Sobolev Type Embeddings for Vector-Valued Besov Spaces

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Chapter 1

Introduction

In this thesis, we discuss several results concerning *Besov spaces* of vector valued functions in a Banach space Y defined over an interval $I \subseteq \mathbb{R}$. Specifically, we study $B_\lambda^{s,p}(I;Y)$ and the effects of the parameters s, p, λ on various types of embeddings.

These spaces are close relatives to both classical *Sobolev spaces* and *fractional Sobolev spaces*. As a refresher, for $m \in \mathbb{N}$, a Sobolev space $W^{m,p}(I;Y)$ is the space of all functions $f \in L^p(I;Y)$ s.t. $\forall 1 \leq n \leq m$ there exists a function $g_n \in L^p(I;Y)$ s.t. $\forall \phi \in C_c^\infty(I)$

$$\int_I f \frac{\mathrm{d}^n \phi}{\mathrm{d}x^n} \,\mathrm{d}x = (-1)^n \int_I g_n \phi \,\mathrm{d}x.$$

The function g_n is called the *weak derivative* of f w.r.t. x^n and is denoted as $\frac{\mathrm{d}^n \phi}{\mathrm{d}x^n}$. Sobolev spaces are incredibly important in the study of Partial Differential Equations (PDE) as they allow for the existence of "weak" solutions. For more on Sobolev spaces and PDE, see [Eva10].

For $0 < s < 1$, we can also define the fractional Sobolev space $W^{s,p}(I;Y)$ as the space of all functions $f \in L^p(I;Y)$ s.t.

$$\int_I \int_I \frac{\|f(x) - f(y)\|_Y^p}{|x-y|^{1+sp}} \,\mathrm{d}x\mathrm{d}y < \infty.$$

One way to interpret fractional Sobolev spaces is as the interpolation between pairs of classical Sobolev spaces. Specifically, it is well-known that

$$W^{s,p}(I;Y) = (L^p(I;Y), W^{1,p}(I;Y))_{s,p}$$

so that the fractional Sobolev space can be thought of as "between" a regular L^p space and the space of first order Sobolev functions $W^{1,p}$. For $p = 2$, fractional Sobolev spaces can also be understood in terms of Fourier transforms, as for a function $f \in L^2(I;Y)$, it is well-known that

$$f \in W^{1,2}(I;Y) = H^1(I;Y) \iff \int_I |x|^2 \left\|\hat{f}(x)\right\|_Y^2 \,\mathrm{d}x < \infty$$

and for $0 < s < 1$

$$f \in W^{s,2}(I;Y) = H^s(I;Y) \iff \int_I |x|^{2s} \left\|\hat{f}(x)\right\|_Y^2 \,\mathrm{d}x < \infty.$$

Here we have that $|x|^s \left\|\hat{f}(x)\right\|_Y$ serves the role of a "fractional derivative," so that rather than define Sobolev and Fractional Sobolev spaces via the previous integrability of the weak derivative and difference quotients respectively, we can instead define it based on the integrability of the Fourier transform.

Fractional Sobolev spaces are commonly used in various areas of PDE. For functions in fractional Sobolev spaces of one variable, their regularity properties are used frequently in evolution problems where I is a time interval and Y is function space over some domain $\Omega \subseteq \mathbb{R}^N$. For more on fractional Sobolev spaces, see [Leo23].

Besov spaces are a further generalization of fractional Sobolev spaces, where for $f \in L^p(I;Y), 0 < s < 1$, and $1 \leq \lambda < \infty$ we say that

$$f \in B^{s,p}_\lambda(I;Y) \iff \left(\int_0^\infty \frac{1}{h} \left(\frac{1}{h^s} \left(\int_{I_h} \|f(x+h) - f(x)\|_Y^p \, \mathrm{d}x \right)^{\frac{1}{p}} \right)^\lambda \mathrm{d}h \right)^{\frac{1}{\lambda}} < \infty.$$

At a high level p, λ are parameters for integrability, and s is a parameter for how regular the function is. A useful fact is that

$$(L^p(I;Y), W^{1,p}(I;Y))_{s,\lambda} = B^{s,p}_\lambda(I;Y)$$

so that in the case $p = \lambda$, we have that

$$B^{s,\lambda}_p(I;Y) = (L^p(I;Y), W^{1,p}(I;Y))_{s,p} = W^{s,p}(I;Y).$$

This can also be observed by just taking $\lambda = p$ in the $B^{s,p}_\lambda(I;Y)$ seminorm and noting that it is the same as the $W^{s,p}(I;Y)$ seminorm. By understanding the effects of the parameters s, p, λ on $B^{s,p}_\lambda$, we are able to prove various embedding theorems and use those to show classical Sobolev embedding theorems as well.

The goal of this thesis is to first summarize the various embedding results for Besov spaces where $0 < s < 1$. We examine the effects of the parameters s, p, λ on containments between pairs of spaces, and analogous to Sobolev and fractional Sobolev spaces, we are able to characterize subcritical, critical, and supercritical embeddings. Then we move on to define and explore Besov spaces for the limit case $s = 1$ and are able to transfer some of the results from the $0 < s < 1$ case. The main feature of this thesis is that the methods used are very elementary, relying mostly on inequalities like Hölder's, Hardy's, and Young's. While some of the results can be shown via interpolation or Fourier analysis, these methods are avoided as much as possible.

In Chapter 2, we review some basic definitions and theorems about Bochner integration that allow us to integrate vector valued functions over an arbitrary measure space.

In Chapters 3, we give the integral definition functions in Besov spaces and present various Sobolev like embedding theorems for spaces where $0 < s < 1$. We state and prove results from Simon [Sim90], most notably

1. $B^{s,p}_\lambda(I;Y)$ increases with λ, i.e. $\lambda \leq \mu \implies B^{s,p}_\lambda(I;Y) \subseteq B^{s,p}_\mu(I;Y)$.
2. $B^{s,p}_\lambda(I;Y)$ decreases with s, i.e. $s \geq r \implies B^{s,p}_\lambda(I;Y) \subseteq B^{r,p}_\lambda(I;Y)$.
3. In the subcritical case $s < \frac{1}{p}$, $B^{s,p}_\lambda(I;Y) \subseteq L^q(I;Y)$ for $p \leq q < \frac{p}{1-ps}$.
4. In the critical case $s = \frac{1}{p}$, $B^{s,p}_\lambda(I;Y) \subseteq L^q(I;Y)$ for $p \leq q < \infty$.
5. In the supercritical case $s > \frac{1}{p}$, $B^{s,p}_\lambda(I;Y) \subseteq C^{0,s-\frac{1}{p}}(I;Y)$.

In Chapter 4 we study Besov spaces for $s = 1$ largely motivated by the results of Simon [Sim90]. We first explain why $B^{1,p}_\lambda(I;Y)$ must have a different seminorm than $B^{s,p}_\lambda(I;Y)$ for $0 < s < 1$, but then via the use of some equivalent seminorms, we are able to inherit useful properties of the $0 < s < 1$ Besov functions and apply them to $s = 1$ Besov functions. Using them, we are able to show that in the critical and subcritical cases, Besov functions with $s = 1$ have analogous embedding properties as in the $0 < s < 1$ case.

Chapter 2

Preliminaries

2.1 Bochner Integration

In order to work with vector-valued Besov functions, we need to be able to define integration for vector-valued functions extending beyond basic Lebesgue integration. With this in mind, we will be using Bochner integration throughout this thesis and begin by defining simple functions.

Definition 2.1 (Simple Function)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \|\cdot\|)$ be a Banach space. A simple function $s : X \to Y$ is a function of the form

$$s(x) = \sum_{i=1}^{l} c_i \chi_{E_i}(x)$$

where $l \in \mathbb{N}$, $E_i \in \mathcal{M}$, and $c_i \in Y$.

Definition 2.2 (Strongly Measurable)**.** A function $f : X \to Y$ is strongly measurable if $\exists \{s_n\}_n$ a sequence of simple functions s.t.

$$\|s_n(x) - f(x)\|_Y \to 0 \text{ for } \mu\text{-a.e.}$$

Definition 2.3 (Bochner Integrable, Simple functions)**.** A simple function $s : X \to Y$ where $s(x) = \sum_{i=1}^{l} c_i \chi_{E_i}(x)$ is Bochner integrable if $c_i = 0$ whenever $\mu(E_i) = \infty$, and the Bochner integral of s over $E \in \mathcal{M}$ is defined as

$$\int_E s \, \mathrm{d}\mu := \sum_{i=1}^{l} c_i \mu(E_i \cap E),$$

where $c_i \mu(E_i \cap E) = 0$ whenever $c_i = 0$ and $\mu(E_i \cap E) = \infty$.

Definition 2.4 (Bochner Integrable, Strongly measurable functions)**.** A strongly measurable function $f : X \to Y$ is Bochner integrable if $\exists \{s_n\}_n$ simple Bochner integrable functions s.t.

$$\|s_n(x) - f(x)\|_Y \to 0 \text{ for } \mu\text{-a.e.}$$

and

$$\lim_{n\to\infty} \int_X \|s_n - f\|_Y \, \mathrm{d}\mu = 0.$$

Then for $E \in \mathcal{M}$ the Bochner integral of f over E is defined as

$$\int_E f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_E s_n \, \mathrm{d}\mu.$$

The limit is provably unique regardless of the sequence.

Here is a useful result that nicely links Bochner integrability to Lebesgue integrability.

Theorem 2.5. Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \|\cdot\|)$ be a Banach space. A strongly measurable function $f : X \to Y$ is Bochner integrable iff $\|f\|$ is Lebesgue integrable over X. Moreover, we have that if f is Bochner integrable, then $\forall E \in \mathcal{M}$

$$\left\| \int_E f \, \mathrm{d}\mu \right\|_Y \leq \int_E \|f\|_Y \, \mathrm{d}\mu.$$

As with the Lebesgue integral, we also have the following incredibly useful convergence theorems for Bochner integration.

Theorem 2.6 (Fatou)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \|\cdot\|)$ be a Banach space. Let $f_n, f : X \to Y$ be strongly measurable with $f_n(x) \rightharpoonup f(x)$ as $n \to \infty$ for μ-a.e. $x \in X$. Then if

$$\sup_n \int_X \|f_n\|_Y \, \mathrm{d}\mu < \infty$$

we have that f is Bochner integrable with

$$\int_X \|f\|_Y \, \mathrm{d}\mu \leq \liminf_{n \to \infty} \int_X \|f_n\|_Y \, \mathrm{d}\mu.$$

Theorem 2.7 (Dominated Convergence Theorem)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \|\cdot\|)$ be a Banach space. Let $f_n : X \to Y$ and $f : X \to Y$ be strongly measurable and assume $\|f_n(x)\| \leq v(x)$ $\forall n \in \mathbb{N}$ and for μ-a.e., where $v : X \to [0, \infty)$ is some Lebesgue integrable function. If $f_n \to f$ μ-a.e., then f is Bochner integrable and

$$\lim_{n \to \infty} \int_X \|f_n - f\|_Y \, \mathrm{d}\mu = 0 \implies \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu = \int_X f \, \mathrm{d}\mu.$$

Theorem 2.8 (Egoroff)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space where μ is finite, and $(Y, \|\cdot\|)$ be a Banach space. Let $f, f_n : X \to Y$ be strongly measurable s.t.

$$\lim_{n \to \infty} \|f_n(x) - f(x)\|_Y = 0 \text{ for } \mu\text{-a.e.}$$

Then $\forall \epsilon > 0$ $\exists E \in \mathcal{M}$ with $\mu(X \setminus E) \leq \epsilon$ s.t.

$$\lim_{n \to \infty} \sup_{x \in E} \|f_n(x) - f(x)\|_Y = 0.$$

The following result shows that the Bochner integral commutes with elements in the dual Y', or the linear operators of Y.

Theorem 2.9. Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \|\cdot\|)$ be a banach space. If $f : X \to Y$ is Bochner integrable, then $\forall T \in Y'$,

$$T\left(\int_X f \, \mathrm{d}\mu \right) = \int_X T(f) \, \mathrm{d}\mu.$$

Finally with our construction of the Bochner integral, we can define L^p spaces.

Definition 2.10 (L^p space)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \|\cdot\|)$ be a Banach space. For $1 \leq p < \infty$ we define

$$L^p(X; Y) := \{f : X \to Y : f \text{ is strongly measurable, } \|f\|_{L^p(X;Y)} < \infty\}$$

where

$$\|f\|_{L^p(X;Y)} = \left(\int_X \|f\|_Y^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

For $p = \infty$, we define

$$L^\infty(X; Y) := \{f : X \to Y : f \text{ is strongly measurable, } \|f\|_{L^\infty(X;Y)} < \infty\}$$

where

$$\|f\|_{L^\infty(X;Y)} = \operatorname*{ess\,sup}_{x \in X} \|f(x)\|_Y.$$

The following result states that compactly supported smooth funtions are dense in L^p, allowing us to prove some theorems for these functions and then extend them to general L^p via density.

Theorem 2.11 (Density of C_c^∞)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and $(Y, \|\cdot\|)$ be a Banach space. For any $f \in L^p(X;Y)$ and $\epsilon > 0$, we have that

$$\exists g \in C_c^\infty(X;Y) \implies g \in L^p(X;Y) \text{ s.t. } \|f-g\|_{L^p(X;Y)} < \epsilon.$$

2.2 Useful L^p Results

The following results on L^p functions will be useful later for working with Besov spaces.

