Modeling Stability of Ranked Market Weights

by

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Author's Declaration

I declare that this thesis has been composed solely by myself. Unless otherwise stated by reference or acknowledgment, the work presented is entirely my own.

I understand that my thesis may be made electronically available to the public.

Abstract

The Capital Distribution Curve refers to the log-log plot of market weights with respect to their ranks. It is a well-known empirical fact that the Capital Distribution Curve has remained stable throughout history. This phenomenon, alongside models that account for it, has been studied extensively, such as in Fernholz (2002) [6]. Based on the work by Brown and Resnick (1977) [1], we will propose a new model that is consistent with this observation using Poisson Random Measures and show relevant finiteness properties.

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Chapter 1

Introduction

In 1611, the Amsterdam Stock Exchange, widely regarded as the first modern stock market, was formed. Since then, various stock markets have opened up around the world, playing an important role in the global financial system.

The primary purpose of the stock market is to provide a central, regulated environment where participants can buy or sell stakes, or *shares* of publically traded companies, also known as *stocks*. For such firms, this system allows them to raise capital by issuing additional shares to the market. For investors, the stock market allows them easily enter or exit investments.

As these markets grew to be more technologically advanced, participants gained access to more information about various stocks traded. Both long-term investors and short-term speculators of the stock market began to study historical data such as price movements of stocks listed on the market. From this, they hoped to be able to come up with best trading strategies.

1.1 Capital Distribution Curve

One data of interest from as early as 1931 is the distribution of *market capitalization* of firms. The market capitalization of a firm refers to the product of its share price and the total number of shares issued. This hence is an indicator of the total value of the firm.

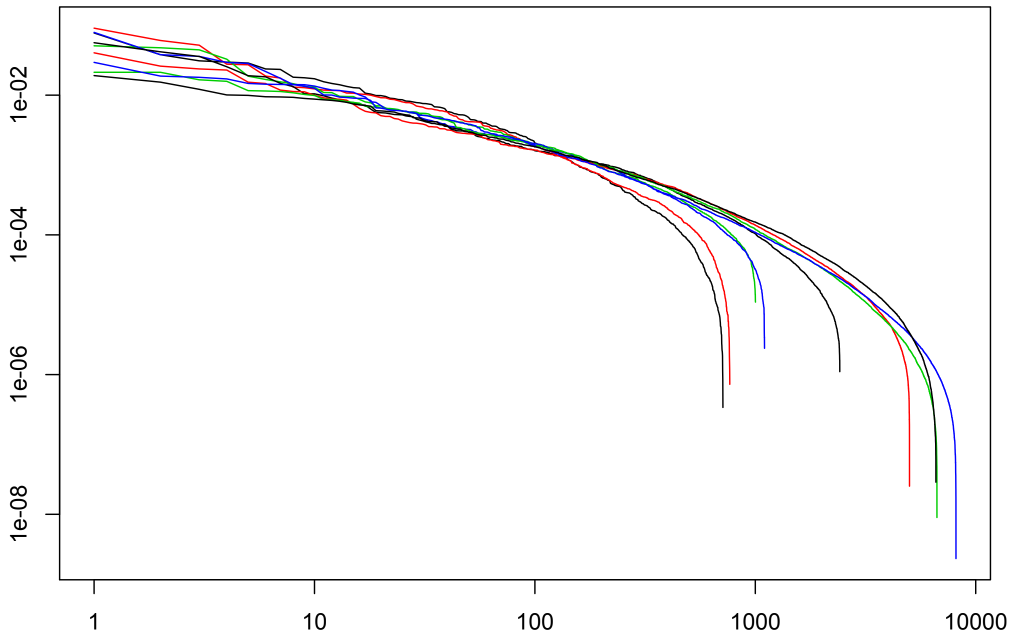

Figure 1.1: Capital Distribution Curves (one per decade) from 1929 to 2009 [2]

For each firm, we obtain its *market weight* by taking a ratio of its market capitalization to the sum of capitalizations of all firms.

Suppose we rank the market weights of all firms, and plot the log market weight against the log ranks. This plot, known as the *Capital Distribution Curve*, is empirically observed to follow a stable structure throughout time [6], as can be seen in fig. 1.1. We shall refer to this as the *stability of ranked market weights*.

Resultantly, mathematicians and financial analysts grew interested in this phenomenon, and began to find ways to incorporate this feature into models of the stock market. Fernholz, for instance, discussed to great lengths various diffusion processes that are consistent with this model [6]. He would go on to establish *Intech Investment* in 1987, presumably making use of some of his results.

In this paper, however, we shall concentrate on a new type of model that is based on *Poisson Random Measures* (PRM), inspired by the work of Brown and Resnick [1].

1.2 Introduction to PRM Model

In its most general form, a Poisson Random Measure (PRM) N over some measure space $(X, \mathcal{X}, \mu)$ is a function such that given pairwise disjoint measurable sets $A_1, ..., A_n \in X$ then

- $N(A_k)$ is a Poisson random variable with rate $\mu(A_k)$, for $k = 1, ..., n$.
- $N(A_1), ..., N(A_n)$ are independent.

Hence, very roughly speaking, N is a model of random points in X, where once we fix a 'region' A, we get $N(A)$, a random variable that tells us how many points exist within the region.

The original paper by Brown and Resnick considers a collection of points $\{X_i(0)\}_i$ from a PRM over the real line. We then model the *evolution* of each point by considering stochastic processes

$$X_i(t) = X_i(0) + W_i(t) - \frac{1}{2}t \tag{1.1}$$

The main result of the paper shows that the distribution of points is *stationary*, in particular that the overall distribution of all the points is not dependent on time. Consequently, it proves that that distribution of the maximum point is stationary.

We interpret this as a model of the capital distribution as follows.

- We consider a collection of stocks with log market capitalizations modelled by the processes X_i. The market capitalizations of these firms at time t is hence the collection $\{Y_i(t)\}$, where
$$Y_i(t) = e^{X_i(t)}$$
- Under this model, there is a countably infinite set of firms. Our calculations yield a uniquely largest Y_i, second largest Y_i and so on. As such, at each time $t \geq 0$, we can let $Y_{(1)}, Y_{(2)}, \ldots$ be the order statistics of $\{Y_i(0)\}$, that is, $Y_{(1)} \geq Y_{(2)} > \ldots$
- The results above give us that $\{Y_{(i)}(t)\}$ form a stationary stochastic process. As such, the ranked market weights are stationary as well, justifying the *stability of ranked market weights.*
- While rank by market capitalizations exists from the top, there are infinitely many Y_i's close to 0. We interpret these as a 'reservoir' of potential firms that are either too small, or merely exist as business ideas.

1.3 Summary of Results and Structure of Thesis

In this thesis, we shall extend the aforementioned results, keeping in mind our model interpretation. However, we will first need to develop this model rigorously, assuming only knowledge of analysis, topology, probability and stochastic processes at an undergraduate level.

Chapter 2 will focus on the theory of *Point Processes* on a 'nice' state space. The assumptions made are typical of texts on the topic. As these results are sometimes left as exercises for the reader, we shall explain their intuition and present full proofs within this chapter.

In Chapter 3, we shall find that the standard setup is too restrictive for our needs. Fortunately, similar results exist for a more general state space, namely *Polish spaces.* Complete proofs of these results can be found in our references. As such, we shall focus on a presentation of these results.

We can then state the existence and uniqueness of PRMs in Chapter 4.

With these tools, we will justify in full the results of Brown and Resnick in Chapter 5.

The results of our research will be presented in Chapter 6. We shall first modify the process described in eq. (1.1) by introducing a contraction parameter $a > 0$. That is, consider stock prices whose log market capitalizations

$$X_i^a(t) = X_i^a(0) + W_i(t) - \frac{1}{2}at$$

where $\{X_i^a(0)\}$ is a collection of points from a modified PRM over the real line. Correspondingly, let

$$Y_i^a(t) = e^{X_i^a(t)}$$

be their market capitalizations, and $Y_{(i)}^a(t)$ be the ranked market capitalizations. We will first show that under this model, we retain the stationarity properties.

The main result of this model concerns the finiteness of market capitalizations, so that ranked market weights can be well defined. Let

$$S_\infty^a(t) = \sum_{k=1}^{\infty} Y_{(k)}^a(t)$$

	$0 < a < 1$	$a = 1$	$a > 1$
$\mathbb{E}[Y^a_{(1)}]$	∞	∞	$\Gamma\left(1 - \frac{1}{a}\right)$
S^a_∞	$< \infty$ a.s.	∞ a.s.	∞ a.s.
$\mathbb{E}[S^a_\infty]$	∞	∞	∞

Table 1.1: Summary of Model Properties for Different Values of a.

be the total market capitalization. We will then study the following:

- Expectation of the leading market capitalization, $Y^a_{(1)}(t)$.
- Expectation of the total market capitalization.
- Almost-sure finiteness of total market capitalization.

We summarize our results in table 1.1. For different values of a, branching out at $a = 1$, we obtain different behavior in the finiteness of these objects. To finish our discussion, we will highlight some potential topics for future extension.

Chapter 2

Point Measures on SLCH Spaces

2.1 Space of Point Measures

We begin the discussion on the construction of the Poisson Random Measure (PRM) by first establishing the existence of point measures. These will be the codomain of the random elements that PRM describes.

We want to construct a probability space that can model random points on a state space X. In this section, this shall be a *second-countable locally compact Hausdorff* (SLCH) space. One way to accomplish this may be a define a sequence of i.i.d. random elements $\{x_i\}$ where we have $x_i : \Omega \to X$ measurable with respect to σ-algebras $\mathcal{F}$ and $\mathcal{X}$ on Ω and X respectively.

Using this setup, we will need to define the law

$$L_{x_i}(A) = \mathbb{P}(x_i \in A) \qquad (A \in \mathcal{X})$$

Yet, there is nothing special about individual random elements x_i. Often, we are more interested in how our random elements are spread out, such as through a random counting function

$$N(A) = \sum_{i=1}^{\infty} \mathbb{1}_{x_i}(A) \tag{2.1}$$

Immediately, we see a problem with such a setup. Since x_i are i.i.d., we have an object $N(A)$ that has expected value $+\infty$ or 0 for any set $A \in \mathcal{X}$.

As such, another idea here is to directly define the object in eq. (2.1). This will hence be a single random element taking values in some space of measures on X. Suppose we can specifically set up a space $M(X)$ of 'nice' measures on X, then we can define the law of N as a measure on $M(X)$:

$$L_N(A) = \mathbb{P}(N \in A) \qquad (A \in \sigma(M(X)))$$

The key challenge here is hence to define a suitable σ-algebra of $M(X)$ such that we can study objects of interest. Specifically, since we are interested in counting functions as in eq. (2.1), we will need $N(A)$ to be random variables. Therefore, we want the smallest σ-algebra on $M(X)$ for which the evaluation maps $\mu \mapsto \mu(A)$ are measurable for every $\mu \in M(X), A \in \mathcal{X}$. A convenient way to do so is to introduce a topology on $M(X)$ whose Borel σ-algebra gives us the desired properties.

First, some definitions.

Definition 2.1 (Second-Countable Topology). *A topological space (X, τ_X) is called* second-countable *if τ_X has a countable base, that is, there exists countable family $\{U_i\}_{i\in\mathbb{N}}$ of elements in τ_X such that for all $U \in \tau_X$,*

$$U = \bigcup_{i \in I_U} U_i$$

for some index set $I_U \subseteq \mathbb{N}$.

Definition 2.2 (Local Compactness). *A topological space (X, τ_X) is* locally compact *if for any $x \in X$, there exists $U \in \tau_X$ and $K \subseteq X$ compact such that*

$$x \in U \subseteq K$$

Definition 2.3 (Space of Radon Measures). *Given SLCH space X, a Radon measure is a measure $\mu : \mathcal{B}(X) \to \mathbb{R}^+ \cup \{\infty\}$ such that*

- *For every compact subset $K \subseteq X$, $\mu(K) < \infty$.*
- *(Outer regularity) For every $A \in \sigma(X)$,*

$$\mu(A) = \inf\{\mu(U) : U \supseteq A,\ U \text{ open}\}$$

- *(Inner regularity) For every $A \in \sigma(X)$,*

$$\mu(A) = \sup\{\mu(K) : K \subseteq A,\ K \text{ compact}\}$$

We define $M(X)$ as the space of all Radon measures.

Definition 2.4 (Space of Point Measures)**.** *The space of point measures $M_p(X)$ is a subspace of $M(X)$ such that every $\mu \in M_p(X)$ can be expressed as*

$$\mu = \sum_{i=1}^{\infty} \delta_{x_i} \qquad \text{(for } \{x_i\} \subseteq X\text{)}$$

where δ_x is the Dirac measure concentrated on $x \in X$.

2.2 Vague Convergence

To come up with the desired topology, we consider the following observation. The map $\mu \mapsto \mu(A)$ is a linear operation on $M(X)$, for fixed A. It can be written as

$$\mu \mapsto \int_X \mathbb{1}_A \, d\mu$$

Suppose there exists some space $\mathcal{C}$ of test functions that can approximate indicator functions of sets $A \in \mathcal{X}$, but can also be approximated by such indicators. Then the evaluation map is measurable if and only if

$$\mu \mapsto \int_X f \, d\mu$$

is measurable for every $f \in \mathcal{C}$. In other words, we need a topology induced by some notion of 'weak convergence'. As such, we shall construct a notion of 'weak convergence' alongside its induced topology.

Recall that when working with probability measures, we can define weak convergence using different families of test functions, many of which are equivalent. Since Radon measures need not be finite, we need to be more specific when defining such a convergence. For instance, suppose we take $C_b(X)$, the set of bounded continuous functions from $X \to \mathbb{R}$ and attempt to define

$$\mu_n \to \mu \iff \int_X f(x) \, d\mu_n(x) \to \int_X f(x) \, d\mu(x) \qquad \text{(for all } f \in C_b(X)\text{)}$$

Then we face a problem: the (Lebesgue) integrals might not exist since it may be possible for $\int_X f^-(x) \, d\mu(x) = \int_X f^+(x) \, d\mu(x) = \infty$. For instance, we can take $X = \mathbb{R}$, μ to be the

Lebesgue measure and $f = \sin(x)$.

