Indian Institute of Science
Department of Electrical Communication Engineering

E1 244: Detection and Estimation

February-May 2021

Homework 3 (deadline 28 May 23.59)

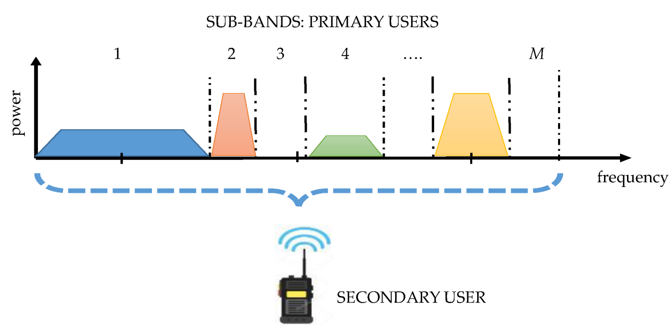
This is a hard deadline. Reports received after the deadline will not be evaluated or counted for the final grade.

Any extension request to the deadline needs to be approved. We will do that on a case-by-case basis. This will also be the deadline for those who have not yet submitted Homework 1 and 2, but would like them to be evaluated for the final grade.

This homework consists of two parts: (a) Developing an energy detector and a detector based on cyclostationarity for an OFDM based cognitive radio system and (b) implementing and evaluating the performance of these detectors. Make a short report using LaTeX containing the required explanations, answers, plots, and Matlab/Python scripts, and turn it in by the deadline using Microsoft Teams. Only PDF files will be evaluated.

Spectrum sensing for cognitive radio

The concept of cognitive radio is to exploit the underutilized spectral resources by reusing unused spectrum in an opportunistic manner. A cognitive radio system generally involves primary users of the spectrum, who are incumbent licensees, and secondary users who seek to opportunistically use the spectrum when the primary users are idle. The cognitive radios must sense the spectrum to detect whether it is available or not.



Primary user signal model

The primary user signal is an orthogonal frequency division multiplexing (OFDM) signal. OFDM is used by many of the current and also future wireless communications systems. A discrete time OFDM symbol block is first constructed by applying the inverse Fast Fourier Transform (IFFT) to N_d data symbols, and the output of the IFFT is given by

$$x[n] = \frac{1}{\sqrt{N_d}} \sum_{k=0}^{N_d-1} s[k] e^{j2\pi n \frac{k}{N_d}}, n = 0, 1, \ldots, N_d - 1$$

where N_d is the number of data symbols in one OFDM symbol block, and the $s[k]$ denote the data symbols. The last N_c symbols of $x[n]$ are copied and appended to the beginning of it as a cyclic prefix (CP), which can be used to deal with inter-symbol interference (ISI) at the receiver side. As a result, the length of the transmitted OFDM symbol $x[n]$ is extended to $N_d + N_c$. Let us denote one OFDM symbol block using $\mathbf{x} = [x[0], x[1], \ldots, x[N_d + N_c - 1]]^T$ with a cyclic-prefix (CP) of length N_c. Suppose we transmit $K+1$ such OFDM symbol blocks $\mathbf{x} = [\mathbf{x}_0^T, \mathbf{x}_1^T, \ldots, \mathbf{x}_K^T]^T$ over an AWGN channel, resulting into the output $\mathbf{y} = \mathbf{x} + \mathbf{w}$.

The transmitted data symbols $\mathbf{x}$ are independent and identically distributed (i.i.d.) with zero-mean and unit variance. The autocorrelation function (ACF) of $\mathbf{x}$ is given by

$$r_x[n, \tau] = E[x[n]x^*[n + \tau]]. \tag{1}$$

We can define $r_y[n, \tau]$ and $r_w[n, \tau]$ in a similar way. Due to the insertion of the CP, the OFDM signal is non-stationary. Thus the ACF $r_x[n, \tau]$ is time-varying but periodic ($r_x[n, \tau] = r_x[n + k(N_d + N_c), \tau]$), and the OFDM signal is second-order cyclostationary. Particularly in one period, $r_x[n, N_d], n = 0, 1, \ldots, N_d + N_c - 1$ can be described as

$$r_x[n, N_d] = \begin{cases} 1, & n = 0, 1, \ldots, N_c - 1 \\ 0, & n = N_c, N_c + 1, \ldots, N_c + N_d - 1 \end{cases} \tag{2}$$

Since $\mathbf{w}$ is white and zero-mean, $r_w[n, \tau] = 0$ for any $\tau \neq 0$ and $r_w[n, 0] = \sigma_w^2$.

