(1.3) P(child is Xx | child has brown eyes + parents have brown eyes)

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HW1

$$= \frac{0(1-p)^4 + \frac{1}{2}4p(1-p)^3 + \frac{1}{2}4p^2(1-p)^2}{1(1-p)^4 + 1\cdot 4p(1-p)^3 + \frac{3}{4}\cdot 4p^2(1-p)^2}$$

$$= \frac{2p(1-p) + 2p^2}{(1-p)^2 + 4p(1-p) + 3p^2}$$

$$= \frac{2p}{1+2p}.$$

$$P(\text{Judy } Xx \mid n \text{ children all have brown eyes + all previous information}) = \frac{\frac{2p}{1+2p}\left(\frac{3}{4}\right)^n}{\frac{2p}{1+2p}\left(\frac{3}{4}\right)^n + \frac{1}{1+2p}\cdot 1}$$

$$P(\text{Judy } Xx \mid \text{all the given info}) = \frac{\frac{2p}{1+2p}\left(\frac{3}{4}\right)^n}{\frac{2p}{1+2p}\left(\frac{3}{4}\right)^n + \frac{1}{1+2p}}\left(\frac{2}{3}\right) + \frac{\frac{1}{1+2p}}{\frac{2p}{1+2p}\left(\frac{3}{4}\right)^n + \frac{1}{1+2p}}\left(\frac{1}{2}\right)$$

$$P(\text{Grandchild is } xx \mid \text{all the given info}) = \frac{\frac{2}{3}\frac{2p}{1+2p}\left(\frac{3}{4}\right)^n + \frac{1}{2}\left(\frac{1}{1+2p}\right)}{\frac{2p}{1+2p}\left(\frac{3}{4}\right)^n + \frac{1}{1+2p}}\left(\frac{1}{4}2p(1-p) + \frac{1}{2}p^2\right)$$

$$\neq \boxed{\frac{2p}{(1+p)^2}}$$

(1.7) The contestant should <u>switch</u>. At first there is a uniform probability of 1/3 that the prize is in each box. After one box is removed you still have only a 1/3 chance of your box being correct, but the remaining 2/3's probability falls to the remaining box that is unopened. Therefore it is in your best interest to switch to win the prize.

1.9 (a) Simulation Results:

Office Closed: 4:20 PM

Patients Seen: 47 patients.

Average Wait Time: 0 minutes

(b) Simulation Results (100x):

Office Closed: 4:12 PM

Patients Seen: 43 patients

Average Wait Time: 0 minutes w/ a 50% CI of [0, .5]

(2.4) We start by computing the mean for each of the approximated normal distr.