Theorem 2.12 (Lemma 5 in [Sim90], Young's Inequality)**.** Let $f \in L^p(I;Y)$ and $g \in L^r((0,a),\mathbb{R})$ where $a > 0$ and $\frac{1}{p} + \frac{1}{r} > 1$. If we defined $F(x) = \int_0^a f(x+t)g(t)\,\mathrm{d}t$ for $x \in I_a$, then we have $F \in L^q(I_a;Y)$, where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ with

$$\|F\|_{L^q(I_a;Y)} \leq \|f\|_{L^p(I;Y)} \|g\|_{L^r((0,a),\mathbb{R})}.$$

Proof. We first assume that $f \in C_c^\infty(I;Y)$. Then we have that

$$\|F(x)\| \leq \int_0^a \|f(x+t)\| \, |g(t)| \, \mathrm{d}t = \int_0^a \left(\|f(x+t)\|^{\frac{p}{q}} |g(t)|^{\frac{r}{q}}\right) \|f(x+t)\|^{1-\frac{p}{q}} |g(t)|^{1-\frac{r}{q}} \, \mathrm{d}t$$

so by generalized Hölder's inequality (with $\frac{1}{q} + \left(\frac{1}{p} - \frac{1}{q}\right) + \left(\frac{1}{r} - \frac{1}{q}\right) = 1$), we get that

$$\|F(x)\| \leq \left(\int_0^a \|f(x+t)\|^p |g(t)|^r \, \mathrm{d}t\right)^{\frac{1}{q}} \left(\int_0^a \|f(x+t)\|^p \, \mathrm{d}t\right)^{\frac{1}{p}-\frac{1}{q}} \left(\int_0^a |g(t)|^r \, \mathrm{d}t\right)^{\frac{1}{r}-\frac{1}{q}}.$$

As such, by raising both sides to q and noting that $t \in (0,a) \implies x+t \in I$ for $x \in I_a$, we have that

$$\|F(x)\|^q \leq \left(\int_0^a \|f(x+t)\|^p |g(t)|^r \, \mathrm{d}t\right) \left(\int_I \|f(y)\|^p \, \mathrm{d}y\right)^{\frac{q}{p}-1} \left(\int_0^a |g(t)|^r \, \mathrm{d}t\right)^{\frac{q}{r}-1}$$

so that by integrating,

$$\begin{aligned}
\int_{I_a} \|F(x)\|^q \, \mathrm{d}x &\leq \left(\int_I \|f(y)\|^p \, \mathrm{d}y\right)^{\frac{q}{p}-1} \left(\int_0^a |g(t)|^r \, \mathrm{d}t\right)^{\frac{q}{r}-1} \left(\int_0^a \int_{I_a} \|f(x+t)\|^p |g(t)|^r \, \mathrm{d}x\mathrm{d}t\right) \\
&\leq \|f\|_{L^p(I;Y)}^{q-p} \|g\|_{L^p((0,a);\mathbb{R})}^{q-r} \int_0^a \|f\|_{L^p(I;Y)}^p |g(t)|^r \, \mathrm{d}t \\
&= \|f\|_{L^p(I;Y)}^q \|g\|_{L^p((0,a);\mathbb{R})}^q
\end{aligned}$$

as desired. The general case follows by theorem 2.11, density of C_c^∞ functions in L^p. □

Theorem 2.13 (Lemma 6 in [Sim90], Hardy's Inequality)**.** Let g be a real nonnegative measurable function on $[0,T]$ where $T \leq \infty, s > 0, 1 \leq \lambda \leq \infty$. Then

$$\left(\int_0^T \frac{1}{t^{1+s\lambda}} \left(\int_0^T \frac{g(h)}{h} \, \mathrm{d}h\right)^\lambda \mathrm{d}t\right)^{\frac{1}{\lambda}} \leq \frac{1}{s} \left(\int_0^T \frac{g(t)^\lambda}{t^{1+s\lambda}} \, \mathrm{d}t\right)^{\frac{1}{\lambda}}$$

$$\sup_{0<t<T} t^{-s} \int_0^t \frac{g(h)}{h} \, \mathrm{d}h \leq \frac{1}{s} \operatorname*{ess\,sup}_{0<t<T} t^{-s} g(t).$$

Proof. First suppose that $g = 0$ in a neighborhood of 0, and let $G(t) = \int_0^t \frac{g(h)}{h} \, \mathrm{d}h$. Note then that G is absolutely continuous, and by chain rule we can write

$$\frac{1}{\lambda} \frac{\mathrm{d}}{\mathrm{d}t} (t^{-s} G(t))^\lambda = -st^{-s\lambda-1} G(t)^\lambda + t^{-s\lambda-1} g(t) G(t)^{\lambda-1}$$

Since $g = 0$ in a neighborhood of 0, G is also 0 in that same neighborhood. As such, integrating both sides from 0 to T, we have that

$$\int_0^T \frac{1}{\lambda}\frac{\mathrm{d}}{\mathrm{d}t}(t^{-s}G(t))^\lambda \,\mathrm{d}t = \frac{1}{\lambda}T^{-s\lambda}G(T)^\lambda \geq 0$$

as g and therefore G are nonnegative. As such,

$$\int_0^T st^{-s\lambda-1}G(t)^\lambda \,\mathrm{d}t \leq \int_0^T t^{-s\lambda-1}g(t)G(t)^{\lambda-1}\,\mathrm{d}t,$$

where we can rewrite the RHS as

$$\int_0^T \left(t^{-s\lambda-1}G(t)^\lambda\right)^{\frac{\lambda-1}{\lambda}}\left(t^{-s\lambda-1}g(t)^\lambda\right)^{\frac{1}{\lambda}}\,\mathrm{d}t.$$

Now by Hölder's inequality, we have that this can be bounded by

$$\leq \left(\int_0^T t^{-s\lambda-1}G(t)^\lambda\,\mathrm{d}t\right)^{\frac{\lambda-1}{\lambda}}\left(\int_0^T t^{-s\lambda-1}g(t)^\lambda\,\mathrm{d}t\right)^{\frac{1}{\lambda}}.$$

Finally we get that

$$\int_0^T st^{-s\lambda-1}G(t)^\lambda\,\mathrm{d}t \leq \left(\int_0^T t^{-s\lambda-1}G(t)^\lambda\,\mathrm{d}t\right)^{\frac{\lambda-1}{\lambda}}\left(\int_0^T t^{-s\lambda-1}g(t)^\lambda\,\mathrm{d}t\right)^{\frac{1}{\lambda}}$$
$$\implies s\left(\int_0^T t^{-s\lambda-1}G(t)^\lambda\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \leq \left(\int_0^T t^{-s\lambda-1}g(t)^\lambda\,\mathrm{d}t\right)^{\frac{1}{\lambda}}.$$

In the case that g is not zero in a neighborhood of 0, we can use theorem 2.7 (DCT) on the functions $g_\epsilon = \mathbb{1}_{(\epsilon,\infty)}g$ and send $\epsilon \to 0^+$. For the second inequality, let $c = \operatorname{ess\,sup}_{0<t<T} t^{-s}g(t)$. Then we have $g(h) \leq ch^s$ a.e. and

$$\sup_{0<t<T} t^{-s}\int_0^t \frac{g(h)}{h}\,\mathrm{d}h \leq \sup_{0<t<T} t^{-s}\int_0^t ch^{s-1}\,\mathrm{d}h = \frac{c}{s}$$

as desired. □

Theorem 2.14 (Lemma 7 in [Sim90]). Suppose $f \in L^p(I;Y)$ for $1 \leq p < \infty$ and let $a > 0$. Then we have that

$$\int_0^a \frac{\|f(x+h)-f(x)\|_{L^p(I_h;Y)}}{h}\,\mathrm{d}h < \infty \implies f(x) = \frac{1}{a}\int_0^a f(x+h)\,\mathrm{d}h + \int_0^a\int_0^{a-h}\frac{f(x+t)-f(x+t+h)}{(t+h)^2}\,\mathrm{d}t\mathrm{d}h$$

in $L^p(I_a;Y)$. It follows that both summands and $f \in L^p(I_a;Y)$.

Proof. First note that the operator $F : [0,a] \to L^p(I_a;Y)$ defined via $F(h)(x) = (\tau_h f)(x) = f(x+h)$ is continuous, as translation in L^p is continuous via density of smooth functions. As such, it follows that the Bochner integral

$$\frac{1}{a}\int_0^a F(h)\,\mathrm{d}h = \frac{1}{a}\int_0^a \tau_h f\,\mathrm{d}h < \infty$$

so that the first summand converges and is well-defined. For the other summand, we note that $\frac{f(x+t)-f(x+t+h)}{(t+h)^2}$ is continuous in $L^p(I;Y)$ w.r.t. t, h when $t, h > 0$, so it follows that the function is measurable. Moreover, it is integrable as

$$\int_0^a\int_0^{a-h}\frac{\|f(x+t)-f(x+t+h)\|_{L^p(I_a;Y)}}{(t+h)^2}\,\mathrm{d}t\mathrm{d}h \leq \int_0^a\int_0^{a-h}\frac{\|f(x)-f(x+h)\|_{L^p(I_h;Y)}}{(t+h)^2}\,\mathrm{d}t\mathrm{d}h$$
$$\leq \int_0^a \|f(x)-f(x+h)\|_{L^p(I_h;Y)}\left(\frac{1}{h}-\frac{1}{a}\right)\mathrm{d}h$$

where the first inequality follows from the fact that $0 \leq t \leq a - h \implies \{x + t : x \in I_a\} \subseteq I_h$. This is finite by the lemma assumption, so we have that the other summand converges and is well-defined.

Now we show equality. Fix $\alpha > 0$, and note that for $a \leq \alpha$ we have that the second summand equals

$$\int_0^a \int_0^k \frac{f(x+u) - f(x+k)}{k^2} \,\mathrm{d}u\mathrm{d}k$$

in $L^p(I_a; Y)$ via the change of variables $u = t, k = t + h$. Moreover, since $a \leq \alpha$ this equality holds in $L^p(I_\alpha; Y)$ as $I_\alpha \subseteq I_a$. Now by FTC, we have that the mapping

$$F_1(a) = \int_0^a \int_0^{a-h} \frac{f(x+t) - f(x+t+h)}{(t+h)^2} \,\mathrm{d}t\mathrm{d}h, F_1 : [0, \alpha] \to L^p(I_\alpha; Y)$$

is differentiable with

$$\frac{\partial}{\partial a} F_1(a) = \int_0^a \frac{f(x+u) - f(x+a)}{a^2} \,\mathrm{d}u$$

in $L^p(I_\alpha, Y)$. Letting $F_2 : [0, \alpha] \to L^p(I_\alpha; Y)$ be defined via

$$F_2(a) = \frac{1}{a} \int_0^a f(x+h) \,\mathrm{d}h,$$

we see that F_2 is differentiable with

$$\frac{\partial}{\partial a} F_2(a) = -\frac{1}{a^2} \int_0^a f(x+h) \,\mathrm{d}h + \frac{1}{a} f(x+a) = -\int_0^a \frac{f(x+u) - f(x+a)}{a^2} \,\mathrm{d}u = -\frac{\partial}{\partial a} F_1(a).$$

As such, we have that $F_1(a) + F_2(a)$ is constant in $L^p(I_\alpha; Y)$ for $a \in [0, \alpha]$. Since $F_2(a) \to f(x)$ as $a \to 0$ by continuity and $F_1(a) \to 0$ as $a \to 0$ via DCT it follows that $F_1(a) + F_2(a) = f(x)$ in $L^p(I_\alpha, Y)$ for $a \leq \alpha$ and therefore $a = \alpha$. Since α was arbitrary, we are done. □

2.3 Besov Spaces

With our construction of Bochner integration and vector-valued L^p functions, we are ready to formally define Besov spaces and their respective norms and seminorms.

Definition 2.15. For $h > 0$ and an interval $I \subseteq \mathbb{R}$ we define

$$I_h = \{x \in I : x + h \in I\}.$$

Definition 2.16. For $h > 0$ we define the translation operator τ_h as follows

$$(\tau_h f)(x) = f(x+h).$$

Note that for $f \in L^p(I; Y)$, we have that $f, \tau_h f \in L^p(I_h; Y)$.

Definition 2.17. For $h > 0$ we inductively define the forward difference operator as follows

$$(\Delta_h^1 f)(x) := (\Delta_h f)(x) = (\tau_h f)(x) - f(x)$$

for $m = 1$ and

$$(\Delta_h^m f)(x) := (\Delta_h)(\Delta_h^{m-1} f)(x)$$

for $m \geq 2$.

Definition 2.18 (Besov space)**.** For $0 < s, 1 \le p \le \infty, 1 \le \lambda \le \infty$, and a Banach space $(Y, \|\cdot\|)$, we define

$$B_{\lambda}^{s,p}(I;Y) = \{f \in L^p(I;Y) : \|f\|_{B_{\lambda}^{s,p}} < \infty\}$$

where

$$\|f\|_{B_{\lambda}^{s,p}} = \begin{cases} \|f\|_{L^p(I;Y)} + \left(\int_0^\infty \left\|\Delta_h^{\lfloor s\rfloor+1} f\right\|_{L^p(I_{\lfloor s\rfloor h+h};Y)}^{\lambda} \frac{\mathrm{d}h}{h^{1+s\lambda}}\right)^{\frac{1}{\lambda}} & \lambda < \infty \\ \|f\|_{L^p(I;Y)} + \operatorname{ess\,sup}_{0<h<\infty} \frac{1}{h^s} \left\|\Delta_h^{\lfloor s\rfloor+1} f\right\|_{L^p(I_{\lfloor s\rfloor h+h};Y)} & \lambda = \infty \end{cases}.$$

In the case that we are only interested in the Besov space seminorm, we can utilize the following definition.

Definition 2.19 (Homogenous Besov space)**.** For $0 < s, 1 \le p \le \infty, 1 \le \lambda \le \infty$ and a Banach space $(Y, \|\cdot\|)$, we define

$$\dot{B}_{\lambda}^{s,p}(I;Y) = \{f \in L^p_{\text{loc}}(I;Y) : \|f\|_{\dot{B}_{\lambda}^{s,p}(I;Y)} < \infty\},$$

where

$$\|f\|_{\dot{B}_{\lambda}^{s,p}(I;Y)} = \begin{cases} \left(\int_0^\infty \left\|\Delta_h^{\lfloor s\rfloor+1} f\right\|_{L^p(I_{\lfloor s\rfloor h+h};Y)}^{\lambda} \frac{\mathrm{d}h}{h^{1+s\lambda}}\right)^{\frac{1}{\lambda}} & \lambda < \infty \\ \operatorname{ess\,sup}_{0<h<\infty} \frac{1}{h^s} \left\|\Delta_h^{\lfloor s\rfloor+1} f\right\|_{L^p(I_{\lfloor s\rfloor h+h};Y)} & \lambda = \infty \end{cases}.$$

Chapter 3

Besov Spaces for $0 < s < 1$

3.1 Embeddings

We begin by showing pairwise embeddings for Besov spaces as well as q-integrability and Hölder continuity ones.