A secondary user must sense the spectrum to detect whether the OFDM signals of the primary user are present or not. If the primary user is not active, the secondary user can take the chance to make use of the available spectrum. In the presence of a primary user, the received OFDM signal which is corrupted by an additive white Gaussian noise (AWGN) channel at the secondary user can simply be modeled as $y[n] = x[n] + w[n]$, where $w[n]$ stands for the AWGN noise. Hence, we wish to decide between the following two hypotheses:

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{y} = \mathbf{w} \\ \mathcal{H}_1 &: \mathbf{y} = \mathbf{x} + \mathbf{w} \end{aligned}$$

where $\mathbf{w} \sim \mathcal{C}N(\mathbf{0}, \sigma_n^2 \mathbf{I})$, and $\mathbf{w}$ is independent of $\mathbf{x}$.

Part A: Derivation

1. Consider a data symbol sequence $\mathbf{s} = [s[0], s[1], \ldots, s[N_d - 1]]^T$ of length N_d, where the entries are QPSK with variance $\sigma_s^2 = 1$, i.e., $s[k] \in \{\pm 1/\sqrt{2} \pm j/\sqrt{2}\}$. According to the central limit theorem and assuming a sufficiently large IFFT, the OFDM symbol blocks $\mathbf{x}$ can be assumed to be zero-mean Gaussian with covariance matrix $E[\mathbf{x}\mathbf{x}^H] = \sigma_s^2 \mathbf{I}_{(K+1)(N_c+N_d)}$. Assume that $\mathbf{w}$ is a zero-mean complex Gaussian random process, where $E[\mathbf{w}\mathbf{w}^H] = \sigma_w^2 \mathbf{I}_{(K+1)(N_d+N_c)}$. Derive a Neyman-Person detector for this problem. Given a fixed P_{FA}, what is the theoretical P_D?

2. The detector in the previous part does not exploit the special structure of the OFDM signal with a CP. In this question, we derive a detector based on the cyclostationary ACF of the OFDM signal with a CP. We design a new test statistic as

$$T(\mathbf{y}) = \sum_{n=0}^{N_c-1} \hat{R}[n]$$

and decide $\mathcal{H}_1$ if

$$|T(\mathbf{y})| > \gamma,$$

where the ACF $r_y[n, N_d]$ as

$$\hat{R}[n] = \frac{1}{K} \sum_{k=0}^{K-1} \hat{r}[n + k(N_c + N_d), N_d], n = 0, 1, \ldots, N_c + N_d - 1$$

with $\hat{r}[n, N_d] = y[n]y^*[n + N_d]$. Note that $y[n], n = 0, 1, \ldots, K(N_c + N_d) + N_d - 1$ are used to generate $\hat{R}[n], n = 0, 1, \ldots, N_c + N_d - 1$. According to the central limit theorem and assuming a sufficiently large IFFT, $T(\mathbf{y})$ can be assumed to be a complex Gaussian random variable.

a. Show that the mean and variance of $T(\mathbf{y}) = \bar{T}(\mathbf{y}) + j\widetilde{T}(\mathbf{y})$ under $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, are

