Derived equivalences as derived functors

Each quiver q defines a category $\mathrm{FreeCategory}(q)$ whose objects are the vertices of q and whose mo paths of q. A finite set of paths in $\mathrm{FreeCategory}(q)$ is called uniform if they share the same source anc

For a field k, the k-linear closure of $\mathrm{FreeCategory}(q)$ is the category $k[\mathrm{FreeCategory}(q)]$ whose ob of $\mathrm{FreeCategory}(q)$ and whose morphisms are formal k-linear combinations of uniform morphisms in Obviously, $k[\mathrm{FreeCategory}(q)]$ is a k-linear category.

Suppose ρ is a finite set of morphisms in $k[\mathrm{FreeCategory}(q)]$. We denote by $I = \langle \rho \rangle$ the two-sided ic generated by ρ. The associated quotient categoy $k[\mathrm{FreeCategory}(q)]/I$ will be called the k-algebroic set of relations ρ. This means, a morphism in $k[\mathrm{FreeCategory}(q)]$ (resp. $k[\mathrm{FreeCategory}(q)]/I$) is uniform element in the path algebra kq (resp. kq/I).

The [Gap (https://gap-system.org)](https://gap-system.org) package [QPA (https://github.com/sunnyquiver/QPA2)](https://github.com/sunnyquiver/QPA2) enables us to co k-algebras kq and their quotients by two-sided ideals. That is, we can check equality of morphisms $k[\mathrm{F}$ (resp. $k[\mathrm{FreeCategory}(q)]/I$) by checking the equality of the corresponding algebra elements in kq (realized by the theory of noncommuative Gröbner bases.

Let ρ be a set of relations and let $\mathbb{A} = kq/\langle \rho \rangle$. We denote by $\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ the category of k-linear functo category $k\text{-}\mathrm{vec}$ of finite dimensional vector spaces. That is

1. an object F in $\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ is a functor $F\colon \mathbb{A}_{\mathrm{oid}} \to k\text{-}\mathrm{vec}$ a its data structure is a pair of lists: a list of (reprsents the images of the objects of $\mathbb{A}_{\mathrm{oid}}$ under F) and a list of k-linear maps (represents the im generating morphisms of $\mathbb{A}_{\mathrm{oid}}$ under F);
2. a morphism $\psi\colon F \to G$ is a natural transformation and its data structure is a list of morphisms (repr of the objects of $\mathbb{A}_{\mathrm{oid}}$ under ψ).

The category $\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ is also known as the category $\mathrm{reps}_k(q, \rho)$ of the ρ-bounded quiver k-represen well-known that

$$\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}} \cong \mathrm{fdmod}\text{-}\mathbb{A}$$

where $\mathrm{fdmod}\text{-}\mathbb{A}$ is the category of finite dimensional right $\mathbb{A}$-modules. Furthermore, if $\mathbb{A}$ is a finite dimen then $\mathrm{fdmod}\text{-}\mathbb{A}$ and $\mathrm{mod}\text{-}\mathbb{A}$ are identical.

This notebook is an illustration of the following constructions:

1. Create a quiver q, its path $\mathbb{Q}$-algebra $\mathbb{Q}\,q$ and an admissible quiver $\mathbb{Q}$-algebra $\mathbb{A} = \mathbb{Q}\,q/I$ with a fini
2. Construct the categories $\mathbb{A}_{\mathrm{oid}}$ and $\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$.
3. Construct the Yoneda embedding $\mathbb{Y}\colon \mathbb{A}_{\mathrm{oid}}^{\mathrm{op}} \hookrightarrow \mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ and the Yoneda equivalence $\mathbb{Y}\colon \mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus} \xrightarrow{\sim}$
4. Construct the categories $\mathrm{Ch}^b(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}),\ \mathrm{K}^b(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}})$ and $\mathrm{D}^b(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}})$ and extend the Y to get equivalences
$$K^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus}) \cong K^b(\mathrm{proj}(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}})) \cong \mathrm{D}^b(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}})\,.$$
5. [illegible] $K^b(\mathbb{A}^{\mathrm{op},\oplus})$ [illegible] $\mathrm{D}^b($ [illegible] $)$

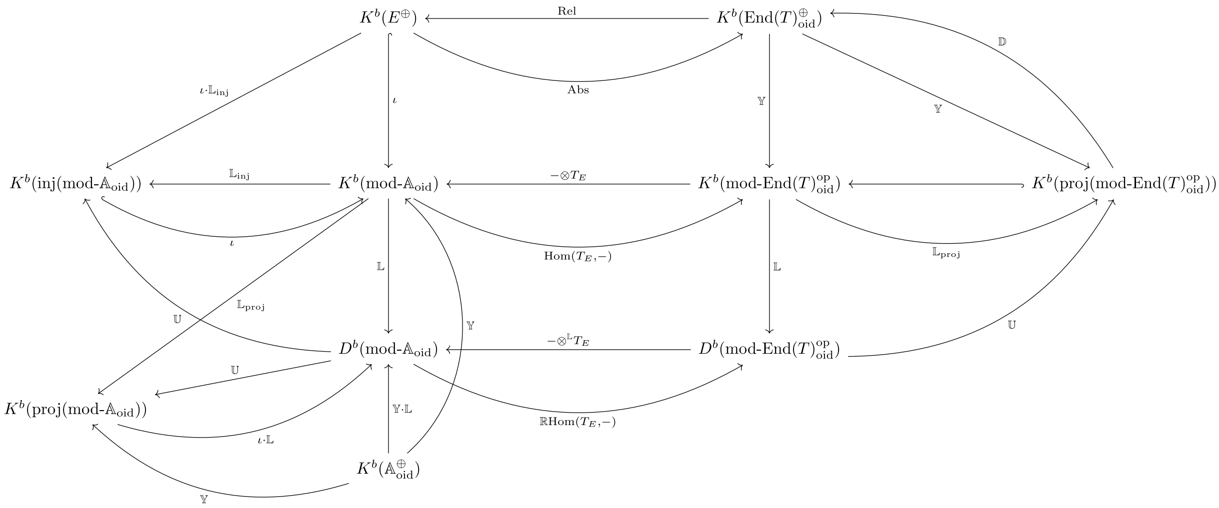
9. Construct the adjoint functors

$$- \otimes_{\mathrm{End}(T_E)^{\mathrm{op}}} T_E \ : \mathrm{mod}\text{-}\mathrm{End}(T_E)^{\mathrm{op}}_{\mathrm{oid}} \to \mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}} \ : \mathrm{Hom}(T_E, -)$$

10. Construct the adjoint derived equivalences

$$- \otimes^{\mathbb{L}}_{\mathrm{End}(T_E)^{\mathrm{op}}} T_E \ : D^b(\mathrm{mod}\text{-}\mathrm{End}(T_E)^{\mathrm{op}}_{\mathrm{oid}}) \xrightarrow{\sim} D^b(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}) \ : \mathbb{R}\mathrm{Hom}(T_E, -$$

and use it to compute an E-replancement of an object $D^b(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}})$.



In [1]:
```
using CapAndHomalg
```

```
                 GAP 4.11.1 of 2021-03-02
   GAP           https://www.gap-system.org (https://www.gap-system.org)
                 Architecture: x86_64-pc-linux-gnu-julia64-kv7
 Configuration:  gmp 6.1.2, Julia GC, Julia 1.5.2, readline
 Loading the library and packages ...
 Packages:   GAPDoc 1.6.3, IO 4.7.1, JuliaInterface 0.5.2, PrimGrp 3.4.0,
             SmallGrp 1.4.1, TransGrp 2.0.5
 Try '??help' for help. See also '?copyright', '?cite' and '?authors'
CapAndHomalg v1.1.3
Imported OSCAR's components GAP and Singular_jll
Type: ?CapAndHomalg for more information
```

In [2]:
```
LoadPackage( "DerivedCategories" )
```

In [3]:
```
SetSpecialSettings( )
EnhanceAllPackages( )
```

Out[3]:
```
GAP: [ "GradedRingForHomalg", "MatricesForHomalg", "FreydCategoriesForCAP",
RingsForHomalg", "ModulePresentationsForCAP", "GradedModulePresentationsForC
ebraForCAP", "FunctorCategories", "CategoryConstructor" ]
```

1. Create a quiver q, its path $\mathbb{Q}$-algebra $\mathbb{Q}\,q$ and an admissible quiver $\mathbb{Q}$-alg

$\mathbb{A} = \mathbb{Q}\,q/I$ with a finite global dimension.

ideal generated by the relation $\rho = \{ab - cd\}$.

Using QPA (https://github.com/sunnyquiver/QPA2), we can create the quiver q, its paths algebra $\mathbb{Q}q$ and quiver $\mathbb{Q}$-algebra $\mathbb{A} := \mathbb{Q}q/\langle ab - cd\rangle$:

In [4]:
```
vertices = [ "v1", "v2", "v3", "v4" ];
arrows   = [ "a", "b", "c", "d" ];
sources  = [  1 ,  2 ,  1 ,  3  ];
ranges   = [  2 ,  4 ,  3 ,  4  ];
```

In [5]:
```
q = RightQuiver( "quiver", vertices, arrows, sources, ranges )
```

Out[5]:
```
GAP: quiver(v1,v2,v3,v4)[a:v1->v2,b:v2->v4,c:v1->v3,d:v3->v4]
```

The following aims for better LaTeX strings for `Show( - )` methods:

In [6]:
```
SetLabelsAsLaTeXStrings( q,
        [ "v_1", "v_2", "v_3", "v_4" ],
        [ "a", "b", "c", "d" ]
    );
```

In [7]:
```
q_op = OppositeQuiver( q )
SetLabelsAsLaTeXStrings( q_op,
        [ "v_1", "v_2", "v_3", "v_4" ],
        [ "a", "b", "c", "d" ]
    );
```

Defining the field of rationals $\mathbb{Q}$ requires the Gap package RingsForHomalg (https://github.com/homalg-project/homalg_project)

In [8]:
```
ℚ = HomalgFieldOfRationals( )
```

Out[8]:
```
GAP: Q
```

In [9]:
```
ℚq = PathAlgebra( ℚ, q )
```

Out[9]:
```
GAP: Q * quiver
```

In [10]:
```
Dimension( ℚq )
```

Out[10]:
```
10
```

In [11]:
```
ρ = [
        ℚq.a * ℚq.b - ℚq.c * ℚq.d,
    ]
```

Out[11]:
```
1-element Array{GAP_jll.MPtr,1}:
 GAP: -1*(c*d) + 1*(a*b)
```

In [12]:
```
𝔸 = ℚq / ρ
```

Out[12]:
```
GAP: (Q * quiver) / [ -1*(c*d) + 1*(a*b) ]
```

In [13]:
```
Dimension( 𝔸 )
```

Out[13]:
```
9
```

It is obvious that $\mathbb{A}$ is admissible because every relation in ρ is a linear combination of paths of length at

In [15]:
```
ℚ vec = MatrixCategory( ℚ )
```
Out[15]:
```
GAP: Category of matrices over Q
```

Creating algebroids of quiver algebras requires the Gap package Algebroids (https://github.com/homalg-

In [16]:
```
𝔸oid = Algebroid( 𝔸, range of HomStructure = ℚ vec )
```
Out[16]:
```
GAP: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] )
```

In [17]:
```
InfoOfInstalledOperationsOfCategory( 𝔸oid )
```
```
22 primitive operations were used to derive 63 operations for this category w
* IsLinearCategoryOverCommutativeRing
```

We can construct the objects of $\mathbb{A}_{\mathrm{oid}}$ by using their labels as vertices in the quiver q:

In [18]:
```
v1 = 𝔸oid."v1"
```
Out[18]:
```
GAP: <(v1)>
```

In [19]:
```
v2 = 𝔸oid."v2"
```
Out[19]:
```
GAP: <(v2)>
```

In [20]:
```
v3 = 𝔸oid."v3"
```
Out[20]:
```
GAP: <(v3)>
```

In [21]:
```
v4 = 𝔸oid."v4"
```
Out[21]:
```
GAP: <(v4)>
```

The list of all objects of $\mathbb{A}_{\mathrm{oid}}$:

In [22]:
```
SetOfObjects( 𝔸oid )
```
Out[22]:
```
GAP: [ <(v1)>, <(v2)>, <(v3)>, <(v4)> ]
```

The algebroid $\mathbb{A}_{\mathrm{oid}}$ is Hom-computable over the category $\mathbb{Q}\text{-vec}$.