Theorem 3.1 (Lemma 8 in [Sim90])**.** Suppose $0 < s < 1$ and $1 \le p < q \le \infty$ with $s - \frac{1}{p} = -\frac{1}{q}$. Then $B_1^{s,p}(I;Y) \subseteq L^q(I;Y)$. Moreover, $\forall a \le \frac{|I|}{2}$ and $f \in B_1^{s,p}(I;Y)$, we have that

$$\|f\|_{L^q(I;Y)} \le 2^{\frac{1}{q}} \left(a^{-s} \|f\|_{L^p(I;Y)} + \int_0^a \frac{\|f(x+h) - f(x)\|_{L^p(I_h;Y)}}{h^{1+s}} \, \mathrm{d}h \right).$$

For unbounded I, $2^{\frac{1}{q}}$ is replaced by 1, and the inequality holds for any a. Note that for $a = \frac{|I|}{2}$ we have that

$$\|f\|_{L^q(I;Y)} \le 2^{\frac{1}{q}} \left(\|f\|_{B_1^{s,p}(I;Y)} + \left(\frac{2}{|I|}\right)^s \|f\|_{L^p(I;Y)} \right)$$

Proof. We make use of the previous lemma, first noting that if $f \in B_1^{s,p}(I;Y)$, then we have that

$$\begin{aligned}
\int_0^\infty \frac{\|f(x+h) - f(x)\|_{L^p(I_h;Y)}}{h^{1+s}} \, \mathrm{d}h < \infty &\implies \int_0^a \frac{\|f(x+h) - f(x)\|_{L^p(I_h;Y)}}{h^{1+s}} \, \mathrm{d}h < \infty \\
&\implies \int_0^a \frac{a^s}{h^s} \frac{\|f(x+h) - f(x)\|_{L^p(I_h;Y)}}{h} \, \mathrm{d}h < \infty \\
&\implies \int_0^a \frac{\|f(x+h) - f(x)\|_{L^p(I_h;Y)}}{h} \, \mathrm{d}h < \infty
\end{aligned}$$

as clearly $\frac{a^s}{h^s} \ge 1$ when $h \in [0,a]$. So the hypotheses of the previous lemma are satisfied, and we may rewrite

$$f(x) = \frac{1}{a} \int_0^a f(x+h) \, \mathrm{d}h + \int_0^a \int_0^{a-h} \frac{f(x+t) - f(x+t+h)}{(t+h)^2} \, \mathrm{d}t$$

in $L^p(I_a;Y)$. By Young's Inequality (lemma 2.12), we get that

$$\left\| \frac{1}{a} \int_0^a f(x+h) \, \mathrm{d}h \right\|_{L^q(I_a;Y)} \le \frac{1}{a} \|f\|_{L^p(I;E)} \|1\|_{L^r((0,a),\mathbb{R})} = a^{-s} \|f\|_{L^p(I;Y)}$$

where $\frac{1}{r} = 1-s$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$. For the other term, we use Young's Inequality on $f(x+h) - f(x) \in L^p(I_h;Y)$

and $\frac{1}{(t+h)^2} \in L^r((0,a),\mathbb{R})$ to get

$$\begin{aligned}
\left\| \int_0^{a-h} \frac{f(x+t)-f(x+t+h)}{(t+h)^2} \, \mathrm{d}t \right\|_{L^q(I_a;Y)} &\leq \|f(x)-f(x+h)\|_{L^p(I_h;Y)} \|g\|_{L^r((0,a-h);\mathbb{R})} \\
&= \|f(x)-f(x+h)\|_{L^p(I_h;Y)} \left(\left(\frac{1}{1-2r} \frac{1}{(t+h)^{2r-1}} \right) \Big|_0^{a-h} \right)^{\frac{1}{r}} \\
&= \|f(x)-f(x+h)\|_{L^p(I_h;Y)} \left(\frac{h^{1-2r}-a^{1-2r}}{2r-1} \right)^{\frac{1}{r}} \\
&\leq \|f(x)-f(x+h)\|_{L^p(I_h;Y)} h^{\frac{1}{r}-2} \\
&= \|f(x)-f(x+h)\|_{L^p(I_h;Y)} h^{-s-1}
\end{aligned}$$

since we know that $r \geq 1 \implies 2r-1 \geq 1 \implies (2r-1)^{-\frac{1}{r}} \leq 1$. As such, we have that

$$\int_0^a \left\| \int_0^{a-h} \frac{f(x+t)-f(x+t+h)}{(t+h)^2} \, \mathrm{d}t \right\|_{L^q(I_a;Y)} \mathrm{d}h \leq \int_0^a \frac{\|f(x)-f(x+h)\|_{L^p(I_h;Y)}}{h^{1+s}} \, \mathrm{d}h < \infty$$

as $f \in B_1^{s,p}(I;Y)$. As such, the other term is well-defined in $L^q(I_a;Y)$, and

$$\|f\|_{L^q(I_a;Y)} \leq a^{-s} \|f\|_{L^p(I;Y)} + \int_0^a \frac{\|f(x)-f(x+h)\|_{L^p(I_h;Y)}}{h^{1+s}} \, \mathrm{d}h.$$

When I is unbounded on the right, we get that $I_a = I$, and we are done. When I is unbounded on the left, we can just consider $f^*(t) = f(-t)$ and flip I, I_a to $-I, -I_a$ to get the same result. Finally for I bounded, with $\inf I = \alpha, \sup I = \beta$, we have that for $\forall a \leq \frac{\beta-\alpha}{2}$,

$$\|f\|_{L^q(I;Y)} \leq \left(\int_\alpha^{\beta-a} \|f(x)\|^q \, \mathrm{d}x + \int_{\alpha+a}^{\beta} \|f(x)\|^q \, \mathrm{d}x \right)^{\frac{1}{q}} = \left(\|f\|_{L^q(I_a;Y)}^q + \|f\|_{L^q(-I_a;Y)}^q \right)^{\frac{1}{q}}.$$

Then using the previous inequality, this is bounded by

$$\begin{aligned}
&\left(2a^{-s} \|f\|_{L^p(I;Y)} + 2 \int_0^a \frac{\|f(x)-f(x+h)\|_{L^p(I_h;Y)}}{h^{1+s}} \, \mathrm{d}h \right)^{\frac{1}{q}} \\
=&2^{\frac{1}{q}} \left(a^{-s} \|f\|_{L^p(I;Y)} + \int_0^a \frac{\|f(x)-f(x+h)\|_{L^p(I_h;Y)}}{h^{1+s}} \, \mathrm{d}h \right)
\end{aligned}$$

as desired. $\square$

Theorem 3.2 (Lemma 9 in [Sim90])**.** Suppose $0 < r < s < 1, 1 \leq p \leq \infty$, and $1 \leq \lambda \leq \infty$. Then we have that

$$B_\lambda^{s,p}(I;Y) \subseteq B_1^{r,p}(I;Y)$$

and moreover

$$\|f\|_{B_1^{r,p}(I;Y)} \leq \begin{cases} \frac{|I|^{s-r}}{s-r} \|f\|_{B_\lambda^{s,p}(I;Y)} & |I| < \infty \\ \frac{1}{s-r} \|f\|_{B_\lambda^{s,p}(I;Y)} + \frac{2}{r} \|f\|_{L^p(I;Y)} & \forall I \end{cases}.$$

Proof. Let $f \in B^{s,p}_\lambda(I;Y)$ and suppose $a > 0$. Then by Hölder's inequality for $p = \lambda, q = \frac{\lambda}{\lambda-1}$, we have that

$$\begin{aligned}
&\int_0^a h^{s+\frac{1}{\lambda}-r-1} \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^{s+\frac{1}{\lambda}}}\,\mathrm{d}h \\
\leq &\left(\int_0^a h^{\left(s+\frac{1}{\lambda}-r-1\right)\left(\frac{\lambda}{\lambda-1}\right)}\,\mathrm{d}h\right)^{\frac{\lambda-1}{\lambda}} \left(\int_0^a \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+s\lambda}}\,\mathrm{d}h\right)^{\frac{1}{\lambda}} \\
= &\left(\left(\frac{\lambda-1}{(s-r)\lambda}\right)^{\frac{\lambda-1}{\lambda}} a^{s-r}\right) \left(\int_0^a \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+s\lambda}}\,\mathrm{d}h\right)^{\frac{1}{\lambda}} \\
\leq &\frac{a^{s-r}}{s-r} \left(\int_0^a \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+s\lambda}}\,\mathrm{d}h\right)^{\frac{1}{\lambda}}
\end{aligned}$$

where the last step follows from the fact that

$$\left(\frac{1}{s-r}\frac{\lambda-1}{\lambda}\right)^{\frac{\lambda-1}{\lambda}} \leq \frac{1}{s-r}(s-r)^{\frac{1}{\lambda}}\left(\frac{\lambda-1}{\lambda}\right)^{\frac{\lambda-1}{\lambda}} \leq \frac{1}{s-r}$$

as $0 < s-r < 1$ and $1 \leq \lambda \leq \infty \implies 0 \leq \frac{\lambda-1}{\lambda} \leq 1$. In the case that $|I| < \infty$, we can set $a = |I|$ to get the first inequality, and for arbitrary I, we can instead bound

$$\begin{aligned}
\|f\|_{B^{r,p}_1(I;Y)} &= \int_0^\infty \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^{1+r}}\,\mathrm{d}h \\
&= \int_0^1 \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^{1+r}}\,\mathrm{d}h + \int_1^\infty \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^{1+r}}\,\mathrm{d}h \\
&\leq \frac{1}{s-r}\left(\int_0^1 \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+s\lambda}}\,\mathrm{d}h\right)^{\frac{1}{\lambda}} + 2\,\|f\|_{L^p(I;Y)}\left(-\frac{h^{-r}}{r}\right)\Bigg|_1^\infty \\
&\leq \frac{1}{s-r}\,\|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{2}{r}\,\|f\|_{L^p(I;Y)}
\end{aligned}$$

as desired. Note that for $\lambda = \infty$ the same steps and bounds work, except the integrand is replaced with a sup. ☐

The next theorem shows that given fixed $s - \frac{1}{p}$, $B^{s,p}_\lambda$ decreases as s and p increase.

Theorem 3.3 (Theorem 10 in [Sim90]). Suppose $0 < r \leq s < 1, 1 \leq p \leq q \leq \infty$, and $1 \leq \lambda \leq \infty$. Then if $s - \frac{1}{p} = r - \frac{1}{q}$ we have that

$$B^{s,p}_\lambda(I;Y) \subseteq B^{r,q}_\lambda(I;Y)$$

with

$$\|f\|_{B^{r,p}(I;Y)} \leq \begin{cases} \frac{3\cdot 2^{\frac{1}{q}}3^{1-r}}{r}\,\|f\|_{B^{s,p}_\lambda(I;Y)} & |I| < \infty \\ \frac{3}{r}\,\|f\|_{B^{s,p}_\lambda(I;Y)} & |I| = \infty \end{cases}.$$

Proof. When $s = r$ the bounds are obvious. Otherwise, suppose $s > r$ with $s - \frac{1}{p} = r - \frac{1}{q}$. Let $f \in B^{s,p}_\lambda(I;Y)$, so that by theorems 3.1 and 3.2 we have

$$B^{s,p}_\lambda(I;Y) \subseteq B^{s-r,p}_1(I;Y) \subseteq L^q(I;Y) \implies f \in L^q(I;Y).$$

Fix $t > 0$ so that clearly $f \in B^{s,p}_\lambda(I_t;Y)$ and $\tau_t f \in B^{s,p}_\lambda(I_t;Y)$. As such, we have that

$$\tau_t f - f \in B^{s,p}_\lambda(I_t;Y) \implies \tau_t f - f \in B^{s-r,p}_1(I_t;Y).$$

Now we apply theorem 3.1 to $\tau_t f - f \in B^{s-r,p}_1(I_t;Y)$ with $(s-r) - \frac{1}{p} = -\frac{1}{q}$ and $a = t$.

1. Case $|I| = \infty$: We have that

$$\|\tau_t f - f\|_{L^q(I_t;Y)} \leq \int_0^t h^{r-s-1} \|\tau_{t+h} f - \tau_h f - \tau_t f + f\|_{L^p(I_{t+h};Y)} \, \mathrm{d}h + t^{r-s} \|\tau_t f - f\|_{L^p(I_t;Y)} .$$

When $\lambda < \infty$, we can observe that

$$\|\tau_{t+h} f - \tau_h f - \tau_t f + f\|_{L^p(I_{t+h};Y)} \leq 2 \|\tau_h f - f\|_{L^p(I_{t+h};Y)}$$

and then integrate with Minkowski's inequality to see that

$$\left(\int_0^\infty \frac{\|\tau_t f - f\|^\lambda_{L^q(I_t;Y)}}{t^{1+r\lambda}} \, \mathrm{d}t\right)^{\frac{1}{\lambda}} \leq 2 \left(\int_0^\infty \frac{1}{t^{1+r\lambda}} \left(\int_0^t \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^{1+s-r}} \, \mathrm{d}h\right)^\lambda \mathrm{d}t\right)^{\frac{1}{\lambda}} + \left(\int_0^\infty \frac{\|\tau_t f - f\|^\lambda_{L^p(I_t;Y)}}{t^{1+s\lambda}} \, \mathrm{d}t\right)^{\frac{1}{\lambda}} .$$

We can then apply Hardy's inequality (2.13) to the first integral on the RHS to get that

$$\left(\int_0^\infty \frac{\|\tau_t f - f\|^\lambda_{L^q(I_t;Y)}}{t^{1+r\lambda}} \, \mathrm{d}t\right)^{\frac{1}{\lambda}} \leq \left(\frac{2}{r} + 1\right) \left(\int_0^\infty \frac{\|\tau_t f - f\|^\lambda_{L^p(I_t;Y)}}{t^{1+s\lambda}} \, \mathrm{d}t\right)^{\frac{1}{\lambda}}$$

which yields the desired bound since $r \leq 1 \implies \frac{2}{r} + 1 \leq \frac{3}{r}$.

