Neuromorphic Sampling

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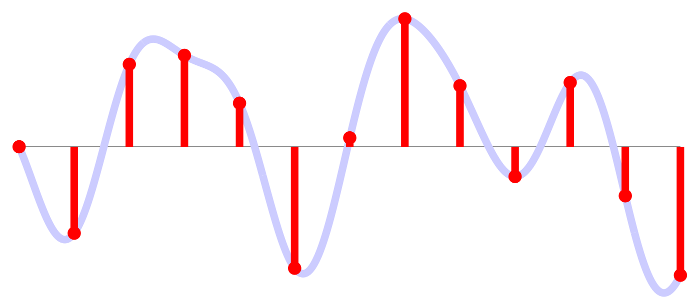
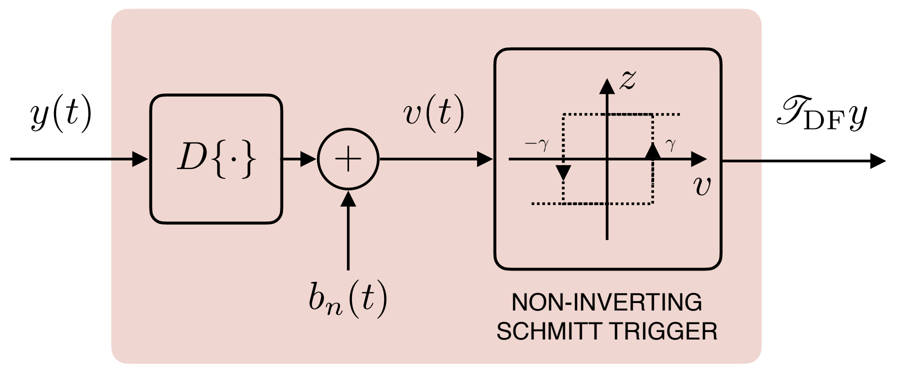
SPECTRUM LAB

Overview

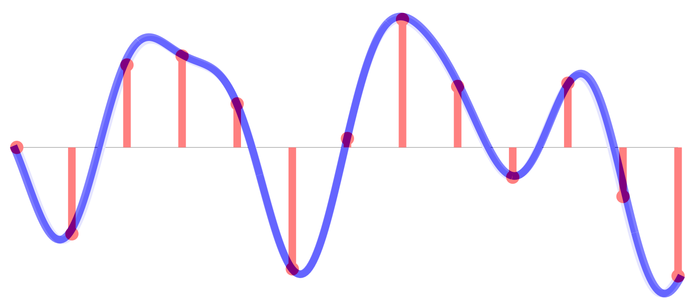
1 Introduction



2 Time-Encoding Machines



3 Signal Reconstruction



4 Examples

$$\mathbf{y} = \mathbf{M}\mathbf{c}$$

1 Introduction

Sampling and Reconstruction

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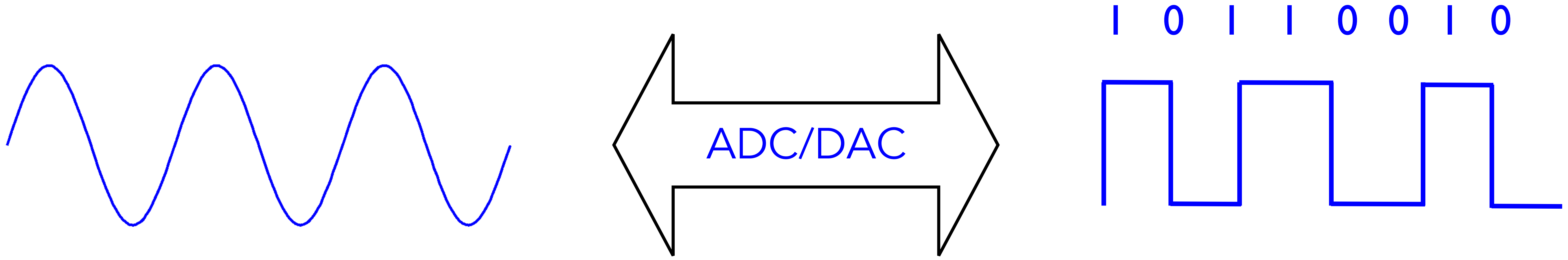

