

THE EDGECONFLICT PREDICATE IN THE 3D APOLLONIUS DIAGRAM

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ABSTRACT. In this paper we study one of the fundamental predicates required for the construction of the 3D Apollonius diagram (also known as the 3D Additively Weighted Voronoi diagram), namely the `EDGECONFLICT` predicate: given five sites S_i, S_j, S_k, S_l, S_m that define an edge e_{ijklm} in the 3D Apollonius diagram, and a sixth query site S_q , the predicate determines the portion of e_{ijklm} that will disappear in the Apollonius diagram of the six sites due to the insertion of S_q .

Our focus is on the algorithmic analysis of the predicate with the aim to minimize its algebraic degree. We decompose the main predicate into three sub-predicates, which are then evaluated with the aid of four additional primitive operations. We show that the maximum algebraic degree required to answer any of the sub-predicates and primitives, and, thus, our main predicate is 10.

Among the tools we use is the 3D inversion transformation. In the scope of this paper and due to space limitations, only non-degenerate configurations are considered, i.e. different Voronoi vertices are distinct and the predicates never return a “degenerate” answer. Most of our analysis is carried out in the inverted space, which is where our geometric observations and analysis is captured in algebraic terms.

1. INTRODUCTION

Voronoi diagrams have been among the most studied constructions in computational geometry since their inception [14, 9, 4, 15], due to their numerous applications, including motion planning and collision detection, communication networks, graphics, and growth of microorganisms in biology.

Date: November 15, 2018.

2010 *Mathematics Subject Classification.* Primary 68U05, 65D99; Secondary 68W30, 68Q25.

Key words and phrases. computational geometry, algebraic computing, geometric predicates, Euclidean Apollonius diagram, EdgeConflict predicate.

This work is part of author’s Ph.D. thesis and was supported by Onassis Foundation and Project Thales. A simpler version of this paper was accepted at *The Sixth International Conference on Analytic Number Theory and Spatial Tessellations* held in Kyiv, Ukraine at September 2018. This work was supported by Onassis Foundation and Project Thales.

Despite being a central topic in research for many years, generalized Voronoi diagrams, and especially the Voronoi diagram of spheres (also known as the 3D Apollonius diagram) have not been explored sufficiently [23]; this is also pointed out by Aurenhammer et al. [5]. Moreover, due to recent scientific discoveries in biology and chemistry, 3D Apollonius diagrams are becoming increasingly important for representing and analysing the molecular 3D structure and surface [18] or the structure of the protein [22].

The methods used to calculate the Apollonius diagram usually rely on the construction of a different diagram altogether. Some methods include the intersection of cones [3] with the lifted power diagram and lower envelope calculations [28, 27, 16]. Boissonnat et al. use the convex hull to describe its construction [8, 7]. Aurenhammer’s lifting method has also been implemented for two dimensions [1]. Karavelas and Yvinec [19] create the 2d Apollonius diagram from its dual, using the predicates developed in [12]. In [12], it is also reported that the Apollonius diagram can be obtained as a concrete case of the abstract Voronoi diagrams of Klein et al. [24].

Kim et al. made a major research contribution in the domain of the Voronoi diagrams of spheres including one patent [21] for the computation of 3D Voronoi diagrams. Their work provides many new algorithms related to the Voronoi diagrams including the computation of three-dimensional Voronoi diagrams [21], Euclidean Voronoi diagram of 3D balls and its computation via tracing edges [20] and the Euclidean Voronoi diagrams of 3D spheres and applications to protein structure analysis [22].

Hanniel and Elber [16] provide an algorithm for computation of the Voronoi diagrams for planes, spheres and cylinders in \mathbb{R}^3 . Their algorithm relies on computing the lower envelope of the bisector surfaces similar to the algorithm of Will [27]. However, none of the current research efforts provide the exact method for computing the Apollonius diagram (or its dual Delaunay graph) of spheres.

In this paper, we are inspired by the the approach presented by Emiris and Karavelas in [12] for the evaluation of the 2D Apollonius diagram. In order to extend their work towards an algorithm that would incrementally construct the Apollonius diagram for 3D spheres, we develop equivalent predicates as the ones presented in their paper for the 2D case. Our main goal is to implement the most degree demanding predicate, called the `EDGECONFLICT` predicate: given five sites S_i, S_j, S_k, S_l, S_m that define a finite edge e_{ijklm} in the 3D Apollonius diagram, and a sixth query site S_q , the predicate determines the portion of e_{ijklm} that will disappear in the Apollonius diagram of the six sites due to the insertion of S_q .

In order to accomplish this task, we developed various subpredicates and primitives. The creation of these tools was made taking into consideration the modern shift of predicate design towards lower level algorithmic issues. Specifically, a critical factor that influenced our designs was our goal to minimize the algebraic degree of the tested quantities (in terms of the input parameters) during a predicate evaluation. Such a minimization problem has become a main concern that influences algorithm design especially in geometric predicates, where zero tolerance in all intermediate computations is needed to obtain an accurate result [10, 13, 6, 26, 29].

Our main contribution in the research area is the development of a list of subpredicates that were not implemented, either explicitly or implicitly, in the current bibliography and can be used within the scope of an incremental algorithm that constructs the 3D Apollonius diagram of a set of spheres. Our most outstanding result is the fact that all subpredicates presented in this paper along with the EDGECONFLICT predicate require at most 10-fold degree demanding operation (with respect to the input quantities). This is quite an unexpected result since the equivalent EDGECONFLICT predicate in the 2D Apollonius diagram required 16-fold operations [12, 24]. Our approach of resolving the “master” predicate and especially the observations made in the inverted plane, could also be implemented in the 2D case to yield lower algebraic degrees.

This paper is organised as follows. In Section 2, we review the preliminaries of the Apollonius diagram of 3D spheres and the orientation of hyperbolic trisectors in such a diagram. An introduction to the inversion technique is also made along with useful remarks regarding the correlation between the original and the inverted space. In Section 3, we present in detail the EDGECONFLICT predicate along with the assumptions made in the scope of this paper. An outline of the subpredicates developed is then provided along with the geometric properties that derive from each one. Finally, we provide the main algorithm that ultimately combines all the aforementioned tools to answer the EDGECONFLICT predicate. Section 4 is devoted to the implementation and algebraic analysis for each subpredicate. Finally, in Section 5, we conclude the paper.

2. PRELIMINARIES AND DEFINITIONS

Let \mathcal{S} be a set of closed spheres S_n (also referred as *sites*) in \mathbb{E}^3 , with centers $C_n = (x_n, y_n, z_n)$ and radii r_n . In this paper, we will assume that no one of these sites is contained inside another. Define the Euclidean distance $\delta(p, S)$ between a point $p \in \mathbb{E}^3$ and a sphere $S = \{C, r\}$ as $\delta(p, S) = \|p - C\| - r$, where $\|\cdot\|$ stands for the Euclidean norm. The *Apollonius diagram* is then

defined as the subdivision of the plane induced by assigning each point $p \in \mathbb{E}^3$ to its nearest neighbor with respect to the distance function $\delta(\cdot, \cdot)$.

For each $i \neq j$, let $H_{ij} = \{y \in \mathbb{E}^3 : \delta(y, S_i) \leq \delta(y, S_j)\}$. Then the (closed) *Apollonius cell* V_i of S_i is defined to be $V_i = \cap_{i \neq j} H_{ij}$. The set of points that belong to exactly two Apollonius cells are called the *Apollonius faces*, whereas the connected set of points that belong to exactly three Apollonius cells are called *Apollonius edges*. Points that belong to more than three Apollonius cells are called *Apollonius vertices*; the *Apollonius diagram* $\mathcal{VD}(S)$ of S is defined as the collection of the Apollonius cells, faces, edges and vertices.

An Apollonius vertex v is a point that belongs to 4 or more Apollonius cells. Without loss of generality we may assume that an Apollonius vertex is tangent to exactly 4 sites S_i, S_j, S_k, S_l , since otherwise we may apply a perturbation scheme to resolve the degeneracy. The sphere centered at v and tangent to S_i, S_j, S_k, S_l is either externally tangent to them, or is contained inside all 4 sites; in the former case it is called an *external Apollonius sphere*, while in the latter an *internal Apollonius sphere*. Let T_n , $n = i, j, k, l$, be the point of tangency of the Apollonius sphere and S_n . The tetrahedron $T_i T_j T_k T_l$ can either be positively or negatively oriented, or even flat [11]. We assume below that $T_i T_j T_k T_l$ is not flat; otherwise we may employ the perturbation scheme described in [11] and, thus, consider it as non-flat. The Apollonius vertex corresponding to a positively (resp., negatively) oriented tetrahedron $T_i T_j T_k T_l$ will be denoted v_{ijkl} (resp., v_{ikjl}). Observe that a cyclic permutation of the indices does not affect our choice of Apollonius vertex.

The *trisector* τ_{ijk} of three different sites S_i, S_j and S_k is the locus of points that are equidistant from the three sites. In the absence of degeneracies its Hausdorff dimension is 1, and it is either (a branch of) a hyperbola, a line, an ellipse, a circle, or a parabola [28]; in this paper, due to space limitations, we focus on the cases where the trisector is an open curve, and more specifically hyperbolic or linear. An *Apollonius edge* e_{ijklm} is a connected subset of the trisector τ_{ijk} of three different sites and is defined by five sites S_i, S_j, S_k, S_l and S_m . The first three sites define the supporting trisector τ_{ijk} of the edge, whereas the last two define its endpoints v_{ijkl} and v_{ikjm} .

2.1. Inversion. The 3-dimensional *inversion transformation* is a mapping from \mathbb{R}^3 to \mathbb{R}^3 that maps a point $z \in \mathbb{R}^3$ to the point $W(z) = (z - z_0) / \|z - z_0\|^2$. The point z_0 is called the *pole of inversion*. Inversion maps spheres that do not pass through the pole to spheres, and spheres that pass through the pole to planes.

In the Apollonius diagram context we call \mathcal{Z} -space the space where the sites live. Since the Apollonius diagram does not change when we add to the radii of all spheres the same quantity, we will, most of the times, reduce the

radii of the spheres S_i, S_j, S_k, S_l, S_m by the radius of one of them, the sphere S_l . The new spheres have obviously the same centers, whereas their radii become spheres become $r_n^\star = r_n - r_l$, $n \in \{i, j, k, l, m\}$. For convenience, we call the image space of this radius-reducing transformation the \mathcal{Z}^\star -space. We may then apply inversion, with C_l as the pole, to get a new set of spheres or planes; we call \mathcal{W} -space the space where the radius-reduced, inverted sites live.

Since the sites S_i, S_j, S_k, S_l, S_m are not contained inside each is tangent to another, the image of the sphere S_n in \mathcal{W} -space is a sphere S_n^\star , centered at $C_n^\star = (u_n, v_n, w_n)$ with radius ρ_n , where

$$u_n = \frac{x_n^\star}{p_n^\star}, \quad v_n = \frac{y_n^\star}{p_n^\star}, \quad w_n = \frac{z_n^\star}{p_n^\star}, \quad \rho_n = \frac{r_n^\star}{p_n^\star},$$

$$x_n^\star = x_n - x_l, \quad y_n^\star = y_n - y_l, \quad z_n^\star = z_n - z_l, \quad r_n^\star = r_n - r_l,$$

and $p_n^\star = (x_n^\star)^2 + (y_n^\star)^2 + (z_n^\star)^2 - (r_n^\star)^2$. Note that p_n^\star is positive due to the non-inclusion assumption. We also define the quantities

$$D_{ijk}^{\pi\theta} = \begin{vmatrix} \pi_i & \theta_i & 1 \\ \pi_j & \theta_j & 1 \\ \pi_k & \theta_k & 1 \end{vmatrix}, \quad D_{ijk}^{\pi\theta\eta} = \begin{vmatrix} \pi_i & \theta_i & \eta_i \\ \pi_j & \theta_j & \eta_j \\ \pi_k & \theta_k & \eta_k \end{vmatrix}, \quad D_{ijkl}^{\pi\theta\eta} = \begin{vmatrix} \pi_i & \theta_i & \eta_i & 1 \\ \pi_j & \theta_j & \eta_j & 1 \\ \pi_k & \theta_k & \eta_k & 1 \\ \pi_l & \theta_l & \eta_l & 1 \end{vmatrix},$$

$$D_{ijkl}^{\pi\theta\eta\zeta} = \begin{vmatrix} \pi_i & \theta_i & \eta_i & \zeta_i \\ \pi_j & \theta_j & \eta_j & \zeta_j \\ \pi_k & \theta_k & \eta_k & \zeta_k \\ \pi_l & \theta_l & \eta_l & \zeta_l \end{vmatrix}, \quad E_{ijk}^{\pi\theta\eta} = \begin{vmatrix} \pi_i^\star & \theta_i^\star & \eta_i^\star \\ \pi_j^\star & \theta_j^\star & \eta_j^\star \\ \pi_k^\star & \theta_k^\star & \eta_k^\star \end{vmatrix}, \quad E_{ijkl}^{\pi\theta\eta\zeta} = \begin{vmatrix} \pi_i^\star & \theta_i^\star & \eta_i^\star & \zeta_i^\star \\ \pi_j^\star & \theta_j^\star & \eta_j^\star & \zeta_j^\star \\ \pi_k^\star & \theta_k^\star & \eta_k^\star & \zeta_k^\star \\ \pi_l^\star & \theta_l^\star & \eta_l^\star & \zeta_l^\star \end{vmatrix},$$

for $\pi, \theta, \eta, \zeta \in \{x, y, z, r, u, v, w, \rho\}$, and it holds that

$$D_{ijk}^{\pi\theta} = E_{ijk}^{klp} (p_i^\star p_j^\star p_k^\star)^{-1}, \quad D_{ijk}^{\pi\theta\eta} = E_{ijk}^{klm} (p_i^\star p_j^\star p_k^\star)^{-1}$$

for $\pi, \theta \in \{u, v, w, \rho\}$ and $k, l, m \in \{x, y, z, r\}$.

2.2. Orientation of a hyperbolic or linear trisector. Under the assumption that the trisector τ_{ijk} of the sites S_i, S_j, S_k is a line or a hyperbola, the three centers C_i, C_j, C_k cannot be collinear [28]. A natural way of orienting τ_{ijk} is accomplished via the well-known “right-hand rule”; if we fold our right hand to follow the centers C_i, C_j and C_k (in that order), our thumb will be showing the positive “end” of τ_{ijk} (see Figure 1).

By orienting τ_{ijk} , we clearly define an ordering on the points of τ_{ijk} , which we denote by \prec . Let o_{ijk} be the intersection of τ_{ijk} and the plane Π_{ijk} going through the centers C_i, C_j and C_k . We can now parametrize τ_{ijk} as follows: if $o_{ijk} \prec p$ then $\zeta(p) = \delta(p, S_i) - \delta(o_{ijk}, S_i)$; otherwise $\zeta(p) = -(\delta(p, S_i) - \delta(o_{ijk}, S_i))$. The function $\zeta(\cdot)$ is a 1-1 and onto mapping from

τ_{ijk} to \mathbb{R} . Moreover, we define $\zeta(S)$, where S is an external tangent sphere to the sites S_i, S_j and S_k , to be $\zeta(c)$, where $c \in \tau_{ijk}$ is the center of S .

We also use τ_{ijk}^+ (resp. τ_{ijk}^-) to denote the positive (resp. negative) semi-trisector i.e., the set of points $p \in \tau_{ijk}$ such that $o_{ijk} \prec p$ (resp. $p \prec o_{ijk}$).

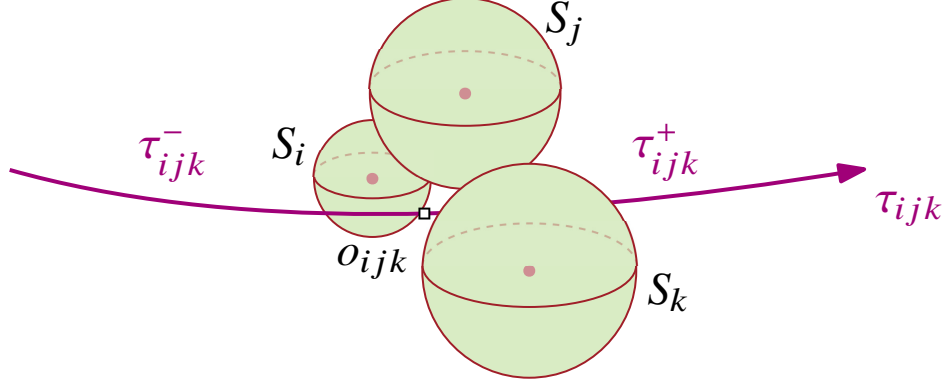


FIGURE 1. The case where the trisector τ_{ijk} of the spheres S_i, S_j and S_k is *hyperbolic*. Notice the orientation of τ_{ijk} based on the “right-hand rule”. The point o_{ijk} of the trisector, which is coplanar with the centers C_i, C_j and C_k , separates the τ_{ijk} into two semi-trisectors, τ_{ijk}^- and τ_{ijk}^+ .

3. NON-DEGENERATE CASE ANALYSIS FOR HYPERBOLIC TRISECTORS

3.1. Voronoi Edges on Hyperbolic Trisectors. In order to better understand our initial problem, more insight regarding the properties of a Voronoi diagram is required. Let us look closer at an edge e_{ijklm} (we drop the subscript for convenience) of $\mathcal{V}(\mathcal{S})$, where \mathcal{S} is a set of given sites that includes S_n for $n \in \{i, j, k, l, m\}$ and does not include S_q . This edge e lies on the trisector τ_{ijk} , the locus of points that are equidistant to the sites S_i, S_j and S_k . In the scope of this paper, we assume that τ_{ijk} is of Hausdorff dimension 1 and is either (a branch of) a hyperbola or a line. To ensure that the spheres S_i, S_j and S_k meet this criteria, the predicate $\text{TRISECTOR}(S_i, S_j, S_k)$, described in Section 3.3.2 and analyzed in Section 4.2, must return “hyperbolic”.

We now focus on the edge e , i.e. the open continuous subset of τ_{ijk} whose closure is bounded by the Voronoi vertices v_{ijkl} and v_{ikjm} . If $\mathcal{T}(t)$ denotes the external Apollonius sphere of the sites S_i, S_j and S_k that is centered at t , the most crucial property of a point $t \in e$ is that $\mathcal{T}(t)$ does not intersect with any other site of \mathcal{S} . We call this *the Empty Sphere Principle* since it is a property that derives from the empty circle principle of a generic Voronoi diagram and its basic properties.

Using this property, we can show that the left and right endpoint of e are the Apollonius vertices v_{ijkl} and v_{ikjm} respectively. To prove that the left endpoint is indeed v_{ijkl} and not v_{ikjl} , consider a point $t \in \tau_{ijk}$ such that $t \equiv v_{ikjl}$ and move it infinitesimally on the trisector towards its positive direction. The initial Apollonius sphere $\mathcal{T}(v_{ikjl})$ was tangent to S_l , and assuming v_{ikjl} was the left endpoint of e , the sphere $\mathcal{T}(t)$ should not longer be tangent nor intersect S_l since $t \in e$ and the Empty Sphere Principle must hold. However, due to the negative orientation of the tetrahedron $T_i T_j T_k T_l$, where T_n is the tangency point of the sphere S_n and $\mathcal{T}(v_{ikjl})$ for $n = i, j, k, l$, and the orientation of τ_{ijk} , the sphere $\mathcal{T}(t)$ contains T_l and therefore intersects S_l , yielding a contradiction. Therefore, we have proven that the left point of e is necessarily v_{ijkl} and we can prove that the right endpoint is v_{ikjm} (and not v_{ijkm}) in a similar way.

3.2. Problem Outline and Assertions. For clarity reasons, we restate the EDGECONFLICT predicate, highlighting its input, output as well as the assertions we are making for the rest of this paper.

The EDGECONFLICT predicate, one of the fundamental predicates required for the construction of the 3D Apollonius diagram (also known as the 3D Additively Weighted Voronoi diagram), takes as input five sites S_i, S_j, S_k, S_l and S_m that define an edge e_{ijklm} in the 3D Apollonius diagram as well as a sixth query site S_q . The predicate determines the portion of e_{ijklm} (we drop the subscript for convenience) that will disappear in the Apollonius diagram

of the six sites due to the insertion of S_q and therefore its output is one of the following

- **NoConflict**: no portion of e is destroyed by the insertion of S_q in the Apollonius diagram of the five sites.
- **EntireEdge**: the entire edge e is destroyed by the addition of S_q in the Apollonius diagram of the five sites.
- **LeftVertex**: a subsegment of e adjacent to its origin vertex (v_{ijkl}) disappears in the Apollonius diagram of the six sites.
- **RightVertex**: is the symmetric case of the **LeftVertex** case; a subsegment of e adjacent to the vertex v_{ikjm} disappears in the Apollonius diagram of the six sites.
- **BothVertices**: subsegments of e adjacent to its two vertices disappear in the Apollonius diagram of the five sites.
- **Interior**: a subsegment in the interior of e disappears in the Apollonius diagram of the five sites.

In Section 3.4, we prove that these are indeed the only possible answers to the studied predicate, under the assumption that no degeneracies occur. Specifically, all analysis presented in this paper is done under the following two major assertions:

- The trisector τ_{ijk} of the sites S_i, S_j and S_k is “hyperbolic” i.e., it is either a branch of a hyperbola or a straight line. Therefore, the spheres must lie in *convex position*; in other words, there must exist two distinct planes commonly tangent to all three spheres.
- None of the subpredicates called during the algorithm presented in Section 3.4 returns a degenerate answer. Mainly, this is equivalent to the statement: *All of the existing Apollonius vertices defined by the sites S_i, S_j, S_k and S_n , for $n \in \{l, m, q\}$, are distinct and the respective Apollonius spheres are all finite i.e., they are not centered at infinity.* Such assertion dictates that the edge e is finite as none of its bounding vertices v_{ijkl} and v_{ikjm} can lie at infinity.

3.3. SubPredicates and Primitives. In this section, we describe the various subpredicates used throughout the evaluation of the **EDGECONFLICT** predicate via the main algorithm presented in Section 3.4. For convenience, only the input, output and specific geometric observations is provided in this section, whereas a detailed implementation along with a algebraic degree analysis of each subpredicate is found in Section 4.

3.3.1. The *INSPHERE* predicate. The $\text{INSPHERE}(S_i, S_j, S_k, S_a, S_b)$ predicate returns $-$, $+$ or 0 if and only if the sphere S_b intersects, does not intersect or is tangent to the external Apollonius sphere of the sites S_i, S_j, S_k and S_b , centered

at v_{ijka} . It is assumed that v_{ijka} exists and none of the first four input sites are contained inside one another. In [17], it is shown that the evaluation of the InSPHERE predicate requires operations of maximum algebraic degree 10, whereas in [2] an implicit InSPHERE predicate could be evaluated via the Delaunay graph, using 6-fold degree operations (although it is not clear if we could easily distinguish if we are testing against the Apollonius sphere centered at v_{ijka} or v_{ikja}).

