, 3);
    g.addEdge(3, 1);

    cout << "Following is a Topological Sort of\n";
    g.topologicalSort();

    return 0;
}
```

Java

```
// A Java program to print topological sorting of a graph
// using indegrees
import java.util.*;
```

```
//Class to represent a graph
class Graph
{
 int V;// No. of vertices

 //An Array of List which contains
 //references to the Adjacency List of
 //each vertex
 List <Integer> adj[];
 public Graph(int V)// Constructor
 {
  this.V = V;
  adj = new ArrayList[V];
  for(int i = 0; i < V; i++)
   adj[i]=new ArrayList<Integer>();
 }

 // function to add an edge to graph
 public void addEdge(int u,int v)
 {
  adj[u].add(v);
 }
 // prints a Topological Sort of the complete graph
 public void topologicalSort()
 {
  // Create a array to store indegrees of all
  // vertices. Initialize all indegrees as 0.
  int indegree[] = new int[V];

  // Traverse adjacency lists to fill indegrees of
  // vertices. This step takes O(V+E) time
  for(int i = 0; i < V; i++)
  {
   ArrayList<Integer> temp = (ArrayList<Integer>) adj[i];
   for(int node : temp)
   {
    indegree[node]++;
   }
  }

  // Create a queue and enqueue all vertices with
  // indegree 0
  Queue<Integer> q = new LinkedList<Integer>();
  for(int i = 0;i < V; i++)
  {
   if(indegree[i]==0)
    q.add(i);
  }

  // Initialize count of visited vertices
  int cnt = 0;

  // Create a vector to store result (A topological
  // ordering of the vertices)
  Vector <Integer> topOrder=new Vector<Integer>();
  while(!q.isEmpty())
  {
   // Extract front of queue (or perform dequeue)
   // and add it to topological order
   int u=q.poll();
   topOrder.add(u);

   // Iterate through all its neighbouring nodes
   // of dequeued node u and decrease their in-degree
   // by 1
   for(int node : adj[u])
   {
    // If in-degree becomes zero, add it to queue
    if(--indegree[node] == 0)
     q.add(node);
   }
```

```
   cnt++;
  }

  // Check if there was a cycle
  if(cnt != V)
  {
   System.out.println("There exists a cycle in the graph");
   return ;
  }

  // Print topological order
  for(int i : topOrder)
  {
   System.out.print(i+" ");
  }
 }
}
// Driver program to test above functions
class Main
{
 public static void main(String args[])
 {
  // Create a graph given in the above diagram
  Graph g=new Graph(6);
  g.addEdge(5, 2);
     g.addEdge(5, 0);
     g.addEdge(4, 0);
     g.addEdge(4, 1);
     g.addEdge(2, 3);
     g.addEdge(3, 1);
     System.out.println("Following is a Topological Sort");
     g.topologicalSort();

 }
}
```

Output :

```
Following is a Topological Sort
4 5 2 0 3 1
```