In the case that $\lambda = \infty$ we replace integrands with supremums to get that

$$\begin{aligned}\sup_{t>0} t^{-r} \|\tau_t f - f\|_{L^q(I_t;Y)} &\leq 2 \sup_{t>0} \left(t^{-r} \int_0^t \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^{1+s-r}} \, \mathrm{d}h\right) + \sup_{t>0} \frac{\|\tau_t f - f\|_{L^p(I_t;Y)}}{t^s} \\ &\leq \left(\frac{2}{r} + 1\right) \sup_{t>0} \frac{\|\tau_t f - f\|_{L^p(I_t;Y)}}{t^s}\end{aligned}$$

where the second step follows from the second inequality of lemma 2.13.

2. Case $|I| < \infty$: Let I be bounded with $\inf I = a, \sup I = b$, so that our application of theorem 3.1 yields

$$\|\tau_t f - f\|_{L^q(I_t;Y)} \leq 2^{\frac{1}{q}} \left(\int_0^t h^{r-s-1} \|\tau_h(\tau_t f - f) - (\tau_t f - f)\|_{L^p(I_{t+h};Y)} \, \mathrm{d}h + t^{r-s} \|\tau_t f - f\|_{L^q(I_t;Y)}\right)$$

for $t \leq \frac{\beta - \alpha}{3}$, as

$$t \leq \frac{\beta - \alpha}{3} \iff 3t \leq |I| \iff a = t \leq \frac{|I_t|}{2}.$$

For $\lambda < \infty$, we once again note

$$\|\tau_{t+h} f - \tau_h f - \tau_t f + f\|_{L^p(I_{t+h};Y)} \leq 2 \|\tau_h f - f\|_{L^p(I_{t+h};Y)}$$

so that upon integrating with Minkowski's inequality again, we have that

$$\left(\int_0^{\frac{\beta-\alpha}{3}} \frac{\|\tau_t f - f\|^\lambda_{L^p(I_t;Y)}}{t^{1+r\lambda}} \, \mathrm{d}t\right)^{\frac{1}{\lambda}} \leq 2^{\frac{1}{q}} \Bigg(2 \left(\int_0^\infty \frac{1}{t^{1+r\lambda}} \left(\int_0^t h^{r-s-1} \|\tau_h f - f\|_{L^p(I_h;Y)}\right)^\lambda \mathrm{d}t\right)^{\frac{1}{\lambda}} + \left(\int_0^\infty \frac{\|\tau_t f - f\|^\lambda_{L^p(I_t;Y)}}{t^{1+s\lambda}} \, \mathrm{d}t\right)^{\frac{1}{\lambda}} \Bigg).$$

Applying Hardy's inequality (lemma 2.13) again to the first integral on the RHS then yields

$$\left(\int_0^{\frac{\beta-\alpha}{3}} \frac{\|\tau_t f - f\|_{L^q(I_t;Y)}^\lambda}{t^{1+r\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \leq 2^{\frac{1}{q}}\left(\frac{2}{r}+1\right)\left(\int_0^\infty \frac{\|\tau_t f - f\|_{L^p(I_t;Y)}^\lambda}{t^{1+s\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}}.$$

Suppose that $h = 3t$, and note that this implies

$$\begin{aligned}\|\tau_h f - f\|_{L^q(I_h;Y)}^\lambda &= \|\tau_h f - \tau_{2t} f + \tau_{2t} f - \tau_t f + \tau_t f - f\|_{L^q(I_h;Y)} \\ &\leq 3\,\|\tau_t - f\|_{L^q(I_t;Y)}.\end{aligned}$$

As such, via change of variables and the above bound, we have that

$$\begin{aligned}\left(\int_0^\infty \frac{\|\tau_h f - f\|_{L^q(I_h;Y)}^\lambda}{h^{1+r\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} &= \left(\int_0^{\beta-\alpha} \frac{\|\tau_h f - f\|_{L^q(I_h;Y)}^\lambda}{h^{1+r\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ &\leq 3^{1-r}\left(\int_0^{\frac{\beta-\alpha}{3}} \frac{\|\tau_t f - f\|_{L^q(I_t;Y)}^\lambda}{t^{1+r\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ &\leq 3^{1-r}2^{\frac{1}{q}}\left(\frac{2}{r}+1\right)\left(\int_0^\infty \frac{\|\tau_t f - f\|_{L^p(I_t;Y)}^\lambda}{t^{1+s\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \\ &\leq \frac{3\cdot 2^{\frac{1}{q}}\cdot 3^{1-r}}{r}\left(\int_0^\infty \frac{\|\tau_t f - f\|_{L^p(I_t;Y)}^\lambda}{t^{1+s\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}}\end{aligned}$$

as desired. The case for $\lambda = \infty$ is analogous, where we instead replace the integrals with supremums.

□

Next we show that $B_\lambda^{s,p}$ increases with λ.

Theorem 3.4 (Theorem 11 in [Sim90]). Suppose $0 < s < 1, 1 \leq p \leq \infty$, and $1 \leq \lambda \leq \mu \leq \infty$. Then

$$B_\lambda^{s,p}(I;Y) \subseteq B_\mu^{s,p}(I;Y)$$

and

$$\|f\|_{B_\mu^{s,p}(I;Y)} \leq \frac{2}{s}\,\|f\|_{B_\lambda^{s,p}(I;Y)}.$$

The proof of this relies on the equivalence of the following seminorm. Given $f \in L^p(I;Y)$, we set

$$\omega_p(h) = \sup_{0\leq t\leq h} \|f(x+t) - f(x)\|_{L^p(I_t;Y)}$$

and

$$\|f\|_{B_\lambda^{s,p}(I;Y)}^* = \begin{cases}\left(\int_0^\infty \frac{(h^{-s}\omega_p(h))^\lambda}{h}\,\mathrm{d}h\right)^{\frac{1}{\lambda}} & \lambda < \infty \\ \sup_{h>0} h^{-s}\omega_p(h) & \lambda = \infty\end{cases}$$

Lemma 3.5 (Seminorm Equivalence). Suppose $0 < s < 1, 1 \leq p \leq \infty$, and $1 \leq \lambda \leq \infty$. Then $\forall f \in B_\lambda^{s,p}(I;Y)$, we have

$$\|f\|_{B_\lambda^{s,p}(I;Y)} \leq \|f\|_{B_\lambda^{s,p}(I;Y)}^* \leq \frac{1}{2^s - 1}\,\|f\|_{B_\lambda^{s,p}(I;Y)}.$$

Proof. The first inequality follows from the fact that $\omega_p(h) \geq \|f(x+h) - f(x)\|_{L^p(I_t;Y)}$. For the second, first note that

$$h \leq t \leq 2h \implies \tau_t f - f = \tau_h f - f + \tau_h(\tau_{t-h} f - f)$$

so that

$$\|\tau_t f - f\|_{L^p(I_t;Y)} \le \|\tau_h f - f\|_{L^p(I_h;Y)} + w_p(h)$$

by taking the L^p norm of both sides and noting that $0 \le t - h \le h$. This inequality is also clearly true for $0 \le t \le h$ as then $\|\tau_t f - f\|_{L^p(I_t;Y)} \le w_p(h)$, so taking the supremum of t over $(0, 2h]$ we see that

$$w_p(2h) \le \|\tau_h f - f\|_{L^p(I_h;Y)} + w_p(h).$$

1. Case $\lambda < \infty$: Integrating with Minkowski's inequality, we have that

$$\left(\int_0^\infty \frac{(h^{-s} w_p(2h))^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \le \left(\int_0^\infty \frac{(h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)})^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} + \left(\int_0^\infty \frac{(h^{-s} w_p(h))^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}}$$

   so that changing variables in the LHS (sending $2h \to h$) yields that

$$\left(\int_0^\infty \frac{(h^{-s} w_p(h))^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \le \frac{1}{2^s - 1}\left(\int_0^\infty \frac{(h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)})^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}}$$

   as desired.

2. Case $\lambda = \infty$: Taking the supremum of both sides, we have that

$$\begin{aligned} 2^s \sup_{h>0} h^{-s} w_p(h) = \sup_{h>0} h^{-s} w_p(2h) &\le \sup_{h>0} h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)} + \sup_{h>0} h^{-s} w_p(h) \\ \implies \sup_{h>0} h^{-s} w_p(h) &\le \frac{1}{2^s - 1}\sup_{h>0} h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)} \end{aligned}$$

   as desired.

□

Proof of Theorem 3.4 (Theorem 11 in [Sim90]). We first prove the theorem in the case that $\mu = \infty$. Suppose that $f \in B^{s,p}_\lambda(I;Y)$, $t > 0$, so that

$$\begin{aligned} \left(\int_0^\infty \frac{(h^{-s} w_p(h))^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} &\ge \left(\int_t^\infty \frac{(h^{-s} w_p(h))^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\ &\ge w_p(t)\left(\int_t^\infty h^{-s\lambda-1}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\ &= (s\lambda)^{-\frac{1}{\lambda}} t^{-s} w_p(t) \end{aligned}$$

as $w_p(h)$ is an increasing function. So taking the supremum over $t > 0$, we have that

$$\|f\|^*_{B^{s,p}_\infty} = \sup_{t>0} t^{-s} w_p(t) \le (s\lambda)^{\frac{1}{\lambda}}\left(\int_0^\infty \frac{(h^{-s} w_p(h))^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} = (s\lambda)^{\frac{1}{\lambda}}\|f\|^*_{B^{s,p}_\lambda(I;Y)}$$

so that $B^{s,p}_\lambda(I;Y) \subseteq B^{s,p}_\infty(I;Y)$.

For general $\mu \ge \lambda$, we have that for $f \in B^{s,p}_\lambda$,

$$\begin{aligned} \left(\int_0^\infty \frac{(h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)})^\mu}{h}\, \mathrm{d}h\right)^{\frac{1}{\mu}} &\le \left(\int_0^\infty (h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)})^{\mu-\lambda}\frac{(h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)})^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\mu}} \\ &\le \left(\sup_{h>0} h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)}\right)^{\frac{\mu-\lambda}{\mu}}\left(\int_0^\infty \frac{(h^{-s}\|\tau_h f - f\|_{L^p(I_h;Y)})^\lambda}{h}\, \mathrm{d}h\right)^{\frac{1}{\mu}} \end{aligned}$$

or equivalently,

$$\|f\|_{B^{s,p}_\mu(I;Y)} \leq \left(\|f\|_{B^{s,p}_\infty(I;Y)}\right)^{\frac{\mu-\lambda}{\mu}} \left(\|f\|_{B^{s,p}_\lambda(I;Y)}\right)^{\frac{\lambda}{\mu}}.$$

But we know from our equivalent seminorm and $\mu = \infty$ case that this can be bounded by

$$\begin{aligned}
\left(\|f\|_{B^{s,p}_\infty(I;Y)}\right)^{\frac{\mu-\lambda}{\mu}} \left(\|f\|_{B^{s,p}_\lambda(I;Y)}\right)^{\frac{\lambda}{\mu}} &\leq \left(\|f\|^*_{B^{s,p}_\infty(I;Y)}\right)^{\frac{\mu-\lambda}{\mu}} \left(\|f\|_{B^{s,p}_\lambda(I;Y)}\right)^{\frac{\lambda}{\mu}} \\
&\leq \left((s\lambda)^{\frac{1}{\lambda}} \|f\|^*_{B^{s,p}_\lambda(I;Y)}\right)^{\frac{\mu-\lambda}{\mu}} \left(\|f\|_{B^{s,p}_\lambda(I;Y)}\right)^{\frac{\lambda}{\mu}} \\
&\leq \frac{(s\lambda)^{\frac{\mu-\lambda}{\mu\lambda}}}{(2^s-1)^{\frac{\mu-\lambda}{\mu}}} \|f\|_{B^{s,p}_\lambda(I;Y)}
\end{aligned}$$

so that $B^{s,p}_\lambda(I;Y) \subseteq B^{s,p}_\mu(I;Y)$. We can improve the constant bound by noting that

$$(s\lambda)^{\frac{1}{\lambda}} = ((s\lambda)^{\frac{1}{s\lambda}})^s \leq e^{\frac{s}{e}}$$

as in general the function $x^{\frac{1}{x}}$ is maximized at $x = e$ for $x > 0$. Then we note that $\frac{e^{\frac{s}{e}}}{2^s-1} \leq \frac{2}{s}$ for $s \in (0,1)$. This is because at $s = 0$ we have $0 = se^{\frac{s}{e}} = 2^{s+1} - 2$, and for $0 < s < 1$ the function $2^{s+1} - 2$ grows faster than $se^{\frac{s}{e}}$ (as $e^{\frac{1}{e}} < 2$). As such, the constant bound can be replaced by $\left(\frac{2}{s}\right)^{\left(1-\frac{\lambda}{\mu}\right)}$ as desired. $\square$

We now show that for fixed p, λ, $B^{s,p}_\lambda$ decreases as s increases.

Theorem 3.6 (Theorem 14 in [Sim90])**.** Suppose $0 < r \leq s < 1, 1 \leq p \leq \infty$, and $1 \leq \lambda \leq \infty$. Then

$$B^{s,p}_\lambda(I;Y) \subseteq B^{r,p}_\lambda(I;Y)$$

and

$$\|f\|_{B^{r,p}_\lambda(I;Y)} \leq \begin{cases} |I|^{s-r} \|f\|_{B^{s,p}_\lambda(I;Y)} & |I| < \infty \\ \|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{2}{r} \|f\|_{L^p(I;Y)} & \forall I \end{cases}$$

Proof. Suppose that $f \in B^{s,p}_\lambda(I;Y)$ and let $a > 0$. We have two cases.