Since degenerate configurations are beyond the scope of this paper, the InSPHERE tests evaluated during the main algorithm (see Section 3.4) will always return $+$ or $-$. We should also remark that, in bibliography, the InSPHERE predicate is also referred to as the VertexConflict predicate to reflect the fact that a negative (resp. positive) outcome of $\text{InSPHERE}(S_i, S_j, S_k, S_a, S_b)$ amounts to the Apollonius vertex v_{ijka} in $\mathcal{VD}(\Sigma)$ vanishing (resp. remaining) in $\mathcal{VD}(\Sigma \cup \{S_b\})$, where Σ contains S_i, S_j, S_k and S_a but not S_b .

Lemma 1. *The InSPHERE predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

3.3.2. The InCONE and TriSector predicates. Given three spheres S_a, S_b and S_c , such that S_a and S_b are not contained one inside the other, we want to determine the relative geometric position of S_c with respect to the uniquely defined closed semi-cone $\mathcal{K}(S_a, S_b)$ that is tangent to both S_a and S_b and includes their centers (see Figure 2). We shall call this the $\text{InCONE}(S_a, S_b; S_c)$ predicate.

In case the radii of S_a and S_b are equal, $\mathcal{K}(a, b)$ (we drop the parenthesis for convenience) degenerates into a cylinder without this having an impact to the predicate. If S_c° is used to denote the open sphere that corresponds to S_c , then all possible answers of the predicate $\text{InCONE}(S_a, S_b; S_c)$ are

- **Outside**, if at least one point of S_c is outside \mathcal{K} ,
- **Inside**, if S_c° lies inside \mathcal{K} and $S_c \cap \mathcal{K} = \emptyset$,
- **OnePointTouch**, if S_c° lies inside \mathcal{K} and $S_c \cap \mathcal{K}$ is a point,
- **CircleTouch**, if S_c° lies inside \mathcal{K} and $S_c \cap \mathcal{K}$ is a circle.

The last two answers are considered “degenerate” and therefore, we may consider that whenever InCONE is called during the algorithm presented in Section 3.4, it will either return **Outside** or **Inside**.

This predicate is basic tool used in various other sub-predicates such as the TriSector , which returns the trisector type of a set of three spheres. It is known ([28]) that if the trisector τ_{abc} of S_a, S_b, S_c has Hausdorff dimension 1, it can either be a branch of a “hyperbola”, a “line”, an “ellipse”, a “circle” or a “parabola”; these are the possible answers of the $\text{TriSector}(S_a, S_b, S_c)$ predicate. However, since the “line” and the “circle” type are sub-case of the

“hyperbolic” and “elliptic” trisector types respectively, we can characterize a trisector as either “hyperbolic”, “elliptic” or “parabolic”.

During the execution of the main algorithm of Section 3.4, the `TRISECTOR` (S_i, S_j, S_k) has to be evaluated. Being able to distinguish the type of the trisector τ_{ijk} is essential since all the analysis presented in this paper assumes that τ_{ijk} is hyperbolic.

The analysis followed to determine the outcome of the `INCONE` or the `TRISECTOR` predicate can be found in Sections 4.1 and 4.2 respectively, where the following lemma is proved.

Lemma 2. *The `INCONE` and `TRISECTOR` predicates can be evaluated by determining the sign of quantities of algebraic degree at most 4 (in the input quantities).*

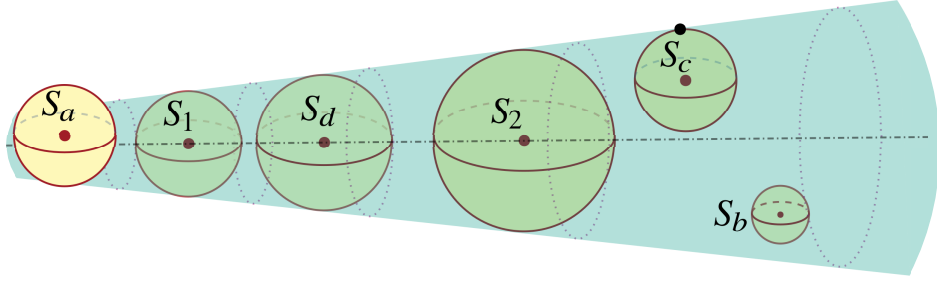


FIGURE 2. Some of the possible locations of a sphere S_n against the cone \mathcal{K} defined by S_1 and S_2 . The `INCONE` $(S_1, S_2; S_n)$ returns `Outside`, `Inside`, `OnePointTouch` and `CircleTouch` for $n = a, b, c$ and d respectively.

3.3.3. The `DISTANCE` predicate. When the trisector τ_{ijk} is a hyperbola or a line, there exist two distinct planes, denoted by Π_{ijk}^- and Π_{ijk}^+ , such that each one is commonly tangent to the sites S_i, S_j, S_k and leave their centers on the same halfspace.

Observe that Π_{ijk}^- and Π_{ijk}^+ correspond to the two Apollonius spheres *at infinity*, in the sense that they are centered at *infinity* and are cotangent to the spheres S_i, S_j and S_k . These planes are considered as oriented, and subdivide \mathbb{R}^3 into a positive and a negative halfspace, the positive being the halfspace containing the centers of the spheres.

Given a point p on τ_{ijk} , we denote by $\mathcal{T}(p)$ the tritangent Apollonius sphere of S_i, S_j and S_k centered at p . If we move p on τ_{ijk} such that $\zeta(p)$ goes towards $-\infty$ or $+\infty$, the sphere $\mathcal{T}(p)$ becomes the corresponding Apollonius sphere at infinity i.e., the plane Π_{ijk}^- or Π_{ijk}^+ (see Figure 3).

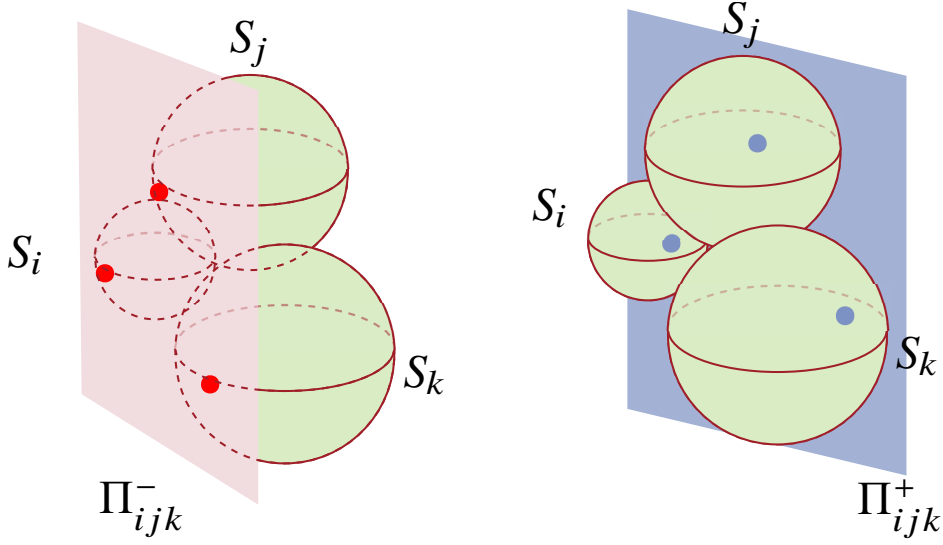


FIGURE 3. If the spheres S_i, S_j and S_k lie in a convex position there exist two distinct planes, Π_{ijk}^- and Π_{ijk}^+ , cotangent to all spheres. These planes are considered as the Apollonius sphere of the sites S_i, S_j and S_k , centered at $p \in \tau_{ijk}$, as $\zeta(p)$ goes to $\pm\infty$ respectively.

Given the sites S_i, S_j, S_k and S_α , the $\text{DISTANCE}(S_i, S_j, S_k, S_\alpha)$ predicate determines whether S_α intersects, is tangent to, or does not intersect the (closed) negative halfspaces delimited by the two planes Π_{ijk}^- and Π_{ijk}^+ . The “tangency” case is considered as degenerate and is beyond the scope of this paper. This predicate is used in the evaluation of the SHADOW predicate, and is equal to $\text{DISTANCE}(S_i, S_j, S_k, S_\alpha) = (\text{sign}(\delta(S_\alpha, \Pi_{ijk}^-)), \text{sign}(\delta(S_\alpha, \Pi_{ijk}^+)))$, where $\delta(S, \Pi) = \delta(C, \Pi) - r$, and $\delta(C, \Pi)$ denotes the signed Euclidean of C from the plane Π and S is a sphere of radius r , centered at C . As for the EXISTENCE predicate, we reduce it to the computation of the signs of the two roots of a quadratic equation and prove the following lemma (see Section 4.4 for this analysis).

Lemma 3. *The DISTANCE predicate can be evaluated by determining the sign of quantities of algebraic degree at most 6 (in the input quantities).*

3.3.4. *The EXISTENCE predicate.* The next primitive operation we need for answering the EDGECONFLICT predicate is what we call the EXISTENCE predicate: given four sites S_a, S_b, S_c and S_n , we would like to determine the number of Apollonius spheres of the quadruple S_a, S_b, S_c, S_n . In general, given four sites there can be “0”, “1”, “2” or “infinite” Apollonius spheres (cf. [11]) including

the Apollonius sphere(s) at infinity. The $\text{EXISTENCE}(S_a, S_b, S_c, S_n)$ predicate only counts the Apollonius spheres that are *not* centered at infinity and since degenerate configurations of the input sites are beyond the scope of this paper, it is safe to assume that the outcome will always be “0”, “1” or “2”. It is also clear that in case of a “1” outcome, the corresponding Apollonius center will either be v_{abcn} or v_{acbn} but not both; the case where $v_{abcn} \equiv v_{acbn}$ is ruled out by our initial no-degeneracies assumption.

The analysis of the EXISTENCE predicate can be found in Section 4.3 where we prove the following lemma.

Lemma 4. *The EXISTENCE predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

3.3.5. *The SHADOW predicate.* We now come to the second major subpredicate used by the EDGECONFLICT predicate: the SHADOW predicate.

Given three sites S_i, S_j and S_k , we define the *shadow region* $\mathcal{SR}(S_\alpha)$ of a site S_α , with respect to the trisector τ_{ijk} , to be the locus of points p on τ_{ijk} such that $\delta(C_\alpha, \mathcal{T}(p)) < r_\alpha$. The shadow region of the sites S_l, S_m and S_q play an important role when answering $\text{EDGECONFLICT}(S_i, S_j, S_k, S_l, S_m, S_q)$ (see Figure 4 and Section 3.4).

The $\text{SHADOW}(S_i, S_j, S_k, S_\alpha)$ predicate returns the type of $\mathcal{SR}(S_\alpha)$ seen as an interval, or union of intervals, in \mathbb{R} . More precisely, the SHADOW predicate returns the topological structure of the set $\zeta(\mathcal{SR}(S_\alpha)) = \zeta(\{p \in \tau_{ijk} \mid \delta(C_\alpha, \mathcal{T}(p)) < r_\alpha\})$, which we denote by $\text{SRT}(S_\alpha)$.

Clearly, the boundary points of $\text{SHADOW}(S_i, S_j, S_k, S_\alpha)$ are the points p on τ_{ijk} for which $\delta(C_\alpha, \mathcal{T}(p)) = r_\alpha$. These points are nothing but the centers of the Apollonius spheres of the four sites S_i, S_j, S_k and S_α , and, as such, there can only be 0, 1 or 2 (assuming no degeneracies). This immediately suggests that $\text{SRT}(S_\alpha)$ can have one of the following 6 types: \emptyset , $(-\infty, \infty) = \mathbb{R}$, $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) , or $(-\infty, \phi) \cup (\chi, +\infty)$, where $\phi, \chi \neq \pm\infty$.

For convenience, we will use the $\mathcal{SR}(S_a)$ notation instead of $\text{SRT}(S_a)$; for example, the statement “ $\mathcal{SR}(S_a) = (-\infty, \phi)$ ” will be often used instead of “ $\mathcal{SR}(S_a)$ ’s type is $(-\infty, \phi)$ ” or “ $\text{SRT}(S_a) = (-\infty, \phi)$ ” (see Figure 5 for an example). This notation change further highlights the fact that we are only interested in the topological structure of $\mathcal{SR}(S_a)$ rather than the actual set itself.

In Section 4.5, we prove that the evaluation of the SHADOW predicate only requires the call of the respective DISTANCE and EXISTENCE predicate, yielding the following lemma.

Lemma 5. *The SHADOW predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

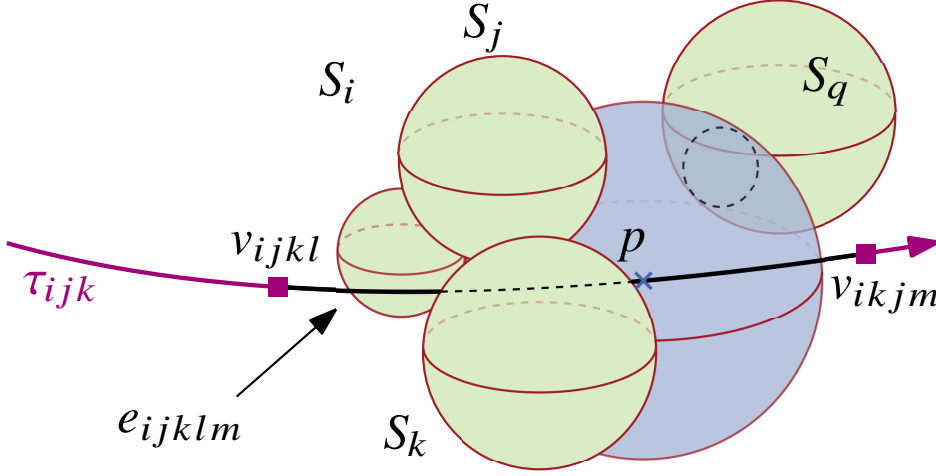


FIGURE 4. A finite edge e_{ijklm} of the Voronoi diagram $\mathcal{VD}(S)$ of the set $S = \{S_n : n = i, j, k, l, m\}$ is the locus of points $p \in \tau_{ijk}$ such that $v_{ijkl} \prec p \prec v_{ikjm}$. Any sphere $\mathcal{T}(p)$, centered at p and cotangent to S_i, S_j and S_k does not intersect any sphere of S (*Empty Sphere Principle*). However, after inserting S_q in the existing Voronoi diagram, $\mathcal{T}(p)$ may intersect it. In $\mathcal{VD}(S \cup \{S_q\})$, all points $p \in \mathcal{SR}(S_q)$ will no longer exist on the “updated” edge e'_{ijklm} .

3.3.6. *The ORDER predicate.* The most important sub-predicate used to evaluate the EDGECONFLICT predicate is what we call the ORDER predicate. When $\text{ORDER}(S_i, S_j, S_k, S_a, S_b)$ is called, it returns the order of appearance of any of the existing Apollonius vertices $v_{ijka}, v_{ikja}, v_{ijkb}$ and v_{ikjb} on the oriented trisector τ_{ijk} .

This sub-predicate is called during the main algorithm that answers the $\text{EDGECONFLICT}(S_i, S_j, S_k, S_l, S_m, S_q)$, for $(a, b) \in \{(l, q), (m, q)\}$, only in the case that either v_{ijkq}, v_{ikjq} or both exist. Let us also recall that, in this paper, the trisector τ_{ijk} is “hyperbolic” and that e_{ijklm} is a valid finite Apollonius edge; the Apollonius vertices v_{ijkl} and v_{ikjm} both exist on (the oriented) τ_{ijk} and $v_{ijkl} \prec v_{ikjm}$.

In order to answer the ORDER predicate, we first call the $\text{SHADOW}(S_i, S_j, S_k, S_n)$ predicate, for $n \in \{a, b\}$, to obtain the type of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$. From the shadow region types, two pieces of information is easily obtained; firstly, we determine which of the Apollonius vertices v_{ijka}, v_{ikja} and v_{ijkb}, v_{ikjb} actually exist and secondarily, if both v_{ijkn} and v_{ikjn} exist for some $n \in \{a, b\}$, then their ordering on the oriented trisector is also retrieved. Such deductions derive from the study of the shadow region, as shown in Section 4.6.3 (see

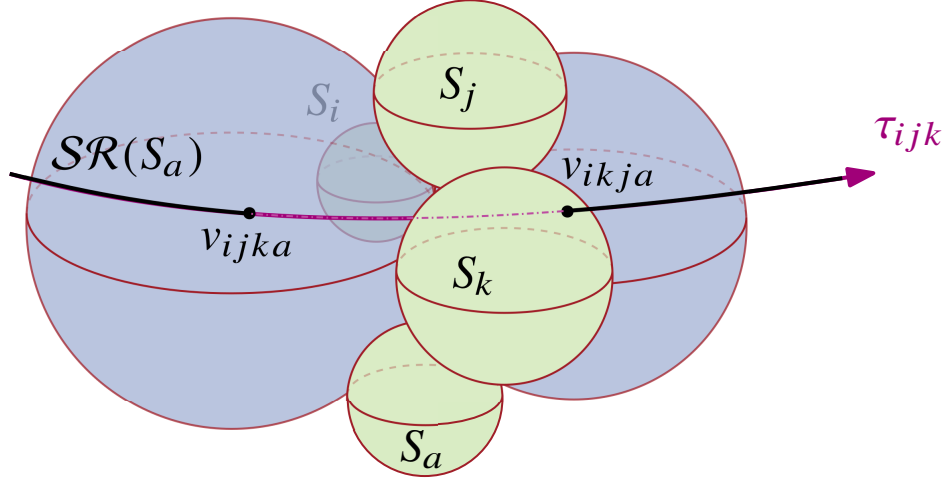


FIGURE 5. Since there are two Apollonius spheres of the sites S_n , for $n \in \{i, j, k, a\}$, centered at v_{ijka} and v_{ikja} , the $\mathcal{SR}(S_a)$ on τ_{ijk} must have two endpoints. In this specific configuration notice that, for every point p on the segments of τ_{ijk} painted black, the sphere $\mathcal{T}(p)$ will intersect S_a . Therefore, the black segments are indeed the shadow region $\mathcal{SR}(S_a)$ of the sphere S_a on the trisector τ_{ijk} .

Lemma 15). For example, if $\mathcal{SR}(S_a) = (\chi, \phi)$, then both v_{ijka} and v_{ikja} exist and appear on the oriented trisector in this order: $v_{ikja} \prec v_{ijka}$.

Now that the existence and partial ordering of the Apollonius vertices v_{ijka} and v_{ikja} (resp. v_{ijkb} and v_{ikjb}) is known, we must provide a way of “merging” them into a complete ordering. For this reason, we examine all possible complete orderings of the Apollonius vertices on the oriented trisector τ_{ijk} . The study of these orderings is seen in the inverted plane \mathcal{W} -space. In Section 4.6.2, we present the strong geometric relationship that holds between the spheres of the original \mathcal{Z} -space and their images in the inverted plane. The observations we make regarding the connection of the two spaces allow us to interpret geometric configurations on one space to equivalent ones on the other. A full analysis of how we tackle all possible configurations is presented in Section 4.6.

In our analysis, we prove that the ORDER predicate, in the worst case, amounts to evaluate up to 4 INSPHERE predicates plus some auxiliary tests of lesser algebraic cost. We have therefore proven the following lemma.

Lemma 6. *The ORDER predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

3.4. The main algorithm. In this section, we describe in detail how the predicate $\text{EDGECONFLICT}(S_i, S_j, S_k, S_l, S_m, S_q)$ is resolved with the use of the subpredicates INCONE , TRISECTOR , EXISTENCE , SHADOW and ORDER .

We begin by determining the type of the trisector τ_{ijk} ; this is done via the call of the $\text{TRISECTOR}(S_i, S_j, S_k)$ predicate. Recall that in the scope of this paper, it is assumed that the τ_{ijk} is a hyperbola (or a line) and that none of the subpredicates called return a degenerate answer.

To answer the EDGECONFLICT predicate, one must determine which “part” of the edge e_{ijklm} remains in the Voronoi diagram after the insertion of the site S_q . This is plausible by identifying the set of points of e_{ijklm} that still remain in the updated Voronoi Diagram; each of these points must satisfy the “empty-sphere property”: a sphere, centered at that point and tangent to the spheres S_i, S_j, S_k , must not intersect any other sites of the Voronoi Diagram. As an immediate result, a point p of the edge e_{ijklm} in $\mathcal{VD}(S)$ remains in $\mathcal{VD}(S \cup \{S_q\})$ if and only if $\mathcal{T}(p)$ does not intersect S_q . Since the shadow region of the sphere S_q with respect to the trisector τ_{ijk} consists of all points p such that $\mathcal{T}(p)$ intersects S_q , it must hold that the part of the edge e_{ijklm} that no longer remains in $\mathcal{VD}(S \cup \{S_q\})$ is actually $e_{ijklm} \cap \mathcal{SR}(S_q)$ (see Figure 4). In conclusion, the result of the EDGECONFLICT predicate is exactly the set $e_{ijklm} \cap \mathcal{SR}(S_q)$ seen as an interval or union of intervals of \mathbb{R} .

To determine the intersection type of $e_{ijklm} \cap \mathcal{SR}(S_q)$, we first take into account that the finite edge e_{ijklm} consists of all points p on the oriented trisector τ_{ijk} bounded by the points v_{ijkl} and v_{ikjm} from left and right respectively (see Section 3.1). Next, we consider the type of $\mathcal{SR}(S_q)$ which can be evaluated as shown in Section 4.5 and is one of the following: $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) , $(-\infty, \phi) \cup (\chi, +\infty)$, \emptyset or \mathbb{R} .

If the edge e_{ijklm} is seen as the interval (λ_1, μ_2) , evidently the intersection type of $E' = e_{ijklm} \cap \mathcal{SR}(S_q)$ must be one of the following 6 types, each corresponding to a different answers of the EDGECONFLICT predicate.

- If E' is of type \emptyset , the predicate returns **NoConflict**.
- If E' is of type e_{ijklm} , the predicate returns **EntireEdge**.
- If E' is of type (λ_1, ϕ) , the predicate returns **LeftVertex**.
- If E' is of type (χ, μ_2) , the predicate returns **RightVertex**.
- If E' is of type $(\lambda_1, \phi) \cup (\chi, \mu_2)$, the predicate returns **BothVertices**.
- If E' is of type (χ, ϕ) , the predicate returns **Interior**.

This observation suggests that, if we provide a way to identify the type of E' , we can answer the EDGECONFLICT predicate. Taking into consideration that

- λ_1 and μ_2 correspond to v_{ijkl} and v_{ikjm} respectively as shown in Section 3.1, and

- χ, ϕ correspond to v_{ikjq} and v_{ijkq} respectively as stated in Lemma 15 that we prove in Section 4.6.3,

it becomes apparent that if we order all Apollonius vertices v_{ijkl}, v_{ikjm} and any of the existing among v_{ijkq}, v_{ikjq} , bearing in mind the type of $\mathcal{SR}(S_q)$, we can deduce the type of E' .

