RWTH Aachen University

Master Thesis

Constructive Boij-Söderberg Theory

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Declaration of Authorship

I hereby declare that I have created this work completely on my own and used no other sources or tools than the ones listed, and that I have marked any citations accordingly.

Hiermit versichere ich, dass ich die vorliegende Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe.

Name: ______________________________

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Date: ______________________________

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Abstract

In 2006 Mats Boij and Johans Söderberg conjectured that every Betti table of a graded finitely generated Cohen-Macaulay module (CM-module) should be a positive rational linear combination of Betti tables of modules with pure resolutions. In 2008 David Eisenbud and Frank-Olaf Schreyer proved constructively a strengthened form of these conjectures for CM-modules and they used the same tools to show that every cohomology table of any vector bundle on projective space is also positive rational combination of cohomology tables of so called *supernatural* vector bundles. Methods and results of Eisenbud and Schreyer inspired Boij and Söderberg in 2008 to prove the same conjectures for finitely generated graded modules that are not CM, and used it to prove the Multiplicity Conjecture of Herzog, Huneke and Srinivasan for general finitely generated graded modules. Again in 2009 Eisenbud and Schreyer extended the theory and proved that the cohomology table of any coherent sheaf is a convergent - possibly infinite - sum of positive real multiples of cohomology tables of what called *supernatural* sheaves.

The main goal of this work is to set up framework to handle some combinatorial structures of the theory in the computer algebra system Gap. Chapter 1 contains the basic definitions and properties of Betti tables of graded modules and cohomology tables of coherent sheaves. Chapter 2 is devoted to explain the basic theory and constructions used to prove the conjectures of Boij-Söderberg for graded CM-modules. Chapter 3. is devoted to the construction of pure resolutions and supernatural vector bundles. In Chapter 4 we discuss the analog of Boij-Söderberg theory in the case of vector bundles.

In each subsection we give examples using the `BoijSoederberg` package,written by the author of this thesis. The package `BoijSoederberg` is basically inspired and guided by the philosophy of the `homalg` project, written by my supervisor Mohamed Barakat. The packages of this project are also used to handle some rings and most of the modules.

Contents

Chapter 1

Preliminaries

1.1 Betti numbers of graded modules

1.1.1 Graded rings and modules

Definition 1.1. A ring R is called graded (or more precisely, $\mathbb{Z}$-graded) if there exists a family of subgroups $\{R_n\}_{n\in\mathbb{Z}}$ of R such that

1. $R = \bigoplus_n R_n$ (as abelian groups), and
2. $R_n R_m \subseteq R_{m+n}$.

A non-zero element $x \in R_n$ is called a homogeneous element of R of degree n.

Definition 1.2. Let R be a graded ring and M an R-module. We say that M is a graded R-module if there exists a family of subgroups $\{M_n\}_{n\in\mathbb{Z}}$ of M such that

1. $M = \bigoplus_n M_n$ as abelian groups;
2. $R_n M_m \subseteq M_{m+n}$ for all n, m.

Given any graded R-module M, we can form a new graded module by twisting the grading on M as follows: If n is any integer, define $M(n)$ to be equal to M as an R-module, but with its grading defined by $M(n)_k = M_{n+k}$.

Example 1.1. The module $R(-d)$ is a free R-module generated by an element in degree d.

Let M be a graded R-module and $k := R_0$ a field, then it follows that M_i is a k-vector space because $R_0 M_i \subseteq M_i$. A basis of this vector space is caled a **basis in degree** i. In addition if M is finitely generated then $\dim(M_i) < \infty$ for all $i \in \mathbb{Z}$ and $M_i = 0$ for $i \ll 0$. In this case, the generating function $i \to \dim_k(M_i)$ is called the Hilbert function of M and the Hilbert series is defined by

$$\mathrm{Hilb}_M(t) = \textstyle\sum_{i\in\mathbb{Z}} \dim_k(M_i) t^i.$$

If $M(-d)$ is the module M twisted by d degrees, then the Hilbert series is

$$\mathrm{Hilb}_{M(-d)}(t) = t^d \mathrm{Hilb}_M(t).$$

Definition 1.3. Let M, N be graded R-modules. We say that a homomorphism $\phi : M \to N$ is graded of **degree** i if $\phi(m)$ is homogeneous and $\deg(\phi(m)) = i + \deg(m)$ for each homogeneous element $m \in M$. We denote by $\mathrm{Hom}(M, N)_i$ the abelian group of all such homomorphisms.

Example 1.2. Let $R = k[x, y]$. The homomorphism

$$\phi : \quad R(-2) \oplus R(-4) \oplus R(-3) \xrightarrow{\begin{pmatrix} x^2 \\ y^4 \\ xy^2 \end{pmatrix}} R$$

is graded of degree 0.

For instance, $v = (xy \quad 1 \quad y) \in R(-2) \oplus R(-4) \oplus R(-3)$ has degree 4, and $\phi(v) = x^3 y + y^4 + xy^3$ has also degree 4 in R.

Let M, N be graded modules. The **graded** Hom from M to N is

$$\mathcal{H}om(M, N) := \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_i(M, N)$$

In general, $\mathcal{H}om(M, N)$ is an R-submodule of $\mathrm{Hom}(M, N)$.

Remark 1.4. If M is a finitely generated graded R-module, and N is a graded R-module, then $\mathcal{H}om(M,N) = \mathrm{Hom}(M,N)$.

Theorem 1.5 (15, Theo.2.10)**.** *The following properties are equivalent:*

1. *M is a finitely generated graded R-module.*

2. *$M \cong L/N$, where L is a finite direct sum of twisted free R-modules, N is graded submodule of L (called the module of relations), and the isomorphism has degree 0.*

Proof. We have to show that 1 implies 2. Let $m_1, m_2, \ldots, m_j$ be homogeneous generators of M and $a_1, a_2, \ldots, a_j$ be their degrees, respectively.

Set $L = R(-a_1) \oplus R(-a_2) \oplus \cdots \oplus R(-a_j)$. For $1 \leq i \leq j$ denote by e_j the generator of $R(-a_i)$ of degree a_i. Consider the homomorphism

$$\phi : \begin{cases} R(-a_1) \oplus R(-a_2) \oplus \cdots \oplus R(-a_j) \longrightarrow M \\ \qquad\qquad e_i \longmapsto m_i \text{ for } 1 \leq i \leq j \end{cases}$$

It is graded of degree 0. set $N = \ker(\phi)$, so N is graded and $M \cong L/N$. □

1.1.2 Graded complexes

Definition 1.6. A complex $\mathbf{F}$ over R is a sequence of homomorphisms of graded R-modules,

$$\mathbf{F} : \cdots \longleftarrow F_{i-1} \xleftarrow{d_i} F_i \xleftarrow{d_{i+1}} \cdots$$

such that $d_i d_{i+1} = 0$ and $\deg d_i = 0$ for $i \in \mathbb{Z}$. The collection of maps $d = \{d_i\}$ is called the differential of $\mathbf{F}$.

Since every F_i is graded, we have,

$$F_i = \bigoplus_{j \in \mathbb{Z}} F_{i,j} \text{ for all } i \in \mathbb{Z}.$$

An element in $F_{i,j}$ is said to have homological degree i and internal degree j. and we say d has a homological degree -1 and internal degree 0.

If each module F_i is a free finitely generated graded R-module, then we can write it as

$$F_i = \bigoplus_{p \in \mathbb{Z}} R(-p)^{c_{i,p}}$$

Therefore, a graded complex of free finitely generated modules has the form:

$$\mathbf{F} : \cdots \longleftarrow \bigoplus_{p \in \mathbb{Z}} R(-p)^{c_{i-1,p}} \xleftarrow{d_i} \bigoplus_{p \in \mathbb{Z}} R(-p)^{c_{i,p}} \xleftarrow{d_{i+1}} \cdots$$

The numbers $c_{i,p}$ are called the **graded Betti numbers** of the complex. We say that $c_{i,p}$ is the betti number in homological degree i and internal degree p, or the i'th Betti number in internal degree p.

1.1.3 Free resolutions

A free resolution of a finitely generated R-module M is a sequence of homomorphisms of R-modules

$$\mathbf{F} : F_0 \xleftarrow{d_1} \cdots \xleftarrow{d_{i-1}} F_{i-1} \xleftarrow{d_i} F_i \longleftarrow \cdots$$

such that

1. $\mathbf{F}$ is a complex of finitely generated free R-modules F_i;
2. $\mathbf{F}$ is exact;
3. $M \cong F_0 / \operatorname{im}(d_1)$.

A resolution is graded if M is graded, $\mathbf{F}$ is a graded complex, and the isomorphism $M \cong F_0 / \operatorname{im}(d_1)$ has degree 0. In this case the differential has homological degree -1 and internal degree 0. If we fix a homogeneous basis of each free module F_i, then the differential d_i is given by a matrix D_i, whose entries are homogeneous elements in R.

Given a graded finitely generated R-module M, we will construct a graded free resolution of M by induction on homological degree[15, Construction 4.2].

Step 0: Set $M_0 = M$. Choose homogeneous generators $m_1, m_2, \ldots, m_r$ of M_0. Let $a_1, a_2, \ldots, a_r$ be their degrees, respectively. Set $F_0 = R(-a_1) \oplus \cdots \oplus R(-a_r)$.

For $1 \leq j \leq r$ denote by f_j a homogeneous generator of $R(-a_j)$. Thus, $\deg(f_j) = a_j$. Define

$$d_0 : \begin{cases} F_0 \to M \\ f_j \longmapsto m_j \text{ for } 1 \leq j \leq r \end{cases}$$

This is a homomorphism of degree 0.

Assume by induction, that F_i and d_i are defined.

Step i+1: Set $U_{i+1} = \ker(d_i)$. Choose homogeneous generators $l_1, l_2, \ldots, l_s$ of U_{i+1}. Let $c_1, c_2, \ldots, c_s$ be their degrees respectively.

Set $F_{i+1} = R(-c_1) \oplus \cdots \oplus R(-c_s)$. For $1 \leq j \leq s$ denote by g_j a homogeneous generator of $R(-c_j)$. Thus, $\deg(g_j) = c_j$. Define in the same way

$$d_{i+1} : \begin{cases} F_{i+1} \to U_i \subseteq F_i \\ g_j \longmapsto l_j \text{ for } 1 \leq j \leq s \end{cases}$$

This is surjective homomorphism of degree 0. The constructed complex is also exact since $\ker(d_i) = \operatorname{im}(d_{i+1})$ by construction.

1.1.4 Minimal free resolutions

Let $n \in \mathbb{Z}$ and $R = k[x_1, x_2, \ldots, x_n]$ be the polynomial ring over the field k. We denote the maximal homogeneous ideal $\langle x_1, x_2, \ldots, x_n \rangle$ by $\mathbf{m}$.

Definition 1.7. A graded free resolution of a graded finitely generated R-module M is minimal if $d_{i+1}(F_{i+1}) \subseteq \mathbf{m}F_i$. That is, no invertible elements appear in the matrices of the differentials.

Theorem 1.8. *The graded free resolution constructed in the construction above is minimal if and only if at each step we choose a minimal homogeneous system of generators of the kernel of the differential.*

Definition 1.9. A complex of the form

$$0 \longleftarrow R(-p) \xleftarrow{\;1\;} R(-p) \longleftarrow 0$$

is called a **short trivial complex**. If $(\mathbf{F}, d)$ and $(\mathbf{G}, \delta)$ are complexes, then their **direct sum** is the complex $\mathbf{F} \oplus \mathbf{G}$ with modules $(\mathbf{F} \oplus \mathbf{G})_i = F_i \oplus G_i$ and the differentials $d \oplus \delta$. A direct sum of short trival complexes is called a **trivial complex**.

Example 1.3. The trivial complex

$$0 \longleftarrow R(-q) \longleftarrow R(-q) \oplus R(-p) \longleftarrow R(-p) \longleftarrow 0$$

is the direct sum of two short trivial complexes.

Theorem 1.10 (15, Theorem 7.5)**.** *Let M be a graded finitely generated R-module.*

1. *There exists a minimal graded free resolution of M.*
2. *Let $\boldsymbol{F}$ be a minimal graded free resolution of M. Any graded free resolution $\boldsymbol{G}$ of M has the form $\boldsymbol{G} = \boldsymbol{F} \oplus \boldsymbol{T}$ for some trival complex $\boldsymbol{T}$.*
3. *Every two minimal graded free resolutions of M are isomorphic.*

1.1.5 Betti numbers

Often it is difficult to obtain a description of the differentials in a graded free resolution. In such cases, we try to obtain some information about the numericals invariants of the resolution (Betti numbers, the projective dimension, and the Poincarè series).

Definition 1.11. Let M be an $R = k[x_1, \ldots, x_n]$-module and let its minimal free resolution be

$$\mathbf{F} : F_0 \xleftarrow{d_1} \cdots \xleftarrow{d_{i-1}} F_{i-1} \xleftarrow{d_i} F_i \longleftarrow \cdots .$$

We define the i'th Betti number of M over R to be $\beta_i^R = \operatorname{rank}(F_i)$.

By the uniqness of the minimal free resolutions up to isomorphism, the Betti numbers only depend on the isomorphism type of M.

If we tensor $\mathbf{F}$ by $k \cong R/\mathbf{m}$ we get

$$k \otimes_R F_i = k \otimes_R R^{b_i^R} = k^{b_i^R}.$$

Since $d_{i+1}(F_{i+1}) \subseteq \mathbf{m}F_i$, it follows that the induces differentials in $k \otimes_R \mathbf{F}$ are zeros. Therefore,

$$k \otimes_R \mathbf{F} : \; k \otimes_R F_0 \xleftarrow{0} \cdots \xleftarrow{0} k \otimes_R F_{i-1} \xleftarrow{0} k \otimes_R F_i \longleftarrow \cdots$$

Hence

$$\beta_i^R = \dim_k \operatorname{Tor}_i^R(k, M).$$

1.1.6 Graded Betti numbers

In the following we define the graded Betti numbers, which are a refined version of the Betti numbers. If F is graded, each free module F_i is a direct sum of modules of the form $R(-p)$. We define the graded Betti numbers of M by

$$\beta_{i,j}^R = \text{number of summands in } F_i \text{ of the form } R(-j).$$

It can be shown that

$$\beta_{ij}^R(M) = \dim_k \operatorname{Tor}_i^R(k, M)_j.$$

The length of a graded free resolution $\mathbf{F}$ is $\max\{i|\ F_i \neq 0\}$. We say that $\mathbf{F}$ is finite resolution if its length is finite, otherwise we say that F is an infinite resolution. The projective dimension of M is $\operatorname{pd}_R(M) = \max\{i|\beta_i^R(M) \neq 0\}$. Thus, $\operatorname{pd}_R(M)$ is the length of the minimal free resolution F of M. The Poincare series of M over R is

$$P_M^R(t) = \sum_{i \geq 0} \beta_i^R(M) t^i.$$

Theorem 1.12. *(Hilbert Syzygy Theorem) If M is finitely generated module over the polynomial ring $R = k[x_1, x_2, \ldots, x_n]$ then M has a free resolution of length at most n.*

The Betti numbers can be given in a table, the labels of the rows and the columns increase downwards and to the right, respectively. The entry in the i'th column and the j'th row is $\beta_{i,i+j}$.

$j \setminus i$	0	1	2	$\cdots$
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	$\cdots$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	$\cdots$
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	$\cdots$
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Example 1.4. Let $S = k[x, y]$ and $M = S/\langle x^3, xy^2, y^3\rangle$. The minimal free resolution of M is:

$$0 \longleftarrow M \longleftarrow S \xleftarrow{\begin{pmatrix} y^3 \\ xy^2 \\ x^3 \end{pmatrix}} S(-3)^3 \xleftarrow{\begin{pmatrix} x & -y & 0 \\ 0 & x^2 & -y^2 \end{pmatrix}} S(-5) \oplus S(-4) \longleftarrow 0.$$

Thus the Betti table $\beta(M)$ is:

$j \setminus i$	0	1	2
0	1	.	.
1	.	.	.
2	.	3	1
3	.	.	1

Gap Code

```
gap>  LoadPackage( "GradedModules" );;
gap>  S:= GradedRing( HomalgFieldOfRationalsInSingular() * "x,y" );;
gap>  I:= LeftSubmodule( "x^3, x*y^2, y^3", S );
<A graded torsion-free (left) ideal given by 3 generators>
gap>  M:= FactorObject( I );
<A graded cyclic torsion left module presented by yet unknown relations
for a cyclic generator>
gap>  Display( M );
Q[x,y]/< y^3, x*y^2, x^3 >

(graded, degree of generator: 0)
gap>  R:= Resolution( M );
<A right acyclic complex containing 2 morphisms of graded left modules
at degrees [ 0 .. 2 ]>
gap>  Display( R );
```

```
-------------------------
at homology degree: 2
Q[x,y]^(1 x 2)

(graded, degrees of generators: [ 4, 5 ])
-------------------------
x,-y, 0,
0,x^2,-y^2

the graded map is currently represented by the above 2 x 3 matrix

(degrees of generators of target: [ 3, 3, 3 ])
------------v------------
at homology degree: 1
Q[x,y]^(1 x 3)

(graded, degrees of generators: [ 3, 3, 3 ])
-------------------------
y^3,
x*y^2,
x^3

the graded map is currently represented by the above 3 x 1 matrix

(degree of generator of target: 0)
------------v------------
at homology degree: 0
Q[x,y]^(1 x 1)

(graded, degree of generator: 0)
-------------------------
gap>  B:= BettiTable( R );
<A Betti diagram of <A right acyclic complex containing 2 morphisms of
graded left modules at degrees [ 0 .. 2 ]>>
gap>  Display( B );
 total:  1 3 2
--------------
     0:  1 . .
     1:  . . .
```

```
     2:   . 3 1
     3:   . . 1
--------------
degree:  0 1 2
```

1.2 Cohen-Macaulay modules

Let $R = k[x_1, \ldots, x_n]$ and M be an R-module. We say that the sequence $f_1, \ldots, f_s \in R$ is an M-**regular sequence** if multiplication by f_1 is an injective map $M \to M$, and for $i > 1$, multiplication by f_i is an injective map $M/(f_1, \ldots, f_{i-1})M \to M/(f_1, \ldots, f_{i-1})M$, that is f_i is a nonzerodivisor on the module $M/(f_1, \ldots, f_{i-1})M$. The **depth**(denoted by $\mathrm{depth}_R(M)$) of M is the maximal length of an M-regular sequence. The dimension of a ring is the longest chain (if finite) of proper prime ideals, $p_0 \subset p_1 \subset \cdots \subset p_k \subset R$. In case $R = k[x_1, \ldots, x_n]$ the dimension of R is n. The **dimension**(denoted by $\dim_R(M)$) of the module M over R can be defined to be $\dim(R/\mathrm{Ann}_R(M))$, where $\mathrm{Ann}_R(M) = \{r \in R | rM = 0\}$.

Definition 1.13 (13, Definition 1.5)**.** The graded R-module M is called **Cohen-Macaulay** if

$$\mathrm{depth}_R(M) = \dim_R(M).$$

Using Auslander Buchsbaum theorem in the graded case, which states that

$$\dim(R) = \mathrm{depth}_R(M) + \mathrm{pd}_R(M),$$

we can rephrase this condition as $\mathrm{codim}_R(M) = \mathrm{pd}_R(M)$, where $\mathrm{codim}_R(M) = \dim(R) - \dim_R(M)$.

1.3 Cohomology of coherent sheaves

1.3.1 Construction of coherent sheaves

Let R be commutative ring with one. The set

$$\mathrm{Spec}(R) := \{\mathfrak{p} \trianglelefteq R | \mathfrak{p} \text{ prime}\}$$

is called the prime spectrum of R. For any ideal I in R we define the vanishing locus by

$$V(I) := \{\mathfrak{p} \in \mathrm{Spec}(S) | I \subset \mathfrak{p}\}.$$

We can define a topology on $\mathrm{Spec}(R)$ in which the closed sets are the sets of the form $V(\sqrt{I})$. We can now define the **structure sheaf** $\mathcal{O}_U$ over $U := \mathrm{Spec}(R)$. For every $f \in R$ let $D(f)$ be the open set $U \backslash V(\langle f \rangle)$. The sets of this form are called distinguished open sets and they form a basis for the Zariski topology of U. We define $\mathcal{O}_U(D(f))$ to be $R_f = R[\frac{1}{f}]$. The stalks of $\mathcal{O}_U$ are the local rings $\mathcal{O}_U|_{\mathfrak{p}} := R_{\mathfrak{p}} := (R \backslash \mathfrak{p})^{-1} R$ for all $\mathfrak{p} \in U$. This sheaf is called **sheafification** of the ring R. For an R-module M we define the sheafification $\widetilde{M}$ to be the sheaf on $U = \mathrm{Spec}(R)$ satisfying

$$\widetilde{M}|_{\mathfrak{p}} := M_{\mathfrak{p}} := R_{\mathfrak{p}} \otimes_R M$$

and

$$\widetilde{M}(D(f)) := M_f := R_f \otimes_R M.$$

Consider the graded polynomial ring $S = k[x_1, \ldots, x_n] = \bigoplus_{i \geq 0} S_i$, and let $\mathfrak{m} := \bigoplus_{i > 0} S_i$.

Define the set

$$\mathrm{Proj}(S) := \{\mathfrak{p} \triangleleft S | \mathfrak{p} \text{ homogeneous prime and } \mathfrak{m} \not\subset \mathfrak{p}\}.$$

For a homogeneous ideal $I \triangleleft S$ set

$$V(I) := \{\mathfrak{p} \in \mathrm{Proj}(S) | I \subset \mathfrak{p}\}.$$

For $\mathfrak{p} \in \mathrm{Proj}(S)$ define the localization $S_{(\mathfrak{p})} := (S \backslash \mathfrak{p})^{-1}_{\mathrm{hom}} S$. For a homogeneous $f \in \mathfrak{m}$ define $S_{(f)} := (S_f)_0$ and $D(f) := \mathrm{Proj}(S) \backslash V(\langle f \rangle)$. We can use this information to construct the structure sheaf $\mathcal{O}_X$. A projective scheme X is a scheme of the form $X := \mathrm{Proj}(S)$ together with its structure sheaf $\mathcal{O}_X$ for some graded ring S.

Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is called quasi-coherent if X can be covered by open affine subsets $U_i = \mathrm{Spec}(S_i)$ with $\mathcal{F}|_{U_i} \cong \widetilde{M}_i$ (where M_i is an R_i-module). We call $\mathcal{F}$ coherent if the M_i's are finitely presented. If the covering can be chosen such that M_i is a free $\mathcal{O}_X(U_i)$-module then we call $\mathcal{F}$ a locally free sheaf or a vector bundle. If all the M_i's have rank 1 then $\mathcal{F}$ is called an invertible sheaf or line bundle.

For a graded S-module M define the sheafification $\widetilde{M}$ to be the quasi-coherent sheaf on $X = \mathrm{Proj}(S)$ satisfying

$$\widetilde{M}_{\mathfrak{p}} := M_{(\mathfrak{p})} := ((S \setminus \mathfrak{p})^{-1}_{\mathrm{hom}} M)_0$$

and

$$\widetilde{M}(D(f)) := M_{(f)} := (M_f)_0 := (S_f \otimes_S M)_0$$

for any $\mathfrak{p} \in \mathrm{Proj}(S)$ and any homogeneous $f \in \mathfrak{m}$.