In [23]:
```
RangeCategoryOfHomomorphismStructure( 𝔸oid )
```
Out[23]:
```
GAP: Category of matrices over Q
```

In [24]:
```
HomStructure( v1, v4 )
```
Out[24]:
```
GAP: <A vector space object over Q of dimension 1>
```

So, $\mathrm{Hom}_{\mathbb{A}_{\mathrm{oid}}}(v_1, v_4)$ is a 1 dimensional $\mathbb{Q}$-vector space. Its basis is given by:

In [25]:
```
B v1 v4 = BasisOfExternalHom( v1, v4 )
```
Out[25]:
```
GAP: [ (v1)-[{ 1*(a*b) }]->(v4) ]
```

The list of all generating morphisms in $\mathbb{A}_{\mathrm{oid}}$:

In [28]:
```
SetOfGeneratingMorphisms( 𝔸oid )
```

Out[28]:
```
GAP: [ (v1)-[{ 1*(a) }]->(v2), (v2)-[{ 1*(b) }]->(v4), (v1)-[{ 1*(c) }]->(v3
(d) }]->(v4) ]
```

Currently, there are two models for the category $\text{mod-}\mathbb{A}_{\mathrm{oid}}$:

1. By using the category constructor $\mathrm{Hom}(\mathbb{A}_{\mathrm{oid}}, \mathbb{Q}\text{-vec})$ provided by the Gap package FunctorCatego (https://github.com/homalg-project/FunctorCategories).
2. By using the category constructor $\mathrm{CategoryOfQuiverRepresentations}(\mathbb{A})$ provided by the Gap (https://github.com/sunnyquiver/QPA2).

As we mentioned in the introduction, we have the following equivalences of categories:

$$\text{mod-}\mathbb{A}_{\mathrm{oid}} \cong \text{fdmod-}\mathbb{A} \cong \mathrm{reps}_{\mathbb{Q}}(q, \rho)$$

An object $F\colon \mathbb{A}_{\mathrm{oid}} \to \mathbb{Q}\text{-vec}$ in $\text{mod-}\mathbb{A}_{\mathrm{oid}}$ corresponds in $\text{fdmod-}\mathbb{A}$ to the right $\mathbb{A}$-module $M_F := \bigoplus_i^4$ is a unifrom element and $x \in F(v)$ for some $v \in \{v_1, v_2, v_3, v_4\}$, we define $x \cdot r$ by $x \cdot F(v)$ if $v =$ otherwise. This operation can be extended to an action $M_F \times \mathbb{A} \to M_F$ giving M_F a right $\mathbb{A}$-module s

So, let us construct $\text{mod-}\mathbb{A}_{\mathrm{oid}}$:

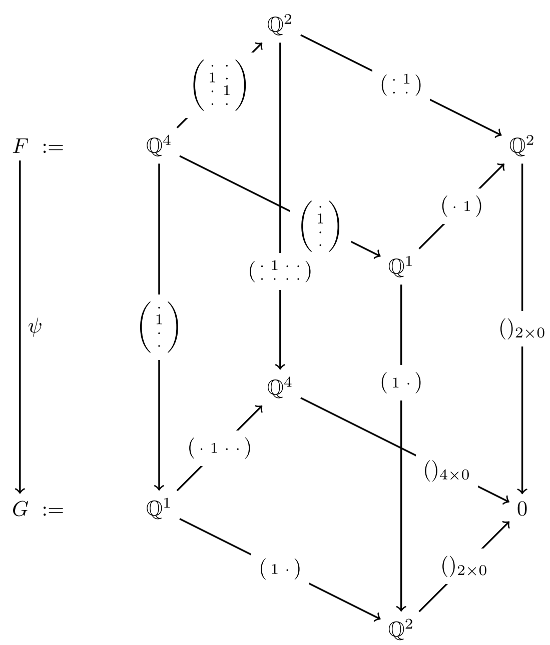
In [29]:
```
mod_𝔸oid = Hom( 𝔸oid, ℚ_vec )
```

Out[29]:
```
GAP: The category of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b
ry of matrices over Q
```

In [30]:
```
InfoOfInstalledOperationsOfCategory( mod_𝔸oid )
```

```
120 primitive operations were used to derive 312 operations for this categor
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughInjectives
* IsAbelianCategoryWithEnoughProjectives
```

Let us create the following morphism $\psi\colon F \to G$



$\mathbb{Q}$-linear maps:

In [31]:
```
F_v1 = 4 / ℚ_vec
F_v2 = 2 / ℚ_vec
F_v3 = 1 / ℚ_vec
F_v4 = 2 / ℚ_vec
```

Out[31]: GAP: <A vector space object over Q of dimension 2>

In [32]:
```
F_a = HomalgMatrix( "[ [ 0, 0 ],
                       [ 1, 0 ],
                       [ 0, 1 ],
                       [ 0, 0 ] ]", 4, 2, ℚ ) / ℚ_vec

F_b = HomalgMatrix( "[ [  0,  1 ],
                       [  0,  0 ] ]", 2, 2, ℚ ) / ℚ_vec

F_c = HomalgMatrix( "[ [ 0 ],
                       [ 1 ],
                       [ 0 ],
                       [ 0 ] ]", 4, 1, ℚ ) / ℚ_vec

F_d = HomalgMatrix( "[ [ 0, 1 ] ]", 1, 2, ℚ ) / ℚ_vec
```

Out[32]: GAP: <A morphism in Category of matrices over Q>

In [33]:
```
F = AsObjectInHomCategory(
        𝔸oid,
        [ F_v1, F_v2, F_v3, F_v4 ],
        [ F_a, F_b, F_c, F_d ]
    )
```

Out[33]: GAP: <(v1)->4, (v2)->2, (v3)->1, (v4)->2; (a)->4x2, (b)->2x2, (c)->4x1, (d)-:

In [34]:
```
Show( F )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 4} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 2} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 2} \\ \hline \end{array}$$

$$a \mapsto \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \\ \cdot & \cdot \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} \cdot & 1 \\ \cdot & \cdot \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix}$$

In [36]:
```
m = PreCompose( 𝔸oid."a", 𝔸oid."b" )
```
Out[36]: GAP: (v1)-[{ 1*(a*b) }]->(v4)

In [37]:
```
Show( m )
```

$$v_1 - (ab) \rightarrow v_4$$

In [38]:
```
F_m = F( m )
```
Out[38]: GAP: <A morphism in Category of matrices over Q>

In [39]:
```
Show( F_m )
```

$$\mathbb{Q}^{1\times 4} \xrightarrow{\begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \\ \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}} \mathbb{Q}^{1\times 2}$$

In [40]:
```
G_v1 = 1 / ℚ_vec
G_v2 = 4 / ℚ_vec
G_v3 = 2 / ℚ_vec
G_v4 = 0 / ℚ_vec
```
Out[40]: GAP: <A vector space object over Q of dimension 0>

In [41]:
```
G_a = HomalgMatrix( "[ [ 0, 1, 0, 0 ] ]", 1, 4, ℚ ) / ℚ_vec

G_b = HomalgZeroMatrix( 4, 0, ℚ ) / ℚ_vec

G_c = HomalgMatrix( "[ [ 1, 0 ] ]", 1, 2, ℚ ) / ℚ_vec

G_d = HomalgZeroMatrix( 2, 0, ℚ ) / ℚ_vec
```
Out[41]: GAP: <A morphism in Category of matrices over Q>

In [42]:
```
G = AsObjectInHomCategory(
        𝔸oid,
        [ G_v1, G_v2, G_v3, G_v4 ],
        [ G_a, G_b, G_c, G_d ]
    )
```
Out[42]: GAP: <(v1)->1, (v2)->4, (v3)->2, (v4)->0; (a)->1x4, (b)->4x0, (c)->1x2, (d)-:

In [43]:
```
Show( G )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 4} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 2} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 0} \\ \hline \\ a & \mapsto & \begin{pmatrix} \cdot & 1 & \cdot & \cdot \end{pmatrix} \end{array}$$

The data structure of a morphism in $\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ is a list of $\mathbb{Q}$-linear maps:

In [44]:
```
ψ_v1 = HomalgMatrix( "[ [ 0 ],
                        [ 1 ],
                        [ 0 ],
                        [ 0 ] ]", 4, 1, ℚ ) / ℚ_vec

ψ_v2 = HomalgMatrix( "[ [ 0, 1, 0, 0 ],
                        [ 0, 0, 0, 0 ] ]", 2, 4, ℚ ) / ℚ_vec

ψ_v3 = HomalgMatrix( "[ [ 1, 0 ] ]", 1, 2, ℚ ) / ℚ_vec

ψ_v4 = HomalgZeroMatrix( 2, 0, ℚ ) / ℚ_vec
```

Out[44]: `GAP: <A morphism in Category of matrices over Q>`

In [45]:
```
ψ = AsMorphismInHomCategory( F, [ ψ_v1, ψ_v2, ψ_v3, ψ_v4 ], G )
```

Out[45]: `GAP: <(v1)->4x1, (v2)->2x4, (v3)->1x2, (v4)->2x0>`

In [46]:
```
Show( ψ )
```

$$v_1 \mapsto \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} 1 & \cdot \end{pmatrix}$$

$$v_4 \mapsto ()_{2\times 0}$$

In [47]:
```
IsMonomorphism( ψ )
```

Out[47]: `false`

In [48]:
```
IsEpimorphism( ψ )
```

Out[48]: `false`

The category $\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ is abelian with enough projectives and injectives. Let us compute the kernel ob embedding of ψ:

In [49]:
```
K_ψ = KernelObject( ψ )
```

Out[49]: `GAP: <(v1)->3, (v2)->1, (v3)->0, (v4)->2; (a)->3x1, (b)->1x2, (c)->3x0, (d)->:`

In [50]:
```
Show( K ψ )
```

$$\begin{array}{lcl} v_1 & \mapsto & \mathbb{Q}^{1\times 3} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 2} \\ \hline \\ a & \mapsto & \begin{pmatrix} \cdot \\ 1 \\ \cdot \end{pmatrix} \\ \\ b & \mapsto & \begin{pmatrix} \cdot & \cdot \end{pmatrix} \\ \\ c & \mapsto & ()_{3\times 0} \\ \\ d & \mapsto & ()_{0\times 2} \end{array}$$

In [51]:
```
κ ψ = KernelEmbedding( ψ )
```

Out[51]: GAP: <(v1)->3x4, (v2)->1x2, (v3)->0x1, (v4)->2x2>

In [52]:
```
Show( κ ψ )
```

$$\begin{array}{lcl} v_1 & \mapsto & \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix} \\ \\ v_2 & \mapsto & \begin{pmatrix} \cdot & 1 \end{pmatrix} \\ \\ v_3 & \mapsto & ()_{0\times 1} \\ \\ v_4 & \mapsto & \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix} \end{array}$$

Of course, the category $\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ is Hom-computable over $\mathbb{Q}\text{-}\mathrm{vec}$:

In [53]:
```
RangeCategoryOfHomomorphismStructure( mod Aoid )
```

Out[53]: GAP: Category of matrices over Q

In [57]:
```
τ = -5 * Hom_GF[3] + 2 * Hom_GF[5] + 15 * Hom_GF[6]
```

Out[57]: GAP: <(v1)->1x4, (v2)->4x2, (v3)->2x1, (v4)->0x2>

In [58]:
```
Show( τ )
```

$$v_1 \mapsto \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} \cdot & -5 \\ \cdot & \cdot \\ \cdot & 2 \\ \cdot & 15 \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} \cdot \\ \cdot \end{pmatrix}$$

$$v_4 \mapsto ()_{0 \times 2}$$

In [59]:
```
CoefficientsOfMorphism( τ )
```

Out[59]: GAP: [0, 0, -5, 0, 2, 15]

In [60]:
```
P_F = SomeProjectiveObject( F )
```

Out[60]: GAP: <(v1)->4, (v2)->4, (v3)->4, (v4)->5; (a)->4x4, (b)->4x5, (c)->4x4, (d)-:

In [61]:
```
IsProjective( P_F )
```

Out[61]: true

In [62]:
```
Show( P F )
```

$$
\begin{array}{lll}
v_1 & \mapsto & \mathbb{Q}^{1\times 4} \\
v_2 & \mapsto & \mathbb{Q}^{1\times 4} \\
v_3 & \mapsto & \mathbb{Q}^{1\times 4} \\
v_4 & \mapsto & \mathbb{Q}^{1\times 5} \\
\hline
\end{array}
$$

$$
a \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}
$$

$$
b \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}
$$

$$
c \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}
$$

$$
d \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot \end{pmatrix}
$$

In [63]:
```
π F = EpimorphismFromSomeProjectiveObject( F )
```

Out[63]:
```
GAP: <(v1)->4x4, (v2)->4x2, (v3)->4x1, (v4)->5x2>
```

In [64]:
```
Show( π F )
```

$$v_1 \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \\ \cdot & \cdot \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} \cdot \\ 1 \\ \cdot \\ \cdot \end{pmatrix}$$

$$v_4 \mapsto \begin{pmatrix} \cdot & \cdot \\ \cdot & 1 \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \cdot \end{pmatrix}$$

In [65]:
```
I F = SomeInjectiveObject( F )
```

Out[65]:
```
GAP: <(v1)->5, (v2)->3, (v3)->2, (v4)->2; (a)->5x3, (b)->3x2, (c)->5x2, (d)-:
```

In [66]:
```
IsInjective( I F )
```

Out[66]:
```
true
```

In [67]:
```
Show( I F )
```

$$\begin{array}{lll} v_1 & \mapsto & \mathbb{Q}^{1\times 5} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 3} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 2} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 2} \\ \hline \end{array}$$

$$a \mapsto \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot \\ \cdot & 1 & \cdot \\ \cdot & \cdot & 1 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

$$d \mapsto \begin{pmatrix} 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

In [68]:
```
ι F = MonomorphismIntoSomeInjectiveObject( F )
```

Out[68]: GAP: <(v1)->4x5, (v2)->2x3, (v3)->1x2, (v4)->2x2>

In [69]:
```
Show( ι F )
```

$$v_1 \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} \cdot & \cdot & 1 \\ 1 & \cdot & \cdot \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} \cdot & 1 \end{pmatrix}$$

$$\mathbb{Y}\colon \mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus} \xrightarrow{\sim} \mathrm{proj}(\text{mod-}\mathbb{A}_{\mathrm{oid}}).$$

The Yoneda embedding $\mathbb{Y}\colon \mathbb{A}_{\mathrm{oid}}^{\mathrm{op}} \hookrightarrow \text{mod-}\mathbb{A}_{\mathrm{oid}}$ sends an object $v \in \mathbb{A}_{\mathrm{oid}}^{\mathrm{op}}$ to the functor $\mathbb{Y}(v) := \mathrm{Hom}_{\mathbb{A}_{\mathrm{oid}}}(v,\text{-})\colon \mathbb{A}_{\mathrm{oid}} \to \mathbb{Q}\text{-vec}$. It is well known that the images of the Yoneda embedding are in $\text{mod-}\mathbb{A}_{\mathrm{oid}}$.

We start by creating the opposite algebroid $\mathbb{A}_{\mathrm{oid}}^{\mathrm{op}}$:

In [70]:
```
𝔸oid_op = OppositeAlgebroidOverOppositeQuiverAlgebra( 𝔸oid )
```
Out[70]:
```
GAP: Algebroid( (Q * quiver_op) / [ -1*(d*c) + 1*(b*a) ] )
```

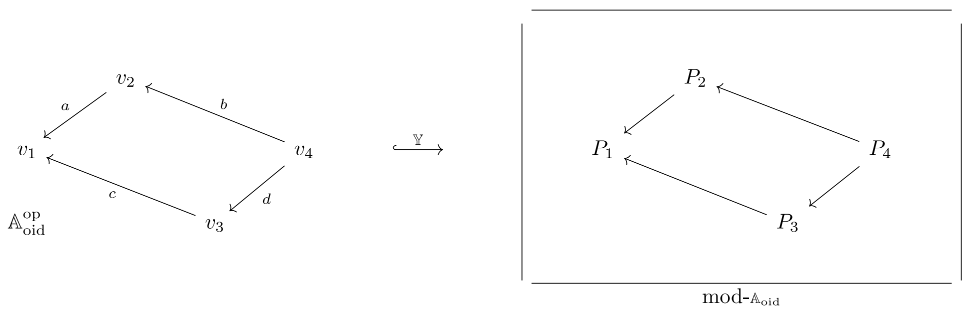
In [71]:
```
𝕐 = YonedaEmbedding( 𝔸oid_op )
```
Out[71]:
```
GAP: Yoneda embedding functor
```

In [72]:
```
Display( 𝕐 )
```
```
Yoneda embedding functor:

Algebroid( (Q * quiver_op) / [ -1*(d*c) + 1*(b*a) ] )
  |
  V
The category of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] )
matrices over Q
```

In [73]:
```
IsIdenticalObj( RangeOfFunctor( 𝕐 ), mod_𝔸oid )
```
Out[73]:
```
true
```

Since $\mathbb{A}$ is an admissible quiver algebra, the images of the Yoneda embedding are, up to isomorphism, t projective objects in $\text{mod-}\mathbb{A}_{\mathrm{oid}}$.



In [74]:
```
P1 = 𝕐( 𝔸oid_op."v1" )
```
Out[74]:
```
GAP: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a)->1x1, (b)->1x1, (c)->1x1, (d)-
```

In [75]:
```
Show( P1 )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 1} \\ \hline a & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \\ b & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \\ c & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \\ d & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \end{array}$$

In [76]:
```
P2 = 𝕐( 𝔸oid_op."v2" )
```

Out[76]: GAP: <(v1)->0, (v2)->1, (v3)->0, (v4)->1; (a)->0x1, (b)->1x1, (c)->0x0, (d)-:

In [77]:
```
Show( P2 )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 1} \\ \hline a & \mapsto & ()_{0\times 1} \\ b & \mapsto & \begin{pmatrix} 1 \end{pmatrix} \\ c & \mapsto & ()_{0\times 0} \\ d & \mapsto & ()_{0\times 1} \end{array}$$

In [78]:
```
P3 = 𝕐( 𝔸oid_op."v3" )
```

Out[78]: GAP: <(v1)->0, (v2)->0, (v3)->1, (v4)->1; (a)->0x0, (b)->0x1, (c)->0x1, (d)-:

In [79]:
```
Show( P3 )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 1} \\ \hline a & \mapsto & ()_{0\times 0} \\ b & \mapsto & ()_{0\times 1} \\ c & \mapsto & ()_{0\times 1} \\ d & \mapsto & (1) \end{array}$$

In [80]:
```
P4 = 𝕐( 𝔸oid_op."v4" )
```

Out[80]:
```
GAP: <(v1)->0, (v2)->0, (v3)->0, (v4)->1; (a)->0x0, (b)->0x1, (c)->0x0, (d)-:
```

In [81]:
```
Show( P4 )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 1} \\ \hline a & \mapsto & ()_{0\times 0} \\ b & \mapsto & ()_{0\times 1} \\ c & \mapsto & ()_{0\times 0} \\ d & \mapsto & ()_{0\times 1} \end{array}$$

In the following we apply $\mathbb{Y}$ on the morphism $\mathbb{A}^{\mathrm{op}}_{\mathrm{oid}} \ni \alpha = ba\colon v_4 \to v_1$

In [82]:
```
α =  PreCompose( 𝔸oid_op.b, 𝔸oid_op.a )
```

Out[82]:
```
GAP: (v4)-[{ 1*(b*a) }]->(v1)
```

$$
\begin{array}{ccc}
v_1 & \mapsto & ()_{0\times 1} \\
v_2 & \mapsto & ()_{0\times 1} \\
v_3 & \mapsto & ()_{0\times 1} \\
v_4 & \mapsto & (1)
\end{array}
$$