For example, let us assume that $\mathcal{SR}(S_q)$ type is $(-\infty, \chi) \cup (\phi, +\infty)$. If $v_{ijkl} \prec v_{ikjq} \prec v_{ikjm} \prec v_{ijkq}$ on the oriented trisector τ_{ijk} , or equivalently $\lambda_1 < \chi < \mu_2 < \phi$, we can conclude that E' is of type (λ_1, ϕ) and the `EDGECONFLICT` predicate would return `LeftVertex`.

Therefore, it is essential that we are able to provide an ordering of the Apollonius vertices v_{ijkl}, v_{ikjm} and any of the existing among v_{ijkq}, v_{ikjq} . Such a task is accomplished via the call of the `ORDER` (S_i, S_j, S_k, S_a, S_b) predicate (a, b) = (l, q) and (m, q). The outcomes of these predicates consist of the orderings of all possible Apollonius vertices of the sites S_i, S_j, S_k, S_a and S_i, S_j, S_k, S_b on the trisector τ_{ijk} . These partial orderings can then be merged into a complete ordering, which contains the desired one. The results' combination principle is identical to the one used when we have to order a set of numbers but we can only compare two at a time.

A detailed algorithm that summarizes the analysis of this Section and can be followed to answer the `EDGECONFLICT`($S_i, S_j, S_k, S_l, S_m; S_q$) is described in the following steps.

Step 1: The `TRISECTOR` (S_i, S_j, S_k) is called to determine the type of the trisector τ_{ijk} . In this Section, we assume that τ_{ijk} return “hyperbolic”.

Step 2: We evaluate $SRT(q) = \text{SHADOW}(S_i, S_j, S_k, S_q)$. If $SRT(q) = \emptyset$ or \mathbb{R} , we return `NoConflict` or `EntireEdge` respectively. Otherwise, we know that $SRT(q)$ has one of the forms $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$, where ϕ and χ correspond to the Apollonius vertices v_{ijkq} and v_{ikjq} respectively.

Step 3: We evaluate `ORDER` (S_i, S_j, S_k, S_l, S_q) and break down our analysis depending on how many of the vertices v_{ijkq} and v_{ikjq} exist; if only one vertex exist (which is equivalent to $SRT(q)$ being $(-\infty, \phi)$ or $(\chi, +\infty)$), denote it by v_q and go to Step 3a. Otherwise, if both exist, go to Step 3b.

Step 3a: If only v_{ikjq} or v_{ijkq} exist, then $SRT(q)$, which was evaluated in Step 2, is of the form $(-\infty, \phi)$ or $(\chi, +\infty)$ respectively. From the outcome of `ORDER` (S_i, S_j, S_k, S_l, S_q), we know whether $v_q \prec v_{ikjl}$ or $v_{ikjl} \prec v_q$. If $v_q \prec v_{ikjl}$, and therefore $v_q \prec v_{ijkl} \prec v_{ikkm}$, the `EDGECONFLICT` predicate returns `NoConflict` if $SRT(q) = (-\infty, \phi)$ or `EntireEdge` if $SRT(q) = (\chi, +\infty)$.

If $v_{ijkl} \prec v_q$, we evaluate $\text{ORDER}(S_i, S_j, S_k, S_m, S_q)$ and determine whether $v_q \prec v_{ikjm}$ or $v_{ikjm} \prec v_q$. In the first case, we conclude that $v_{ijkl} \prec v_q \prec v_{ikjm}$ and the EDGECONFLICT predicate returns **LeftVertex** if $SRT(q) = (-\infty, \phi)$ or **RightVertex** if $SRT(q) = (\chi, +\infty)$. In the second case, we get that $v_{ijkl} \prec v_{ikjm} \prec v_q$ and the predicate returns **EntireEdge** if $SRT(q) = (-\infty, \phi)$ or **NoConflict** if $SRT(q) = (\chi, +\infty)$.

Step 3b: If both v_{ijkq} and v_{ikjq} exist, then $SRT(q)$ is of the form (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$, where ϕ and χ correspond to v_{ijkq} and v_{ikjq} respectively. We now call the $\text{ORDER}(S_i, S_j, S_k, S_l, S_q)$ predicate. If it returns that $v_{ijkq}, v_{ikjq} \prec v_{ijkl}$, the EDGECONFLICT predicate immediately returns **NoConflict** if $SRT(q) = (\chi, \phi)$, otherwise it returns **EntireEdge**.

If this is not the case, we have to call the $\text{ORDER}(S_i, S_j, S_k, S_m, S_q)$ predicate to acquire the ordering of v_{ijkq}, v_{ikjq} and v_{ikjm} . If $v_{ikjm} \prec v_{ijkq}, v_{ikjq}$, the EDGECONFLICT predicate immediately returns **NoConflict** if $SRT(q) = (\chi, \phi)$, otherwise it returns **EntireEdge**. In any other case, we must combine the information from the two ORDER predicates with the fact that $v_{ikjl} \prec v_{ikjm}$, to obtain the complete ordering of $v_{ikjl}, v_{ikjm}, v_{ijkq}$ and v_{ikjq} . The list of possible EDGECONFLICT predicate answers for the cases that have not been already handled is found in this list.

NoConflict: If $SRT(q) = (-\infty, \phi) \cup (\chi, +\infty)$,

$v_{ikjq} \prec v_{ijkl} \prec v_{ikjm}$ and $v_{ijkl} \prec v_{ikjm} \prec v_{ijkq}$.

EntireEdge: If $SRT(q) = (\chi, \phi)$, $v_{ijkl} \prec v_{ikjm} \prec v_{ikjq}$ and

$v_{ijkq} \prec v_{ijkl} \prec v_{ikjm}$.

LeftVertex: If $SRT(q) = (\chi, \phi)$, $v_{ijkl} \prec v_{ikjq} \prec v_{ikjm}$ and $v_{ijkq} \prec v_{ijkl} \prec v_{ikjm}$, or $SRT(q) = (-\infty, \phi) \cup (\chi, +\infty)$,

$v_{ijkl} \prec v_{ikjq} \prec v_{ikjm}$ and $v_{ijkl} \prec v_{ikjm} \prec v_{ijkq}$.

RightVertex: If $SRT(q) = (\chi, \phi)$ $v_{ijkl} \prec v_{ikjm} \prec v_{ikjq}$ and $v_{ijkq} \prec v_{ijkl} \prec v_{ikjm}$, or $SRT(q) = (-\infty, \phi) \cup (\chi, +\infty)$,

$v_{ikjq} \prec v_{ijkl} \prec v_{ikjm}$ and $v_{ijkl} \prec v_{ijkq} \prec v_{ikjm}$.

BothVertices: If $SRT(q) = (-\infty, \phi) \cup (\chi, +\infty)$ and

$v_{ijkl} \prec v_{ijkq}, v_{ikjq} \prec v_{ikjm}$

Interior: If $SRT(q) = (\chi, \phi)$ and $v_{ijkl} \prec v_{ijkq}, v_{ikjq} \prec v_{ikjm}$.

A sketch of the subpredicates used when answering the EDGECONFLICT predicate is shown in Figure 6. Since the highest algebraic degree needed in the evaluation of the subpredicates used is 10, we have proven the following theorem.

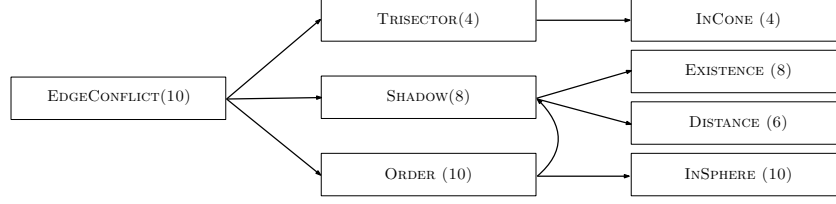


FIGURE 6. The layout of predicates and their subpredicates used to answer the `EDGECONFLICT` predicate. The number next to each predicate corresponds to its algebraic cost. It is assumed that every subpredicate returns a non-degenerate answer.

Theorem 1. *The `EDGECONFLICT` predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

4. IMPLEMENTATION AND ANALYSIS OF SUBPREDICATES

In this Section, we provide a detailed description on how to answer every subpredicate involved in the algorithm presented in Section 3.4. For each primitive, beside analyzing how we derive the outcome, we also compute its algebraic degree i.e., the maximum algebraic degree of all quantities that have to be evaluated to obtain the subpredicate's result.

4.1. The `INCONE` predicate. To answer the `INCONE` (S_a, S_b, S_c) predicate we first determine the number of possible tangent planes to the sites S_a, S_b , and S_c that leave them all on the same side; there can be either 0, 1, 2 or ∞ such planes. Bear in mind that the `INCONE` predicate can be called only if no one of the spheres S_a and S_b are contained inside another.

If S_a and S_b have different radii, then \mathcal{K} will denote the cone that contains and is tangent to these spheres, whereas \mathcal{K}^- will symbolize the symmetric cone with the same axis and apex. Let us now consider each of the four possible cases regarding the number of cotangent planes, since it is indicative of the relative position of the three spheres.

- (1) If no such plane exists, there are three cases to consider;
 - S_c lies strictly inside the cone \mathcal{K} ; the predicate returns **Inside**.
 - S_c lies strictly inside the cone \mathcal{K}^- ; the predicate returns **Outside**.
 - S_c fully intersects the cone \mathcal{K} in the sense that there is a circle on S_c that is outside \mathcal{K} . In this case, the predicate returns **Outside**.
 - There is a circle on S_c that is outside \mathcal{K} ; S_c could lie strictly inside the cone \mathcal{K}^- or fully intersect the cone $\mathcal{K}, \mathcal{K}^-$ or both. In all these cases, the predicate returns **Outside**.

- (2) If there is only one such plane, then S_c touches \mathcal{K} in a single point. There are two cases to consider;
 - S_c lies strictly inside the cone \mathcal{K} ; the predicate returns **OnePointTouch**.
 - S_c fully intersects the cone \mathcal{K} (there is a circle in S_c that is outside \mathcal{K}). In this case, the predicate returns **Outside**.
- (3) If there are two such planes, the spheres must lie in convex position, hence the predicate returns **Outside**.
- (4) If there are infinite such planes, the spheres S_a, S_b and S_c have collinear centers and the points of tangency of each sphere with the cone is a single circle. The predicate returns **CircleTouch** in this scenario.

In the case no cotangent plane to all sites S_a, S_b and S_c exist, we must be able to tell if S_c lies inside the cone \mathcal{K}^- . However, this check is only needed in the case $r_a \neq r_b$; if $r_a = r_b$, the cone C_{ij} degenerates into a cylinder and \mathcal{K}^- does not exist.

Let us consider the case $r_a \neq r_b$ in detail. First, observe that we can assume without loss of generality that $r_a < r_b$ as this follows from the definition of the **INCONE** predicate. Indeed, since **INCONE** (S_a, S_b, S_c) and **INCONE** (S_b, S_a, S_c) represent the same geometric inquiry, we can exchange the notation of the spheres S_a and S_b in case $r_a > r_b$.

Taking this into consideration, we denote K to be the apex of the cone \mathcal{K} and Π_c to be the plane that goes through K and perpendicular to axis ℓ of the cone. We also denote by Π_c^+ the half-plane defined by the plane Π_c and the centers C_a and C_b , whereas its compliment half-plane is denoted by Π_c^- .

It is obvious that if the center of the sphere S_c does not lie in Π_c^+ the predicate must return **Outside** since S_c has at least one point outside the cone \mathcal{K} . Moreover, if the center C_c lies in Π_c^+ then S_c cannot lie inside \mathcal{K}^- and this case is ruled out. To check if C_c lies on Π_c^+ , we first observe that C_a and C_b define the line ℓ hence for every point P of ℓ stands that $\overrightarrow{OP} = \overrightarrow{OC_a} + t\overrightarrow{C_aC_b}$ for some $t \in \mathbb{R}$. We now claim that for a sphere with center $P(t)$ to be tangent to the cone it must have radius $r(t)$ that is linearly dependent with t , i.e. $r(t) = k_1t + k_0$. To evaluate k_1 and k_0 , we observe that for $t = 0$, $P(0) \equiv C_a$ hence $r(0) = r_a$ and respectively for $t = 1$, $P(1) \equiv C_b$ hence $r(1) = r_b$. We conclude that $r(t) = t \cdot (r_b - r_a) + r_a$, $t \in \mathbb{R}$. The cone apex lies on ℓ so $\overrightarrow{OK} = \overrightarrow{OC_a} + t_c\overrightarrow{C_aC_b}$ for $t_c \in \mathbb{R}$ such that $r(t_c) = 0$ or equivalently $t_c = r_a/(r_a - r_b)$. In this way we have evaluated the cone apex coordinates which derive from the relation $\overrightarrow{OK} = \overrightarrow{OC_a} + \overrightarrow{C_bC_a} \cdot r_a/(r_b - r_a)$. Since Π_c is perpendicular to ℓ and therefore to $\overrightarrow{C_aC_b}$, and $\overrightarrow{C_aC_b}$ points towards

the positive side of Π_c , the point C_c lies on the positive half plane Π_c^+ iff the quantity $M = \overrightarrow{C_a C_b} \cdot \overrightarrow{K C_c}$ is strictly positive.

To evaluate the sign of M we have

$$\begin{aligned} \text{sign}(M) &= \text{sign}(\overrightarrow{C_a C_b} \cdot \overrightarrow{K C_c}) = \text{sign}(\overrightarrow{C_a C_b} \cdot \overrightarrow{O C_c} - \overrightarrow{C_a C_b} \cdot \overrightarrow{O K}) \\ &= \text{sign}(\overrightarrow{C_a C_b} \cdot (\overrightarrow{O C_c} - \overrightarrow{O C_a} - \frac{r_a}{r_b - r_a} \overrightarrow{C_b C_a})) \\ &= \text{sign}(\overrightarrow{C_a C_b} \cdot ((r_b - r_a) \overrightarrow{C_a C_c} + r_a \overrightarrow{C_a C_b})) \text{sign}(r_b - r_a) \\ &= \text{sign}((r_b - r_a) \overrightarrow{C_a C_c} \cdot \overrightarrow{C_a C_b} + r_a \overrightarrow{C_a C_b} \cdot \overrightarrow{C_a C_b}) \end{aligned}$$

so determining $\text{sign}(M)$ requires operations of degree 3.

If M is non-positive the predicate returns **Outside**, however more analysis is need if this is not the case. For the rest of this section, we assume that M is positive and break the analysis depending on the collinearity of the centers of the spheres S_a, S_b and S_c . Remember that C_a, C_b and C_c are collinear if and only if the cross product $\overrightarrow{C_a C_b} \times \overrightarrow{C_a C_c}$ is the zero vector, which is a 2-degree demanding operation in the input quantities.

4.1.1. The Centers C_a, C_b, C_c are Collinear. If C_a, C_b and C_c are collinear, they all lie on the line ℓ , and therefore $\overrightarrow{O C_c} = \overrightarrow{O C_a} + t_o \overrightarrow{C_a C_b}$ for some $t_o \in \mathbb{R}$. Equivalently, we get that $t_o \overrightarrow{C_a C_b} = \overrightarrow{C_a C_c}$ and since C_a and C_b cannot be identical we can evaluate $t_o = X/Y$ where $(X, Y) = (x_c - x_a, x_b - x_a)$ or $(y_c - y_a, y_b - y_a)$ or $(z_c - z_a, z_b - z_a)$, if $x_b - x_a \neq 0$ or $y_b - y_a \neq 0$ or $z_b - z_a \neq 0$ respectively.

Denote $r(t)$ as before, we evaluate the sign S of $r_c - r(t_o)$,

$$S = \text{sign}(r_c - r(t_o)) = \text{sign}((r_a - r_c)Y + (r_b - r_a)X)$$

which requires operations of degree 2.

We can now answer the predicate because if $r_c < r(t_o)$, ie. S is negative, then S_c lies strictly on the cone; otherwise, if $r_c > r(t_o)$, ie. S is positive, then S_c intersects the cone. If $r_c = r(t_o)$, ie. S is zero, then S_c touches \mathcal{K} in a circle. In conclusion, we get that

$$\text{INCONE}(S_a, S_b, S_c) = \begin{cases} \text{Inside}, & \text{if } S < 0, \\ \text{CircleTouch}, & \text{if } S = 0, \\ \text{Outside}, & \text{if } S > 0, \end{cases}$$

4.1.2. Non-Collinear Centers. If C_a, C_b and C_c are not collinear and must examine the number of possible tritangent planes to the sites S_n for $n \in \{i, j, k\}$. Denote $\Pi : ax + by + cz + d = 0$ a plane tangent to S_a, S_b and S_c that leaves the spheres on the same half-plane, and assume without loss of

generality that $a^2 + b^2 + c^2 = 1$. Since the sphere S_n for $n \in \{i, j, k\}$ touches the plane Π , we get that $\delta(S_n, \Pi) = ax_n + by_n + cz_n + d = r_n$. We examine the resulting system of equations

$$\begin{aligned} ax_a + by_a + cz_a &= r_a - d \\ ax_b + by_b + cz_b &= r_b - d \\ ax_c + by_c + cz_c &= r_c - d \\ a^2 + b^2 + c^2 &= 1 \end{aligned}$$

and distinguish the following cases

- if $D_{abc}^{xyz} \neq 0$, we can express a, b and c linearly in terms of d . Substituting these expressions in the last equation, we get a quadratic equation that vanishes at d ; the sign of the discriminant Δ' of this quadratic reflects the number of possible d 's and therefore tangent planes to the spheres S_a, S_b and S_c .
- if $D_{abc}^{xyz} = 0$ and since the centers of the spheres S_a, S_b and S_c are not collinear, one of the quantities $D_{abc}^{xy}, D_{abc}^{xz}$ or D_{abc}^{yz} is non-zero, without loss of generality assume $D_{abc}^{xy} \neq 0$. In this case, we can express a and b linearly in terms of c , whereas $d = D_{abc}^{xyr}/D_{abc}^{xy}$. From the last equation, we get a quadratic equation that vanishes at c , and the sign of the discriminant Δ'' again reflects the number of possible c 's and therefore tangent planes to the spheres S_a, S_b and S_c .

Writing down the expressions of the discriminants, we finally evaluate that $\Delta' = 4(D_{abc}^{xyz})^2\Delta$ and $\Delta'' = 4(D_{abc}^{xy})^2\Delta$, where

$$\Delta = (D_{abc}^{xy})^2 + (D_{abc}^{xz})^2 + (D_{abc}^{yz})^2 - (D_{abc}^{xr})^2 - (D_{abc}^{yr})^2 - (D_{abc}^{zr})^2.$$

Since the signs of the discriminants Δ_1, Δ_2 and Δ are identical, we evaluate $\text{sign}(\Delta)$ and proceed as follows:

- (1) If $\Delta > 0$, there are two planes tangent to all three spheres S_a, S_b and S_c ; the predicate returns **Outside**.
- (2) If $\Delta = 0$, there is a single plane tangent to the spheres S_a, S_b and S_c and we have to distinguish between the two possible cases; S_c lies strictly inside \mathcal{K} or S_c intersects \mathcal{K} .

Assume we are in the latter case, observe that for a proper $\epsilon < 0$, the deflated sphere \tilde{S}_c , with radius $\tilde{r}_c = r_c + \epsilon$, would point-touch the cone \mathcal{K} . (Note that the tangency points of the cone \mathcal{K} and the spheres S_c and \tilde{S}_c are not the same!) Therefore, if we consider the analysis of the predicate $\text{INCONE}(S_a, S_b, \tilde{S}_c)$, we would get that the “perturbed” discriminant $\tilde{\Delta}$ would vanish for this $\epsilon < 0$!

For the evaluation of $\tilde{\Delta}$, we simply substitute r_c in Δ with $\tilde{r}_c = r_c + \epsilon$ and rewrite $\tilde{\Delta} = \tilde{\Delta}(\epsilon)$ as a polynomial in terms of ϵ : $\tilde{\Delta}(\epsilon) = \Delta_2\epsilon^2 + \Delta_1\epsilon + \Delta_0$, where $\Delta_0 = \Delta = 0$ and

$$\begin{aligned}\Delta_2 &= -[(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2](< 0), \\ \Delta_1 &= -2((x_b - x_a)D_{abc}^{xr} + (y_b - y_a)D_{abc}^{yr} + (z_b - z_a)D_{abc}^{zr}).\end{aligned}$$

- Since $\epsilon = 0$ is a root of the quadratic (in terms of ϵ) $\tilde{\Delta}(\epsilon)$, a simple use of Vieta's formula shows that $\tilde{\Delta}(\epsilon)$ has a negative root, if and only if $\text{sign}(\Delta_1)$ is strictly negative. In conjunction with our previous remarks, the predicate should return **Outside** if $\Delta_1 < 0$; otherwise it must return **OnePointTouch**. Indeed, if S_c point touches the cone \mathcal{K} from the “inside”, one can inflate S_c to \tilde{S}_c so that \tilde{S}_c point-touches \mathcal{K} !
- (3) If $\Delta < 0$, there is no plane tangent to the spheres S_a, S_b and S_c . Since S_c lying inside the cone \mathcal{K} is ruled out, we have to distinguish between the two possible cases; S_c lies strictly inside \mathcal{K} or S_c intersects \mathcal{K} .

It follows that, either S_c lies strictly within the cone and the predicate must return **Inside**, or S_c fully intersects the cone and the predicate must return **Outside**. Using the same analysis as in case $\Delta = 0$, we observe that if we inflate (or deflate) S_c , the perturbed sphere \tilde{S}_c with radius $\tilde{r}_c = r_c + \epsilon$ will touch the cone \mathcal{K} for two different values ϵ_1 and ϵ_2 . The predicate must return **Inside** if we must inflate S_c to touch \mathcal{K} , ie. if $\epsilon_1, \epsilon_2 > 0$ whereas the predicate must return **Outside** if we must deflate S_c , to point-touch \mathcal{K} , ie. if $\epsilon_1, \epsilon_2 < 0$. As shown in the case $\Delta = 0$, the perturbed discriminant that will appear during the evaluation of $\text{InCone}(S_a, S_b, \tilde{S}_c)$ is $\tilde{\Delta}(\epsilon) = \Delta_2\epsilon^2 + \Delta_1\epsilon + \Delta_0$ where

$$\begin{aligned}\Delta_2 &= -[(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2](< 0), \\ \Delta_1 &= -2((x_b - x_a)D_{abc}^{xr} + (y_b - y_a)D_{abc}^{yr} + (z_b - z_a)D_{abc}^{zr}), \\ \Delta_1 &= \Delta < 0.\end{aligned}$$

Since the sphere \tilde{S}_c point-touches the cone \mathcal{K} for $\epsilon = \epsilon_1, \epsilon_2$, the discriminant $\tilde{\Delta}(\epsilon)$ must vanish for these epsilons, as mentioned in the previous case. Therefore, ϵ_1 and ϵ_2 are the roots of the quadratic (in terms of ϵ) $\Delta_2\epsilon^2 + \Delta_1\epsilon + \Delta_0$ and we know, using Vieta's rule and the fact that $\Delta_0 < 0$, that ϵ_1, ϵ_2 are both negative (resp. positive) if and only if Δ_1 is negative (resp. positive).