This example is not even particularly esoteric. As such, we explore *vague convergence*. First for any $f \in C_c(X)$ define the shorthand

$$\mu(f) = \int_X f(x)\, d\mu(x)$$

Definition 2.5 (Vague Convergence on $M(X)$)**.** *Let $C_c(X)$ be the space of compactly supported continuous functions $f : X \to \mathbb{R}$. A sequence of measures $\{\mu_n\}_{n\in\mathbb{N}} \in M(X)$ is said to converge vaguely to $\mu \in M(X)$, written as $\mu_n \xrightarrow{v} \mu$, if and only if*

$$\mu_n(f) \to \mu(f)$$

in $\mathbb{R}$ for all $f \in C_c(X)$. Since each μ_n, μ is Radon, $\mu_n(supp(f)), \mu(supp(f)) < \infty$. Alongside the fact that f is bounded, each $\mu_n(f), \mu(f)$ is well-defined and finite.

We want to describe a topology on $M(X)$ where the canonical convergence gives rise to *vague convergence*. Literature on this topic commonly achieves this by showing $M(X)$ is metrizable as a complete, separable metric space. This is not often described in full, so in this thesis we shall break down the construction in [10].

First we note the following.

Definition 2.6 (Relatively Compact Sets)**.** *A set A of a topological space is called relatively compact if its closure, $\overline{A}$, is compact.*

Remark 2.2.1. *As a corollary of Lemma A.1.2, we know that for SLCH space X, there exists a basis $\mathcal{U}$ of relatively compact open sets. As setup for a later theorem, we shall further consider two modifications.*

First, let

$$\mathcal{V} = \left\{ \bigcup_{k=1}^{n} U_k : n \in \mathbb{N}, U_k \in \mathcal{U} \text{ for } k = 1, ..., n \right\}$$

Next, we can assume that $\mathcal{V}$ is a π-system (as per Theorem A.1.7). Otherwise, we can instead take

$$\mathcal{V}' = \left\{ \bigcap_{k=1}^{n} V_k : n \in \mathbb{N}, V_k \in \mathcal{V} \text{ for } k = 1, ..., n \right\}$$

Since the family of finite subsets of $\mathbb{N}$ is countable, we end up with a family of open sets $\mathcal{V}$ with the following properties:

(i) $\mathcal{V}$ is a countable basis of relatively compact sets.

(ii) $\mathcal{V}$ is a π-system.

(iii) For any $U \in \tau_X$ open, there exists an increasing sequence of open sets $\{V_n\}_n \subseteq \mathcal{V}$ such that

$$U = \bigcup_{n=1}^{\infty} V_n$$

We shall now prove a version of the Portmanteau theorem for vague convergence. To do so, we will make use of the fact that SLCH spaces are *normal*, which allows us to use equivalences in Urysohn's lemma (Theorem A.1.6).

Lemma 2.2.2 (Modification of Theorem 3.1. in [1])**.**

a) Let $K \subseteq X$ be a compact set. Then there exists a sequence of compact sets $\{K_n\}_n$ with $K_{n+1} \subseteq K_n$ such that

$$K = \bigcap_{n=1}^{\infty} K_n$$

alongside open sets W_n and $\{f_n\}_n \in C_c(X)$ such that

$$\mathbb{1}_K \leq f_n \leq \mathbb{1}_{W_n} \leq \mathbb{1}_{K_n} \downarrow \mathbb{1}_K$$

b) Let $V \subseteq X$ be a relatively compact open set. Then there exists a sequence of open sets $\{V_n\}$ with $V_n \subseteq V_{n+1}$ such that

$$V = \bigcup_{n=1}^{\infty} V_n$$

alongside compact sets K_n and $\{f_n\}_n \in C_c(X)$ such that

$$\mathbb{1}_V \geq f_n \geq \mathbb{1}_{K_n} \geq \mathbb{1}_{V_n}$$

Proof. a) By Remark 2.2.1, there exists a countable basis $\mathcal{V}$ of relatively compact open sets.

Since X is Hausdorff, for any $x \in K$ and $y \in K^c$ there exists an open set that contains x but not y. We can hence find $V_{x,y} \in \mathcal{V}$ such that $x \in V_{x,y}$ but $y \notin V_{x,y}$.

As such, for each fixed $y \in K^c$, $\bigcup_{x \in K} V_{x,y}$ is an open cover of K. By compactness, there exists a finite index set I_y of points in K such that

$$K \subseteq \bigcup_{x \in I_y} V_{x,y}$$

Correspondingly, we have

$$K = \bigcap_{y \in K^c} \bigcup_{x \in I_y} V_{x,y} \tag{2.2}$$

Now consider the family

$$\mathcal{U} = \left\{ \bigcup_{i=1}^{k} V_i : V_i \in \mathcal{V}, k \in \mathbb{N} \right\}$$

Since $\mathcal{V}$ is a countable set, there exists an injection from sets of $\mathcal{U}$ to the family of all finite subsets of $\mathbb{N}$, which is countable. Thus $\mathcal{U}$ is a countable family of sets. We can hence rewrite eq. (2.2) as

$$K = \bigcap_{i=1}^{\infty} U_i$$

where $\{U_i\}_i$ is a family of sets in $\mathcal{U}$. We now take

$$S_n = \bigcap_{i=1}^{n} U_i$$

Then each S_n is an open neighborhood of K. By Urysohn's lemma (ii) (Theorem A.1.6), there must exist open W_n and closed K_n such that

$$K \subseteq W_n \subseteq K_n \subseteq S_n$$

Since S_n is a finite intersection of finite unions of relatively compact sets, it is relatively compact. Thus K_n is in fact compact, with

$$K = \bigcap_{n=1}^{\infty} K_n$$

Furthermore, by Urysohn's lemma (iv) (Theorem A.1.6), there exist continuous functions $f_n \in C_c(X)$ with

$$\mathbb{1}_K \leq f_n \leq \mathbb{1}_{W_n} \leq \mathbb{1}_{K_n}$$

so we are done.

b) Singletons are closed in Hausdorff spaces. As such by Urysohn's lemma (ii) (Theorem A.1.6), for every $x \in V$ there exists open V_x and closed K_x such that

$$x \in V_x \subseteq K_x \subseteq V$$

By taking a subset of V_x belonging to $\mathcal{V}$ if necessary, we can assume that $V_x \in \mathcal{V}$. Since $\mathcal{V}$ is countable, the set

$$\{V_x : x \in V\}$$

Is also countable. Consider an enumeration of this set, and identify each set with some $x_i \in V$, i.e. V_{x_i}. Then we have

$$V = \bigcup_{x \in V} V_x = \bigcup_{i=1}^{\infty} V_{x_i}$$

Correspondingly, let

$$V_n = \bigcup_{i=1}^{n} V_{x_i}$$

We hence have

$$V = \bigcup_{n=1}^{\infty} V_n$$

Since V is relatively compact, each V_n is also relatively compact. Furthermore, we have

$$V \supseteq \bigcup_{i=1}^{n} K_{x_i} =: K_n \supseteq \bigcup_{i=1}^{n} V_{x_i} = V_n$$

Thus, by Urysohn's lemma (iv) (Theorem A.1.6) there exists continuous functions $f_n \in C_c(X)$ with

$$\mathbb{1}_V \geq f_n \geq \mathbb{1}_{K_n} \geq \mathbb{1}_{V_n}$$

as desired.

$\square$

Theorem 2.2.3 (Theorem 3.2. in [1]).
Let $\{\mu_n\}_n$ and μ be in $M(X)$. The following are equivalent:

a) $\mu_n \xrightarrow{v} \mu$

b) $\mu_n(V) \to \mu(V)$ *for all relatively compact V satisfying $\mu(\partial V) = 0$.*

c) For all K compact, we have

$$\limsup_{n\to\infty} \mu_n(K) \leq \mu(K)$$

and for all V relatively compact and open, we have

$$\liminf_{n\to\infty} \mu_n(V) \geq \mu(V)$$

Proof. (c) is the easiest to work with, hence we show (c) $\iff$ (a) and (c) $\iff$ (b).

(a) $\implies$ (c).

Consider K compact. By Lemma 2.2.2 a) there exists functions $\{f_\ell\}_\ell \in C_c(X)$ and decreasing sequence $\{K_\ell\}_\ell$ of compact sets such that

$$\mathbb{1}_K \leq f_\ell \leq \mathbb{1}_{K_\ell} \downarrow \mathbb{1}_K$$

thus

$$\limsup_{n\to\infty} \mu_n(K) \leq \lim_{n\to\infty} \int_X f_\ell \, d\mu_n = \int_X f_\ell \, d\mu \downarrow \mu(K) \tag{2.3}$$

by continuity of measures, since each $\mu(f_\ell) < \infty$.

On the other hand, consider V relatively compact and open. Then by Lemma 2.2.2 b) there exists functions $\{g_m\}_m \in C_c(X)$ and increasing sequence $\{V_m\}_m$ of relative compact open sets such that

$$\mathbb{1}_V \geq g_m \geq \mathbb{1}_{V_m} \uparrow \mathbb{1}_V$$

thus

$$\liminf_{n\to\infty} \mu_n(V) \geq \lim_{n\to\infty} \int_X g_m \, d\mu_n = \int_X g_m \, d\mu \uparrow \mu(V) \tag{2.4}$$

by continuity of measures.

(c) $\implies$ (a).

Let $f \in C_c(X)$. Then, by Fatou's lemma, combined with the fact that continuity implies sets $f^{-1}(t,\infty)$ are open, we have

$$\begin{aligned}\liminf_{n\to\infty}\int_X f\,d\mu_n &= \liminf_{n\to\infty}\int_0^\infty \mu_n(f^{-1}(t,\infty))\,dt \\ &\geq \int_0^\infty \liminf_{n\to\infty}\mu_n(f^{-1}(t,\infty))\,dt \\ &\geq \int_0^\infty \mu(f^{-1}(t,\infty))\,dt && \text{(using (c))} \\ &= \int_X f\,d\mu\end{aligned}$$

Note that we can use (c) in the second last step because $f \in C_c(X)$, so $\mu(f^{-1}(t,\infty)) \subseteq \operatorname{supp}(f)$ which is compact. We can apply the same argument to $-f$, which gives us

$$\int_X f\,d\mu \leq \liminf_{n\to\infty}\int_X f\,d\mu_n \leq \limsup_{n\to\infty}\int_X f\,d\mu_n \leq \int_X f\,d\mu$$

We note here that is it important to assume $\mu \in M(X)$, which gives us both sides of the inequality are finite. Thus we can conclude $\mu_n(f) \to \mu(f)$. Since the choice of f is arbitrary, we are done.

(b) $\implies$ (c).

By Lemma 2.2.2 a) there exists compact sets $\{K_m\}_m$ and $\{f_m\}_m \in C_c(X)$ such that

$$\mathbb{1}_K \leq f_m \leq \mathbb{1}_{K_m} \downarrow \mathbb{1}_K$$

For each m, consider for $0 < \delta < \frac{1}{2}$,

$$A_m(\delta) = f_m^{-1}\left[1-\delta, 2\right]$$

which has to be a compact set (closed set contained in K_m). Also consider

$$U_m(\delta) = f_m^{-1}\left(1-\delta, 2\right)$$

which has to be an open set contained in $A_m(\delta)$. Hence,

$$\partial A_m(\delta) \subseteq A_m(\delta) \setminus U_m(\delta) = f_m^{-1}\{1-\delta\} \qquad \text{(since } f_m(x) \neq 2 \text{ for all } x\text{)}$$

Since $\partial A_m(\delta)$ are disjoint for distinct $\delta \in \left(0, \frac{1}{2}\right)$, we must have $\mu(\partial A_m(\delta)) = 0$ for all but countably many δ. For each m, pick some δ such that $\mu(\partial A_m(\delta)) = 0$, and set

$$V_m := A_m(\delta)$$

Then

$$K = \bigcap_{m=1}^{\infty} V_m$$

and we can apply (b) on V_m. In particular,

$$\limsup_{n\to\infty} \mu_n(K) \le \lim_{n\to\infty} \mu_n(V_m) = \mu(A_m)$$

Now taking $m \to \infty$ we are done by continuity of measures. A similar reasoning gives us

$$\liminf_{n\to\infty} \mu_n(V) \ge \mu(V)$$

for relatively compact V, using Lemma 2.2.2 b).

(c) $\implies$ (b).

For any V relatively compact with $\mu(\partial V) = 0$, we can take $\overline{V}$ which is compact and V^{o} which is relatively compact and open. Then by (c) we have

$$\begin{aligned}\liminf_{n\to\infty} \mu_n(V) &\ge \liminf_{n\to\infty} \mu_n(V^{\mathrm{o}}) \\ &\ge \mu(V^{\mathrm{o}}) \\ &= \mu(\overline{V}) \\ &\ge \limsup_{n\to\infty} \mu_n(\overline{V}) \\ &\ge \limsup_{n\to\infty} \mu_n(V)\end{aligned}$$

Thus we are done. $\square$

2.3 Topology of $M(X)$

These theorems give us a lot of tools to work with this space, but specifically allow us to define a countable 'dense' subset of $C_c(X)$ for the sake of vague convergence. This allows us to metrize $M(X)$.

Corollary 2.3.1. *Let $\mu, \nu \in M(X)$. For every $V_n \in \mathcal{V}$ (as defined in Remark 2.2.1), there exists a sequence of functions $\{f_{n,k}\}_k \in C_c(X)$ such that*

$$\mu(f_{n,k}) = \nu(f_{n,k}) \ \forall k \implies \mu(V_n) = \nu(V_n)$$

Proof. In view of Lemma 2.2.2 b), there exists an increasing sequence of relatively compact open sets $\{V_{n,k}\}_k$ and sequence of functions $\{f_{n,k}\}_k \in C_c(X)$ such that

$$V_n = \bigcup_{k=1}^{\infty} V_{n,k}$$

and

$$\mathbb{1}_{V_n} \geq f_{n,k} \geq \mathbb{1}_{V_{n,k}}$$

Similar to past calculations, taking limits of integrals with respect to $d\mu$ and $d\nu$ respectively we have

$$\mu(V_n) = \lim_{k\to\infty} \int_X f_{n,k}\, d\mu = \lim_{k\to\infty} \int_X f_{n,k}\, d\nu = \nu(V_n)$$

□

Corollary 2.3.2. *Let $\mu, \nu \in M(X)$. There exists a countable family $\mathcal{F}$ of functions $\{f_i\}_i \in C_c(X)$, such that $\mu = \nu$ if and only if $\mu(f_i) = \nu(f_i)$ for $i \in \mathbb{N}$.*

Proof. We claim that we can simply take the collection

$$\{f_{n,k} : n, k \in \mathbb{N}\} \qquad \text{(as per Corollary 2.3.1)}$$

Suppose $\mu(f_{n,k}) = \nu(f_{n,k})\ \forall n, k$. By Corollary 2.3.1 we know that

$$\mu(V) = \nu(V) \qquad \text{(for all } V \in \mathcal{V})$$

As noted in Remark 2.2.1, we can assume that our basis $\mathcal{V}$ is a π-system. Thus μ, ν are σ-finite measures that agree on a generating π-system. Applying a corollary of Dynkin's theorem (Corollary A.1.8), we have $\mu = \nu$. □

Finally, we have enough setup to construct the following.