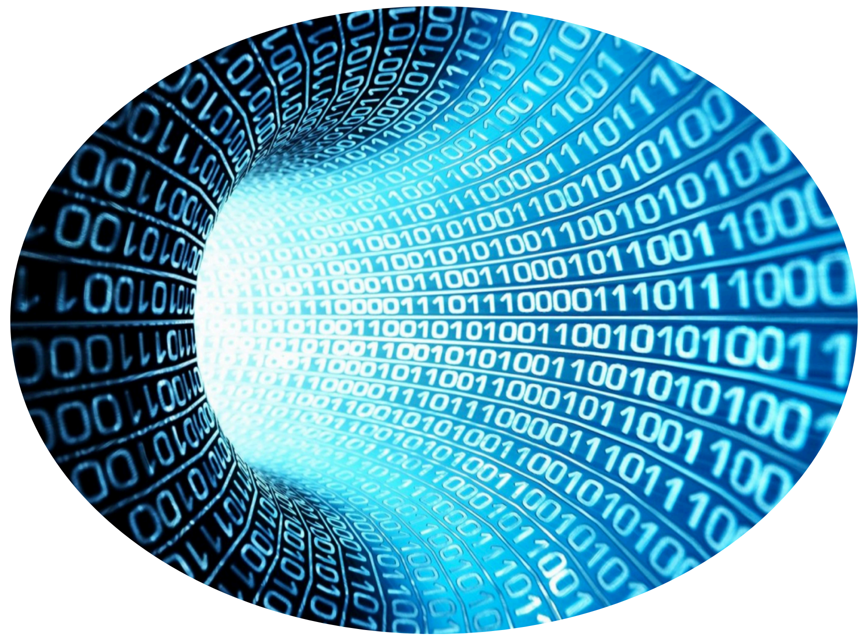

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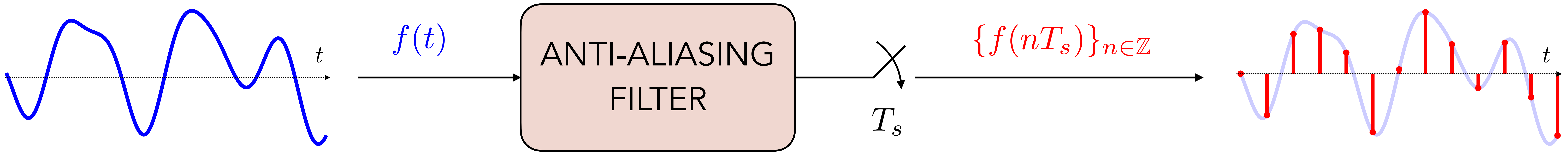
Uniform Sampling