Theorem 1.14 (7,Theorem 2.3)**.** *Any quasi-coherent sheaf on a projective scheme $X := \mathrm{Proj}(S)$ is the sheafification $\mathrm{Proj}(M)$ of some graded S-module M.*

Define the twisting sheaf or twisting line bundle by:

$$\mathcal{O}_X(1) = \widetilde{S(1)}$$

More generally define the twisted line bundles

$$\mathcal{O}_X(n) = \widetilde{S(n)}$$

for all $n \in \mathbb{Z}$. For $\mathcal{F} = \widetilde{M}$ define the twisted sheaves

$$\mathcal{F}(n) = \widetilde{M(n)} = S\widetilde{(n) \otimes_S} M.$$

If S is generated by S_1 as an S_0-algebra, then

$$\mathcal{F}(n) \cong \mathcal{O}_X(n) \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Unfortunately the sheafification doesn't yield an equivalence of categories

$$\{\text{graded } S\text{-modules}\} \to \{\text{quasi-coh. sheaves on} \operatorname{Proj}(S)\}$$

For $S = k[x_1, \ldots, x_{m+1}]$, the above construction has the following properties [20 ,Remark 7.1.6],[7]:

1. $\widetilde{S(d)} = \mathcal{O}_{\mathbb{P}^m_k}(d)$.
2. $\widetilde{M}(d) = \widetilde{M(d)}$.
3. If I is an ideal in S defining a subvariety $X \subseteq \operatorname{Proj}(S) = \mathbb{P}^m$ then $\widetilde{I} = \mathcal{I}_X$ (the ideal sheaf on $\mathbb{P}^m$). Furthermore, if $R = S/I$ then $\mathcal{O}_X(d) = \widetilde{R(d)}$.
4. If M is a finitely generated graded R-module such that $M_d = 0$ for all $d \gg 0$ then $\widetilde{M} = 0$.
5. Every coherent $\mathcal{O}_{\mathbb{P}^m_k}$-module is isomorphic to $\widetilde{M}$, for some finitely generated graded S-module M.

Theorem 1.15 (Serre,7, Theorem.2.6)**.** *Let M and N be two graded S-modules. They define the same quasi-coherent sheaf iff $M_{\geq d} \cong N_{\geq d}$ for some $d \in \mathbb{Z}$.*

Q: Is there a way to find a canonical representative in an equivalence class of graded modules which are isomorphic in high degrees?

Answer: Yes, it is

$$M := \bigoplus_{d \in \mathbb{Z}} H^0 \mathcal{F}(d).$$

1.3.2 Cohomology of coherent sheaves

We say that an abelian category has enough injectives, if each object can be "embedded" into an injective object. According to [16,Prop.III.2.2, Prop.III.2.3], the category of abelian sheaves has enough injectives, and for every ringed space $(X, \mathcal{O}_X)$, i.e., a topological space with a sheaf of rings, the category of $\mathcal{O}_X$-modules also has enough injectives.

Definition 1.16. Let X be a topological space and $\mathcal{F}$ a sheaf of abelian groups on X. Define the i-th sheaf cohomology $H^i(\mathcal{F}) = H^i(X, \mathcal{F})$ to be i-th right derived functor $R^i\Gamma$ applied to $\mathcal{F}$. Γ is the global sections functor which maps a sheaf $\mathcal{F}$ to its global sections $\mathcal{F}(X)$. This is defined by choosing an injective resolution $0 \to \mathcal{F} \to I^\bullet$ of $\mathcal{F}$ and then setting

$$H^i(\mathcal{F}) := (R^i\Gamma)(\mathcal{F}) := H^i(\Gamma(I^\bullet)).$$

The definition does not depend on the choice of the injective resolution ([6, Theorem III.1.1A]).

Remark 1.17 (6,Lemma III.2.10). Any sheaf $\mathcal{F}$ defined on a projective variety or closed subscheme $X \subseteq \mathbb{P}_k^m$ can be thought of as a sheaf $i_*\mathcal{F}$ on $\mathbb{P}_k^m$ which coincides with $\mathcal{F}$ on X and is the 0-sheaf outside of X when i is the inclusion map $i : X \to \mathbb{P}_k^m$. The sheaf $i_*\mathcal{F}$ has isomorphic cohomologies to those of $\mathcal{F}$:

$$H^i(X, \mathcal{F}) \cong H^i(\mathbb{P}_k^m, i_*\mathcal{F}).$$

Computing the cohomology of a sheaf $\mathcal{F}$ could mean, for example, to compute one of the dimensions $h^i(\mathcal{F}(j)) := \dim_k(H^i(\mathcal{F}(j)))$, or to compute these dimensions in a range of twists, or to compute the graded S-modules

$$H^i_*(\mathcal{F}) := \bigoplus_{j\in\mathbb{Z}} H^i(\mathcal{F}(j)).$$

Several methods can be used to compute the cohomology groups of a coherent sheaf. In the following we describe two of them. The first method uses *local cohomology* and the second relies on the *Bernstein, Gelfand, and Gelfand correspondence.*

The first method can be summarized in the following theorem:

Theorem 1.18 (11, Chapter 8)**.** *Let* $\mathcal{F} \cong \widetilde{M}$ *be a coherent sheaf on* $\mathbb{P}_k^m = \mathrm{Proj}(S)$, *then*

1. *For all* $i > 0$

$$H^i_*\widetilde{M} \cong \mathrm{Ext}_S^{m-i}(M, S(-m-1))^\vee.$$

2. There exists an exact sequence

$$0 \to \operatorname{Ext}_S^{m+1}(M, S(-m-1))^\vee \to M \to H^0_*\widetilde{M} \to \operatorname{Ext}_S^m(M, S(-m-1))^\vee \to 0.$$

The graded vector space dual $(\bigoplus_{j\in\mathbb{Z}} M_j)^\vee = \bigoplus_{j\in\mathbb{Z}} \operatorname{Hom}_k(M_{-j}, k)$ *is endowed with its natural structure as a graded* S*-module. Hence for all* $i > 0$*, we have*

$$h^i\widetilde{M}(d) = \dim_k \operatorname{Ext}_S^{m-i}(M, S)_{-m-1-d},$$

and the dimension of the space of the global sections of $\widetilde{M}(d)$ *is*

$$h^0\widetilde{M}(d) = \dim_k M_d + \dim_k \operatorname{Ext}_S^m(M, S)_{-m-1-d} - \dim_k \operatorname{Ext}_S^{m+1}(M, S)_{-m-1-d}.$$

Example 1.5. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^m_k = \operatorname{Proj}(S)$, and $\mathcal{E}^*$ be the dual bundle $\mathcal{H}om(\mathcal{E}, \mathcal{O}_{\mathbb{P}^m_k})$. Let $M = H^0_*(\mathcal{E}^*)$, then M is free graded S-module. Let

$$E : 0 \leftarrow E_0 \leftarrow E_1 \leftarrow \cdots \leftarrow E_{m+1} \leftarrow 0$$

be the free resolution of M. If we apply the functor $\operatorname{Hom}_S(-, S)$ to the complex E, we get its dual

$$\check{E} : 0 \to \check{E}^0 \to \check{E}^1 \to \cdots \to \check{E}^{m+1} \to 0,$$

where $\check{E}^j := \operatorname{Hom}_S(E_j, S)$ for $j = 0, \ldots, m+1$.

For all $i = 0, \ldots, m+1$, we have $H^i(\check{E}) = \operatorname{Ext}^i_S(M, S)$. Using the theorem above, we get $h^i\widetilde{M}(d) \cong \dim_k \operatorname{Ext}_S^{m-i}(M, S)_{-m-1-d}$ for all $i > 0$. But $h^i\widetilde{M}(d) = h^i\mathcal{E}^*(d) = h^{m-i}\mathcal{E}(-m-1-d)$, hence

$$h^{m-i}\mathcal{E}(-m-1-d) = \dim_k \operatorname{Ext}_S^{m-i}(M, S)_{-m-1-d}$$

for all $i > 0$ and $d \in \mathbb{Z}$. Hence

$$h^i\mathcal{E}(d) = \dim_k \operatorname{Ext}^i_S(M, S)_d = \dim_k H^i(\check{E})_d$$

for $i = 0, \ldots, m-1$ and $d \in \mathbb{Z}$.

Since $M \cong H^0_*\mathcal{E}^*$ and the sequence in the above theorem is exact, we get $\operatorname{Ext}^m_S(M, S) = \operatorname{Ext}^{m+1}_S(M, S) = 0$. Hence $H^m(\check{E}) = H^{m+1}(\check{E}) = 0$.

The second method is due to Eisenbud, Fløystad, and Schreyer. It is a constructive version of the Bernstein, Gelfand, and Gelfand correspondence (BGG for short). This correspondence consists of a pair of adjoint functors R and L which define an equivalence between the bounded derived category of finitely generated graded S-modules and the bounded derived category of finitely generated graded E-modules, where E is the exterior algebra of an $(m+1)$-dimensional vector space V and $S := \mathrm{Sym}_k(W) = k[x_1, \ldots, x_{m+1}]$ where $W := V^*$. We grade S and E by taking elements of V to have degree -1 and the elements of W to have degree 1.

Let $M = \bigoplus_i M_i$ be a finitely generated graded S-module, considered as a complexe concentrated in cohomological degree 0. Then we define the complex $R(M)$ to be the sequence of free E-modules and maps

$$R(M): \quad \cdots \longrightarrow F^{i-1} \xrightarrow{\phi^{i-1}} F^i \xrightarrow{\phi^i} F^{i+1} \longrightarrow \cdots$$

defined as follows

$$F^i := \mathrm{Hom}_k(E, M_i) = M_i \otimes_k \omega_E$$

where

$$\omega_E = \mathrm{Hom}_k(E, k) = E \otimes_k \bigwedge^{m+1} W \cong E(-m-1),$$

and M_i is considered as a vector space concentrated in degree i. Furthermore, let $\phi^i : F^i \to F^{i+1}$ be the map taking $\alpha \in \mathrm{Hom}_k(E, M_i)$ to

$$(e \mapsto \sum_j x_j \alpha(e_j \wedge e)) \in \mathrm{Hom}_k(E, M_{i+1})$$

where x_j and e_j are dual bases of W and V respectively. It can be shown that $R(M)$ is indeed a complex. In [3, Corollary 2.4] we find that there is an integer $r \in \mathbb{Z}$ such that

$$R_{\geq r}(M): \qquad F^r \xrightarrow{\phi^r} F^{r+1} \xrightarrow{\phi^{r+1}} F^{r+2} \xrightarrow{\phi^{r+2}} \cdots$$

is exact.

The smallest integer r at which exactness of $R_{\geq r}(M)$ occurs is called the **Castelnuovo-Mumford regularity** of M. The regularity can be characterzed as follows. If $M = \bigoplus_i M_i$ is a finitely generated graded S-module, then for large enough integers r, the truncated module $M_{\geq r}$ is generated in degree r and has a linear free resolution, that is, it first syzygies are generated in degree $r+1$, its second syzygies in degree $r+2$, and so on. The Castelnuovo-Momford regularity of M is the least integer r for which this occurs (Eisenbud(1995), chapter 20).

Starting from $T^{>r} := R(M_{>r})$, we get a doubly infinite, exact, E-free complex $T(M)$, the **Tate resolution of** M, by adjoining a minimal free resolution of the kernel of

$$\mathrm{Hom}_k(E, M_{r+1}) \to \mathrm{Hom}_k(E, M_{r+2}).$$

Theorem 1.19 (3, Theorem 4.1, Corollary 4.2)**.** *Let M be a finitely generated graded S-module, and let $\mathcal{F} = \widetilde{M}$ be the associated coherent sheaf on $\mathbb{P}^m$. Then the term of the complex $T(\mathcal{F})$ with cohomological degree j is*

$$\bigoplus_{i=0}^{m} H^i\mathcal{F}(j-i) \otimes_k \omega_E$$

where $H^i(\mathcal{F}(j-i))$ is regarded as a vector space concentrated in degree $j-i$, so that the summand $H^i(\mathcal{F}(j-i)) \otimes_k \omega_E$ is isomorphic to a direct sum of copies of $\omega_E(i-j)$. Moreover the subquotient complex

$$\to H^i(\mathcal{F}(j-i)) \otimes_k \omega_E \to H^i(\mathcal{F}(j+1-i)) \otimes_k \omega_E \to$$

is $R(H^i_\bullet(\mathcal{F}))$.

So each cohomology group of each twist of the sheaf $\mathcal{F}$ accurs exactly once in a term of $T(M)$. When we compute a part of $T(M)$, we are automatically computing the sheaf cohomology of various twists of the associated sheaf.

Remark 1.20. The above theorem implies that the term of the Tate resolution $T(M)$ with degree j is isomorphic to

$$\bigoplus_{i=0}^{m} E(-m-1-j+i)^{h^i\mathcal{F}(j-i)}$$

Example 1.6. Let $S = k[x,y,z]$. Using Gap we can compute the Tate resolution of the S-module $M = S^{1\times 3}/(x \;\; x \;\; y)$:

Gap code

```
gap> LoadPackage("GradedModules");;
gap> S:= GradedRing(HomalgFieldOfRationalsInSingular()*"x,y,z");;
gap> m:= HomalgMatrix("x,x,y", 1, 3, S);
```

```
<A 1 x 3 matrix over a graded ring>
gap> M:= LeftPresentationWithDegrees( m );
<A graded non-torsion left module presented by 1
relation for 3 generators>
gap> T:= TateResolution(M, -3,1);
<An acyclic cocomplex containing 4 morphisms of graded
left modules at degrees [ -3 .. 1 ]>
gap> Display(last);
-------------------------
at cohomology degree: 1
Q{ e0,e1,e2} ^(1 x 8)

(graded, degrees of generators: [ 4, 4, 4, 4, 4, 4, 4, 4 ])
------------^------------
e0, e1,e2,0, 0, 0, 0,  0,
-e0,0, 0, e1,e2,0, -e0,0,
0,  0, 0, 0, 0, e0,e1, e2

the graded map is currently represented by the above
3 x 8 matrix

(degrees of generators of target: [ 4, 4, 4, 4, 4, 4, 4, 4 ])
-------------------------
at cohomology degree: 0
Q{ e0,e1,e2} ^(1 x 3)

(graded, degrees of generators: [ 3, 3, 3 ])
------------^------------
-e1*e2,  -e1*e2,e0*e2,
e0*e1*e2,0,     0

the graded map is currently represented by the above
2 x 3 matrix

(degrees of generators of target: [ 3, 3, 3 ])
-------------------------
```

```
at cohomology degree: -1
Q{ e0,e1,e2} ^(1 x 2)

(graded, degrees of generators: [ 1, 0 ])
------------^------------
e2,   0,
0,    e2,
0,    e1,
0,    e0,
e0*e1,0

the graded map is currently represented by the
above 5 x 2 matrix

(degrees of generators of target: [ 1, 0 ])
-------------------------
at cohomology degree: -2
Q{ e0,e1,e2} ^(1 x 5)

(graded, degrees of generators: [ 0, -1, -1, -1, -1 ])
------------^------------
e2,   0, 0, 0, 0,
0,    e2,0, 0, 0,
0,    e1,e2,0, 0,
0,    0, e1,0, 0,
0,    0, 0, 0, e1,
0,    e0,0, e2,0,
0,    0, e0,e1,0,
0,    0, 0, e0,0,
0,    0, 0, 0, e0,
e0*e1,0, 0, 0, -e2

the graded map is currently represented by the
above 10 x 5 matrix

(degrees of generators of target: [ 0, -1, -1, -1, -1 ])
```

```
-------------------------
at cohomology degree: -3
Q{ e0,e1,e2} ^(1 x 10)

(graded, degrees of generators:
[ -1, -2, -2, -2, -2, -2, -2, -2, -2, -2 ])
-------------------------
```

The above computations shows that the term of the complex $T(M)$ with cohomological degree -3 is

$$T^{-3}(M) = E(1) \oplus E(2)^{1\times 9}.$$

In the same time

$$T^{-3}(M) = \bigoplus_{i=0}^{m} E(-m-1-j+i)^{h^i\mathcal{F}(j-i)} = \bigoplus_{i=0}^{2} E(i)^{h^i\mathcal{F}(-3-i)}.$$

Hence $h^0\mathcal{F}(-3) = 0$, $h^1\mathcal{F}(-4) = 1$ and $h^2\mathcal{F}(-5) = 9$.

Gap code

```
gap> Display(BettiTable( T ));
total:  10  5  2  3  8  ?  ?
---------|--|--|--|--|--|--|
    2:   9  4  1  .  .  0  0
    1:   *  1  1  1  .  .  0
    0:   *  *  .  .  .  3  8
---------|--|--|--|--|--S--|
twist:  -5 -4 -3 -2 -1  0  1
----------------------------
Euler:   8  3  0 -1  0  3  8
```

Theorem 1.21 (Serre vanishing theorem, [6,Theo.III.5.2])**.** *Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^m_k$. Then*

1. *For all $0 \le i \le m$: $H^i\mathcal{F}$ is a finite dimensional vector space over k.*
2. *$H^i\mathcal{F} = 0$ for all $i > \dim \operatorname{supp}(\mathcal{F})$.*

3. *$H^i(\mathcal{F}(d)) = 0$ for all $j > 0$ and $d \gg 0$.*

Theorem 1.22 (Serre, [16,Theo.8.2.5]). *Let $X = \mathbb{P}^m_k = \mathrm{Proj}(k[x_1, \ldots, x_{m+1}])$, then*

1. *$H^0(X, \mathcal{O}_X(d)) \cong k[x_1, \ldots, x_{m+1}]_d$.*
2. *$H^m(X, \mathcal{O}_X(d)) \cong H^0(X, \mathcal{O}_X(-d-m-1))$.*
3. *$H^p(X, \mathcal{O}_X(d)) = 0$ if $0 < p < m$ or $p > m$.*

Example 1.7. Let $S = k[x_1, \ldots, x_4]$ then $X = \mathbb{P}^3_k$ and

1. $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3_k}) \cong k$.
2. $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(1)) \cong \langle x_1, x_2, x_3, x_4 \rangle_k \cong k^4$.
3. $H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(d)) \cong k^{\binom{d+3}{d}}$.
4. $H^3(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(-8)) \cong H^0(\mathbb{P}^3_k, \mathcal{O}_{\mathbb{P}^3_k}(4)) \cong k^{\binom{7}{4}} \cong k^{35}$.

Definition 1.23. For a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^m_k$, we define $\gamma_{i,d}(\mathcal{F})$ to be the cohomological dimensions $\gamma_{i,d}(\mathcal{F}) = \dim_k H^i(\mathbb{P}^m_k, \mathcal{F}(d))$. The indexed set $(\gamma_{i,d})_{i=0,\ldots,m, d\in\mathbb{Z}}$ is the cohomology table of $\mathcal{F}$, which lives in the vector space $\mathbb{T} = \prod_{d\in\mathbb{Z}} \mathbb{Q}^{m+1}$.

We shall display an element (a table) of the vector space $\mathbb{T}$ as follows:

$\cdots$	$\gamma_{m,-1}$	$\gamma_{m,0}$	$\gamma_{m,1}$	$\cdots$	m
	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$\cdots$	$\gamma_{1,-1}$	$\gamma_{1,0}$	$\gamma_{1,1}$	$\cdots$	1
$\cdots$	$\gamma_{0,-1}$	$\gamma_{0,0}$	$\gamma_{0,1}$	$\cdots$	0
$\cdots$	-1	0	1	$\cdots$	$d \setminus i$

Example 1.8. The cohomology table of the structure sheaf $\mathcal{O}_{\mathbb{P}^3_k} = \widetilde{S}$, where $S = k[x_1, x_2, x_3, x_4]$, looks as follows

$\cdots$	35	20	10	4	1	.	.	.	.	.	.	.	.	.	$\cdots$	3
$\cdots$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$\cdots$	2
$\cdots$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	$\cdots$	1
$\cdots$	.	.	.	.	.	.	.	.	1	4	10	20	35	56	$\cdots$	0
$\cdots$	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5		$d \setminus i$

Gap Code

```
gap> LoadPackage( "GradedModules" );;
gap> S:= GradedRing( HomalgFieldOfRationalsInSingular()*"x1,x2,x3,x4");
Q[x1,x2,x3,x4]
(weights: yet unset)
gap> S^0;
<The graded free left module of rank 1 on a free generator>
gap> T:= TateResolution( S^0, -5, 5 );
================================================================
                     SINGULAR::SCA
The SINGULAR Subsystem for Super-Commutative Algebras
     by: G.-M. Greuel, O. Motsak, H. Schoenemann
FB Mathematik der Universitaet, D-67653 Kaiserslautern
================================================================
<An acyclic cocomplex containing 10 morphisms of graded left modules at
degrees [ -5 .. 5 ]>
gap> B:= BettiTable( T );
<A Betti diagram of <An acyclic cocomplex containing 10 morphisms of
graded left modules at degrees [ -5 .. 5 ]>>
gap> Display( B );
total:   35  20  10   4   1   1   4  10  20  35  56   ?   ?   ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
    3:   35  20  10   4   1   .   .   .   .   .   .   0   0   0
    2:    *   .   .   .   .   .   .   .   .   .   .   .   0   0
    1:    *   *   .   .   .   .   .   .   .   .   .   .   .   0
    0:    *   *   *   .   .   .   .   .   1   4  10  20  35  56
----------|---|---|---|---|---|---|---|---S---|---|---|---|---|
twist:   -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4   5
---------------------------------------------------------------
Euler:  -35 -20 -10  -4  -1   0   0   0   1   4  10  20  35  56
gap> I:= LeftSubmodule( "x1^2, x2*x3, x4^2", S );
<A graded torsion-free (left) ideal given by 3 generators>
gap> M:= FactorObject( I );
<A graded cyclic torsion left module presented by yet unknown relations
for a cyclic generator>
gap> Display( M );
Q[x1,x2,x3,x4]/< x4^2, x2*x3, x1^2 >

(graded, degree of generator: 0)
```

```
gap> T1:= TateResolution( M, -4, 3 );
<An acyclic cocomplex containing 7 morphisms of graded left modules
at degrees [ -4 .. 3 ]>
gap> B1:=BettiTable( T1 );
<A Betti diagram of <An acyclic cocomplex containing 7 morphisms of
graded left modules at degrees [ -4 .. 3 ]>>
gap> Display( B1 );
    0:   8  8  8  8  8  8  8  8
---------|--|--|--|--|--|--|--M
twist:  -4 -3 -2 -1  0  1  2  3
-------------------------------
Euler:   8  8  8  8  8  8  8  8
```

It is often convenient to approximate the information about all the cohomology of $\mathcal{F}$ in a single integer

$$\chi(X,\mathcal{F}) = \sum_i (-1)^i \mathrm{dim}_k H^i(X,\mathcal{F}),$$

called the Euler characteristic of $\mathcal{F}$.

For example for the structure sheaf $\mathcal{O}_{\mathbb{P}^m_k}(d)$ the **Euler characteristic** is given by

$$\chi(\mathbb{P}^m_k, \mathcal{O}_{\mathbb{P}^m_k}(d)) = \frac{(d+m)(d+m-1)\cdots(d+1)}{m!},$$

which is polynomial in d of degree $m = \dim \mathbb{P}^m_k$.

Theorem 1.24 (Hilbert polynomial of a coherent sheaf, [17,Theo.18.6.1])**.** *Let $\mathcal{F}$ be a coherent sheaf on a projective variety X. Then there exists a polynomial $P_{\mathcal{F}}(t) \in \mathbb{Q}[t]$ such that $\chi(X,\mathcal{F}(d)) = P_{\mathcal{F}}(d)$ for all $d \in \mathbb{Z}$. This polynomial is called the **Hilbert polynomial** of the sheaf $\mathcal{F}$.*

Definition 1.25. By the **rank** of a coherent sheaf $\mathcal{E}$ on $\mathbb{P}^m$ we mean the normalized leading coefficient of the Hilbert polynomial,

$$\chi(\mathcal{E}(d)) = \frac{\text{rank } \mathcal{E}}{s!} d^s + O(d^{s-1}).$$

s is called the dimension of the sheaf and is denoted by $\dim \mathcal{E}$.

Definition 1.26. Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^m_k$ and $a \in \mathbb{Z}$. We say $\mathcal{F}$ is a-regular if $H^i(\mathbb{P}^m_k, \mathcal{F}(a-i)) = 0$ whenever $i > 0$.

Theorem 1.27. *Let $m \geq 0$ and $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^m_k$ with Hilbert polynomial $P_{\mathcal{F}}(t) \in \mathbb{Q}[t]$. Then*

$$P_{\mathcal{F}}(d) = \dim_k H^0(\mathbb{P}^m_k, \mathcal{F}(d))$$

for all $d \gg 0$.

This follows from the vanishing of the higher cohomology of high enough twists of $\mathcal{F}$ (Serre vanishing theorem).