Theorem 2.3.3. *Let $\mathcal{F}$ be a countable family of functions as described in Corollary 2.3.2. Then*

$$d(\mu,\nu) = \sum_{i=1}^{\infty} \frac{1 \wedge |\mu(f_i) - \nu(f_i)|}{2^i}$$

metrizes vague convergence on $M(X)$.

Proof. Clearly $d(\mu,\mu) = 0$, $d(\mu,\nu) = d(\nu,\mu)$ and the triangle inequality holds, since they hold for each summand. By Corollary 2.3.2, we know

$$\mu = \nu \iff \mu(f_i) = \nu(f_i) \ \forall i \in \mathbb{N} \iff d(\mu,\nu) = 0$$

Finally,

$$d(\mu,\nu) \leq \sum_{i=1}^{\infty} \frac{1}{2^i} < \infty \qquad \text{(for all } \mu,\nu \in M(X))$$

thus showing that $d(\cdot,\cdot)$ is a metric.

To show that this metrizes vague convergence, consider measures $\{\mu_n\}_{n\in\mathbb{N}}, \mu \in M(X)$. Since each $f_i \in C_c(X)$, if $\mu_n \xrightarrow{v} \mu$ then $\mu_n(f_i) \to \mu(f_i)$ for all $f_i \in \mathcal{F}$, so $d(\mu_n,\mu) \to 0$.

On the other hand, suppose $d(\mu_n,\mu) \to 0$. Then for each $f_i \in \mathcal{F}$, $\mu_n(f_i) \to \mu(f_i)$.

First consider approximation on open sets. Fix any $\epsilon > 0$. By Remark 2.2.1 (iii), for every U relatively compact and open, there exists increasing sequence $\{V_i\}_i \in \mathcal{V}$ such that

$$U = \bigcup_{i=1}^{\infty} V_i$$

As such, there exists some $k \in \mathbb{N}$ such that

$$\mu(V_k) \geq \mu(U) - \epsilon$$

We now recall that $\mathcal{F}$ was constructed based on functions derived in Lemma 2.2.2. As such, given this V_k, there exists an increasing sequence of open sets $\{W_{k,i}\}_i$ with $V_k = \bigcup_{i=1}^{\infty} W_{k,i}$ and $\{f_{k_i}\}_i \subseteq \mathcal{F}$ such that

$$\mathbb{1}_{V_k} \geq f_{k_i} \geq \mathbb{1}_{W_{k,i}} \uparrow \mathbb{1}_{V_k}$$

Correspondingly, there exists some ℓ such that

$$\mu(W_{k,\ell}) \geq \mu(V_k) - \epsilon$$

Now we have

$$\begin{aligned}
\liminf_{n\to\infty} \mu_n(U) &\geq \liminf_{n\to\infty} \mu_n\left(f_{k_\ell}\right) \\
&= \mu(f_{k_\ell}) && (f_{k_\ell} \in \mathcal{F}) \\
&\geq \mu(W_{k,\ell}) \\
&\geq \mu(U) - 2\epsilon
\end{aligned}$$

Since ϵ was arbitrary, taking $\epsilon \to 0$ we get the second criterion of Theorem 2.2.3 (c).

Now consider approximation on compacts. Fix any $\epsilon > 0$. For every compact K, recall in our proof of Lemma 2.2.2 a) that there exists a decreasing sequence of compact sets K_i with

$$K = \bigcap_{i=1}^{\infty} K_i$$

such that

$$K \subseteq U_i \subseteq K_i$$

for some sequence of $\{U_i\}_i$ open (and relatively compact). As such, there exists some k such that

$$\mu(U_k) \leq \mu(K) + \epsilon$$

Fixing this U_k, by Remark 2.2.1 (iii) there exists an increasing sequence $\{V_i\}_i \in \mathcal{V}$ such that

$$U_k = \bigcup_{i=1}^{\infty} V_i$$

But $K \subseteq U_k$, thus by compactness, there is some L for which $K \in V_i$ for all $i \geq L$. Furthermore, by continuity of measures there exists some $\ell \geq L$ for which

$$\mu(V_\ell) \leq \mu(U_k) + \epsilon, \quad K \subseteq V_\ell$$

Similar to the previous part, by Lemma 2.2.2, given this V_ℓ, there exists an increasing sequence of open sets $\{W_{\ell,i}\}_i$ with $V_\ell = \bigcup_{i=1}^{\infty} W_{\ell,i}$ and $\{f_{\ell_i}\}_i \subseteq \mathcal{F}$ such that

$$\mathbb{1}_{V_\ell} \geq f_{\ell_i} \geq \mathbb{1}_{W_{\ell,i}} \uparrow \mathbb{1}_{V_\ell}$$

Similar to the reasoning above, by compactness of K, there exists some m such that

$$K \subseteq W_{\ell,m}$$

Summarizing, we have

- $\mu(V_\ell) \leq \mu(K) + 2\epsilon$
- $K \subseteq W_{\ell,m} \subseteq V_\ell$
- $\mathbb{1}_K \leq \mathbb{1}_{W_{\ell,m}} \leq f_{\ell_m} \leq \mathbb{1}_{V_\ell}$

Thus we have

$$\begin{aligned}
\limsup_{n\to\infty} \mu_n(K) &\leq \limsup_{n\to\infty} \mu_n(f_{\ell_m}) \\
&= \mu(f_{\ell_m}) && (f_{\ell_m} \in \mathcal{F}) \\
&\leq \mu(V_\ell) \\
&= \mu(K) + 2\epsilon
\end{aligned}$$

Sending $\epsilon \to 0$ we get the first criterion of Theorem 2.2.3 (c).

Thus by Theorem 2.2.3 a) we must have $\mu_n \xrightarrow{v} \mu$. $\square$

As such, the topology that we would like on $M(X)$ can in fact be induced by a suitable metric. It can further be shown that $M(X)$ is a Polish space, and that $M_p(X)$ is a closed subspace of $M(X)$.

Instead of proving these extensions for this section, we shall refer to more general results in the next section.

Chapter 3

Point Measures on Polish Spaces

3.1 Topology of Compact Convergence on C and C^0

In our discussion of the results of Brown and Resnick, we will need to define a PRM on the space $\mathbb{R} \times C^0(\mathbb{R}^+)$ and $C(\mathbb{R}^+)$.

Definition 3.1. *For the rest of this chapter, we refer to $C = C(\mathbb{R}^+; \mathbb{R})$ as the space of continuous functions $f : \mathbb{R}^+ \to \mathbb{R}$ and $C^0 = C^0(\mathbb{R}^+; \mathbb{R})$ as the subspace of such functions with $f(0) = 0$. These spaces will be equipped with the topology of compact convergence.*

That is, for a sequence $\{f_n\} \subset C$, $f_n \to f$ compactly if and only if $f_n \to f$ uniformly over every compact set in $\mathbb{R}^+$. It is clear to see that this topology can be induced by the metric

$$d_C(f, g) = \sum_{n=1}^{\infty} \frac{1 \wedge \sup_{[0,n]} |f - g|}{2^n}$$

This will tell us that this space is a complete metric space. However, a major flaw prevents us from using the results from the previous section.

Theorem 3.1.1. *C is not locally compact.*

Proof. Assume for the sake of contradiction that it is locally compact. Then, for every $f \in C$ there exists open set U and compact set K such that $f \in U \subseteq K$. Since our space is clearly nontrivial pick any such f and a corresponding U and K. By Lemma A.1.9, K is sequentially compact.

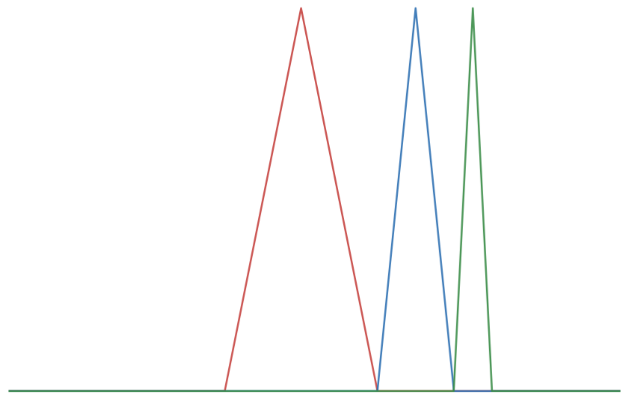

Figure 3.1: Intuition for counterexample in Theorem 3.1.1.

Since U is open there exists $\epsilon > 0$ such that

$$B_d(f, \epsilon) \subseteq U$$

It now suffices to construct a sequence $\{f_n\}_n \in B_d(f, \epsilon)$ that has no Cauchy subsequences, which will contradict our assumption .

The idea is as such. We shall create a sequence of functions supported say on $[0, 1]$, with disjoint and increasingly 'sharper' spikes.

First define function

$$S(a, b) = \begin{cases} 0 & \text{on } [0,1] \setminus (a, b) \\ \frac{a+b}{2}(x - a) & \text{on } \left(a, \frac{a+b}{2}\right) \\ 1 - \frac{a+b}{2}\left(x - \frac{a+b}{2}\right) & \text{on } \left(\frac{a+b}{2}, b\right) \end{cases}$$

Now define

$$f_n = f + \frac{\epsilon}{2} S\left(\frac{1}{n+1}, \frac{1}{n}\right)$$

so that $f_n \in B_d(f, \epsilon)$, but then we have

$$d_{C^0}(f_m, f_n) = \frac{\epsilon}{4}$$

for all $m, n \in \mathbb{N}$. Hence there cannot be any Cauchy subsequence of $\{f_n\}_n$. $\square$

3.2 Weak Hash Topology

In this section, we shall instead consider a complete, separable metric space (X, d_X). Such spaces are also known as *Polish metric spaces*, which we shall call *Polish spaces* or *Polish* for short. Similar to the previous section, we will need define a space of point measures on X and a suitable topology for our purposes. The details and complete proofs of these results can be found in *An Introduction to the Theory of Point Processes* by D. J. Daley and D. Vere-Jones ([3], [4]).

Definition 3.2 (Boundedly Finite Measures)**.** *A Borel measure μ on a Polish space X is boundedly finite if $\mu(A) < \infty$ for all A bounded, that is, $A \subseteq B(x, r)$ for some $x \in X$ and $r \in \mathbb{R}^+$.*

Let $\mathcal{M}(X)$ be the space of boundedly finite Borel measures on X. Similar to chapter 2, the goal is to equip $\mathcal{M}(X)$ with the smallest σ-algebra for which the evaluation map

$$\mu \mapsto \mu(A)$$

is measurable for all $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{X}$ bounded. As noted before, we do so by identifying a suitable topology on $\mathcal{M}(X)$ that encapsulates some notion of weak convergence. We will find that we can then turn $\mathcal{M}(X)$ into a Polish space in its own right.

In the previous section, we modified weak convergence to *vague convergence*, using $C_c(X)$ as test functions due to the local compactness property of $M(X)$. In this case, we do not have local compactness, but we do have bounded finiteness. As such, we can define a version of weak convergence using a different space of test functions.

Definition 3.3. *Let (X, d_X) be a Polish space. We define $C_v(X)$ to be the space of bounded continuous functions with bounded support, that is,*

$$C_v(X) = \left\{ f \in C_b(X; \mathbb{R}) : f = 0 \text{ in } B(x_0, r)^c \text{ for some } x_0 \in X, r \in \mathbb{R}^+ \right\}$$

We note here that these were in essence the properties needed for linear functions induced by C_v to be well-defined. We also recall the importance of Lemma 2.2.2 in proving properties of vague convergence. What we lack in terms of compactness, however, we can make up for using the existence of a metric. This inspires the following definition.

Definition 3.4 (Halo Sets)**.** *Let X be Polish and F be a closed set. We define Halo Sets of F to be the family of open sets $\{F_\epsilon\}_\epsilon$ indexed by $\epsilon > 0$ given by*

$$F_\epsilon = \{x \in X : d(x, F) < \epsilon\}$$

They have the property that $F_\epsilon \downarrow F$.

Definition 3.5 (Prokhorov Distance)**.** *The Prokhorov distance between two finite measures μ, ν is defined as*

$$d(\mu, \nu) = \inf \{\epsilon \geq 0 : \textit{for all closed } F \subseteq X,\ \mu(F) \leq \nu(F_\epsilon) + \epsilon \textit{ and } \nu(F) \leq \mu(F_\epsilon) + \epsilon\}$$

Where F_ϵ are Halo sets of F. Then $d(\cdot, \cdot)$ is a metric ([3] p.398).

We draw a parallel of this notion of distance to the summands of the metric defined in Theorem 2.3.3. In particular, Lemma 2.2.2 told us that the indicator function on every compact set or relatively compact set can be approximated by some sequence $\{f_n\}_n \subseteq C_c(X)$. In Theorem 2.3.3, we composed the metric using terms

$$|\mu(f_i) - \nu(f_i)|$$

which is analogous to the approximation

$$\mu(F) \leq \nu(F_\epsilon) + \epsilon \text{ and } \nu(F) \leq \mu(F_\epsilon) + \epsilon$$

on Halo sets in the Prokhorov distance. We now explicitly define a metric on $\mathcal{M}(X)$.

Definition 3.6 (Weak Hash Metric)**.** *Let X be Polish. Fix any $x_0 \in X$, and let $B_r := B(x_0, r)$ be open, bounded balls in X. For any $\mu \in \mathcal{M}(X)$, let μ^r be the unique measure given by*

$$\mu^r(B_r^c) = 0, \quad \mu^r(A) = \mu(A) \textit{ for all } A \subseteq B_r$$

As such, μ^r is always a finite measure. Let

$$d_r(\mu, \nu) = d(\mu^r, \nu^r)$$

be the Prohorov distance of μ, ν restricted to our finite ball. Then, the Weak-Hash metric is defined ([3] p.403) as

$$d^{\#}(\mu, \nu) = \int_0^\infty e^{-r} \frac{d_r(\mu, \nu)}{1 + d_r(\mu, \nu)}\, dr$$

The Weak-Hash Topology is hence the metric topology induced by $d^{\#}$.