1. Case $|I| < \infty$. For $\lambda < \infty$ we have that

$$\left(\int_0^a \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+r\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} \leq a^{s-r} \left(\int_0^a \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+s\lambda}}\right)^{\frac{1}{\lambda}}$$

and for $\lambda = \infty$ we have that

$$\sup_{0<h<a} \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^r} \leq a^{s-r} \sup_{0<h<a} \frac{\|\tau_h f - f\|_{L^p(I_h;Y)}}{h^s}$$

so that by picking $a = |I|$, we get that $\forall\lambda$ we have $\|f\|_{B^{r,p}_\lambda(I;Y)} \leq |I|^{s-r} \|f\|_{B^{s,p}_\lambda(I;Y)}$ as desired.

2. Case $\forall I$. In general (including bounded I) we can note that

$$\begin{aligned}
\left(\int_0^\infty \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+r\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} &\leq \left(\int_0^1 \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+r\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} + 2\|f\|_{L^p(I;Y)} \left(\int_1^\infty \frac{1}{h^{1+r\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\leq 1^{s-r} \left(\int_0^1 \frac{\|\tau_h f - f\|^\lambda_{L^p(I_h;Y)}}{h^{1+s\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} + 2\|f\|_{L^p(I;Y)} \left(\frac{1}{r}\right)^{\frac{1}{\lambda}} \\
&\leq \|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{2}{r} \|f\|_{L^p(I;Y)}
\end{aligned}$$

as desired.

In both cases, we have the desired bounds, and $B^{s,p}_\lambda(I;Y) \subseteq B^{r,p}_\lambda(I;Y)$. □

In fact, combining all previous results gives us that $B^{s,p}_\lambda$ decreases when s increases regardless of λ.

Theorem 3.7 (Corollary 15 in [Sim90])**.** Suppose $0 < r < s < 1, 1 \le p \le \infty, 1 \le \lambda \le \infty$, and $1 \le \mu \le \infty$. Then

$$B^{s,p}_\lambda(I;Y) \subseteq B^{r,p}_\mu(I;Y)$$

and

$$\|f\|_{B^{r,p}_\mu(I;Y)} \le \begin{cases} \frac{2|I|^{s-r}}{r(s-r)} \|f\|_{B^{s,p}_\lambda(I;Y)} & |I| < \infty \\ \frac{2}{r(s-r)} \|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{4}{r^2} \|f\|_{L^p(I;Y)} & \forall I \end{cases}.$$

In the case that $\lambda \le \mu$ we have the stronger bounds

$$\|f\|_{B^{r,p}_\mu(I;Y)} \le \begin{cases} \frac{2}{s} |I|^{s-r} \|f\|_{B^{s,p}_\lambda(I;Y)} & |I| < \infty \\ \frac{2}{s} \|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{2}{r} \|f\|_{L^p(I;Y)} & \forall I \end{cases}.$$

Proof. In the case that $\lambda \le \mu$, we have from theorems 3.6 and 3.4 that

$$\|f\|_{B^{r,p}_\mu(I;Y)} \le |I|^{s-r} \|f\|_{B^{s,p}_\mu(I;Y)} \le \frac{2|I|^{s-r}}{s} \|f\|_{B^{s,p}_\lambda(I;Y)}$$

for $|I| < \infty$ and

$$\|f\|_{B^{r,p}_\mu(I;Y)} \le \|f\|_{B^{s,p}_\mu(I;Y)} + \frac{2}{r} \|f\|_{L^p(I;Y)} \le \frac{2}{s} \|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{2}{r} \|f\|_{L^p(I;Y)}$$

for arbitrary I. When $\lambda > \mu$, we have from theorem 3.2 that

$$\|f\|_{B^{r,p}_1(I;Y)} \le \begin{cases} \frac{|I|^{s-r}}{s-r} \|f\|_{B^{s,p}_\lambda(I;Y)} & |I| < \infty \\ \frac{1}{s-r} \|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{2}{r} \|f\|_{L^p(I;Y)} & \forall I \end{cases}$$

and then applying theorem 3.4 yields

$$\|f\|_{B^{r,p}_\lambda(I;Y)} \le \frac{2}{r} \|f\|_{B^{r,p}_1(I;Y)} \le \begin{cases} \frac{2|I|^{s-r}}{r(s-r)} \|f\|_{B^{s,p}_\lambda(I;Y)} & |I| < \infty \\ \frac{2}{r(s-r)} \|f\|_{B^{s,p}_\lambda(I;Y)} + \frac{4}{r^2} \|f\|_{L^p(I;Y)} & \forall I \end{cases}$$

as desired. In any case, we have that $B^{s,p}_\lambda(I;Y) \subseteq B^{r,p}_\mu(I;Y)$. □

The following theorem shows that $B^{s,p}_\lambda$ decreases as $s, s - \frac{1}{p}$ increase, and as p decreases.

Theorem 3.8 (Theorem 16 in [Sim90])**.** Let $s \ge r$ and $p \le q$ so that either $s - \frac{1}{p} > r - \frac{1}{q}$ or $s - \frac{1}{p} = r - \frac{1}{q}$ with $\lambda \le \mu$. In either case, we have that

$$B^{s,p}_\lambda(I;E) \subseteq B^{r,q}_\mu(I;E).$$

When $s - \frac{1}{p} > r - \frac{1}{q}$, we have that

$$\|f\|_{B^{r,q}_\mu} \le \begin{cases} \frac{36|I|^{s-r-\frac{1}{p}+\frac{1}{q}}}{r^2\left(s-r-\frac{1}{p}+\frac{1}{q}\right)} \|f\|_{B^{s,p}_\lambda} & |I| < \infty \\ \frac{6}{r^s\left(s-r-\frac{1}{p}+\frac{1}{q}\right)} \|f\|_{B^{s,p}_\lambda} + \frac{12}{r^3} \|f\|_{L^p} & \forall I \end{cases}.$$

In the case that $s - \frac{1}{p} = r - \frac{1}{q}$ and $\lambda \le \mu$, we instead get that

$$\|f\|_{B^{r,q}_\mu} \le \begin{cases} \frac{36|I|^{s-r-\frac{1}{p}+\frac{1}{q}}}{rs} \|f\|_{B^{s,p}_\lambda} & |I| < \infty \\ \frac{6}{rs} \|f\|_{B^{s,p}_\lambda} + \frac{6}{r^2} \|f\|_{L^p} & \forall I \end{cases}$$

Proof. Let $S = r - \frac{1}{q} + \frac{1}{p}$ so that $r \leq S \leq s$ with $S - \frac{1}{p} = r - \frac{1}{q}$. In the case that $\lambda \leq \mu$, theorems 3.4, 3.6, and 3.3 yield

$$B_\lambda^{s,p}(I;Y) \subseteq B_\mu^{s,p}(I;Y) \subseteq B_\mu^{S,p}(I;Y) \subseteq B_\mu^{r,q}(I;Y)$$

and the desired inequalities. If $s - \frac{1}{p} > r - \frac{1}{q}$, theorems 3.2, 3.3, and 3.4 yield

$$B_\lambda^{s,p}(I;Y) \subseteq B_1^{s,p}(I;Y) \subseteq B_1^{r,q}(I;Y) \subseteq B_\mu^{r,q}(I;Y)$$

and the desired inequalities. $\square$

Note that in the above theorem, the embeddings contain all previous embeddings, but the coefficients are worse here than in each particular case.

We now show Lipschitz embedding properties of $B_\lambda^{s,p}$.

Lemma 3.9 (Equivalence of fractional Sobolev and Besov spaces)**.** For $0 < s < 1$ and $1 \leq p \leq \infty$ we have that

$$W^{s,p}(I;Y) = B_p^{s,p}(I;Y)$$

and in the case $p = \infty$,

$$C^{0,s}(I;Y) = W^{s,\infty}(I;Y) = B_\infty^{s,\infty}(I;Y)$$

Proof. Note that for $p < \infty$, we have that for $f \in L^p(I;Y)$,

$$\|f\|_{W^{s,p}(I;Y)} = \left(\int_I \int_{I_h} \frac{\|f(y) - f(x)\|_Y^p}{|y-x|^{sp+1}}\, \mathrm{d}y\mathrm{d}x\right)^{\frac{1}{p}} = \left(\int_0^\infty \int_I \frac{\|f(x+h) - f(x)\|_Y^p}{h^{sp+1}}\, \mathrm{d}x\mathrm{d}h\right)^{\frac{1}{p}} = \|f\|_{B_p^{s,p}(I;Y)}$$

via change of variables, so that the spaces and their norms coincide. The case for $p = \infty$ is analogous. Moreover, we have that

$$\|f\|_{C^{0,s}(I;Y)} = \operatorname*{ess\,sup}_{x,y \in I} \frac{\|f(y) - f(x)\|_Y}{|y-x|^s} = \|f\|_{W^{s,p}(I;Y)}$$

so that these spaces and their norms coincide as well. $\square$

Theorem 3.10 (Corollary 26 in [Sim90])**.** Let $s > \frac{1}{p}$ where $0 < s < 1, 1 < p \leq \infty$, and $1 \leq \lambda \leq \infty$. Then we have that

$$B_\lambda^{s,p}(I;Y) \subseteq C^{0,s-\frac{1}{p}}(I;Y)$$

with

$$\|f\|_{C^{0,s-\frac{1}{p}}(I;Y)} \leq \frac{36}{s\left(s - \frac{1}{p}\right)} \|f\|_{B_\lambda^{s,p}(I;Y)}.$$

Proof. By theorems 3.4 and 3.3, we have that

$$B_\lambda^{s,p}(I;Y) \subseteq B_\infty^{s,p}(I;Y) \subseteq B_\infty^{s-\frac{1}{p},\infty}(I;Y)$$

with the inequalities

$$\|f\|_{B_\infty^{s-\frac{1}{p},\infty}(I;Y)} \leq \frac{18}{s - \frac{1}{p}} \|f\|_{B_\infty^{s,p}(I;Y)} \leq \frac{36}{s\left(s - \frac{1}{p}\right)} \|f\|_{B_\lambda^{s,p}(I;Y)}$$

for any $f \in B_\lambda^{s,p}(I;Y)$. We then note that by the previous lemma, this implies

$$\|f\|_{C^{0,s-\frac{1}{p}}} = \|f\|_{B_\infty^{s-\frac{1}{p},\infty}(I;Y)} \leq \frac{18}{s - \frac{1}{p}} \|f\|_{B_\infty^{s,p}(I;Y)} \leq \frac{36}{s\left(s - \frac{1}{p}\right)} \|f\|_{B_\lambda^{s,p}(I;Y)}$$

with $B_\lambda^{s,p}(I;Y) \subseteq C^{0,s-\frac{1}{p}}(I;Y)$, as desired. $\square$

Let $C_u(I;Y)$ denote the space of uniformly continuous functions from $I \to Y$. Then we have the following trace results for Besov spaces.

Theorem 3.11 (Theorem 29 in [Sim90]). Suppose that either $s > \frac{1}{p}$ or $s = \frac{1}{p}$ with $\lambda = 1$. Then we have that

$$B_\lambda^{s,p}(I;Y) \subseteq C_u(I;Y)$$

and that for $f \in B_\lambda^{s,p}(I;Y), t \in \bar{I}$, $f(t)$ is uniquely defined with

$$\|f(t)\|_Y \leq \begin{cases} \|f\|_{B_1^{\frac{1}{p},p}(I;Y)} + \left(\frac{2}{|I|}\right)^{\frac{1}{p}} \|f\|_{L^p(I;Y)} & s = \frac{1}{p} \\ \frac{1}{s-\frac{1}{p}} \|f\|_{B_\lambda^{s,p}(I;Y)} + \left(2p + \left(\frac{2}{|I|}\right)^{\frac{1}{p}}\right) \|f\|_{L^p(I;Y)} & s > \frac{1}{p} \end{cases}$$

where $\frac{2}{|I|} = 0$ for unbounded I.

Proof. By theorem 3.1, we have that $B_1^{\frac{1}{p},p}(I;Y) \subseteq L^\infty(I;Y)$ with

$$\|f\|_{L^\infty(I;Y)} \leq \|f\|_{B_1^{\frac{1}{p},p}(I;Y)} + \left(\frac{2}{|I|}\right)^{\frac{1}{p}} \|f\|_{L^p(I;Y)}$$
$$\implies \|f(t)\|_Y \leq \|f\|_{B_1^{\frac{1}{p},p}(I;Y)} + \left(\frac{2}{|I|}\right)^{\frac{1}{p}} \|f\|_{L^p(I;Y)}$$

for $f \in B_1^{\frac{1}{p},p}(I;Y)$ and a.e. $t \in I$. Then by theorem 3.2, we have that $B_\lambda^{s,p}(I;Y) \subseteq B_1^{\frac{1}{p},p}(I;Y)$ for $s > \frac{1}{p}$ and also

$$\|f\|_{B_1^{\frac{1}{p},p}(I;Y)} \leq \frac{1}{s - \frac{1}{p}} \|f\|_{B_\lambda^{s,p}(I;Y)} + 2p \|f\|_{L^p(I;Y)}.$$

Putting this together with the previous inequality yields that for $f \in B_\lambda^{s,p}(I;Y)$ with $s > \frac{1}{p}$ and $t \in I$ a.e., we have that

$$\|f(t)\|_Y \leq \frac{1}{s - \frac{1}{p}} \|f\|_{B_\lambda^{s,p}(I;Y)} + \left(2p + \left(\frac{2}{|I|}\right)^{\frac{1}{p}}\right) \|f\|_{L^p(I;Y)}$$

yielding the desired inequality in the second case. □

Previously we showed an embedding from $B_\lambda^{s,p}$ into L^q for $s - \frac{1}{p} = -\frac{1}{q}$ in the case $\lambda = 1$. We now extend this result for $\lambda \leq q$.