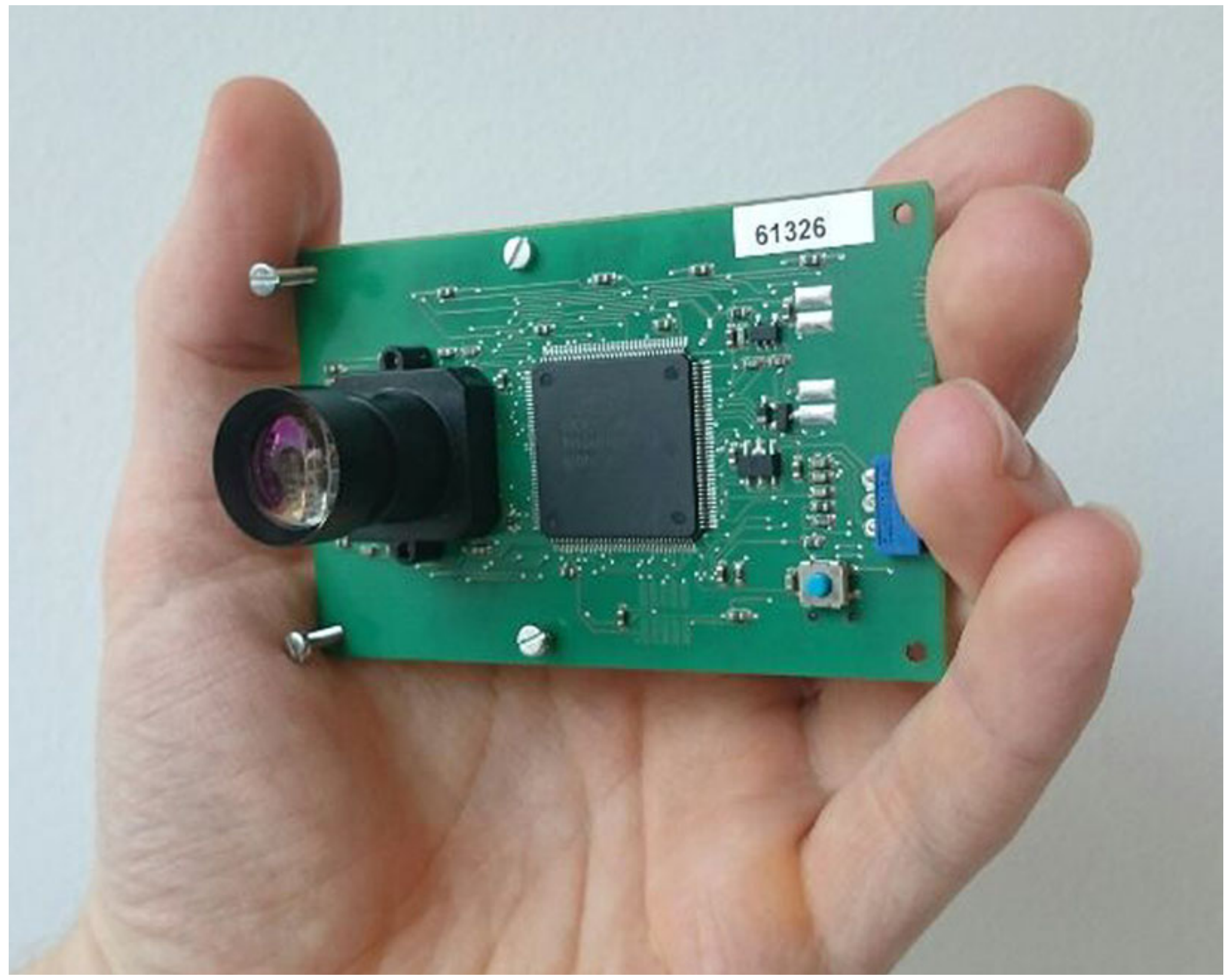
- Finite-energy bandlimited functions: $f \in (L^2 \cap B_\Omega)(\mathbb{R})$

