Indian Institute of Science
Department of Electrical Communication Engineering

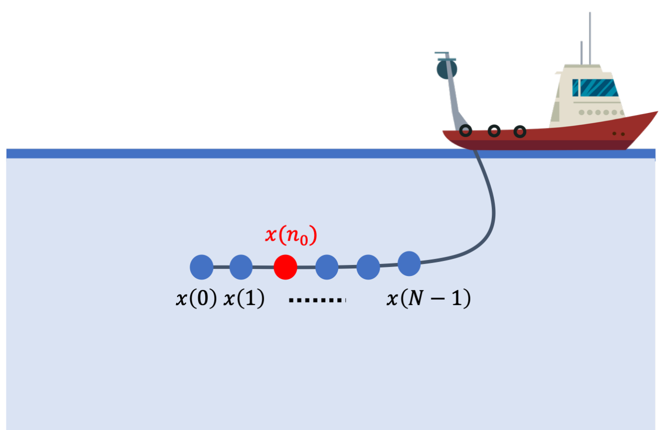
E1 244: Detection and Estimation

February-May 2021

Homework 2 (deadline 10 May 23:59 hrs)

This homework consists of two parts: (a) Developing a Wiener (LMMSE) filter and Kalman filter for interpolation of a faulty sensor in a uniform sensor array and (b) implementing and evaluating the performance of the estimators that you have derived. Make a short report using LaTeX containing the required explanations, answers, plots, and Matlab/Python scripts, and turn it in by the deadline using Microsoft Teams. Only PDF files will be evaluated.

Interpolation of a faulty sensor/data sample



We are interested in a real-valued signal $x(n)$. Assume that $x(n)$ is a zero-mean real-valued wide-sense stationary (WSS) random process with autocorrelation sequence $r_x(k)$. More specifically, the signal $x(n)$ is AR(1) [i.e., Autoregressive process of order 1] that is generated by the difference equation

$$x(n) = \alpha x(n-1) + w(n) \tag{1}$$

where $w(n)$ is white noise with variance σ_w^2. We have access to the data set $\mathbf{x} = [x(0), x(1), \cdots, x(n_0-1), x(n_0+1), \cdots, x(N-1)]^T$ or its noisy version. We wish to estimate or interpolate $x(n_0)$ as one of the sensors in the array is faulty (indicated in red color).

Note that for a real-valued zero-mean wide-sense stationary (WSS) random process $x(n)$, we have $E\{x(k)x(l)\} = r_x(k-l)$.

Part A: Derivation and modeling

1. Let the LMMSE (or the Wiener) interpolator of $x(n_0)$ be

$$\hat{x}(n_0) = \sum_{\substack{i=0 \\ i \neq n_0}}^{N-1} a_i x(i).$$

   Find the coefficients $\{a_i\}$ that minimize the Bayesian mean-square error by solving a linear system, which is referred to as the Wiener-Hopf equations. Also derive the expression for the minimum mean-square error.

2. Instead of using all the data samples, suppose we wish to interpolate $x(n_0)$ using only $x(n_0 - 1)$ and $x(n_0 + 1)$ as

$$\hat{x}(n_0) = a_1 x(n_0 - 1) + a_2 x(n_0 + 1).$$

   Derive an explicit expression for a_1 and a_2 in terms of α. Also, what will be the expression for the minimum mean-square error.

3. We may also use a causal Wiener (i.e., LMMSE) predictor, which uses only the past samples to predict the missing sample as

$$\hat{x}(n_0) = \sum_{i=0}^{n_0-1} a_i x(i).$$

   Find the coefficients $\{a_i\}$ that minimize the Bayesian mean-square error by solving a linear system, and derive the expression for the minimum mean-square error.

4. Suppose we make measurements of the process as

$$y(n) = x(n) + v(n), \quad n = 0, 1, \ldots, n_0 - 1,$$

   where $v(n)$ is white noise having zero mean and variance σ_v^2. Derive the Kalman filter to estimate $x(n_0)$ using observations $y(n)$ up to n_0.

Part B: Implementation

1. Make a subroutine to implement the Wiener interpolator that you derived in Part A.Q1

```
function x_1(n_0) = wiener_interpolator1(x, n_0, alpha).
```

2. Modify the above subroutine to implement the Wiener interpolator in Part A.Q2

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function x_2(n_0) = wiener_interpolator2(x, n_0 alpha).
```

3. Make a subroutine to implement the Wiener predictor that you derived in Part A.Q3

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function x_3(n_0) = wiener_predictor(x, n_0, alpha).
```

4. For the above estimators, use N=100, $n_0 = 40$, and $\sigma_w^2 = 0.36$. Plot the theoretical mean-squared error and the Bayesian mean squared error, $(x(n_0) - \hat{x}(n_0))^2$, averaged over 1000 realizations for different values of $\alpha \in \{0.1, 0.2, 0.3, \cdots, 0.9\}$. Comment on your observations.

5. Make a subroutine to implement the Kalman filter that you derived in part A

```
function [x(n|n),P(n|n),P(n|n-1)]
= kf(y(n), sigma_v, sigma_w, x(n-1|n-1), P(n-1|n-1)).
```

   Use $n_0 = 40$, $\sigma_v^2 = 1$, $\sigma_w^2 = 0.36$, and $\alpha = 0.8$. Plot the true and estimated signal $x(n)$ up to n_0. Also, plot the true and predicted error for different n. Comment on your observations.

6. Compare the prediction error of the Kalman filter to the prediction error that would get with a causal Wiener (i.e., LMMSE) predictor, which uses only the past samples but not the measurements as

$$\hat{x}(n_0) = \sum_{i=0}^{n_0-1} a_i x(i).$$

   As before, use $\sigma_w^2 = 0.36$, and $\alpha = 0.8$ to generate the process. Also, compare the results for different values of n_0, e.g., $n_0 = \{10, 40\}$.