1.4 Convex geometry

Let V be a $\mathbb{Q}$-vector space.

Definition 1.28 (Convex cone, [21, Appendix A])**.** A subset $C \subseteq V$ is a **cone** if it is closed under multiplication by elements of $\mathbb{Q}_{\geq 0}$. A cone C is called **convex** if it is closed under addition of its elements.

Definition 1.29 (Positive hull)**.** For a subset $B \subseteq V$, $\sigma(B)$ denotes the convex hull of B, defined as

$$\sigma(B) := \{\sum_{b \in B} \lambda_b b | \lambda_b \in \mathbb{Q}_{\geq 0}\}$$

which is clearly a convex cone.

Definition 1.30 (ray)**.** A **ray** in V is a cone that consists only of a non-zero vector v and all its positive multiples λv ($\lambda \geq 0$), i.e., every ray has the form $\sigma(v) := \sigma(\{v\})$ for some $0 \neq v \in V$.

It results from the definitions above that every cone is just the union of all rays it contains.

Definition 1.31 (Extremal ray)**.** A ray $\sigma(v)$ in a positive hull $\sigma(B)$ is called **extremal ray** if $v = u + w$ with $u, w \in \sigma(B)$ implies either u or w is in $\sigma(v)$.

Definition 1.32 (Simplicial cone, face, facet)**.** A cone C is called n-dimensional **simplicial cone** if $C = \sigma(B)$ for a set of n linearly independent vectors B. An m-dimensional **face** of C is a subset of the form $\sigma(B')$ for B' a subset of m vectors of B. A **facet** is an $(n-1)$-dimensional face.

Definition 1.33 (Simplicial fan)**.** A **simplicial fan** Σ is a collection of simplicial cones $\{C_i\}$ such that:

1. Each face of a simplicial cone in the fan is also in the fan.

2. Any pair of simplicial cones in the fan intersect in a common face.

We refer to $\bigcup_i C_i$ as the **support** of Σ. We say that a subset Σ of V has a **structure of a simplicial fan** if Σ is the support of some simplicial fan.

Definition 1.34 (Equidimensional simplicial fan)**.** A simplicial fan that is a finite union of cones is m-**equidimensional** if each maximal cone has dimension m. A facet of an equidimensional fan is a facet of any maximal cone, and it is a **exterior facet** if it is contained in only one maximal cone.

Remark 1.35. If $\dim(V)$ is finite and Σ is $(\dim V)$-equidimensional, then each exterior facet F determines a unique, up to scalar, functional $L : V \to \mathbb{Q}$ such that L vanishes along F and is nonnegative on the unique maximal cone containing F. We will call this functional the **hyperplane** which defines F. We refer to the halfspace of V that makes $L \geq 0$ as a **boundary halfspace** of the fan(denoted L^+).

Let P be a poset and let us assume there is a map $\phi : P \longrightarrow V$ such that $\phi(p_1), \ldots, \phi(p_s)$ are linearly independent in V for all chains $p_1 < \cdots < p_s$ in P and such that the union of simplicial cones

$$\Sigma(P, \phi) := \{\sigma(\{\phi(p_1), \ldots, \phi(p_s)\}) | s \in \mathbb{Z}_{\geq 0} \text{ and } p_1 < \cdots < p_s \text{ is a chain in } P\}$$

is a simplicial fan. When P is finite, this fan is referred to as a **geometric realisation** of P. In our case, P is the poset of R-degree sequences, and ϕ is the map $d \mapsto \pi_d$ where π_d is the Betti table associated to the sequence d, and it is the smallest integer solution of the Herzog-Kühl Equations associated to the sequence d, as we will see later. If the $\dim(V)$ is finite, then maximal cones of $\Sigma(P, \phi)$ are in bijection with maximal chains in P, and submaximal chains in P are in bijection with facets of $\Sigma(P, \phi)$.

Let V be an m-dimensional $\mathbb{Q}$-vector space, P be a finite poset, $\phi : P \to V$ as above, and $\Sigma(P, \phi)$ be an m-equidimensional simplicial fan. Then there is a bijective map between the set of submaximal chains of P thet lie in a unique maximal chain of P and the set of all boundary facets of $\Sigma(P, \phi)$.

This map is given by sending the submaximal chain $p_1 < p_2 < \cdots < p_{m-1}$ to $\sigma(\{\phi(p_1), \ldots, \phi(p_{m-1})\})$.

In addition, since $p_1, \ldots, p_{m-1}$ lies in a unique maximal cone, there is a unique $q \in P$ which extends this to a maximal chain. The boundary halfspace determined by this submaximal chain is the halfspace L^+, where $L(\phi(p_i)) = 0$ for $i = 1, \ldots, m-1$ and $L(\phi(q)) > 0$. Though more than one submaximal chain may determine the same boundary halfspace, each boundary halfspace corresponds to at least one such chain.

Let V be an m-dimensional $\mathbb{Q}$-vector space, and let Σ be an m-equidimensional simplicial fan. Let $\{L_k^+\}$ be the set of the boundary halfspaces of Σ. The convex cone $\bigcap_k \{L_k^+\}$ is the largest convex cone contained in the support of Σ.

Chapter 2

Boij-Söderberg theory of graded modules

Let $S = k[x_1, \ldots, x_n]$ and M be a finitely generated graded S-module, then the Hilbert syzygy theorem states that M has finite free resolution of length at most n, so we can consider the Betti tables to be elements of the vector space $\mathbb{D} := \bigoplus_{i \in \mathbb{Z}} \mathbb{Q}^{n+1}$.

2.1 The positive cone of the Betti tables

We want to make the Betti tables live in a finite dimenstional vector space. So let $c \leq n$ and $\mathbb{Z}^{c+1}_{deg}$ be the set of strictly increasing integer sequences $(a_0, a_1, .., a_c) \in \mathbb{Z}^{c+1}$. Such an element is called a **degree sequence**, and $\mathbb{Z}^{c+1}_{deg}$ is a partially ordered set, with $a \leq b$ if $a_i \leq b_i$ for $i \in \{0, .., c\}$.

Definition 2.1. For $a, b \in \mathbb{Z}^{c+1}_{deg}$, let $\mathbb{D}(a, b)$ be the $\mathbb{Q}$-vector space of the tables $(\beta_{ij})_{i=0,..,n;j\in\mathbb{Z}}$ such that β_{ij} maybe nonzero only in the range $0 \leq i \leq c$ and $a_i \leq j \leq b_i$.

The dimension of this vector space is $\sum_{i=0}^{c}(b_i - a_i + 1)$. To illustrate the definition of $\mathbb{D}(a, b)$ let for example $a = (0, 1, 2, 3), b = (2, 3, 5, 6)$. Then the elements of the vector space are of the form:

$j \setminus i$	0	1	2	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-1	0	0	0	0
0	$*$	$*$	$*$	$*$
1	$*$	$*$	$*$	$*$
2	$*$	$*$	$*$	$*$
3	0	0	$*$	$*$
4	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Definition 2.2. Let $a, b \in \mathbb{Z}^{c+1}_{deg}$, then we define $L(a, b)$ to be the $\mathbb{Q}$-vector space in $\mathbb{D}(a, b)$ spanned by tables in $\mathbb{D}(a, b)$ that represent Betti tables of CM-modules of codimension c. The set of nonnegative rays spanned by such Betti tables will be denoted by $B(a, b)$.

Lemma 2.3 (8, Lemma 1.5)**.** *B(a,b) is a convex cone.*

Proof. We should prove that $B(a, b)$ is closed under addition and multiplication with positive rational numbers. Let β_1, β_2 be two elements in $B(a, b)$, then there exists $r_1, r_2 \geq 0$ with $\beta_1 = r_1\beta(M_1)$ and $\beta_2 = r_2\beta(M_2)$ where M_1, M_2 are two CM-modules of codimension c. Now for any $q_1, q_2 \in \mathbb{Q}_{\geq 0}$ we have

$$q_1\beta_1 + q_2\beta_2 = q_1r_1\beta(M_1) + q_2r_2\beta(M_2).$$

We have $q_1r_1, q_2r_2 \in \mathbb{Q}$, say, $q_1r_1 = \frac{u_1}{v_1}, q_2r_2 = \frac{u_2}{v_2}$ then we have

$$q_1\beta_1 + q_2\beta_2 = \frac{1}{v_1v_2}(u_1v_2\beta(M_1)) + u_2v_1\beta(M_2)) = \frac{1}{v_1v_2}\beta(M_1^{u_1v_2} \oplus M_2^{u_2v_1}),$$

where $M_1^{u_1v_2} \oplus M_2^{u_2v_1}$ is also a CM-module of codimension c. $\square$

2.2 Herzog-Kühl equations

Let $S = k[x_1, \ldots, x_n]$ and let M be a finitely generated S-module. Let $\mathbf{F}$ be the minimal free resolution of M and let c denote the codimension of M, then the

Hilbert-Serre theorem states that the Hilbert series of M can be written as follows

$$h_M(t) = \frac{p(t)}{(1-t)^{n-c}},$$

where $p(t)$ is some polynomial and $\dim_R M = n - c$ is the smallest integer for which $(1-t)^p h_M(t)$ is a polynomial.

Using the additivity property of the Hilbert series on $\mathbf{F}$ we get

$$h_M(t) = \frac{1}{(1-t)^n} \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i \beta_{ij} t^j,$$

thus

$$(1-t)^c p_M(t) = \sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i \beta_{ij} t^j.$$

Differentiating and setting $t = 1$ gives the equations:

$$\sum_{i=0}^{n} \sum_{j \in \mathbb{Z}} (-1)^i j^p \beta_{ij} = 0, p = 0, .., c-1.$$

These equations are called the **Herzog-Kühl Equations** for the Betti table $\beta(M)$ of the module M.

Definition 2.4. The $\mathbb{Q}$-linear subspace of $\mathbb{D}(a,b)$ satisfying the Herzog-Kühl equations is denoted by $L^{HK}(a,b)$.

It is clear that $L(a,b)$ is a subspace of $L^{HK}(a,b)$, and later we will prove that are both equal.

Let $d = (d_0, \ldots, d_l) \in \mathbb{Z}^{c+1}_{deg}$. A free resolution of a module M is called **pure** of type d if it has the form

$$0 \leftarrow M \leftarrow R(-d_0)^{\beta_{0,d_0}} \leftarrow R(-d_1)^{\beta_{1,d_1}} \leftarrow \cdots \leftarrow R(-d_l)^{\beta_{l,d_l}} \leftarrow 0.$$

In other words $\beta_{i,j} = 0$ except when $j = d_i$.

And by a **pure Betti table** we shall mean a table such that for each column i there is exactly one nonzero entry β_{i,d_i} and the d_i form an increasing sequence. We immediately see that a pure resolution gives a pure Betti table.

Theorem 2.5 (12, Theorem 1). *Let M be an CM-module over R having a pure resolution of type $d = (d_0, \ldots, d_c)$, where c is the projective dimension of M. Then we have $\beta_{i,d_i} = (-1)^n t \prod_{j \neq i} \frac{1}{d_j - d_i}$ for $0 \leq i \leq c$ and $t \in \mathbb{Q}_{\geq 0}$.*

Proof. Since M is Cohen-Macaulay, we have $\operatorname{codim}(M) = c$. The Herzog-Kühl equations can be written as follows

$$\begin{cases} \sum_{i=0}^{c}(-1)^i \beta_{i,d_i} & =0 \\ \sum_{i=0}^{c}(-1)^i d_i \beta_{i,d_i} & =0 \\ \sum_{i=0}^{c}(-1)^i d_i^2 \beta_{i,d_i} & =0 \\ \qquad \vdots & \\ \sum_{i=0}^{c}(-1)^i d_i^{c-1} \beta_{i,d_i} & =0 \end{cases}$$

This system contains c equations in $c+1$ unknowns. The solutions are given by the kernel of the matrix

$$\begin{bmatrix} 1 & -1 & \cdots & (-1)^c \\ d_0 & -d_1 & \cdots & (-1)^c d_c \\ \vdots & \vdots & \ddots & \vdots \\ d_0^{c-1} & -d_1^{c-1} & \cdots & (-1)^c d_c^{c-1} \end{bmatrix}$$

This is a $c \times (c+1)$ matrix of rank c because the $c \times c$ minor given by the first c columns is nonzero. This minor is the determinant of the Vandermonde-matrix of $-d_0, \ldots, -d_{c-1}$. Hence there is only a one-dimensional $\mathbb{Q}$-vector of solutions. The solutions maybe found by computing the maximal minors and we find

$$\beta_{i,d_i} = (-1)^i t \prod_{j \neq i} \frac{1}{d_j - d_i}$$

with $t \in \mathbb{Q}$. When $t \geq 0$ then the solutions are all positive because the sequence consists of strictly increasing integers. □

We define $\pi(d)$ be the smallest integer positive solution of the equations above.

Remark 2.6. We will see later that the rays generated by $\pi(d)$ for all $d \in \mathbb{Z}_{deg}^{c+1}$ turn out to be exactly the extremal rays in the cone $B(a, b)$. Thus any Betti table is a positive linear combination of pure Betti tables.

With $\mathbb{Z}^{c+1}_{deg}$ equipped with "$\leq$", we define for $a, b \in \mathbb{Z}^{c+1}_{deg}$ the interval $[a, b]_{deg}$ consisting of all degree sequences $d \in [a, b]_{deg}$ with $a \leq d \leq b$. So the tables $\pi(d)$ where $d \in [a, b]_{deg}$ are the pure Betti tables in the window determined by a and b.

Example 2.1. Let $a = (0, 2, 3)$ and $b = (0, 3, 4)$. The elements of the vector space $\mathbb{D}(a, b)$ are of the form:

$$\begin{array}{c|ccc} j \setminus i & 0 & 1 & 2 \\ \hline 0 & * & 0 & 0 \\ 1 & 0 & * & * \\ 2 & 0 & * & * \end{array},$$

where $*$ represent the positions where the elements may be nonzero. Hence it is five dimensional.

The Herzog-Kühl equations for the tables are the following two equations:

$$\beta_{00} - (\beta_{12} + \beta_{13}) + (\beta_{23} + \beta_{24}) = 0,$$
$$-(2\beta_{12} + 3\beta_{13}) + (3\beta_{23} + 4\beta_{24}) = 0.$$

They are linearly independent and so $L^{HK}(a, b)$ will be three dimensional. On the other hand the tables $\pi((0, 2, 3)), \pi((0, 2, 4))$ and $\pi((0, 3, 4))$ are clearly linearly independent in this vector space, so they form a basis for it. This is not a coincidence:

Proposition 2.7 (8, Prop.1.8)**.** *For any maximal chain $a = d^1 < d^2 < \cdots < d^r = b$ in $[a, b]_{deg}$, the pure Betti tables $\{\pi(d^1), \pi(d^2), .., \pi(d^r)\}$ form a basis for $L^{HK}(a, b)$. The length of such a chain and hence the dimension of the vector space is $r = 1 + \sum_i (b_i - a_i)$.*

Proof. Let $\beta \in L^{HK}(a, b)$. We should write it as a linear combination of the associated pure Betti tables. The sequences d^1 and d^2 differ in one coordinate, otherwise the chain is not maximal. Suppose it is the i'th coordinate, i.e, $d^1 = (\ldots, d^1_i, \ldots)$ and $d^2 = (\ldots, d^1_i + 1, \ldots)$. Let $c_1 \in \mathbb{Q}$ be such that $\beta_1 = \beta - c_1\pi(d^1)$ is zero in the position (i, d^1_i). Then β_1 is contained in the vector spave $\mathbb{D}(d^2, b)$. We may proceed by induction and in the end, we get β_{r-1} contained in $\mathbb{D}(b, b)$. This means β_{r-1} is pure and so is a multiple of $\pi(d^r)$. In conclusion

$$\beta = \sum_{i=1}^{r} c_i \pi(d^i).$$

Every pure Betti table $\pi(d^i)$ is nonzero in a position, in which all other pure Betti tables $\pi(d^j), j \neq i$ are zero . This implies the linear independency of the pure Betti tables $\pi(d^i)$ for $i = 1, \dots, r$. □

For example, the Betti table of the module $k[x,y]/\langle x^2, xy, y^3\rangle$ can be written as a linear combination:

$$\beta = \tfrac{1}{2}\pi(0,2,3) + \tfrac{1}{4}\pi(0,2,4) + \tfrac{1}{4}\pi(0,3,4).$$

2.3 Boij-Söderberg conjectures

Here are the conjectures for which we will reproduce the proof below:

Conjecture 1 For every degree sequence $d = (d_0, d_1, \dots, d_c)$, there exists a CM-module M of codimension c with pure resolution of type d.

Conjecture 2 For each CM-module M of codimension c with Betti table $\beta(M)$ in $\mathbb{D}(a,b)$ there is a unique chain $d^1 < d^2 < \cdots < d^r$ in $[a,b]_{deg}$ such that $\beta(M)$ is uniquely a linear combination $c_1\pi(d^1) + \cdots + c_r\pi(d^r)$, where the c_i's are positive rationals.

2.3.1 Geometric interpretation

Since for any chain $D : d^1 < \cdots < d^r$ in $[a,b]_{deg}$ the Betti tables $\pi(d^i)$ for $i = 1, \dots, r$ are linearly independent tables in $\mathbb{D}(a,b)$, their positive rational linear combination gives a simplical cone in $\mathbb{D}(a,b)$. Two such cones intersect along faces, which is the content of the following proposition:

Proposition 2.8 (8, Prop.1.15)**.** *The set of simplical cones $\sigma(D)$ where D ranges all over chains $d^1 < \cdots < d^r$ in $[a,b]_{deg}$ form a simplical fan, which we denote by $\Sigma(a,b)$.*

Proof. Let D be a chain like above and E another chain $e^1 < \cdots < e^s$ in $[a,b]_{deg}$. We shall show that $\sigma(D)$ and $\sigma(E)$ intersect in $\sigma(D \cap E)$. So consider

$$\beta = \sum c_i\pi(d^i) = \sum c_i'\pi(e^i)$$

in the intersection. By omitting elements in the chain we may assume all c_i and c_i' positive. Then the lower bound of β, which we denoted $\underline{d}(\beta)$, will be d^1. But it will also be e^1, and so $e^1 = d^1$. Now we assume $c_1 \leq c_1'$. Let $\beta' = \beta - c_1\pi(d^1)$. Then β' is in $\sigma(D \backslash \{d^1\})$ and in $\sigma(E)$. By induction on the sum of the coordinates of D and E, we get that β' is in $\sigma(D \cap E \backslash \{d^1\})$ and so β is in $\sigma(D \cap E)$. □

Since all maximal chains in $[a,b]_{deg}$ have the same length, then the fan is equidimensional.

We define

$$\pi : \begin{cases} [a,b]_{deg} \rightarrow L^{HK}(a,b) \\ \quad d \longmapsto \pi(d) \end{cases}$$

The map π defines a geometric realisation of $[a,b]_{deg}$, and this turns out to be the fan $\Sigma(a,b)$.

Theorem 2.9. *With the same notations as above, we have*

1. $|\Sigma(a,b)| \subseteq B(a,b)$ *if and only if the first conjecture above is true.*

2. $B(a,b) \subseteq |\Sigma(a,b)|$ *if and only if the second conjecture above is true.*

Proof. Suppose that $|\Sigma(a,b)| \subseteq B(a,b)$, and let $d = (d_0,\ldots,d_c)$ be a degree sequence, then this sequence alone is a chain consisting of one element, therefore $\sigma(d) \in |\Sigma(a,b)|$, which implies that $\pi(d) \in B(a,b)$. This means there is a CM-module M of codimension c, and $\pi(d) = t\beta(M)$ where $t \in \mathbb{Q}$, that is M has the type d. Now we prove the inverse. Suppose the first conjecture is true, and let us prove that $|\Sigma(a,b)| \subseteq B(a,b)$. Let $D : d^1 < d^2 < \cdots < d^r$ be a chain of degree sequences, and let $\beta = \sum_{i=1}^r c_i\pi(d^i)$ with $c_i \geq 0$. Now for every degree sequence $d^i, i = 1,\ldots,r$, there is a CM-module M_i of codimension c and of type d^i. This implies $c_i\pi(d^i) \in B(a,b)$, but $B(a,b)$ is a cone, which implies that $\beta \in B(a,b)$, so $|\Sigma(a,b)| \subseteq B(a,b)$.

To prove the second equivalence, suppose that $B(a,b) \subseteq |\Sigma(a,b)|$ and let M be a CM-module of codimension c with Betti table $\beta(M)$ in $\mathbb{D}(a,b)$. So $\beta(M) \in |\Sigma(a,b)|$ which implies the existence of a chain $d^1 < d^2 < \cdots < d^r$ in $[a,b]_{deg}$ such that $\beta(M) = \sum_{i=1}^r c_i\pi(d^i)$ with $c_i > 0$, and the uniqueness follows from the fact that the pure Betti tables assosiated to a chain of degree sequences are linear independent. We prove now the inverse. Suppose that $\beta(M) \in B(a,b)$ is the Betti table of a CM-module of codimension c. Then based on the second conjecture there is a chain $d^1 < d^2 < \cdots < d^r$, where $\beta(M) = \sum_{i=1}^{r} c_i \pi(d^i)$. This implies that $\beta(M)$ is already in some simplical cone in the simplical fan. $\square$

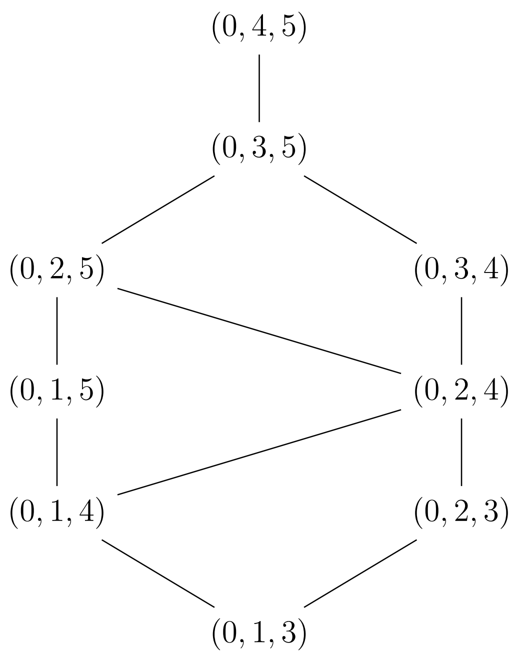
In order to prove the second conjecture $B(a, b) \subseteq |\Sigma(a, b)|$, we should describe the exterior facets of the Boij-Söderberg fan $\Sigma(a, b)$ and their supporting hyperplanes.

2.4 The exterior facets of the Boij-Söderberg fan

Let $D : d^1 < d^2 < \cdots < d^r$ be a maximal chain in the interval $[a, b]_{deg}$. The set of all positive rational combinations of the pure Betti tables $\pi(d^1), \ldots, \pi(d^r)$ defines a maximal simplical cone $\sigma(D)$ in the Boij Söderberg fan $\Sigma(a, b)$.

The facets of the cone $\sigma(D)$ are cones $\sigma(D\backslash\{d^i\})$ for $i \in \{1, 2, .., r\}$. A facet is called exterior if it lies on exactly one simplical cone in the fan $\Sigma(a, b)$.

Example 2.2. Let $a = (0, 1, 3), b = (0, 4, 5)$, then the length of any maximal chain is $\sum_i (b_i - a_i) + 1 = 6$. The Hasse table of the poset $[a, b]_{deg}$ is the table:



and there are five maximal chains. So the realization of the fan consists of the union of five simplical cones of dimension 6. Two of these five maximal sequences are

$$D : (0,1,3) < (0,2,3) < (0,2,4) < (0,3,4) < (0,3,5) < (0,4,5)$$
$$E : (0,1,3) < (0,1,4) < (0,1,5) < (0,2,5) < (0,3,5) < (0,4,5)$$

- $D\backslash\{(0,4,5)\}$: here we just omit the maximal element b, and clearly this can be completed to a maximal chain in exactly one way giving an exterior facet.
- $E\backslash\{(0,3,5)\}$: here the chain contains $(0,2,5)$ and $(0,4,5)$, and the only way to complete this to maximal chain is by including $(0,3,5)$ giving an exterior facet.
- $D\backslash\{(0,2,4)\}$: here the chain contains $(0,2,3)$ and $(0,3,4)$, and the only way to complete this to maximal chain is by including $(0,2,4)$ giving an exterior facet.