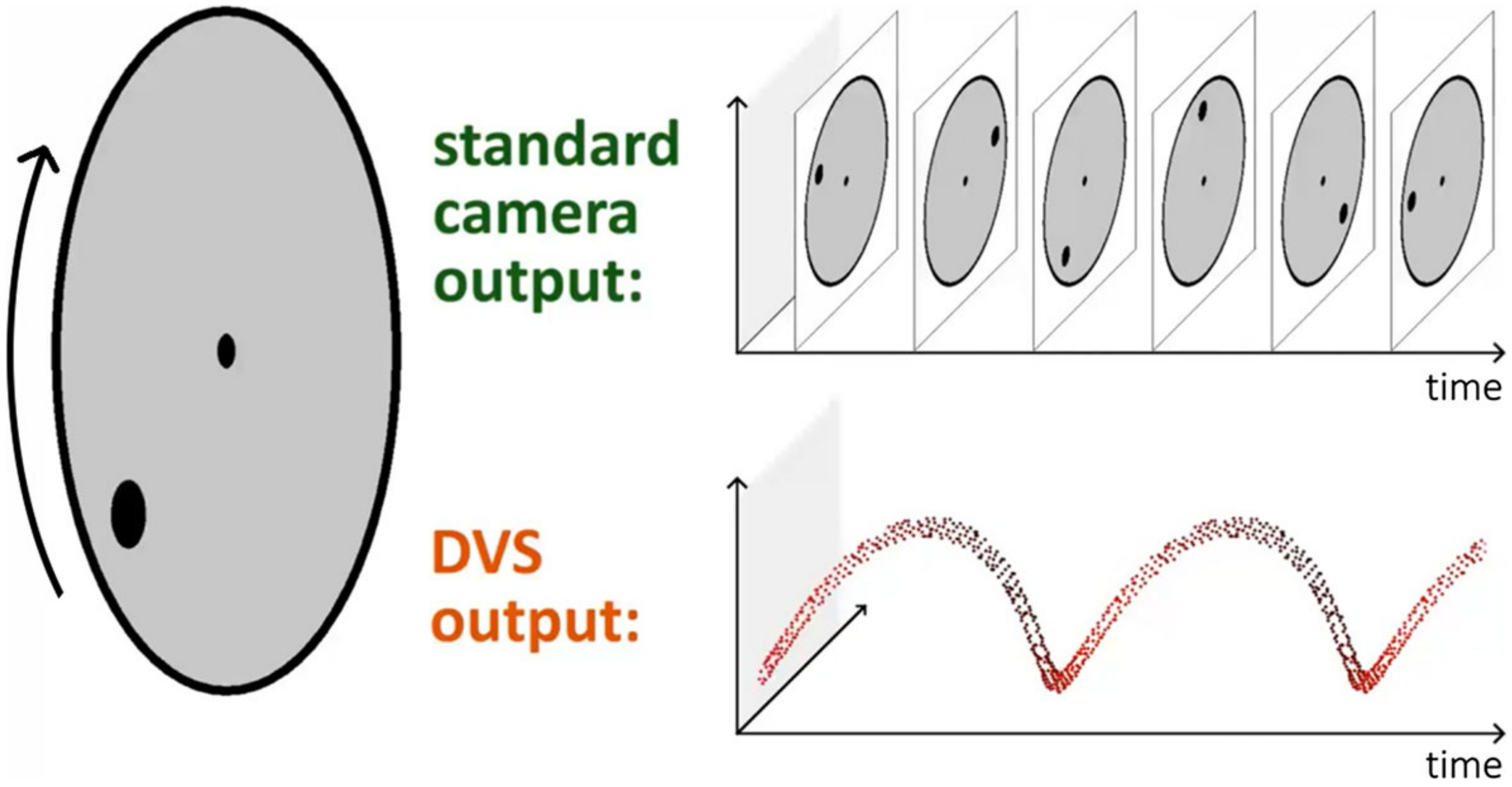
$$\boxed{f(t) = \sum_{n\in\mathbb{Z}} f(nT_s)\,\text{sinc}\left(\frac{t-nT_s}{T_s}\right)}$$

Shannon's Sampling Theorem

- The samples $\{f(nT_s)\}_{n\in\mathbb{Z}}$ completely specify the signal.

Dynamic Vision Sensors





Rebecq *et al.* 2017

2 Time-Encoding Machines

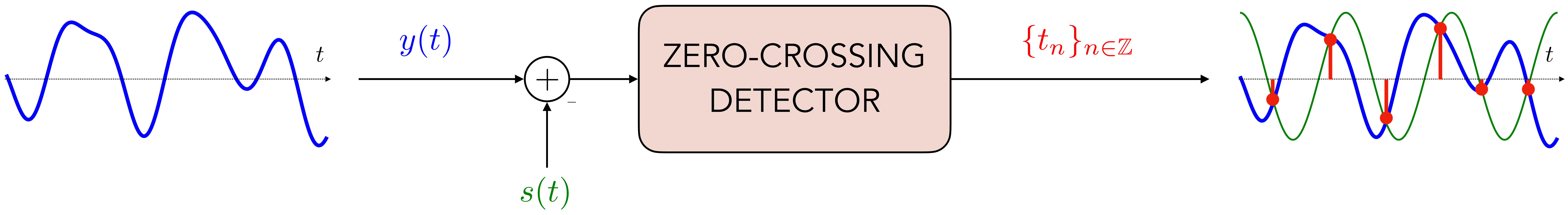
Time-Encoding Machine

Definition 1: Time-Encoding Machines

A time-encoding machine with an event operator $\mathscr{E} : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{R}}$ and references $\{r_n \in \mathbb{R}^{\mathbb{R}}\}_{n\in\mathbb{Z}}$ is a map $\mathscr{T} : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{Z}}$ such that $\mathbb{R}^{\mathbb{R}} \ni y \mapsto \mathscr{T}y$, with

- $\mathscr{T}y = \{t_i \in \mathbb{R} \mid t_i > t_j,\ \forall i > j, i \in \mathbb{Z}\}$,
- $\lim\limits_{n\to\pm\infty} t_n = \pm\infty$, and
- $(\mathscr{E}y)(t_n) = r_n(t_n), \forall t_n \in \mathscr{T}y$.