$$\begin{aligned}
\mathcal{H}_0 \quad &: \quad E(\bar{T}) = 0 \\
& \quad\;\; \mathrm{var}(\bar{T}) = \frac{N_c}{2K}\sigma_w^4 \\
& \quad\;\; E(\widetilde{T}) = 0 \\
& \quad\;\; \mathrm{var}(\widetilde{T}) = \frac{N_c}{2K}\sigma_w^4 \\
& \quad\;\; \mathrm{cov}(\bar{T}\widetilde{T}) = 0 \\
\mathcal{H}_1 \quad &: \quad E(\bar{T}) = N_c \\
& \quad\;\; \mathrm{var}(\bar{T}) = \frac{N_c}{K}(1 + \sigma_w^2 + \sigma_w^4/2) \\
& \quad\;\; E(\widetilde{T}) = 0 \\
& \quad\;\; \mathrm{var}(\widetilde{T}) = \frac{N_c}{K}(\sigma_w^2 + \sigma_w^4/2) \\
& \quad\;\; \mathrm{cov}(\bar{T}\widetilde{T}) = 0
\end{aligned} \tag{3}$$

b. Derive the Neyman-Pearson detector. Given a fixed P_{FA}, find out the threshold γ. *Hint:* Use the fact that the decision rule $|T(\mathbf{y})| > \gamma$ is equivalent to $|T(\mathbf{y})|^2 > \gamma^2$.

Part B: Implementation

1. Energy detector: Generate 1000 test statistics under $\mathcal{H}_1$ for $N_d = 32$, $N_c = 8$, $K = 50$, and a fixed signal-to-noise ratio (SNR). Recall that the data symbol sequence $\mathbf{s} = [s[0], s[1], \ldots, s[N_d - 1]]^T$ of length N_d, contains QPSK entries with variance $\sigma_s^2 = 1$, i.e., $s[k] \in \{\pm 1/\sqrt{2} \pm j/\sqrt{2}\}$. The variance σ_w^2 should be chosen according to the SNR, which is defined as the ratio of the signal energy over the noise energy expressed in dB, i.e., $\text{SNR} = 10 \log(\sigma_s^2/\sigma_w^2)$, where $\sigma_s^2 = 1$.

   a. Compare the test statistics with the threshold ($P_{FA} = 0.05$) and count the number of detections to calculate P_D. Assume perfect knowledge of σ_w^2, and let the SNR vary in the range of $[-20$ dB, 4 dB$]$ with a 2 dB step. Plot the numerical P_D and the theoretical P_D vs. SNR. In the same plot, count the number of false alarms to calculate P_{FA} and verify if it is always 0.05.

   b. How will the energy detector behave, if we use an inaccurate $\widetilde{\sigma}_w^2$ instead of the true σ_w^2? Let us assume that $\widetilde{\sigma}_w^2$ is 1 dB larger than σ_w^2. Plot P_D vs. SNR for inaccurate $\widetilde{\sigma}_w^2$ and compare it with the previous results.

   c. Suppose we assume that the primary user is mostly inactive with $P(\mathcal{H}_1) = 0.2$, illustrate and comment on how the threshold is modified compared to the one obtained with $P_{FA} = 0.05$.

2. Cyclostationarity detector: As before, generate 1000 $T(\mathbf{y})$ under $\mathcal{H}_1$ for $N_d = 32$, $N_c = 8$, $K = 50$ and a fixed SNR.

   a. Generate 1000 $T(\mathbf{y})$ for SNR = 100 dB under the two hypotheses, respectively. Plot the histograms of $T(\mathbf{y})$ to verify whether they are matched with a complex Gaussian distribution having mean and variances that we derived in Part A (Question 2).

   b. For a fixed SNR, compare the test statistics with the threshold ($P_{FA} = 0.05$) and count the number of detections to calculate P_D. Assume perfect knowledge of σ_w^2, and let the SNR vary in the range of $[-20$ dB, 4 dB$]$ with a 2 dB step. Plot the numerical P_D vs. SNR, and compare it with the P_D of the energy detector from Question 1. In the same plot, count the number of false alarms to calculate P_{FA} and verify if it is always 0.05.

   c. How will this detector behave, if we use an inaccurate $\widetilde{\sigma}_w^2$ instead of the true σ_w^2? Let us assume that $\widetilde{\sigma}_w^2$ is 1 dB larger than σ_w^2. Plot P_D vs. SNR for inaccurate $\widetilde{\sigma}_w^2$ and compare it with the previous results.