Note that, since evaluating the sign of Δ and Δ_1 require operations of degree 4 and 3 (in the input quantities) respectively, we have shown that the maximum algebraic cost of the InCone predicate is 4, yielding the following lemma.

Lemma 7. *The INCONE predicate can be evaluated by determining the sign of quantities of algebraic degree at most 4 (in the input quantities).*

4.2. The TRISECTOR predicate. Assuming the trisector τ_{ijk} exists and has Hausdorff dimension 1, observe that the trisector's type and the relative position of the spheres are closely related. Specifically,

- If the spheres are in *convex position*, i.e. there exist two distinct commonly tangent planes that leave them on the same side, then their trisector can either be a hyperbola or a line, in the special case $r_i = r_j = r_k$. The predicate returns “hyperbolic” in this scenario.
- If the spheres are in *strictly non-convex position*, i.e. one of them lies strictly inside the cone defined by the other two, then their trisector can either be an ellipse or a circle, in the special case where C_i, C_j and C_k are collinear, The predicate returns “elliptic” in this scenario.
- If the spheres are in *degenerate non-convex position*, i.e. they are in non-convex position and the closure of all three touch their convex hull, then their trisector is a parabola and the predicate returns “parabolic”.

Therefore, one can answer the TRISECTOR predicate if the relative position of the spheres three input spheres is identified. We accomplish such task by combining the outcomes of the three INCONE predicates with inputs (S_i, S_j, S_k) , (S_i, S_k, S_j) and (S_j, S_k, S_i) . Since the trisector τ_{ijk} is assumed to have Hausdorff dimension 1, we know that no one of the spheres S_i, S_j and S_k are contained inside another and therefore we may call the INCONE predicate on these inputs.

- If at least one outcome is **Inside** or **CircleTouch**, then the spheres are in *strictly non-convex position* and the trisector's type is “elliptic”.
- If at least one outcome is **OnePointTouch**, then the spheres are in *degenerate non-convex position* and the TRISECTOR predicate returns “parabolic”.
- Finally, if all three outcomes are **Outside**, then the sites are in *convex position* and the trisector's type is “hyperbolic”.(see Figure 7).

We have proven that the TRISECTOR predicate can be resolved by calling the INCONE predicate at most three times (for example, if the first INCONE returns **Inside** the trisector must be “elliptic”). Since the INCONE predicate is a 4-degree demanding operation in the input quantities, we have proven the following lemma.

Lemma 8. *The TRISECTOR predicate can be evaluated by determining the sign of quantities of algebraic degree at most 4 (in the input quantities).*

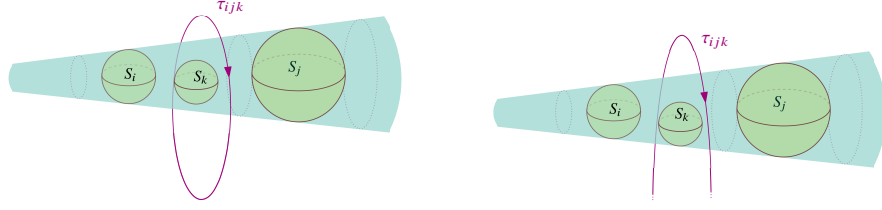


FIGURE 7. The case of an *elliptic* (Left) and a *parabolic* trisector (Right). In either case, we can orient these trisectors using the “right-hand rule”; if our thumb points from C_i towards C_j the direction pointed by our hand if it “wraps” τ_{ijk} is the positive direction.

4.3. The EXISTENCE predicate. To answer the EXISTENCE (S_i, S_j, S_k, S_a) predicate, we first break up our analysis depending on whether the radii of all sites S_n , for $n \in \{i, j, k, a\}$, are equal; if r is indeed their common radius, we deflate them by r . The existence of a sphere tangent to the original spheres amounts to the existence of a sphere tangent to the centers of the sites where the latter cotangent sphere is the former inflated by r . Regarding the existence of the Apollonius sphere of the centers C_n for $n \in \{i, j, k, a\}$, we consider the tetrahedron $C_i C_j C_k C_a$ formed and check it for flatness; this is a 3-degree demanding operation. If it is not flat, there are is a single sphere S passing through all the centers, hence there are two spheres tangent to the sites, one internal and one external (inflate or deflate S by r) and the EXISTENCE predicate returns 1 (since we only count external Apollonius spheres). If the tetrahedron is flat, i.e. the centers lie on the same plane Π_{ijka} , we check them for co-circularity (this is a 5-degree demanding operation); if co-circular, there are infinite number of spheres tangent to the sites (either internal or external), hence the EXISTENCE predicate returns ∞ (degenerate answer). If they are not co-circular, there can not be a sphere tangent to all sites and the EXISTENCE predicate returns 0. Note that if the EXISTENCE predicate would return ∞ we apply a QSP scheme to resolve the degeneracy.

Let us now consider the case where the radii of the sites S_n for $n \in \{i, j, k, a\}$ are not all equal. Since a name exchange of the sites does not affect the EXISTENCE predicate, we name exchange them such that $r_a = \min\{r_i, r_j, r_k, r_a\}$. We now deflate all spheres by r_a and then invert all sites with respect to the point C_a ; we call this “inversion through the sphere S_a ”.

In the inverted \mathcal{W} -space, a plane tritangent to the inverted spheres S_i^*, S_j^*, S_k^* amounts to a sphere tangent to all sites S_i, S_j, S_k and S_l in the original \mathcal{Z} -space. For the corresponding sphere to be an external Apollonius sphere of the sites, the following conditions must stand:

- (1) The plane must leave all inverted spheres on one side, called the positive side of the plane, and
- (2) The origin $O = (0, 0, 0)$ of the \mathcal{W} -space, that corresponds to the “point at infinity” in the \mathcal{Z} -space, must also lie on the positive side of the plane.

Considering the Condition 1, we denote $\Pi_{ijk}^* : au + bv + cw + d = 0$ a plane tangent to S_i^*, S_j^* and S_k^* in the inverted space and assume without loss of generality that $a^2 + b^2 + c^2 = 1$. Since the signed Euclidean distance of a point $P(u_p, v_p, w_p)$ from the plane Π_{ijk}^* is $\delta(P, \Pi_{ijk}) = au_p + bv_p + wz_p + d$ and the sphere S_n^* , for $n \in \{i, j, k\}$, touches Π_{ijk}^* and lies on its positive side, we get that $\delta(C_n^*, \Pi_{ijk}) = au_n + bv_n + wz_n + d = \rho_n$.

A tuple (a, b, c, d) that satisfies the resulting system of equations, amounts to a tangent plane in \mathcal{W} -space and an Apollonius sphere in \mathcal{Z} -space. In order for the condition 2 to be valid, the point $O = (0, 0, 0)$ of the \mathcal{W} -space must lie on the positive side of Π_{ijk} i.e. the signed distance of O from the plane Π_{ijk} must be positive. Equivalently, we want $\delta(O, \Pi_{ijk}) = d$ to be positive, hence we are only interested in the solutions (a, b, c, d) that satisfy $d > 0$.

The rest of this section is devoted to the algebraic analysis of the aforementioned system of equations to determine the number of such solutions with the minimum algebraic cost. The conclusion of our analysis is that such a task is possible by evaluating expressions of algebraic degree at most 8 (in the input quantities) yielding the lemma at the end of this section.

Our main tool is Crammer’s rule and therefore two major cases rise during the analysis of the system

$$\begin{aligned} au_i + bv_i + cw_i &= \rho_i - d, \\ au_j + bv_j + cw_j &= \rho_j - d, \\ au_k + bv_k + cw_k &= \rho_k - d, \\ a^2 + b^2 + c^2 &= 1. \end{aligned}$$

If $D_{ijk}^{uvw} \neq 0$, we can express a, b and c in terms of d as follows

$$a = \frac{D_{ijk}^{vw\rho} - dD_{ijk}^{vw}}{D_{ijk}^{uvw}}, \quad b = -\frac{D_{ijk}^{uw\rho} - dD_{ijk}^{uw}}{D_{ijk}^{uvw}}, \quad c = \frac{D_{ijk}^{uv\rho} - dD_{ijk}^{uv}}{D_{ijk}^{uvw}}.$$

We will then substitute the expressions of a , b and c to the equation $a^2 + b^2 + c^2 = 1$ and conclude that d is a root of $M(d) = M_2d^2 + M_1d + M_0$, where

$$\begin{aligned} M_2 &= (D_{ijk}^{uv})^2 + (D_{ijk}^{uw})^2 + (D_{ijk}^{vw})^2, \\ M_1 &= D_{ijk}^{vw\rho} D_{ijk}^{vw} + D_{ijk}^{uw\rho} D_{ijk}^{uw} + D_{ijk}^{uv\rho} D_{ijk}^{uv}, \\ M_0 &= (D_{ijk}^{vw\rho})^2 + (D_{ijk}^{uw\rho})^2 + (D_{ijk}^{uv\rho})^2 - (D_{ijk}^{uvw})^2. \end{aligned}$$

The signs of M_1 and M_0 are determined using the the following equalities

$$\begin{aligned} \text{sign}(M_1) &= -\text{sign}(D_{ijk}^{vw\rho} D_{ijk}^{vw} + D_{ijk}^{uw\rho} D_{ijk}^{uw} + D_{ijk}^{uv\rho} D_{ijk}^{uv}) \\ &= -\text{sign}(E_{ijk}^{yzr} E_{ijk}^{yzp} + E_{ijk}^{x zr} E_{ijk}^{x zp} + E_{ijk}^{xyr} E_{ijk}^{x yp}), \\ \text{sign}(M_0) &= \text{sign}((D_{ijk}^{vw\rho})^2 + (D_{ijk}^{uw\rho})^2 + (D_{ijk}^{uv\rho})^2 - (D_{ijk}^{uvw})^2) \\ &= \text{sign}((E_{ijk}^{yzr})^2 + (E_{ijk}^{x zr})^2 + (E_{ijk}^{xyr})^2 - (E_{ijk}^{x yz})^2), \end{aligned}$$

whereas M_2 always positive unless E_{ijk}^{xyr} , $E_{ijk}^{x zr}$ and E_{ijk}^{yzr} are all zero; in this case $M_2 = 0$. The expressions that appear in the evaluation of M_2 , M_1 and M_0 have maximum algebraic degree 7 in the input quantities.

First, we shall consider the case $M_2 \neq 0$; this is geometrically equivalent to the non-collinearity of the inverted centers C_n^* for $n \in \{i, j, k\}$. Since M_2 is assumed to be strictly positive, the quadratic $M(d)$ has 0, 1 or 2 real roots depending on whether the sign of the discriminant $\Delta_M = M_1^2 - 4M_2M_0$ is negative, zero or positive respectively. We evaluate the discriminant Δ_M of $M(d)$ to be

$$\Delta_M = 4(D_{ijk}^{uvw})^2((D_{ijk}^{uv})^2 + (D_{ijk}^{uw})^2 + (D_{ijk}^{vw})^2 - (D_{ijk}^{u\rho})^2 - (D_{ijk}^{v\rho})^2 - (D_{ijk}^{w\rho})^2),$$

hence determining $\text{sign}(\Delta_M)$ requires 8-fold algebraic operations since

$$\text{sign}(\Delta_M) = \text{sign}\left((E_{ijk}^{x yp})^2 + (E_{ijk}^{x zp})^2 + (E_{ijk}^{yzp})^2 - (E_{ijk}^{x rp})^2 - (E_{ijk}^{y rp})^2 - (E_{ijk}^{z rp})^2\right).$$

If $\Delta_M < 0$, the predicate returns “0”, whereas if $\Delta_M = 0$ we get that $M(d)$ has a double root $d = -M_1/(2M_2)$; the predicate returns “1 double” if $\text{sign}(d) = -\text{sign}(M_1)$ is positive, otherwise it returns “0”. Finally, if $\Delta_M > 0$, $M(d)$ has two distinct roots $d_1 < d_2$, whose sign we check for positiveness. Using Vieta’s rules and since $\text{sign}(M_1) = -\text{sign}(d_1 + d_2)$ and $\text{sign}(M_0) = \text{sign}(d_1)\text{sign}(d_2)$ we conclude that, if M_0 is negative the predicate returns “1”. Otherwise, the predicate’s outcome is “0” if $\text{sign}(M_1)$ is positive or “2” if $\text{sign}(M_1)$ is negative.

The signs of M_1 and M_0 are determined using the the following equalities

$$\begin{aligned}\text{sign}(M_1) &= -\text{sign}(D_{ijk}^{vw\rho} D_{ijk}^{vw} + D_{ijk}^{uw\rho} D_{ijk}^{uw} + D_{ijk}^{uv\rho} D_{ijk}^{uv}) \\ &= -\text{sign}(E_{ijk}^{yzr} E_{ijk}^{yzp} + E_{ijk}^{x zr} E_{ijk}^{x zp} + E_{ijk}^{x yr} E_{ijk}^{x yp}), \\ \text{sign}(M_0) &= \text{sign}((D_{ijk}^{vw\rho})^2 + (D_{ijk}^{uw\rho})^2 + (D_{ijk}^{uv\rho})^2 - (D_{ijk}^{uvw})^2) \\ &= \text{sign}((E_{ijk}^{yzr})^2 + (E_{ijk}^{x zr})^2 + (E_{ijk}^{x yr})^2 - (E_{ijk}^{xyz})^2).\end{aligned}$$

The expressions that appear in the evaluation of M_1 and M_0 have maximum algebraic degree 7 in the input quantities.

We now analyse the case where $M_2 = 0$. Since the quantities D_{ijk}^{uv} , D_{ijk}^{uw} and D_{ijk}^{vw} are all zero, we get that $a = D_{ijk}^{vw\rho} / D_{ijk}^{uvw}$, $b = D_{ijk}^{uw\rho} / D_{ijk}^{uvw}$ and $c = D_{ijk}^{uv\rho} / D_{ijk}^{uvw}$. Therefore, since $d = \rho_i - (au_i + bv_i + cw_i)$ we can evaluate the sign of d as

$$\begin{aligned}\text{sign}(d) &= \text{sign}(D_{ijk}^{uvw}) \text{sign}(D_{ijk}^{uvw} \rho_i - D_{ijk}^{vw\rho} u_i - D_{ijk}^{uw\rho} v_i - D_{ijk}^{uv\rho} w_i) \\ &= \text{sign}(E_{ijk}^{xyz}) \text{sign}(E_{ijk}^{xyz} r_i^* - E_{ijk}^{yzr} x_i^* - E_{ijk}^{x zr} y_i^* - E_{ijk}^{x yr} z_i^*)\end{aligned}$$

The evaluation of the sign of d therefore demands operations of algebraic degree 4 (in the input quantities). If $d > 0$ the predicate returns “1” otherwise it returns “0”.

Last, we consider the case $D_{ijk}^{uvw} = 0$. We can safely assume that at least one of the quantities D_{ijk}^{uv} , D_{ijk}^{uw} and D_{ijk}^{vw} since otherwise the centers C_n^* for $n \in \{i, j, k\}$ would be collinear¹, yielding a contradiction. Assume without loss of generality that $D_{ijk}^{uv} \neq 0$, we can solve the system of equations in terms of c and we get that

$$a = \frac{-D_{ijk}^{v\rho} + cD_{ijk}^{vw}}{D_{ijk}^{uv}}, \quad b = \frac{D_{ijk}^{u\rho} - cD_{ijk}^{uw}}{D_{ijk}^{uv}}, \quad d = \frac{D_{ijk}^{uv\rho}}{D_{ijk}^{uv}}.$$

If we substitute a and b in the equation $a^2 + b^2 + c^2 = 1$, we get that c is a root of a quadratic polynomial $L(c) = L_2 c^2 + L_1 c + L_0$, where

$$\begin{aligned}L_2 &= (D_{ijk}^{uv})^2 + (D_{ijk}^{uw})^2 + (D_{ijk}^{vw})^2, \\ L_1 &= -2(D_{ijk}^{v\rho} D_{ijk}^{vw} + D_{ijk}^{u\rho} D_{ijk}^{uw}), \\ L_0 &= (D_{ijk}^{u\rho})^2 + (D_{ijk}^{v\rho})^2 - (D_{ijk}^{uv})^2.\end{aligned}$$

¹If $D_{ijk}^{uv} = D_{ijk}^{uw} = D_{ijk}^{vw} = 0$, the projections of the points C_n^* , $n \in \{i, j, k\}$, on all three planes $w = 1$, $v = 1$ and $u = 1$ would form a flat triangle. For each projection, this is equivalent to either some of the projection points coinciding or all three being collinear. Since the original centers C_n^* are distinct points for $n \in \{i, j, k\}$, they must be collinear for such a geometric property to hold.

We evaluate the discriminant of $L(c)$ to be

$$\Delta_L = 4(D_{ijk}^{xy})^2 \left[(D_{ijk}^{xy})^2 + (D_{ijk}^{xz})^2 + (D_{ijk}^{yz})^2 - (D_{ijk}^{xr})^2 - (D_{ijk}^{yr})^2 - (D_{ijk}^{zy})^2 \right],$$

and therefore $\text{sign}(\Delta_L) = \text{sign}(\Delta_M)$. The evaluation of $\text{sign}(\Delta_M)$ is known to require 8-fold algebraic operations as shown in a previous case.

If $\Delta_M < 0$, the predicate returns “0”; there is no tangent plane in the inverted space. Otherwise, we determine the sign of d to be

$$\text{sign}(d) = \text{sign}(D_{ijk}^{uv\rho}/D_{ijk}^{uv}) = \text{sign}(E_{ijk}^{xyr}/E_{ijk}^{xyp}) = \text{sign}(E_{ijk}^{xyr})\text{sign}(E_{ijk}^{xyp})$$

and the following cases arise

- (1) If $\Delta_M = 0$, the predicate returns “0” if $d < 0$ or “1” if $d > 0$.
- (2) If $\Delta_M > 0$, the predicate returns “0” if $d < 0$ or “2” if $d > 0$.

Note that in the last case, the algebraic degrees of Δ_M and d are 8 and 4 respectively and that, in every possible scenario, 8 was the maximum algebraic degree of any quantity we had to evaluate.

Lemma 9. *The EXISTENCE predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

4.4. The DISTANCE predicate. In this section, we provide a detailed analysis regarding the evaluation of the $\text{DISTANCE}(S_i, S_j, S_k, S_a)$ predicate, as this was defined in Section 3.3.3. As stated there, the outcome of this primitive is the tuple $(\text{sign}(\delta(S_a, \Pi_{ijk}^-)), \text{sign}(\delta(S_a, \Pi_{ijk}^+)))$, where the planes Π_{ijk}^- and Π_{ijk}^+ are commonly tangent to the sites S_i, S_j and S_k . The existence of these planes is guaranteed since the trisector τ_{ijk} is assumed to be “hyperbolic” (see Section 3.3.2). Also take into consideration that in the scope of this paper, this subpredicate never returns a degenerate answer i.e., neither $(\text{sign}(\delta(S_a, \Pi_{ijk}^-)))$ nor $\text{sign}(\delta(S_a, \Pi_{ijk}^+))$ can equal zero.

We shall now consider such a plane $\Pi : ax + by + cz + d = 0$ tangent to all sites S_i, S_j and S_k , that leaves them all on the same side (this site is denoted as the *positive* side). If we assume without loss of generality that $a^2 + b^2 + c^2 = 1$, it must stand that $\delta(C_n, \Pi) = ax_n + by_n + cz_n + d = r_n$ for $n \in \{i, j, k\}$, where $\delta(C_n, \Pi)$ denotes the signed Euclidean of the center C_n from the plane Π . If we consider the distance $\epsilon = \delta(S_a, \Pi) = \delta(C_a, \Pi) - r_a$ of the sphere S_a from this plane, then the following system of equations must hold.

$$\begin{aligned}
ax_i + by_i + cz_i + d &= r_i, \\
ax_j + by_j + cz_j + d &= r_j, \\
ax_k + by_k + cz_k + d &= r_k, \\
ax_a + by_a + cz_a + d &= r_a + \epsilon, \\
a^2 + b^2 + c^2 &= 1
\end{aligned}$$

Due to the initial assumption of a hyperbolic trisector, only two such planes, Π_{ijk}^- and Π_{ijk}^+ , are cotangent to the spheres S_i, S_j and S_k and therefore algebraically satisfy the system of equations above. In other words, there exist only two distinct algebraic solutions $(a_v, b_v, c_v, d_v, \epsilon_v)$ for $v \in \{1, 2\}$, and apparently $\{\epsilon_1, \epsilon_2\} = \{\delta_a^+, \delta_a^-\}$, where $\delta_a^+ = \delta(S_a, \Pi_{ijk}^+)$ and $\delta_a^- = \delta(S_a, \Pi_{ijk}^-)$.

Bearing in mind that the answer to the DISTANCE predicate is actually the tuple (δ_a^+, δ_a^-) , we want to determine the signs of ϵ_1 and ϵ_2 and correspond them to δ_a^+ and δ_a^- . Regarding the signs of $\{\epsilon_1, \epsilon_2\}$ we algebraically study the system of equations above.

First, we consider the case $D_{ijka}^{xyz} \neq 0$ in detail. Under this assumption and with the use of Crammer's rule, we may express a, b and c with respect to ϵ as

$$a = \frac{D_{ijka}^{yzr} - \epsilon D_{ijk}^{yz}}{D_{ijka}^{xyz}}, \quad b = \frac{-D_{ijka}^{x zr} + \epsilon D_{ijk}^{xz}}{D_{ijka}^{xyz}}, \quad c = \frac{D_{ijka}^{xyr} + \epsilon D_{ijk}^{xy}}{D_{ijka}^{xyz}}.$$

After substituting these expressions in the equation $a^2 + b^2 + c^2 = 1$, we obtain that $\Lambda(\epsilon) = \Lambda_2 \epsilon^2 + \Lambda_1 \epsilon + \Lambda_0 = 0$,

$$\begin{aligned}
\Lambda_2 &= (D_{ijk}^{xy})^2 + (D_{ijk}^{yz})^2 + (D_{ijk}^{xz})^2, \\
\Lambda_1 &= -2(D_{ijka}^{yzr} D_{ijk}^{yz} + D_{ijka}^{x zr} D_{ijk}^{xz} - D_{ijka}^{xyr} D_{ijk}^{xy}), \\
\Lambda_0 &= (D_{ijka}^{xyr})^2 + (D_{ijka}^{x zr})^2 + (D_{ijka}^{yzr})^2 - (D_{ijka}^{xyz})^2.
\end{aligned}$$

Take into consideration that Λ_2 cannot be zero since otherwise the centers C_i, C_j and C_k would be collinear, yielding a contradiction; we have assumed that τ_{ijk} is hyperbolic. Since $\Lambda(\epsilon)$, which is definitely a quadratic in terms of ϵ , has the aforementioned ϵ_1 and ϵ_2 as roots, we may use Vieta's formula to determine the signs of ϵ_1, ϵ_2 . All we need is the signs of Λ_1 and Λ_0 , quantities of algebraic degree 5 and 6 respectively (we already proved Λ_2 is positive).