The exterior facets are described as follows:

Proposition 2.10 (9, Prop.2.12)**.** *Let D be a maximal chain in $[a,b]_{deg}$ and $f^- < f < f^+$ a subchain of D, then $\sigma(D\backslash\{f\})$ is an exterior facet if and only if one of the following holds:*

1. *f is either a or b.*
2. *f^-, f^+ differ in exactly one position, i.e there is an integer r such that:*
$$f^- = (\dots, r-1, \dots), f = (\dots, r, \dots), f^+ = (\dots, r+1, \dots).$$
3. *f^-, f^+ differ in exactly two adjacent positions, i.e., there is an integer r such that*
$$f^- = (\dots, r-1, r, \dots), f = (\dots, r-1, r+1, \dots), f^+ = (\dots, r, r+1, \dots).$$

Proof. $\sigma(D\backslash\{f\})$ is an exterior facet if it belongs to only one simplical cone in the fan. This happens if and only if we can extend the chain $D\backslash\{f\}$ to a maximal chain in only one way.

If $f = a$ or $f = b$ then $D\backslash\{f\}$ can be completed to a maximal chain only by adding f, thus $\sigma(D\backslash\{a\})$ and $\sigma(D\backslash\{b\})$ are exterior facets. Now, if $f \neq a$ and $f \neq b$ then f^- and f^+ differ in at most two positions because if they differ in three positions or more, then there are $l < k < m$, with $f^-_l < f^+_l, f^-_k < f^+_k, f^-_m < f^+_m$ and we have $f^- = (\ldots, f_l^-, \ldots, f_k^-, \ldots, f_m^-, \ldots) < (\ldots, f_l^-, \ldots, f_k^-, \ldots, f_m^- + 1, \ldots) < (\ldots, f_l^-, \ldots, f_k^- + 1, \ldots, f_m^- + 1, \ldots) < (\ldots, f_l^+, \ldots, f_k^+, \ldots, f_m^+, \ldots) = f^+$, which contradicts that D is a maximal chain.

If f^- and f^+ differ in only one position i then $f_i^- + 2 = f_i^+$ and thus $f_i = f_i^- + 1$.

If f^- and f^+ differ in two positions k, m with $k < m$, then $f_k^+ = f_k^- + 1, f_m^+ = f_m^- + 1$, and whenever $f_m^- \neq f_k^- + 1$, there are two elements between f^- and f^+, namely $(\ldots, f_k^- + 1, \ldots, f_m^-, \ldots)$ and $(\ldots, f_k^-, \ldots, f_m^- + 1, \ldots)$. This implies that $\sigma(D \backslash \{f\})$ is not an exterior facet. Hence $f_m^- = f_k^- + 1$, and we get $m = k + 1$ since f^- is strictly increasing. Hence the three cases listed above are the only cases where we get exterior facets. □

In case 3. We denote the exterior facet by $\mathbf{facet}(f, \tau)$ where τ is the position of the number $r - 1$ in f. For example the exterior facet $D \backslash \{(0, 2, 4)\}$ will be denoted by $\mathbf{facet}((0, 2, 4), 1)$.

2.5 The supporting hyperplanes

If σ is a full dimensional simplical cone in a vector space L, then each facet of σ is contained in a unique hyperplane, which is the kernel of a nonzero linear functional $h : L \to k$.

We shall apply this to the cones $\sigma(D)$ in $L^{HK}(a, b)$ where D is a maximal chain. Then we will find the equations of the hyperplanes H defining the exterior facets of $\sigma(D)$.

Actually we consider the inclusion $\sigma(D) \subseteq L^{HK}(a, b) \subseteq \mathbb{D}(a, b)$ and rather find a hyperplane H' in $\mathbb{D}(a, b)$ with $H = H' \cap L^{HK}(a, b)$. The equation of such a hyperplane is not unique up to a constant. Since $L^{HK}(a, b)$ is cut out by the Herzog-Kühl equations, we may add any linear combination of these equations, say l, so we get a new equation $h'' = h' + l$ defining another hyperplane $H'' \subseteq \mathbb{D}(a, b)$ which will intersect $L^{HK}(a, b)$ in H.

In cases 1. and 2. of Proposition 2.10, there exists a unique natural choice for the hyperplane. In example 2.2 the natural hyperplane equation defining the facet $\sigma(D \backslash \{(0, 4, 5)\})$ is $\beta_{1,4} = 0$ and for the facet $\sigma(E \backslash \{(0, 3, 5)\})$ the natural equation is $\beta_{1,3} = 0$; while in case 3. of the same proposition, there are two distingushed hyperplanes.

Lemma 2.11 (8, Lemma 2.5)**.** *Consider facets of type 3 (cf. Proposition 2.10).*

1. *If D and E are two maximal chains in $[a,b]_{deg}$ which both contain the subsequence $f^- < f < f^+$, then the exterior facets $\sigma(D\backslash\{f\})$ and $\sigma(E\backslash\{f\})$ define the same hyperplane in $L^{HK}(a,b)$.*

2. *Let $a' \leq a \leq b \leq b'$ and suppose D' is a maximal chain in $[a',b']_{deg}$ restricting to D in $[a,b]_{deg}$. If H' in $\mathbb{D}(a',b')$ is a hyperplane defining $\sigma(D'\backslash\{f\})$, then $H' \cap \mathbb{D}(a,b)$ is a hyperplane defining $\sigma(D\backslash\{f\})$.*

Proof. If D is a maximal chain

$$a = d^1 < \cdots < d^{p-1} = f^- < f < f^+ = d^{p+1} < \cdots < d^r = b$$

then $d^1 < \cdots < d^{p-1}$ is maximal chain in $\mathbb{D}(a,f^-)$. Hence $\pi(d^1),\ldots,\pi(d^{p-1})$ spans $L^{HK}(a,f^-)$. Similarly $L^{HK}(f^+,b)$ is generated by $\pi(d^{p+1}),\ldots,\pi(d^r)$. Thus the hyperplane of $\sigma(D\backslash\{f\})$ is $\langle \pi(d^1),\ldots,\pi(d^{p-1}),\pi(d^{p+1}),\ldots,\pi(d^r)\rangle_{\mathbb{Q}} = L^{HK}(a,f^-) + L^{HK}(f^+,b)$, which depends only on f, f^-, f^+, a and b.

For part 2. note that $\sigma(D\backslash\{f\}) \subseteq \sigma(D'\backslash\{f\})$, thus $\sigma(D\backslash\{f\}) \subseteq H'$. But H' does not contain $L^{HK}(a,b)$ since $\pi(f) \notin H'$. So $H' \cap L^{HK}(a,b)$ is hyperplane in $L^{HK}(a,b)$. Now the remark below shows that $H' \cap \mathbb{D}(a,b)$ is a hyperplane in $\mathbb{D}(a,b)$ defining $\sigma(D\backslash\{f\})$. $\square$

Remark 2.12. Let V be vector space, and $U \subseteq W \subseteq V$. If L is hyperplane in V, and $L \cap U$ is hyperplane in U then $L \cap W$ is also hyperplane in W.

Proof. We have $\dim(U \cap L) = \dim(U) - 1$. So

$$\begin{aligned}\dim(U+L) &= \dim(U) + \dim(L) - \dim(U \cap L)\\ &= \dim(U) + (\dim(V) - 1) - (\dim(U) - 1) \quad = \dim(V).\end{aligned}$$

Hence $U + L = V$, which implies that $W + L = V$, thus

$$\begin{aligned}\dim(W \cap L) &= \dim(W) + \dim(L) - \dim(W + L)\\ &= \dim(W) + (\dim(V) - 1) - \dim(V) \quad = \dim(W) - 1. \quad \square\end{aligned}$$

Let $a, b \in \mathbb{Z}^{c+1}_{deg}$ and L be a hyperplane in $\mathbb{D}(a, b)$ whose equation is

$$\sum_{\substack{0\le i\le c\\ a_i\le j\le b_i}} \lambda_{ij}\beta_{ij} = 0.$$

We will represent the equation by giving the coefficients λ_{ij} of β_{ij} in a table. For instance, the equation

$$2\beta_{00} + 5\beta_{12} + 3\beta_{13} + 4\beta_{23} - 3\beta_{24} + 7\beta_{34} - 8\beta_{35} = 0$$

will be represented as follows

$$\begin{array}{r|cccc} -1 & \cdot & \cdot & \cdot & \cdot \\ 0 & 2 & \cdot & \cdot & \cdot \\ 1 & \cdot & 5 & 4 & 7 \\ 2 & \cdot & 3 & -3 & -8 \end{array}$$

If $f^- < f < f^+$ is as in type 3, then the hyperplane equation is called **upper equation** if the values in its table can be nonzero only in the upper part of f^+. Similarly the equation is called **lower equation** if the values in its table can be nonzero only in the lower part of f^-.

Example 2.3. Let

$$f^- = (0, 2, 3) < f = (0, 2, 4) < f^+ = (0, 3, 4).$$

The upper equation of the facet has a form as in table 1, where the possible nonzero coefficients are marked by $*$ and the nonzero positions of f^+ are labelled by $+$. Similarly the lower equation has the form as in table 2, and the nonzero positions of f^- are labelled by $-$.

$$\begin{array}{r|ccc} \vdots & \vdots & \vdots & \vdots \\ -2 & * & * & * \\ -1 & * & * & * \\ 0 & 0^+ & * & * \\ 1 & 0 & * & * \\ 2 & 0 & 0^+ & 0^+ \\ 3 & 0 & 0 & 0 \end{array} \qquad \begin{array}{r|ccc} -2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0^- & 0 & 0 \\ 1 & * & 0^- & 0^- \\ 2 & * & * & * \\ 3 & * & * & * \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Upper equation Lower equation

The upper hyperplane equation can be algorithmically determined [1, Prop.2.2] as follows:

1. We begin by putting zeros in all entries of the table corresponding to degree sequences that are $\geq f^+$, that is we put a zero in column i and each row with index $j \geq f_i^+ - i$ for each $i = 0, \ldots, n+1$.

2. Suppose that the Betti numbers corresponding to f^- are $\beta_0, \beta_1, \ldots, \beta_{n+1}$. In column τ and row $f_\tau^- - \tau$ we put $\beta_{\tau+1}$ and in column $\tau + 1$ and row $f_{\tau+1}^- - (\tau + 1)$ we put $-\beta_\tau$; this ensures that our functional will vanish on the Betti tables corresponding to f^- as well as on that corresponding to f^+, and will be positive on that corresponding to f.

3. Whatever we put in the other entries of the table, this functional will vanish on every pure Betti table with degree sequence $\geq f^+$. We can now solve for the remaining (as much as we want) coefficients inductivily. We start by f^- and at each stage, we choose the next smaller degree sequence in our chain. The corresponding Betti table contains only one nonzero entry in the region that is not yet determined, so the vanishing condition allow us to determine uniquely all elements of the table.

The same idea could be used to find the lower hyperplane equation.

Let us go back to the example above to find the upper equation h_{up} of the **facet**$((0, 2, 4), 1)$. Applying the first step gives a table as above. The pure Betti table corresponding to f^- is

$$\pi(f^-) = \pi((0,2,3)) = \begin{array}{r|ccc|} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & 2 \end{array}.$$

After applying the second step, we get the following equation table

$$\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ -2 & * & * & * \\ -1 & * & * & * \\ 0 & 0^+ & * & * \\ 1 & 0 & 2 & -3 \end{array}$$

In step 3, let us choose $d = (0, 1, 3)$; its corresponding pure Betti table is

$$\pi(d) = \pi((0,1,3)) = \begin{array}{c|ccc|} 0 & 2 & 3 & 0 \\ 1 & 0 & 0 & 1 \end{array}.$$

Since $\pi(d) \in \mathbf{facet}(f, 1)$, we get $0(2)+\lambda_{11}(3)-3(1) = 0$, hence $\lambda_{11} = 1$, so equation table is now

$$\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ -2 & * & * & * \\ -1 & * & * & * \\ 0 & 0^+ & 1 & * \\ 1 & 0 & 2 & -3 \end{array}$$

Again in step 3, let us choose $d = (0, 1, 2)$; its corresponding pure Betti table is

$$\pi(d) = \pi((0,1,2)) = \begin{array}{c|ccc|} 0 & 1 & 2 & 1 \end{array}.$$

Since $\pi(d) \in \mathbf{facet}(f, 1)$, we get $0(1) + (1)(2) + \lambda_{22}(1) = 0$, hence $\lambda_{22} = -2$, so the equation table is now

$$\begin{array}{c|ccc} \vdots & \vdots & \vdots & \vdots \\ -2 & * & * & * \\ -1 & * & * & * \\ 0 & 0^+ & 1 & -2 \\ 1 & 0 & 2 & -3 \end{array}$$

In this way we may proceed to find all coefficients of the hyperplane in the window determined by a and b. We find that the coefficients of h_{up} is given by the table

$$\begin{array}{r|ccc}
-3 & 3 & -2 & 1 \\
-2 & 2 & -1 & 0 \\
-1 & 1 & 0 & -1 \\
0 & 0^+ & 1 & -2 \\
1 & 0 & 2 & -3 \\
2 & 0 & 0^+ & 0^+
\end{array}$$

Gap Code

```
gap>  LoadPackage("GradedModules");;
gap>  LoadPackage("BoijSoederberg");
gap>  d:= DegreeSequence([ 0, 2, 4]);
< Degree sequence of Betti table >
gap>  U:=UpperEquation(d, 1, -3, 2);
<Upper equation of a facet>
gap>  Display(U);
   0   1   2
------------
-3|3   -2  1
-2|2   -1  .
-1|1    .  -1
 0|.    1  -2
 1|.    2  -3
 2|.    .   .
gap>  L:= LowerEquation(d, 1, 0, 5);
<Lower equation of a facet>
gap>  Display(L);
  0   1   2
------------
0|.   .   .
1|-1  .   .
2|-2  3   -4
3|-3  4   -5
4|-4  5   -6
5|-5  6   -7
```

We notice in the example above that the diagonals from lower left to upper right have the same absolute values but alternating signs in the range where they are nonzero. This is not a special case as the following lemma shows.

Lemma 2.13 ([1, Coro.2.3]). *Let λ_{ij} be the coefficient of β_{ij} in the upper equation h_{up}. If $j < f_i^+$ then $\lambda_{i+1,j} = -\lambda_{ij}$.*

Proof. The upper equation differs from the lower equation by a linear combination of Herzog-Kühl equations, otherwise the Betti tables corresponding to some degree sequences $< f$ will be uniquely determined. But in these equations the coefficients of $\beta_{i+1,j}$ and $\beta_{i,j}$ always have the same absolute value but different signs. □

Remark 2.14. The above construction method shows that the upper hyperplane is uniquely determined by the condition $\beta_{i,j} = 0$ for all $j \geq f_i$. The same holds for the lower hyperplane of the facet.

2.6 Pairing between Betti tables and cohomology tables

Let $S = k[x_1, \ldots, x_n]$ and $m = n-1$, then the Betti table of any finitely generated S-module M lies in $\mathbb{D} = \bigoplus_{i\in\mathbb{Z}} \mathbb{Q}^{n+1}$. We defined the cohomology table $\gamma(\mathcal{F})$ of a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}_k^m = \operatorname{Proj}(S)$ to be the table $(\gamma_{i,d})_{\substack{i=0,\ldots,m\\ d\in\mathbb{Z}}} \in \mathbb{T} = \prod_{i\in\mathbb{Z}} \mathbb{Q}^{m+1}$ where

$$\gamma_{i,d} = \dim_k H^i(\mathbb{P}_k^m, \mathcal{F}(d)).$$

In the following we define the cohomology table of a complex $\mathbf{E}$ of S-modules:

Definition 2.15. Let $\mathbf{E} : 0 \to E^0 \to E^1 \to \cdots \to E^n \to 0$ be a complex of graded free S-modules. We define the cohomology table $\gamma(\mathbf{E})$ of the complex $\mathbf{E}$ to be the table $(\gamma_{i,d})_{\substack{i=0,\ldots,m\\ d\in\mathbb{Z}}} \in \mathbb{T} = \prod_{i\in\mathbb{Z}} \mathbb{Q}^{m+1}$ where

$$\gamma_{i,d} = \dim_k H^i(\mathbf{E})_d,$$

i.e., the value of the Hilbert function of the homology module $H^i(\mathbf{E})$ in degree d.

Remark 2.16. Example 1.5 shows that for every vector bundle $\mathcal{E}$ on $\mathbb{P}_k^m = \operatorname{Proj}(S)$, there is a complex $\mathbf{E}$ of graded S-modules, for which $\gamma(\mathcal{E})$ and $\gamma(\mathbf{E})$ coincide up to the top row m. Serre duality can be used to recover the m'th row of $\gamma(\mathcal{E})$ from $\gamma(\mathbf{E})$.

Definition 2.17. Using the same notation as above we define the bilinear form $\langle -, - \rangle : \mathbb{D} \times \mathbb{T} \to k$ as follows

$$\langle \beta, \gamma \rangle := \sum_{i \geq j} (-1)^{i-j} \sum_k \beta_{i,k} \gamma_{j,-k},$$

for $\beta \in \mathbb{D}$ and $\gamma \in \mathbb{T}$.

Remark 2.18. Let

$$\mathbf{F} : 0 \leftarrow F_0 \leftarrow \cdots \leftarrow F_r \leftarrow 0$$

be a free resolution of a finitely generated graded S-module M and let $\mathcal{E}$ be any coherent sheaf on $\mathbb{P}^m_k = \mathrm{Proj}(S)$. If $F_i = \bigoplus_k S(-k)^{\beta_{i,k}}$ then

$$\begin{aligned} h^j(\widetilde{F_i} \otimes \mathcal{E}) &= h^j(\bigoplus_k \mathcal{O}(-k)^{\beta_{i,k}} \otimes \mathcal{E}) = h^j(\bigoplus_k (\mathcal{O}(-k) \otimes \mathcal{E})^{\beta_{i,k}}) \\ &= h^j(\bigoplus_k \mathcal{E}(-k)^{\beta_{i,k}}) = \sum_k h^j(\mathcal{E}(-k)^{\beta_{i,k}}) = \sum_k \beta_{i,k} h^j \mathcal{E}(-k) = \sum_k \beta_{i,k} \gamma_{j,-k}. \end{aligned}$$

Hence the above pairing can be simplified as follows:

$$\langle \beta(\mathbf{F}), \gamma(\mathcal{E}) \rangle = \sum_{i \geq j} (-1)^{i-j} h^j(\widetilde{F_i} \otimes \mathcal{E}).$$

Theorem 2.19 (5, 3.2)**.** *Let* $\mathbf{F}$ *be a free resolution of a finitely generated graded* S*-module* M *and* $\mathcal{E}$ *is a vector bundle on* $\mathbb{P}^m_k$*, we have* $\langle \beta(\mathbf{F}), \gamma(\mathcal{E}) \rangle \geq 0$.

Proof. Since $\mathcal{E}$ is a vector bundle and the sheafification functor is exact, the following complex

$$0 \leftarrow \mathcal{M}_0 \leftarrow \widetilde{F_0} \otimes \mathcal{E} \leftarrow \widetilde{F_1} \otimes \mathcal{E} \leftarrow \cdots \leftarrow \widetilde{F_r} \otimes \mathcal{E} \leftarrow 0$$

with $\mathcal{M}_0 = \widetilde{M} \otimes \mathcal{E}$ is exact. Since the complex is exact, we can break it in short exact sequences as follows:

$$0 \leftarrow \mathcal{M}_0 \leftarrow \widetilde{F_0} \otimes \mathcal{E} \leftarrow \mathcal{M}_1 \leftarrow 0,$$

$$0 \leftarrow \mathcal{M}_1 \leftarrow \widetilde{F_1} \otimes \mathcal{E} \leftarrow \mathcal{M}_2 \leftarrow 0,$$

$$0 \leftarrow \mathcal{M}_2 \leftarrow \widetilde{F_2} \otimes \mathcal{E} \leftarrow \mathcal{M}_3 \leftarrow 0,$$

$$\vdots$$

For every short exact sequence we write the initial part of the long exact seqeunce of cohomologies, we get the following diagrams:

$$\begin{tikzcd}
0 & \operatorname{coker}(\alpha_1) \arrow[l] & H^0(\widetilde{F}_0 \otimes \mathcal{E}) \arrow[l] & H^0(\mathcal{M}_1) \arrow[l, "\alpha_1"'] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & \operatorname{coker}(\alpha_2) \arrow[l] & H^1(\widetilde{F}_1 \otimes \mathcal{E}) \arrow[l] & H^1(\mathcal{M}_2) \arrow[l, "\alpha_2"'] & \\
 & H^0(\mathcal{M}_1) \arrow[urr] & H^0(\widetilde{F}_1 \otimes \mathcal{E}) \arrow[l] & H^0(\mathcal{M}_2) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$$\begin{tikzcd}
0 & \operatorname{coker}(\alpha_3) \arrow[l] & H^2(\widetilde{F}_2 \otimes \mathcal{E}) \arrow[l] & H^2(\mathcal{M}_3) \arrow[l, "\alpha_3"'] & \\
 & H^1(\mathcal{M}_2) \arrow[urr] & H^1(\widetilde{F}_2 \otimes \mathcal{E}) \arrow[l] & H^1(\mathcal{M}_3) \arrow[l] & \\
 & H^0(\mathcal{M}_2) \arrow[urr] & H^0(\widetilde{F}_2 \otimes \mathcal{E}) \arrow[l] & H^0(\mathcal{M}_3) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

$$\vdots$$

Considering the remark above, the desired functional is the alternating sum of the Euler characteristics of the sequences above. Hence,

$$\langle \beta(\mathbf{F}), \gamma(\mathcal{E}) \rangle = \sum_{j=0}^{r} \dim \operatorname{coker}(\alpha_j) \geq 0$$

. $\square$

Remark 2.20. $\langle \beta, \gamma \rangle = \sum_{i \geq j} (-1)^{i-j} \sum_k \beta_{i,k} \gamma_{j,-k} = \sum_{i,k} \beta_{i,k} \sum_{j \leq i} (-1)^{i-j} \gamma_{j,-k}$. So from now on we will denote the sum $\sum_{j \leq i} (-1)^{i-j} \gamma_{j,-k}$ by $\lambda_{i,k}$, i.e.,

$$\lambda_{i,k} = \gamma_{i,-k} - \gamma_{i-1,-k} + \ldots + (-1)^i \gamma_{0,-k}.$$

Thus the value of the functional $\langle -, \gamma \rangle$ on the Betti table β is given by the dot product of β with the matrix

$$
\begin{array}{r|llll}
\vdots & \vdots & \vdots & \vdots & \\
-2 & \lambda_{0,-2} & \lambda_{1,-1} & \lambda_{2,0} & \cdots \\
-1 & \lambda_{0,-1} & \lambda_{1,0} & \lambda_{2,1} & \cdots \\
0 & \lambda_{0,0} & \lambda_{1,1} & \lambda_{2,2} & \cdots \\
1 & \lambda_{0,1} & \lambda_{1,2} & \lambda_{2,3} & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
$$

This functional is interesting because it has some common values with the upper and lower facet equations for some facet $\mathbf{facet}(f, \tau)$.

Example 2.4. The upper and lower equations of the facet $\mathbf{facet}(f = (-1, 0, 2, 3), 1)$ have coefficients as indicated in the following tables.

$$
\begin{array}{r|llll}
\vdots & \vdots & \vdots & \vdots & \vdots \\
-4 & 21 & -12 & 5 & \cdot \\
-3 & 12 & -5 & \cdot & 3 \\
-2 & 5 & \cdot & -3 & 4 \\
-1 & 0^* & 3 & -4 & 3 \\
0 & \cdot & 0^+ & 0^+ & 0^* \\
1 & \cdot & \cdot & \cdot & \cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\qquad\qquad
\begin{array}{r|llll}
\vdots & \vdots & \vdots & \vdots & \vdots \\
-2 & \cdot & \cdot & \cdot & \cdot \\
-1 & 0^* & 0^- & 0^- & \cdot \\
0 & 3 & -4 & 3 & 0^* \\
1 & 4 & -3 & \cdot & 5 \\
2 & 3 & \cdot & -5 & 12 \\
3 & \cdot & 5 & -12 & 21 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Upper equation   Lower equation

The nonzero positions of f^+ are labelled by $+$ and nonzero positions of f^- are labelled by $-$. The positions labelled by $*$ are the common nonzero positions of f^+ and f^-.