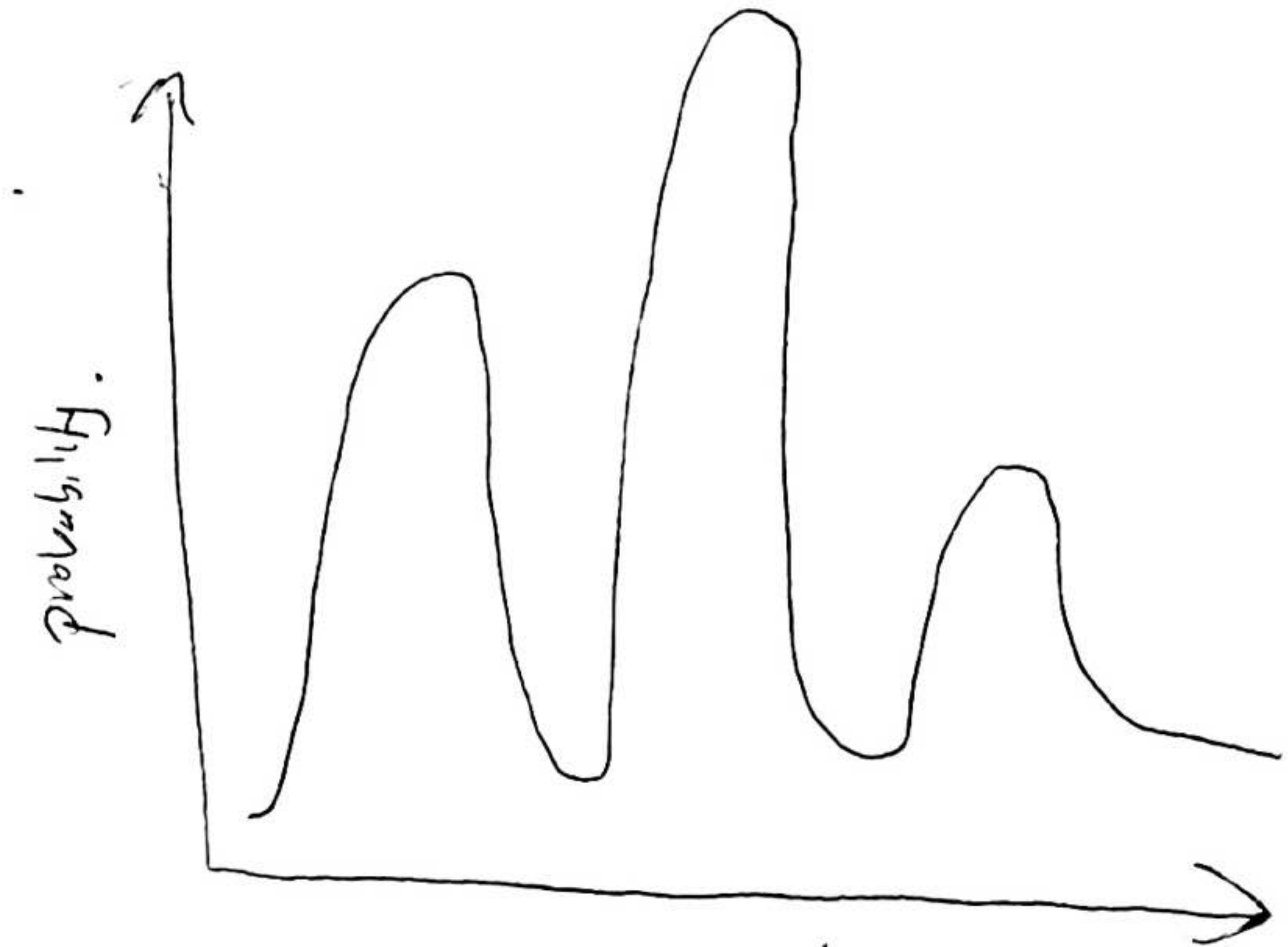
(a) $\mu_1 = \frac{1}{12}(1{,}000) = 83.3$; $\mu_2 = \frac{1}{6}(1{,}000) = 166.7$; $\mu_3 = \frac{1}{4}(1{,}000) = 250$.

Next we compute the SDs.

$\sigma_1 = \sqrt{83.3 \cdot \frac{11}{12}} = 8.74$; $\sigma_2 = \sqrt{166.7 \cdot \frac{5}{6}} = 11.79$; $\sigma_3 = \sqrt{250 \cdot \frac{3}{4}} = 13.69$

Thus $p(\theta) \propto \frac{1}{4} N(83.3, 8.74) + \frac{1}{2} N(166.7, 11.79) + \frac{1}{4} N(250, 13.69)$

Distribution sketch.



(b) The distribution is a sum of three normal curves with half in the middle. There is 25% under the leftmost curve, and 25% under the rightmost curve. Therefore the 5%, 25%, 50%, 75%, and 95% points occur at at the 20% point to the right of the leftmost distribution, point at the center of the middle distribution, point to the right of the centermost distribution, and at the 20% point of the rightmost curve, respectively.

percent point	y
5%	75.9
25%	126
50%	167
75%	210
95%	262

(2.7)

(a) $y \sim Bin(n, \theta) \Rightarrow y \propto \theta^y (1-\theta)^{n-y} = \dfrac{\theta^y (1-\theta)^n}{(1-\theta)^y} = \left(\dfrac{\theta}{1-\theta}\right)^y (1-\theta)^n = (1-\theta)^n \cdot e^{y \ln\left(\frac{\theta}{1-\theta}\right)}$

Now let $g(\theta) = (1-\theta)^n$ and $u(y) = y$, we now have a distribution in the form of an exponential with natural parameter $\ln\left(\frac{\theta}{1-\theta}\right)$. Lets use Jeffrey's Invariance Principle to find a uniform prior distr. for θ.

Let ~~θ~~ $\phi = \ln\left(\frac{\theta}{1-\theta}\right)$. Solving for θ, we find that $\theta = \dfrac{e^\phi}{1+e^\phi}$.

Suppose we have a uniform distr on ϕ, meaning that $p(\phi) \propto 1$.

