Example about BGG correspondence

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1 The functors R and L

In the following S is the graded polynomial ring $\mathbb{Q}[x_0, \dots, x_n]$ with $\deg(x_i) = 1, i = 0, \dots, n$ and A is its dual graded ring, i.e., the exterior algebra generated by $e_i, i = 0, \dots, n$ with $\deg(e_i) = -1$ and $\omega_A := \mathrm{Hom}_k(E, k) \cong A(n+1)$.

Definition 1. Given a graded S-module $M = \bigoplus_{d \in \mathbb{Z}} M_d$, we construct the following cochain complex of graded A-modules:

$$\mathbf{R}(M) : \dots \to M_{i-1} \otimes_k \omega_A \to M_i \otimes_k \omega_A \to \dots$$

where the term $M_i \otimes_k \omega_A$ has cohomological degree i.

Definition 2. Given a graded A-module $P = \bigoplus_{d \in \mathbb{Z}} P_d$, we construct the following cochain complex of graded S-modules:

$$\mathbf{L}(P) : \dots \to S \otimes_k P_j \to S \otimes_k P_{j-1} \to \dots$$

where the term $S \otimes_k P_j$ has cohomological degree $-j$.

Theorem 1. *If M is finitely generated graded S-module and P is a finitely generated graded A-module, then $\mathbf{L}(P)$ is free resolution of M if and only if $\mathbf{R}(M)$ is an injective resolution of P.*

Idea 1. If M is finitely generated graded S-module and $r \geq \mathrm{reg}(M)$, then $\mathbf{L}(\mathbf{H}^r(\mathbf{R}^{>r-1}(M)))$ is free resolution of $M_{\geq r}$. Here $\mathbf{R}^{>r-1}(M) = \mathbf{trunc}^{>r-1}_{below}(\mathbf{R}(M))$. Of course, we can replace $\mathbf{R}$ by $\mathbf{T}$ (the Tate functor).

Gap Code

```
S;
A;
m := RandomMatrixBetweenGradedFreeLeftModules( [ 5, 4 ],[ 4, 2, 3, 1 ], S );
M := AsGradedLeftPresentation( m, [ 4, 2, 3, 1 ] );
Display( M );
r := Maximum( 1, CastelnuovoMumfordRegularity( M ) ) + 1;
M_geq_r := GradedLeftPresentationGeneratedByHomogeneousPart( M, r );
R := RFunctor( S );
trunc_g_rm1_below := BrutalTruncationBelowFunctor( cochains_graded_lp_cat_ext, r - 1 );
H_r := CohomologyFunctorAt( cochains_graded_lp_cat_ext, graded_lp_cat_ext, r );
L := LFunctor( S );
Free_res := PreCompose( [ R, trunc_g_rm1_below, H_r, L ] );
F := ApplyFunctor( Free_res, M_geq_r );
RM_geq_r := ApplyFunctor(R,M_geq_r);;
P := Source( CyclesAt( RM_geq_r, r ) );;
P_leq_r := GradedLeftPresentationGeneratedByHomogeneousPart(P,r);;
emb_P_leq_r_in_P := EmbeddingInSuperObject( P_leq_r );
h := PreCompose( emb_P_leq_r_in_P, CyclesAt( RM_geq_r, r ) );
mat := UnderlyingMatrix(h);
mat := DecompositionOfHomalgMat(mat)[2^(1+1)][2]*S;
t := GradedPresentationMorphism( F[ -r ], mat, M_geq_r );
IsZero( PreCompose( F^(-r-1), t ) );
iso := CokernelColift( F^(-r-1), t );
IsIsomorphism( iso );
```

Idea 2. If M is finitely generated graded S-module and $r \geq \mathrm{reg}(M)$, Then the exactness of $\mathbf{T}(M)$ implies

$$\mathbf{H}^{r-1}(\mathbf{T}^{\leq r-1}(M)) \cong \mathbf{H}^r(\mathbf{T}^{>r-1}(M)),$$

hence,

$$\mathbf{L}(\mathbf{H}^{r-1}(\mathbf{T}^{\leq r-1}(M))) \cong \mathbf{L}(\mathbf{H}^r(\mathbf{T}^{>r-1}(M)))$$

are isomorphic. In particular $\mathbf{L}(\mathbf{H}^{r-1}(\mathbf{T}^{\leq r-1}(M)))$ is free resolution of $M_{\geq r}$. Here $\mathbf{T}^{\leq r-1}(M) = \mathbf{trunc}^{\leq r-1}_{above}(\mathbf{T}(M))$.

