TiltingequivalenceAsDerivedFunctors

July 31, 2021

1 Derived equivalences as derived functors

Each quiver q defines a category FreeCategory(q) whose objects are the vertices of q and whose morphisms are the paths of q. A finite set of paths in FreeCategory(q) is called uniform if they share the same source and range.

For a field k, the k-linear closure of FreeCategory(q) is the category $k[\text{FreeCategory}(q)]$ whose objects are the objects of FreeCategory(q) and whose morphisms are formal k-linear combinations of uniform morphisms in FreeCategory(q). Obviously, $k[\text{FreeCategory}(q)]$ is a k-linear category.

Suppose ρ is a finite set of morphisms in $k[\text{FreeCategory}(q)]$. We denote by $I = \langle \rho \rangle$ the two-sided ideal of morphisms generated by ρ. The associated quotient categoy $k[\text{FreeCategory}(q)]/I$ will be called the k-algebroid of q defined by the set of relations ρ. This means, a morphism in $k[\text{FreeCategory}(q)]$ (resp. $k[\text{FreeCategory}(q)]/I$) is nothing but a uniform element in the path algebra kq (resp. kq/I).

The Gap package QPA enables us to construct path k-algebras kq and their quotients by two-sided ideals. That is, we can check equality of morphisms $k[\text{FreeCategory}(q)]$ (resp. $k[\text{FreeCategory}(q)]/I$) by checking the equality of the corresponding algebra elements in kq (resp. kq/I) which is realized by the theory of noncommuative Gröbner bases.

Let ρ be a set of relations and let $\mathbb{A} = kq/\langle \rho \rangle$. We denote by mod-$\mathbb{A}_{\text{oid}}$ the category of k-linear functors from $\mathbb{A}_{\text{oid}}$ to the category k-vec of finite dimensional vector spaces. That is 1. an object F in mod-$\mathbb{A}_{\text{oid}}$ is a functor $F : \mathbb{A}_{\text{oid}} \to k\text{-vec}$ a its data structure is a pair of lists: a list of vector spaces (reprsents the images of the objects of $\mathbb{A}_{\text{oid}}$ under F) and a list of k-linear maps (represents the images of the generating morphisms of $\mathbb{A}_{\text{oid}}$ under F); 2. a morphism $\psi : F \to G$ is a natural transformation and its data structure is a list of morphisms (represents the images of the objects of $\mathbb{A}_{\text{oid}}$ under ψ).

The category mod-$\mathbb{A}_{\text{oid}}$ is also known as the category $\text{reps}_k(q, \rho)$ of the ρ-bounded quiver k-representations of q. It is a well-known that

$$\text{mod-}\mathbb{A}_{\text{oid}} \cong \text{fdmod-}\mathbb{A}$$

where fdmod-$\mathbb{A}$ is the category of finite dimensional right $\mathbb{A}$-modules. Furthermore, if $\mathbb{A}$ is a finite dimensional k-algebra, then fdmod-$\mathbb{A}$ and mod-$\mathbb{A}$ are identical.

This notebook is an illustration of the following constructions:

1. Create a quiver q, its path $\mathbb{Q}$-algebra $\mathbb{Q}q$ and an admissible quiver $\mathbb{Q}$-algebra $\mathbb{A} = \mathbb{Q}q/I$ with a finite global dimension.
2. Construct the categories $\mathbb{A}_{\text{oid}}$ and mod-$\mathbb{A}_{\text{oid}}$.
3. Construct the Yoneda embedding $\mathbb{Y} : \mathbb{A}_{\text{oid}}^{\text{op}} \hookrightarrow \text{mod-}\mathbb{A}_{\text{oid}}$ and the Yoneda equivalence $\mathbb{Y} : \mathbb{A}_{\text{oid}}^{\text{op},\oplus} \xrightarrow{\sim} \text{proj}(\text{mod-}\mathbb{A}_{\text{oid}})$.