The Weak-Hash metric hence takes the role of the metric in Theorem 2.3.3 in the previous section. Similarly, by now establishing a Portmanteau theorem for this topology, we can relate it to linear functionals using the space $C_v(X)$ of test functions as per Definition 3.3.

Theorem 3.2.1 (Theorem A2.6.II in [3] p.403)**.** *Let* $\{\mu_n\}_n, \mu \in \mathcal{M}(X)$*. The following are equivalent:*

a) $\mu_n \to \mu$ *under* $d^{\#}$*.*

b) $\mu_n(f) \to \mu(f)$ *for all* $f \in C_v(X)$*.*

c) There exists an increasing sequence $\{r_k\}_k$ *of reals with* $B_{r_k} \uparrow X$ *such that* $\mu_n^{r_k} \to \mu^{r_k}$ *weakly for every* k*.*

d) $\mu_n(A) \to \mu(A)$ *for all bounded* $A \in \sigma(X)$ *with* $\mu(\partial A) = 0$*.*

Finally, by formalizing our intuition at the beginning of section 3.2, we can show exactly what we needed.

Theorem 3.2.2 (Theorem A2.6.III in [3] p.404)**.**

a) $\mathcal{M}(X)$ *is Polish with respect to the weak hash metric (Definition 3.6).*

b) $\mathcal{B}(\mathcal{M}(X))$*, the Borel* σ*-algebra generated by the weak hash topology, is the smallest* σ*-algebra under which the evaluation maps*

$$\mu \mapsto \mu(A)$$

are measurable for all $A \in \mathcal{X}$ *bounded.*

3.3 Random Point Measures

We proceed to define *random point measures*. The space of measures we are interested in is a subspace of $\mathcal{M}(X)$ known as counting measures.

Definition 3.7 (Counting Measures)**.** *Let* X *be Polish. The space* $\mathcal{N}(X)$ *of counting measures is a subspace of* $\mathcal{M}(X)$ *whose elements only take on values in* $\mathbb{N}_0 \cup \{\infty\}$*.*

Indeed, we eventually hope to define a random element N which, when given a set $A \in \mathcal{X}$, produces a discrete random variable. In Daley and Vere-Jones ([4] p. 6), the following key observation is made.

Lemma 3.3.1 (Lemma 9.1.V in [4])**.** *$\mathcal{N}(X)$ is a closed subset of $\mathcal{M}(X)$.*

As such, Theorem 3.2.2 can be extended to the following.

Theorem 3.3.2 (Proposition 9.1.IV in [4])**.**

a) Under the subspace topology induced by $d^{\#}$, $\mathcal{N}(X)$ is Polish.

b) $\mathcal{B}(\mathcal{N}(X))$, the Borel σ-algebra generated by the above, is the smallest σ-algebra under which the evaluation maps

$$\mu \mapsto \mu(A)$$

are measurable for all $A \in \mathcal{X}$ bounded.

We can now define random point measures.

Definition 3.8 (Random Point Measure)**.** *A random point measure N on a Polish state space X is a measurable map from a probability space into $\mathcal{N}(X)$ equipped with the weak-hash topology.*

Consequently, when dealing with a random measure $N \in \mathcal{N}(X)$, we can fix any $A \subseteq X$ bounded and Borel and instead deal with each $N(A)$ as a random variable. The *mean measure* of N, then, is the expected number of points we can find in each A.

Definition 3.9 (Mean Measure)**.** *Given a random point measure N on a Polish space X, the mean measure λ is a Borel measure on X given by*

$$\lambda(B) = \mathbb{E}[N(B)]$$

Chapter 4

Poisson Random Measure on Polish Spaces

We are now ready to state the construction of PRMs on a Polish space X with mean measure λ.

Definition 4.1. *Let X be Polish and μ be a σ-finite measure on $(X, \mathcal{B}(X))$. A PRM on X is defined as a random element N taking values in $\mathcal{N}(X)$ equipped with the weak-hash topology, such that*

a) (Poisson Property) For every bounded $A \in \mathcal{B}(X)$, $N(A)$ is a discrete random variable with

$$\mathbb{P}(N(A) = k) = \frac{\mu(A)^k}{k!} e^{-\mu(A)}$$

b) (Complete Independence) For every finite collection $A_1, ..., A_k$ of pairwise disjoint bounded sets in $\mathcal{B}(X)$, $N(A_1), ..., N(A_k)$ are independent random variables.

The goal of this section is to motivate and state the existence and uniqueness of PRMs, following Last and Penrose lecture notes on the topic [8].

4.1 Existence and Uniqueness

We want to prove that the PRM is unique in the sense of distributions. That is, we want to show that any random element N taking values in $\mathcal{N}(X)$ satisfying the properties above must have the same law with respect to $\mathcal{N}(X)$.

In standard probability spaces, to show a real random variable X satisfying some properties is unique, we can make use of the characteristic function. Indeed, for any X, X' satisfying the same properties, we can show $\varphi_X(t) = \varphi_{X'}(t)$ to show that their distributions are identical. The proof of uniqueness of PRMs uses a similar tool in the *Laplace Functional*.

But first, we need to set up the concept of a *random integral*, as well as a good way to compute them.

Definition 4.2 (Random Integral)**.** *Let N be a random point measure (Definition 3.8) on a Polish space X. Then for each $\omega \in \Omega$, $N(\omega) \in \mathcal{N}(X)$ is a fixed measure on X. Correspondingly, we can define a Lebesgue integral for all functions $f : X \to \mathbb{R}^+$. We claim that the mapping*

$$\omega \mapsto \int_X f\, dN(\omega)$$

is a random variable. We write it as

$$N(f) = \int_X f\, dN$$

Proof. First suppose $f = \mathbb{1}_A$ for some $A \in \mathcal{B}(X)$. Then the mapping $\omega \mapsto N(\omega)(A)$ is a random variable by Theorem 3.3.2. Correspondingly, for f simple, the mapping is a linear combination of random variables and is hence a random variable as well. Finally, there exists increasing sequence $\{f_n\}_n$ simple with $f_n \uparrow f$ pointwise. By the Monotone Convergence Theorem we have

$$\lim_{n\to\infty} \int_X f_n\, dN(\omega) = \int_X f\, dN(\omega)$$

But the LHS is a pointwise limit of random variables, which has to be a random variable as well. □

Lemma 4.1.1 (Campbell)**.** *Let N be a random point measure (Definition 3.8) on a Polish space X with mean measure μ. Let $f : X \to \mathbb{R}^+$ be a measurable function. Then we have*

$$\mathbb{E}\left[\int_X f\, dN\right] = \int_X f\, d\mu$$

Proof. We apply the same technique. If f is simple, then we have

$$\mathbb{E}\left[\int_X f\,dN\right] = \sum_{i=1}^{k} a_i \mathbb{E}\left[N(A_i)\right] = \sum_{i=1}^{k} a_i \mu(A_i)$$

Else, there exists increasing, non-negative simple $\{f_n\}_n$ such that $f_n \uparrow f$ pointwise. By the Monotone Convergence Theorem,

$$\begin{aligned}\mathbb{E}\left[N(f)\right] &= \mathbb{E}\left[\lim_{n\to\infty} N(f_n)\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[N(f_n)\right] \\ &= \lim_{n\to\infty} \int_X f_n\,d\mu = \int_X f\,d\mu\end{aligned}$$

□

Definition 4.3 (Laplace Functional)**.**

$$\mathcal{L}_N(f) = \mathbb{E}\left[\exp\left(-\int_X f\,dN\right)\right]$$

It turns out, the two random point measures have the same distribution if and only if they have the same Laplace Functional. We have the following important result.

Theorem 4.1.2 (Characterization of Law of Point Measure)**.** *Let X be Polish and M, N be random point measures on X. Then the following are equivalent:*

a) $L_M = L_N$, where $L_{(\cdot)}$ indicates the law of a random point measure.

b) $(M(A_1), ..., M(A_k))$ and $(N(A_1), ..., N(A_k))$ are equal in distribution for all $k \in \mathbb{N}$ and $A_1, ..., A_k \in \mathcal{B}(X)$ pairwise disjoint.

c) $\mathcal{L}_M(f) = \mathcal{L}_N(f)$ for all $f : X \to \mathbb{R}^+$ measurable.

d) For all $f : X \to \mathbb{R}^+$ measurable, $N(f)$ and $M(f)$ are equal in distribution.

Corollary 4.1.3 (Uniqueness)**.** *If M, N are two PRMs on X with the same mean measure μ, then $L_M = L_N$.*

On the other hand, the proof of existence of a PRM with any given mean measure is a more involved process. Roughly speaking, it involves building up PRMs using sums of smaller PRMs. The justification comes in the form of a *superposition principle.*

Theorem 4.1.4 (Superposition). *Let $\{N_i\}_i$ be a sequence of independent PRMs on X with mean measure μ_i, then*

$$N = \sum_{i=1}^{\infty} N_i$$

is a PRM with mean measure $\mu = \sum_{i=1}^{\infty} \mu_i$.

For our purposes, we only need a weaker version of the existence theorem as introduced in *Lectures on the Poisson Process* ([8] p.21).

Theorem 4.1.5 (Existence Theorem). *For any σ-finite measure μ on X, there exists a PRM on X with mean measure μ.*

4.2 Induced PRMs

For our model, we want to rigorously define the distribution of points that originate from a simple PRM on the real line, then propogates through time according to a stochastic process.

The main mechanism for this, as well as discussions of stationarity, is the notion of push-forward measures and the mapping of one PRM onto another. We shall first define the transfer of measures.

Definition 4.4. *Given measurable spaces $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_T)$, a measurable function $f : X \to Y$ and a non-negative measure μ on $(X, \mathcal{A}_X)$, then the pushforward measure of μ by f is a measure on $(Y, \mathcal{A}_Y)$ given by*

$$\mu \circ f^{-1}(A) = \mu(f^{-1}(A)) \qquad \text{(for all } A \in \mathcal{A}_Y)$$

By the definition of measurability, if A is Y-measurable then f^{-1} is X-measurable and as such the expressions in the above equation are all well defined. Furthermore, properties of measures extend to pre-images of measurable functions, thus $\mu \circ f^{-1}$ produces a measure on $(Y, \mathcal{A}_Y)$.

It turns out, by taking the pushforward of a mean measure onto another Polish space, we can define an induced PRM on the output space, assuming we preserve the properties that allow us to define a PRM. This will come in handy when defining the objects relevant to our discussion later.

Theorem 4.2.1 (Mapping Theorem)**.** *Let N be a PRM on a Polish space X with σ-finite mean measure μ and let f be a measurable map from X to Y, another Polish space. Suppose further that the pushforward measure $\mu \circ f^{-1}$ is also σ-finite (need not be true in general).*

Then N^*, the point measure on Y defined by

$$N^*(A) = N(f^{-1}(A)) \qquad \text{(for } A \in \mathcal{B}(Y))$$

is a PRM on Y with mean measure $\mu \circ f^{-1}$.

We now have all the theoretical underpinnings to establish our model.

Chapter 5

Results of Brown & Resnick

5.1 PRM on C

We shall first discuss the main results of Brown & Resnick [1].

Definition 5.1. *We let $N_{\mathbb{R}}$ be a PRM (Definition 4.1) on $\mathbb{R}$ under the standard topology with mean measure*

$$\mu(A) = \int_A e^{-u}\, du \qquad (A \in \mathcal{B}(\mathbb{R}) \text{ bounded})$$

where du is the Lebesgue measure.

We shall also consider the stochastic process $Z_t = W_t - \frac{1}{2}t$, where W_t is a standard Brownian motion. The standard construction of a Brownian motion gives us a probability space where W_t (and hence Z_t) can be defined. For each outcome ω in this space, we consider the function induced by the path of the process Z_t. That is, take $z(\omega) : \mathbb{R}_0^+ \to \mathbb{R}$ to be an element of C^0 (first introduced in chapter 3) defined by

$$z(\omega)(t) = Z_t(\omega)$$

Eventually, we will convert this probability space to a measure on C^0, then define

$$X_t = X_0 + Z_t \tag{5.1}$$

where X_0 are atoms of $N_{\mathbb{R}}$. As such, X_t can be viewed as a function in C. Both C^0 and C in our discussion share the topology of compact convergence (as discussed in chapter 3).

Therefore, to model processes based on eq. (5.1), we shall define a PRM on C^0 instead, then get a PRM on $\mathbb{R} \times C^0$ and finally C. The technical challenge we then face is to define a suitable mean measure on C^0 such that the calculations we would like to make are measurable.

First of all, we verify that C^0 is Polish.

Lemma 5.1.1. *C^0 is Polish under the compact convergence metric (Definition 3.1).*

Proof. First we prove that C^0 is complete. Suppose we have $\{f_n\} \subseteq C^0$ Cauchy. Then on each $[0,k]$ for $k \in \mathbb{N}$, f_n restricted to $[0,k]$ is Cauchy with respected to the supremum norm on $C([0,k])$. Correspondingly, there exist a continuous pointwise limit $f^{(k)}$ along every $[0,k]$.

Let f be a continuous function on $\mathbb{R}^+$ that corresponds to $f^{(k)}$ on every $[0,k]$. We claim that this is the d_C-limit of $\{f_n\}$. Indeed, for any ϵ there exists k large enough such that $2^{-k} < \epsilon/4$. Then along each $[0,\ell]$ for $\ell = 1, 2, ..., k-1$, find n_ℓ large enough such that

$$\sup_{[0,\ell]} |f - f_n| < \frac{\epsilon}{2} \qquad \text{(for all } n \geq n_\ell\text{)}$$

Therefore, for all $n \geq \max_{\ell=1,\ldots,k-1} n_\ell$, we have

$$d_C(f_n, f) \leq \sum_{i=1}^{k-1} \frac{\epsilon/2}{2^i} + \sum_{i=k}^{\infty} \frac{1}{2^i} \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Next, we show separability. Fix any $f \in C^0$ and $\epsilon > 0$. Thus, there exists k large enough such that $2^{-k} < \epsilon/4$. Then by Stone-Weierstrass theorem, any continuous function on $[0, k-1]$ can be uniformly approximated by polynomials.