This isomorphism can be simply computed by applying the functor H^{r-1} on the natural cochain morphism

$$\psi : \mathbf{T}^{\leq r-1}(M) \to \mathbf{T}^{>r-1}(M)[1]^{\text{unsigned}}.$$

Gap Code

```
m := RandomMatrixBetweenGradedFreeLeftModules( [ 5, 4 ],[ 4, 2, 3, 1 ], S );;
M := AsGradedLeftPresentation( m, [ 4, 2, 3, 1 ] );;
r := Maximum( 1, CastelnuovoMumfordRegularity( M ) ) + 1;;
M_geq_r := GradedLeftPresentationGeneratedByHomogeneousPart( M, r );;
Display( M_geq_r );;
T := TateFunctor(S);;
trunc_leq_rm1 := BrutalTruncationAboveFunctor( cochains_graded_lp_cat_ext, r-1 );;
trunc_g_rm1 := BrutalTruncationBelowFunctor( cochains_graded_lp_cat_ext, r-1 );;
unsigned_shift_by_1 := UnsignedShiftFunctor( cochains_graded_lp_cat_ext, 1 );;
coh_rm1 := CohomologyFunctorAt( cochains_graded_lp_cat_ext, graded_lp_cat_ext, r-1 );;
psi := CochainMorphism(
        ApplyFunctor( PreCompose([T,trunc_leq_rm1]), M_geq_r ),
        ApplyFunctor( PreCompose([T,trunc_g_rm1, unsigned_shift_by_1]), M_geq_r ),
        [ ApplyFunctor(T,M_geq_r)^(r-1) ],
        r-1 );
iso := ApplyFunctor( coh_rm1, psi );;
IsIsomorphism(iso);
```

Idea 3. If M is finitely generated graded S-module and $r \geq \operatorname{reg}(M)$.

Gap Code

```
m := RandomMatrixBetweenGradedFreeLeftModules( [ 5, 4 ],[ 4, 2, 3, 1 ], S );;
M := AsGradedLeftPresentation( m, [ 4, 2, 3, 1 ] );;
r := Maximum( 1, CastelnuovoMumfordRegularity( M ) )+1;;
M_geq_r := GradedLeftPresentationGeneratedByHomogeneousPart( M, r );;
trunc_leq_rm1 := BrutalTruncationAboveFunctor( cochains_graded_lp_cat_ext, r-1 );;
T := TateFunctor(S);;
trunc_leq_m1 := BrutalTruncationAboveFunctor( cochains_graded_lp_cat_sym, -1 );;
ch_trunc_leq_m1 := ExtendFunctorToCochainComplexCategoryFunctor(trunc_leq_m1 );;
complexes_sym := CochainComplexCategory( cochains_graded_lp_cat_sym );;
bicomplxes_sym := AsCategoryOfBicomplexes(complexes_sym);;
complexes_to_bicomplex := ComplexOfComplexesToBicomplexFunctor(complexes_sym, bicomplxes_sym );;
L := LFunctor(S);;
chL := ExtendFunctorToCochainComplexCategoryFunctor(L);;
trunc_leq_rm1_TM_geq_r := ApplyFunctor( PreCompose(T,trunc_leq_rm1), M_geq_r );;
phi := CochainMorphism(
    trunc_leq_rm1_TM_geq_r,
    StalkCochainComplex( CokernelObject( trunc_leq_rm1_TM_geq_r^(r-2) ), r-1 ),
    [ CokernelProjection( trunc_leq_rm1_TM_geq_r^(r-2) ) ],
    r-1 );
IsWellDefined( phi,2,4);;
mor := ApplyFunctor( PreCompose( [ chL, ch_trunc_leq_m1, complexes_to_bicomplex ] ), phi );;
tau := ComplexMorphismOfHorizontalCohomologiesAt(mor,r-1);;
```

Idea 4. In the previous right and above bounded bicomplex, the Beilinson Monad is the cochain complex of the vertical cohomologies at the line with cohomological index -1.

$$
\begin{tikzcd}
 & & i & i+1 & \cdots & 0 & \cdots & r-1 & r \\
0 & & 0 & 0 & \cdots & 0 & & 0 & \\
-1 & \cdots \arrow[r] & S \otimes_k T_1^i \arrow[r] \arrow[u] & S \otimes_k T_1^{i+1} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & S \otimes_k T_1^0 \arrow[r] \arrow[u] & \cdots \arrow[r] & S \otimes_k T_1^{r-1} (= 0) \arrow[r] \arrow[u] & 0 \\
 & {} \arrow[r] & \vdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & \vdots \arrow[r] \arrow[u] & \cdots \arrow[r] & \cdots (= 0) \arrow[u] & \\
-r+1 & {} \arrow[r] & S \otimes_k T_{r-1}^i \arrow[r] \arrow[u] & S \otimes_k T_{r-1}^{i+1} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & S \otimes_k T_{r-1}^0 \arrow[r] \arrow[u] & \cdots \arrow[r] & S \otimes_k T_{r-1}^{r-1} \arrow[r] \arrow[u] & 0 \\
-r & {} \arrow[r] & S \otimes_k T_r^i \arrow[r] \arrow[u] & S \otimes_k T_r^{i+1} \arrow[r] \arrow[u] & \cdots \arrow[r] \arrow[u] & S \otimes_k T_r^0 \arrow[r] \arrow[u] & \cdots \arrow[r] & S \otimes_k T_r^{r-1} \arrow[r] \arrow[u] & 0 \\
 & & \vdots & \vdots & \vdots & \vdots & & \vdots \arrow[u] & \\
-r-n & & \vdots & \vdots & \vdots & \vdots & & S \otimes_k T_{n+r}^{r-1} \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$