If we remove f_τ and $f_{\tau+1}$ from f and multiply the remaining entries by -1 we get a sequence of strictly decreasing integers $Z : z_1 = 1 > -3 = z_2$. We call it a **root sequence**. Applying Theorem 3.9 on Z provides a vector bundle $\mathcal{E}$ over $\mathbb{P}^2_k$ of rank 2 with supernatural cohomology and Theorem 3.12 allows us to compute its cohomology table.

$$
\gamma_{0,d} = \begin{cases} |p(d)| & \text{if } d > 1 \\ 0 & \text{otherwise} \end{cases}
\quad
\gamma_{1,d} = \begin{cases} |p(d)| & \text{if } 1 > d > -3 \\ 0 & \text{otherwise} \end{cases}
\quad
\gamma_{2,d} = \begin{cases} |p(d)| & \text{if } -3 > d \\ 0 & \text{otherwise} \end{cases}
$$

where $p(d) = (d-1)(d+3)$ is the Hilbert polynomial of $\mathcal{E}$.

21	12	5	·	·	·	·	·	·	·	·	·	·	2
·	·	·	·	3	4	3	·	·	·	·	·	·	1
·	·	·	·	·	·	·	·	5	12	21	32	45	0
-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	$d \setminus i$

Hence the coefficients of the functional $\langle -, \gamma \rangle$ are

$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	
-4	21	-12	5	·	$\cdots$
-3	12	-5	·	3	$\cdots$
-2	5	·	-3	4	$\cdots$
-1	·	3	-4	3	$\cdots$
0	·	4	-3	·	$\cdots$
1	·	3	·	-5	$\cdots$
2	·	·	5	-12	$\cdots$
3	·	·	12	-21	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The coefficients of the functional $\langle -, \gamma \rangle$ and the upper hyperplane equation are almost the same. To make them equal we define for a cohomological index τ and a degree bound c the truncated functional([1, Chapter.4]):

$$\langle \beta, \gamma \rangle_{c,\tau} = \sum_{\substack{\{i,j,k \mid j \leq i \,\wedge \\ (j<\tau \,\vee\, j \leq i-2)\}}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k} + \sum_{\substack{\{i,j,k,\epsilon \mid 0 \leq \epsilon \leq 1, \\ j=\tau, i=j+\epsilon, \\ k \leq c+\epsilon\}}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k}.$$

The formula of the coefficient $\mu^{\tau,c}_{i,i+\ell}$ of $\beta_{i,i+\ell}$ (the entry in column i and row ℓ) in the functional $\langle -, \gamma \rangle_{c,\tau}$ depends on which region of the Betti table the index (i, ℓ) belongs:

	$i < \tau$	$\tau \leq i \leq \tau + 1$	$\tau + 1 < i$
$\ell \leq c - \tau$	$\lambda_{i,i+\ell}$	$(-1)^{i-\tau} \lambda_{\tau,i+\ell}$	$\lambda_{i-2,i+\ell}$
$\ell > c - \tau$	$\lambda_{i,i+\ell}$	$(-1)^{i-\tau+1} \lambda_{\tau-1,i+\ell}$	$\lambda_{i-2,i+\ell}$

Now if $\tau = 1$ and $c = f_\tau$ then the coefficients of the truncated functional $\langle -, \mathcal{E}\rangle_{0,1}$ are

$$\begin{array}{r|ccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
-4 & 21 & -12 & 5 & \cdot & \cdots \\
-3 & 12 & -5 & \cdot & 3 & \cdots \\
-2 & 5 & \cdot & -3 & 4 & \cdots \\
-1 & \cdot & 3 & -4 & 3 & \cdots \\
0 & \cdot & \cdot & \cdot & \cdot & \cdots \\
1 & \cdot & \cdot & \cdot & \cdot & \cdots \\
2 & \cdot & \cdot & \cdot & \cdot & \cdots \\
3 & \cdot & \cdot & \cdot & \cdot & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

We notice that the coefficients of the functional $\langle -, \mathcal{E}\rangle_{0,1}$ are exactly the coefficients of the upper hyperplane equation of $\mathbf{facet}((-1, 0, 2, 3), 1)$. We will see later that this is always the case.

Remark 2.21. The construction above explains why the diagonals from lower left to upper right in the upper hyperplane equation have the same absolute values but alternating signs in the range where they are not zero.

Gap Code

```
gap>  LoadPackage("BoijSoederberg");;
gap>  d:= DegreeSequence( [-1, 0, 2, 3] );
< Degree seqeunce of virtual Betti table >
gap>  UpperEquation(d, 1, -4, 1);
< Upper equation Of a facet >
gap>  Display(last);
   0    1    2    3
--------------------
-4|42   -24  10   .
-3|24   -10  .    6
-2|10   .    -6   8
-1|.    6    -8   6
 0|.    .    .    .
 1|.    .    .    .
gap>  LowerEquation(d, 1, -2, 3);
< Lower equation Of a facet >
gap>  Display(last);
   0    1    2    3
```

```
--------------------
-2|.    .    .    .
-1|.    .    .    .
 0|-6   8    -6   .
 1|-8   6    .    -10
 2|-6   .    10   -24
 3|.    -10  24   -42
gap>  Z:= RootSequence( [1, -3] );
< Root sequence of virtual cohomology table >
gap>  V:=VirtualCohomologyTable(Z, -4, 6);
< Virtual cohomology table >
gap>  Display(V);
total:  21/2    6  5/2  3/2    2  3/2  5/2    6 21/2   16 45/2
-----------|----|----|----|----|----|----|----|----|----|----|----|----|
    2:  21/2    6  5/2    .    .    .    .    .    .    .    .    .    .
    1:     .    .    .    .  3/2    2  3/2    .    .    .    .    .    .
    0:     .    .    .    .    .    .    .    .  5/2    6 21/2   16 45/2
-------------------------------------------------------------------------
twist:    -6   -5   -4   -3   -2   -1    0    1    2    3    4    5    6
gap>  V1:= CohomologyTableOfStandardSheafwithRootSequence(Z, -4, 6);
< Virtual cohomology table >
gap>  Display(V1);
total:  21 12  5  3  4  3  5 12 21 32 45
---------|--|--|--|--|--|--|--|--|--|--|--|--|
    2:  21 12  5  .  .  .  .  .  .  .  .  .  .
    1:   .  .  .  .  3  4  3  .  .  .  .  .  .
    0:   .  .  .  .  .  .  .  .  5 12 21 32 45
---------------------------------------------
twist:  -6 -5 -4 -3 -2 -1  0  1  2  3  4  5  6
gap>  FunctionalOfCohomologyTableInSpecificPosition(V1, 0, 0);
0
gap>  FunctionalOfCohomologyTableInSpecificPosition(V1, 1, 1);
4
gap>  FunctionalOfCohomologyTable(V1, -4, 3);
< functional of cohomology table >
gap>  Display(last);
   0    1    2    3
--------------------
-4|21   -12  5    .
```

```
-3|12   -5    .    3
-2|5    .    -3    4
-1|.    3    -4    3
 0|.    4    -3    .
 1|.    3     .   -5
 2|.    .     5   -12
 3|.    .    12   -21
gap>  TruncatedFunctionalOfCohomologyTable(V1, 1, 0, -4, 3);
< Truncated functional of cohomology table >
gap>  Display(last);
   0    1    2    3
--------------------
-4|21   -12  5    .
-3|12   -5   .    3
-2|5    .    -3   4
-1|.    3    -4   3
 0|.    .    .    .
 1|.    .    .    .
 2|.    .    .    .
 3|.    .    .    .
```

Using spectral sequences, Eisenbud and Schreyer proved the following theorem:

Theorem 2.22 ([1, Theo.3.1, 4.1]). *Let* $\mathbf{E} : 0 \to E^0 \to \cdots \to E^n \to 0$ *be a complex of graded free* S*-modules and let* $\mathbf{F}$ *be a free resolution of a graded* S*-module* M *then*

1. $\langle \beta(\mathbf{F}), \gamma(\mathbf{E}) \rangle \geq 0$
2. *if* M *and the modules* $H^j(E)$ *for* $j > 0$ *have finite length and*

$$\begin{aligned} 0 &> \operatorname{reg} M + \operatorname{reg} E^0 \\ 0 &> \operatorname{reg} F_{j-1} + \operatorname{reg} H^j(E) \quad \textit{for every } j > 0 \end{aligned}$$

   then $\langle \beta(\mathbf{F}), \gamma(\mathbf{E}) \rangle = 0$.
3. *if* $\mathbf{F}$ *is minimal then* $\langle \beta(\mathbf{F}), \gamma(\mathbf{E}) \rangle_{c,\tau} \geq 0$.

Chapter 3

Existence of pure resolutions and supernatural vector bundles

3.1 The existence of pure resolutions

Proposition 3.1 ([1, Prop.5.2]). *Let $m_0, \ldots, m_k$ be non-negative integers, and let the homogeneous coordinates on $\mathbb{P}^{m_j}$ be $x_0^{(j)}, \ldots, x_{m_j}^{(j)}$. Then the multilinear forms*

$$x_\ell = \sum_{\mu_0+\mu_1+\cdots+\mu_k=\ell} \prod_{j=0}^{k} x_{\mu_i}^{(j)} \text{ for } \ell = 0, \ldots, \sum_{j=0}^{k} m_j$$

have no common zero in $\mathbb{P}_k^{m_0} \times \mathbb{P}_k^{m_1} \times \cdots \times \mathbb{P}_k^{m_k}$.

Proof. We do induction on $M = \sum_{j=0}^{k} m_j$. The case $M = 0$ is clear because there is only one multilinear form in this case. Suppose that $M > 0$ and x_ℓ all vanish at a point $P \in \mathbb{P}_k^{m_0} \times \mathbb{P}_k^{m_1} \times \cdots \times \mathbb{P}_k^{m_k}$. In particular,

$$x_M(P) = \prod_{j=0}^{k} x_{m_j}^{(j)}(P) = 0.$$

We cannot have $x_{m_j}^{(j)}(P) = 0$ for j and $m_j = 0$, so $x_{m_j}^{(j)}(P) = 0$ for some j with $m_j > 0$. Write $\mathbb{P}^{m_j-1}$ for the subspace of $\mathbb{P}^{m_j}$ where $x_{m_j}^{(j)}$ vanishes. The forms $x_0, \ldots, x_{M-1}$ restrict to the corresponding set of forms on

$$\mathbb{P}_k^{m_0} \times \cdots \times \mathbb{P}_k^{m_j-1} \times \cdots \times \mathbb{P}_k^{m_k}$$

and vanish at P, contradicting the inductive hypothesis. $\square$

For example if $m_0 = 1, m_1 = 2, m_2 = 3$ then the coordinates are

$$[x_0^{(0)}, x_1^{(0)}], [x_0^{(1)}, x_1^{(1)}, x_2^{(1)}], [x_0^{(2)}, x_1^{(2)}, x_2^{(2)}, x_3^{(2)}]$$

and the multilinear forms are

$$\begin{aligned}
x_0 &= x_0^{(0)} x_0^{(1)} x_0^{(2)} \\
x_1 &= x_0^{(0)} x_0^{(1)} x_1^{(2)} + x_0^{(0)} x_1^{(1)} x_0^{(2)} + x_1^{(0)} x_0^{(1)} x_0^{(2)} \\
x_2 &= x_0^{(0)} x_0^{(1)} x_2^{(2)} + x_0^{(0)} x_2^{(1)} x_0^{(2)} + x_0^{(0)} x_1^{(1)} x_1^{(2)} + x_1^{(0)} x_0^{(1)} x_1^{(2)} + x_1^{(0)} x_1^{(1)} x_0^{(2)} \\
x_3 &= x_0^{(0)} x_0^{(1)} x_3^{(2)} + x_0^{(0)} x_1^{(1)} x_2^{(2)} + x_0^{(0)} x_2^{(1)} x_1^{(2)} + x_1^{(0)} x_1^{(1)} x_1^{(2)} + x_1^{(0)} x_2^{(1)} x_0^{(2)} + x_1^{(0)} x_0^{(1)} x_2^{(2)} \\
x_4 &= x_0^{(0)} x_1^{(1)} x_3^{(2)} + x_0^{(0)} x_2^{(1)} x_2^{(2)} + x_1^{(0)} x_0^{(1)} x_3^{(2)} + x_1^{(0)} x_1^{(1)} x_2^{(2)} + x_1^{(0)} x_2^{(1)} x_1^{(2)} \\
x_5 &= x_0^{(0)} x_2^{(1)} x_3^{(2)} + x_1^{(0)} x_1^{(1)} x_3^{(2)} + x_1^{(0)} x_2^{(1)} x_2^{(2)} \\
x_6 &= x_1^{(0)} x_2^{(1)} x_3^{(2)}
\end{aligned}$$

Definition 3.2. For any product $X_1 \times X_2$ of projective varieties $X1$ and $X2$ with projections $p : X_1 \times X_2 \to X_1$, $q : X_1 \times X_2 \to X_2$ and sheaves $\mathcal{L}_i$ on X_i, we define the exterior product of $\mathcal{L}_1$ and $\mathcal{L}_2$ as

$$\mathcal{L}_1 \boxtimes \mathcal{L}_2 = p^*\mathcal{L}_1 \otimes q^*\mathcal{L}_2,$$

where $p^*\mathcal{L}_1$ and $q^*\mathcal{L}_2$ are the pullback of along p and q the sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$. In particular, $\mathcal{O}_{\mathbb{P}_k^m \times \mathbb{P}_k^n}(a, b)$ is defined to be $\mathcal{O}_{\mathbb{P}_k^m}(a) \boxtimes \mathcal{O}_{\mathbb{P}_k^n}(b)$.

Proposition 3.3 ([1, Prop.5.3])**.** *Let $\mathcal{F}$ be a sheaf on $X \times \mathbb{P}^m$, and let $p : X \times \mathbb{P}^m \to X$ be the projection. Suppose that $\mathcal{F}$ has a resolution of the form*

$$\mathcal{K} : 0 \leftarrow \mathcal{F} \leftarrow \mathcal{K}_0 \boxtimes \mathcal{O}(-e_0) \leftarrow \cdots \leftarrow \mathcal{K}_N \boxtimes \mathcal{O}(-e_N) \leftarrow 0$$

with degrees $e_0 < \cdots < e_N$. If this sequence contains the subsequence $(e_{k+1}, \ldots, e_{k+m}) = (1, \ldots, m)$ for some $k \geq -1$ then $R^\ell p_*\mathcal{F} = 0$ for $\ell > 0$ and $p_*\mathcal{F}$ has a resolution of the form

$$\mathcal{K}_0 \otimes H^0\mathcal{O}(-e_0) \leftarrow \cdots$$

$$\leftarrow \mathcal{K}_k \otimes H^0\mathcal{O}(-e_k) \leftarrow \mathcal{K}_{k+m+1} \otimes H^m\mathcal{O}(-e_{k+m+1})$$

$$\cdots \leftarrow \mathcal{K}_N \otimes H^m \mathcal{O}(-e_N) \leftarrow 0.$$

Lemma 3.4 (Acyclicity Lemma, [18, Lemma 20.11])**.** *Let*

$$\mathbf{F} : 0 \leftarrow F_0 \leftarrow \cdots \leftarrow F_n \leftarrow 0$$

be a complex of free finitely generated R-modules with $n \leq \operatorname{depth}(R)$. If the homology modules of $\mathbf{F}$ are of a finite length ($\ell(H_i(\mathbf{F})) < \infty$) for all $i > 0$, then $H_i(\mathbf{F}) = 0$ for all $i > 0$.

The above proposition provides a tool to construct a pure resolution with specific degree sequence.

Example 3.1.

Let $d = (0, 2, 5, 6)$ be a degree seqeunce. Let $m_0 := 3-1 = 2, m_1 := d_1-d_0-1 = 1, m_2 := d_2 - d_1 - 1 = 2, m_3 := d_3 - d_2 - 1 = 0$($m_i$'s are defined according to the next theorem). We want to construct a pure free resolution of type d. Using Proposition 3.1 we can find 6 multilinear forms of multidegree $(1, 1, 1)$ without common zero on $\mathbb{P} = \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$. Let $\mathcal{G}$ be the Koszul complex of these forms over the Cox ring S of $\mathbb{P}$, and let $\mathcal{K}$ be the tensor product of this complex with the line bundle $\mathcal{O}_{\mathbb{P}}(0, 0, 2)$ then

$$\mathcal{K}_i = \mathcal{O}_{\mathbb{P}}(-i, -i, 2-i)^{\binom{6}{i}} = \mathcal{O}_{\mathbb{P}^2}(-i)^{\binom{6}{i}} \boxtimes \mathcal{O}_{\mathbb{P}^1}(-i) \boxtimes \mathcal{O}_{\mathbb{P}^2}(2-i)$$

$\mathcal{K}$	$\mathbb{P}^2$	$\times$	$\mathbb{P}^1$	$\times$	$\mathbb{P}^2$
$\mathcal{K}_0$	$\mathcal{O}^1_{\mathbb{P}^2}$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^2}(2)$
$\mathcal{K}_1$	$\mathcal{O}_{\mathbb{P}^2}(-1)^6$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-1)$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^2}(1)$
$\mathcal{K}_2$	$\mathcal{O}_{\mathbb{P}^2}(-2)^{15}$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-2)$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^2}(0)$
$\mathcal{K}_3$	$\mathcal{O}_{\mathbb{P}^2}(-3)^{20}$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-3)$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^2}(-1)$
$\mathcal{K}_4$	$\mathcal{O}_{\mathbb{P}^2}(-4)^{15}$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-4)$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^2}(-2)$
$\mathcal{K}_5$	$\mathcal{O}_{\mathbb{P}^2}(-5)^6$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-5)$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^2}(-3)$
$\mathcal{K}_6$	$\mathcal{O}_{\mathbb{P}^2}(-6)^1$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-6)$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^2}(-4)$

Applying the above proposition on the resolution $\mathcal{K}$ (of the zero sheaf $\mathcal{F}$) along the projection

$$p_1 : \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^1$$

we get the resolution $\mathcal{K}'$ of the bundle $p_{1*}\mathcal{F}$.

$\mathcal{K}$	$\mathbb{P}^2$	$\times$	$\mathbb{P}^1$		
$\mathcal{K}'_0$	$\mathcal{O}^1_{\mathbb{P}^2}$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}$	$\otimes$	$H^0\mathcal{O}_{\mathbb{P}^2}(2)$
$\mathcal{K}'_1$	$\mathcal{O}_{\mathbb{P}^2}(-1)^6$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-1)$	$\otimes$	$H^0\mathcal{O}_{\mathbb{P}^2}(1)$
$\mathcal{K}'_2$	$\mathcal{O}_{\mathbb{P}^2}(-2)^{15}$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-2)$	$\otimes$	$H^0\mathcal{O}_{\mathbb{P}^2}(0)$
–	–	–			–
–	–	–			–
$\mathcal{K}'_3$	$\mathcal{O}_{\mathbb{P}^2}(-5)^6$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-5)$	$\otimes$	$H^2\mathcal{O}_{\mathbb{P}^2}(-3)$
$\mathcal{K}'_4$	$\mathcal{O}_{\mathbb{P}^2}(-6)^1$	$\boxtimes$	$\mathcal{O}_{\mathbb{P}^1}(-6)$	$\otimes$	$H^2\mathcal{O}_{\mathbb{P}^2}(-4)$

Then we apply it again to $\mathcal{K}'$ along the projection

$$p_2 : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$$

and get the resolution $\mathcal{K}''$ of the bundle $p_{2*}(p_{1*}\mathcal{F})$.

$\mathcal{K}$	$\mathbb{P}^2$				
$\mathcal{K}''_0$	$\mathcal{O}^1_{\mathbb{P}^2}$	$\otimes$	$H^0\mathcal{O}_{\mathbb{P}^1}$	$\otimes$	$H^0\mathcal{O}_{\mathbb{P}^2}(2)$
–	–	–			–
$\mathcal{K}''_1$	$\mathcal{O}_{\mathbb{P}^2}(-2)^{15}$	$\otimes$	$H^1\mathcal{O}_{\mathbb{P}^1}(-2)$	$\otimes$	$H^0\mathcal{O}_{\mathbb{P}^2}(0)$
–	–	–			–
–	–	–			–
$\mathcal{K}''_2$	$\mathcal{O}_{\mathbb{P}^2}(-5)^6$	$\otimes$	$H^1\mathcal{O}_{\mathbb{P}^1}(-5)$	$\otimes$	$H^2\mathcal{O}_{\mathbb{P}^2}(-3)$
$\mathcal{K}''_3$	$\mathcal{O}_{\mathbb{P}^2}(-6)^1$	$\otimes$	$H^1\mathcal{O}_{\mathbb{P}^1}(-6)$	$\otimes$	$H^2\mathcal{O}_{\mathbb{P}^2}(-4)$

We then have

$$\mathcal{K}''_0 = \mathcal{O}^6_{\mathbb{P}^2}, \mathcal{K}''_1 = \mathcal{O}_{\mathbb{P}^2}(-2)^{15}, \mathcal{K}''_2 = \mathcal{O}_{\mathbb{P}^2}(-5)^{24}, \mathcal{K}''_3 = \mathcal{O}_{\mathbb{P}^2}(-6)^{15}.$$

The resolution $\mathcal{K}''$ is then

$$\mathcal{K}'' : \mathcal{O}^6_{\mathbb{P}^2} \leftarrow \mathcal{O}_{\mathbb{P}^2}(-2)^{15} \leftarrow \mathcal{O}_{\mathbb{P}^2}(-5)^{24} \leftarrow \mathcal{O}_{\mathbb{P}^2}(-6)^{15} \leftarrow 0.$$

Taking the global sections in all twists, we get the complex

$$S_1^6 \leftarrow S_1(-2)^{15} \leftarrow S_1(-5)^{24} \leftarrow S_1(-6)^{15} \leftarrow 0,$$

where $S_1 = k[x_1, x_2, x_3]$. Its homology is of finite length, and since the length of the complex is only $3 \leq \mathrm{depth}(S_1)$ the complex is also exact by the acyclicity lemma. This implies that the complex is a pure minimal resolution, with the degree sequence $d = (0, 2, 5, 6)$.

Theorem 3.5. *Let k be any field, and let $d = (d_0 < \cdots < d_n)$ be a sequence of integers. There exists a graded $k[x_1, \ldots, x_n]$-module of finite length, whose minimal free resolution is pure of degree sequence d.*

Proof. Without loss of generality we may assume that $d_0 = 0$. Let $m_0 = m = n-1$, and for $i = 1, \ldots, n$ we set $m_i = d_i - d_{i-1} - 1$, and $M = \sum_{j=0}^{n} m_j = d_n - 1$. We can now find $M+1$ homogeneous forms of multidegree $(1, \ldots, 1)$ without common zero on $\mathbb{P} := \mathbb{P}^m \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n}$.