If we restricte the Yoneda embedding $\mathbb{Y}:\mathbb{A}_{\mathrm{oid}}^{\mathrm{op}} \hookrightarrow \mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}}$ to its image, we get an isomorphism

$$\mathbb{Y}:\mathbb{A}_{\mathrm{oid}}^{\mathrm{op}} \xrightarrow{\sim} \mathrm{proj}_0(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}})$$

where $\mathrm{proj}_0(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}})$ is the skeletal of the full subcategory of indecomposable projective objects in

In the following we construct this isomorphism:

In [86]:
```
projs_0 = FullSubcategoryGeneratedByIndecProjectiveObjects( mod_Aoid )
```

Out[86]:
```
GAP: Full subcategory generated by the 4 indecomposable projective objects(
functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of 
)
```

In [87]:
```
projs_0[ 1 ]
```

Out[87]:
```
GAP: An object in full subcategory given by: <(v1)->1, (v2)->1, (v3)->1, (v4
(b)->1x1, (c)->1x1, (d)->1x1>
```

In [88]:
```
IsEqualForObjects( P1, UnderlyingCell( projs_0[ 1 ] ) )
```

Out[88]:
```
true
```

In [89]:
```
KnownFunctors( Aoid_op, projs_0 )
```

```
1: Yoneda isomorphism
```

In [90]:
```
𝕐 = Functor( Aoid_op, projs_0, 1 )
```

Out[90]:
```
GAP: Isomorphism functor from algebroid onto full subcategory generated by i
rojective objects
```

In [91]:
```
Display( 𝕐 )
```

```
Isomorphism functor from algebroid onto full subcategory generated by indeco
tive objects:

Algebroid( (Q * quiver_op) / [ -1*(d*c) + 1*(b*a) ] )
  |
  V
Full subcategory generated by the 4 indecomposable projective objects( The c
tors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matr
```

The forward equivalence is the extension of Yoneda isomorphism to additive closures and the backward decomposition functor of projective objects into direct sums of indecomposable projective objects resp. c

There is currently two

In [93]:
```
𝔸op = OppositeAlgebra( 𝔸 )
```

Out[93]:
```
GAP: (Q * quiver_op) / [ -1*(d*c) + 1*(b*a) ]
```

Currently, there are two models for the category $\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus}$

1. **AdditiveClosure**$(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op}})$ or
2. **QuiverRows**$(\mathbb{A}^{\mathrm{op}})$

both of which are provided by the Gap package [FreydCategoriesForCAP (https://github.com/homalg-proj/tree/master/FreydCategoriesForCAP#readme)](https://github.com/homalg-proj/tree/master/FreydCategoriesForCAP#readme).

In [94]:
```
𝔸oid_op_plus = AdditiveClosure( 𝔸oid_op )
```

Out[94]:
```
GAP: Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*c) + 1*(b*a) ] )
```

In [95]:
```
InfoOfInstalledOperationsOfCategory( 𝔸oid_op_plus )
```

```
23 primitive operations were used to derive 113 operations for this category
* IsLinearCategoryOverCommutativeRing
* IsAdditiveCategory
```

In [96]:
```
projs = FullSubcategoryGeneratedByProjectiveObjects( mod_𝔸oid )
```

Out[96]:
```
GAP: Full additive subcategory generated by projective objects( The category
lgebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices ov
```

In [97]:
```
InfoOfInstalledOperationsOfCategory( projs )
```

```
53 primitive operations were used to derive 119 operations for this category
* IsLinearCategoryOverCommutativeRing
* IsAdditiveCategory
```

The above categories are also Hom-computable over $\mathbb{Q}\text{-vec}$:

In [98]:
```
RangeCategoryOfHomomorphismStructure( 𝔸oid_op_plus )
```

Out[98]:
```
GAP: Category of matrices over Q
```

In [99]:
```
RangeCategoryOfHomomorphismStructure( projs )
```

Out[99]:
```
GAP: Category of matrices over Q
```

In the following we create the equivalences between $\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus} \simeq \mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})$

In [100]:
```
KnownFunctors( 𝔸oid_op_plus, projs )
```

```
1: Yoneda embedding
```

In [101]:
```
𝕐 = Functor( 𝔸oid_op_plus, projs, 1 )
```

In [103]:
```
KnownFunctors( projs, Aoid_op_plus )
```

```
1: Decomposition of projective objects
```

In [104]:
```
𝔻 = Functor( projs, Aoid_op_plus, 1 )
```

Out[104]:
```
GAP: Decomposition of projective objects
```

In [105]:
```
Display( 𝔻 )
```

```
Decomposition of projective objects:

Full additive subcategory generated by projective objects( The category of f
oid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q
  |
  V
Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*c) + 1*(b*a) ] ) )
```

So, let us decompose some projective object $P \in \mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})$ by using the isomorphism $\mathbb{D}\colon \mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}) \xrightarrow{\sim} \mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus}$:

In [106]:
```
K = DirectSum( KernelObject( ψ ), CokernelObject( ψ ) )
```

Out[106]:
```
GAP: <(v1)->3, (v2)->4, (v3)->1, (v4)->2; (a)->3x4, (b)->4x2, (c)->3x1, (d)->
```

In [107]:
```
IsProjective( K )
```

Out[107]:
```
false
```

In [108]:
```
P = SomeProjectiveObject( K )
```

Out[108]:
```
GAP: <(v1)->3, (v2)->6, (v3)->4, (v4)->9; (a)->3x6, (b)->6x9, (c)->3x4, (d)->
```

In [109]:
```
Show( P )
```

$$\begin{array}{lll} v_1 & \mapsto & \mathbb{Q}^{1\times 3} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 6} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 4} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 9} \\ \hline \end{array}$$

$$a \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \end{pmatrix}$$

$$d \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \end{pmatrix}$$

In [110]:
```
P =  P / projs
```

Out[110]:
```
GAP: An object in full subcategory given by: <(v1)->3, (v2)->6, (v3)->4, (v4
(b)->6x9, (c)->3x4, (d)->4x9>
```

In [111]:
```
𝔻 P = 𝔻( P )
```

Out[111]:
```
GAP: <An object in Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*c)
) defined by 9 underlying objects>
```

In [112]:
```
Show( 𝔻 P )
```

$$v_1{}^{\oplus 3} \oplus v_2{}^{\oplus 3} \oplus v_3 \oplus v_4{}^{\oplus 2}$$

In the following, we apply the Yoneda isomorphism on a morphism φ: 𝔻_P $\rightarrow$ 𝔻_P

In [115]:
```
Show( φ )
```

$$v_1{}^{\oplus 3} \oplus v_2{}^{\oplus 3} \oplus v_3 \oplus v_4{}^{\oplus 2} \xrightarrow{\begin{pmatrix} v_1 & v_1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & v_1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v_1 & v_1 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & a & a & v_2 & v_2 & v_2 & 0 & 0 & 0 \\ a & a & a & v_2 & v_2 & v_2 & 0 & 0 & 0 \\ a & a & a & v_2 & v_2 & v_2 & 0 & 0 & 0 \\ c & c & c & 0 & 0 & 0 & v_3 & 0 & 0 \\ ba & ba & ba & b & b & b & d & v_4 & v_4 \\ ba & ba & ba & b & b & b & d & v_4 & v_4 \end{pmatrix}} v_1{}^{\oplus 3} \oplus v_2{}^{\oplus 3} \oplus v$$

In [116]:
```
𝕐 φ = 𝕐( φ )
```

Out[116]: GAP: A morphism in full subcategory given by: <(v1)->3x3, (v2)->6x6, (v3)->4

In [117]:
```
Show( UnderlyingCell( 𝕐 φ ) )
```

$$v_1 \mapsto \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$v_3 \mapsto \begin{pmatrix} 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & \cdot \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$v_4 \mapsto \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The equivalence $\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus} \cong \mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})$ can be lifted to an equivalence between the (bounded) co

$$\mathrm{Ch}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus}) \cong \mathrm{Ch}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$$

and the (bounded) homotopy categories:

$$\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus}) \cong \mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}))\,.$$

Since the quiver q has no loops, the global dimension of $\mathbb{A}$ is finite and bounded above by the the length in $\mathbb{Q}\,q$. In this example the global dimension of $\mathbb{A}$ is 2.

Since the global dimension is finite, we get the equivalence:

$$\mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \cong \mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod})\,.$$

To sum up, we get the following equivalences:

$$\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus}) \cong \mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \cong \mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod})\,.$$

The package [QPA (https://github.com/sunnyquiver/QPA2)](https://github.com/sunnyquiver/QPA2) can be used to check whether an integer n is dimension of $\mathbb{A}$:

In [119]:
```
GlobalDimensionOfAlgebra( 𝔸, 1 )
```
Out[119]:
```
false
```

In [120]:
```
GlobalDimensionOfAlgebra( 𝔸, 2 )
```
Out[120]:
```
2
```

We start by creating the homotopy categories $\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus})$ and $\mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$:

In [121]:
```
K_𝔸oid_op_plus = HomotopyCategoryByCochains( 𝔸oid_op_plus )
```
Out[121]:
```
GAP: Homotopy^• category( Additive closure( Algebroid( (Q * quiver_op) / [ -
a) ] ) ) )
```

In [122]:
```
K_projs = HomotopyCategoryByCochains( projs )
```
Out[122]:
```
GAP: Homotopy^• category( Full additive subcategory generated by projective 
tegory of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Ca
ces over Q ) )
```

Of course both categories are Hom-computable over $\mathbb{Q}\text{-vec}$:

In [123]:
```
RangeCategoryOfHomomorphismStructure( K_𝔸oid_op_plus )
```
Out[123]:
```
GAP: Category of matrices over Q
```

In [124]:
```
RangeCategoryOfHomomorphismStructure( K_projs )
```
Out[124]:
```
GAP: Category of matrices over Q
```

In [126]:
```
K 𝕐 = ExtendFunctorToHomotopyCategoriesByCochains( 𝕐 )
```
Out[126]:
```
GAP: Extension of a functor to homotopy categories
```
In [127]:
```
Display( K 𝕐 )
```
```
Extension of a functor to homotopy categories:

Homotopy^• category( Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*
) ) )
  |
  V
Homotopy^• category( Full additive subcategory generated by projective objec
y of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Categor
ver Q ) )
```
In [128]:
```
IsIdenticalObj( SourceOfFunctor( K 𝕐 ), K 𝔸oid_op_plus ) && IsIdenticalObj(
```
Out[128]:
```
true
```
In [129]:
```
Display( 𝔻 )
```
```
Decomposition of projective objects:

Full additive subcategory generated by projective objects( The category of fu
oid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q
  |
  V
Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*c) + 1*(b*a) ] ) )
```
In [130]:
```
K 𝔻 = ExtendFunctorToHomotopyCategoriesByCochains( 𝔻 )
```
Out[130]:
```
GAP: Extension of a functor to homotopy categories
```
In [131]:
```
Display( K 𝔻 )
```
```
Extension of a functor to homotopy categories:

Homotopy^• category( Full additive subcategory generated by projective objec
y of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Categor
ver Q ) )
  |
  V
Homotopy^• category( Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*
) ) )
```
In [132]:
```
IsIdenticalObj( SourceOfFunctor( K 𝔻 ), K projs ) && IsIdenticalObj( K 𝔸oid
```
Out[132]:
```
true
```

The equivalence $\mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \cong \mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod})$ is the composition:

$$\mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \hookrightarrow \mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \xrightarrow{\mathbb{L}} \mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod})$$

where $\mathbb{L}$ is the natural localization functor. That is, $\mathbb{L}$ sends a morphism $\beta\colon B \to C$ in $\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$
$\mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$ represented by the roof $(B \xleftarrow{\mathrm{id}_B} B \xrightarrow{\beta} C) : B \to C$.