Theorem 3.12 (Theorem 30 in [Sim90]). Suppose that $s - \frac{1}{p} = -\frac{1}{q}, s < \frac{1}{p}$, and $\lambda \leq q < \infty$ $(0 < s < 1, 1 \leq p < q < \infty, 1 \leq \lambda \leq q)$. Then we have that

$$\|f\|_{L^q(I;Y)} \preceq \|f\|_{B_\lambda^{s,p}(I;Y)} \implies B_\lambda^{s,p}(I;Y) \subseteq L^q(I;Y)$$

Proof. The proof relies heavily on interpolation and is ommitted for this thesis. For a full proof see [Sim90]. □

As such, we can compile our embedding reults for $B_\lambda^{s,p}$ into L^q to get the following:

Theorem 3.13 (Corollary 31 in [Sim90]). Let s, p, q, λ be as regularly defined $(0 < s < 1, 1 \leq p \leq q \leq \infty, 1 \leq \lambda \leq \infty)$. Then in the following three cases:

1. Supercritical Case: $s > \frac{1}{p}$. $p \leq q \leq \infty$.
2. Critical Case: $s = \frac{1}{p}$. $p \leq q < \infty$ or $q = \infty, \lambda = 1$.
3. Subcritical Case: $s < \frac{1}{p}$. $p \leq q < p^*$ or $q = p^*, \lambda \leq p^*$, where $-\frac{1}{p^*} = s - \frac{1}{p}$.

We have that $B^{s,p}_\lambda(I;Y) \subseteq L^q(I;Y)$ with $\|f\|_{L^q(I;Y)} \preceq \|f\|_{B^{s,p}_\lambda(I;Y)}$.

Proof. For the supercritical case, the critical case when $p \leq q < \infty$, and the subcritical case when $p \leq q < p^*$, we can proceed as follows. Let $r = \frac{1}{p} - \frac{1}{q} < s$, so that by theorem 3.2 we have $B^{s,p}_\lambda(I;Y) \subseteq B^{r,p}_1(I;Y)$. Then by theorem 3.1 we have that $B^{r,p}_1(I;Y) \subseteq L^q(I;Y)$ as desired.

In the critical case when $q = \infty, \lambda = 1$ this result follows directly by theorem 3.1.

In the subcritical case where $q = p^*, \lambda \leq p^*$, this result follows directly by theorem 3.12. $\square$

Chapter 4

Besov Spaces for $s = 1$

It is first important to understand why when $s \geq 1$, the order of the difference in the integrand of the Besov space seminorm must change. Let $s \geq 1$ and consider some $f \in C^1(\mathbb{R}; Y)$ s.t.

$$\left(\int_0^\infty \frac{1}{h^{1+s\lambda}} \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_Y^p \, \mathrm{d}x \right)^{\frac{\lambda}{p}} \mathrm{d}h \right)^{\frac{1}{\lambda}} < \infty$$

where this quantity is just $\|f\|_{\dot{B}_\lambda^{s,p}(\mathbb{R};Y)}$, but there is a first order difference instead of a second order one. Suppose that for some $x_0 \in \mathbb{R}$ we have that $f'(x_0) \neq 0$. Then by continuity, $\exists \epsilon, \delta > 0$ s.t. $x \in (x_0 - \epsilon, x_0 + \epsilon) \implies \|f'(x)\|_Y > \delta$. As such, we can bound

$$\begin{aligned}
&\left(\int_0^\infty \frac{1}{h^{1+s\lambda}} \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_Y^p \, \mathrm{d}x \right)^{\frac{\lambda}{p}} \mathrm{d}h \right)^{\frac{1}{\lambda}} \\
\geq &\left(\int_0^{\frac{\epsilon}{2}} \frac{1}{h^{1+s\lambda}} \left(\int_{x_0 - \frac{\epsilon}{2}}^{x_0 + \frac{\epsilon}{2}} \|f(x+h) - f(x)\|_Y^p \, \mathrm{d}x \right)^{\frac{\lambda}{p}} \mathrm{d}h \right)^{\frac{1}{\lambda}} \\
\succeq &\left(\int_0^{\frac{\epsilon}{2}} \frac{1}{h^{1+s\lambda-\lambda}} \right)^{\frac{1}{\lambda}}
\end{aligned}$$

by MVT. But since $s \geq 1$, this integral diverges and the seminorm is infinite. Therefore if we keep the first order difference seminorm for $s \geq 1$, the space $B_\lambda^{s,p}$ would just be constant functions.

4.1 Some Equivalent Seminorms

Theorem 4.1 (Equivalent Seminorms)**.** For $0 < s < 1, 1 \leq p \leq \infty, 1 \leq \lambda < \infty$ and a Banach space $(Y, \|\cdot\|)$, we have that

$$\|f\|^*_{B_\lambda^{s,p}(\mathbb{R};Y)} = \|f\|_{L^p(\mathbb{R};Y)} + \left(\int_0^\infty \frac{\left\|\Delta_h^2 f\right\|^\lambda_{L^p(\mathbb{R};Y)}}{h^{1+s\lambda}} \, \mathrm{d}h \right)^{\frac{1}{\lambda}}$$

is an equivalent norm for $B_\lambda^{s,p}(\mathbb{R}; Y)$. Similarly for $\lambda = \infty$ we have that

$$\|f\|^*_{B_\infty^{s,p}(\mathbb{R};Y)} = \|f\|_{L^p(\mathbb{R};Y)} + \sup_{h>0} \frac{\left\|\Delta_h^2 f\right\|_{L^p(\mathbb{R};Y)}}{h^s}$$

is an equivalent seminorm for $B_\infty^{s,p}(\mathbb{R}; Y)$.

Proof. We show that

$$\left(\int_0^\infty \frac{\|\Delta_h f\|_{L^p(\mathbb{R};Y)}^\lambda}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \sim \left(\int_0^\infty \frac{\left\|\Delta_h^2 f\right\|_{L^p(\mathbb{R};Y)}^\lambda}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}}.$$

Clearly we have that

$$\begin{aligned}
&\left(\int_0^\infty \frac{\left(\int_{\mathbb{R}} \|f(x+2h) - 2f(x+h) + f(x)\|^p\, \mathrm{d}x\right)^{\frac{\lambda}{p}}}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\preceq \left(\int_0^\infty \frac{\left(\int_{\mathbb{R}} \|f(x+2h) - f(x+h)\|^p + \|f(x+h) - f(x)\|^p\, \mathrm{d}x\right)^{\frac{\lambda}{p}}}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\preceq \left(\int_0^\infty \frac{\left(\int_{\mathbb{R}} \|f(x+2h) - f(x+h)\|^p\, \mathrm{d}x\right)^{\frac{\lambda}{p}} + \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|^p\, \mathrm{d}x\right)^{\frac{\lambda}{p}}}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\preceq \left(\int_0^\infty \frac{\|\Delta_h f\|_{L^p(\mathbb{R};Y)}^\lambda}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}}
\end{aligned}$$

for one direction. For the other, we first note that

$$2\Delta_h u(x) = \Delta_{2h} u(x) - \Delta_h^2 u(x)$$

so that

$$\begin{aligned}
2\left(\int_0^\infty \frac{\|\Delta_h f\|_{L^p(\mathbb{R};Y)}^\lambda}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} &= 2\left(\int_0^\infty \frac{\left\|\Delta_{2h} u(x) - \Delta_h^2 u(x)\right\|_{L^p(\mathbb{R};Y)}^\lambda}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\le \left(\int_0^\infty \frac{\left(\|\Delta_{2h} u(x)\|_{L^p(\mathbb{R};Y)} + \left\|\Delta_h^2 u(x)\right\|_{L^p(\mathbb{R};Y)}\right)^\lambda}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&= \left\| \frac{\|\Delta_{2h} u(x)\|_{L^p(\mathbb{R};Y)} + \left\|\Delta_h^2 u(x)\right\|_{L^p(\mathbb{R};Y)}}{h^{\frac{1}{\lambda}+s}} \right\|_{L^\lambda((0,\infty),\mathbb{R})} \\
&\le \left\| \frac{\|\Delta_{2h} u(x)\|_{L^p(\mathbb{R};Y)}}{h^{\frac{1}{\lambda}+s}} \right\|_{L^\lambda((0,\infty),\mathbb{R})} + \left\| \frac{\left\|\Delta_h^2 u(x)\right\|_{L^p(\mathbb{R};Y)}}{h^{\frac{1}{\lambda}+s}} \right\|_{L^\lambda((0,\infty),\mathbb{R})} \\
&= \left(\int_0^\infty \frac{\|\Delta_{2h} u(x)\|_{L^p(\mathbb{R};Y)}}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} + \left(\int_0^\infty \frac{\left\|\Delta_h^2 u(x)\right\|_{L^p(\mathbb{R};Y)}}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&= 2^s \left(\int_0^\infty \frac{\|\Delta_l u(x)\|_{L^p(\mathbb{R};Y)}}{l^{1+s\lambda}}\, \mathrm{d}l\right)^{\frac{1}{\lambda}} + \left(\int_0^\infty \frac{\left\|\Delta_h^2 u(x)\right\|_{L^p(\mathbb{R};Y)}}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}}
\end{aligned}$$

where the last step follows from the change of variables $l = 2h$. From here, we can simplify to get that

$$\left(\int_0^\infty \frac{\|\Delta_h f\|_{L^p(\mathbb{R};Y)}^\lambda}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}} \le \frac{1}{2-2^s}\left(\int_0^\infty \frac{\left\|\Delta_h^2 u(x)\right\|_{L^p(\mathbb{R};Y)}}{h^{1+s\lambda}}\, \mathrm{d}h\right)^{\frac{1}{\lambda}}$$

as desired. The proof for $\lambda = \infty$ is analogous via replacing the integrals with supremums. $\square$

Lemma 4.2. Given $f \in L^p(\mathbb{R}; Y)$ we set

$$w_p(h) = \sup_{0 \le t \le h} \|f(x+2t) - 2f(x+t) + f(x)\|_{L^p(\mathbb{R};Y)}$$

and

$$\|f\|^*_{B^{1,p}_\lambda(\mathbb{R};Y)} = \left(\int_0^\infty \frac{(w_p(h))^\lambda}{h^{1+\lambda}} \,\mathrm{d}h \right)^{\frac{1}{\lambda}}$$

For $f \in B^{1,p}_\lambda(\mathbb{R}; Y)$ we have that

$$\|f\|_{B^{1,p}_\lambda(\mathbb{R};Y)} \sim \|f\|^*_{B^{1,p}_\lambda(\mathbb{R};Y)}$$

as equivalent seminorms.

Proof. The first direction follows from the fact that $\omega_p(h) \ge \|f(x+2h) - 2f(x+h) + f(x)\|_{L^p(\mathbb{R};Y)}$. For the other direction, note that for $0 \le t' \le t$ we have that

$$\begin{aligned} f(x+2t) - 2f(x+t) + f(x) =& + (f(x+2t') - 2f(x+t') + f(x)) \\ & - (f(x+2t) - 2f(x+t+t') + f(x+2t')) \\ & + 2\left(f(x+2t) - 2f\left(x+t+\frac{t'}{2}\right) + f(x+t') \right) \\ & - 2\left(f(x+t+t') - 2f\left(x+t+\frac{t'}{2}\right) + f(x+t) \right) \\ =& 2\Delta^2_{t-\frac{t'}{2}} f(x+t') - 2\Delta^2_{\frac{t'}{2}} f(x+t) + \Delta^2_{t'} f(x) - \Delta^2_{t-t'} f(x+2t'). \end{aligned}$$

Now by triangle inequality and then averaging over $t' \in (0, t)$ we get that

$$\left\|\Delta^2_t f\right\|_{L^p(\mathbb{R};Y)} \le \frac{1}{t} \left(\int_0^t 2\left\|\Delta^2_{t-\frac{t'}{2}} f\right\|_{L^p(\mathbb{R};Y)} + 2\left\|\Delta^2_{\frac{t'}{2}} f\right\|_{L^p(\mathbb{R};Y)} + \left\|\Delta^2_{t'} f\right\|_{L^1(\mathbb{R};Y)} + \left\|\Delta^2_{t-t'} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t' \right).$$

Since t' ranges from 0 to t, we have that

$$\int_0^t \left\|\Delta^2_{t-t'} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t' = \int_0^t \left\|\Delta^2_{t'} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t'$$

and

$$\int_0^t \left\|\Delta^2_{t-\frac{t'}{2}} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t' = \int_0^t \left\|\Delta^2_{\frac{t'}{2}} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t' = 2\int_0^{\frac{t}{2}} \left\|\Delta^2_{t'} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t'$$

by change of variables. Now clearly $\int_0^{\frac{t}{2}} \left\|\Delta^2_{t'} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t' \le \int_0^t \left\|\Delta^2_{t'} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t'$, so putting these with the previous bound on $\left\|\Delta^2_t f\right\|_{L^p(\mathbb{R};Y)}$ we get that

$$\begin{aligned} & \left\|\Delta^2_t f\right\|_{L^p(\mathbb{R};Y)} \le \frac{10}{t} \int_0^t \left\|\Delta^2_{t'} f\right\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}t' \le 10 \int_0^t \frac{\left\|\Delta^2_{t'} f\right\|_{L^p(\mathbb{R};Y)}}{t'} \,\mathrm{d}t' \\ \implies & \sup_{0 \le t \le h} \left\|\Delta^2_t f\right\|_{L^p(\mathbb{R};Y)} \le 10 \int_0^h \frac{\left\|\Delta^2_{t'} f\right\|_{L^p(\mathbb{R};Y)}}{t'} \,\mathrm{d}t' \end{aligned}$$

as the RHS integral is monotonically increasing. As such, we have that

$$
\begin{aligned}
\|f\|^*_{B^{1,p}_\lambda(\mathbb{R};Y)} = \left(\int_0^\infty \frac{(w_p(h))^\lambda}{h^{1+\lambda}}\,\mathrm{d}h\right)^{\frac{1}{\lambda}} &\preceq \left(\int_0^\infty \frac{1}{h^{1+\lambda}}\left(\int_0^h \frac{\left\|\Delta_t^2 f\right\|_{L^p(\mathbb{R};Y)}}{t}\,\mathrm{d}t\right)^\lambda \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\leq \left(\int_0^\infty \frac{1}{h^{1+\lambda}}\left(\left(\int_0^h \frac{\left\|\Delta_t^2 f\right\|^\lambda_{L^p(\mathbb{R};Y)}}{t^\lambda}\,\mathrm{d}t\right)^{\frac{1}{\lambda}}\left(\int_0^h 1\,\mathrm{d}t\right)^{\frac{\lambda-1}{\lambda}}\right)^\lambda \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&= \left(\int_0^\infty \frac{1}{h^2}\left(\int_0^h \frac{\left\|\Delta_t^2 f\right\|^\lambda_{L^p(\mathbb{R};Y)}}{t^\lambda}\,\mathrm{d}t\right)\mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&= \left(\int_0^\infty \int_t^\infty \frac{\left\|\Delta_t^2 f\right\|^\lambda_{L^p(\mathbb{R};Y)}}{h^2 t^\lambda}\,\mathrm{d}h\mathrm{d}t\right)^{\frac{1}{\lambda}} \\
&= \left(\int_0^\infty \frac{\left\|\Delta_t^2 f\right\|^\lambda_{L^p(\mathbb{R};Y)}}{t^{1+\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \\
&= \|f\|_{B^{1,p}_\lambda(\mathbb{R};Y)}
\end{aligned}
$$

by Hölder's inequality and Fubini. This completes the other direction, and as such, the two seminorms are equivalent. $\square$

This result allows us to prove a useful connection between Besov spaces and Sobolev spaces.