Let

$$\pi : \mathbb{P}^m \times \mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_n} \to \mathbb{P}^m$$

be the projection onto the first factor and

$$\mathcal{K} : 0 \leftarrow \mathcal{K}_0 \leftarrow \cdots \leftarrow \mathcal{K}_{M+1} \leftarrow 0$$

be the tensor product of the Koszul complex of these forms on $\mathbb{P}$ and the line bundle $\mathcal{O}_{\mathbb{P}}(0, 0, d_1, \ldots, d_{n-1})$, so

$$\mathcal{K}_i = \mathcal{O}_{\mathbb{P}}(-i, -i, \ldots, d_{n-1} - i)^{\binom{d_n}{i}} =$$

$$\mathcal{O}_{\mathbb{P}^m}(-i)^{\binom{d_n}{i}} \boxtimes \mathcal{O}_{\mathbb{P}^{m_1}}(-i) \boxtimes \cdots \boxtimes \mathcal{O}_{\mathbb{P}^{m_n}}(d_{n-1} - i).$$

The complex $\mathcal{K}$ is exact because the forms have no common zero. If we think of $\mathcal{K}$ as a resolution of the zero sheaf $\mathcal{F} = 0$, we can use the Proposition 3.3 repeatedly to get resolution of $\pi_*\mathcal{F} = 0$ of the form

$$\mathcal{O}^{\beta_0} \leftarrow \mathcal{O}^{\beta_1}(-d_1) \leftarrow \cdots \leftarrow \mathcal{O}^{\beta_n}(-d_n) \leftarrow 0.$$

Taking the global sections in all twists, we get the complex

$$S_1^{\beta_0} \leftarrow S_1^{\beta_1}(-d_1) \leftarrow \cdots \leftarrow S_1^{\beta_n}(-d_n) \leftarrow 0,$$

where $S_1 = k[x_1, x_2, \ldots, x_n]$. Its homology is of finite length and since the length of the complex is only $n \leq \operatorname{depth}(S_1)$, it is also exact by the acyclicity lemma 3.4. This implies that the complex is a pure minimal resolution with the degree sequence d. $\square$

Remark 3.6. If we follow the construction above carefully we find that

$$\beta_i = \left| \prod_{j \neq i, j \neq 0} \frac{d_j - d_0}{d_j - d_i} \right| \cdot \prod_{i=1}^{n} \binom{d_i - d_0 - 1}{d_i - d_{i-1} - 1}, \text{ for } i = 0, \dots, n.$$

Gap Code

```
gap> LoadPackage("BoijSoederberg");;
gap> d:= DegreeSequence( [ 0, 2, 5, 6 ] );
< Degree seqeunce of virtual Betti table >
gap> BettiNumbersOfDegreeSequence( d );
[ 6, 15, 24, 15 ]
gap> B:= VirtualPureBettiTable( d );
< Virtual Betti table >
gap> Display( last );
 total:  2 5 8 5
----------------
     0:  2 . . .
     1:  . 5 . .
     2:  . . . .
     3:  . . 8 5
----------------
degree:  0 1 2 3
```

3.2 Existence of vector bundles with a given root sequence

Definition 3.7. Let $\mathcal{F}$ be a sheaf on $\mathbb{P}_k^m$. We say that $\mathcal{F}$ has a natural cohomology if for each integer d, the cohomology $H^i(\mathcal{F}(d))$ is nonzero for at most one value i. We say $\mathcal{F}$ has supernatural cohomology if in addition, the Hilbert polynomial $\chi(\mathcal{F}(d))$ has distinct integral roots. In this case we define the root sequence of $\mathcal{F}$ to be the sequence of these roots in decreasing order, $z_1 > z_2 > \cdots > z_s$.

Remark 3.8. If $s = m$, then any sheaf with supernatural cohomology is locally free, so we will generally speak of supernatural vector bundles.

Theorem 3.9 (1,Theo.6.1)**.** *Let $z = (z_1 > z_2 > \cdots > z_m)$ be a seqeunce of strictly decreasing integers, consists of k disjoint subsequences of consecutive integers, of* *length $m_1, m_2, \ldots, m_k$. Let $m := \sum_{i=1}^{k} m_i$, then there exists a supernatural vector bundle $\mathcal{E}$ on $\mathbb{P}_k^m$ with root sequence z and rank $\binom{m}{m_1, m_2, \ldots, m_k}$.*

Proof. Let ν_j denote the starting index of the j-th subsequence, so that

$$z_{\nu_j}, \ldots, z_{\nu_j + m_j - 1}$$

are consecutive. Consider the product

$$\mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_k}$$

of k projective spaces and the line bundle

$$\mathcal{L} := \mathcal{O}(-z_{\nu_1} - 1, \ldots, -z_{\nu_k} - 1) := p_1^* \mathcal{O}(-z_{\nu_1} - 1) \otimes \cdots \otimes p_k^* \mathcal{O}(-z_{\nu_k} - 1)$$

on it.

Let

$$\pi : \mathbb{P}^{m_1} \times \mathbb{P}^{m_2} \times \cdots \times \mathbb{P}^{m_k} \to \mathbb{P}^m$$

be defined by the $m+1$ multilinear forms as in Proposition 3.1. Then π is a finite morphism, hence we have

$$H^i \mathcal{E}(d) \cong H^i(\mathbb{P}^{m_1} \times \cdots \times \mathbb{P}^{m_k}, \mathcal{L}(d, \ldots, d))$$

where $\mathcal{E} = \pi_* \mathcal{L}$.

Using the Künneth formula[19, Prop.4] we can by induction on k prove that $\mathcal{E}$ has supernatural cohomology as desired and the rank of it is $\binom{m}{m_1, \ldots, m_k}$. $\square$

Example 3.2. Let $\mathbf{z} = (5, 4, 3, 1, 0)$ then $m_1 = 3, m_2 = 2, m = 5, \nu_1 = 1, \nu_2 = 4$.

Let $\mathcal{L} := \mathcal{O}_{\mathbb{P}^3}(-6) \boxtimes \mathcal{O}_{\mathbb{P}^2}(-2) = p_1^* \mathcal{O}(-6) \otimes p_2^* \mathcal{O}(-2)$. If $\pi : \mathbb{P}^3 \times \mathbb{P}^2 \to \mathbb{P}^5$ and $\mathcal{E} := \pi_* \mathcal{L}$ then

$$\begin{aligned} H^i \mathcal{E}(d) &\cong H^i(\mathbb{P}^3 \times \mathbb{P}^2, \mathcal{L}(d, d)) \\ &\cong H^i(\mathbb{P}^3 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^3}(d-6) \boxtimes \mathcal{O}_{\mathbb{P}^2}(d-2)) \\ &\cong \bigoplus_{j+k=i} H^j(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-6)) \otimes H^k(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-2)) \end{aligned}$$

we have the following cases

1. If $d > 5$ then $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-6)) \neq 0$ and $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-2)) \neq 0$, hence $H^0\mathcal{E}(d) \neq 0$.

2. If $d \in \{5, 4, 3\}$ then $d - 6 \in \{-1, -2, -3\}$ which are the roots of the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^3}$, thus all summands in the direct sum above are zero, and in this case we have $H^i\mathcal{E}(d) = 0$ for all i's.

3. If $d = 2$ then $H^3\mathcal{E}(2) \cong H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(-4)) \otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}) \neq 0$.

4. If $d \in \{1, 0\}$ then $d - 2 \in \{-1, -2\}$ which are the roots of the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^2}$, thus all summands in the direct sum above are zero, and in this case we have $H^i\mathcal{E}(d) = 0$ for all i's.

5. If $d < 0$ then $H^3(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-6)) \neq 0$ and $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d-2)) \neq 0$, hence $H^5\mathcal{E}(d) \neq 0$.

Thus $\mathbf{z}$ is the root sequence of $\mathcal{E}$. Furthermore $H^i\mathcal{E}(d) \neq 0$ for at most one $i \in \{0, \ldots, 5\}$ as for $\mathcal{O}_{\mathbb{P}^3}$ and $\mathcal{O}_{\mathbb{P}^2}$.

Remark 3.10 (19, Prop.7). The statement about the rank of the constructed bundle above follows immediately from the fact that

$$P_{\mathcal{F}\boxtimes\mathcal{G}}(d) = P_{\mathcal{F}}(d) \cdot P_{\mathcal{G}}(d),$$

where $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves on some projective schemes X, Y respectively.

Gap Code

```
gap>  LoadPackage("BoijSoederberg");;
gap>  z:= RootSequence( [ 5, 4, 3, 1, 0 ] );
< Root sequence of virtual cohomology table >
gap>  V:= VirtualCohomologyTable(z, 3, 7);
< Virtual cohomology table >
gap>  Display( V );
 total:   21/2     2 1/10  3/2 42/5    ?    ?    ?    ?    ?
------------|----|----|----|----|----|----|----|----|----|----|
     5:   21/2     2    .    .    .    .    .    .    .    .
     4:      .     .    .    .    .    .    .    .    .    .
     3:      .     .    .    . 1/10    .    .    .    .    .
     2:      .     .    .    .    .    .    .    .    .    .
     1:      .     .    .    .    .    .    .    .    .    .
     0:      .     .    .    .    .    .    .    .  3/2 42/5
```

```
----------------------------------------------------------
 twist:    -2   -1    0    1    2    3    4    5    6    7
gap>  rank:= RankOfStandardSheafWithRootSequence( z );
10
gap>  V1:= CohomologyTableOfStandardSheafwithRootSequence( z, 3, 7 );
< Virtual cohomology table >
gap>  Display( V1 );
 total:   105  20   1  15  84   ?   ?   ?   ?   ?
-----------|---|---|---|---|---|---|---|---|---|---|
     5:   105  20   .   .   .   .   .   .   .   .
     4:     .   .   .   .   .   .   .   .   .   .
     3:     .   .   .   .   1   .   .   .   .   .
     2:     .   .   .   .   .   .   .   .   .   .
     1:     .   .   .   .   .   .   .   .   .   .
     0:     .   .   .   .   .   .   .   .  15  84
------------------------------------------------
 twist:   -2  -1   0   1   2   3   4   5   6   7
```

The following theorem will play a central rule for finding bounds for cohomology tables of a vector bundles. Using these bounds it will be enough to study tables in a finite vector space $\subseteq \mathbb{T}$.

Theorem 3.11 (1, Theo.6.3)**.** *If $\mathcal{E}$ is a nonzero coherent sheaf on $\mathbb{P}^m$ then for each integer d some $\gamma_{i,d-i} := h^i\mathcal{E}(d-i) \neq 0$, i.e., the diagonals in its cohomology table from lower right to upper left must have at least a nonzero value. Furthermore,*

$$d \mapsto M_d := \max\{i|\gamma_{i,d-i} \neq 0\} \quad \textit{and} \quad d \mapsto m_d := \min\{i|\gamma_{i,d-i} \neq 0\}$$

are weakly decreasing functions of d.

Proof. The term of Tate resolution of $\mathcal{E}$ in cohomological degree d is

$$\bigoplus_j H^j(\mathcal{E}(d-j)) \otimes \omega_E.$$

Since the construction of Tate resolution is a recusive procedure, the resolution terminates at the first term equals to zero, but this contradicts the fact that $h^{\dim \operatorname{Supp}(\mathcal{E})}\mathcal{E}(d) \neq 0$ for $d \ll 0$, which gives the first statment.

Since the generators of the exterior algebra are of negative degree, there are nonzero maps $f : E(-d) \to E(-e)$ only if $d < e$. To explain this, say $u \in E(-d)_q$ then $f(u)$ should have also degree q, i.e., $f(u) \in E(-e)_q$, but we have $u \in E(-d)_q = E_{q-d}$ and so $f(u) \in E_{\dim f+q-d} = E(\dim f - d)_q$, which implies $-e = \dim f - d$, and therefore $e = d - \dim f$. But $\dim f < 0$, so $d < e$. The dimension of any map in $T(\mathcal{E})$ can't be zero because this contradicts that $T(\mathcal{E})$ is a minimal resolution (if the matrix contains constants, then for some $r \in \mathbb{Z}$ we can cancel $E(r)$ from two consecutive terms in $T(\mathcal{E})$. Hence the sequence M_d is weakly decreasing.

We do the same discussion for the dual complex of $T(\mathcal{E})$ to get the second desired statment. □

Theorem 3.12. *If $\mathcal{E}$ is a supernatural sheaf of dimension s with root sequence $z = (z_1 > z_2 > \cdots > z_s)$, and if we set $z_0 = \infty$ and $z_{s+1} = -\infty$ then, for each $0 \leq j \leq s$,*

$$h^j\mathcal{E}(d) = \begin{cases} \frac{\operatorname{rank}\mathcal{E}}{s!}\prod_{i=1}^{s}|d - z_i| & \text{if } z_j > d > z_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let M_d, m_d be as in the previous theorem. Serre vanishing theorem states that $h^j\mathcal{E}(d) = 0$ for $d \gg 0$ and $j > 0$. In addition, $P_{\mathcal{E}}(d) = h^0\mathcal{E}(d)$ for $d \gg 0$, hence $M_d = m_d = 0$ for $d \gg 0$. Since the Hilbert polynomial has s distinct zeros, the dimension of the support of $\mathcal{E}$ is s. It follows that $H^j\mathcal{E}(d) = 0$ for $j > s$ and any d, while if $d \ll 0$ then $H^s\mathcal{E}(d) \neq 0$. Thus $M_d = s$ for $d \ll 0$. Since $\mathcal{E}$ has natural cohomology this implies $m_d = s$ for $d \ll 0$ as well. By the theorem above the sequences M_d and m_d are weakly decreasing from s to 0.

If we think of the cohomology table as a coordinate system, then $d - M_d$ and $d - m_d$ indicate the d-coordinates of the cells from which we read off M_d and m_d, respectively.

Since the M_d are weakly decreasing, the sequence of numbers $d - M_d$ is strictly increasing. It omits precisely those values z such that $z = d - i$ with $M_d > i \geq M_{d+1}$. This means that precisely s distinct values are omitted from the sequence $d - M_d$. Exactly the same consideration apply to the sequence $d - m_d$.

If $\mathcal{E}$ has natural cohomology, then the vanishing of $\chi(\mathcal{E}(z))$ implies the vanishing of all $H^j\mathcal{E}(z)$, so the integral roots of the Hilbert polynomial must be among the omitted values of the sequences $d - M_d$ and $d - m_d$. If E has supernatural cohomology, then there are s distinct roots, which thus give all the omitted values. It follows that the omitted values are the same for $d - M_d$ and $d - m_d$. Since these two sequences are the same for $d \ll 0$, they must be the same for all d; that is, $M_d = m_d$ for all d.

Moreover, $M_{d+1} = M_d - k$ if and only if there are exactly k roots of the Hilbert polynomial between $d - M_d$ and $d + 1 - M_{d+1}$. By induction we see that the value of M_d is equal to the number of roots above d.

The condition of natural cohomology implies that the value of $|\chi(\mathcal{E}(d))|$ is the value of some $h^j\mathcal{E}(d)$. The formulas above tell us about the value of j. The zeros determine the Hilbert polynomial as $\chi(\mathcal{E}(d)) = C \cdot \prod_{i=1}^{s}(d - z_i)$ for some constant C, and C can be computed by comparing leading coefficients, yielding the formulas above. □

The previous theorem shows that for every full dimensional supernatural sheaf $\mathcal{E}$ over $\mathbb{P}^m$, the intermediate cohomology modules $\bigoplus_d H^j\mathcal{E}(d)$ for $1 \leq j \leq m-1$ have finite length. Hence $\mathcal{E}$ is a vector bundle.

Proposition 3.13 (1, Prop.6.8)**.** *Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^m$, and let a be an interger. If $\mathcal{E}^*$ is a-regular then there exists a linear complex*

$$0 \to E^0 \to E^1 \to \cdots \to E^m \to 0$$

with $E^k = S(a+k)^{b_k}$, with homology $H^iE = \sum_d H^i\mathcal{E}(d)$ for $i \leq m-1$, and $H^mE = \sum_{d \geq -a-m} H^m\mathcal{E}(d)$.

Proof. Since $\mathcal{E}^*$ is a-regular, $\sum_{d \geq a} H^0\mathcal{E}^*(d)$ has a linear resolution

$$0 \leftarrow \sum_{d \geq a} H^0\mathcal{E}^*(d) \leftarrow S(-a)^{b_0} \leftarrow \cdots \leftarrow S(-a-m)^{b_m} \leftarrow 0$$

Since $\mathrm{Hom}_S(S(-d), S) \cong S(d)$, the dual complex is the desired complex E. Its sheafification $\widetilde{E}$ has homology $H^0(\widetilde{E}) \cong \mathcal{E}$ and is exact otherwise. The statement about the homology follows by chasing sheaf cohomology through the sheafified complex(cf. Theorem 1.18). □

Proposition 3.14 (1, Prop.6.9)**.** *Let $\mathcal{E}$ on $\mathbb{P}^m$ be a vector bundle with supernatural cohomology with root sequence $z_1 > z_2 > \cdots > z_m$. Let $a \geq -z_m - m$ be* *an integer. Then $\mathcal{E}^*$ is a-regular, and the complex constructed from $\mathcal{E}$ as in the previous proposition has supernatural cohomology.*

Proof. By Theorem 3.12 the dual bundle $\mathcal{E}^*$ is a-regular because

$$h^i\mathcal{E}^*(a-i) = h^{m-i}\mathcal{E}(i-a-m-1) = 0 \ for \ i \geq 1$$

because $i-a-m-1 \leq i-1+z_{m-1} \leq z_{m-i}$. □

Remark 3.15. Let $\mathcal{E}$ be a vector bundle on $\mathbb{P}^m_k$ and $\mathbf{E}$ be the above construted complex. Then both can be used to compute an upper hyperplane equation for a facet of type 3(cf. Example 2.4). These computations can be done by using the rows from cohomological degree 0 to $m-1$ in the cohomology table. Hence, in this context, the both $\gamma(\mathcal{E})$ and $\gamma(\mathbf{E})$ play the same rule.

3.3 Proof of the Boij-Söderberg conjectures

Since any module M over $k[x_1, \dots, x_c]$ can be given a structure of a module over $k[x_1, \dots, x_{c'}]$ for any $c' > c$, both Conjectures reduce at once to the case $n = c$.

J. Herzog and M. Kühl proved the following theorem:

Theorem 3.16 ([12, Theo.1]). *Let M be an S-module having pure resolution of type $d = (d_0 = 0, d_1, d_2, ..., d_c)$ and Betti numbers $\beta_0, \beta_1, ..., \beta_c$ where c is the projective dimension of M. Then the following two conditions are equivalent:*

1. *M is Cohen-Macaulay.*
2. *$\beta_i = b_i\beta_0$ for $i = 1, ..., c$ where $b_i := \left|\prod_{j\neq i, j\neq 0}(\frac{d_j}{d_j-d_i})\right|$.*

Theorem 3.17. *The cone defined by the upper facets equations contains the Betti tables of minimal free resolutions of all finitely generated graded S-modules.*

Proof. Suppose $f = (f_0 < f_1 < \cdots < f_n)$ is a degree sequence and τ is an integer with $0 \leq \tau \leq n-1$ such that $f_{\tau+1} = f_\tau + 2$, so the facet **facet**(f, τ) of type 3(cf. Prop.2.10) is defined. By Theorem 3.9 there is a vector bundle $\mathcal{E}$ on $\mathbb{P}^{n-1}$ with supernatural cohomology whose Hilbert polynomial $\chi(\mathcal{E}(d))$ has roots

$$(z_1 > z_2 > ... > z_{n-1}) = (-f_0 > \cdots > -f_{\tau-1} > -f_{\tau+2} > \cdots > -f_n).$$

Let

$$\mathbf{E} : 0 \to E^0 \to \cdots \to E^{n-1} \to 0$$

be the linear complex made from $\mathcal{E}$ with $a = f_n - n + 1$ as in Proposition 3.13, so that $E^0 = S(a)^{b_0}$. Set $c = f_\tau$. Theorem 2.22 proves that $\langle -, \mathbf{E} \rangle_{c,\tau}$ is non-negative on the cone of Betti tables of all minimal resolutions. Thus it suffices to prove that facet(f, τ) is defined by the vanishing of this functional.

Let us look closer to the coefficients of $\beta_{i,j}$ for $j \geq f_i^+$. We have only 4 cases:

1. If $i < \tau$ and $j \geq f_i^+ = f_i$, then $-j \leq z_{i+1} < z_i < ... < z_1$; this implies that $\gamma_{i,-j} = \gamma_{i-1,-j} = \cdots = \gamma_{0,-j} = 0$, hence $\mu_{i,j}^{\tau,f_\tau} = \lambda_{i,j} = 0$.
2. If $i = \tau$, and $j \geq f_\tau^+ = f_\tau + 1$ then $\mu_{\tau,j}^{\tau,f_\tau}$ lies in the row with index $j - \tau \geq f_\tau + 1 - \tau > c - \tau$, thus $\mu_{\tau,j}^{\tau,f_\tau} = -\lambda_{\tau-1,j}$, and on the other hand we have $-j < -f_\tau < z_\tau < z_{\tau-1} < \cdots < z_1$; thus $\gamma_{\tau-1,-j} = \gamma_{\tau-2,-j} = \cdots = \gamma_{0,-j} = 0$ and $\mu_{\tau,j}^{\tau,f_\tau} = 0$.
3. If $i = \tau + 1$, and $j \geq f_{\tau+1}^+ = f_{\tau+1}$ then $\mu_{\tau+1,j}^{\tau,f_\tau}$ lies in the row with index $j - (\tau + 1) \geq f_{\tau+1} - \tau - 1 = f_\tau + 2 - \tau - 1 = f_\tau - \tau + 1 > c - \tau$, thus $\mu_{\tau+1,j}^{\tau,f_\tau} = \lambda_{\tau-1,j}$, and on the other hand $-j \leq -f_{\tau+1} < z_\tau < z_{\tau-1} < \cdots < z_1$; thus $\gamma_{\tau-1,-j} = \gamma_{\tau-2,-j} = \cdots = \gamma_{0,-j} = 0$ and $\mu_{\tau+1,j}^{\tau,f_\tau} = 0$.
4. If $i > \tau + 1$ and $j \geq f_i^+ = f_i$, then $-j \leq -f_i = z_{i-1} < z_{i-2} < ... < z_1$; and this implies that $\gamma_{i-2,-j} = \gamma_{i-3,-j} = \cdots = \gamma_{0,-j} = 0$, hence $\mu_{i,j}^{\tau,f_\tau} = \lambda_{i-2,j} = 0$.

The previous discussion shows that the coefficient $\mu_{i,j}^{\tau,f_\tau}$ of $\beta_{i,j}$ for $j \geq f_i^+$ in the functional $\langle -, \mathbf{E} \rangle_{c,\tau}$ is zero, so $\langle \mathbf{F}, \mathbf{E} \rangle_{c,\tau} = 0$ for all resolutions $\mathbf{F}$ such that $\beta_{i,j} = 0$ for all i, j with $j < f_i^+$. In other words if $f' \geq f^+$ and $\mathbf{F}'$ is pure resolution of type f', then $\langle \mathbf{F}', \mathbf{E} \rangle_{c,\tau} = 0$.

On the other hand we have

$$\langle \mathbf{F}, \mathbf{E} \rangle_{c,\tau} = \langle \mathbf{F}, \mathbf{E} \rangle$$

for a resolution $\mathbf{F}$ of a module M such that $\beta_{i,j} = 0$ for all i, j with $j > f_i^-$. For these modules we check the vanishing criterion of Theorem 2.22.

We have reg $M \leq f_n - n$ and reg $E^0 = -f_n + n - 1$, We also note that if $j > 0$ then reg $F_{j-1} \leq f_{j-1}$ and reg $H^j(\mathbf{E}) = z_{j-1} - 1$. This means that $\langle \mathbf{F}', \mathbf{E} \rangle_{c,\tau} = 0$

for any pure resolution $\mathbf{F}'$ of type $f' \leq f^-$. Thus $\langle -, \mathbf{E}\rangle_{c,\tau}$ vanishes on $\mathbf{facet}(f, \tau)$. Finally we observe that $\langle \mathbf{F}, \mathbf{E}\rangle_{c,\tau} = \beta_{\tau,f_\tau}\mu^{\tau,f_\tau}_{\tau,f_\tau} = \beta_{\tau,f_\tau}\lambda_{\tau,f_\tau} = \beta_{\tau,f_\tau}\gamma_{\tau,-f_\tau} \neq 0$ for the pure resolution $\mathbf{F}$ of type f. Hence the truncated functional $\langle -, \mathbf{E}\rangle_{c,\tau} = 0$ is the supporting equation of this facet. □

Now we can state the conjectures as a theorems:

Theorem 3.18. *For every degree sequence $d = (d_0, d_1, .., d_c)$, there exists a CM-module M of codimension c with pure resolution of type d.*

Proof. It follows directly from Theorem 3.5 and the Theorem 3.16. □

Theorem 3.19. *Let M be Cohen Macaulay S-module of codimension c with Betti table $\beta(M)$ in $\mathbb{D}(a, b)$; There is a unique chain $d^1 < d^2 < ... < d^r$ in $[a, b]_{deg}$ such that $\beta(M)$ is uniquely a linear combination $c_1\pi(d^1) + ... + c_r\pi(d^r)$ where the c_i's are positive rationals.*

Proof. It follows from Theorem 3.17 that $B(a, b) \subseteq |\Sigma(a, b)|$, and Theorem 2.9 implies the desired statement. □

Now we describe the algorithm used to decompose any Betti table of CM-module. But first we need the following definition.