Then, $p(\theta) = p\left(\dfrac{e^\phi}{1+e^\phi}\right) \left| \dfrac{d}{d\theta} \ln\left(\dfrac{\theta}{1-\theta}\right) \right| \propto \dfrac{d}{d\theta} \left| \ln(\theta) + \ln(1-\theta) \right| = \dfrac{1}{\theta} - \dfrac{1}{1-\theta}$

$= \dfrac{1}{\theta(1-\theta)} = \theta^{-1}(1-\theta)^{-1}$. Equivalent under Jeffrey's Invariance Principle,

so $Beta(-1,-1)$ is the uniform prior distribution on the natural parameter of the binomial distribution.

(b) Given Prior ~~k~~ $Beta(\alpha, \beta)$ and likelihood $Bin(y \mid \theta, n)$, the posterior distribution is proportional to $Beta(\alpha+y, \beta+n-y)$. Taking the uniform prior $Beta(-1,1)$, if we have $y=0$ we get the posterior $Beta(-1,n)$. Similarly, if $y=n$ we get the posterior $Beta(n,-1)$. Similarly, if $y=n$ we get the posterior $Beta(n,-1)$. We note

$Beta(-1, n) \propto \theta^{-2}(1-\theta)^{n-1}$

$Beta(n,-1) \propto \theta^{n-1}(1-\theta)^{-2}$

Take the transformation $\hat{\theta} = 1-\theta$ for the $y=n$ case, we see that these distributions are both proportional to $\dfrac{x^{n-1}}{(1-x)^2}$ which means the result ~~[illegible]~~ ~~[illegible]~~ is infinite, and therefore improper.

(2.8) Prior is $N(180, 40)$.

(a) $p(\theta|y) \sim N(\mu_1, \tau_1^2)$. conjugate distr for normal distr. w/ unknown mean is a normal distr. we start w/ μ_1, calculating μ_1.

$$\mu_1 = \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} = \frac{\frac{180}{1600} + \frac{150n}{400}}{\frac{1}{1600} + \frac{n}{400}} = \frac{180+60n}{1+4n}$$

$$\tau_1 : \frac{1}{\tau_1^2} = \frac{1}{\tau_0^2} + \frac{n}{\sigma^2} = \frac{1}{1600} + \frac{n}{400} = \frac{1+4n}{1600}.$$

$$\Rightarrow \tau_1^2 = \frac{1600}{1+4n}$$

Posterior is proportional to $N(\mu_1, \tau_1)$

(b) $E[\tilde{y}|y] = \mu_1 = \frac{180+600n}{1+4n}$ $\quad$ $Var[\tilde{y}|y] = \sigma^2 + \tau_1^2 = 400 + \frac{1600}{1+4n}$

(c) $n=10$, so our posterior for θ is $N(\mu_1, \tau_1) = N\left(\frac{180+600(10)}{1+4(10)}, \frac{1600}{1+4(10)}\right)$

$$= N\left(\frac{6180}{41}, \frac{1600}{41}\right)$$

95% CI for θ: $\frac{6180}{41} \pm 2 \cdot \sqrt{\frac{1600}{41}} = [138, 163]$

95% CI for $\tilde{y}$: $\frac{6180}{41} \pm 2\sqrt{400 + \frac{1600}{41}} = [108, 193]$

(2.8) (cont.) (d) $n=100$:

posterior becomes $N\left(\frac{60160}{401}, \frac{1600}{401}\right)$ ~~or~~

95% CI for ~~$\tilde{y}$~~ $= \theta$ is: $\frac{60160}{401} \pm 2\sqrt{400 + \frac{1600}{401}} = [109, 191]$.

2.10 (a) $p(\text{data}|N) = \begin{cases} 1/N & \text{if } N \geq 203 \\ 0 & \text{otherwise} \end{cases}$

$$p(N|\text{data}) \propto p(N)\,p(\text{data}|N)$$

$$= \frac{1}{N}(.01)(.99)^{N-1} \quad \text{for } N \geq 203$$

$$\propto \frac{1}{N}(.99)^N \quad \text{for } N \geq 203.$$

(b) $p(N|\text{data}) = c \cdot \frac{1}{N}(.99)^N$. Below we compute the normalizing constant c

$$\sum_N p(N|\text{data}) = 1, \cancel{so}$$

$$\frac{1}{c} = \sum_{N=203}^{\infty} \frac{1}{N}(.99)^N$$

Computer approximation: $\sum_{N=203}^{1000} \frac{1}{N}(.99)^N = .04658$.