A roof in $\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$ is by definition a pair of morphisms $(A \xleftarrow{\alpha} B \xrightarrow{\beta} C)$ where α is a quasi-isom

In [135]:
```
D mod 𝔸oid = DerivedCategory( mod 𝔸oid, true )
```
Out[135]:
```
GAP: Derived^• category( The category of functors: Algebroid( (Q * quiver) /
*(a*b) ] ) -> Category of matrices over Q )
```

In [136]:
```
IsIdenticalObj( mod 𝔸oid, AmbientCategory( projs ) )
```
Out[136]:
```
true
```

In [137]:
```
ι = InclusionFunctor( projs );
```

In [138]:
```
𝕀 = ExtendFunctorToHomotopyCategoriesByCochains( ι )
```
Out[138]:
```
GAP: Extension of a functor to homotopy categories
```

In [139]:
```
Display( 𝕀 )
```
```
Extension of a functor to homotopy categories:

Homotopy^• category( Full additive subcategory generated by projective objec
y of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Categor
ver Q ) )
  |
  V
Homotopy^• category( The category of functors: Algebroid( (Q * quiver) / [ -
b) ] ) -> Category of matrices over Q )
```

In [140]:
```
IsIdenticalObj( K mod 𝔸oid, RangeOfFunctor( 𝕀 ) )
```
Out[140]:
```
true
```

In [141]:
```
𝕃 = LocalizationFunctor( K mod 𝔸oid )
```
Out[141]:
```
GAP: Localization functor in derived category
```

In [142]:
```
Display( 𝕃 )
```
```
Localization functor in derived category:

Homotopy^• category( The category of functors: Algebroid( (Q * quiver) / [ -
b) ] ) -> Category of matrices over Q )
  |
  V
Derived^• category( The category of functors: Algebroid( (Q * quiver) / [ -1
b) ] ) -> Category of matrices over Q )
```

In [143]:
```
IsIdenticalObj( D mod 𝔸oid, RangeOfFunctor( 𝕃 ) )
```
Out[143]:
```
true
```

On the other hand, the equivalence $\mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}) \xrightarrow{\mathbb{U}} \mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$ can be computed by the
of derived categories. More precisely, the functor

$$\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \xrightarrow{\mathbb{P}} \mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$$

which maps cells in $\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$ to their projective replacements in $\mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$ is a loca
hence factors uniquely along $\mathbb{L}$ via the functor $\mathbb{U}$ which maps a morphism $A \xleftarrow{\alpha} B \xrightarrow{\beta} C$ in $\mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-m}$
$(\mathbb{P}(\alpha))^{-1} \cdot \mathbb{P}(\beta) \colon \mathbb{P}(A) \to \mathbb{P}(C)$ in $\mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod}))$.

```
Localization functor by projective objects:

Homotopy^• category( The category of functors: Algebroid( (Q * quiver) / [ -
b) ] ) -> Category of matrices over Q )
  |
  V
Homotopy^• category( Full additive subcategory generated by projective objec
y of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Categor
ver Q ) )
```

In [146]:
```
𝕌 = UniversalFunctorFromDerivedCategory( ℙroj )
```

Out[146]:
```
GAP: Universal functor from derived category onto a localization category
```

In [147]:
```
Display( 𝕌 )
```

```
Universal functor from derived category onto a localization category:

Derived^• category( The category of functors: Algebroid( (Q * quiver) / [ -1
b) ] ) -> Category of matrices over Q )
  |
  V
Homotopy^• category( Full additive subcategory generated by projective objec
y of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Categor
ver Q ) )
```

Now, we can compute the composition

$$\mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}) \xrightarrow{\mathbb{U}} \mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \xrightarrow{K\_\mathbb{D}} \mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus})$$

In [148]:
```
𝕌K_𝔻 = PreCompose( 𝕌, K_𝔻 )
```

Out[148]:
```
GAP: Composition of Universal functor from derived category onto a localizat
d Extension of a functor to homotopy categories
```

In [149]:
```
Display( 𝕌K_𝔻 )
```

```
Composition of Universal functor from derived category onto a localization c
ension of a functor to homotopy categories:

Derived^• category( The category of functors: Algebroid( (Q * quiver) / [ -1
b) ] ) -> Category of matrices over Q )
  |
  V
Homotopy^• category( Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*
) ) )
```

and the other way around

$$\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus}) \xrightarrow{K\_\mathbb{Y}} \mathrm{K}^b(\mathrm{proj}(\mathbb{A}_{\mathrm{oid}}\text{-mod})) \hookrightarrow \mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}) \xrightarrow{\mathbb{L}} \mathrm{D}^b(\mathbb{A}_{\mathrm{oid}}\text{-mod}$$

In [150]:
```
K_𝕐𝕀𝕃 = PreCompose( [ K_𝕐, 𝕀, 𝕃 ] )
```

Out[150]:
```
GAP: Composition of Composition of Extension of a functor to homotopy catego
ion of a functor to homotopy categories and Localization functor in derived
```

In [151]:
```
Display( K_𝕐𝕀𝕃 )
```

5. Create an object in $K^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus})$ and compute its image in $\mathrm{D}^b(\mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oi}}$

In the following we want to apply the functor $K_{\mathbb{Y}} \cdot \mathbb{I} \cdot \mathbb{L}$ on the object C in $\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus})$ defined by

$$C \ := \qquad 0 \longrightarrow v_4 \xrightarrow{\begin{pmatrix} b & d \end{pmatrix}} v_2 \oplus v_3 \longrightarrow 0$$

and whose lower bound is -1.

In [152]:
```
C_m1 = AdditiveClosureObject(
            [ Aoid_op."v4" ],
            Aoid_op_plus
        )
C_0 = AdditiveClosureObject(
            [ Aoid_op."v2", Aoid_op."v3" ],
            Aoid_op_plus
        )
∂_m1 = AdditiveClosureMorphism(
            C_m1,
            [
                [ Aoid_op."b", Aoid_op."d" ],
            ],
            C_0
        )
```

Out[152]:
```
GAP: <A morphism in Additive closure( Algebroid( (Q * quiver_op) / [ -1*(d*c
) defined by a 1 x 2 matrix of underlying morphisms>
```

In [153]:
```
Show( ∂_m1 )
```

$$v_4 \xrightarrow{\begin{pmatrix} b & d \end{pmatrix}} v_2 \oplus v_3$$

In [154]:
```
C = HomotopyCategoryObject( K_Aoid_op_plus, [ ∂_m1 ], -1 )
```

Out[154]:
```
GAP: <An object in Homotopy^• category( Additive closure( Algebroid( (Q * qu
*(d*c) + 1*(b*a) ] ) ) ) with active lower bound -1 and active upper bound 0:
```

In [155]:
```
Show( C )
```

$$\begin{array}{c} v_2 \oplus v_3 \\ \uparrow \\ \begin{pmatrix} b & d \end{pmatrix} \\ |_{-1} \\ v_4 \end{array}$$

In [156]:
```
IsWellDefined( C )
```

Out[156]:
```
true
```

In [157]:
```
W = K_YIL( C )
```

Out[157]:
```
GAP: <An object in Derived^• category( The category of functors: Algebroid(
```

In [159]:
```
ObjectAt( W, -1 )
```

Out[159]: GAP: <(v1)->0, (v2)->0, (v3)->0, (v4)->1; (a)->0x0, (b)->0x1, (c)->0x0, (d)-:

In [160]:
```
ObjectAt( W, 0 )
```

Out[160]: GAP: <(v1)->0, (v2)->1, (v3)->1, (v4)->2; (a)->0x1, (b)->1x2, (c)->0x1, (d)-:

In [161]:
```
∂ m1 = DifferentialAt( W, -1 )
```

Out[161]: GAP: <(v1)->0x0, (v2)->0x1, (v3)->0x1, (v4)->1x2>

In [162]:
```
Show( ∂ m1 )
```

$$\begin{array}{ccc} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & ()_{0\times 1} \\ v_3 & \mapsto & ()_{0\times 1} \\ v_4 & \mapsto & \begin{pmatrix} 1 & 1 \end{pmatrix} \end{array}$$

In [163]:
```
CohomologySupport( W )
```

Out[163]: GAP: [0]

Since 0 is an upper bound of W and its cohomology support is $[0]$, we can create the following acyclic c

$$B := \qquad 0 \to W^{-1} \xrightarrow{\partial^{-1}} W^0 \xrightarrow{\mathrm{CokernelProjection}(\partial^{-1})} \mathrm{CokernelObject}(\partial^{-1}) \cong H^($$

In [164]:
```
H 0 = CohomologyAt( W, 0 )
```

Out[164]: GAP: <(v1)->0, (v2)->1, (v3)->1, (v4)->1; (a)->0x1, (b)->1x1, (c)->0x1, (d)-:

In [165]:
```
Show( H 0 )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 0} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 1} \\ \hline a & \mapsto & ()_{0\times 1} \\ b & \mapsto & \begin{pmatrix} -1 \end{pmatrix} \end{array}$$

In [167]:
```
Show( ∂ 0 )
```

$$\begin{array}{ccc} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & (1) \\ v_3 & \mapsto & (1) \\ v_4 & \mapsto & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{array}$$

In [168]:
```
IsEqualForObjects( H 0, Range( ∂ 0 ) )
```

Out[168]: true

In [169]:
```
B = DerivedCategoryObject( D mod 𝔸oid, [ ∂ m1, ∂ 0 ], -1 )
```

Out[169]:
```
GAP: <An object in Derived^• category( The category of functors: Algebroid(
[ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower l
tive upper bound 1>
```

In [170]:
```
IsWellDefined( B )
```

Out[170]: true

In [171]:
```
CohomologySupport( B )
```

Out[171]: GAP: [ ]

Since B is an acyclic complex, it vanishes in the derived category. In the following, we check that applyi $\mathbb{U}\,K$_$\mathbb{D}$ on B returns an object which also vanishes in $\mathrm{K}^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus})$

In [172]:
```
IsZero( B )
```

Out[172]: true

In [173]:
```
𝕌K 𝔻 B = 𝕌K 𝔻( B )
```

Out[173]:
```
GAP: <An object in Homotopy^• category( Additive closure( Algebroid( (Q * qu
*(d*c) + 1*(b*a) ] ) ) ) with active lower bound -1 and active upper bound 1
```

In [174]:
```
Show( 𝕌K 𝔻 B )
```

$$\begin{array}{c} v_2 \oplus v_3 \\ \uparrow \\ \begin{pmatrix} v_2 & 0 \\ 0 & v_3 \\ -b & -d \end{pmatrix} \\ | \end{array}$$

In [175]:
```
IsZero( 𝕌𝕂 𝔻 B )
```

Out[175]:
```
true
```

6. Construct a full strong exceptional collection $E = (E_1, E_2, E_3, E_4)$ ir $\operatorname{mod-}\mathbb{A}_{\text{oid}}$.

Consider the following objects $E_1 := P_2,\ E_2 := P_3, E_3 := H^0(W), E_4 := P_1$ and let $T := E_1 \oplus I$

In [176]:
```
E_1 = P2
E_2 = P3
E_3 = CohomologyAt( W, 0 )
E_4 = P1
```

Out[176]:
```
GAP: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a)->1x1, (b)->1x1, (c)->1x1, (d)-:
```

We can rewrite the acyclic complex B as follows:

$$B := \qquad 0 \to P_4 \xrightarrow{\partial^{-1}} E_1 \oplus E_2 \xrightarrow{\text{CokernelProjection}(\partial^{-1})} E_3 \to 0$$

The above acyclic complex says that we can coresolve P_4 in terms of direct sums of E_1, E_2, E_3. That r module over itself can also be coresolved by direct sums of E_1, E_2, E_3 because $\mathbb{A} \cong P_1 \oplus P_2 \oplus P_3 \oplus$

In [177]:
```
T = DirectSum( E_1, E_2, E_3, E_4 )
```

Out[177]:
```
GAP: <(v1)->1, (v2)->3, (v3)->3, (v4)->4; (a)->1x3, (b)->3x4, (c)->1x3, (d)-:
```

In [178]:
```
Show( T )
```

$$\begin{array}{lll} v_1 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 3} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 3} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 4} \\ \hline \end{array}$$

$$a \mapsto \begin{pmatrix} \cdot & \cdot & 1 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} \cdot & \cdot & 1 \end{pmatrix}$$

$$d \mapsto \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \end{pmatrix}$$

In the following we want to prove that $\mathrm{Ext}^n(T,T) = 0$ for all $n \geq 1$.

Since the global dimension of $\mathbb{A}$ is 2, we have $\mathrm{Ext}^n(T,T) = 0$ for all $n \geq 3$. It remains to show that E
$\mathrm{Ext}^2(T,T) = 0$.

It is well known that

$$\mathrm{Ext}^n(T,T) \cong \mathrm{Hom}_{\mathrm{D}^b(\text{mod-}\mathbb{A}_{\mathrm{oid}})}(T, \Sigma^n T)$$

where $\Sigma : \mathrm{D}^b(\text{mod-}\mathbb{A}_{\mathrm{oid}}) \xrightarrow{\sim} \mathrm{D}^b(\text{mod-}\mathbb{A}_{\mathrm{oid}})$ is the shift autoequivalence on $\mathrm{D}^b(\text{mod-}\mathbb{A}_{\mathrm{oid}})$.

In [180]:
```
T = T / Ch_mod_𝔸oid / K_mod_𝔸oid / D_mod_𝔸oid
```

Out[180]:
```
GAP: <An object in Derived^• category( The category of functors: Algebroid(
[ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower l
ive upper bound 0>
```

In [181]:
```
Shift( T, 1 )
```

Out[181]:
```
GAP: <An object in Derived^• category( The category of functors: Algebroid(
[ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower l
tive upper bound -1>
```

In [182]:
```
HomStructure( T, Shift( T, 0 ) )
```

Out[182]:
```
GAP: <A vector space object over Q of dimension 9>
```

In [183]:
```
HomStructure( T, Shift( T, 1 ) )
```

Out[183]:
```
GAP: <A vector space object over Q of dimension 0>
```

In [184]:
```
HomStructure( T, Shift( T, 2 ) )
```

Out[184]:
```
GAP: <A vector space object over Q of dimension 0>
```

To sum up,

- T admits a finite projective resolution,
- $\mathrm{Ext}^n(T,T) \cong 0$ for all $n \geq 1$ and
- $\mathbb{A}$ can be coresolved by direct summands of direct sums of T.

We might also do those computations in $K^b(\mathbb{A}_{\mathrm{oid}}^{\mathrm{op},\oplus})$:

In [185]:
```
𝕌K_𝔻T = 𝕌K_𝔻( T )
```

Out[185]:
```
GAP: <An object in Homotopy^• category( Additive closure( Algebroid( (Q * qu
*(d*c) + 1*(b*a) ] ) ) ) with active lower bound -1 and active upper bound 0
```

In [186]:
```
Show( 𝕌K_𝔻T )
```

$$\begin{array}{c} v_1 \oplus {v_2}^{\oplus 2} \oplus {v_3}^{\oplus 2} \\ \uparrow \\ \begin{pmatrix} 0 & 0 & b & 0 & d \end{pmatrix} \end{array}$$

In [188]:
```
HomStructure( UK DT, Shift( UK DT, 1 ) )
```
Out[188]:
```
GAP: <A vector space object over Q of dimension 0>
```

In [189]:
```
HomStructure( UK DT, Shift( UK DT, 2 ) )
```
Out[189]:
```
GAP: <A vector space object over Q of dimension 0>
```

It turns out that the collection E_1, E_2, E_3, E_4 defines a full strong exceptional collection in $\text{mod-}\mathbb{A}_{\text{oid}}$ a
$T_E = E_1 \oplus E_2 \oplus E_3 \oplus E_4$ is a generalized tilting object in $\text{mod-}\mathbb{A}_{\text{oid}} \cong \text{mod-}\mathbb{A}$. Hence, it induces t
equivalence

$$- \otimes^{\mathbb{L}} T_E : \mathbb{D}^b(\text{mod-End}(T_E)^{\text{op}}) \xrightarrow{\sim} \mathbb{D}^b(\text{mod-}\mathbb{A}) : \mathbb{R}\text{Hom}(T_E, -)$$

where $\text{End}(T_E)$ is the endomorphism algebra of T_E and the multiplication in $\text{End}(T_E)$ is the **precom**
morphisms.

In the following we create this strong exceptional collection. For a better readability, we label each object
by its dimension vector:

In [190]:
```
E = CreateStrongExceptionalCollection( [ E 1, E 2, E 3, E 4 ], [ "[0101]", "
```
Out[190]:
```
GAP: <An exceptional collection defined by the objects of the Full subcatego
4 objects in The category of functors: Algebroid( (Q * quiver) / [ -1*(c*d)
> Category of matrices over Q>
```

7. Compute the endomorphism $\mathbb{Q}$-algebra $\text{End}(T_E)$ as a quiver algebra

The endomorphism $\mathbb{Q}$-algebra of the tilting object $T_E = \bigoplus_1^4 E_i$ is isomorphic to an admissible quiver $\mathbb{Q}$

In [191]:
```
EndT = EndomorphismAlgebra( E )
```
Out[191]:
```
GAP: End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )
```

In [192]:
```
qEndT = QuiverOfAlgebra( EndT )
```
Out[192]:
```
GAP: quiver([0101],[0011],[0111],[1111])[m1_3_1:[0101]->[0111],m2_3_1:[0011]
1:[0111]->[1111]]
```

That is, the quiver of $\text{End}(T_E)$ consists of 4 vertices and 3 arrows:

$$\begin{tikzcd}
{[0101]} \arrow[dr, "m^1_{1,3}"] & & \\
& {[0111]} \arrow[r, "m^1_{3,4}"] & {[1111]} \\
{[0011]} \arrow[ur, "m^1_{2,3}"] & &
\end{tikzcd}$$

The vertices of the quiver are labeled by the strings we assigned to the objects of E and the arrows are
which means that the arrow is the k'th arrow from v_i to v_j.

In [196]:
```
RelationsOfAlgebra( EndT )
```
Out[196]:
```
GAP: [  ]
```

I.e., the endomorphism algebra of T_E is the path $\mathbb{Q}$-algebra of the above quiver. Let us compute its dime

In [197]:
```
Dimension( EndT )
```
Out[197]:
```
9
```

In [198]:
```
IsAdmissibleQuiverAlgebra( EndT )
```
Out[198]:
```
true
```

8. Construct the algeborid $\mathrm{End}(T_E)_{\mathrm{oid}}$, the isomorphism functor $E \cong \mathrm{E}$ and the equivalences

$$K^b(E^{\oplus}) \cong \ K^b(\mathrm{End}(T_E)^{\oplus}_{\mathrm{oid}}) \cong K^b(\mathrm{proj}(\text{mod-}\mathrm{End}(T_E)^{\mathrm{op}}_{\mathrm{oid}})) \cong \mathrm{D}^b(\text{mod-}\mathrm{End}(T_I$$

The algebroid category $\mathrm{End}(T)_{\mathrm{oid}}$ can be considered as an abstraction of the full subcategory of mod $\{E_1, E_2, E_3, E_4\}$. That is, the objects of E (regardless of the complexity of their data strucutres) are n $\mathrm{End}(T)_{\mathrm{oid}}$.

In particular, the two categories are isomorphic. We call the isomorphism functors between them *the abs* and *the realization functor* Rel

$$\mathrm{Abs} : E \xrightarrow{\sim} \mathrm{End}(T_E)_{\mathrm{oid}} : \mathrm{Rel}$$

In [199]:
```
EndT_oid = Algebroid( E )
```
Out[199]:
```
GAP: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] ) )
```

In [200]:
```
Abs = IsomorphismOntoAlgebroid( E )
```
Out[200]:
```
GAP: Isomorphism functor from exceptional collection onto algebroid
```

In [201]:
```
Abs( E[ 1 ] )
```
Out[201]:
```
GAP: <([0101])>
```

In [202]:
```
Rel = IsomorphismFromAlgebroid( E )
```
Out[202]:
```
GAP: Isomorphism functor from algebroid onto exceptional collection
```

In [203]:
```
Rel( EndT_oid."[1111]" )
```
Out[203]:
```
GAP: An object in full subcategory given by: <(v1)->1, (v2)->1, (v3)->1, (v4
(b)->1x1, (c)->1x1, (d)->1x1>
```

In [207]:
```
Show( UnderlyingCell( m ) )
```

$$\begin{array}{ccc} v_1 & \mapsto & ()_{0\times 1} \\ v_2 & \mapsto & (-1) \\ v_3 & \mapsto & (1) \\ v_4 & \mapsto & (1) \end{array}$$

In [208]:
```
m = Rel( EndT oid."m2 3 1" )
```

Out[208]: GAP: A morphism in full subcategory given by: <(v1)->0x0, (v2)->0x1, (v3)->1:

In [209]:
```
Source( m ) == E[ 2 ] && Range( m ) == E[ 3 ] && m == BasisOfPaths( E, 2, 3
```

Out[209]: true

In [210]:
```
Show( UnderlyingCell( m ) )
```

$$\begin{array}{ccc} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & ()_{0\times 1} \\ v_3 & \mapsto & (1) \\ v_4 & \mapsto & (1) \end{array}$$

In [211]:
```
m = Rel( EndT oid."m1 3 1" )
```

Out[211]: GAP: A morphism in full subcategory given by: <(v1)->0x0, (v2)->1x1, (v3)->0:

In [212]:
```
Source( m ) == E[ 1 ] && Range( m ) == E[ 3 ] && m == BasisOfPaths( E, 1, 3
```

Out[212]: true

In [213]:
```
Show( UnderlyingCell( m ) )
```

$$\begin{array}{ccc} v_1 & \mapsto & ()_{0\times 0} \\ v_2 & \mapsto & (-1) \\ v_3 & \mapsto & ()_{0\times 1} \end{array}$$

$$E^{\oplus} \simeq \mathrm{End}(T)^{\oplus}_{\mathrm{oid}} \simeq \mathrm{proj}(\text{mod-}\mathrm{End}(T)^{\mathrm{op}}_{\mathrm{oid}})$$

In [214]:
```
E plus = AdditiveClosure( E )
```

Out[214]:
```
GAP: Additive closure( Full subcategory generated by 4 objects in The catego
Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices o
```

In [215]:
```
EndT oid plus = AdditiveClosure( EndT oid )
```

Out[215]:
```
GAP: Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] ) )
```

In [216]:
```
Abs plus = ExtendFunctorToAdditiveClosures( Abs )
```

Out[216]:
```
GAP: Extension of Isomorphism functor from exceptional collection onto algeb
e closures
```

In [217]:
```
Rel plus = ExtendFunctorToAdditiveClosures( Rel )
```

Out[217]:
```
GAP: Extension of Isomorphism functor from algebroid onto exceptional collec
e closures
```

In [218]:
```
EndT oid op = OppositeAlgebroidOverOppositeQuiverAlgebra( EndT oid )
```

Out[218]:
```
GAP: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op )
```

In [219]:
```
mod EndT oid op = Hom( EndT oid op, ℚ vec )
```

Out[219]:
```
GAP: The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1
Category of matrices over Q
```

In [220]:
```
InfoOfInstalledOperationsOfCategory( mod EndT oid op )
```