Crossing-Time-Encoding Machine [Gontier, '14]



D. Gontier and M. Vetterli, "Sampling based on timing: Time encoding machines on shift-invariant subspaces," *Appl. Comput. Harmon. Anal.*, 2014.

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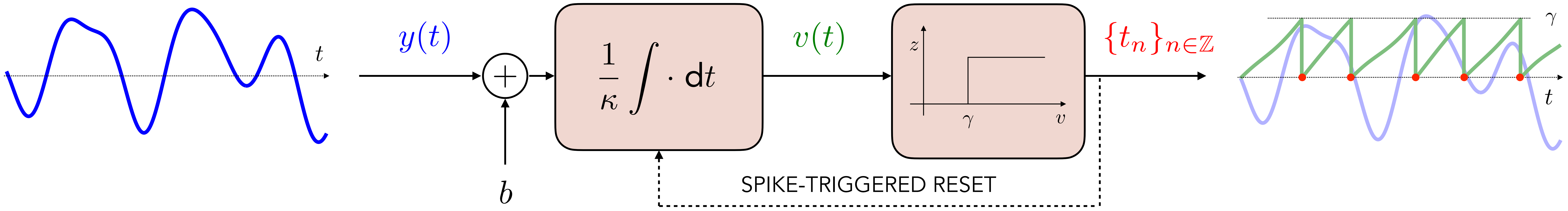
Integrate-and-Fire Time-Encoding Machine [Lazar, '04]



A.A. Lazar, "Time encoding with an integrate-and-fire neuron with a refractory period," *Neurocomput.*, 2004.

Time-Based Sampling

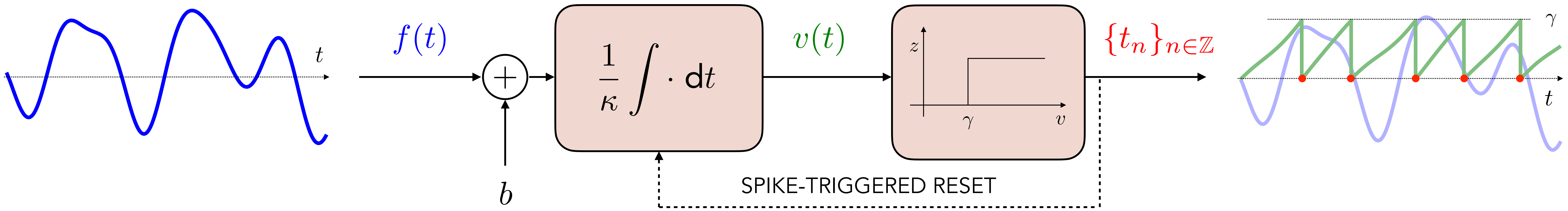
Advantages

- Time-encoding mimics representation of sensory signals.
- Time-encoding is asynchronous $\rightsquigarrow$ low power.
- Nonuniform sampling $\rightsquigarrow$ sparse measurements.
- Event-driven sampling $\rightsquigarrow$ no redundancy.

Disadvantages

- Sophisticated sampling devices.
- Digital processing of continuous-domain signal is not possible.
- Iterative reconstruction techniques.