Definition 3.20. Let β be a Betti table. The lower bound of β is the sequence $\underline{\mathbf{d}}(\beta) = (d_0, d_1, ..., d_c)$ where $d_i = \min\{j|\beta_{i,j} \neq 0\}$. We define $\mathbf{q}(\beta) > 0$ to be the maximal number such that $\beta' := \beta - \mathbf{q}(\beta)\pi(\underline{\mathbf{d}}(\beta))$ is nonnegative.

Remark 3.21. The algorithm presented in the first chapter for constructing free resolutions ensures that $\underline{\mathbf{d}}(\beta)$ is indeed strictly increasing.

The algorithm proposed by Boij and Söderberg [9, Conj.2.10] is the following:

Input: Betti table $\beta(M)$ of a CM-module of codimension c.
Output: Chain D of degree sequences, and set L positive real numbers $(q_{\mathbf{d}})_{\mathbf{d}\in D}$, such that $\beta(M) = \sum_{\mathbf{d}\in D} q_{\mathbf{d}}\pi(\mathbf{d})$.

$D := [\,]$;
$L := [\,]$;
$\beta := \beta(M)$;
while $\beta \neq 0$ **do**
  $d := \underline{\mathbf{d}}(\beta)$;
  $q := \mathbf{q}(\beta)$;
  $\mathrm{Add}(D, d)$;
  $\mathrm{Add}\ (L, q)$;
  $\beta := \beta - q\pi(\mathbf{d})$;
end
return D, L;

Example 3.3. The Betti table in Example 1.4 can be decomposed as

$$\beta(S/\langle x^3, xy^2, y^3\rangle) = \frac{1}{3}\cdot\pi(\ (0,3,4)\ ) + \frac{1}{3}\cdot\pi(\ (0,3,5)\ ).$$

Gap Code
```
gap>  LoadPackage("GradedModules");;
gap>  LoadPackage("BoijSoederberg");;
gap>  S:= GradedRing( HomalgFieldOfRationalsInSingular() * "x,y" );;
gap>  M:= FactorObject( LeftSubmodule( "x^3,x*y^2,y^3", S ) );;
gap>  IsCohenMacaulay( M );
true
```

Since M is a CM-module, its Betti table is decomposable as linear combination of pure Betti tables with degree sequences of the same length.

Gap Code
```
gap>  B:= BettiTable( Resolution( M ) );
<A Betti diagram of <A right acyclic complex containing 2 morphisms of
graded left modules at degrees [ 0 .. 2 ]>>
gap>  Display(B);
 total:   1 3 2
```

```
--------------
     0:  1 . .
     1:  . . .
     2:  . 3 1
     3:  . . 1
--------------
degree:  0 1 2
gap>  DecomposeBettiTable(B);
[ [ 1/3, [ 0, 3, 4 ] ], [ 1/3, [ 0, 3, 5 ] ] ]
gap>  1/3*VirtualPureBettiTable( [ 0, 3, 4 ] )+
>1/3*VirtualPureBettiTable( [ 0, 3, 5 ] );
<Virtual Betti table >
gap>  Display( last );
 total:  1 3 2
--------------
     0:  1 . .
     1:  . . .
     2:  . 3 1
     3:  . . 1
--------------
degree:  0 1 2
```

3.4 Betti tables of graded modules in general

To extend the Boij-Söderberg theorem to graded modules in general we need to do some simple modifications. Let $\mathbb{Z}^{\leq n+1}_{deg}$ be the set of increasing sequences of integers $d = (d_0, \dots, d_s)$ with $s \leq n$ and consider a partial order on this set by letting

$$(a_0, \dots, a_t) \leq (b_0, \dots, b_s)$$

if $t \geq s$ and $a_i \leq b_i$ for $i = 0, \dots, s$. Using the same methods and results as in the previous sections we can prove the following theorem:

Theorem 3.22 ([10, Theo.2]). *Let $S = k[x_1, \dots, x_n]$ and $\beta(M)$ be the Betti table of a graded S-module M. Then there exists positive rational numbers c_i and a* *chain of sequences* $d^1 < d^2 < \cdots < d^p$ *in* $\mathbb{Z}_{deg}^{\leq n+1}$ *such that*

$$\beta(M) = c_1\pi(d^1) + \cdots + c_p\pi(d^p).$$

The algorithm used to decompose Betti tables in this case is exactly as the one given for CM-modules.

Example 3.4. Let $S = k[x, y, z]$ and $M = S/\langle x^3, x^2y^2, z^3\rangle$. Then

Gap Code

```
gap>  LoadPackage("GradedModules");
gap>  LoadPackage("BoijSoederberg");
gap>   S:= GradedRing(HomalgFieldOfRationalsInSingular( )* "x, y, z" );;
gap>   I:= LeftSubmodule( " x^3 , x^2*y^2 , z^3 ", S );
<A graded torsion-free (left) ideal given by 3 generators>
gap>  M:= FactorObject( I );
<A graded cyclic torsion left module presented by yet unknown relations
for a cyclic generator>
gap>  Display( M );
Q[x,y,z]/< z^3, x^3, x^2*y^2 >
gap>  IsCohenMacaulay( M );
false
```

Since M is not CM-module, its Betti table can be written as a positive linear combination of pure Betti tables with degree sequences of different lengths.

Gap Code

```
gap>  R:= Resolution( M );
<A right acyclic complex containing 3 morphisms of graded left modules
at degrees [ 0 .. 3 ]>
gap>  Display( R )
(graded, degree of generator: 0)
-------------------------
at homology degree: 3
Q[x,y,z]^(1 x 1)


(graded, degree of generator: 8)
-------------------------
z^3,y^2,-x
```

```
the graded map is currently represented by the above 1 x 3 matrix

(degrees of generators of target: [ 5, 6, 7 ])
------------v------------
at homology degree: 2
Q[x,y,z]^(1 x 3)

(graded, degrees of generators: [ 5, 6, 7 ])
-------------------------
0,      y^2, -x,
x^3,    -z^3,0,
x^2*y^2,0,   -z^3

the graded map is currently represented by the above 3 x 3 matrix

(degrees of generators of target: [ 3, 3, 4 ])
------------v------------
at homology degree: 1
Q[x,y,z]^(1 x 3)

(graded, degrees of generators: [ 3, 3, 4 ])
-------------------------
z^3,
x^3,
x^2*y^2

the graded map is currently represented by the above 3 x 1 matrix

(degree of generator of target: 0)
------------v------------
at homology degree: 0
Q[x,y,z]^(1 x 1)

(graded, degree of generator: 0)
-------------------------
gap>  B:= BettiTable( R );;
gap>  Display(B);
 total:  1 3 3 1
```

```
----------------
     0:  1 . . .
     1:  . . . .
     2:  . 2 . .
     3:  . 1 1 .
     4:  . . 1 .
     5:  . . 1 1
----------------
degree:  0 1 2 3
gap>  L:= DecomposeBettiTable(last);
[ [ 1/4, [ 0, 3, 5, 8 ] ], [ 1/20, [ 0, 3, 6, 8 ] ],
[ 1/70, [ 0, 3, 7, 8 ] ], [ 1/7, [ 0, 4, 7 ] ] ]
gap>  s:=VirtualPureBettiTable([0]);
< Virtual Betti table >
gap>  Display(last);
     0:  1
----------
degree:  0
gap>  s:=0*s;
< Virtual Betti table >
gap>  Display(last);
     0:  .
----------
degree:  0
gap> for i in L do
> s:= s+i[1]*VirtualPureBettiTable(i[2]);
> od;
gap>  Display( s );
 total:  1 3 3 1
----------------
     0:  1 . . .
     1:  . . . .
     2:  . 2 . .
     3:  . 1 1 .
     4:  . . 1 .
     5:  . . 1 1
----------------
degree:  0 1 2 3
```

The module M can be defined in Gap by another Gode as follows:

Gap Code
```
gap>  m := HomalgMatrix( "x^3,x^2*y^2,z^3", 3, 1, S );
<A 3 x 1 matrix over a graded ring>
gap>  M := LeftPresentationWithDegrees( m );
<A graded cyclic left module presented by 3 relations for a cyclic
generator>
gap>   Display( M );
Q[x,y,z]/< x^3, x^2*y^2, z^3 >
(graded, degree of generator: 0)
```

Example 3.5. Let C be the matrix

$$\begin{bmatrix} 20 & 24 & \cdot \\ 3 & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & 6 & \cdot \\ \cdot & \cdot & 4 \\ \cdot & \cdot & 3 \end{bmatrix}$$

Let us find if a multiple of C can be a Betti table of some CM-module M on $S = k[x_1, x_2]$.

Gap Code
```
gap>  LoadPackage("BoijSoederberg");
gap>  C:= [ [ 20, 24, 0 ], [ 3, 0, 0 ], [ 0, 0, 0 ],
>[ 0, 6, 0 ], [ 0, 0, 4 ], [ 0, 0, 3 ] ];;
gap>  B:= ConvertToVirtualBettiTable(
> HomalgBettiTable( C, [ 0..5 ],[ 0..2 ], rec( String:= "" ) ),
> true, false);
< Virtual Betti table >
gap>  Display( B );
 total:   23 30  7
-----------------
     0:   20 24  .
     1:    3  .  .
     2:    .  .  .
     3:    .  6  .
```

```
     4:   .  .  4
     5:   .  .  3
-----------------
degree:   0  1  2
gap>  DecomposeBettiTable( B );
[ [ 4, [ 0, 1, 6 ] ], [ 3, [ 1, 4, 7 ] ] ]
```

Since B can be written as a positive linear combination of pure Betti tables with degree sequences of the same length, it results that a multiple of B is Betti table for some CM-module T. Indeed, there are infinite number of multiples of B that represents Betti tables of CM-modules. Here we find one of them:

Gap Code

```
gap>  B1:=VirtualPureBettiTable( [ 0, 1, 6 ] );;Display( B1 );
 total:  5 6 1
--------------
     0:  5 6 .
     1:  . . .
     2:  . . .
     3:  . . .
     4:  . . 1
--------------
degree:  0 1 2
gap>  BettiNumbersOfDegreeSequence( [ 0, 1, 6 ] );
[ 5, 6, 1 ]
```

The command BettiNumbersOfDegreeSequence(d) returns the Betti numbers of the module constructed in Theorem 3.5. Thus, B1 is the Betti table of some CM-module, i.e, B1$= \beta(M)$ for some CM-module M.

Gap Code

```
gap>  B2:= VirtualPureBettiTable( [ 1, 4, 7 ] );;Display( B2 );
 total:  1 2 1
--------------
     1:  1 . .
     2:  . . .
     3:  . 2 .
     4:  . . .
```

```
     5:  . . 1
--------------
degree:  0 1 2
gap> BettiNumbersOfDegreeSequence( [ 1, 4, 7 ] );
[ 10, 20, 10 ]
```

This means `10·B2` is the Betti table of some CM-module, i.e., `10·B2`$= \beta(N)$ for some CM-module N. Now `B`$= 4\beta(M) + \frac{3}{10}\beta(N) = \frac{1}{10}(40\beta(M) + 3\beta(N)) = \frac{1}{10}\beta(M^{40} \oplus N^3)$.

Chapter 4

Boij-Söderberg theory of vector bundles

4.1 The cone of cohomology tables

For a coherent sheaf $\mathcal{F}$ on the projective space $\mathbb{P}^m$. The cohomology table $\gamma(\mathcal{F})$ of $\mathcal{F}$ is the indexed set $(\gamma_{i,d})_{\substack{i=0,\dots,m,\\ d\in\mathbb{Z}}}$, which lies in the vector space $\mathbb{T} = \prod_{d\in\mathbb{Z}} \mathbb{Q}^{m+1}$, where

$$\gamma_{i,d} = \dim_k H^i\mathcal{F}(d).$$

Let $\mathbb{Z}^m_{root}$ be the set of strictly decreasing integer sequences $\mathbf{a} = (a_1, a_2, \dots, a_m)$. Such sequences are called root sequences. This set is a partially ordered set with $\mathbf{a} \leq \mathbf{b}$ iff $a_i \leq b_i$ for $i = 1, \dots, m$. The interval $[\mathbf{a}, \mathbf{b}]_{root}$ is the set of all root sequences $\mathbf{z}$ such that $\mathbf{a} \leq \mathbf{z} \leq \mathbf{b}$.

Definition 4.1. We define $\mathbb{T}(\mathbf{a}, \mathbf{b})$ to be the subspace of $\mathbb{T}$ consisting of all tables where,

1. $\gamma_{i,d} = 0$ for $i = 1, \dots, m$ and $d \geq b_i$.
2. $\gamma_{i,d} = 0$ for $i = 0, \dots, m-1$ and $d \leq a_{i+1}$.
3. For all d the alternating sum $\gamma_{0,d} - \gamma_{1,d} + \cdots + (-1)^m\gamma_{m,d}$ is a polynomial in d of degree $\leq m$.

and $C(\mathbf{a},\mathbf{b})$ to be the set of all positive rational multiples of cohomology tables of vector bundes whose tables are in $\mathbb{T}(\mathbf{a},\mathbf{b})$.

The vector space $\mathbb{T}(\mathbf{a},\mathbf{b})$ is of finite dimension, since the values of a polynomial of degree $\leq m$ is determined by any of $m+1$ successive values.

Theorem 4.2. *If $\mathcal{E}$ is a vector bundle on the projective space $\mathbb{P}^m$, then there exist $\mathbf{a}(\mathcal{E})$ and $\mathbf{b}(\mathcal{E})$ in $\mathbb{Z}^m_{root}$ such that $\gamma(\mathcal{E}) \in \mathbb{T}(\mathbf{a}(\mathcal{E}),\mathbf{b}(\mathcal{E}))$.*

Proof. By Serre vanishing theorem and Serre duality only the 0-th and m-th row of the cohomology table of $\mathcal{E}$ can have infinitely many non-zeros entries. Thus it make sense to define the following two sequences:

$$r(\mathcal{E}) = (r_1(\mathcal{E}) \geq r_2(\mathcal{E}) \geq \cdots \geq r_m(\mathcal{E})),$$

$$R(\mathcal{E}) = (R_1(\mathcal{E}) \geq R_2(\mathcal{E}) \geq \cdots \geq R_m(\mathcal{E})),$$

such that $r_i(\mathcal{E}) = \sup\{d \mid M_d \geq i\}$ and $R_i(\mathcal{E}) = \inf\{d \mid m_d < i\}$, where M_d and m_d for $d \in \mathbb{Z}$ are defined as in Theorem 3.11. We have $r_i(\mathcal{E}) \geq r_{i+1}(\mathcal{E})$ and $R_i(\mathcal{E}) \geq R_{i+1}(\mathcal{E})$ for all $i = 1,\cdots,m$ because $d \mapsto M_d$ and $d \mapsto m_d$ are weakly decreasing functions of $d \in \mathbb{Z}$. We have also $R_i(\mathcal{E}) \leq r_i(\mathcal{E})+1$ for $i = 1,\ldots,m$. To prove this, let us assume that $r_i(\mathcal{E}) = d'$. Then by definition $M_{d'+1} < i$, but we have $m_i \leq M_i$ for all $i = 1,\ldots,m$, thus $m_{d'+1} < i$, which implies that $R_i(\mathcal{E}) \leq d'+1$. Hence $R_i(\mathcal{E}) \leq r_i(\mathcal{E})+1$.

Now we define the sequence $\mathbf{a}(\mathcal{E}) = (a_1,\ldots,a_m)$ and $\mathbf{b}(\mathcal{E}) = (b_1,\ldots,b_m)$ by the formulas $a_i = R_i(\mathcal{E}) - i$ and $b_i = r_i(\mathcal{E}) - i + 1$.

It is clear that $\mathbf{a}(\mathcal{E})$ and $\mathbf{b}(\mathcal{E})$ are strictly decreasing sequences which implies that both of them are root sequences.

We should now show that $\gamma_{i,d} = 0$ for $i = 1,\ldots,m$ and $d \geq b_i$. We have $d \geq b_i = r_i(\mathcal{E}) - i + 1$, thus $d+i \geq r_i(\mathcal{E})+1$, thus $M_{d+i} < i$, and on the other hand if $\gamma_{i,d} \neq 0$, then $\gamma_{i,(d+i)-i} \neq 0$ which implies that $M_{d+i} \geq i$, a contradiction.

We also need to show that $\gamma_{i,d} = 0$ for $i = 0,\ldots,m-1$ and $d \leq a_{i+1}$. We have $d \leq a_{i+1} = R_{i+1}(\mathcal{E}) - i - 1 \leq R_i(\mathcal{E}) - i - 1$, thus $d+i \leq R_i(\mathcal{E}) - 1$, hence $m_{d+i} \geq i$, and on the other hand if $\gamma_{i,d} \neq 0$ then $\gamma_{i,(d+i)-i} \neq 0$, which implies that $m_{d+i} < i$, a contradiction. $\square$

Example 4.1. The cohomology table of the Horrocks–Mumford bundle $\mathcal{F}_{HM}$ over $\mathbb{P}^4$ is

100	35	4	$\lfloor .$	.	.	.	.	.	.	.	.	.	.	.	4
.	$.\rceil$	2	10	10	5	$\lfloor .$	.	.	.	.	.	.	.	.	3
.	.	.	.	.	.	$.\rceil$	2	$\lfloor .$	.	.	.	.	.	.	2
.	.	.	.	.	.	.	.	$.\rceil$	5	10	10	2	$\lfloor .$	.	1
.	.	.	.	.	.	.	.	.	.	.	$.\rceil$	4	35	100	0
-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$d \setminus i$

In this case the sequences defined in the last theorem are

$$r(\mathcal{F}_{HM}) = (4, 0, -1, -3), R(\mathcal{F}_{HM}) = (3, 1, 0, -4)$$

and

$$\mathbf{a}(\mathcal{E}) = (2, -1, -3, -8), \mathbf{b}(\mathcal{E}) = (4, -1, -3, -6)$$

Example 4.2. If $\mathcal{E}$ is a supernatural vector bundle on $\mathbb{P}^m$ then

$$\mathbf{a}(\mathcal{E}) = \mathbf{b}(\mathcal{E}) = \mathbf{z}(\mathcal{E})$$

where $\mathbf{z}(\mathcal{E})$ is the root sequence of $\mathcal{E}$.

Remark 4.3. . Let $\mathcal{F}$ be a coherent sheaf on $\mathbb{P}^m$, then Serre vanishing theorem and Theorem 3.11 imply the existence of $\mathbf{b}(\mathcal{F})$ as $r(\mathcal{E})$ has always entries $< \infty$, however, $R(\mathcal{F})$ may contain some entries $= \infty$. We will call $\mathbf{b}(\mathcal{F})$ the root sequence of $\mathcal{F}$ and denote it by $\mathbf{z}(\mathcal{F})$.

Theorem 4.4 (1, Chapter.0)**.** *$C(\mathbf{a}, \mathbf{b})$ is a a convex cone.*

Proof. The sum of cohomology tables of vector bundles is the cohomology table of the direct sum of these vector bundles. □

For a root sequence $\mathbf{z} = (z_1, z_2, \ldots, z_m)$ we associate a table γ^z given by

$$\gamma^{\mathbf{z}}_{i,d} = \begin{cases} \frac{1}{m!} \prod_{i=1}^{m} |d - z_i| & \text{if } z_j > d > z_{j+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.5 (8, Lemma.4.5)**.** *If $Z : \mathbf{z}^1 > \mathbf{z}^2 > \cdots > \mathbf{z}^r$ is a chain of roots sequences, then $\gamma^{\mathbf{z}^1}, \gamma^{\mathbf{z}^2}, \ldots, \gamma^{\mathbf{z}^r}$ are linearly independent.*

Hence these supernatural tables span a simplicial cone $\sigma(Z)$ in $\mathbb{T}$. The simplicial cones defined by a chain of supernatural cohomology tables intersect only in common faces. This follows by the same argument as for pure resolutions of CM-modules(Proposition 2.8). Hence the set of simplicial cones $\sigma(Z)$ where Z ranges over the chains in $[\mathbf{a},\mathbf{b}]_{root}$ form a simplicial fan in $\mathbb{T}(\mathbf{a},\mathbf{b})$ which we denote by $\Gamma(\mathbf{a},\mathbf{b})$. Since all maximal chains in $[\mathbf{a},\mathbf{b}]_{root}$ have the same length, the fan is equidimensional. We define

$$\gamma : \begin{cases} [\mathbf{a},\mathbf{b}]_{root} \to \mathbb{T}(\mathbf{a},\mathbf{b}) \\ \qquad \mathbf{z} \longmapsto \gamma^{\mathbf{z}} \end{cases}$$

The map γ defines a geometric realisation of $[\mathbf{a},\mathbf{b}]_{root}$, and this turns out to be the fan $\Gamma(\mathbf{a},\mathbf{b})$.

Here is the analog of the Boij-Södeberg conjectures:

Theorem 4.6. *With the same notation as above, we have*

1. $|\Gamma(\mathbf{a},\mathbf{b})| \subseteq C(\mathbf{a},\mathbf{b})$.
2. $C(\mathbf{a},\mathbf{b}) \subseteq |\Gamma(\mathbf{a},\mathbf{b})|$.

The first statement of the above theorem is a direct consequence of the existence of a vector bundle with supernatural cohomology of a specific root sequence treated in Theorem 3.9. In order to proof the second statement, we describe the exterior facets of the fan and proceed exactly as in the previous sections.

4.2 Facets equations

Proposition 4.7. *Let $Z : \mathbf{b} = \mathbf{z}^1 > \mathbf{z}^2 > \cdots > \mathbf{z}^q = a$ be a maximal chain in $[a,b]_{root}$ and $\mathbf{z}^+ > \mathbf{z} > \mathbf{z}^-$ be a subchain in Z. Then $\sigma(Z\backslash\{\mathbf{z}\})$ is an exterior facet of $\Gamma(\mathbf{a},\mathbf{b})$ if either of the following holds*

1. *$\mathbf{z}$ is either $\mathbf{a}$ or $\mathbf{b}$. The facet equation in this case is $\gamma_{i,d} = 0$ for appropriate i and d.*

2. *The root sequences* $\mathbf{z}^-$ *and* $\mathbf{z}^+$ *differ in exactly one position. So for some* r *we have*

$$\mathbf{z}^+ = (\dots, -(r-1), \dots) > \mathbf{z} = (\dots, -r, \dots) > \mathbf{z}^- = (\dots, -(r+1), \dots).$$

3. *The root sequences* $\mathbf{z}^-$ *and* $\mathbf{z}^+$ *differ in two consecutive positions. So for some* r *we have*

$$\begin{aligned} \mathbf{z}^+ = (\dots, -(r-1), -r, \dots) \; &> \; \mathbf{z} = (\dots, -(r-1), -(r+1), \dots) \\ &> \; \mathbf{z}^- = (\dots, -r, -(r+1), \dots). \end{aligned}$$

Letting i *be the position of* $-(r-1)$ *in* $\mathbf{z}$*, the facet equation is* $\gamma_{i,-r} = 0$.

Proof. That we have only these three cases follows immediately from Proposition 2.10 when we multiply the entries of the sequences in $[\mathbf{a}, \mathbf{b}]_{root}$ by -1.

Now we find the exterior facet equation for cases 1. and 3.