If both roots are positive, negative or zero, the predicate returns $(+, +)$, $(-, -)$ or $(0, 0)$ respectively. If the sign of the roots differ, we must consider a way of distinguishing which of them corresponds to δ_a^+ and δ_a^- . Since the ϵ_1

and ϵ_2 have different signs, the set $\{\epsilon_1, \epsilon_2\} = \{\delta_a^+, \delta_a^+\}$ is one of the following: $\{+, -\}$, $\{0, -\}$ or $\{0, +\}$.

The cases we now consider are products of geometric observations based on the three possible configurations of the centers C_i, C_j, C_k and C_a :

- If D_{ijka}^{xyz} is positive, then C_a lies on the positive (resp. negative) side of the plane Π_{ijk} . In this case, only the geometric configurations $(\delta_a^-, \delta_a^+) = (+, -), (0, -)$ or $(+, 0)$ are possible.
- If D_{ijka}^{xyz} is negative, then C_a lies on the negative side of the plane Π_{ijk} . In this case, only the geometric configurations $(\delta_a^-, \delta_a^+) = (-, +), (-, 0)$ or $(0, +)$ are possible.

For example, if the roots of $\Lambda(\epsilon)$ turn out to be one positive and one negative, the predicate will return $(+, -)$ if D_{ijka}^{xyz} is positive or $(-, +)$ if D_{ijka}^{xyz} is negative.

Lastly, we consider the case $D_{ijka}^{xyz} = 0$, where the centers C_n , for $n \in \{i, j, k, a\}$, are coplanar. In this context, it is both algebraically and geometrically apparent that $\epsilon_1 = \epsilon_2 = \epsilon$; the sphere S_a either intersects, is tangent or does not intersect both planes Π_{ijk}^- and Π_{ijk}^+ . Specifically, if $D_{ijk}^{xyz} \neq 0$, then $\epsilon = -D_{ijka}^{xyzr}/D_{ijk}^{xyz}$ and we immediately evaluate $\text{sign}(\epsilon) = -\text{sign}(D_{ijka}^{xyzr}) \cdot \text{sign}(D_{ijk}^{xyz})$. If $D_{ijk}^{xyz} = 0$ then at least one of the quantities $D_{ijk}^{xy}, D_{ijk}^{xz}, D_{ijk}^{yz}$ does not equal zero, since the centers C_n for $n \in \{i, j, k\}$ are not collinear. Assume without loss of generality that $D_{ijk}^{xy} \neq 0$, then $\epsilon = D_{ijka}^{xyr}/D_{ijk}^{xy}$ and we evaluate $\epsilon = \text{sign}(D_{ijka}^{xyr})\text{sign}(D_{ijk}^{xy})$. In every case the predicate returns $\text{sign}(\epsilon), \text{sign}(\epsilon)$.

Taking into consideration that the evaluation of Λ_1 is the most degree-demanding operation to resolve the DISTANCE predicate, we have proven the following lemma.

Lemma 10. *The DISTANCE predicate can be evaluated by determining the sign of quantities of algebraic degree at most 6 (in the input quantities).*

4.5. The SHADOW Predicate. In this section, we provide a way of resolving the SHADOW $(S_i, S_j, S_k, S_\alpha)$ predicate as this was described in Section 3.3.5. Assuming that the trisector τ_{ijk} is “hyperbolic” and no degeneracies occur, we have shown that the outcome, which is the topological structure of $\mathcal{SR}(S_\alpha)$ on τ_{ijk} , is one of the following: \emptyset , $(-\infty, \infty) = \mathbb{R}$, $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) , or $(-\infty, \phi) \cup (\chi, +\infty)$, where $\phi, \chi \neq \pm\infty$.

Initially, the predicates $\text{EXISTENCE}(S_i, S_j, S_k, S_\alpha)$ and $\text{DISTANCE}(S_i, S_j, S_k, S_\alpha)$ are called; let us denote by E and (σ_1, σ_2) their respective outcomes. The geometric interpretation of these two quantities leads to the resolution of the shadow region $\mathcal{SR}(S_\alpha)$ with respect to τ_{ijk} .

Regarding the meaning of the signs σ_1 and σ_2 , assume that $\sigma_1 = -$. In this case, the sphere S_α does not intersect the plane Π_{ijk}^- which is in fact the Apollonius sphere $\mathcal{T}(\zeta^{-1}(-\infty))$. In other words, $\zeta^{-1}(-\infty)$ does not belong to $\mathcal{SR}(S_\alpha)$ i.e., $-\infty$ does not “shows up” in the predicate’s outcome. Using a similar argument, if $\sigma_2 = -$ then $+\infty$ shows up in the outcome whereas, if σ_1 or σ_2 is positive then $-\infty$ or $+\infty$ shows up respectively. Note that $\sigma_1, \sigma_2 \in \{+, -\}$ under the assumption of no degeneracies.

Regarding the geometric interpretation of E , we have mentioned that the boundary points of $\mathcal{SR}(S_\alpha)$ correspond to centers of Apollonius spheres of the sites S_n , for $n \in \{i, j, k, \alpha\}$ that are not centered at infinity. We have shown in Section 4.3 that the cardinality of these Apollonius spheres is in fact E and assuming no degeneracies there are either 0, 1 or 2.

The combined information of σ_1, σ_2 and E is used to determine the type of $\mathcal{SR}(S_\alpha)$, as follows:

- If $E = 0$, the shadow region $\mathcal{SR}(S_\alpha)$ has no boundary points, hence it’s type is either $(-\infty, +\infty)$ or \emptyset . If $\sigma_1 = -$ (or $\sigma_2 = -$), the predicate returns $(-\infty, +\infty)$ otherwise, if $\sigma_1 = +$ (or $\sigma_2 = +$), it returns \emptyset .
- If $E = 1$, then $\mathcal{SR}(S_\alpha)$ has one boundary points, hence it’s type is either $(-\infty, \phi)$ or $(\chi, +\infty)$. If $\sigma_1 = -$ (or $\sigma_2 = +$), the predicate returns $(-\infty, \phi)$ otherwise, if $\sigma_1 = +$ (or $\sigma_2 = -$) it returns $(\chi, +\infty)$.
- Finally, if $E = 2$ then $\mathcal{SR}(S_\alpha)$ has two boundary points, hence it’s type is either (χ, ϕ) or $(-\infty, \phi) \cup (\chi, +\infty)$. If $\sigma_1 = +$ (or $\sigma_2 = +$) the predicate returns (χ, ϕ) otherwise, if $\sigma_1 = -$ (or $\sigma_2 = -$), it returns $(-\infty, \phi) \cup (\chi, +\infty)$.

Since the evaluation of the SHADOW predicate only requires the call of the DISTANCE and the EXISTENCE predicates, which demand operations of maximum algebraic degree 6 and 8 respectively, we proved the following lemma.

Lemma 11. *The SHADOW predicate can be evaluated by determining the sign of quantities of algebraic degree at most 8 (in the input quantities).*

4.6. The ORDER predicate. The major subpredicate called during the evaluation of the EDGECONFLICT predicate via the algorithm described in Section 3.4 is the so called ORDER predicate. As already described in Section 3.3.6, the ORDER $(S_i, S_j, S_k, S_a, S_b)$ returns the order of appearance on the oriented tri-sector τ_{ijk} of any Apollonius vertices defined by the sites S_i, S_j, S_k and S_a or S_b . This primitive needs to be evaluated only if both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are not of the form \emptyset or \mathbb{R} , with $\{a, b\} \in \{l, m, q\}$. Indeed, for $a \in \{l, m\}$ we know that v_{ijkl} and v_{ikjm} exist and therefore $\mathcal{SR}(S_l)$ and $\mathcal{SR}(S_m)$ can not be of that type. Moreover, $\mathcal{SR}(S_q)$ can not be of this form since otherwise,

the ORDER predicate is not called at all. To conclude, the predicates SHADOW (S_i, S_j, S_k, S_n) for $n \in \{a, b\}$ are called in advance and their outcomes are considered to be known for the rest of the analysis. As already mentioned, τ_{ijk} is also assumed to be a “hyperbolic” trisector.

The analysis of the ORDER predicate consists most of our breakthrough and contribution in this research area. We demonstrate the usefulness of the inversion technique by proving the strong connection between the original and the inverted space. By exploiting this relation, we are able to create useful tools which can also be used in both 2D and 3D Apollonius diagrams to improve results of the current bibliography.

The following section is organised as follows. In Section 4.6.1 we make an introduction to the inverted space and some initial observations on how it is connected with the original space. Afterwards, in Section 4.6.2, we define a 2-dimensional sub-space of the inverted space, to make useful geometric observations effortless. In the last three sections, we break up our analysis of the ORDER predicate according to the shadow region types $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$. Ultimately, we prove the following lemma.

Lemma 12. *The ORDER predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

4.6.1. *The \mathcal{W} -space.* The original space where the sites S_i, S_j, S_k, S_a and S_b lie is called the \mathcal{Z} -space. However, most of our analysis is carried in the inverted \mathcal{W} -space, as defined in Section 2.1 with a slight modification.

Specifically, observe that the definition of the \mathcal{W} -space depends on the choice of the sphere S_I . Since a cyclic permutation of the sites S_i, S_j and S_k does not alter the outcome of the ORDER predicate, we assume that $r_k = \min\{r_i, r_j, r_k\}$; otherwise we name exchange the sites so it does. We now select to invert \mathcal{Z} -space “through the sphere S_k ” i.e., we reduce the radii of all initial sites by r_k (and obtain \mathcal{Z}^* -space) and then invert all points with C_k as the pole.

In the resulting space, called the \mathcal{W} -space, is where most of our geometric observations and analysis takes place. Notice that when we reduce the sites S_a or S_b by r_k we may end up with a sphere of negative radius if $r_a < r_k$ or $r_b < r_k$. Although the existence (or not) of spheres with negative radius in \mathcal{W} -space makes the geometric configurations quite different to handle, the algebraic methods we present here can handle both cases without modifications. For this reason, we shall assume for the rest of the Section that all sites lying in \mathcal{W} -space have positive radii.

The analysis that follows is based on the strong relation that holds between the geometric configuration of the sites S_i, S_j, S_k, S_a and S_b in \mathcal{Z} -space and

the corresponding configuration of the inverted sites S_i^*, S_j^*, S_a^* and S_b^* in \mathcal{W} -space.

In \mathcal{W} -space, O denotes the point $(0, 0, 0)$ which is the image of the “point at infinity” of \mathcal{Z} -space. Given a point p that lies on τ_{ijk} , $\mathcal{T}(p)$ denotes the external Apollonius sphere tangent to the sites S_i, S_j and S_k , centered at p . Such a sphere $\mathcal{T}(p)$ of \mathcal{Z} -space corresponds in \mathcal{W} -space to a plane, denoted $\Pi^*(p)$, that is tangent to the inverted sites S_i^* and S_j^* and therefore tangent to the cone defined by them. Notice that, since S_i^* and S_j^* are distinct spheres of \mathcal{W} -space due to their pre-images S_i and S_j also being distinct in \mathcal{Z} -space, the cone $\mathcal{K}(S_i^*, S_j^*)$ is well defined. For the rest of this Section, we define \mathcal{K}^* to be the semi cone (or cylinder if $\rho_i = \rho_j$) that contains S_i^* and S_j^* .

Let us observe what happens in \mathcal{W} -space when we consider this point p moving on τ_{ijk} such that $\zeta(p)$ goes from $-\infty$ towards $+\infty$. The corresponding plane $\Pi^*(p)$ rotates remaining tangent to \mathcal{K}^* , with starting and ending positions the planes denoted by $\Pi^*(-\infty)$ and $\Pi^*(+\infty)$ respectively.

It is obvious that these two planes correspond to the two Apollonius spheres of S_i, S_j and S_k “at infinity” i.e., the planes $\{\Pi_{ijk}^-, \Pi_{ijk}^+\}$. These planes of \mathcal{Z} -space must be distinct in the case of a “hyperbolic” trisector τ_{ijk} as shown in Sections 4.1 and 4.4, and, as a result, their images in \mathcal{W} -space must also be distinct.

Moreover, each of $\Pi^*(-\infty)$ and $\Pi^*(+\infty)$ must go through O because the their pre-images are planes that go through the “point at infinity” in \mathcal{Z} -space. Combining these last two remarks, we conclude that the points C_i^*, C_j^* and O are not collinear and O lies strictly outside the semi cone \mathcal{K}^* . It is of great importance to understand that the last fact holds only because τ_{ijk} is “hyperbolic”; if we were studying the “elliptic” trisector type, O would lie strictly inside the semi cone \mathcal{K}^* and in the degenerate case of a “parabolic” trisector, O would lie on the boundary of \mathcal{K}^* .

For every point $p \in \tau_{ijk} \setminus \{\pm\infty\}$, the sphere $\mathcal{T}(p)$ is an external Apollonius sphere and therefore does not contain the “point at infinity” in \mathcal{Z} -space. Correspondingly, its image in \mathcal{W} -space i.e., the plane $\Pi^*(p)$, must leave the point O and the centers of the spheres S_i^* and S_j^* on the same side. This side of the plane $\Pi^*(p)$ is called *positive* whereas the other is referred to as *negative*.

Let us now consider the plane Π^* that goes through the points S_i^*, S_j^* and the point O of \mathcal{W} -space. The well-definition of Π^* follows from the non-collinearity of the three points we proved earlier. This plane turns out to be the inversion image of the plane Π_{ijk} that goes through the centers C_i, C_j and C_k (and apparently the point at infinity) in \mathcal{Z} -space.

The latter plane separates \mathcal{Z} -space into two half-spaces, \mathcal{H}_+ and \mathcal{H}_- , where \mathcal{H}_+ (resp. \mathcal{H}_-) denotes the set of points N such that $\text{ORIENT3D}(N, C_i, C_j, C_k)$ is positive (resp. negative). The plane Π^\star also separates \mathcal{W} -space into two half-spaces, \mathcal{H}_+^\star and \mathcal{H}_-^\star , where \mathcal{H}_+^\star (resp. \mathcal{H}_-^\star) denotes the set of points M such that $\text{ORIENT3D}(M, C_i^\star, C_j^\star, O)$ is positive (resp. negative).

If we now consider a point C_n of \mathcal{Z} -space and its inversion image C_n^\star in \mathcal{Z} -space, we can easily prove that

$$\begin{aligned} \text{ORIENT3D}(C_n^\star, C_i^\star, C_j^\star, O) &= \text{sign}(p_i^\star p_j^\star p_n^\star D_{nij}^{uvw}) = \text{sign}(D_{nij}^{xyz}) \\ &= \text{ORIENT3D}(C_n, C_i, C_j, C_k). \end{aligned}$$

This result indicates that \mathcal{H}_+^\star and \mathcal{H}_-^\star are in fact the inversion images of the open semi-spaces \mathcal{H}_+ and \mathcal{H}_- .

This simple correspondence of semi-spaces yields a remarkable result. If we consider a point $p \in \tau_{ijk}^+$ (resp. τ_{ijk}^-), then the tangency points of the Apollonius sphere $\mathcal{T}(p)$ with the spheres S_i and S_j must lie on \mathcal{H}_+ (resp. \mathcal{H}_-). If this fact is considered through inversion, the tangency points of the plane $\Pi^\star(p)$ with the semi cone \mathcal{K}^\star must lie on \mathcal{H}_+^\star (resp. \mathcal{H}_-^\star). Furthermore, if we let p move on τ_{ijk}^+ (resp. τ_{ijk}^-) such that $\zeta(p)$ goes to $+\infty$ (resp. $-\infty$), we can deduce that the plane $\Pi^\star(+\infty)$ (resp. $\Pi^\star(-\infty)$) is the one that goes through the point O and its tangency points with the spheres S_i^\star and S_j^\star lie on \mathcal{H}_+^\star (resp. \mathcal{H}_-^\star).

4.6.2. The \mathcal{Y} -space. All these observations are indicative of the strong connection of the original and the inverted space. However, since a three-dimensional space such as \mathcal{W} -space makes observations and case breakdowns too complex, we will now consider a sub-space to carry our analysis. For this reason, we consider a (random) plane Π^\perp in \mathcal{W} -space that is perpendicular to the axis of the semi-cone \mathcal{K}^\star at point $\hat{\mathcal{A}}$ and intersects it at a full circle \mathcal{K}' ; the intersection of \mathcal{W} -space and Π^\perp is called the \mathcal{Y} -space. Notice that in every figure representing the \mathcal{Y} -space, we always depict the Π^\perp plane such that the vector $\overrightarrow{C_i^\star C_j^\star}$ points “towards” the reader (see Figure 8).

In \mathcal{Y} -space, we will use the following notation:

- $\hat{\ell}(p)$, $\hat{\ell}(\pm\infty)$, $\hat{\ell}$ and $\hat{\ell}(o_{ijk})$ denote the intersection of the plane Π^\perp with the planes $\Pi^\star(p)$, $\Pi^\star(\pm\infty)$, Π^\star and $\Pi^\star(o_{ijk})$ respectively.
- $\hat{\eta}$, $\hat{\theta}$, \hat{o} and \hat{p} denote the points of tangency of \mathcal{K}' and the lines $\hat{\ell}(-\infty)$, $\hat{\ell}(+\infty)$, $\hat{\ell}(o_{ijk})$ and $\hat{\ell}(p)$ respectively (for $p \in \tau_{ijk}$).
- The intersection of \mathcal{H}_+^\star (resp. \mathcal{H}_-^\star) with the Π^\perp plane is called the positive (resp. negative) half-plane $\hat{\mathcal{H}}_+$ (resp. $\hat{\mathcal{H}}_-$).

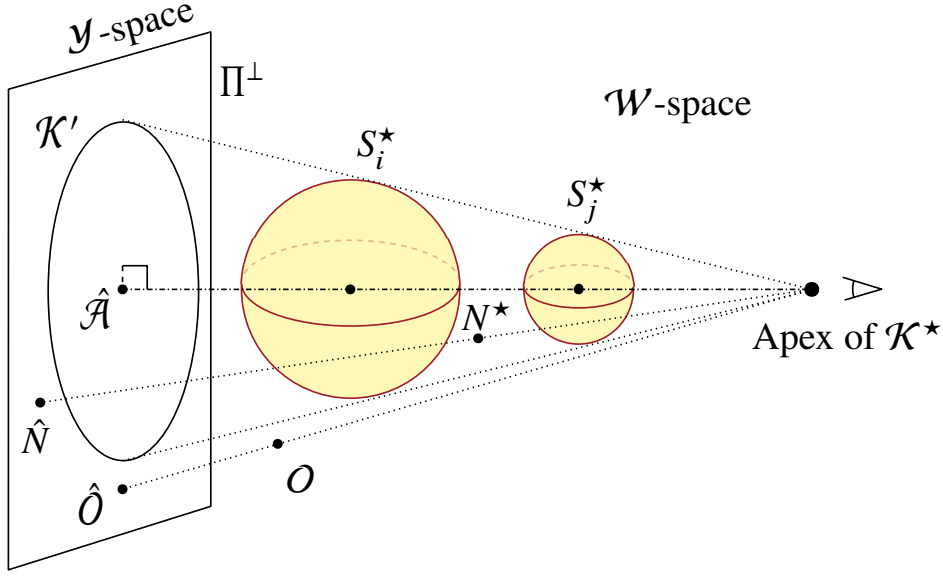


FIGURE 8. The \mathcal{Y} -space, where most of the analysis of the ORDER predicate is carried on, is in essence a projection of \mathcal{W} -space to a plane Π^\perp via the apex of the cone \mathcal{K}^\star . Since \mathcal{Y} -space is a 2-dimensional space, an observation regarding a \mathcal{W} -space geometric configuration is made easier if we consider the corresponding configuration in \mathcal{Y} -space.

- The positive (resp. negative) side of the line $\hat{\ell}(p)$ for a point $p \in \tau_{ijk}$ to be the side that contains (resp. does not contain) the point $\hat{\mathcal{A}}$.
- \hat{O} denotes the point of intersection of the lines $\hat{\ell}(-\infty)$ and $\hat{\ell}(+\infty)$.

We shall now prove an equivalency relation between the trisector τ_{ijk} and an arc of \mathcal{K}' , which is the biggest idea upon which the rest of our analysis is based. If a point p moves on τ_{ijk} such that $\zeta(p)$ goes from $-\infty$ to $+\infty$ then, in \mathcal{Y} -space, the corresponding point \hat{p} moves on \mathcal{K}' from the point $\hat{\eta}$ to the point $\hat{\theta}$, going through the point \hat{o} . Observe that there is a 1-1 correspondence between the oriented trisector τ_{ijk} and the oriented arc $(\hat{\eta}\hat{o}\hat{\theta})$. We denote this 1-1 and onto mapping from τ_{ijk} to the arc $(\hat{\eta}\hat{o}\hat{\theta})$ by $\psi(\cdot)$, such that $\psi(p) = \hat{p}$.

What naturally follows is that the order of appearance of the vertices v_{ijka} , v_{ikja} , v_{ijkb} and v_{ikjb} on the oriented trisector amounts to the order of appearance of the points $\psi(v_{ijka})$, $\psi(v_{ikja})$, $\psi(v_{ijkb})$ and $\psi(v_{ikjb})$ on the oriented arc $(\psi(\eta), \psi(o), \psi(\theta))$ (see Figure 9). Consider however that we only need to order the Apollonius vertices that actually exist among v_{ijka} , v_{ikja} , v_{ijkb} and v_{ikjb} .

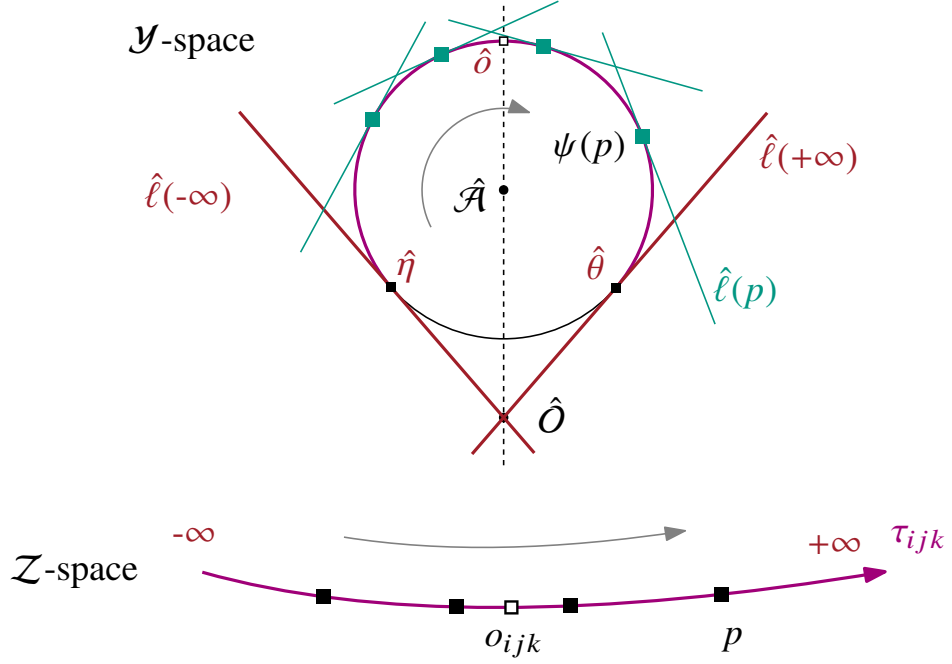


FIGURE 9. There is a strong relation between the \mathcal{Z} -space and the \mathcal{Y} -space. When the point p of \mathcal{Z} -space traverses the trisector τ_{ijk} towards its positive direction, the corresponding point $\psi(p)$ of \mathcal{Z} -space traverses the arc $(\psi(\eta), \psi(o), \psi(\theta))$. At the same time, $\hat{\ell}(p)$ rotates in \mathcal{Y} -space, remaining tangent to \mathcal{K}' , with starting and ending positions the lines $\hat{\ell}(-\infty)$ and $\hat{\ell}(+\infty)$ respectively.