Theorem 4.3. For $1 \leq \lambda \leq \infty$ we have that

$$W^{2,1}(\mathbb{R};Y) \subseteq B^{1,1}_\lambda(\mathbb{R};Y).$$

In the case where $\lambda = 1$, we have the additional containment that

$$B^{1,1}_1(\mathbb{R};Y) \subseteq W^{1,1}(\mathbb{R};Y).$$

Proof. We first show the first containment, letting $f \in W^{2,1}(\mathbb{R};Y)$ with first and second weak derivatives f', f'' respectively. We wish to show that

$$\|f\|_{\dot{B}^{1,1}_\lambda(\mathbb{R};Y)} = \left(\int_0^\infty \frac{1}{h^{1+\lambda}}\left(\int_\mathbb{R} \|f(x+2h) - 2f(x+h) + f(x)\|_Y\,\mathrm{d}x\right)^\lambda \mathrm{d}h\right)^{\frac{1}{\lambda}} < \infty$$

but since we know that $f \in L^1(\mathbb{R};Y)$, we can drop the part of the integrand where $h \geq 1$ as $\int_1^\infty \frac{1}{h^{1+\lambda}}\,\mathrm{d}h < \infty$ and $\left(\int_\mathbb{R} \|f(x+2h) - 2f(x+h) + f(x)\|_Y\,\mathrm{d}x\right)^\lambda \preceq \|f\|_{L^1(\mathbb{R};Y)} < \infty$. Now by the Fundamental Theorem of Calculus, we have that

$$
\begin{aligned}
&\left(\int_0^1 \frac{1}{h^{1+\lambda}}\left(\int_\mathbb{R} \|f(x+2h) - 2f(x+h) + f(x)\|_Y\,\mathrm{d}x\right)^\lambda \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&= \left(\int_0^1 \frac{1}{h^{1+\lambda}}\left(\int_\mathbb{R} \left\|\int_x^{x+h}\int_y^{y+h} f''(z)\,\mathrm{d}z\mathrm{d}y\right\|_Y \mathrm{d}x\right)^\lambda \mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\leq \left(\int_0^1 \frac{1}{h^{1+\lambda}}\left(\int_\mathbb{R}\int_x^{x+h}\int_y^{y+h} \|f''(z)\|_Y\,\mathrm{d}z\mathrm{d}y\,\mathrm{d}x\right)^\lambda \mathrm{d}h\right)^{\frac{1}{\lambda}}.
\end{aligned}
$$

Now by Fubini, this can be rewritten as

$$\begin{aligned}
&\left(\int_0^1 \frac{1}{h^{1+\lambda}}\left(\int_{\mathbb{R}}\int_{z-h}^{z}\int_{y-h}^{y}\|f''(z)\|_Y\ \mathrm{d}x\mathrm{d}y\mathrm{d}z\right)^{\lambda}\ \mathrm{d}h\right)^{\frac{1}{\lambda}}\\
=&\left(\int_0^1 \frac{1}{h^{1-\lambda}}\left(\int_{\mathbb{R}}\|f''(z)\|_Y\ \mathrm{d}z\right)^{\lambda}\ \mathrm{d}h\right)^{\frac{1}{\lambda}}\\
\preceq&\|f''\|_{L^1(\mathbb{R};Y)}\\
<&\infty.
\end{aligned}$$

For the second containment, suppose that $f\in B_1^{1,1}(\mathbb{R};Y)$ so that $f\in L^1(\mathbb{R};Y)$ and by the previous equivalent seminorm, we have that

$$\int_0^\infty \left(\sup_{0\le t\le h}\int_{\mathbb{R}}\left\|\Delta_t^2 f(x)\right\|_Y\ \mathrm{d}x\right)\frac{1}{h^{1+\lambda}}\ \mathrm{d}h<\infty.$$

Now define $f_k(x)=2^k\left(f(x+2^{-k})-f(x)\right)=2^k\Delta_{2^{-k}}f(x)$. Then we have that $\forall k,l\in\mathbb{N}$,

$$\begin{aligned}
f_{k+l}(x)-f_k(x)&=2^{k+l}(f(x+2^{-k-l})-f(x))-2^k(f(x+2^{-k})-f(x))\\
&=\sum_{j=k}^{k+l-1}2^j(f(x+2^{-j})-f(x))-2^{j+1}(f(x+2^{-j-1})-f(x))\\
&=\sum_{j=k}^{k+l-1}2^j\Delta^2_{2^{-j-1}}f(x).
\end{aligned}$$

As such, we can bound

$$\begin{aligned}
\|f_{k+l}-f_k\|_{L^1(\mathbb{R})}&\le\sum_{j=k}^{k+l-1}2^j\left\|\Delta^2_{2^{-j-1}}f\right\|_{L^1(\mathbb{R})}\\
&\le\sum_{j=k}^{\infty}2^j\left\|\Delta^2_{2^{-j-1}}f\right\|_{L^1(\mathbb{R})}\\
&\le\sum_{j=k}^{\infty}2^j\sup_{0\le t\le 2^{-j-1}}\left\|\Delta^2_t f\right\|_{L^1(\mathbb{R})}.
\end{aligned}$$

Now observing that

$$\int_{2^{-j-1}}^{2^{-j}}\frac{1}{h^2}\ \mathrm{d}h=\left(-\frac{1}{h}\right)\Big|_{2^{-j-1}}^{2^{-j}}=2^j$$

we can further bound

$$\begin{aligned}
\|f_{k+l}-f_k\|_{L^1(\mathbb{R})}&\le\sum_{j=k}^{\infty}\int_{2^{-j-1}}^{2^{-j}}\frac{\sup_{0\le t\le h}\left\|\Delta_t^2 f\right\|_{L^1(\mathbb{R})}}{h^2}\ \mathrm{d}h\\
&=\int_0^{2^{-k}}\frac{\sup_{0\le t\le h}\left\|\Delta_t^2 f\right\|_{L^1(\mathbb{R})}}{h^2}\ \mathrm{d}h.
\end{aligned}$$

where the last step follows by Monotone Convergence. But since we know that

$$\int_0^\infty\frac{\sup_{0\le t\le h}\left\|\Delta_t^2 f\right\|_{L^1(\mathbb{R})}}{h^2}\ \mathrm{d}h<\infty$$

it follows by Dominated Convergence that

$$\lim_{k\to\infty}\int_0^{2^{-k}} \frac{\sup_{0\le t\le h}\left\|\Delta_t^2 f\right\|_{L^1(\mathbb{R})}}{h^2}\,\mathrm{d}h = 0$$

so that for sufficiently large k, $\|f_{k+l} - f_k\|_{L^1(\mathbb{R})}$ is sufficiently small, and $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy in L^1. Since L^1 is Banach, $f_k \to g$ for some $g \in L^1(\mathbb{R})$, and for any $\phi \in C_c^\infty(\mathbb{R})$ we have that

$$\begin{aligned}\int_{\mathbb{R}} f_k(x)\phi(x)\,\mathrm{d}x &= \int_{\mathbb{R}} \frac{f(x+2^{-k}) - f(x)}{2^{-k}}\phi(x)\,\mathrm{d}x\\ &= \frac{1}{2^{-k}}\left(\int_{\mathbb{R}} f(x+2^{-k})\phi(x)\,\mathrm{d}x \;-\int_{\mathbb{R}} f(x)\phi(x)\,\mathrm{d}x\right)\\ &= -\int_{\mathbb{R}} f(x)\frac{\phi(x+2^{-k})-\phi(x)}{2^{-k}}\,\mathrm{d}x\end{aligned}$$

where the last step follows by chyange of variables. Since $\phi \in C_c^\infty(\mathbb{R})$ and $f \in L^1(\mathbb{R})$, taking $k \to \infty$ gives us

$$-\int_R f(x)\phi'(x)\,\mathrm{d}x$$

by Dominated Convergence. But also since $f_k \to g$ in $L^1(\mathbb{R})$, we have that

$$\lim_{k\to\infty}\int_R f_k(x)\phi(x)\,\mathrm{d}x = \int_R g(x)\phi(x)\,\mathrm{d}x$$

again by Dominated Convergence. It follows then that g is the integrable weak derivative of $f \in L^1(\mathbb{R})$ so that $f \in W^{1,1}(\mathbb{R})$, as desired. □

4.2 Critical Case

We now study the critial case where $1 = s = \frac{1}{p}$.

Theorem 4.4 ($s = 1$ critical case L^q embedding)**.** Suppose that $1 \le p < \infty$. Then we have that $\forall p \le q < \infty$

$$L^p(\mathbb{R};Y) \cap \dot{B}_\lambda^{1,1}(\mathbb{R};Y) \subseteq L^q(\mathbb{R};Y).$$

Proof. We proceed by induction, showing that if $f \in L^p(\mathbb{R};Y) \cap \dot{B}_\lambda^{1,1}(\mathbb{R};Y)$ then it must be that $f \in L^{p+1}(\mathbb{R};Y)$. Suppose $f \in L^p(\mathbb{R};Y) \cap \dot{B}_\lambda^{1,1}(\mathbb{R};Y)$ and that $\lambda < \infty$. Then we have that for any $r > 0$,

$$\|f(x)\|_Y \le \frac{1}{r}\int_{-r}^{r} \|f(x+h) - 2f(x) + f(x-h)\|_Y\,\mathrm{d}h + \frac{1}{r}\int_{-r}^{r}\|f(x+h) - f(x-h)\|_Y\,\mathrm{d}h$$

by the triangle inequality. Now by Hölder's inequality on the second term, we have that this quantity is

$$\begin{aligned}&\le\frac{1}{r}\int_{-r}^{r}\|f(x+h)-2f(x)+f(x-h)\|_Y\,\mathrm{d}h + \frac{1}{r}(2r)^{\frac{1}{p'}}\left(\int_{-r}^{r}\|f(x+h)-f(x-h)\|_Y^p\,\mathrm{d}h\right)^{\frac{1}{p}}\\ &\le\frac{1}{r}\int_{-r}^{r}\|f(x+h)-2f(x)+f(x-h)\|_Y\,\mathrm{d}h + \frac{2(2r)^{\frac{1}{p'}}}{r}\|f\|_{L^p(\mathbb{R};Y)}.\end{aligned}$$

By Hölder and using the fact that $|h| \le r$, we get that this is

$$\begin{aligned}&\le \frac{1}{r}(2r)^{1-\frac{1}{\lambda}}\left(\int_{-r}^{r}\|f(x+h)-2f(x)+f(x-h)\|_Y^\lambda\,\mathrm{d}h\right)^{\frac{1}{\lambda}} + \frac{2(2r)^{\frac{1}{p'}}}{r}\|f\|_{L^p(\mathbb{R};Y)}\\ &= \frac{2^{1-\frac{1}{\lambda}}}{r^{\frac{1}{\lambda}}}\left(\int_{-r}^{r}\frac{h^{1+\lambda}}{h^{1+\lambda}}\|f(x+h)-2f(x)+f(x-h)\|_Y^\lambda\,\mathrm{d}h\right)^{\frac{1}{\lambda}} + \frac{2^{1+\frac{1}{p'}}}{r^{\frac{1}{p}}}\|f\|_{L^p(\mathbb{R};Y)}\\ &\le 2^{1-\frac{1}{\lambda}}r\left(\int_{-r}^{r}\frac{\|f(x+h)-2f(x)+f(x-h)\|_Y^\lambda}{h^{1+\lambda}}\,\mathrm{d}h\right)^{\frac{1}{\lambda}} + \frac{2^{1+\frac{1}{p'}}}{r^{\frac{1}{p}}}\|f\|_{L^p(\mathbb{R};Y)}.\end{aligned}$$

Assuming $f(x) \neq 0$ (0 case is trivial), we can then pick $r = \left(\frac{2^{2+\frac{1}{p'}} \|f\|_{L^p(\mathbb{R};Y)}}{\|f(x)\|_Y} \right)^p$ so that the inequality simplifies to

$$\|f(x)\|_Y \leq \frac{2^{1-\frac{1}{\lambda}+2p+\frac{p}{p'}} \|f\|^p_{L^p(\mathbb{R};Y)}}{\|f(x)\|^p_Y} \left(\int_{-r}^{r} \frac{\|f(x+h) - 2f(x) + f(x-h)\|^\lambda_Y}{h^{1+\lambda}} \, \mathrm{d}h \right)^{\frac{1}{\lambda}} + \frac{\|f(x)\|_Y}{2}$$

so that we have

$$\|f(x)\|^{p+1}_Y \leq 2^{2-\frac{1}{\lambda}+2p+\frac{p}{p'}} \|f\|^p_{L^p(\mathbb{R};Y)} \left(\int_{-r}^{r} \frac{\|f(x+h) - 2f(x) + f(x-h)\|^\lambda_Y}{h^{1+\lambda}} \, \mathrm{d}h \right)^{\frac{1}{\lambda}}.$$

Integrating over x yields

$$\|f\|_{L^{p+1}(\mathbb{R};Y)} \leq \left(2^{2-\frac{1}{\lambda}+2p+\frac{p}{p'}} \|f\|^p_{L^p(\mathbb{R};Y)} \|f\|_{\dot{B}^{1,1}_\lambda(\mathbb{R};Y)} \right)^{\frac{1}{p+1}} < \infty$$

so that $f \in L^{p+1}(\mathbb{R};Y)$ as desired. In the case that $\lambda = \infty$, we can replace the integrals with esssups and the same method works. $\square$

Corollary 4.5. We have that $\forall 1 \leq q < \infty$ that $B^{1,1}_\lambda(\mathbb{R};Y) \subseteq L^q(\mathbb{R};Y)$. In the case that $\lambda = 1$, we also have that $B^{1,1}_1(\mathbb{R};Y) \subseteq L^\infty(\mathbb{R};Y)$.