1. If $\mathbf{z} = \mathbf{b}$ and i is the position where $\mathbf{z}$ differs from $\mathbf{z}^2$ then $z_i = z_i^2 + 1$ and $z_{i+1} = z_{i+1}^2 < z_i^2 = z_i - 1 < z_i$, hence $\gamma^{\mathbf{z}}_{i,z_i-1} \neq 0$. Since $z_i - 1 = z_i^2 \geq z_i^3 \geq \dots \geq z_i^q$, it follows that $\gamma^{\mathbf{z}^j}_{i,z_i-1} = 0$ for $j \geq 2$. Thus the exterior facet equation is $\gamma_{i,z_i-1} = 0$.

2. If $\mathbf{z} = \mathbf{a}$ and i is the position where $\mathbf{z}$ differs from $\mathbf{z}^{q-1}$ then with the same discussion as above we find that the facet equation is $\gamma_{i,z_i+1} = 0$.

3. Let $\sigma(Z \backslash \{\mathbf{z}\})$ be an exterior facet of type 3, and $\mathbf{z} = \mathbf{z}^j$ for some $1 \geq j \geq q$. We have $z_i^j > -r > z_{i+1}^j$ thus $\gamma^{\mathbf{z}^j}_{i,-r} \neq 0$. On the other hand we have $z^1_{i+1} \geq z^2_{i+1} \geq \dots \geq z^{j-1}_{i+1} \geq -r \geq z_i^{j+1} \geq z_i^{j+2} \geq \dots \geq z_i^q$ which implies that $\gamma^{\mathbf{z}^1}_{i,-r} = \dots = \gamma^{\mathbf{z}^{j-1}}_{i,-r} = \gamma^{\mathbf{z}^{j+1}}_{i,-r} = \gamma^{\mathbf{z}^{j+2}}_{i,-r} = \dots = \gamma^{\mathbf{z}^q}_{i,-r} = 0$. Thus the facet equation is $\gamma_{i,-r} = 0$. □

Remark 4.8. Let $\mathbf{z}^+ > \mathbf{z} > \mathbf{z}^-$ be three root sequences as in type 2. in the above proposition. Then given $\mathbf{z}$ and the position τ where they differ, we can recover $\mathbf{z}^-$ and $\mathbf{z}^-$. We denote the exterior facets abtained by this way by $\mathbf{facet}(\mathbf{z}, \tau)$. We can find the equation for the supporting hyperplane of the facet using an algorithm that is completely analogous to that in Example 2.3.

Example 4.3. Let $\mathbf{z}^+ = (3, 1, -4), \mathbf{z} = (3, 0, -4), \mathbf{z}^- = (3, -1, -4)$, then the facet is denoted by $\mathbf{facet}(\mathbf{z}, 2)$. Let f be the degree sequence that is negative of the union of the root sequences $\mathbf{z}^+, \mathbf{z}, \mathbf{z}^-$, i.e., $f = (-3, -1, 0, 1, 4)$.

Gap Code

```
gap>  LoadPackage("BoijSoederberg");;
gap>  z:= RootSequence([ 3, 0, -4 ] );
< Root sequence of virtual cohomology table >
gap>  U:=  UpperEquation( z, 2, -4, 4 );
< Upper equation Of a facet >
gap>  Display( U );
  -4   -3   -2   -1   0    1    2    3    4
 ---------------------------------------------
3|.    .    .    .    .    .    .    .    .
2|2    .    .    .    .    .    .    .    .
1|-2   .    .    35   -70  42   .    .    .
0|2    .    .    -35  70   -42  .    5    .
gap>  f:= ConstructDegreeSequenceFromRootSequence( z, 2 );
< Degree seqeunce of virtual Betti table >
gap>  Display( f );
[ -3, -1, 0, 1, 4 ]
gap>  B:= VirtualPureBettiTable( f );
< Virtual Betti table >
gap>  Display( B );
 total:   5 42 70 35  2
-----------------------
    -3:   5  .  .  .  .
    -2:   . 42 70 35  .
    -1:   .  .  .  .  .
     0:   .  .  .  .  2
-----------------------
degree:   0  1  2  3  4
```

The following theorem explains why the Betti table of f and upper equation of the $\mathbf{facet}(\mathbf{z}, 2)$ have some common values.

Theorem 4.9 (1, Theo.8.4)**.** *Suppose that the three root sequences* $\mathbf{z}^+ > \mathbf{z} > \mathbf{z}^-$ *differ only in the* τ*-th spot, where we have* $z_\tau^+ - 1 = z_\tau = z_\tau^- + 1$*. Let* f *be the degree sequence that is the negative of the union of the elements in these three root sequences, then* $f_\tau = -z_\tau$*, the middle value. Let* $c = f_{\tau-1} = f_\tau - 1$*, the smaller value. Let* $\mathbf{F}$ *be a pure resolution corresponding to the sequence* f*. Then the functional* $\langle \mathbf{F}, - \rangle_{c,\tau}$ *is positive on the supernatural bundle* $\mathcal{E}$ *with root sequence* $\mathbf{z}$*, and vanishes on all supernatural bundles* $\mathcal{E}$ *with root sequences* $\leq \mathbf{z}^-$ *or* $\geq \mathbf{z}^+$*.*

Proof. Let $\mathbf{z} = (z_1, z_2, \ldots, z_\tau, \ldots, z_m)$ then

$$f = (-z_1, -z_2, \ldots, -z_{\tau-1}, -(z_\tau + 1), -z_\tau, -(z_\tau - 1), -z_{\tau+1} \ldots, -z_m)$$

So $f_0 = -z_1, \ldots, f_{\tau-2} = -(z_{\tau-1}), f_{\tau-1} = -(z_\tau + 1), f_\tau = -z_\tau, f_{\tau+1} = -(z_\tau - 1), f_{\tau+2} = -z_{\tau+1}, \ldots, f_{m+1} = -z_m$.

If $\mathcal{E}$ is any vector bundle over $\mathbb{P}^m$, then the truncated functional is

$$\langle \mathbf{F}, \mathcal{E} \rangle_{c,\tau} = \sum_{\substack{\{i,j,k | j \leq i \wedge \\ (j<\tau \vee j \leq i-2)\}}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k} + \sum_{\substack{\{i,j,k,\epsilon | 0 \leq \epsilon \leq 1, \\ j=\tau, i=j+\epsilon \\ k \leq c+\epsilon\}}} (-1)^{i-j} \beta_{i,k} \gamma_{j,-k}.$$

Since $\beta_{i,k}$ is always zero unless $k = f_i$, we can rewrite the truncated functional as follows:

$$\langle \mathbf{F}, \mathcal{E} \rangle_{c,\tau} = \sum_{j=i<\tau} (-1)^{i-j} \beta_{i,f_i} \gamma_{j,-f_i} + \sum_{j=i-1<\tau} (-1)^{i-j} \beta_{i,f_i} \gamma_{j,-f_i} + \sum_{j \leq i-2} (-1)^{i-j} \beta_{i,f_i} \gamma_{j,-f_i}.$$

So the coefficient $\eta_{j,d}$ of $\gamma_{j,d}$ in the functional $\langle \mathbf{F}, - \rangle_{c,\tau}$ is given by

$$\eta_{j,d} = \begin{cases} 0 & \text{if } d \neq -f_i \text{ for all } i\text{'s} \\ 0 & \text{if } j > i \\ 0 & \text{if } j = i \geq \tau \\ \beta_{i,f_i} & \text{if } j = i < \tau \\ 0 & \text{if } j = i - 1 \geq \tau \\ -\beta_{i,f_i} & \text{if } j = i - 1 < \tau \\ (-1)^{i-j} \beta_{i,f_i} & \text{if } j \leq i - 2 \end{cases}$$

Let $\mathbf{z}' \geq \mathbf{z}^+$, and let $\gamma^{\mathbf{z}'}$ be the cohomology table corresponding to $\mathbf{z}'$. Then $\gamma^{\mathbf{z}'}_{j,d} \neq 0$ only if $z'_j > d > z'_{j+1}$. So to find $\langle \mathbf{F}, \gamma^{z'} \rangle_{c,\tau}$ we need to find the coefficients $\eta_{j,d}$ for $z'_j > d > z'_{j+1}$.

We have several cases

1. If $\tau - 2 \geq j \geq 0$, and $z'_j > d > z'_{j+1}$, then $d > z_{j+1} = -f_j$, so if $d = -f_i$ for some i, then $i < j$. Thus $\eta_{j,d} = 0$.

2. If $j = \tau - 1$, and $z'_{\tau-1} > d > z'_{\tau}$, then $d > z^+_{\tau} = z_{\tau} + 1 = -f_{\tau-1}$, so if $d = -f_i$ for some i, then $i < \tau - 1 = j$. Thus $\eta_{j,d} = 0$.

3. If $j \geq \tau$, and $z'_j > d > z'_{j+1}$, then $d > z_{j+1} = -f_{j+2}$, so if $d = -f_i$ for some i, then $i < j + 2$. For all possible values of i we get $\eta_{j,d} = 0$.

This implies that the functional vanishes on all supernatural bundles with degree roots $\geq \mathbf{z}^+$.

Now let $\gamma^{\mathbf{z}}$ be the cohomology table corresponding to the root sequence $\mathbf{z}$. With the same discussion as above we find that $\eta_{j,d} = 0$ for all j's and all $z_j > d > z_{j+1}$ except for $j = \tau - 1$ and $i = \tau - 1$, where $\eta_{\tau-1,\tau-1} = \beta_{\tau-1,f_{\tau-1}}$. Thus the functional is positive on γ^z.

If $\mathcal{E}$ is a supernatural bunde with root sequence $\leq \mathbf{z}^-$ then $\langle \mathbf{F}, \mathcal{E} \rangle = \langle \mathbf{F}, \mathcal{E} \rangle_{c,\tau}$. To prove, that this is zero, we check the conditions of Theorem 2.22: The module $M = \operatorname{coker}(F_1 \to F_0)$ has regularity reg $M = f_{m+1} - (m + 1)$. The module E^0 has regularity reg $E^0 = z_m(\mathcal{E}) + m \leq z_m + m = -f_{m+1} + m$. Thus

$$\text{reg } M + \text{reg } E^0 = -1 < 0.$$

Moreover for $j > 0$, we have reg $H^j(E) < z_j(\mathcal{E}) - 1 \leq z_j^- - 1 \leq -f_{j-1} - 1$ and reg $F_{j-1} = f_{j-1}$, hence reg $F_{j-1} +$ reg $H^j(E) \leq -1 < 0$. □

4.3 The proof and algorithm of the decomposition of a cohomology table

We can now proof the second statement of Theorem 4.6.

Proof. (The 2. part of Theorem 4.6)
The equation of any exterior facet of the realisation of the fan is given by the truncated functional $\langle \mathbf{F}, - \rangle_{c,\tau}$ for suitable pure free resolution $\mathbf{F}$ and integers c, τ. By Theorem 2.22, this functional is non-negative on the monad $E = E_a$ obtained from the free resolution of $\bigoplus_{d \geq a} H^0\mathcal{E}^*(d)$ with $a \gg 0$ for any vector bundle $\mathcal{E}$. □

Now we describe the algorithm used to decompose any cohomology table of any vector bundle on $\mathbb{P}^m$. But first we need the following definitions.

Definition 4.10 (8, Def.5.4)**.** Given a root sequence $\mathbf{z} : z_1 > z_2 > \cdots > z_m$. The position $(i, d) = (i, z_i - 1)$ is called a *corner position* if $z_{i+1} < z_i - 1$.

Example 4.4. Let $z = (6, 3, 2, -1)$, then the set of corner positions is $\{(1, 5), (3, 1), (4, -2)\}$. We notice that the quadrant determined by each corner position only consists of zeros except in the corner.

$*$	$\lfloor *$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	4
$\cdot$	$\cdot$	$\cdot$	$*$	$\lfloor *$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	3
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	2
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$*$	$\lfloor *$	$\cdot$	$\cdot$	$\cdot$	1
$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$*$	$*$	0
-3	-2	-1	0	1	2	3	4	5	6	7	8	$d \setminus i$

Definition 4.11. Let $\mathbf{z}$ be the root sequence of the vector bundle $\mathcal{E}$ on $\mathbb{P}^m$, and assume $\alpha_0, \alpha_1, \ldots, \alpha_r$ are the values of these corner positions of $\gamma(\mathcal{E})$, and let $a_0, a_1, \ldots, a_r$ be the values of the corresponding corner positions in $\gamma^{\mathbf{z}}$. Then we define $q_{\mathbf{z}} := \min\{\frac{\alpha_0}{a_0}, \frac{\alpha_1}{a_1}, \ldots, \frac{\alpha_r}{a_r}\}$.

[4, Proposition 2.1] and [8, Definition 4.2] show the following.

1. The table $\gamma(\mathcal{E}) - q_{\mathbf{z}}\gamma^{\mathbf{z}}$ has non-negative entries.

2. The root sequence $\mathbf{z}'$ of the new table is lower than the root sequence $\mathbf{z}$.

For a table γ, we define $\dim(\gamma)$ to be the largest i such that the row i is non-zero.

The algorithm of Eisenbud and Schreyer([4, Algorithm 0.4],[8, Definition 4.2]) reads as follows:

Input: Cohomology table $\gamma(\mathcal{E})$ of a vector bundle $\mathcal{E}$ on $\mathbb{P}^m_k$.
Output: Chain Z of root sequences, and set L of positive real numbers $(q_{\mathbf{z}})_{\mathbf{z}\in Z}$, such that $\gamma(\mathcal{E}) = \sum_{\mathbf{z}\in Z} q_{\mathbf{z}}\gamma^{\mathbf{z}}$.

```
Z := [ ];
L := [ ];
γ := γ(E);
while dim(γ) ≠ 0 do
    z := z(γ);
    Add(Z, z);
    Add (L, q_z);
    γ := γ − q_z γ^z;
end
return Z, L;
```

Since for any vector bundle $\mathcal{E}$ there are upper and lower bounds $\mathbf{a}(\mathcal{E}), \mathbf{b}(\mathcal{E})$ for which $\gamma(\mathcal{E}) \in \mathbb{T}(\mathbf{a}(\mathcal{E}), \mathbf{b}(\mathcal{E}))$, we are guaranteed that the algorithm terminates at latest when $z = a(\mathcal{E})$.

Let us now apply the algorithm to the cohomology table of the Horrocks–Mumford bundle $\mathcal{F}_{HM}$ over $\mathbb{P}^4$. Its cohomology table $\gamma := \gamma(\mathcal{F}_{HM})$ is

100	35	4	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	2	10	10	5	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	2	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	5	10	10	2	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	4	35	100	0
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	$d \setminus i$

The root sequence of γ is $\mathbf{z}^1 = (4, -1, -3, -6)$ and the standard cohomology table with root sequence $\mathbf{z}^1$ is the following table $\gamma^{\mathbf{z}^1}$.

78	35	⌊11	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	·	·	3	⌊2	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	⌊1	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	3	7	10	⌊9	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	·	·	22	0
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	$d \setminus i$

The corner positions are marked by ⌊.
We compute $q_{\mathbf{z}^1} = \min\{\frac{2}{9}, \frac{2}{1}, \frac{5}{2}, \frac{4}{11}\} = \frac{2}{9}$. So $Z = [\mathbf{z}^1], L = [q_{\mathbf{z}^1}]$ and $\gamma := \gamma - \frac{2}{9}\gamma^{\mathbf{z}^1}$ is

248/3	245/9	14/9	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	2	10	28/3	41/9	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	16/9	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	13/3	76/9	70/9	·	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	4	35	856/9	0
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	$d \setminus i$

The root sequence of γ is $\mathbf{z}^2 = (3, -1, -3, -6)$ and the standard cohomology table with root sequence $\mathbf{z}^2$ is $\gamma^{\mathbf{z}^2}$:

72	385/12	⌊10	·	·	·	·	·	·	·	·	·	·	·	·	4
	·	·	·	8/3	⌊7/4	·	·	·	·	·	·	·	·	·	3
	·	·	·	·	·	·	⌊5/6	·	·	·	·	·	·	·	2
	·	·	·	·	·	·	·	·	9/4	14/3	⌊5	·	·	·	1
													175/12	44	0
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	$d \setminus i$

Thus $q_{\mathbf{z}^2} = \min\{\frac{14}{9} \cdot \frac{1}{10}, \frac{41}{9} \cdot \frac{4}{7}, \frac{16}{9} \cdot \frac{6}{5}, \frac{70}{90} \cdot \frac{1}{5}\} = \frac{7}{45}$. So $Z = [\mathbf{z}^1, \mathbf{z}^2]$, $L = [q_{\mathbf{z}^1}, q_{\mathbf{z}^2}]$ and $\gamma := \gamma - \frac{7}{45}\gamma^{\mathbf{z}^2}$ is

1072/15	2401/108	·	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	2	10	1204/135	257/60	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	89/54	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	239/60	1042/135	7	·	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	4	3535/108	1324/15	0
−9	−8	−7	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	$d \setminus i$

The root sequence of γ is $\mathbf{z}^3 = (3, -1, -3, -7)$ and the standard cohomology table with root sequence $\mathbf{z}^3$ is $\gamma^{\mathbf{z}^3}$

-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$d \setminus i$
48	⌊385/24	·	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	·	45/8	16/3	⌊21/8	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	⌊25/24	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	21/8	16/3	⌊45/8	·	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	·	385/24	48	0

Hence $q_{\mathbf{z}^3} = \frac{56}{45}$. So $Z = [\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3]$, $L = [q_{\mathbf{z}^1}, q_{\mathbf{z}^2}, q_{\mathbf{z}^3}]$, and $\gamma := \gamma - \frac{56}{45}\gamma^{\mathbf{z}^3}$ is

-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$d \setminus i$
176/15	245/108	·	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	·	·	308/135	61/60	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	19/54	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	43/60	146/135	·	·	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	4	1379/108	428/15	0

The root sequence of γ is $\mathbf{z}^4 = (2, -1, -3, -7)$ and the standard cohomology table with root sequence $\mathbf{z}^4$ is $\gamma^{\mathbf{z}^4}$

-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$d \setminus i$
44	⌊175/12	·	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	·	·	14/3	⌊9/4	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	⌊5/6	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	7/4	⌊8/3	·	·	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	10	385/12	72	0

Hence $q_{\mathbf{z}^4} = \frac{7}{45}$. So $Z = [\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4]$, $L = [q_{\mathbf{z}^1}, q_{\mathbf{z}^2}, q_{\mathbf{z}^3}, q_{\mathbf{z}^4}]$, and $\gamma := \gamma - \frac{7}{45}\gamma^{\mathbf{z}^4}$ is

-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$d \setminus i$
44/9	·	·	·	·	·	·	·	·	·	·	·	·	·	·	4
·	·	2	20/9	14/9	2/3	·	·	·	·	·	·	·	·	·	3
·	·	·	·	·	·	·	2/9	·	·	·	·	·	·	·	2
·	·	·	·	·	·	·	·	·	4/9	2/3	·	·	·	·	1
·	·	·	·	·	·	·	·	·	·	·	·	22/9	70/9	52/3	0

The root sequence of γ is $\mathbf{z}^5 = (2, -1, -3, -8)$ and the standard cohomology table with root sequence $\mathbf{z}^5$ is $\gamma^{\mathbf{z}^5}$

$\lfloor 22$	.	.	.	.	.	.	.	.	.	.	.	.	.	.	4
.	.	9	10	7	$\lfloor 3$	.	.	.	.	.	.	.	.	.	3
.	.	.	.	.	.	.	$\lfloor 1$	.	.	.	.	.	.	.	2
.	.	.	.	.	.	.	.	.	2	$\lfloor 3$	.	.	.	.	1
.	.	.	.	.	.	.	.	.	.	.	.	11	35	78	0
-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	$d \setminus i$

Hence $q_{\mathbf{z}^5} = \frac{2}{7}$. So $Z = [\mathbf{z}^1, \mathbf{z}^2, \mathbf{z}^3, \mathbf{z}^4, \mathbf{z}^5]$, $L = [q_{\mathbf{z}^1}, q_{\mathbf{z}^2}, q_{\mathbf{z}^3}, q_{\mathbf{z}^4}, q_{\mathbf{z}^5}]$, and $\gamma := \gamma - \frac{2}{9}\gamma^{\mathbf{z}^5} = 0$. So algorithm terminates and gives back Z and L.

Here is the gap code to find the previous decomposition of the Horrocks–Mumford bundle.

Gap Code

```
gap>  SetAssertionLevel( 2 );;
gap>  LoadPackage("BoijSoederberg");;
gap>  LoadPackage( "GradedRingForHomalg" );;
gap>  R := HomalgFieldOfRationalsInDefaultCAS( ) * "x0..x4";;
gap>  S := GradedRing( R );
Q[x0,x1,x2,x3,x4]
(weights: yet unset)
gap>  A := KoszulDualRing( S, "e0..e4" );;
gap>  mat := HomalgMatrix( "[
> e1*e4, e2*e0, e3*e1, e4*e2, e0*e3,
> e2*e3, e3*e4, e4*e0, e0*e1, e1*e2
> ]", 2, 5, A );
<A 2 x 5 matrix over a graded ring>
gap>  phi := GradedMap( mat, "free", "free", "left", A );;
<A "homomorphism" of graded left modules>
gap>  IsMorphism( phi );
true
gap>  M := GuessModuleOfGlobalSectionsFromATateMap( 2, phi );
<A graded left module presented by yet unknown relations for
19 generators>
gap>  B:= BettiDiagram( TateResolution( M, -5, 5 ) );
<A Betti diagram of <An acyclic cocomplex containing 10 morphisms of
graded left modules at degrees [ -5 .. 5 ]>
```

```
gap>  Display(last);
total:  100  37  14  10   5   2   5  10  14  37 100   ?   ?   ?   ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
    4:  100  35   4   .   .   .   .   .   .   .   .   0   0   0   0
    3:    *   .   2  10  10   5   .   .   .   .   .   .   0   0   0
    2:    *   *   .   .   .   .   .   2   .   .   .   .   .   0   0
    1:    *   *   *   .   .   .   .   .   .   5  10  10   2   .   0
    0:    *   *   *   *   .   .   .   .   .   .   .   .   4  35 100
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---S
twist:   -9  -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4   5
-------------------------------------------------------------------
Euler:  100  35   2 -10 -10  -5   0   2   0  -5 -10 -10   2  35 100
gap>  L:=DecomposeCohomologyTable(B);
[ [ 2/9, [ 4, -1, -3, -6 ] ], [ 7/45, [ 3, -1, -3, -6 ] ], [ 56/45,
[ 3, -1, -3, -7 ] ], [ 7/45, [ 2, -1, -3, -7 ] ],
[ 2/9, [ 2, -1, -3, -8 ] ] ]
gap> s:= 0*VirtualCohomologyTable([ 4, 3, 2, 1 ], -5, 5);
< Virtual cohomology table >
gap>  Display( s );
total:   0  0  0  0  0  0  0  0  0  0  0  ?  ?  ?  ?
---------|--|--|--|--|--|--|--|--|--|--|--|--|--|--|
    4:   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
    3:   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
    2:   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
    1:   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
    0:   .  .  .  .  .  .  .  .  .  .  .  .  .  .  .
----------------------------------------------------
twist:  -9 -8 -7 -6 -5 -4 -3 -2 -1  0  1  2  3  4  5
gap>  for i in L do
> s:= s+ i[1]*VirtualCohomologyTable(i[2], -5, 5);
> od;
gap> s;
< Virtual cohomology table >
```

```
gap>  Display(s);
total:  100  37  14  10   5   2   5  10  14  37 100   ?   ?   ?   ?
----------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
    4:  100  35   4   .   .   .   .   .   .   .   .   .   .   .   .
    3:    .   .   2  10  10   5   .   .   .   .   .   .   .   .   .
    2:    .   .   .   .   .   .   .   2   .   .   .   .   .   .   .
    1:    .   .   .   .   .   .   .   .   .   5  10  10   2   .   .
    0:    .   .   .   .   .   .   .   .   .   .   .   .   4  35 100
-------------------------------------------------------------------
twist:   -9  -8  -7  -6  -5  -4  -3  -2  -1   0   1   2   3   4   5

gap> UpperBoundOfCohomologyTable(s);
[ 4, -1, -3, -6 ]
```

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