Lemma 13. *There is a 1-1 correspondence between the order of appearance of the existing vertices among $v_{ijka}, v_{ikja}, v_{ijkb}$ and v_{ikjb} on the oriented trisector τ_{ijk} and the order of appearance of the existing points $\psi(v_{ijka}), \psi(v_{ikja}), \psi(v_{ijkb})$ and $\psi(v_{ikjb})$ on the oriented arc $(\psi(\eta), \psi(o), \psi(\theta))$.*

The lemma suggests that the outcome of the ORDER predicate could return the order of appearance of the images of the aforementioned Apollonius vertices on the arc $(\psi(\eta), \psi(o), \psi(\theta))$ instead of the order of the original vertices on the trisector τ_{ijk} .

Towards our goal of obtaining the ordering of the inverted Apollonius vertices, we denote the circle $\hat{S}_n \in \mathcal{Y}$ -space for $n \in \{a, b\}$ which will be considered as the image of S_n^* of \mathcal{W} -space. We need to define the image of these spheres in a proper way such that they carry their geometric properties from \mathcal{W} -space to \mathcal{Y} -space. For this reason, we consider the center \hat{C}_n of \hat{S}_n to be the intersection of Π^\perp with the line that goes through the apex of the cone

\mathcal{K}^* and the center \hat{C}_n . Observe that such a line is well defined since the latter two points cannot coincide; if they did then $\mathcal{SR}(S_n)$ would be \emptyset or \mathbb{R} yielding a contradiction ². Finally, the radius of \hat{C}_n is such that \hat{C}_n is tangent to each of the existing lines $\hat{\ell}(v_{ijkn})$ and $\hat{\ell}(v_{ikjn})$ (at least one of them exists due to $\mathcal{SR}(S_n)$ not being \emptyset or \mathbb{R}).

Another crucial property we want to point out derives from the inversion mapping we used to go from \mathcal{Z}^* -space to \mathcal{W} -space. The mapping $W(z)$ we used is known to be *inclusion preserving* i.e., the relative position of two spheres in the original space is preserved in the inverted one. For example, consider a sphere S_μ that intersects the (existing) sphere Apollonius sphere $\mathcal{T}(v_{ikjn})$ (resp. $\mathcal{T}(v_{ijkn})$), for $n \in \{a, b\}$ in \mathcal{Z} -space. After reducing both spheres by r_k , their images in \mathcal{Z}^* retain the same relative position i.e., they intersect. Applying the inversion mapping, it must stand that the sphere S_μ^* must intersect the negative side of $\Pi^*(v_{ikjn})$ (resp. $\Pi^*(v_{ijkn})$) since this half space is precisely the inversion image of the interior of $\mathcal{T}(v_{ikjn})$ (resp. $\mathcal{T}(v_{ijkn})$). Finally, if we consider this configuration in \mathcal{Y} -space, we deduce that \hat{S}_μ must intersect the negative side of $\hat{\ell}(v_{ikjn})$ (resp. $\hat{\ell}(v_{ijkn})$).

In a similar way we can show that if S_μ is tangent or does not intersect the sphere Apollonius sphere $\mathcal{T}(v_{ikjn})$ (resp. $\mathcal{T}(v_{ijkn})$) then, in \mathcal{Y} -space, \hat{S}_μ is tangent to $\hat{\ell}(v_{ikjn})$ (resp. $\hat{\ell}(v_{ijkn})$) or does not intersect its negative side (see Figure 10). A fact tightly connected with these observations is that the relative position of S_μ and $\mathcal{T}(v_{ikjn})$ (resp. $\mathcal{T}(v_{ijkn})$) is provided by the **INSPHERE** predicate. Specifically,

- if **INSPHERE** $(S_i, S_k, S_j, S_n, S_m)$ is $-$, 0 or $+$ then S_m intersects, is tangent to or does not intersect the Apollonius sphere $\mathcal{T}(v_{ikjn})$ and,
- if **INSPHERE** $(S_i, S_j, S_k, S_n, S_m)$ is $-$, 0 or $+$ then S_m intersects, is tangent to or does not intersect the Apollonius sphere $\mathcal{T}(v_{ijkn})$.

Since these **INSPHERE** predicates can be evaluated as shown in Section 3.3.1, the relative position of \hat{S}_μ with respect to any of the existing lines $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_{ijkn})$ can be determined in \mathcal{Y} -space, for $n \in \{a, b\}$.

Lemma 14. *The circle \hat{S}_m intersects, is tangent to or does not intersect the negative side of $\hat{\ell}(v_{ikjn})$ (resp. $\hat{\ell}(v_{ijkn})$) if and only if the **INSPHERE** predicate with input $(S_i, S_k, S_j, S_n, S_m)$ (resp. $(S_i, S_j, S_k, S_n, S_m)$) is negative, zero or positive respectively.*

²In such a geometric configuration, there would not exist a plane in \mathcal{W} -space co-tangent to all spheres S_i^*, S_j^* and S_n^* . Equivalently, in \mathcal{Z} -space there would not exist an Apollonius sphere of the sites S_i, S_j, S_k and S_n hence $\mathcal{SR}(S_n)$ would be either \emptyset or \mathbb{R} based on the analysis of Section 4.5.

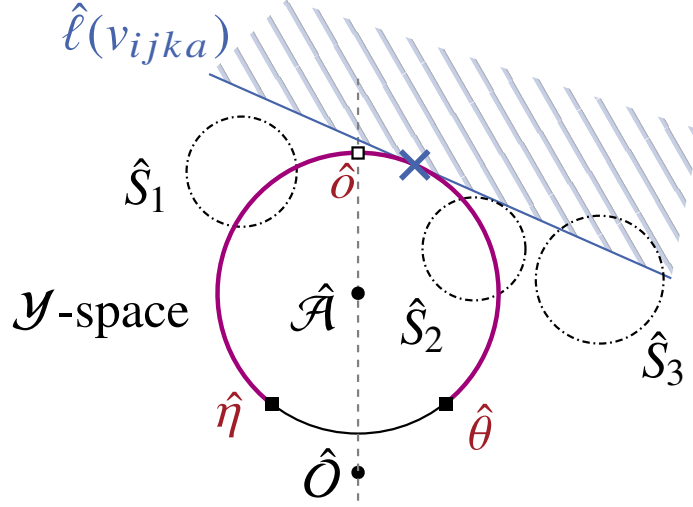


FIGURE 10. As Lemma 14 suggests, the InSphere predicate with input $(S_i, S_j, S_k, S_a, S_n)$ is positive, zero or negative for $n = 0, 1$ or 2 respectively.

4.6.3. *The classic configuration.* When the $\text{ORDER}(S_i, S_j, S_k, S_a, S_b)$ predicate is called, we initially determine the shadow region types of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ via the appropriate SHADOW predicates. If the type of each shadow region is (χ, ϕ) or $(\chi, +\infty)$ (not necessary the same), we say that we are in *classic configuration*. In such a setup, we can distinguish simpler cases regarding the ordering the images of the Apollonius vertices on the oriented arc $(\hat{\eta}\hat{o}\hat{\theta})$.

We therefore break up the analysis the ORDER predicate depending on whether we are in a classic (Section 4.6.3) or non-classic configuration (Section 4.6.5), the latter being reduced to the former using various observations. Let us now study in more detail what kind of information derives from the fact that the sites S_a and S_b satisfy the conditions of a classic configuration.

Suppose S_n , for $n = a$ or b , is (χ, ϕ) and therefore, the endpoints $\{\chi, \phi\}$ must correspond to the two Apollonius vertices $\{\zeta(v_{ijkn}), \zeta(v_{ikjn})\}$ on the trisector τ_{ijk} based on the remarks of Section 3.3.5. Let us consider χ and ϕ as $\zeta(v_n)$ and $\zeta(v'_n)$ respectively, with $\{v_n, v'_n\} = \{v_{ijkn}, v_{ikjn}\}$. Then, for every $p \in \tau_{ijk}$ such that $v_n \prec p \prec v'_n$, the sphere $\mathcal{T}(p)$ must intersect S_n as this follows from the definition of $\mathcal{SR}(S_n)$.

If we consider this point p moving on τ_{ijk} , initially starting from the left-endpoint position v_n , then we observe that the Apollonius sphere $\mathcal{T}(v_n)$ intersects S_n if we move its center infinitesimally towards the positive direction of τ_{ijk} . Taking a closer look at the tangency points T_i, T_j, T_k and T_n of $\mathcal{T}(v_n)$

with the spheres S_i, S_j, S_k and S_n respectively, and since the orientation of τ_{ijk} is based in such a way on the orientation of C_i, C_j and C_k , it must hold that T_n must lie with respect to the plane formed by T_i, T_j and T_k such that $T_i T_j T_k T_n$ be negative oriented. For that reason, v_n is in fact v_{ikjn} and subsequently v'_n is v_{ijkn} .

The same argument can be used to prove that if $\mathcal{SR}(S_n)$ is $(\chi, +\infty)$ then χ corresponds to $\zeta(v_{ikjn})$. If we apply a similar analysis in all shadow region types that contain a finite endpoint, it will lead to the following lemma.

Lemma 15. *If the type of the shadow region $\mathcal{SR}(S_n)$ of a sphere S_n on a hyperbolic trisector is one of the following: $(-\infty, \phi)$, $(\chi, +\infty)$, (χ, ϕ) , or $(-\infty, \phi) \cup (\chi, +\infty)$ where $\phi, \chi \neq \pm\infty$, then $\chi \equiv \zeta(v_{ikjn})$ and $\phi \equiv \zeta(v_{ijkn})$.*

As the lemma suggest, in a classic configuration, for $n \in \{a, b\}$,

- if $\mathcal{SR}(S_n) = (\chi, \phi)$, then both v_{ijkn} and v_{ikjn} exist and $v_{ikjn} \prec v_{ijkn}$, whereas
- if $\mathcal{SR}(S_n) = (\chi, +\infty)$, then v_{ikjn} exists while v_{ijkn} does not.

An equally important result arises when pondering of the possible positions of the circle \hat{S}_n , for $n \in \{a, b\}$ with any of the existing lines $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_{ijkn})$. Firstly, let us consider the scenario where $\mathcal{SR}(S_n) = (\chi, \phi)$ and in consequence, both lines exist. In this case, both points $\psi(v_{ikjn})$ and $\psi(v_{ijkn})$ exist on the oriented arc such that $\psi(v_{ikjn}) \prec \psi(v_{ijkn})$, as this follows from all previous remarks. From the definition of $\mathcal{SR}(S_n) = (\chi, \phi)$, it derives as a result that for a point p on the trisector τ_{ijk} such that $v_{ikjn} \prec pv_{ijkn}$, the sphere $\mathcal{T}(p)$ intersects with S_n . Using the “inclusion preserving” argument, it must stand that, in \mathcal{Y} -space, \hat{S}_n intersects with the negative side of $\hat{\ell}(p)$. Therefore, if \hat{M} is the midpoint $\psi(v_{ikjn})$ and $\psi(v_{ijkn})$ on the arc $(\eta o \theta)$ and \mathbb{V} denotes the open ray from $\hat{\mathcal{A}}$ towards \hat{M} , then the circle \hat{S}_n must be centered at a point on \mathbb{V} i.e., $\hat{C}_n \in \mathbb{V}$ (see Figure 11).

Lastly, let us examine the case where $\mathcal{SR}(S_n) = (\chi, +\infty)$. We begin by observing that S_n must intersect with Π_{ijk}^+ due to the definition of the shadow region and this amounts, in \mathcal{Y} -space, to \hat{S}_n intersecting the negative side of the line $\hat{\ell}(+\infty)$. Moreover, following a similar analysis as in the case of $\mathcal{SR}(S_n) = (\chi, \phi)$, we come to the conclusion that \hat{S}_n must be tangent to $\hat{\ell}(v_{ikjn})$ at a point \hat{T} such that the (counterclockwise) angle $(\hat{\mathcal{A}}, \psi(v_{ikjn}), \hat{T})$ is 90° (not 270° , see Figure 12). This fact must hold for the line $\hat{\ell}(p)$ to intersect \hat{S}_n , for every point $p \in \tau_{ijk}$ with $\psi(v_{ikjn}) \prec \psi(p) \prec \hat{\theta}$.

4.6.4. Ordering the Apollonius vertices in a classic configuration. In a classic configuration, we take for granted that, for $n \in \{a, b\}$, either $\mathcal{SR}(S_n) = (\chi, \phi)$ and therefore $v_{ikjn} \prec v_{ijkn}$ on the trisector τ_{ijk} or $\mathcal{SR}(S_n) = (\chi, +\infty)$ and

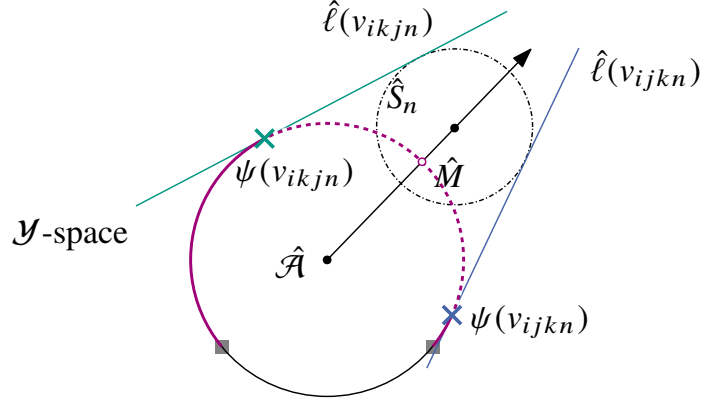


FIGURE 11. If both v_{ikjn} , v_{ijkn} exist and $\mathcal{SR}(S_a)$ is (χ, ϕ) with respect to the trisector τ_{ijk} , \hat{C}_n must lie on the ray (\hat{A}, \hat{M}) . The dotted arc represent the image of $\mathcal{SR}(S_n)$ in \mathcal{Y} -space; it is obvious that, for any $\psi(p)$ in this arc, \hat{S}_n intersects the negative side of the line $\hat{\ell}(p)$. As of Lemma 14, S_n must intersect the sphere $\mathcal{T}(p)$ which is equivalent to $p \in \mathcal{SR}(S_n)$.

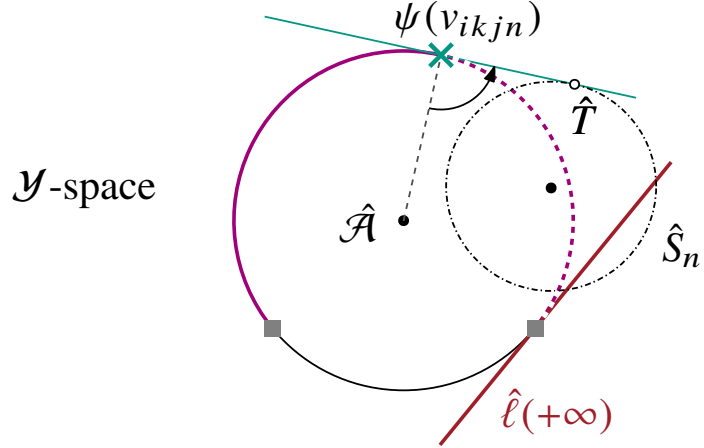


FIGURE 12. If v_{ikjn} exists and v_{ijkn} does not, the shadow region of S_n is known to be $(\chi, +\infty)$. This means that the tangency point \hat{T} of \hat{S}_n with the line $\hat{\ell}(v_{ikjn})$ is on the side of the line that forms a 90° angle with the vector $(\psi(v_{ikjn}), \hat{A})$. The image of $\mathcal{SR}(S_n)$ in \mathcal{Y} -space in this case is the dotted arc which is indeed of the form $(\chi, +\infty)$.

only v_{ikjn} exists on τ_{ijk} . To order all of these existing Apollonius vertices, we break down our analysis into four sub-configurations.

Case A.: All vertices v_{ikja} , v_{ijka} , v_{ikjb} and v_{ijkb} exist i.e., both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are of type (χ, ϕ) .

Case B.: Only the vertices v_{ikja} and v_{ikjb} exist i.e., both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are of type $(\chi, +\infty)$.

Case C.: Only the vertices v_{ijka} , v_{ijkb} and v_{ikjb} exist i.e., the type of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are (χ, ϕ) and $(\chi, +\infty)$ respectively.

Case D.: Only the vertices v_{ikjb} , v_{ijkb} and v_{ikja} exist i.e., the type of $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are $(\chi, +\infty)$ and (χ, ϕ) respectively.

The last Case D is identical with the Case C if we name exchange the spheres S_a and S_b . Therefore, if Case D arises, we evaluate $\text{ORDER}(S_i, S_j, S_k, S_b, S_a)$ instead, which falls in Case C, and return the resulting ordering of v_{ikjb} , v_{ijkb} and v_{ikja} . Consequently, we only need to consider the Cases A, B and C; the analysis of each configuration is deployed separately in the following sections.

Analysis of Case A. Given that all Apollonius vertices v_{ijka} , v_{ikja} , v_{ijkb} , v_{ikjb} exist on τ_{ijk} and $v_{ikja} \prec v_{ijka}$ as well as $v_{ikjb} \prec v_{ijkb}$, the list of all possible orderings (and thus outcomes of the ORDER predicate) is the following

OrderCase 1.: $v_{ikja} \prec v_{ijka} \prec v_{ikjb} \prec v_{ijkb}$,

OrderCase 2.: $v_{ikja} \prec v_{ikjb} \prec v_{ijka} \prec v_{ijkb}$,

OrderCase 3.: $v_{ikjb} \prec v_{ikja} \prec v_{ijka} \prec v_{ijkb}$,

OrderCase 4.: $v_{ikjb} \prec v_{ikja} \prec v_{ijka} \prec v_{ijka}$,

OrderCase 5.: $v_{ikjb} \prec v_{ijkb} \prec v_{ikja} \prec v_{ijka}$,

OrderCase 6.: $v_{ikja} \prec v_{ikjb} \prec v_{ijkb} \prec v_{ijka}$.

Any of these ordering on the trisector is equivalent to the corresponding ordering of $\psi(v_{ikja})$, $\psi(v_{ikjb})$, $\psi(v_{ijkb})$ and $\psi(v_{ijka})$ on the arc $(\hat{\eta}, \hat{\delta}, \hat{\theta})$ as stated in Lemma 13.

We now study separately all these possible cases in \mathcal{Y} -space following the same approach. Firstly, we place the images of the Apollonius vertices on the arc according to the OrderCase we are examining. Then, we consider a possible location for each of the circles \hat{S}_a and \hat{S}_b taking into consideration the remarks made in the Section 4.6.3. Lastly, we draw some conclusions regarding the relative position of \hat{S}_a and \hat{S}_b with the lines $\psi(v_{ikjb})$, $\psi(v_{ijkb})$ and $\psi(v_{ijka})$, $\psi(v_{ikja})$ respectively. The later observations are then translated as INSPHERE test's results based on Lemma 14.

Let us consider one case in detail, for example the OrderCase 2 configuration; a similar approach will be applied to each OrderCase. In Figure 13

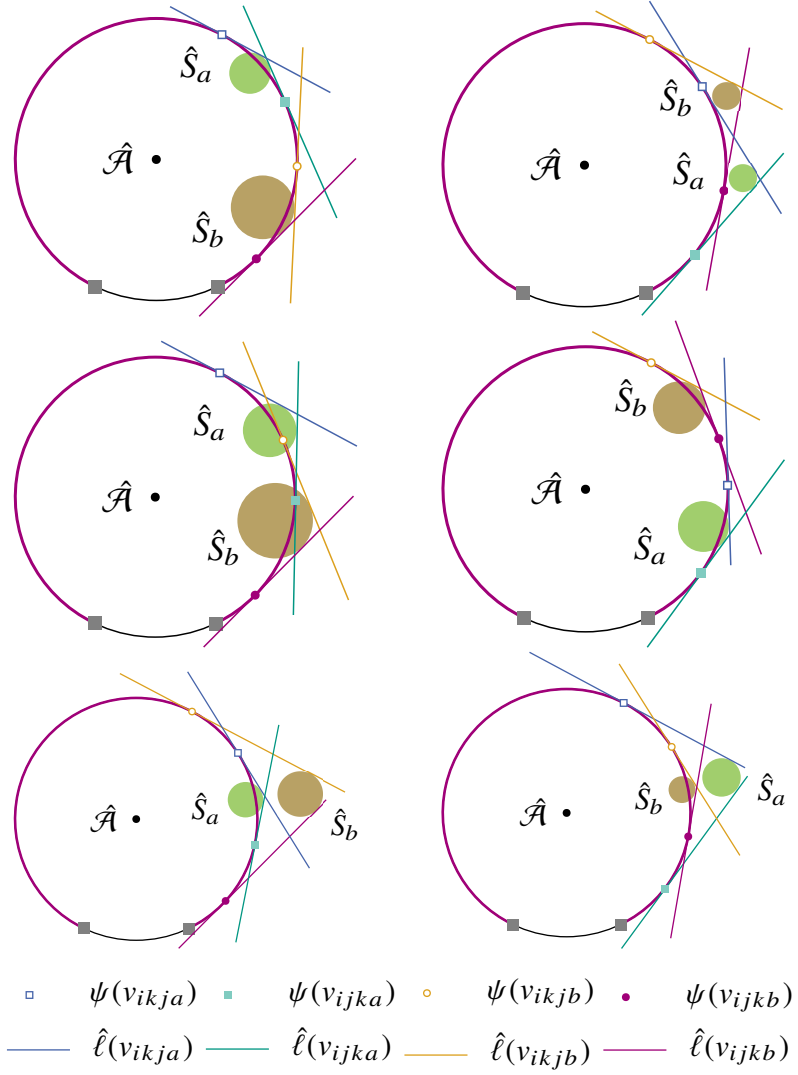


FIGURE 13. Under the assumption that all Apollonius vertices $v_{ikja}, v_{ijka}, v_{ikjb}$ and v_{ijkb} exist on the trisector τ_{ijk} , we consider all possible orderings of these vertices. As of Lemma 13, each of these orderings is equivalent to a respective ordering of the points $\psi(v_{ikja}), \psi(v_{ijka}), \psi(v_{ikjb})$ and $\psi(v_{ijkb})$ on the oriented arc $(\psi(\eta), \psi(o), \psi(\theta))$ of \mathcal{Y} -space. For every possible ordering a possible location of \hat{S}_a and \hat{S}_b is considered, such that the shadow regions $\mathcal{SR}(S_n)$ for $n \in \{a, b\}$ is of type (χ, ϕ) since we examine a classic configuration. From top to bottom, Left: OrderCase 1, 2, 3. From top to bottom, Right: OrderCase 4, 5, 6.