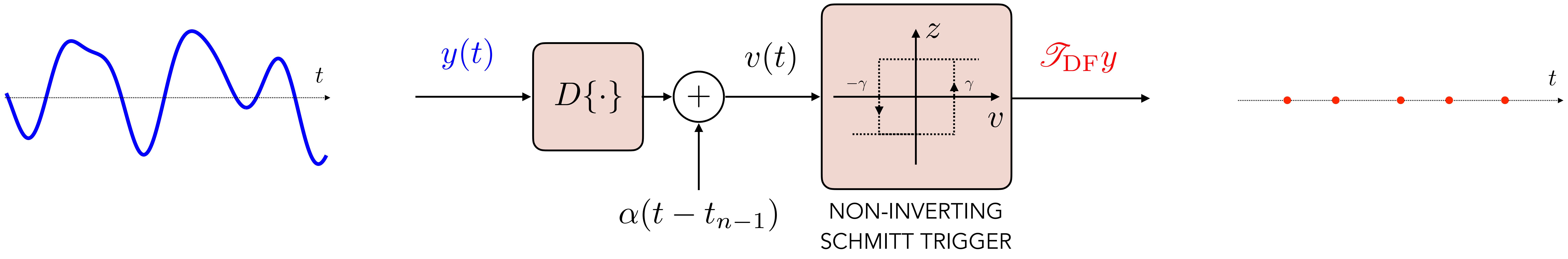
The Integrate-and-Fire Time-Encoding Machine



- The signal is encoded using a set of strictly increasing time-instants that satisfy
  - (Stability) $\dfrac{\kappa\gamma}{b+\|f\|_\infty} < t_{n+1} - t_n < \dfrac{\kappa\gamma}{b-\|f\|_\infty}$,
  - (t-transform) $\displaystyle\int_{t_n}^{t_{n+1}} f(t)\mathrm{d}t = -b(t_{n+1}-t_n) + \kappa\gamma$.
- Are $\{t_n\}_{n\in\mathbb{Z}}$ sufficient for perfect reconstruction of f?

A.A. Lazar, "Time encoding with an integrate-and-fire neuron with a refractory period," *Neurocomput.*, 2004.

Differentiate-and-Fire Time-Encoding Machine

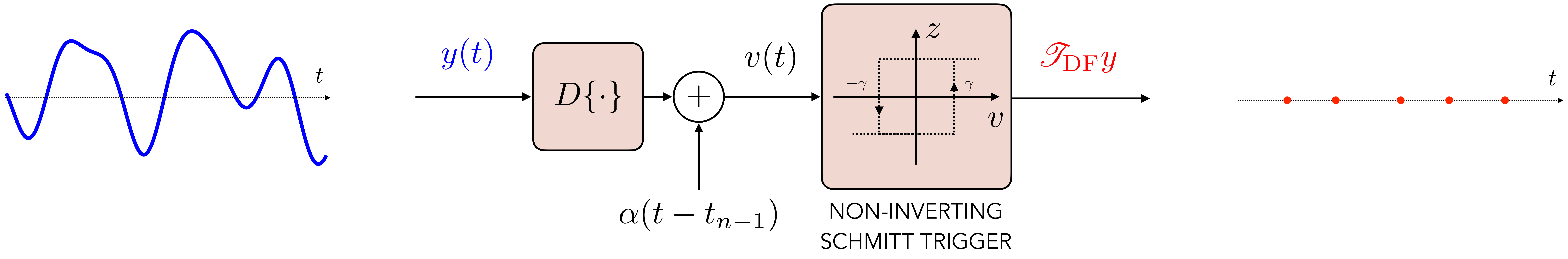


Lemma 1: t-transform

Let $y \in C^1(\mathbb{R})$ be the input to the DF-TEM. The output $\mathscr{T}_{\mathsf{DF}}y = \{t_n\}_{n\in\mathbb{Z}}$ satisfies

$$(Dy)(t_n) = (-1)^{n+1}\left(\gamma - \alpha(t_n - t_{n-1})\right), \ \forall t_n \in \mathscr{T}_{\mathsf{DF}}y.$$

Differentiate-and-Fire Time-Encoding Machine



Corollary 1: Sampling Density of DF-TEM

Let $y \in C^1(\mathbb{R})$ with $\|Dy\|_\infty \leq \beta$ be the input to the DF-TEM. The output $\mathscr{T}_{\mathsf{DF}}y = \{t_n\}_{n\in\mathbb{Z}}$ satisfies

$$d(\mathscr{T}_{\mathsf{DF}}y) \doteq \sup_{n\in\mathbb{Z}} |t_n - t_{n-1}| \leq \frac{\gamma + \beta}{\alpha}.$$

3 Signal Reconstruction

Signals in Shift-Invariant Spaces

- Consider signals in the integer shift-invariant space $V(\varphi)$,

$$y(t) = \sum_{k\in\mathbb{Z}} c_k \varphi(t-k),$$

- $\{\varphi(\cdot - k)\}_{k\in\mathbb{Z}}$ forms a Riesz basis for $V(\varphi)$,
- The coefficient sequence $\tilde{\mathbf{c}} = \{c_k\}_{k\in\mathbb{Z}}$ defines the signal,
- The reconstruction problem: Given $\mathscr{T}_{\mathsf{DF}}y$, find y.