Proof. For $1 \leq q < \infty$ this is a direct result of the previous theorem as $B^{1,1}_1(\mathbb{R};Y) \subseteq L^1(\mathbb{R};Y)$. When $q = \infty$ and $\lambda = 1$, we note that $B^{1,1}_1(\mathbb{R};Y) \subseteq W^{1,1}(\mathbb{R};Y)$ by theorem 4.3. By absolute continuity of $W^{1,1}(\mathbb{R};Y)$ and integrability of the weak derivative, we then have that $W^{1,1}(\mathbb{R};Y) \subseteq L^\infty(\mathbb{R};Y)$ by FTC. $\square$

4.3 Supercritical Case

We now study the supercritical case where $1 = s > \frac{1}{p}$ and begin by reproving some analogous lemmas.

Theorem 4.6. (Theorem 3.2 for $s = 1$) Suppose $0 < r < 1, 1 \leq p \leq \infty, 1 \leq \lambda \leq \infty$. Then we have that

$$B^{1,p}_\lambda(\mathbb{R};Y) \subseteq B^{r,p}_1(\mathbb{R};Y)$$

with

$$\|f\|_{B^{r,p}_1(\mathbb{R};Y)} \preceq \|f\|_{B^{1,p}_\lambda(\mathbb{R};Y)}.$$

Proof. Let $f \in B^{1,p}_\lambda(\mathbb{R};Y)$ so that we can rewrite

$$\begin{aligned}
\|f\|^*_{B^{r,p}_1(\mathbb{R};Y)} &= \int_0^\infty \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^p(\mathbb{R};Y)}}{h^{1+r}} \, \mathrm{d}h \\
&= \int_0^1 \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^p(\mathbb{R};Y)}}{h^{1+r}} \, \mathrm{d}h + \int_1^\infty \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^p(\mathbb{R};Y)}}{h^{1+r}} \, \mathrm{d}h.
\end{aligned}$$

By Minkowski's inequality, the second term can be bounded by

$$4 \|f\|_{L^p(\mathbb{R};Y)} \left(-\frac{h^{-r}}{r} \right) \Bigg|_1^\infty \preceq \|f\|_{L^p(\mathbb{R};Y)} \leq \|f\|_{B^{1,p}_\lambda(\mathbb{R};Y)}.$$

For the first, we can apply Hölder's inequality for $p = \lambda, q = \frac{\lambda}{\lambda-1}$ to get that

$$\begin{aligned}
\int_0^1 \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^p(\mathbb{R};Y)}}{h^{1+r}} \,\mathrm{d}h &= \int_0^1 h^{\frac{1}{\lambda}-r} \frac{\|\tau_{2h} f - 2\tau_h f + f\|_{L^p(\mathbb{R};Y)}}{h^{1+\frac{1}{\lambda}}} \,\mathrm{d}h \\
&\leq \left(\int_0^1 h^{(\frac{1}{\lambda}-r)(\frac{\lambda}{\lambda-1})} \,\mathrm{d}h\right)^{\frac{\lambda-1}{\lambda}} \left(\int_0^1 \frac{\|\tau_{2h} f - 2\tau_h f + f\|^{\lambda}_{L^p(\mathbb{R};Y)}}{h^{1+\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&= \left(\frac{\lambda-1}{(1-r)\lambda}\right)^{\frac{\lambda-1}{\lambda}} \left(\int_0^1 \frac{\|\tau_{2h} f - 2\tau_h f + f\|^{\lambda}_{L^p(\mathbb{R};Y)}}{h^{1+\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\preceq \left(\int_0^\infty \frac{\|\tau_{2h} f - 2\tau_h f + f\|^{\lambda}_{L^p(\mathbb{R};Y)}}{h^{1+\lambda}} \,\mathrm{d}h\right)^{\frac{1}{\lambda}} \\
&\preceq \|f\|_{B^{1,p}_{\lambda}(\mathbb{R};Y)}
\end{aligned}$$

Then by the equivalence of seminorms shown in theorem 4.1, we have that $f \in B^{r,p}_1(\mathbb{R};Y)$ as desired. Note that for $\lambda = \infty$ the same steps and bounds work, except the integrand is replaced with a $\sup$. □

Theorem 4.7 (Theorem 3.3 for $s = 1$)**.** Suppose $0 < r \leq 1, 1 \leq p \leq q \leq \infty$, and $1 \leq \lambda \leq \infty$. Then if $1 - \frac{1}{p} = r - \frac{1}{q}$ we have that

$$B^{1,p}_{\lambda}(\mathbb{R};Y) \subseteq B^{r,q}_{\lambda}(\mathbb{R};Y)$$

with

$$\|f\|_{B^{r,q}_{\lambda}(\mathbb{R};Y)} \preceq \|f\|_{B^{1,p}_{\lambda}(\mathbb{R};Y)} .$$

Proof. When $r = 1$ the bounds are obvious. Otherwise, suppose $r < 1$ with $1-\frac{1}{p} = r-\frac{1}{q}$. Let $f \in B^{1,p}_{\lambda}(\mathbb{R};Y)$, so that by theorem 4.6 we have $B^{1,p}_{\lambda}(\mathbb{R};Y) \subseteq B^{1-r,p}_1(\mathbb{R};Y)$. Then since $0 < 1-r < 1$, we have by theorem 3.1 that

$$B^{1-r,p}_1(\mathbb{R};Y) \subseteq L^q(\mathbb{R};Y) \implies f \in L^q(\mathbb{R};Y).$$

Fix $t > 0$ so that clearly $f \in B^{1,p}_{\lambda}(\mathbb{R};Y)$ and $\tau_t f \in B^{1,p}_{\lambda}(\mathbb{R};Y)$. As such, we have that

$$\tau_{2t} f - 2\tau_t f + f \in B^{1,p}_{\lambda}(\mathbb{R};Y) \implies \tau_{2t} f - 2\tau_t f + f \in B^{1-r,p}_1(\mathbb{R};Y).$$

Now we apply theorem 3.1 to $\tau_{2t} f - 2\tau_t f + f \in B^{1-r,p}_1(\mathbb{R};Y)$ with $(1-r) - \frac{1}{p} = -\frac{1}{q}$ and $a = t$. We have that

$$\begin{aligned}
\|\tau_{2t} f - 2\tau_t f + f\|_{L^q(\mathbb{R};Y)} \leq &\int_0^t h^{r-2} \|\tau_h(\tau_{2t} f - 2\tau_t f + f) - (\tau_{2t} f - 2\tau_t f + f)\|_{L^p(\mathbb{R};Y)} \,\mathrm{d}h \\
&+ t^{r-1} \|\tau_{2t} f - 2\tau_t f + f\|_{L^p(\mathbb{R};Y)} .
\end{aligned}$$

When $\lambda < \infty$, we can observe that

$$\|\tau_h(\tau_{2t} f - 2\tau_t f + f) - (\tau_{2t} f - 2\tau_t f + f)\|_{L^p(\mathbb{R};Y)} \leq 2 \|\tau_{2t} f - 2\tau_t f + f\|$$

and then integrate with Minkowski's inequality to see that

$$\begin{aligned}
\left(\int_0^\infty \frac{\|\tau_{2t} f - 2\tau_t f + f\|^{\lambda}_{L^q(\mathbb{R};Y)}}{t^{1+r\lambda}} \,\mathrm{d}t\right)^{\frac{1}{\lambda}} \leq &2 \left(\int_0^\infty \frac{1}{t^{1+r\lambda}} \left(\int_0^t \frac{\|\tau_{2t} f - 2\tau_t f + f\|_{L^p(\mathbb{R};Y)}}{h^{2-r}} \,\mathrm{d}h\right)^{\lambda} \mathrm{d}t\right)^{\frac{1}{\lambda}} \\
&+ \left(\int_0^\infty \frac{\|\tau_{2t} f - 2\tau_t f + f\|^{\lambda}_{L^p(\mathbb{R};Y)}}{t^{1+\lambda}} \,\mathrm{d}t\right)^{\frac{1}{\lambda}} .
\end{aligned}$$

We can then apply Hardy's inequality (lemma 2.13) to the first integral on the RHS to get that

$$\left(\int_0^\infty \frac{\|\tau_{2t}f - 2\tau_t f + f\|_{L^q(\mathbb{R};Y)}^\lambda}{t^{1+r\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \le \left(\frac{2}{r}+1\right)\left(\int_0^\infty \frac{\|\tau_{2t}f - 2\tau_t f + f\|_{L^p(\mathbb{R};Y)}^\lambda}{t^{1+\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}}.$$

Finally by equivalence of seminorms, we get that

$$\|f\|^*_{B^{r,p}_\lambda(\mathbb{R};Y)} = \left(\int_0^\infty \frac{\|\tau_{2t}f - 2\tau_t f + f\|_{L^q(\mathbb{R};Y)}^\lambda}{t^{1+r\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} \preceq \left(\int_0^\infty \frac{\|\tau_{2t}f - 2\tau_t f + f\|_{L^p(\mathbb{R};Y)}^\lambda}{t^{1+\lambda}}\,\mathrm{d}t\right)^{\frac{1}{\lambda}} = \|f\|_{B^{1,p}_\lambda(\mathbb{R};Y)}$$

as desired.

In the case that $\lambda = \infty$ we replace integrands with supremums to get that

$$\begin{aligned}
\|f\|^*_{B^{r,p}_\infty(\mathbb{R};Y)} &= \sup_{t>0} t^{-r}\,\|\tau_{2t}f - 2\tau_t f + f\|_{L^q(\mathbb{R};Y)} \\
&\le 2\sup_{t>0}\left(t^{-r}\int_0^t \frac{\|\tau_{2t}f - 2\tau_t f + f\|_{L^p(\mathbb{R};Y)}}{h^{1+(1-r)}}\,\mathrm{d}h\right) + \sup_{t>0}\frac{\|\tau_{2t}f - 2\tau_t f + f\|_{L^p(\mathbb{R};Y)}}{t} \\
&\le \left(\frac{2}{r}+1\right)\sup_{t>0}\frac{\|\tau_{2t}f - 2\tau_t f + f\|_{L^p(\mathbb{R};Y)}}{t} \\
&\preceq \|f\|_{B^{1,p}_\infty(\mathbb{R};Y)}
\end{aligned}$$

once again using Hardy's inequality (lemma 2.13) to bound the first term. $\square$

Next we show that $B^{s,p}_\lambda$ increases with λ.

Theorem 4.8 (Theorem 3.4 for $s=1$)**.** Suppose $1 \le p \le \infty$ and $1 \le \lambda \le \mu \le \infty$. Then

$$B^{1,p}_\lambda(\mathbb{R};Y) \subseteq B^{1,p}_\mu(\mathbb{R};Y)$$

and

$$\|f\|_{B^{1,p}_\mu(\mathbb{R};Y)} \preceq \|f\|_{B^{1,p}_\lambda(\mathbb{R};Y)}.$$

Proof. From lemma 4.2, we know that defining

$$\omega_p(h) = \sup_{0\le t\le h}\|f(x+2t) - 2f(x+t) + f(x)\|_{L^p(\mathbb{R};Y)}$$

we have that

$$\|f\|^*_{B^{1,p}_\lambda(\mathbb{R};Y)} = \begin{cases}\left(\int_0^\infty \frac{(h^{-s}\omega_p(h))^\lambda}{h}\,\mathrm{d}h\right)^{\frac{1}{\lambda}} & \lambda<\infty \\ \sup_{h>0} h^{-s}w_p(h) & \lambda=\infty\end{cases}$$

is an equivalent seminorm for $B^{1,p}_\lambda(\mathbb{R};Y)$. Then following analogous steps as in the $s<1$ case (theorem 3.4), we get that $B^{1,p}_\lambda(\mathbb{R};Y) \subseteq B^{1,p}_\mu(\mathbb{R};Y)$ for $1 \le \lambda \le \mu \le \infty$ as desired. $\square$

Theorem 4.9 (Corollary 3.10 for $s=1$)**.** Let $1 < p \le \infty$ and $1 \le \lambda \le \infty$. Then we have that

$$B^{1,p}_\lambda(\mathbb{R};Y) \subseteq C^{0,1-\frac{1}{p}}(\mathbb{R};Y)$$

with

$$\|f\|_{C^{0,s-\frac{1}{p}}(\mathbb{R};Y)} \preceq \|f\|_{B^{s,p}_\lambda(\mathbb{R};Y)}.$$

Proof. By theorems 4.8 and 4.7 we have that

$$B^{1,p}_\lambda(\mathbb{R};Y) \subseteq B^{1,p}_\infty(\mathbb{R};Y) \subseteq B^{1-\frac{1}{p},\infty}_\infty(\mathbb{R};Y)$$

where $\|f\|_{B^{1-\frac{1}{p},\infty}_\infty} \preceq \|f\|_{B^{1,p}_\lambda(\mathbb{R};Y)}$ for any $f \in B^{1,p}_\lambda(\mathbb{R};Y)$. Then we are done since $B^{1-\frac{1}{p},\infty}_\infty(\mathbb{R};Y) = C^{0,1-\frac{1}{p}}(\mathbb{R};Y)$, as desired. $\square$

Chapter 5

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