```
120 primitive operations were used to derive 312 operations for this categor
* IsLinearCategoryOverCommutativeRing
* IsAbelianCategoryWithEnoughInjectives
* IsAbelianCategoryWithEnoughProjectives
```

In [221]:
```
projs = FullSubcategoryGeneratedByProjectiveObjects( mod EndT oid op )
```

Out[221]:
```
GAP: Full additive subcategory generated by projective objects( The category
lgebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] )^op ) -> Category of matri
```

In [222]:
```
KnownFunctors( EndT oid plus, projs )
```

```
1: Yoneda embedding
```

In [223]:
```
KnownFunctors( projs, EndT oid plus )
```

```
1: Decomposition of projective objects
```

The above isomorphisms can in turn be extended to equivalences of categories:

$$K^b(E^{\oplus}) \cong K^b(\mathrm{End}(T)^{\oplus}_{\mathrm{oid}}) \cong K^b(\mathrm{proj}(\text{mod-}\mathrm{End}(T)^{\mathrm{op}}_{\mathrm{oid}})) \cong D^b(\text{mod-}\mathrm{End}(T))$$

In [224]:
```
K Abs plus = ExtendFunctorToHomotopyCategoriesByCochains( Abs plus )
```

Out[224]:
```
GAP: Extension of a functor to homotopy categories
```

In [225]:
```
Display( K Abs plus )
```

In [226]:
```
K_Rel_plus = ExtendFunctorToHomotopyCategoriesByCochains( Rel_plus )
```
Out[226]: GAP: Extension of a functor to homotopy categories

In [227]:
```
Display( K_Rel_plus )
```

```
Extension of a functor to homotopy categories:

Homotopy^• category( Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [01
) ) )
  |
  V
Homotopy^• category( Additive closure( Full subcategory generated by 4 objec
ory of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Categ
over Q ) )
```

On the other hand, we have a natural embedding functor $K^b(E^{\oplus}) \hookrightarrow K^b(\text{mod-}\mathbb{A}_{\text{oid}})$

In [228]:
```
ζ = InclusionFunctor( DefiningFullSubcategory( E ) );
ζ = ExtendFunctorToAdditiveClosureOfSource( ζ );
ζ = ExtendFunctorToHomotopyCategoriesByCochains( ζ )
```
Out[228]: GAP: Extension of a functor to homotopy categories

In [229]:
```
Display( ζ )
```

```
Extension of a functor to homotopy categories:

Homotopy^• category( Additive closure( Full subcategory generated by 4 objec
ory of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] ) -> Categ
over Q ) )
  |
  V
Homotopy^• category( The category of functors: Algebroid( (Q * quiver) / [ -
b) ] ) -> Category of matrices over Q )
```

In [230]:
```
N = RandomObject( SourceOfFunctor( K_Rel_plus ), julia_to_gap( [ -1, 1, 2 ]
```
Out[230]: GAP: <An object in Homotopy^• category(Additive closure(Algebroid(End([0
[0111] ⊕ [1111])))) with active lower bound -1 and active upper bound 1>

In [231]:
```
Show( N )
```

$$\begin{array}{c}
[0101] \oplus [1111] \\
\uparrow \\
\begin{pmatrix} 0 & 3[1111] \\ 0 & 3m_{3,4}^{1} \end{pmatrix} \\
|_{0} \\
[1111] \oplus [0111] \\
\uparrow \\
\begin{pmatrix} -3m_{1,3}^{1}m_{3,4}^{1} & 3m_{1,3}^{1} \\ -3m_{3,4}^{1} & 3[0111] \end{pmatrix} \\
|_{-1}
\end{array}$$

In [233]:
```
N[-1]
```
Out[233]:
```
GAP: <(v1)->0, (v2)->2, (v3)->1, (v4)->2; (a)->0x2, (b)->2x2, (c)->0x1, (d)-:
```
In [234]:
```
N[0]
```
Out[234]:
```
GAP: <(v1)->1, (v2)->2, (v3)->2, (v4)->2; (a)->1x2, (b)->2x2, (c)->1x2, (d)-:
```
In [235]:
```
N[1]
```
Out[235]:
```
GAP: <(v1)->1, (v2)->2, (v3)->1, (v4)->2; (a)->1x2, (b)->2x2, (c)->1x1, (d)-:
```

9. Construct the adjoint functors

$$- \otimes_{\mathrm{End}(T_E)^{\mathrm{op}}} T_E \ : \mathrm{mod}\text{-}\mathrm{End}(T_E)^{\mathrm{op}}_{\mathrm{oid}} \rightarrow \mathrm{mod}\text{-}\mathbb{A}_{\mathrm{oid}} \ : \mathrm{Hom}(T_E, -)$$

In [236]:
```
mod_𝔸oid
```
Out[236]:
```
GAP: The category of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b
ry of matrices over Q
```
In [237]:
```
mod_EndT_oid_op
```
Out[237]:
```
GAP: The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1
Category of matrices over Q
```
In [238]:
```
Hom_T = HomFunctorToCategoryOfFunctors( E )
```
Out[238]:
```
GAP: Hom(T,-) functor
```
In [239]:
```
Display( Hom_T )
```
```
Hom(T,-) functor:

The category of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] )
matrices over Q
  |
  V
The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111]
ory of matrices over Q
```
In [240]:
```
tensor_T = TensorFunctorFromCategoryOfFunctors( E )
```
Out[240]:
```
GAP: - ⊗_{(End T)^op} T functor
```
In [241]:
```
Display( tensor_T )
```
```
- ⊗_{(End T)^op} T functor:

The category of functors: Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111]
ory of matrices over Q
  |
  V
The category of functors: Algebroid( (Q * quiver) / [ -1*(c*d) + 1*(a*b) ] )
matrices over Q
```
In [242]:
```
ϵ = CounitOfTensorHomAdjunction( E, tensor_T, Hom_T )
```
Out[242]:
```
GAP: Hom(T,-) ⊗_{End T} T --> Id
```

In [245]:
```
tensor_T_Hom_T_F = tensor_T( Hom_T( F ) )
```
Out[245]: GAP: <(v1)->4, (v2)->2, (v3)->1, (v4)->1; (a)->4x2, (b)->2x1, (c)->4x1, (d)-:

In [246]:
```
Show( tensor_T_Hom_T_F )
```

$$\begin{array}{ccc} v_1 & \mapsto & \mathbb{Q}^{1\times 4} \\ v_2 & \mapsto & \mathbb{Q}^{1\times 2} \\ v_3 & \mapsto & \mathbb{Q}^{1\times 1} \\ v_4 & \mapsto & \mathbb{Q}^{1\times 1} \\ \hline \end{array}$$

$$a \mapsto \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \\ 1 & \cdot \\ \cdot & 1 \end{pmatrix}$$

$$b \mapsto \begin{pmatrix} \cdot \\ 1 \end{pmatrix}$$

$$c \mapsto \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}$$

$$d \mapsto \begin{pmatrix} 1 \end{pmatrix}$$

In [247]:
```
ϵ_F = ϵ( F )
```
Out[247]: GAP: <(v1)->4x4, (v2)->2x2, (v3)->1x1, (v4)->1x2>

In [248]:
```
Source( ϵ_F ) = tensor_T_Hom_T_F && Range( ϵ_F ) = F
```
Out[248]: true

In [249]:
```
Show( ϵ_F )
```

$$v_1 \mapsto \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & \cdot \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} \cdot & 1 \\ 1 & \cdot \end{pmatrix}$$

$$\begin{tikzcd}
\mathrm{Hom}(T_E, F) \otimes T_E \arrow[r, "\mathrm{Hom}(T_E,\psi)\otimes T_E"] \arrow[d, "\epsilon(F)"'] & \mathrm{Hom}(T_E, G) \otimes T_E \arrow[d, "\epsilon(G)"] \\
F \arrow[r, "\psi"'] & G
\end{tikzcd}$$

In [250]:
```
PreCompose( ϵ( F ), ψ ) == PreCompose( tensor T( Hom T( ψ ) ), ϵ( G ) )
```

Out[250]:
```
true
```

In [251]:
```
Hom_T = ExtendFunctorToHomotopyCategoriesByCochains( Hom_T )
tensor_T = ExtendFunctorToHomotopyCategoriesByCochains( tensor_T )
ϵ = ExtendNaturalTransformationToHomotopyCategories( ϵ, true )
```

Out[251]:
```
GAP: Extention of natural transformation ( Hom(T,-) ⊗_{End T} T --> Id ) : E
unctor to homotopy categories ===> Extension of a functor to homotopy catego
```

In [252]:
```
Display( Hom T )
```