So $\frac{1}{c} = .04658$ and $c = 21.47$

$$E[N|\text{data}] = \sum_{N=203}^{\infty} N \cdot p(N|\text{data})$$

$$= c\sum_{N=203}^{\infty} (.99)^N$$

$$= 21.47\,\frac{(.99)^{203}}{1-.99}$$

$$= \boxed{279}$$

$$SD(N|\text{data}) = \sqrt{\sum_{N=203}^{\infty}(N-279.1)^2 \cdot c \cdot \frac{1}{N}(.99)^N}$$

$$\approx \boxed{79}$$

(2.10) (c) Note that:

- if more than one data point is available, then the posterior distr is proper under all the above prior densities.

- With only one data point we don't need a noninformative prior distr.

2.13 (a) y_i = total # of fatal accidents in year i, for $i \in [10]$.

θ = expected # of accidents / yr. Model for data is $y_i | \theta \sim \text{Poisson}($

Use conjugate family of distributions for convenience

- Prior distr for $\theta \sim \text{Gamma}(\alpha, \beta) \Rightarrow$ posterior $\sim \text{Gamma}(\alpha + 10\bar{y}, \beta + 10)$
- Assume non-informative prior: $(\alpha, \beta) = (0, 0)$ - this should be okay since we have enough information: $n = 10$

- The posterior distr. is $\theta | y \sim \text{Gamma}(238, 10)$.

- $\tilde{y}$: # of fatal accidents in 1986.

- Given θ, the predictive distr for $\tilde{y}$ is Poisson(θ)

we compute the predictive distr for $\tilde{y}$ is Poisson(θ) via simulation.

- Draw θ from $p(\theta|y)$ and $\tilde{y}$ from $p(\tilde{y}|\theta)$: $[14, 35]$

```
theta <- rgamma(1000, 238)/10.
y1996 <- rpois(1,000, theta)
print(sort(y1996)[c(25, 976)])
```

(b). Let x_i = # of passenger miles flown in year i and θ = expected accident rate per passenger mile. Model for data is $y_i | x_i, \theta \sim \text{Poisson}(x_i \theta)$.

Use Gamma$(0,0)$ prior for θ. Then the posterior distr. for θ is

$y | \theta \sim \text{Gamma}(10\bar{y}, 10\bar{x}) = \text{Gamma}(238, 5.716 \times 10^{12})$. Given θ, the predictive distr for $\tilde{y}$ is Poisson$(\tilde{x}\theta)$ = Poisson$(8 \times 10^{10}\theta)$. In the following page we obtain a 95% posterior interval for $\tilde{y}$ via simulation.

2.13 (b) Cont.

- Draw θ from $p(\theta|y)$ and $\tilde{y}$ from $p(\tilde{y}|\tilde{x},\theta)$

computed 95% interval is ~~[22, 48]~~ ~~[22, 48]~~ [22, 48]

```
theta <- rgamma(1000, 238)/5.716e12
y1986 <- rpois(1000, theta * 8e11)
print(sort(y1986)[c(25,976)])
```

(c) Same analysis as in (a), but replace 238 with 6919, the total number of deaths in the data. 1000 simulation draws results in a 95% posterior interval of [638, 750] deaths.

(d) Repeat analysis from (b), replacing 238 by 6919. From 1000 simulation draws we see a 95% posterior interval of [900, 1035] deaths.

(e) Poisson model seems more reasonable w/ rate proportional to passenger miles. Because more miles flown each year yields more accidents. This is in case (b) and (d). While accidents are independent, deaths are not, so the Poisson model should be more reasonable for accidents (models (a) and (b)) than for total deaths.

9 Extra Problem

Prior Distribution is $N(15,5)$. Standard deviation of the prior is large enough relative to the known standard deviation that the distribution is approximately uniform and uninformative.

Posterior Mean: $$\mu_1 = \frac{\frac{\mu_0}{\tau_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} = \frac{\frac{15}{25} + \frac{158.87}{.0144}}{\frac{1}{25} + \frac{9}{.0144}} = \boxed{17.7}$$

Posterior SD: $$\frac{1}{\tau_1^2} = \frac{1}{25} + 9\left(\frac{1}{.0144}\right) = 625.04$$

$$\Rightarrow \tau_1 = .04$$

Therefore our posterior is $N(17.653, .04)$. Using qnorm we get a 99% interval of $[17.33, 18.0]$.