Thus there exists polynomial P such that

$$\sup_{[0,k-1]} |f - P| < \frac{\epsilon}{2}$$

Hence,

$$d_C(f, P) \leq \sum_{i=1}^{k-1} \frac{\epsilon/2}{2^i} + \sum_{i=k}^{\infty} \frac{1}{2^i} = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

To finish, we note that the family of polynomials with rational coefficients is dense in the family of polynomials on each $[0, k-1]$. This family of sets is countable, so taking union over all k we are done. $\square$

In the process of setting up the model, we will need to be able to consider sets of the form

$$\{z(\cdot) \in C^0 : z(t) \in I\}$$

where $t \in \mathbb{R}^+$ is fixed and I is some interval in $\mathbb{R}$. Intuitively, we want to assign to such sets the value

$$\mathbb{P}\{Z_t \in I\}$$

It turns out, these sets are not only naturally measurable, but we can also define a measure over the entire space C^0 by building using this idea.

Lemma 5.1.2 (Measure Induced on C^0)**.**

a) *Consider sets in C^0 of the form*

$$E(k, \{t_i\}, B) = \{z(\cdot) \in C^0 : (z(t_1), ..., z(t_k)) \in B\}$$

for $B \in \mathcal{B}(\mathbb{R}^k)$. Each $E(k, \{t_i\}, B)$ lies in $\mathcal{B}(C^0)$. Let the family of such sets be $\mathcal{E}$.

b) *There is a unique measure $\mathbb{Q}$ on C^0 such that*

$$\mathbb{Q}\left(E(k, \{t_i\}, B)\right) = \mathbb{P}\left((Z_{t_1}, ..., Z_{t_k}) \in B\right) \tag{5.2}$$

for all $E(k, \{t_i\}, B) \in \mathcal{E}$.

Proof.

a) We show that the evaluation map $z(\cdot) \mapsto (z(t_1), ..., z(t_k))$ is continuous. Indeed, for any $x, y \in C^0$ and each t_i, we have

$$|x(t_i) - y(t_i)| \leq 2^{\lceil t_i \rceil} \frac{\sup_{[0,\lceil t_i \rceil]} |x - y|}{2^{\lceil t_i \rceil}} \leq 2^{\lceil t_i \rceil} d_{C^0}(x, y)$$

Therefore, this map is Lipschitz in each component and hence continuous. Resultantly, $E(k, \{t_i\}, B)$, the pre-image of measurable B, is measurable.

b) We claim that the family of sets $E(k, \{t_i\}, B)$ form an algebra of measurable sets in C^0.

Indeed, the emptyset can be expressed as $E(\cdot, \cdot, \emptyset)$. We have closure under complementation due to the fact that

$$C^0 \setminus E(k, \{t_i\}, B) = E(k, \{t_i\}, \mathbb{R} \setminus B)$$

We also have closure under finite intersections. Indeed, consider $s_1 < \ldots < s_k$ and $t_1 < \ldots < t_\ell$, as well as Borel sets $B_1 \in \mathbb{R}^k, B_2 \in \mathbb{R}^\ell$. Suppose they share points $u_1 < ... < u_m$. We consider modifications of B_1, B_2 to B_1', B_2' so that they are in the same space $\mathbb{R}^{k+\ell-m}$ and the first m coordinates describe the values that $u_1, ..., u_m$ can take.

This can be done by a ordering of the coordinates of B_1 and B_2 respectively to put $u_1, ..., u_m$ as the first m coordinates, then appending entire axes as follows:

$$\begin{cases} B_1' = B_1 \times \mathbb{R}^{\ell-m} \\ B_2' = B_2 \times \mathbb{R}^{k-m} \end{cases}$$

Then

$$\begin{cases} E(k, \{s_i\}, B_1) = E(k+\ell-m, \{s_i\} \cup \{t_j\}, B_1') \\ E(\ell, \{t_j\}, B_2) = E(k+\ell-m, \{s_i\} \cup \{t_j\}, B_2') \end{cases}$$

As such,

$$E(k, \{s_i\}, B_1) \cap E(\ell, \{t_j\}, B_2) = E(k+\ell-m, \{s_i\} \cup \{t_j\}, B_1' \cap B_2')$$

as desired.

Now, by defining eq. (5.2) on each set E, we have a pre-measure on this algebra of sets. Indeed, the required properties are simply imported from the fact that $\mathbb{P}$ is a measure.

Finally, we want to show that this ring generates the entire sigma algebra $\mathcal{B}(C^0)$. Consider any open ball

$$B_{d_C}(f, \epsilon) = \{g \in C^0 : d_C(f, g) < \epsilon\}$$

We attempt to write $B_{d_C}(f, \epsilon)$ as an element of the σ-algebra generated by sets E, say call this $\mathcal{E}$. First write for any $r \in \mathbb{R}^+$

$$\begin{aligned} \left\{ g \in C^0 : \sup_{\mathbb{Q} \cap [0,k]} |f - g| < r \right\} &= \bigcap_{q \in \mathbb{Q} \cap [0,k]} \{g \in C^0 : |f(q) - g(q)| < r\} \\ &= \bigcap_{q \in \mathbb{Q} \cap [0,k]} E(1, \{q\}, (f(q) - r, f(q) + r)) \in \mathcal{E} \end{aligned}$$

For each $q \in \mathbb{Q}$, let $S_{n,q}$ be the set of all n-tuples of rationals $\boldsymbol{q}$ such that

$$\boldsymbol{q}_1 \leq \boldsymbol{q}_2 \ldots \leq \boldsymbol{q}_n, \text{ and } \sum_{k=1}^{n} \frac{\boldsymbol{q}_k}{2^k} = q$$

Then each $S_{n,q}$ is the set of finite tuples of a countable set, i.e. it is countable. For convenience, further define

$$\boldsymbol{q}_k = 2^n \left(\epsilon - \sum_{k=1}^{n} \frac{\boldsymbol{q}_k}{2^k} \right) \qquad \text{(for all } k > n\text{)}$$

Then we claim that we can write

$$B_{d_C}(f, \epsilon) = \bigcup_{n \in \mathbb{N}} \bigcup_{q \in \mathbb{Q}} \bigcup_{\boldsymbol{q} \in S_{n,q}} \bigcap_{k=1}^{\infty} \left\{ g \in C^0 : \sup_{\mathbb{Q} \cap [0,k]} |f - g| < \boldsymbol{q}_k \right\} \tag{5.3}$$

Proof of claim: $\supseteq$. Fix n, q and $\boldsymbol{q}$. Then for every

$$g^* \in \bigcap_{k=1}^{\infty} \left\{ g \in C^0 : \sup_{\mathbb{Q} \cap [0,k]} |f - g| < \boldsymbol{q}_k \right\}$$

we have

$$\begin{aligned} d_C(f, g^*) &< \sum_{k=1}^{n} \frac{\boldsymbol{q}_k}{2^k} + \sum_{k=n+1}^{\infty} \frac{\boldsymbol{q}_k}{2^k} \\ &\leq \sum_{k=1}^{n} \frac{\boldsymbol{q}_k}{2^k} + \left(\epsilon - \sum_{k=1}^{n} \frac{\boldsymbol{q}_k}{2^k} \right) = \epsilon \end{aligned}$$

so we are done.

$\subseteq$. On the other hand, take $g^* \in$ LHS, then $d_C(f, g^*) = \epsilon^*$ for some $\epsilon^* < \epsilon$. Let $m_k := 1 \wedge \sup_{[0,k]} |f - g^*|$, then m_k is increasing in k with

$$\epsilon^* = \sum_{k=1}^{\infty} \frac{m_k}{2^k}$$

so there must exist bounded (by 1)

$$m = \lim_{k \to \infty} m_k$$

and large enough n such that

$$\frac{m}{2^n} < \frac{1}{2}\left(\epsilon - \epsilon^*\right)$$

Then, we have

$$\sum_{k=1}^{n} \frac{m_k}{2^k} < \frac{\epsilon + \epsilon^*}{2}$$

so there exists rationals $q_1, ..., q_n$ such that

$$m_k < q_k$$

and

$$\sum_{k=1}^{n} \frac{m_k}{2^k} < \sum_{k=1}^{n} \frac{q_k}{2^k} < \frac{\epsilon + \epsilon^*}{2}$$

As such, we also have

$$\begin{aligned} m_k \leq m &< 2^n \left(\frac{\epsilon - \epsilon^*}{2}\right) \\ &= 2^n \left(\epsilon - \frac{\epsilon + \epsilon^*}{2}\right) \\ &< 2^n \left(\epsilon - \sum_{k=1}^{n} \frac{\boldsymbol{q}_k}{2^k}\right) && \text{(for all } k > n) \end{aligned}$$

Therefore, defining $\boldsymbol{q} = (q_1, ..., q_n)$ and

$$q = \sum_{k=1}^{n} \frac{q_k}{2^k}$$

we have g^* is in the set

$$\bigcap_{k=1}^{\infty} \left\{ g \in C^0 : \sup_{\mathbb{Q}\cap[0,k]} |f - g| < \boldsymbol{q}_k \right\}$$

i.e. in the RHS. Thus eq. (5.3) is justified. In particular, it shows that $B_{d_C}(f, \epsilon)$ is in the σ-algebra generated by $\mathcal{E}$. By Lemma 5.1.1, C^0 is separable, so its topology is countably generated by open balls and as such its borel σ-algebra can be generated by open balls, so we are done.

□

As such, the following exist.

Definition 5.2. *Let N_{C^0} be the PRM on C^0 with mean measure $\mathbb{Q}$.*

Definition 5.3. *Let N' be a PRM on $\mathbb{R} \times C^0$ with mean measure $\mu \times \mathbb{Q}$.*

Lemma 5.1.3. *Let $C = C([0, \infty))$ be the space of continuous functions on $[0, \infty)$. Consider the mapping $T_+ : \mathbb{R} \times C^0 \to C$ (equipped with the same compact convergence topology) given by*

$$T_+(x, z(\cdot)) = x + z(\cdot)$$

Then T_+ is a bijective continuous map.

Proof. T_+ is clearly bijective. To show continuity, take any $(x_1, z_1), (x_2, z_2) \in \mathbb{R} \times C^0$, then

$$\begin{aligned} d_C(x_1 + z_1, x_2 + z_2) &\leq \sum_{n=1}^{\infty} \frac{1 \wedge \left(|x_1 - x_2| + \sup_{[0,n]} |z_1 - z_2|\right)}{2^n} \\ &\leq |x_1 - x_2| + d_C(z_1, z_2) \end{aligned}$$

□

Corollary 5.1.4. *In view of Theorem 4.2.1, there exists a PRM N on C with mean measure ν given by the pushforward $(\mu \times \mathbb{Q}) \circ T^{-1}$.*

5.2 Stationarity

The Brown Resnick model considers the evolution of points with starting points belonging to the atoms of $N_{\mathbb{R}}$, each of which evolves independently according to identically distributed copies of X_t.

This is modelled by the PRM N as described in Corollary 5.1.4. The main advantage of PRMs is that order statistics, the main object of interest, are much easier to evaluate. First define shorthand

$$\{x_t > y\} = \{x(\cdot) \in C : x(t) \in (y, \infty)\}$$

Then for $k = 1, 2, ...$ let

$$X_{(k)}(t) = \inf\{y : N(x_t > y) = k - 1\} \tag{5.4}$$

which are a random variables in their own right. Then we can calculate its distribution in terms of the distribution of the random variable

$$N(x_t > y)$$

To do so, we first need to evaluate $\nu(\{x_t > y\})$. This is the first important calculation in [1].

$$\begin{aligned}
\nu(x_t > y) &= (\mu \times \mathbb{Q})\left(\{(x_0, z(\cdot)) : x_0 + z(t) > y\}\right) \\
&= \int_{C^0} \int_{y-z(t)}^{\infty} e^{-u}\, du\; d\mathbb{Q}(z(\cdot)) \\
&= \int_{C^0} e^{-(y-z(t))}\, d\mathbb{Q}(z(\cdot)) \\
&= \mathbb{E}\left[e^{-(y-W_t+t/2)}\right] \\
&= e^{-y}
\end{aligned} \tag{5.5}$$

Using the moment generating function of a Gaussian with mean 0 and variance t. Importantly, we see why we made this choice of a mean measure for $\mathbb{R}$ - the measure of this set is not dependent on t!

More precisely, suppose we were to now find the distribution of $X_{(1)}$

$$\begin{aligned}
\mathbb{P}(X_{(1)}(t) \leq y) &= \mathbb{P}(N(x_t > y) = 0) \\
&= \frac{\nu(x_t > y)^0}{0!} \exp\left(-\nu(x_t > y)\right) \\
&= \exp\left(-e^{-y}\right)
\end{aligned}$$

which naturally is not dependent on t. As such, the random variable $X_{(1)}(t)$ is also not dependent on t. We shall hence use $X_{(1)}$ to denote this random variable where we are only interested in the distribution.

Now, we introduce a concept of stationarity for points in such a PRM.

Theorem 5.2.1 (Stationarity)**.** *For every $\Delta > 0$, define mapping $T_\Delta : \mathbb{R} \times C^0 \to \mathbb{R} \times C^0$ defined by*

$$T_\Delta(x_0, x(\cdot)) = (x(\Delta), x(\cdot + \Delta) - x(\Delta))$$

Then T_Δ is a measurable map. Let N' be the PRM defined on $\mathbb{R} \times C^0$ with mean measure $\nu \circ T_\Delta^{-1}$. Then $\nu \circ T_\Delta^{-1} \sim \nu$, and as such $N \sim N'$.