(Left Column, 2nd Row), we consider a random³ layout of the points $\psi(v_{ikja})$, $\psi(v_{ikjb})$, $\psi(v_{ijkb})$ and $\psi(v_{ijka})$ (and the respective tangent planes at these points) that appear in the order OrderCase 2 dictates. In the same figure, we provide a possible location of \hat{S}_n , for $n \in \{a, b\}$ with respect to the selected layout; \hat{S}_n must be tangent to both $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_{ijkn})$, and centered according to the analysis of Section 4.6.3.

Finally, we inspect the relative position of \hat{S}_a (resp. \hat{S}_b) with the lines $\psi(v_{ikjb})$ and $\psi(v_{ijkb})$ (resp. $\psi(v_{ikja})$ and $\psi(v_{ijka})$). In any such random layout, it must hold that

- \hat{S}_a intersects the negative side of $\psi(v_{ikjb})$ but does not intersect the negative side of $\psi(v_{ijkb})$ and,
- \hat{S}_b intersects the negative side of $\psi(v_{ijka})$ but does not intersect the negative side of $\psi(v_{ijka})$.

Another way of proving this, is by looking at the shadow regions of S_a and S_b on the arc. For example, in a OrderCase 2 configuration, $v_{ikja} \prec v_{ikjb} \prec v_{ijkb}$ and subsequently $v_{ikjb} \in \mathcal{SR}(S_a)$, since $\mathcal{SR}(S_a)$ consists of all points $p \in \tau_{ijk}$ with $v_{ikja} \prec p \prec v_{ijka}$. As a result the sphere $\mathcal{T}(v_{ikjb})$ must intersect the sphere S_a i.e., \hat{S}_a intersects the negative side of $\hat{\ell}(v_{ikjb})$.

Lastly, we translate the obtained relative positions of circles and lines of \mathcal{Y} -space to INSPHERE tests outcomes. For example, if \hat{S}_a intersects the negative side of $\hat{\ell}(v_{ikjb})$, we conclude that $\text{INSPHERE}(S_i, S_k, S_j, S_b, S_a)$ is negative, as an immediate result of Lemma 9. In conclusion, we get that if the Apollonius vertices we seek to order appear as in OrderCase 2, then

- $\text{INSPHERE}(S_i, S_k, S_j, S_b, S_a) = -$ and $\text{INSPHERE}(S_i, S_j, S_k, S_b, S_a) = +$,
- $\text{INSPHERE}(S_i, S_k, S_j, S_a, S_b) = +$ and $\text{INSPHERE}(S_i, S_j, S_k, S_a, S_b) = -$.

Ultimately, we create a table of the four possible INSPHERE outcomes that hold in each of the OrderCase's 1 to 6 (see Table 1). A simple way of distinguishing the ordering of the Apollonius vertices becomes clear now, due to tuple of outcomes being so different in most OrderCase's. Indeed, if $Q = (Q_1, Q_2, Q_2, Q_4)$ denotes the ordered tuple of the INSPHERE predicate

³In Figure 13, the circles \hat{S}_a and \hat{S}_b always appear to be centered on the same side of the line going through $\hat{\mathcal{A}}$ and $\hat{\mathcal{O}}$. This was done for reasons of consistency and does not always correspond to reality, since it would be equivalent to C_a^* and C_b^* always lying on the same side of the plane going through the points C_i^* , C_j^* and C_k^* .

OrderCase:	1	2	3	4	5	6
$\text{INSPHERE}(S_i, S_k, S_j, S_b; S_a)$	+	−	+	+	+	−
$\text{INSPHERE}(S_i, S_j, S_k, S_b; S_a)$	+	+	+	−	+	−
$\text{INSPHERE}(S_i, S_k, S_j, S_a; S_b)$	+	+	−	−	+	+
$\text{INSPHERE}(S_i, S_j, S_k, S_a; S_b)$	+	−	−	+	+	+

TABLE 1. Case A: Signs of all possible INSPHERE tests that follow from the analysis of each OrderCase. Notice that only the rows that correspond to OrderCase 1 and OrderCase 5 are identical and therefore we only need the signs of these INSPHERE predicates to determine the OrderCase most of the time. If all predicates return positive, we require some auxiliary tests to distinguish between the two cases.

outcomes, where

$$Q_1 = \text{INSPHERE}(S_i, S_k, S_j, S_b, S_a), \quad Q_2 = \text{INSPHERE}(S_i, S_j, S_k, S_b, S_a), \\ Q_3 = \text{INSPHERE}(S_i, S_k, S_j, S_a, S_b), \quad Q_4 = \text{INSPHERE}(S_i, S_j, S_k, S_a, S_b),$$

then the ORDER predicate returns :

- the ordering of OrderCase 2, if $Q = (-, +, +, -)$ or,
- the ordering of OrderCase 3, if $Q = (+, +, -, -)$ or,
- the ordering of OrderCase 4, if $Q = (+, -, -, +)$ or,
- the ordering of OrderCase 6, if $Q = (-, -, +, +)$.

Finally, if $Q = (+, +, +, +)$ then either OrderCase 1 or OrderCase 5 is the correct ordering of the vertices (see Figure 14). To resolve this dilemma, we distinguish cases depending on the ordering of the midpoints M_a and M_b of the arcs $(\psi(v_{ikja}), \psi(v_{ijka}))$ and $(\psi(v_{ikjb}), \psi(v_{ijkb}))$ respectively. Since it must either hold that $\{v_{ikja} \prec v_{ijka}\} \prec \{v_{ikjb} \prec v_{ijkb}\}$ (OrderCase 1) or $\{v_{ikjb} \prec v_{ijkb}\} \prec \{v_{ikja} \prec v_{ijka}\}$ (OrderCase 5), then we are obviously in the former case if $M_a \prec M_b$ or in the latter if $M_b \prec M_a$.

To determine the ordering of M_a and M_b on the arc $(\hat{\eta}, \hat{o}, \hat{\theta})$, we shall use the auxiliary point \hat{o} . Initially, we reflect on the fact that, for $n \in \{a, b\}$, \hat{C}_n is known to lie on the open ray from \mathcal{A} towards M_n . It is also apparent that the points O , \mathcal{A} and \hat{o} are collinear and appear in this order on the line $\hat{\ell}$ they define.

Based on the definition of \mathcal{Y} -space and the remarks of Section 4.6.2, the midpoint M_n , for $n \in \{a, b\}$, satisfies

- $M_n \prec \hat{o}$ if and only if $\text{ORIENT3D}(C_n^*, C_i^*, C_j^*, O) < 0$,
- $\hat{o} \prec M_n$ if and only if $\text{ORIENT3D}(C_n^*, C_i^*, C_j^*, O) > 0$,

- $M_n \equiv \hat{o}$ if and only if $\text{ORIENT3D}(C_n^*, C_i^*, C_j^*, O) = 0$.

Lastly, we notice that $\text{ORIENT3D}(C_b^*, C_i^*, C_j^*, C_a^*) < 0$ is equivalent to \hat{C}_b lying on the “right side” of the oriented line going from \mathcal{A} to \hat{C}_a .

Ultimately, we determine the relative position of M_a and M_b by combining all the information extracted of the ORIENT3D predicates mentioned, using the following algorithm.

Step 1.: We evaluate $\Pi = o_1 \cdot o_2$, where $o_1 = \text{ORIENT3D}(C_a^*, C_i^*, C_j^*, O)$ and $o_2 = \text{ORIENT3D}(C_b^*, C_i^*, C_j^*, O)$. If $\Pi > 0$ go to Step 2a, otherwise go to Step 2b.

Step 2a.: Either $M_a, M_b \prec \hat{o}$ or $\hat{o} \prec M_a, M_b$. In either case, we evaluate $o_3 = \text{ORIENT3D}(C_b^*, C_i^*, C_j^*, C_a^*)$. If $o_3 < 0$ then $M_a \prec M_b$, and the ORDER predicate returns the ordering of OrderCase 1. Otherwise, $M_b \prec M_a$ and the ordering of OrderCase 5 is returned. (see Figure 15).

Step 2b.: Either \hat{o} lies in-between M_a and M_b or is identical with one of them. In both cases, if $o_1 < o_2$ then $M_a \prec M_b$ and the ORDER predicate return the ordering of OrderCase 1, otherwise, $M_b \prec M_a$ and the ordering of OrderCase 5 is returned. (see Figure 16).

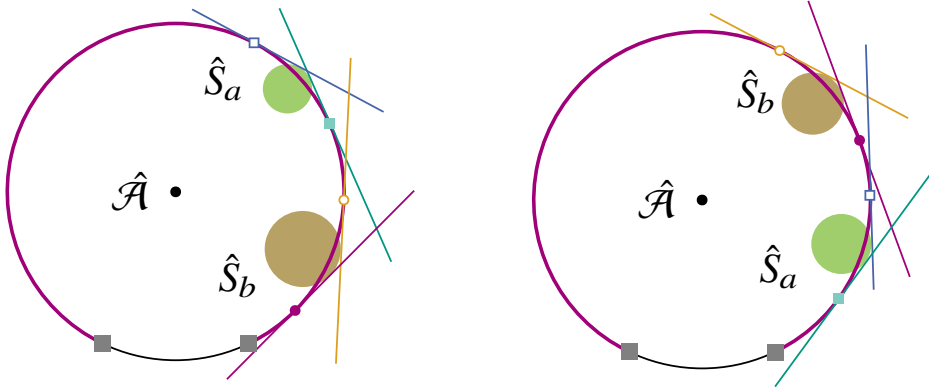


FIGURE 14. If $Q = (+, +, +, +)$ then we must determine if the ordering of the Apollonius vertices correspond to OrderCase 1 (Left) or 5 (Right). It is apparent that we are in the first case if and only if the ray (\mathcal{A}, t) “meets” \hat{C}_a first as t traverses the arc $(\psi(\eta), \psi(o), \psi(\theta))$.

Analysis of Case B. Given that $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ are both of the form $(\chi, +\infty)$ and therefore only the Apollonius vertices v_{ikja} and v_{ikjb} exist on τ_{ijk} , the ordering of these vertices on $(\hat{\eta}, \hat{o}, \hat{\theta})$ is either

OrderCase 1.: $v_{ikja} \prec v_{ikjb}$ or,

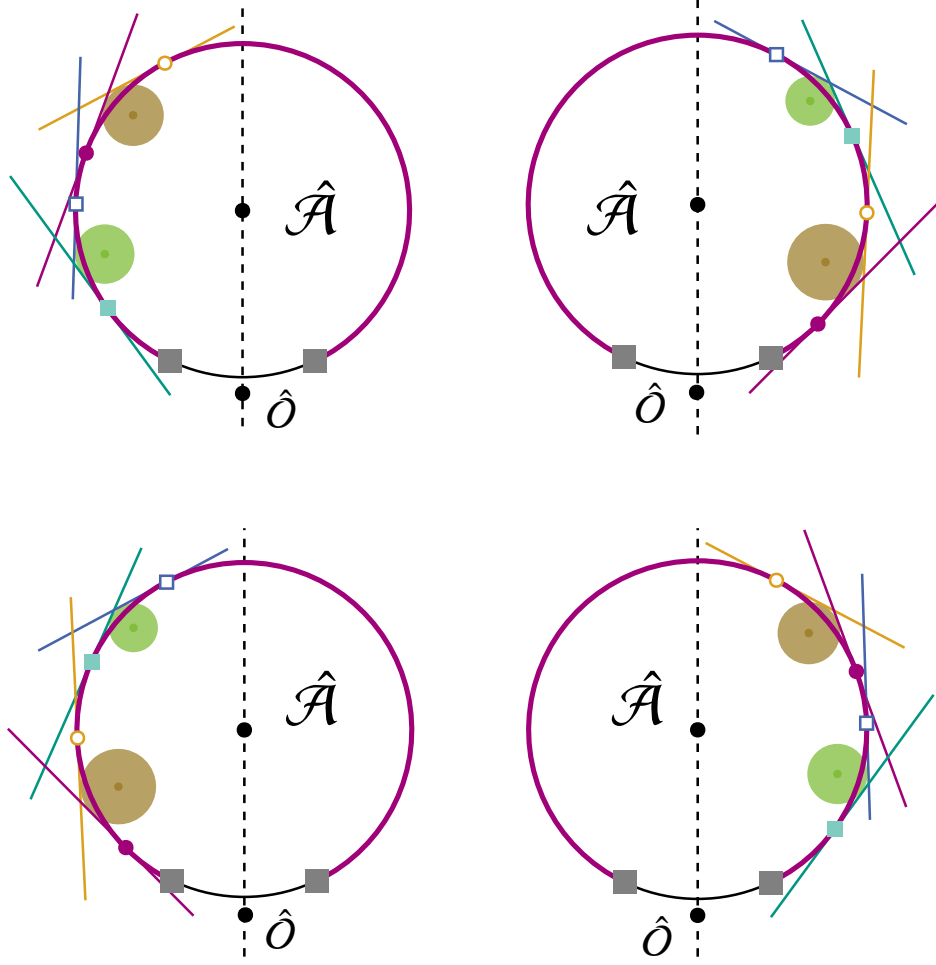


FIGURE 15. If $o_1 \cdot o_2 > 0$, the centers \hat{C}_a and \hat{C}_b must lie on the same side of the line that goes through $\hat{\mathcal{A}}$ and $\hat{\mathcal{O}}$. No matter which side the centers lie on, if $o_3 < 0$ or equivalently \hat{C}_a lies on the left side of the oriented line that goes from $\hat{\mathcal{A}}$ to \hat{C}_b , (Top 2 Figures) then we obtain the ordering described in OrderCase 1. Otherwise, we obtain the ordering described in OrderCase 5 (Bottom 2 Figures).

OrderCase 2.: $v_{ikjb} \prec v_{ikja}$.

A similar analysis with the Case A is used to resolve the predicate in Case B; we create a table regarding the possible outcomes of the INSphere tests with inputs $(S_i, S_k, S_j, S_a, S_b)$ and $(S_i, S_k, S_j, S_b, S_a)$. Recall that the outcome of $Q_1 = \text{INSphere}(S_i, S_k, S_j, S_a, S_b)$ (resp. $Q_2 = \text{INSphere}(S_i, S_k, S_j, S_b, S_a)$)

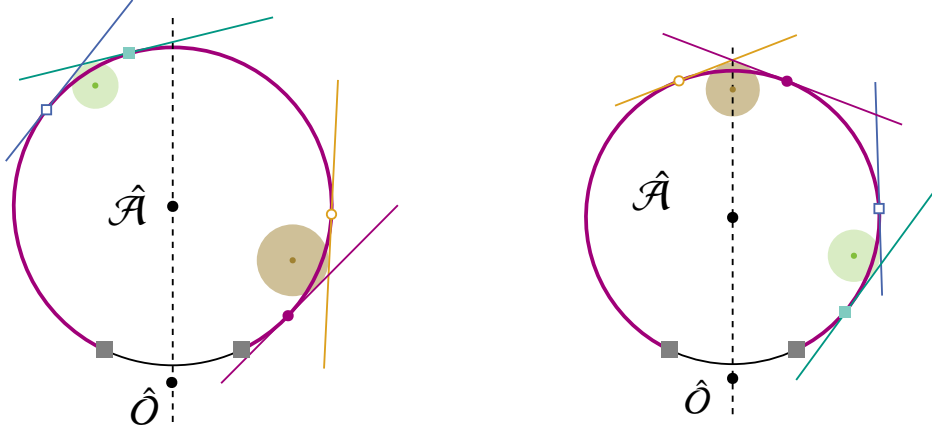


FIGURE 16. If $o_1 \cdot o_2 \leq 0$, the centers \hat{C}_a and \hat{C}_b lie on different sides of the line that goes through \hat{A} and \hat{O} (Left) or only one of them lies on the line (since we are in either OrderCase 1 or 5). No matter which side the centers lie on, if $o_1 < o_2$ we obtain the ordering described in OrderCase 1. Otherwise, we obtain the ordering described in OrderCase 5 (Bottom 2 Figures).

is $-$, 0 or $+$ if the circle \hat{S}_b (resp. \hat{S}_a) intersects, is tangent to or does not intersect the negative side of $\hat{\ell}(v_{ikja})$ (resp. $\hat{\ell}(v_{ikjb})$).

Using a simpler approach, we observe that

- in OrderCase 1, v_{ikja} does not belong to the shadow region of the sphere S_b on τ_{ijk} and therefore $\mathcal{T}(v_{ikja})$ does not intersect S_b or equivalently $Q_1 = +$. Moreover, in this case, v_{ikjb} belongs to the shadow region of the sphere S_a on τ_{ijk} and therefore $\mathcal{T}(v_{ikjb})$ intersects S_a or equivalently $Q_2 = -$.
- In OrderCase 2, v_{ikja} belongs to the shadow region of the sphere S_b on τ_{ijk} and therefore $\mathcal{T}(v_{ikja})$ intersects S_b or equivalently $Q_1 = -$. Furthermore, v_{ikjb} does not belong to the shadow region of the sphere S_a on τ_{ijk} and therefore $\mathcal{T}(v_{ikjb})$ does not intersect S_a or equivalently $Q_2 = +$.

In conclusion we can answer the $\text{ORDER}(S_i, S_j, S_k, S_a, S_b)$ predicate in case B by evaluating Q_1 ; if $Q_1 = +$ then return OrderCase 1 otherwise, if $Q_1 = -$ return OrderCase 2. Equivalently, we could evaluate Q_2 instead of Q_1 ; if $Q_2 = -$ then return OrderCase 1 otherwise if $Q_2 = +$ return OrderCase 2. The following equivalencies are depicted in Table 2 and this concludes the analysis of Case B.

	OrderCase 1	OrderCase 2
$\text{InSPHERE}(S_i, S_k, S_j, S_b; S_a)$	+	-
$\text{InSPHERE}(S_i, S_k, S_j, S_a; S_b)$	-	+

TABLE 2. Case B: Signs of all possible InSPHERE tests that follow from the analysis of each OrderCase. Notice that each column is distinct and therefore we can determine the OrderCase after the outcomes of the InSPHERE predicates.

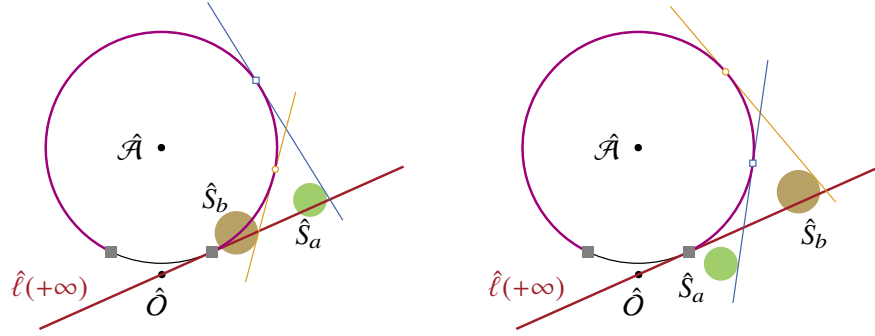


FIGURE 17. In Case B, it is assumed that only the Apollonius vertices v_{ikja} and v_{ijkb} exist on the trisector τ_{ijk} . We consider the two possible orderings of these vertices: OrderCase 1 (Left) and OrderCase 2 (Right). Similar with Case A, we consider the corresponding ordering of the points $\psi(v_{ikja})$ and $\psi(v_{ijkb})$ on the arc $(\psi(\eta), \psi(o), \psi(\theta))$. A possible location for the circles \hat{S}_a and \hat{S}_b is drawn based on the analysis of Section 3.4.

Analysis of Case C. In Case C, it is assumed that $\mathcal{SR}(S_a) = (\chi, \phi)$ hence $v_{ikja} \prec v_{ijka}$ while $\mathcal{SR}(S_b) = (\chi, +\infty)$ and consequently only v_{ikjb} exists on the arc $(\hat{\eta}, \hat{o}, \hat{\theta})$. All three possible orderings of these three Apollonius vertices on the arc are

OrderCase 1.: $v_{ikja} \prec v_{ijka} \prec v_{ikjb}$,

OrderCase 2.: $v_{ikja} \prec v_{ikjb} \prec v_{ijka}$ and

OrderCase 3.: $v_{ikjb} \prec v_{ikja} \prec v_{ijka}$.

The analysis of this Case uses the same tools and analysis presented in the previous two cases with small adjustments, since $\mathcal{SR}(S_a) = (\chi, \phi)$ and $\mathcal{SR}(S_b) = (\chi, +\infty)$ in the case studied. Let us denote by Q_1, Q_2 and Q_3 the results of the InSPHERE predicates with inputs $(S_i, S_k, S_j, S_b, S_a)$, $(S_i, S_k, S_j, S_a, S_b)$ and $(S_i, S_j, S_k, S_b, S_a)$ respectively.

Notice now that

	OrderCase 1	OrderCase 2	OrderCase 3
$\text{INSPHERE}(S_i, S_k, S_j, S_b; S_a)$	+	−	+
$\text{INSPHERE}(S_i, S_k, S_j, S_a; S_b)$	+	+	−
$\text{INSPHERE}(S_i, S_j, S_k, S_a; S_b)$	+	−	−

TABLE 3. Case C: Signs of all possible INSPHERE tests that follow from the analysis of each OrderCase. Notice that each column is distinct and therefore we can determine the OrderCase after the outcomes of the INSPHERE predicates, as in Case B.

- in OrderCase 1, v_{ikjb}, v_{ikja} and v_{ijka} do not belong to the shadow region of S_a , S_b and S_b respectively and therefore it must stand that $Q_1 = Q_2 = Q_3 = +$.
- in OrderCase 2, v_{ikjb} and v_{ijka} belong to the shadow region of S_a and S_b respectively and for this reason $Q_1 = -$ and $Q_3 = -$. On the other hand, v_{ikja} does not belong to the shadow region of S_b and therefore $Q_2 = +$. Finally,
- in OrderCase 3, both v_{ikja} and v_{ijka} belong to the shadow region of S_b and consequently $Q_2 = -$ and $Q_3 = -$ whereas, v_{ikjb} does not belong to the shadow region of S_a and therefore $Q_1 = +$.