```
Extension of a functor to homotopy categories:

Homotopy^• category( The category of functors: Algebroid( (Q * quiver) / [ -
b) ] ) -> Category of matrices over Q )
  |
  V
Homotopy^• category( The category of functors: Algebroid( End( [0101] ⊕ [001
[1111] )^op ) -> Category of matrices over Q )
```

In [253]:
```
Display( tensor T )
```

```
Extension of a functor to homotopy categories:

Homotopy^• category( The category of functors: Algebroid( End( [0101] ⊕ [001
[1111] )^op ) -> Category of matrices over Q )
  |
  V
Homotopy^• category( The category of functors: Algebroid( (Q * quiver) / [ -
b) ] ) -> Category of matrices over Q )
```

In [254]:
```
K = P4 / Ch mod 𝔸oid / K mod 𝔸oid
```

Out[254]:
```
GAP: <An object in Homotopy^• category( The category of functors: Algebroid(
[ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower l
ive upper bound 0>
```

In [255]:
```
Inj K = InjectiveResolution( K, true )
```

Out[255]:
```
GAP: <An object in Homotopy^• category( The category of functors: Algebroid(
[ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower l
ive upper bound 2>
```

In [256]:
```
Inj K[0]
```

Out[256]:
```
GAP: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a)->1x1, (b)->1x1, (c)->1x1, (d)-:
```

In [257]:
```
Inj K[1]
```

Out[257]:
```
GAP: <(v1)->2, (v2)->1, (v3)->1, (v4)->0; (a)->2x1, (b)->1x0, (c)->2x1, (d)-:
```

In [258]:
```
Inj K[2]
```

Out[258]:
```
GAP: <(v1)->1, (v2)->0, (v3)->0, (v4)->0; (a)->1x0, (b)->0x0, (c)->1x0, (d)-:
```

```
GAP: <An object in Homotopy^• category( The category of functors: Algebroid(
```

```
In [261]: q_Proj_Hom_T_Inj_K = QuasiIsomorphismFromProjectiveResolution( Hom_T_Inj_K,

Out[261]: GAP: <A morphism in Homotopy^• category( The category of functors: Algebroid
          [0011] ⊕ [0111] ⊕ [1111] )^op ) -> Category of matrices over Q ) with active
          and active upper bound 2>

In [262]: IsWellDefined( q_Proj_Hom_T_Inj_K ) & IsQuasiIsomorphism( q_Proj_Hom_T_Inj_K

Out[262]: true

In [263]: tensor_T_Proj_Hom_T_Inj_K = tensor_T( ProjectiveResolution( Hom_T( Inj_K ),

Out[263]: GAP: <An object in Homotopy^• category( The category of functors: Algebroid(
          [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lower 
          ive upper bound 2>

In [264]: tensor_T_Proj_Hom_T_Inj_K[0]

Out[264]: GAP: <(v1)->1, (v2)->2, (v3)->2, (v4)->3; (a)->1x2, (b)->2x3, (c)->1x2, (d)-

In [265]: tensor_T_Proj_Hom_T_Inj_K[1]

Out[265]: GAP: <(v1)->2, (v2)->3, (v3)->3, (v4)->3; (a)->2x3, (b)->3x3, (c)->2x3, (d)-

In [266]: tensor_T_Proj_Hom_T_Inj_K[2]

Out[266]: GAP: <(v1)->1, (v2)->1, (v3)->1, (v4)->1; (a)->1x1, (b)->1x1, (c)->1x1, (d)-

In [267]: ι_Inj_K = PreCompose( tensor_T( q_Proj_Hom_T_Inj_K ), ϵ( Inj_K ) )

Out[267]: GAP: <A morphism in Homotopy^• category( The category of functors: Algebroid
          / [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lowe
          ctive upper bound 2>

In [268]: ( Source( ι_Inj_K ) == tensor_T_Proj_Hom_T_Inj_K ) & ( Range( ι_Inj_K ) == I

Out[268]: true

In [269]: IsWellDefined( ι_Inj_K )

Out[269]: true

In [270]: IsQuasiIsomorphism( ι_Inj_K )

Out[270]: true

In [271]: EndT_oid_plus

Out[271]: GAP: Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [0111] ⊕ [1111] ) )

In [272]: K_EndT_oid_plus = HomotopyCategoryByCochains( EndT_oid_plus )

Out[272]: GAP: Homotopy^• category( Additive closure( Algebroid( End( [0101] ⊕ [0011] 
          1] ) ) ) )

In [273]: K_mod_EndT_oid_op = HomotopyCategoryByCochains( mod_EndT_oid_op )

Out[273]: GAP: Homotopy^• category( The category of functors: Algebroid( End( [0101] ⊕
          1] ⊕ [1111] )^op ) -> Category of matrices over Q )
```

```
In [276]: KnownFunctors( K projs mod EndT op, K EndT oid plus )
```

```
1: Apply ExtendFunctorToHomotopyCategoriesByCochains on ( Decomposition of p
ts )
```

```
In [277]: 𝔻 = Functor( K projs mod EndT op, K EndT oid plus, 1 )
```

```
Out[277]: GAP: Extension of a functor to homotopy categories
```

```
In [278]: ℝ = PreCompose( [ Hom T, 𝕃p, 𝔻 ] )
```

```
Out[278]: GAP: Composition of Composition of Extension of a functor to homotopy catego
zation functor by projective objects and Extension of a functor to homotopy
```

```
In [279]: Display( ℝ )
```

```
Composition of Composition of Extension of a functor to homotopy categories
n functor by projective objects and Extension of a functor to homotopy categ

Homotopy^• category( The category of functors: Algebroid( (Q * quiver) / [ -
b) ] ) -> Category of matrices over Q )
  |
  V
Homotopy^• category( Additive closure( Algebroid( End( [0101] ⊕ [0011] ⊕ [01
) ) )
```

```
In [280]: ℝ Inj K = ℝ( Inj K )
```

```
Out[280]: GAP: <An object in Homotopy^• category( Additive closure( Algebroid( End( [0
[0111] ⊕ [1111] ) ) ) ) with active lower bound 0 and active upper bound 2>
```

```
In [281]: Show( ℝ Inj K )
```

$$
\begin{array}{c}
[1111] \\
\uparrow \\
\begin{pmatrix} -m^1_{3,4} \\ -[1111] \\ [1111] \end{pmatrix} \\
|_1 \\
[0111] \oplus [1111]^{\oplus 2} \\
\uparrow \\
\begin{pmatrix} -m^1_{1,3} & 0 & -m^1_{1,3}m^1_{3,4} \\ m^1_{2,3} & -m^1_{2,3}m^1_{3,4} & 0 \\ 0 & [1111] & [1111] \end{pmatrix} \\
|_0 \\
[0101] \oplus [0011] \oplus [1111]
\end{array}
$$

```
In [282]: s = SimplifyObject( ℝ Inj K, infinity )
```

```
Out[282]: GAP: <An object in Homotopy^• category( Additive closure( Algebroid( End( [0
[0111] ⊕ [1111] ) ) ) ) with active lower bound 0 and active upper bound 2>
```

In [283]: `Show( s )`

$$
\begin{array}{c}
0 \\
\uparrow \\
() \\
|_1 \\
[0111] \\
\uparrow \\
\begin{pmatrix} m^1_{1,3} \\ -m^1_{2,3} \end{pmatrix} \\
|_0 \\
[0101] \oplus [0011]
\end{array}
$$

In [284]: `i = SimplifyObject IsoToInputObject( ℝ Inj K, infinity )`

Out[284]: GAP: <A morphism in Homotopy^• category(Additive closure(Algebroid(End([
⊕ [0111] ⊕ [1111])))) with active lower bound 0 and active upper bound 2

In [285]: `Show( i )`

$$
\begin{array}{ccc}
0 & - \quad 0 \quad \rightarrow & [1111 \\
\uparrow & & \uparrow \\
() & & \begin{pmatrix} -m^1_{3,} \\ -[111 \\ [1111 \end{pmatrix} \\
|_1 & & |_1 \\
[0111] & - \quad \begin{pmatrix} [0111] & -m^1_{3,4} & 0 \end{pmatrix} \quad \rightarrow & [0111] \oplus [1 \\
\uparrow & & \uparrow \\
\begin{pmatrix} m^1_{1,3} \\ -m^1_{2,3} \end{pmatrix} & & \begin{pmatrix} -m^1_{1,3} & 0 \\ m^1_{2,3} & -m^1_{2,3}m^1_{3,} \\ 0 & [1111] \end{pmatrix} \\
|_0 & & |_0 \\
[0101] \oplus [0011] & - \quad \begin{pmatrix} -[0101] & 0 & -m^1_{1,3}m^1_{3,4} \\ 0 & -[0011] & 0 \end{pmatrix} \quad \rightarrow & [0101] \oplus [0011
\end{array}
$$

In [286]: `j = InverseForMorphisms( i )`

Out[286]: GAP: <A morphism in Homotopy^• category(Additive closure(Algebroid(End([
⊕ [0111] ⊕ [1111])))) with active lower bound 0 and active upper bound 2

In [287]:
```
Show( i )
```

$$
\begin{array}{ccccc}
[1111] & \quad & - & () & \to \\
\uparrow & & & & \\
\begin{pmatrix} -m^{1}_{3,4} \\ -[1111] \\ [1111] \end{pmatrix} & & & & \\
|_{1} & & & & \\
[0111] \oplus [1111]^{\oplus 2} & & - & \begin{pmatrix} [0111] \\ 0 \\ 0 \end{pmatrix} & \to \\
\uparrow & & & & \\
\begin{pmatrix} -m^{1}_{1,3} & 0 & -m^{1}_{1,3} m^{1}_{3,4} \\ m^{1}_{2,3} & -m^{1}_{2,3} m^{1}_{3,4} & 0 \\ 0 & [1111] & [1111] \end{pmatrix} & & & & \Big(- \\
|_{0} & & & & \\
[0101] \oplus [0011] \oplus [1111] & & - & \begin{pmatrix} -[0101] & 0 \\ 0 & -[0011] \\ 0 & 0 \end{pmatrix} & \to \quad [010
\end{array}
$$

In [288]:
```
ζ K Rel plus i = ζ( K Rel plus( i ) )
```

Out[288]:
```
GAP: <A morphism in Homotopy^• category( The category of functors: Algebroid
/ [ -1*(c*d) + 1*(a*b) ] ) -> Category of matrices over Q ) with active lowe
ctive upper bound 2>
```

In [289]:
```
IsIsomorphism( ζ K Rel plus i )
```

Out[289]:
```
true
```

In [290]:
```
Range( ζ K Rel plus i ) == tensor T Proj Hom T Inj K
```

Out[290]:
```
true
```

In [ ]:
```

```