Proof. It suffices to evaluate

$$\begin{aligned}
\nu \circ T_\Delta^{-1} \left(x_{t_i} > y_i \text{ for } i = 1, ..., k\right) &= \nu \left(x_{t_i + \Delta} > y_i \text{ for } i = 1, ..., k\right) \\
&= \nu \left(x_0 > \max_{i=1,\dots,k} (y_i - z(t_i + \Delta)) \right) \\
&= \int_{C^0} \int_{\max(y_i - z(t_i+\Delta))}^{\infty} e^{-u}\, du\, d\mathbb{Q}(z(\cdot)) \\
&= \int_{C^0} \exp\left[\min(z(t_i + \Delta) - y_i)\right]\, d\mathbb{Q}(z(\cdot)) \\
&= \mathbb{E}\left[\exp\left[\min(Z_{t_i+\Delta} - y_i)\right]\right] \\
&= \mathbb{E}\left[e^{Z_\Delta}\right] \mathbb{E}\left[\exp\left[\min(Z_{t_i+\Delta} - Z_\Delta - y_i)\right]\right] \\
&= \mathbb{E}\left[\exp\left[\min(Z_{t_i} - y_i)\right]\right]
\end{aligned}$$

Since $\mathbb{E}\left[e^{Z_\Delta}\right] = 1$ and $(Z_{t_i+\Delta} - Z_\Delta) \sim Z_{t_i}$ due to the fact that $W_t - \frac{1}{2}t$ is Markov. $\square$

Remark 5.2.2. *Implicit in our calculations above is the fact that the input sets to our measures are measurable sets of $\mathcal{B}(C)$. Indeed, this is the case by construction since they are sets in $\mathcal{E}$ as shown in Lemma 5.1.2*

Chapter 6

Model Properties

We interpret the results developed in the previous section as a model of the stock market. As discussed in chapter 1, we view the processes $X_{(k)}$ in eq. (5.4) as the ranked log market capitalizations of firms. Our PRM, N, can thus be written as

$$N = \sum_{k=1}^{\infty} \delta_{X_{(k)}}$$

where δ_X refers to the random Dirac measure concentrated at X. Given time $t \in \mathbb{R}^+$ and interval $(a, b) \subset \mathbb{R}$, the random variable $N(x_t \in (a, b))$ hence 'counts' the number of firms at time t with log market capitalizations within (a, b).

Moreover, the stationarity result in Theorem 5.2.1 is curious result. Under the context of stochastic processes, a process X_t is called *stationary* if

$$(X_{t_1}, X_{t_2}, ..., X_{t_k}) \sim (X_{t_1+\Delta}, X_{t_2+\Delta}, ..., X_{t_k+\Delta}) \qquad \text{(for any } k \in \mathbb{N}, \Delta > 0)$$

i.e. the joint distribution of finitely many times $\{t_i\}$ is translation-invariant across time. Theorem 5.2.1 gives us this precise property in the context of PRMs. This can also be seen by the fact that the distribution of $X_{(k)}(t)$ does not depend on t. As such, in our investigation of the order statistics $X_{(k)}(t)$ we will at times omit t.

6.1 Order Statistics

In view of eq. (5.5), we have the distribution of $X_{(1)}$. For general k, we shall proceed recursively. Indeed, we have

$$\begin{aligned}
\mathbb{P}(X_{(k+1)} \leq y) &= \mathbb{P}(X_{(k)} \leq y) + \mathbb{P}(X_{(k)} > y \text{ and } N(x_t > y) = k) \\
&= \mathbb{P}(X_{(k)} \leq y) + \mathbb{P}(N(x_t > y) = k) \\
&= \mathbb{P}(X_{(k)} \leq y) + \frac{\nu(x_t > y)^k}{k!} e^{-\nu(x_t > y)} \\
&= \mathbb{P}(X_{(k)} \leq y) + \frac{e^{-yk}}{k!} \exp\left(-e^{-y}\right) \qquad (6.1)
\end{aligned}$$

Once again, there is no surprise that the distribution of each $X_{(k)}$ is not dependent on t, in view of the stationarity (Theorem 5.2.1) property proven in the previous section.

Definition 6.1. *Let*

$$Y_{(k)} = \exp\left(X_{(k)}\right)$$

Then $Y_{(k)}$ represents the k-th ranked market capitalization within the market.

In order to model the ranked market weights, we look at objects of the form

$$W_{(k)} = \frac{Y_{(k)}}{\sum_{k=1}^{\infty} Y_{(k)}} \qquad (6.2)$$

Our results so far tell us that $(X_{(1)}, X_{(2)}, \ldots)$ is stationary, and hence $(Y_{(1)}, Y_{(2)}, \ldots)$ is stationary. Consequently, we know that each $W_{(k)}$ is stationary. The distribution of $(W_{(1)}, W_{(2)}, \ldots)$ is hence not dependent on t, giving us a justification of the stability of ranked market weights.

To get useful calculations out of this model, however, we need finiteness properties for $Y_{(k)}$ and $\sum_{k=1}^{\infty} Y_{(k)}$. Indeed, suppose with nonzero probability that

$$\sum_{k=1}^{\infty} Y_{(k)} = \infty$$

but each $Y_{(k)}$ is finite, then $W_{(k)} = 0$ for all k, contradicting the intuitive and desired property that the weights have to sum to 1. Furthermore, using the same setup as Brown & Resnick, we have no degree of freedom or parameter for model tuning.

As such, we shall first make a rather obvious extension of the results from the previous section.

6.2 Contraction Parameter

Theorem 6.2.1. *Instead of a mean measure of*

$$\mu(A) = \int_A e^{-u}\, du \qquad (A \in \mathcal{B}(\mathbb{R}))$$

Suppose we use a mean measure of

$$\mu(A) = a \int_A e^{-au}\, du \qquad (A \in \mathcal{B}(\mathbb{R}))$$

for some $a > 0$. Furthermore, instead of the process $Z_t = W_t - \frac{1}{2}t$, for each a define the process $Z_t^a = W_t - \frac{1}{2}at$ and let

$$X_t^a = X_0^a + Z_t^a$$

where X_0^a is an atom of the PRM $N_{\mathbb{R}}$. Then, we have the same stationarity property with respect to the pushforward measure T_Δ as in Theorem 5.2.1.

Proof. The proof proceeds almost identically to eq. (5.5) and Theorem 5.2.1. First we have

$$\begin{aligned}
\nu(x_t > y) &= (\mu \times \mathbb{Q})\left(\{(x_0, z(\cdot)) : x_0 + z(t) > y\}\right) \\
&= \int_{C^0} \int_{y-z(t)}^{\infty} ae^{-au}\, du\; d\mathbb{Q}(z(\cdot)) \\
&= \int_{C^0} e^{-a(y-z(t))}\, d\mathbb{Q}(z(\cdot)) \\
&= \mathbb{E}\left[e^{-a(y-W_t+at/2)}\right] \\
&= e^{-ay} \qquad (6.3)
\end{aligned}$$

Using the moment generating function of a Gaussian with mean 0 and variance a^2t. Next, we have

$$
\begin{aligned}
\nu \circ T_\Delta^{-1}\left(x_{t_i} > y_i \text{ for } i = 1, ..., k\right) &= \nu\left(x_{t_i+\Delta} > y_i \text{ for } i = 1, ..., k\right) \\
&= \nu\left(x_0 > \max_{i=1,\ldots,k}(y_i - z(t_i + \Delta))\right) \\
&= \int_{C^0} \int_{\max(y_i - z(t_i+\Delta))}^{\infty} a e^{-au}\, du\, d\mathbb{Q}(z(\cdot)) \\
&= \int_{C^0} \exp\left[a \min(z(t_i + \Delta) - y_i)\right]\, d\mathbb{Q}(z(\cdot)) \\
&= \mathbb{E}\left[\exp\left[a \min(Z^a_{t_i+\Delta} - y_i)\right]\right] \\
&= \mathbb{E}\left[e^{aZ^a_\Delta}\right] \mathbb{E}\left[\exp\left[a \min(Z^a_{t_i+\Delta} - Z^a_\Delta - y_i)\right]\right] \\
&= \mathbb{E}\left[\exp\left[\min(Z^a_{t_i} - y_i)\right]\right] \qquad (6.4)
\end{aligned}
$$

□

We can now present the first finite-ness property.

6.3 Integrability of Market Capitalizations

We will make use of properties of the gamma function.

Definition 6.2 (Gamma Function)**.** *For $z \in \mathbb{R}$, the gamma function $\Gamma(z)$ is defined as*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

In particular, integrating by parts and using induction, we can show $\Gamma(n) = (n-1)!$ for all $n \in \mathbb{N}$.

Theorem 6.3.1 ([5])**.** *Let $\alpha, \beta \in \mathbb{R}$. Then as $z \to \infty$, we have*

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + \frac{(\alpha-\beta)(\alpha+\beta-1)}{2z} + O(|z|^{-2})\right]$$

Theorem 6.3.2. *Let $X^a_{(k)}$ be the corresponding order statistic defined as per eq.* (5.4) *but for the modified model as described in Theorem 6.2.1 instead, for parameter $a > 0$.*

Define

$$Y^a_{(k)} = \exp(X^a_{(k)})$$

Then we have

$$\mathbb{E}[Y^a_{(1)}] = \begin{cases} \Gamma\left(1 - \frac{1}{a}\right) & \text{for } a > 1 \\ +\infty & \text{otherwise} \end{cases}$$

Proof. Firstly, we have

$$\begin{aligned} \mathbb{P}(X^a_{(1)}(t) \leq y) &= \mathbb{P}(N(x_t > y) = 0) \\ &= \frac{\nu(x_t > y)^0}{0!} \exp\left(-\nu(x_t > y)\right) \\ &= \exp\left(-e^{-ay}\right) \end{aligned}$$

As such, we have

$$\begin{aligned} \mathbb{E}[Y^a_{(1)}] &= \int_0^\infty \mathbb{P}(Y^a_{(1)} > t)\, dt \\ &= \int_0^\infty 1 - \mathbb{P}(X^a_{(k+1)} \leq \ln t)\, dt \\ &= \int_0^\infty 1 - \exp\left(-e^{-a \ln t}\right)\, dt \\ &= \int_0^\infty 1 - e^{-t^{-a}}\, dt \end{aligned} \tag{6.5}$$

When $a > 1$, consider change of variables $z = t^{-a}$, then we have $dz = -at^{-a-1}\, dt = -az^{1+\frac{1}{a}}\, dt$, so

$$\begin{aligned} \int_0^\infty 1 - e^{-t^{-a}}\, dt &= \int_0^\infty (1 - e^{-z}) \frac{1}{a} z^{-1-\frac{1}{a}}\, dz \\ &= -(1 - e^{-z}) z^{-\frac{1}{a}} \Big|_0^\infty + \int_0^\infty z^{-\frac{1}{a}} e^{-z}\, dz && \text{(Integration by parts)} \\ &= \Gamma\left(1 - \frac{1}{a}\right) \end{aligned}$$

When $a \in (0,1]$, note that as $t \to \infty$, $t^{-a} \to 0$. As such, using the Taylor expansion of e^x, we have

$$e^z = 1 + z + o(z)$$

As such, $1 - e^{-t^{-a}} \asymp t^{-a}$ for large t, so there exists $T = T(a)$ large enough such that

$$\frac{1}{2}t^{-a} \leq 1 - e^{-t^{-a}} \qquad \text{(for all } t \geq T\text{)}$$

Therefore we have

$$\begin{aligned}
\mathbb{E}[Y^a_{(1)}] &= \int_0^T 1 - e^{-t^{-a}}\,dt + \int_T^\infty 1 - e^{-t^{-a}}\,dt \\
&\geq K + \frac{1}{2}\int_T^\infty t^{-a}\,dt && \text{(some finite } K\text{)} \\
&= +\infty
\end{aligned}$$

Since the integral diverges for $0 < a \leq 1$. $\square$

We are also interested in the total sum of weights. Define

$$S^a_n = \sum_{k=1}^n Y^a_{(k)}$$

Since $Y^a_{(k)}$ is decreasing in k, if $Y^a_{(1)}$ is integrable then each term is integrable as well. However, how about $S^a_\infty = \lim_{n\to\infty} S^a_n$?

Theorem 6.3.3. *Suppose $a > 1$. Then by Theorem 6.3.2 we know $\mathbb{E}[Y^a_{(1)}] < \infty$. However, $\mathbb{E}[S^a_\infty] = \infty$.*

Proof. In view of eq. (6.1), we have

$$\begin{aligned}
\mathbb{P}(X^a_{(k+1)} \leq y) &= \mathbb{P}(X^a_{(k)} \leq y) + \mathbb{P}(X^a_{(k)} > y \text{ and } N(x_t > y) = k) \\
&= \mathbb{P}(X^a_{(k)} \leq y) + \mathbb{P}(N(x_t > y) = k) \\
&= \mathbb{P}(X^a_{(k)} \leq y) + \frac{\nu(x_t > y)^k}{k!} e^{-\nu(x_t > y)} \\
&= \mathbb{P}(X^a_{(k)} \leq y) + \frac{e^{-ayk}}{k!} \exp\left(-e^{-ay}\right) \qquad (6.6)
\end{aligned}$$

As such,

$$
\begin{aligned}
\mathbb{E}[Y^a_{(k)}] - \mathbb{E}[Y^a_{(k+1)}] &= \int_0^\infty \frac{e^{-ak\ln t}}{k!} \exp\left(-e^{-a\ln t}\right)\, dt \\
&= \frac{1}{k!}\int_0^\infty t^{-ak} \exp\left(-t^{-a}\right)\, dt \\
&= \frac{1}{k!}\int_0^\infty z^k e^{-z} \frac{1}{a} z^{-1-\frac{1}{a}}\, dz && (z = t^{-a}) \\
&= \frac{1}{ak!}\int_0^\infty z^{k-1-\frac{1}{a}} e^{-z}\, dz \\
&= \frac{1}{ak!}\Gamma\left(k - \frac{1}{a}\right) =: d_a(k)
\end{aligned}
$$

Where for the change of variables we have $dz = -at^{-a-1}\, dt = -az^{1+\frac{1}{a}}\, dt$. In particular, taking the second line of the above we note that

$$
\begin{aligned}
\sum_{k=1}^\infty d_a(k) &= \sum_{k=1}^\infty \int_0^\infty \frac{t^{-ak}}{k!} \exp\left(-t^{-a}\right)\, dt \\
&= \int_0^\infty e^{-t^{-a}} \left(\sum_{k=1}^\infty \frac{t^{-ak}}{k!}\right) dt && \text{(integrand non-negative)} \\
&= \int_0^\infty e^{-t^{-a}} \left(e^{t^{-a}} - 1\right) dt \\
&= \int_0^\infty 1 - e^{-t^{-a}}\, dt = \mathbb{E}[Y^a_{(1)}]
\end{aligned}
$$

As such, we can now evaluate

$$\begin{aligned}
\mathbb{E}[S_\infty^a] &= \sum_{\ell=1}^{\infty} \mathbb{E}[Y_{(\ell)}^a] \\
&= \sum_{\ell=1}^{\infty} \left(\mathbb{E}[Y_{(1)}^a] - \sum_{k=1}^{\ell-1} d_a(k) \right) \\
&= \sum_{\ell=1}^{\infty} \sum_{k=\ell}^{\infty} d_a(k) \\
&= \sum_{k=1}^{\infty} \sum_{\ell=1}^{k} d_a(k) \\
&= \sum_{k=1}^{\infty} \frac{k}{ak!} \Gamma\left(k - \frac{1}{a}\right) \\
&= \frac{1}{a} \sum_{k=1}^{\infty} \frac{\Gamma\left(k - \frac{1}{a}\right)}{\Gamma(k)}
\end{aligned}$$

By Theorem 6.3.1, taking $\alpha = -\frac{1}{a}$ and $\beta = 0$, there exists N large enough such that for all $z \geq N$, we have

$$\frac{\Gamma\left(k - \frac{1}{a}\right)}{\Gamma(k)} \geq \frac{1}{2} z^{-\frac{1}{a}}$$

As such,

$$\begin{aligned}
\mathbb{E}[S_\infty^a] &= \frac{1}{a} \sum_{k=1}^{N} \frac{\Gamma\left(k - \frac{1}{a}\right)}{\Gamma(k)} + \frac{1}{a} \sum_{k=N}^{\infty} \frac{\Gamma\left(k - \frac{1}{a}\right)}{\Gamma(k)} \\
&\geq K + \frac{1}{2a} \sum_{k=N}^{\infty} z^{-\frac{1}{a}} \qquad \text{(some } K < \infty\text{)}
\end{aligned}$$

which diverges since $a > 1$. □

This suggests that when $a > 1$, the largest point is finite in expectation, but the points are compressed inwards on $(X_{(1)}^a, 0)$, and as such the sum of $Y_{(k)}^a$ diverges in expectation. This means that $\mathbb{E}[S_\infty^a] = \infty$ always.