Since the tuple $Q = (Q_1, Q_2, Q_3)$ is different in each OrderCase 1 to 3, we can answer the predicate by evaluating the three INSPHERE predicates hence Q and correspond it the respective ordering (also see Table 3):

- if $Q = (+, +, +)$, return the ordering of OrderCase 1 or,
- if $Q = (-, +, -)$, return the ordering of OrderCase 2 otherwise,
- if $Q = (+, -, -)$, return the ordering of OrderCase 3.

Algebraic Cost to resolve the Cases A, B or C. The analysis of the Cases A, B and C showed that the answer of the $\text{ORDER}(S_i, S_j, S_k, S_a, S_b)$ predicate in a classic configuration ultimately amounts to determining the outcomes of up to four INSPHERE predicates and, if needed, some auxiliary ORIENT3D tests.

To answer any of the INSPHERE predicates that may require evaluation, we must perform operations of maximum algebraic degree 10 (in the input quantities), as mentioned in Section 3.3.1.

Regarding the auxiliary ORIENT3D primitives, we observe that

$$\begin{aligned} \text{ORIENT3D}(C_b^*, C_i^*, C_j^*, C_a^*) &= \text{sign}(D_{bija}^{uvw}) = \text{sign}(p_i^* p_j^* p_a^* p_b^*) \text{sign}(E_{bija}^{xyzp}) \\ &= \text{sign}(E_{bija}^{xyzp}), \end{aligned}$$

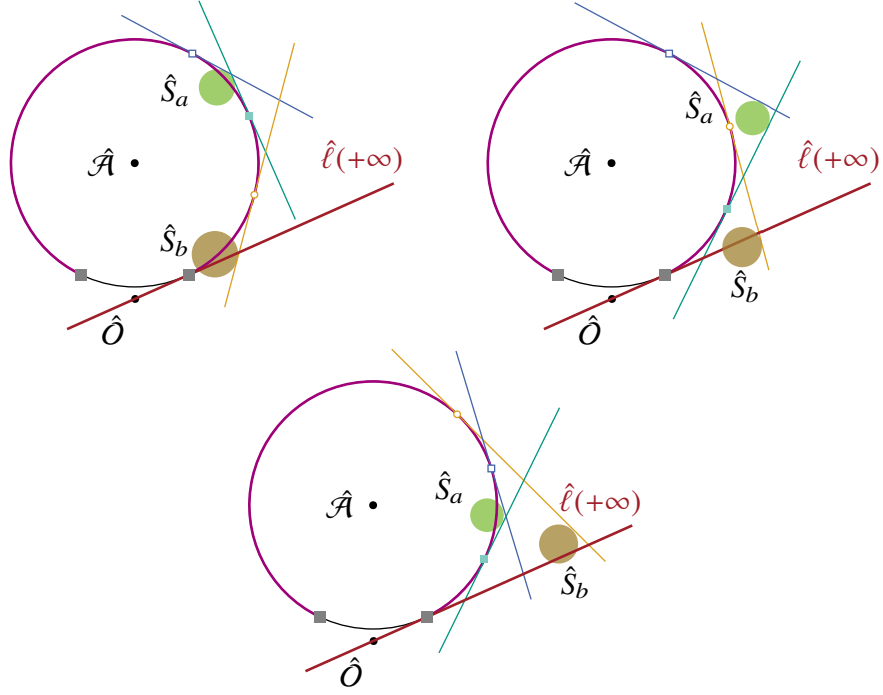


FIGURE 18. In Case C, it is assumed that only the Apollonius vertices v_{ikja}, v_{ijka} and v_{ijkb} exist on the trisector τ_{ijk} . We consider the three possible orderings of these vertices: Order Case 1 (Top Left), Order Case 2 (Top Right) and Order Case 3 (Bottom). Similar with Case A and B, we consider the corresponding ordering of the points $\psi(v_{ikja})$ and $\psi(v_{ijkb})$ on the arc $(\psi(\eta), \psi(o), \psi(\theta))$. A possible location for the circles \hat{S}_a and \hat{S}_b is drawn based on the analysis of Section 3.4.

where the quantity E_{bija}^{xyzp} and is an expression of algebraic degree 5 on the input quantities. The expression $\text{ORIENT3D}(C_n^*, C_i^*, C_j^*, O)$, for $n \in \{a, b\}$ can be evaluated as shown in Section 4.6.2,

$$\begin{aligned} \text{ORIENT3D}(C_n^*, C_i^*, C_j^*, O) &= \text{sign}(p_i^* p_i^* p_n^* D_{nij}^{uvw}) = \text{sign}(D_{nijk}^{xyz}) \\ &= \text{ORIENT3D}(C_n, C_i, C_j, C_k) \end{aligned}$$

and therefore its evaluation requires operations of algebraic degree 4 (in the input quantities).

In conclusion, since the evaluation of the INSPHERE predicates is the most degree-demanding operation throughout the evaluation of the ORDER predicate in a classic configuration, we have proven the following lemma.

Lemma 16. *The ORDER predicate in a classic configuration can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

4.6.5. *Ordering the Apollonius vertices in a non-classic configuration.* In the previous section we presented a way to resolve the $\text{ORDER}(S_i, S_j, S_k, S_a, S_b)$ predicate under the assumption that $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ were either $(\chi, +\infty)$ or (χ, ϕ) (not necessary the same); we called this a *classic configuration*. In this section, we will assume we are in a non-classic configuration i.e., at least one of $\mathcal{SR}(S_a)$ or $\mathcal{SR}(S_b)$ is $(-\infty, \phi)$ or $(-\infty, \phi) \cup (\chi, \phi)$. For convenience, these last two forms of a shadow region are labelled as non-classic whereas the classic forms are $(\chi, +\infty)$ and (χ, ϕ) .

If $\mathcal{SR}(S_n)$ has a non-classic type, for $n = a$ or b , then we claim that there exist a sphere S_N , for $N = A$ or B respectively, such that:

- if $\mathcal{SR}(S_n) = (-\infty, \phi)$ then $\mathcal{SR}(S_N) = (\chi, +\infty)$ and $v_{ijkn} \equiv v_{ikjN}$ or,
- if $(-\infty, \phi) \cup (\chi, \phi)$ then $\mathcal{SR}(S_N) = (\chi, \phi)$ and $v_{ijkn} \equiv v_{ikjN}$ as well as $v_{ikjn} \equiv v_{ijkN}$.

If these conditions hold, we will say that S_n and S_N are *equivalent spheres*. Notice that if $\mathcal{SR}(S_n)$ has a non-classic type then the shadow region of its equivalent sphere has a classic type and vice versa. The utility of this equivalency is that it enable us to make a connection between a classic and a non-classic configuration in the following way.

When the predicate $\text{ORDER}(S_i, S_j, S_k, S_a, S_b)$ is called then

- (1) if $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ have a classic type, we are in a classic and therefore, we resolve the predicate based on the analysis of Section 4.6.4.
- (2) If $\mathcal{SR}(S_a)$ has a classic type and $\mathcal{SR}(S_b)$ does not, then we call $\text{ORDER}(S_i, S_j, S_k, S_a, S_B)$. Since both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_B)$ have a classic type, this predicate can be evaluated using analysis of Section 4.6.4 with some adjustments. The predicate's outcome would be the ordering of v_{ikja} , v_{ikjB} and any of the existing v_{ijka} or v_{ijkB} . Using the property of equivalent spheres, we could answer the initial predicate by substituting v_{ikjB} with v_{ijkb} and, if it exists, v_{ijkB} with v_{ikjb} .
- (3) If $\mathcal{SR}(S_b)$ has a classic type and $\mathcal{SR}(S_a)$ does not, then we follow a similar analysis with the previous case. We evaluate $\text{ORDER}(S_i, S_j, S_k, S_A, S_b)$ and in the resulting ordering of the Apollonius vertices v_{ikjA} , v_{ikjb} and any of the existing v_{ijka} or v_{ijkb} , we will substitute $v_{ikjA} \equiv v_{ijkA}$ and if necessary, $v_{ijka} \equiv v_{ikja}$, to obtain the answer to the initial ORDER predicate.
- (4) Finally, if both $\mathcal{SR}(S_a)$ and $\mathcal{SR}(S_b)$ do not have a classic type we evaluate $\text{ORDER}(S_i, S_j, S_k, S_A, S_B)$. As before, we substitute $v_{ikjA} \equiv$

$v_{ijka}, v_{ikjB} \equiv v_{ijkb}$ and if necessary, $v_{ijkA} \equiv v_{ikja}$ and/or $v_{ijkB} \equiv v_{ikjb}$, and the acquired ordering is the answer of the initial ORDER predicate.

The evaluation of the ORDER predicate called in any of these 4 cases will eventually require determining INSPHERE or ORIENT3D predicates with inputs that involve the sites S_i, S_j, S_k, S_A (or S_a) and S_B (or S_b). The list of all possible predicates that must be evaluated, in the worst case scenario and assuming a classic configuration, would be:

- INSPHERE (S_i, S_k, S_k, S_a, S_b),
- INSPHERE (S_i, S_j, S_k, S_a, S_b),
- INSPHERE (S_i, S_k, S_j, S_b, S_a),
- INSPHERE (S_i, S_j, S_k, S_b, S_a),
- ORIENT3D (C_a, C_i, C_j, C_k),
- ORIENT3D (C_b, C_i, C_j, C_k) and
- ORIENT3D ($C_a^*, C_i^*, C_j^*, C_b^*$).

It is apparent that we must be able to answer these predicates when either one or both of S_a and S_b are substituted by S_A and S respectively.

Firstly, we present a way of defining an equivalent sphere S_N when $\mathcal{SR}(S_n)$ has a non-classic type, for $N = A$ or B and $n = a$ or b respectively. Since \hat{C}_n cannot coincide with $\hat{\mathcal{A}}$ (because there are either 1 or 2 cotangent lines to \mathcal{K}^* and \hat{S}_n), these points define a line $\hat{\ell}_n$. If a random point \hat{C}_N is selected on $\hat{\ell}_n$ such that $\hat{\mathcal{A}}$ lies in-between \hat{C}_N and \hat{C}_n , then we may choose an appropriate radius such that a circle \hat{S}_N , centered at \hat{C}_N , is tangent to any of the existing lines $\hat{\ell}(v_{ikjn})$ and $\hat{\ell}(v_{ijkn})$.

Notice that any sphere S_N of \mathcal{W} -space whose corresponding image in \mathcal{Y} -space is the circle \hat{S}_N has the desired properties of an equivalent sphere of S_n . Indeed, if $\mathcal{SR}(S_n)$ is (χ, ϕ) then it must stand that $\mathcal{SR}(S_N) = (-\infty, \phi) \cup (\chi, +\infty)$ and specifically the actual endpoints of these shadow regions on the trisector τ_{ijk} coincide. To prove this argument, we only need observe in \mathcal{Y} -space that the circle \hat{C}_n intersects the negative side of a line $\hat{\ell}(p)$ only for $v_{ikjn} \prec v_{ijkn}$ whereas, these are the only family of lines $\hat{\ell}(p)$ for $p \in \tau_{ijk}$ that do not intersect \hat{S}_N . As a conclusion the shadow region of S_n and S_N must be complementary i.e., $\mathcal{SR}(S_N) = (-\infty, \phi) \cup (\chi, +\infty)$. From Lemma 15, we deduce that $v_{ijkN} \prec v_{ikjN}$ and since these endpoints coincide with the endpoints of $\mathcal{SR}(S_n)$ it must hold that $v_{ijkN} \equiv v_{ikjn}$ and $v_{ikjN} \equiv v_{ijkn}$, since $v_{ikjn} \prec v_{ijkn}$ (see Figure 19).

Using a similar analysis, one can consider an equivalent sphere S_N of S_n , when $\mathcal{SR}(S_n)$ is assumed to be $(-\infty, \phi)$. The center of the respective circle \hat{C}_N is selected in the same way as above, and the radius of \hat{S}_N is chosen such that the circle is tangent to $\hat{\ell}(v_{ijkn})$. Again, we can conclude that $\mathcal{SR}(S_N)$

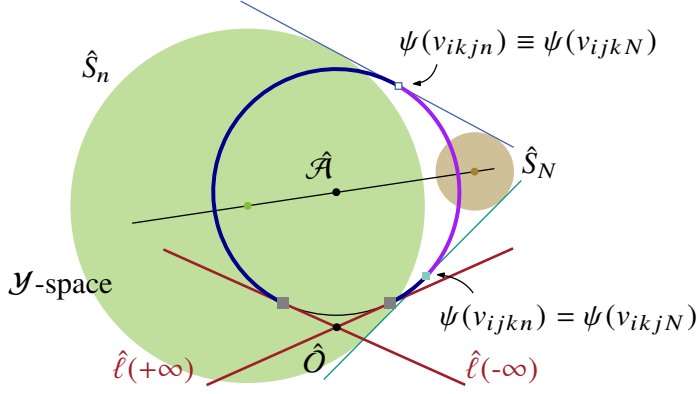


FIGURE 19. The shadow region of S_n is $(-\infty, \phi) \cup (\chi, +\infty)$ as its image in \mathcal{Y} -space is the blue area of the arc. Notice that the respective image of $\mathcal{SR}(S_N)$ is the purple area and therefore $\mathcal{SR}(S_n)$ must equal (χ, ϕ) . Since the endpoints of the two shadow regions coincide and based on Lemma 15, it must hold that $v_{ijkN} \equiv v_{ikjn}$ and $v_{ikjN} \equiv v_{ijkn}$. Therefore, \hat{S}_N and \hat{S}_n are equivalent.

and $\mathcal{SR}(S_n)$ are complementary since the family of lines $\hat{\ell}(p)$ for $\hat{\eta} \prec p$ are the locus of lines $\hat{\ell}(p)$, with $\hat{p} \in (\hat{\eta}, \hat{\delta}, \hat{\theta})$, whose negative side is intersected by \hat{S}_n and simultaneously, whose negative side is not intersected by \hat{S}_N (see Figure 20).

An interesting observation is that S_N is not uniquely defined in the sense that we do not provide its exact coordinates expressed as a function of the input quantities. This is a consequence of the fact that there are infinite spheres S_n that all share the same Apollonius vertices v_{ikjn} and v_{ijkn} .

Resuming the analysis of the properties of the equivalent sphere, we notice that if a point $p \in \tau_{ijk}$ lies on the shadow region of S_n then it must not lie on the shadow region of S_N and vice versa. An equivalent statement would be that a sphere $\mathcal{T}(p)$, for $p \in \tau_{ijk}$, intersects S_n if and only if it does not intersect S_N (see Figure 21). If p is chosen to be either v_{ikjm} or v_{ijkm} , where $m \in \{a, b\} \setminus \{n\}$, we get the following relations

$$\text{INSPIHERE}(S_i, S_j, S_k, S_m, S_N) = -\text{INSPIHERE}(S_i, S_j, S_k, S_m, S_n),$$

$$\text{INSPIHERE}(S_i, S_k, S_j, S_m, S_N) = -\text{INSPIHERE}(S_i, S_k, S_j, S_m, S_n).$$

Moreover, if S_m has a non-classic type and S_M is an equivalent sphere, where $M = B$ if $m = b$ or $M = A$ if $m = a$, it is known that $v_{ikjM} \equiv v_{ijkm}$ and, if v_{ikjm} also exists, then $v_{ijkM} \equiv v_{ikjm}$. Therefore, using the previous

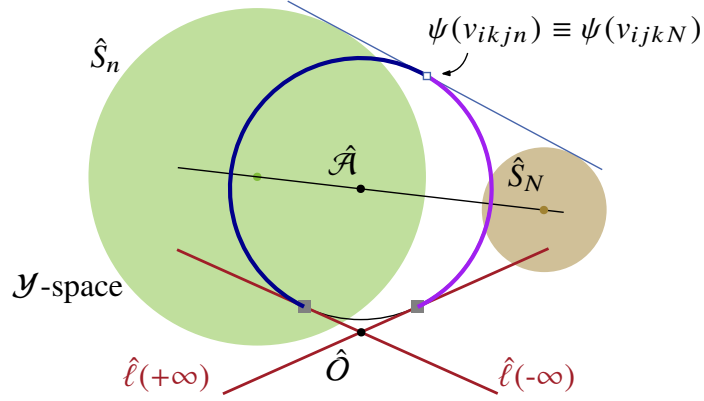


FIGURE 20. The shadow region of S_n is $(-\infty, \phi)$ as its image in \mathcal{Y} -space is the blue area of the arc. Notice that the respective image of $\mathcal{SR}(S_N)$ is the purple area and therefore $\mathcal{SR}(S_n)$ must equal $(\chi, +\infty)$. Since the endpoints of the two shadow regions coincide and based on Lemma 15, it must hold that $v_{ijkN} \equiv v_{ikjn}$. Therefore, \hat{S}_N and \hat{S}_n are equivalent.

observation for $p = v_{ijkM}$ or v_{ikjM} , we obtain the following expressions,

$$\begin{aligned} \text{INSPIRE}(S_i, S_j, S_k, S_M, S_N) &= -\text{INSPIRE}(S_i, S_j, S_k, S_M, S_n) \\ &= -\text{INSPIRE}(S_i, S_k, S_j, S_m, S_n), \\ \text{INSPIRE}(S_i, S_k, S_j, S_M, S_N) &= -\text{INSPIRE}(S_i, S_k, S_j, S_M, S_n) \\ &= -\text{INSPIRE}(S_i, S_j, S_k, S_m, S_n). \end{aligned}$$

These last four equalities can be used to evaluate any **INSPIRE** predicate that arises during the evaluation of the **ORDER** predicate in the case of a non-classic configuration.

Regarding the respective **ORIENT3D** predicates that may have to be evaluated, we consider the fact that $\hat{\mathcal{A}}$, \hat{C}_n and \hat{C}_N are collinear and the latter two lie on opposite sides of $\hat{\mathcal{A}}$. Subsequently, it is also true that \hat{C}_n and \hat{C}_N must lie on opposite sides with respect to any line $\hat{\lambda}$ that goes through $\hat{\mathcal{A}}$ (see Figure 21).

If we choose $\hat{\lambda}$ to be the line $\hat{\ell}$ that goes through \hat{O} and bear in mind that the position of a point of \mathcal{Y} -space with respect to this line corresponds to the position of its pre-image in \mathcal{Z} -space against the plane Π_{ijk} , we infer that C_n and C_N lie on different sides of Π_{ijk} and therefore

$$\text{ORIENT3D}(C_N, C_i, C_j, C_k) = -\text{ORIENT3D}(C_n, C_i, C_j, C_k).$$

If $\hat{\lambda}$ is chosen to be the line that goes through \hat{C}_m for $m \in \{a, b\} \setminus \{n\}$ then it must hold in \mathcal{Y} -space that C_n^* and C_N^* lie on different sides with respect to

the plane that goes through C_i^* , C_j^* and C_m^* , which is equivalent to

$$\begin{aligned}\text{ORIENT3D}(C_N^*, C_i^*, C_j^*, C_m^*) &= -\text{ORIENT3D}(C_n^*, C_i^*, C_j^*, C_m^*), \\ \text{ORIENT3D}(C_m^*, C_i^*, C_j^*, C_N^*) &= -\text{ORIENT3D}(C_m^*, C_i^*, C_j^*, C_n^*).\end{aligned}$$

Finally, combining the last two equations, we obtain that

$$\begin{aligned}\text{ORIENT3D}(C_N^*, C_i^*, C_j^*, C_M^*) &= -\text{ORIENT3D}(C_n^*, C_i^*, C_j^*, C_M^*) \\ &= \text{ORIENT3D}(C_n^*, C_i^*, C_j^*, C_m^*).\end{aligned}$$

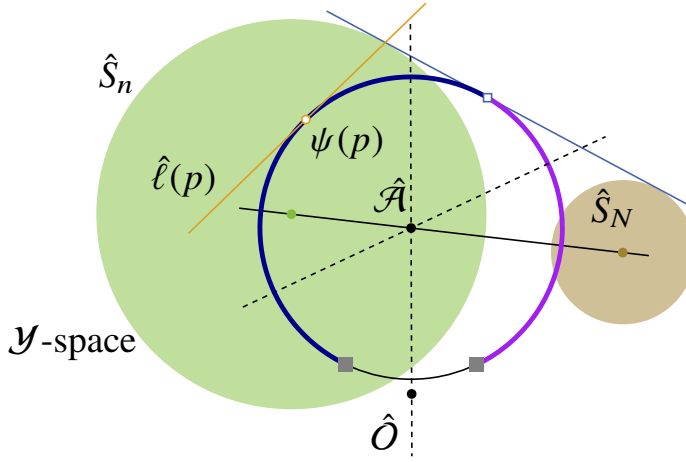


FIGURE 21. If S_N is an equivalent sphere of S_n , then it must hold that the centers \hat{C}_n , \hat{C}_N and $\hat{\mathcal{A}}$ are collinear and the former two points lie on opposite sides with respect to the latter. Observe that they also lie on opposite sides with respect to any line that goes through $\hat{\mathcal{A}}$. Lastly, it is apparent that a point $\psi(p)$ on the arc $(\psi(\eta), \psi(o), \psi(\theta))$ must lie on the image of the shadow region of either S_n or S_N .

In conclusion, we have shown that the evaluation of all 7 `INSPHERE` or `ORIENT3D` predicates, that may involve one or two equivalent spheres, can be amounted to the evaluation of respective predicates that contain only the original spheres S_a and S_b instead. Ultimately, we proved that the algebraic cost of the `ORDER` predicate in a non-classic configuration is the same as in a classic configuration, yielding the following lemmas.

Lemma 17. *The `ORDER` predicate in a non-classic configuration can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

Lemma 18. *The ORDER predicate can be evaluated by determining the sign of quantities of algebraic degree at most 10 (in the input quantities).*

5. CONCLUSION AND FUTURE WORK

In this paper, we presented a clever way of combining various subpredicates in order to answer the *master* EDGECONFLICT predicate. The design of all predicates and primitives was made in such a way such that the maximum algebraic cost of answering them would not exceed 10 (on the input quantities). Based on current bibliography, this is a quite small bound if compared with the respective 2D version of the EDGECONFLICT predicate (16 as shown in [12] and 6 as shown in [25]). It is also remarkable that both the VERTEXCONFLICT (equivalent to INSPHERE in non degenerate configurations) and the EDGECONFLICT predicates share the same algebraic degree.

Through our attempt to answer the master predicate, various useful primitives were also developed. These tools can also be used in the context of an incremental algorithm that evaluates the Apollonius diagram of a set of spheres.

A natural extension of the work presented in this paper involves answering the EDGECONFLICT predicate in the case where the trisector of the first three input sphere is an ellipse (or a circle) or a parabola. One can follow a similar analysis with the hyperbolic case presented here, but several modifications have to be made for the analysis to be complete.

Ultimately, we would like to resolve the EDGECONFLICT predicate even in degenerate configurations, i.e. if one or more sub-predicates return a degenerate answer. This task can be handled in various ways; our intention is to resolve any degeneracies that may arise using a qualitative perturbation scheme, in accordance to the one presented in [11].

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