6.4 Almost-sure Finiteness

Another way to investigate if ranked market weights might make sense is to look at the almost sure behavior of S^a_∞. Indeed, even if $\mathbb{E}[S^a_\infty] = \infty$, we are still able to define ranked market weights $W_{(k)}$ as in eq. (6.2) if we have $S^a_\infty < \infty$ almost surely. We shall show that $a = 1$ is a bifurcation point for which the almost sure behavior diverges.

Theorem 6.4.1 (Almost Sure Finiteness)**.** *When $a \geq 1$, $S^a_\infty = \infty$ almost surely. When $a \in (0, 1)$, $S^a_\infty < \infty$ almost surely.*

Proof. We can fix any $t \in \mathbb{R}^+_0$. First, we investigate the finiteness of S^a_∞ on the event $X_{(1)} \leq K$ for some $K \in \mathbb{N}$, that is, on the event

$$E_K = \{N(x_t > K) = 0\}$$

and let

$$B_i := (\ln b_{i+1}, \ln b_i]$$

for $i = 1, \ldots$, where

$$b_i = \frac{e^N}{i^{1/a}}$$

Note that $b_1 = N, b_i \to 0$ as $i \to \infty$. Consider sets on C given by

$$\{x_t \in B_i\} = \{x(\cdot) \in C : x_t \in B_i\}$$

which are disjoint, so $N(x_t \in B_i)$ are independent random variables independent from E_k. Furthermore, we can calculate using eq. (6.3) that

$$\begin{aligned} \nu(x_t \in B_i) &= b_{i+1}^{-a} - b_i^{-a} \\ &= e^{-aN} =: \lambda \end{aligned}$$

As such

$$\mathbb{P}(N(x_t \in B_i) = 0) = e^{-a\lambda} \in (0, 1)$$

Now we look at $\mathbb{1}_{E_k} S^a_\infty$. We have

$$\begin{aligned}
\mathbb{1}_{E_k} S^a_\infty &= \mathbb{1}_{E_k} \sum_{k=1}^{\infty} Y^a_{(k)} \left(\sum_{i=1}^{\infty} \mathbb{1}_{Y^a_{(k)} \in (b_{i+1}, b_i]} \right) \\
&= \mathbb{1}_{E_k} \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} Y^a_{(k)} \mathbb{1}_{Y^a_{(k)} \in (b_{i+1}, b_i]} \\
&\geq \mathbb{1}_{E_k} \sum_{i=1}^{\infty} b_{i+1} \sum_{k=1}^{\infty} \mathbb{1}_{Y^a_{(k)} \in (b_{i+1}, b_i]} \\
&\geq \mathbb{1}_{E_k} \sum_{i=1}^{\infty} b_{i+1} \mathbb{1}_{N(x_t \in B_i) > 0}
\end{aligned} \tag{6.7}$$

First consider $a \geq 1$. For clarity, let

$$Y_i = b_{i+1} \mathbb{1}_{N(x_t \in B_i) > 0}$$

Then Y_i are i.i.d. non-negative random variables. Furthermore, we calculate

$$\begin{aligned}
\sum_{i=1}^{\infty} \mathbb{E}[Y_i] &= \sum_{i=1}^{\infty} b_{i+1}(1 - e^{-a\lambda}) \\
&= e^N (1 - e^{-a\lambda}) \sum_{i=1}^{\infty} \frac{1}{(i+1)^{1/a}} = \infty
\end{aligned}$$

As such, by Kolmogorov's Three-Series (Theorem A.2.2), taking $C = e^N(1 - e^{-a\lambda})$, the condition b) fails. Consequently, $\sum_{i=1}^{\infty} Y_i$ does not converge almost surely.

However, we note that the event

$$\left\{ \sum_{i=1}^{\infty} Y_i \text{ diverges} \right\}$$

is in the tail sigma-algebra of independent sigma-algebras $\sigma(Y_i)$. As such, by Kolmogorov's 0-1 Law (Theorem A.2.1), its probability must either be 0 or 1 - to it must be 1.

On the other hand, consider $a < 1$. From eq. (6.7) by taking the upper bound of each

$(b_{i+1}, b_i]$ instead, we get

$$\begin{aligned}\mathbb{1}_{E_k} S_\infty^a &\leq \mathbb{1}_{E_k} \sum_{i=1}^{\infty} b_i \\ &= \sum_{i=1}^{\infty} \frac{e^N}{i^{1/a}} < \infty\end{aligned}$$

In conclusion, on each E_k, our series diverges almost surely for $a \geq 1$, and it converges almost surely for $a \in (0,1)$. To finish, note that $X_{(1)} < \infty$ a.s. Indeed, taking the distribution function we have

$$\lim_{y\to\infty} \mathbb{P}(X_{(1)} \leq y) = \lim_{y\to\infty} \exp\left(-e^{-y}\right) = 1$$

As such, our result is true on $\bigcup_{k=1}^{\infty} E_k$, i.e. almost surely. $\square$

6.5 Interpretation and Extensions

Summarizing our results, we have the following:

	$0 < a < 1$	$a = 1$	$a > 1$
$\mathbb{E}[Y_{(1)}^a]$	∞	∞	$\Gamma\left(1 - \frac{1}{a}\right)$
S_∞^a	$< \infty$ a.s.	∞ a.s.	∞ a.s.
$\mathbb{E}[S_\infty^a]$	∞	∞	∞

Figure 6.1: Summary of Model Properties for Different Values of a.

In other words, there is no value of a for which we have both desirable properties.

It helps at this point of the discussion to consider the interpretation of finiteness in our model. Each path in our state space C represents the evolution of the log market capitalization of a firm.

At each t, there exists a countably infinite set of point-masses, since $\nu(x_t > y) \to \infty$ as $y \to 0$. This is in contrast with the finitely many firms we observe in real life.

As mentioned, the reason for the lack of finiteness of total market capitalizations is due to the high concentration of paths taking small values. As such, our model takes into account infinitely many firms with extremely tiny value.

Where the paths produce small values, we interpret these are firms that are not yet public, or even firms that only exist as potential ideas. That is, we allow

$$X_0^a, X_1^a, \ldots$$

to represent the initial values of all the firms in the history of the universe. For large values, they may represent established companies at the point in history we are starting our model. For small ones, they will represent companies that surface later on. When thought this way, a more accurate way to look at market capitalization of current firms, therefore, is to only considers firm that have gone past a certain threshold $\tau > 0$.

An additional comment is that under this model: by Law of Iterated Logarithms, $\lim_{t\to\infty} X_t = -\infty$. In other words, each firm will eventually reach 0 value (almost surely). This appears to be a reasonable assumption with the fall of many giant conglomerates in history.

In view of this discussion we shall make another refinement to our model. Looking at fig. 6.1, consider $a > 1$, where we have $\mathbb{E}[Y_{(1)}^a] < \infty$. Intuitively, S_∞^a diverges almost surely due to the fact that our points are compressed towards the interior of $(0, Y_{(1)}^a)$. As such, by instead considering a threshold cutoff τ as discussed, taking

$$Z_{(k)}^a = Y_{(k)}^a \mathbb{1}_{Y_{(k)}^a > \tau}$$

Then we still have

$$\mathbb{E}[Z_{(1)}^a] < \infty$$

But now the sum of market capitalizations is finite almost surely. Indeed, let

$$S_\infty^{a*} = \sum_{k=1}^{\infty} Z_{(k)}^a \tag{6.8}$$

Then

$$\nu(x_t > \tau) = e^{-a\tau} < \infty$$

Thus $N(x_t > \tau)$ is a Poisson random variable with finite parameter, thus is finite almost surely. We show that this modification solves the additional problem of infinite expected sum of market capitalizations.

Theorem 6.5.1. *Let $a > 1$, and let S_∞^{a*} be defined as in eq. (6.8). Then*

$$\mathbb{E}[S_\infty^{a*}] < \infty$$

Proof. We have

$$\begin{aligned}
\mathbb{E}[S_\infty^{a*}] &\leq \mathbb{E}\left[\sum_{k=1}^{\infty} Y_{(k)}^a \mathbb{1}_{Y_{(k)}^a > \tau}\right] \\
&\leq \mathbb{E}\left[Y_{(1)}^a \sum_{k=1}^{\infty} \mathbb{1}_{Y_{(k)}^a > \tau}\right] \\
&= \mathbb{E}\left[Y_{(1)}^a N(x_t > \tau)\right] \\
&\leq \mathbb{E}\left[\left(Y_{(1)}^a\right)^p\right]^{1/p} \mathbb{E}[N(x_t > \tau)^q]^{1/q} \qquad (6.9)
\end{aligned}$$

By Holder's Inequality, for $p, q > 1$ to be chosen such that $\frac{1}{p} + \frac{1}{q} = 1$.

Since $a > 1$, there exists a' such that $1 < a' < a$. We then choose

$$p = \frac{a}{a'} > 1$$

We now modify the calculations in eq. (6.5):

$$\begin{aligned}
\mathbb{E}\left[\left(Y_{(1)}^a\right)^p\right] &= \int_0^\infty \mathbb{P}\left(\left(Y_{(1)}^a\right)^p > t\right)\, dt \\
&= \int_0^\infty 1 - \mathbb{P}(pX_{(k+1)}^a \leq \ln t)\, dt \\
&= \int_0^\infty 1 - \exp\left(-e^{-\frac{a}{p}\ln t}\right)\, dt \\
&= \int_0^\infty 1 - \exp\left(-e^{-a'\ln t}\right)\, dt \\
&= \int_0^\infty 1 - \exp\left(-t^{-a'}\right)\, dt
\end{aligned}$$

Hence, following the remaining calculations in Theorem 6.3.2, we have

$$\mathbb{E}\left[\left(Y_{(1)}^a\right)^p\right] < \infty$$

On the other hand, $N(x_t > \tau)$ is a Poisson random variable with mean $\lambda := \nu(x_t > \tau)$. We can explicitly calculate:

$$\mathbb{E}[N(x_t > \tau)^q] = e^{-\lambda} \sum_{k=0}^{\infty} k^q \frac{\lambda^k}{k!}$$

By L'Hopital's applied $\lceil q \rceil$ times,

$$\lim_{k \to \infty} \frac{k^q}{e^k} = \lim_{k \to \infty} \frac{q(q-1)...(q - \lceil q \rceil + 1)k^{q - \lceil q \rceil}}{e^k} = 0$$

Thus there exists K large enough such that for all $k \geq K$,

$$\frac{k^q}{e^k} \leq \frac{1}{2}$$

Thus

$$\begin{aligned}
\mathbb{E}[N(x_t > \tau)^q] &\leq e^{-\lambda} \left(\sum_{k=0}^{K-1} k^q \frac{\lambda^k}{k!} + \sum_{k=K}^{\infty} \frac{e^k}{2} \cdot \frac{\lambda^k}{k!} \right) \\
&= M + e^{-\lambda} \sum_{k=K}^{\infty} \frac{(\lambda e)^k}{2k!} && (M < \infty) \\
&\leq \frac{1}{2} e^{\lambda e} < \infty
\end{aligned}$$

Resultantly, the RHS of eq. (6.9) is finite, and we are done. □

6.6 Extensions and Small Pertubation Principle

To end this main chapter, we shall discuss a major possible extension to this work. Having created a PRM model that models the stability of ranked market weights, its usefulness needs to be evaluated based on real-life applications to the market. To do so, we will need to calibrate the two parameters that can be tuned, $a > 1$ and $\tau > 0$.

Values of a may be approximated by studying the paths of individual market capitalizations, as they follow the stochastic process $Z_t = W_t - \frac{1}{2}at$. We might notice, however, that this could be difficult as this assumes a negative long-term drift $-\frac{1}{2}at$, which might be reasonable in the long-run but is unrealistic in the short-term. τ on the other hand can be approximated by taking the mean sum of market capitalizations, however that will involve evaluating

$$\mathbb{E}[S_\infty^{a*}]$$

as a function of a, τ, something we have not done here.

This model may suffer from robustness due to the small number of tunable parameters. One way to solve this problem is to identify other areas of the model where we have some freedom over, without compromising our key stationarity result. For instance, we could add a 'small pertubation' to the mean measure μ on $\mathbb{R}$.

We expect that for a suitably nice function f, changing our mean measure to

$$\mu_f(A) = \int_A ae^{-au} + f\, du \tag{6.10}$$

does not change the long-term behavior of particles of our model. In particular, we expect the distribution of order statistics $X_{(k)}$ to 'revert' to our stationary distribution. For that, we have the following result.

Theorem 6.6.1 (Small Pertubation)**.** *Let f be a compactly supported continuous function on $\mathbb{R}$. Suppose we instead the mean measure described in eq. (6.10) in the construction of our PRM, and let ν_f be the resultant measure on C.*

Then, our distribution on C reverts to the mean measure eq. (6.3) in the sense that

$$\lim_{\Delta\to\infty} \nu_f(x_{t_i+\Delta} > y_i \textit{ for } i = 1, ..., k) = \nu((x_{t_i} > y_i \textit{ for } i = 1, ..., k))$$

Proof. We proceed similar to eq. (6.4).

$$
\begin{aligned}
&\nu_f\left(\{x_{t_i+\Delta} > y_i \text{ for } i = 1, ..., k\}\right) \\
=&\nu_f\left(x_0 > \max_{i=1,\ldots,k}(y_i - z(t_i + \Delta))\right) \\
=&\int_{C^0}\int_{\max(y_i - z(t_i+\Delta))}^{\infty}\left(ae^{-au} + f(u)\right)\,du\,d\mathbb{Q}(z(\cdot)) \\
=&\int_{C^0} e^{a\min(z(t_i+\Delta)-y_i)} + f(a\min(z(t_i+\Delta) - y_i))\,d\mathbb{Q}(z(\cdot)) \\
=&\mathbb{E}\left[e^{a\min(Z^a_{t_i+\Delta}-y_i)}\right] + \mathbb{E}\left[f(a\min(Z^a_{t_i+\Delta} - y_i))\right] \\
=&\mathbb{E}\left[\exp\left[\min(Z^a_{t_i} - y_i)\right]\right] + \mathbb{E}\left[f(a\min(Z^a_{t_i+\Delta} - y_i))\right]
\end{aligned}
$$

The first term is simply $\nu((x_{t_i} > y_i \text{ for } i = 1, ..., k))$ and is not dependent on Δ. As such, it suffices to show that the second term vanishes. But f is a bounded function, so by Dominated Convergence Theorem and continuity of f,

$$
\begin{aligned}
&\lim_{\Delta\to\infty}\mathbb{E}\left[f(a\min(X^a_{t_i+\Delta} - y_i))\right] \\
=&\mathbb{E}\left[\lim_{\Delta\to\infty} f(a\min(X^a_{t_i+\Delta} - y_i))\right] \\
=&\mathbb{E}\left[f\left(\lim_{\Delta\to\infty} a\min(X^a_{t_i+\Delta} - y_i)\right)\right] \\
=&0
\end{aligned}
$$

Since we can fix say t_1, and realize by Law of Iterated Logarithms that

$$\lim_{\Delta\to\infty} X_{t_1+\Delta} = \lim_{\Delta\to\infty}\left(W_{t_1+\Delta} - \frac{1}{2}a(t_1 + \Delta)\right) = -\infty$$

almost surely. In general, any f that allows us to apply Dominated Convergence will suffice. $\square$

References

[1] Bruce M. Brown and Sidney I. Resnick. Extreme values of independent stochastic processes. *Journal of Applied Probability*, 14(4):732–739, Dec. 1977.

[2] Wikimedia Commons. File:u.s. stock market capital distribution curves 1929-2009.pdf — wikimedia commons, the free media repository, 2022. [Online; accessed 17-April-2023].

[3] D. J. Daley and D. Vere-Jones. *Elementary Theory and Methods*, volume I of *An Introduction to the Theory of Point Processes*. Springer New York, NY, 2 edition, 2006.

[4] D. J. Daley and D. Vere-Jones. *General Theory and Structure*, volume II of *An Introduction to the Theory of Point Processes*. Springer New York, NY, 2 edition, 2007.

[5] A. Erdélyi and F. G. Tricomi. The asymptotic expansion of a ratio of gamma functions. *Pacific Journal of Mathematics*, 1(1):133 – 142, 1951.

[6] Robert E. Fernholz. *Stochastic Portfolio Theory*. Stochastic Modelling and Applied Probability. Springer New York, NY, 1 edition, 2002.

[7] John L. Kelley. *General Topology*. Graduate Texts in Mathematics. Springer New York, NY, 1 edition, 1975.

[8] Gunter Last and Mathew Penrose. Lectures on the poisson process, August 2017.

[9] Filip Lindskog, Sidney I. Resnick, and Joyjit Roy. Regularly varying measures on metric spaces: Hidden regular variation and hidden jumps. *Probability Surveys*, 11(none):270 – 314, 2014.

[10] Sidney I. Resnick. *Heavy-Tail Phenomena Probabilistic and Statistical Modeling.* Springer, 2007.

[11] Terence Tao. 245b, notes 12: Continuous functions on locally compact hausdorff spaces, 2009. [Online; accessed 2 April 2023].

[12] Tomasz Tkocz. Probability (graduate class) lecture notes, 2020.

APPENDICES

Appendix A

Supplementary Theorems and Lemmas

A.1 A Collection of Various Topological Results

Lemma A.1.1. *A compact set K in a Hausdorff space is measurable (with respect to its borel sigma algebra).*

Proof. By Hausdorff property, for every $x \neq y$ in X, there exists open set $U_{x,y}$ that contains x but not y. There also exists open $V_{x,y} \in \mathcal{V}$ with $V_{x,y} \subseteq U_{x,y}$. Then

$$K = \bigcap_{y \in K^c} \bigcup_{x \in K} V_{x,y}$$

Thus by compactness, for each fixed $y \in K^c$ we can reduce the inner union to a finite union, that is,

$$K = \bigcap_{y \in K^c} \bigcup_{n=1}^{n_y} V_n$$

For $n_y \in \mathbb{N}$ and $\{V_n\}_{1,\dots,n_y}$ some finite subseteq of $\{V_{x,y}\}_{x \in K}$. But V_n is countable, so K is expressed as an intersection of union of sets from a countable family, so $K \in \sigma(X)$. $\square$

Lemma A.1.2. *Let X be an locally compact Hausdorff space. Then there exists a basis $\mathcal{V}$ of open sets with compact closure.*

Proof. Take a basis $\mathcal{U}$ of τ_X. Let

$$\mathcal{V} = \left\{U \in \mathcal{U} : \overline{U} \text{ is compact}\right\}$$

We claim that $\mathcal{V}$ is also a basis for τ_X. Take any open set W. For every $x \in U$, by local compactness there exists K_x compact and W_x open such that $x \in W_x \subseteq K_x$. Since $\mathcal{U}$ is a basis, there exists $U_x \in \mathcal{U}$ such that $x \in U_x \subseteq W_x$. Since X is Hausdorff, $\overline{U_x} \subseteq K_x$ is compact. Thus $U_x \in \mathcal{V}$. Now we can write

$$W = \bigcup_{x \in W} U_x$$

$\square$

Definition A.1. *A topological space X is regular if and only if for every closed set $F \subseteq X$ and point $x \in F^c$, there exists disjoint neighborhoods U and V such that*

$$F \subseteq U, \;\; x \in V$$

Definition A.2. *A topological space X is normal if and only if for any two disjoint closed sets $F_1, F_2 \subseteq X$ there exists disjoint neighborhoods U and V such that*

$$F_1 \subseteq U, \;\; C_2 \subseteq V$$

Lemma A.1.3. *Let X be a second-countable topological space. If X is regular (Definition A.1), then X is normal (Definition A.2)*

Proof. Let $\mathcal{V}$ be a countable basis for X and let F_1, F_2 be disjoint closed sets in X.

By regularity, for every $x \in F_1$ there exists neighborhoods U_x of x and U'_x of F_1 disjoint. Then by definition of closure, we have

$$x \in U_x \subseteq \overline{U_x} \subseteq (U'_x)^c \subseteq F_2^c$$

Correspondingly there exists $U^*_x \in \mathcal{V}$ such that $x \in U^*_x \subseteq U_x$, which gives us $x \in U^*_x \subseteq \overline{U^*_x} \subseteq F_2^c$. Since $\mathcal{V}$ is countable, this reduces to some countable sequence of open sets $\{U_m\}_m$ with

$$F_1 \subseteq \bigcup_{m \in \mathbb{N}} U_m, \;\; \overline{U_m} \subseteq F_2^c \; \forall m \in \mathbb{N}$$

Similarly, there exists countable sequence $\{V_n\}_n$ of open sets with

$$F_2 \subseteq \bigcup_{n \in \mathbb{N}} V_n, \;\; \overline{V_n} \subseteq F_1^c \; \forall n \in \mathbb{N}$$

Now take

$$U = \bigcup_{i=1}^{\infty} \left(U_i \setminus \bigcup_{k=1}^{i} \overline{V_k} \right)$$

and

$$V = \bigcup_{j=1}^{\infty} \left(V_j \setminus \bigcup_{k=1}^{j} \overline{U_k} \right)$$

Then clearly $F_1 \subseteq U$ and $F_2 \subseteq V$. Now suppose there exists $x \in U \cap V$. Then for some i, j,

$$x \in U_i \setminus \bigcup_{k=1}^{i} \overline{V_k}$$

and

$$x \in V_j \setminus \bigcup_{k=1}^{j} \overline{U_k}$$

WLOG assume $i \leq j$, then $x \in U_i$ and $x \notin \overline{U}_i$ at the same time, contradiction, so we are done. ☐

Lemma A.1.4. *Let X be a compact Hausdorff space. That is, X is Hausdorff, and every open cover of X has a finite subsocver.*

Then, X is regular.

Proof. Let F be a closed set in X and $x \in X \setminus F$. Then F is compact, and for every $y \in F$ we have open sets U_y, V_y with $y \in U_y$, $x \in V_y$ and $U_y \cap V_y = \emptyset$. Now

$$F \subseteq \bigcup_{y \in F} U_y$$

Thus it can be reduced to a finite subcover. Let $y_1, ..., y_k$ be the corresponding $y \in F$ that index this finite subcover. Then

$$F \subseteq \bigcup_{i=1}^{k} U_{y_i}$$

and

$$x \in \bigcap_{i=1}^{k} V_{y_i}$$

But the RHS of the above are disjoint open sets, so we are done. ☐

Theorem A.1.5. *If X is a SLCH space, then it is normal.*

Proof. We show that a locally compact Hausdorff space is regular. Since X is second-countable, then we are done by Lemma A.1.3.

Fix some $F \subseteq X$ closed and $x \in F^c$. By local compactness there exists U, K such that $x \in U \subseteq K$. Consider K as a compact Hausdorff space in its own right, under the subspace topology. Then, by Lemma A.1.4, K is regular as its own topological space.

By taking the open set $U \cap F^c$ instead if necessary, we can assume that U is disjoint from F. It is also an open set in this subspace. Thus the set

$$K \setminus U$$

is closed in K, and thus there exists open sets V_1, V_2 with $V_1 \cap V_2 \cap K = \emptyset$ such that

$$x \in V_1 \cap K$$

and

$$K \setminus U \subseteq V_2 \cap K$$

We claim that $V_1 \cap U$ and $V_2 \cup K^c$ are the desired open sets. Indeed, firstly we have

$$x \in V_1 \cap K, \quad x \in U \implies x \in V_1 \cap K \cap U = V_1 \cap U \qquad (U \subseteq K)$$

On the other hand, we have

$$F \subseteq U^c = (K \setminus U) \cup (K^c \setminus U) \subseteq (V_2 \cap K) \cup K^c \subseteq V_2 \cup K^c$$

Finally, they are disjoint since $V_1 \cap U$ and V_2 are disjoint and $V_1 \cap U$ and K^c are also disjoint. □

Theorem A.1.6 (Urysohn)**.** *Let X be a topological space. The following are equivalent:*

(i) *X is normal.*

(ii) *For every closed set $K \subseteq X$ and open neighborhood U of K, there exists V open and F closed such that*

$$K \subseteq V \subseteq F \subseteq U$$

(iii) *For disjoint sets $F_1, F_2 \subseteq X$, there exists a continuous function $f : X \to [0,1]$ that is 1 on F_1 and 0 on F_2.*

(iv) *For every closed set $K \subseteq X$ and open neighborhood U of K, there exists continuous function $f : X \to [0,1]$ such that*

$$\mathbb{1}_K \leq f \leq \mathbb{1}_U$$

Proof. We refer to remarks in Dr. Terence Tao's blog post [11] and *General Topology* [7] □

Let P, L be families of subsets of X. P is known as a π-sytem if it is closed under finite intersections.

L is known as a λ-system if

- $\emptyset \in L$.
- $A \in L \implies X \setminus A \in L$.

- If $A_1, ..., A_k$ pairwise disjoint are in L then $\bigsqcup_{i=1}^{k} A_i \in L$.

Theorem A.1.7 (Dynkin's Theorem)**.** *Suppose P is a π-system on X and L is a λ-system on X, such that $P \subseteq L$. Then*

$$\sigma(P) \subseteq L$$

Proof. As proven in Measure & Integration (21-720). □

Corollary A.1.8. *Let $\mathcal{X}$ be a σ-algebra on some state space X. Suppose $P \subseteq \mathcal{X}$ is a π-system on X that generates $\mathcal{X}$, that is, $\sigma(P) \supseteq \mathcal{X}$, then $P = \mathcal{X}$. Indeed, a σ-algebra is an λ-system, so by the above we have*

$$\mathcal{X} \subseteq \sigma(P) \subseteq \sigma(\mathcal{X}) = \mathcal{X}$$

Lemma A.1.9. *A set S in a complete metric space (X, d) is compact if and only if it is sequentially compact. That is, for every sequence $\{x_n\}_n \in S$, there exists a subsequence $\{x_{n_k}\}_k$ that converges in S.*

Proof. As proven in Math Studies Analysis I (21-235). □

A.2 A Collection of Various Probability Results

Theorem A.2.1 (Kolmogorov 0-1 Theorem). *Given sequence of random variables $\{X_n\}_n$, its tail σ-algebra is defined by*

$$\mathcal{F} = \bigcap_{n=1}^{\infty} \sigma(X_{n+1}, X_{n+2}, \ldots)$$

Then, we must have $\mathbb{P}(A) = 0$ or $\mathbb{P}(A) = 1$ for every $A \in \mathcal{F}$.

Proof. As proven in Probability (21-721) [12]. $\square$

Theorem A.2.2 (Kolmogorov Three-Series Test). *Let $\{X_n\}_n$ be independent random variables. The sum*

$$\sum_{n=1}^{\infty} X_n$$

converges in $\mathbb{R}$ almost surely if there exists some $C > 0$ for which

a) $\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > C) < \infty$

b) $\sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{1}_{|X_n| \leq C}]$ *converges in* $\mathbb{R}$

c) $\sum_{n=1}^{\infty} \mathit{Var}[X_n \mathbb{1}_{|X_n| \leq C}] < \infty$

Conversely, if the sum converges a.s. then these conditions hold for all $C > 0$.

Proof. As proven in Probability (21-721) [12]. $\square$