Signals in Shift-Invariant Spaces

- The derivative signal lies in the shift-invariant space $V(D\varphi)$

$$Dy(t) = \sum_{k\in\mathbb{Z}} c_k D\varphi(t-k),$$

- *Key idea:* The DF-TEM provides *crossing time-instants* of the derivative signal.
- Reconstruction of the derivative signal Dy is achieved using the method of alternating projections,
- Reconstruction of the signal y can be achieved using an integrator.

Method of Alternating Projections

- Consider the linear operator $\mathcal{V} : \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{R}}$ defined by

$$\mathcal{V}x(t) = \sum_{n\in\mathbb{Z}} x(t_n)\mathbb{1}_{[s_n,s_{n+1}[}(t),$$

where $s_n = \dfrac{t_{n-1}+t_n}{2}$ and $\mathbb{1}_{[s_n,s_{n+1}[}$ is the characteristic function in the interval $[s_n, s_{n+1}[$.

Lemma 2: Invertibility of $\mathcal{V}$

Let $\varphi, D\varphi, D^2\varphi \in L^2(\mathbb{R})$. Let the operator $\mathcal{V}$ be defined with the set $\{t_n\}_{n\in\mathbb{Z}}$ having increasing entries and bounded density $T = d(\{t_n\}_{n\in\mathbb{Z}}) < \infty$. Then, $\forall y \in V(\varphi)$, we have

$$\|Dy - \mathcal{V}Dy\|^2_{L^2(\mathbb{R})} \leq \underbrace{\left(\frac{T}{\pi} \sup_{\omega\in[0,2\pi[} \frac{G_{D^2\varphi}(\omega)}{G_{D\varphi}(\omega)}\right)^2}_{\eta} \|Dy\|^2_{L^2(\mathbb{R})},$$

where $G_\varphi(\omega) = \left(\sum_{k\in\mathbb{Z}} |\hat{\varphi}(\omega+2\pi k)|^2\right)^{1/2}$.

D. Gontier and M. Vetterli, "Sampling based on timing: Time encoding machines on shift-invariant subspaces," *Appl. Comput. Harmon. Anal.*, 2014.

Method of Alternating Projections

- The operator $\mathcal{V}$ is invertible when $\eta < 1$,
- The inversion can be performed with $\Pi = \Pi_{V(D\varphi)}$, $x_1 = \Pi\mathcal{V}Dy$ and

$$x_{\ell+1} = x_1 + (\mathrm{Id} - \Pi\mathcal{V})x_\ell.$$

- We have $\|Dy - x_k\|^2_{L^2(\mathbb{R})} \leq \eta^k \|Dy\|^2_{L^2(\mathbb{R})} \underset{k\to+\infty}{\longrightarrow} 0.$
- $\|x - \Pi\mathcal{V}x\|_{L^2(\mathbb{R})} = \|\Pi x - \Pi\mathcal{V}x\|_{L^2(\mathbb{R})} = \|x - \mathcal{V}x\|_{L^2(\mathbb{R})}, \forall x \in V(D\varphi)$, and

$$\begin{aligned}
Dy - x_k &= Dy - x_1 - (\mathrm{Id} - \Pi\mathcal{V})x_{k-1}, \\
&= (\mathrm{Id} - \Pi\mathcal{V})Dy - (\mathrm{Id} - \Pi\mathcal{V})x_{k-1}, \\
&= (\mathrm{Id} - \Pi\mathcal{V})(Dy - x_{k-1}), \\
&= (\mathrm{Id} - \Pi\mathcal{V})^k Dy.
\end{aligned}$$

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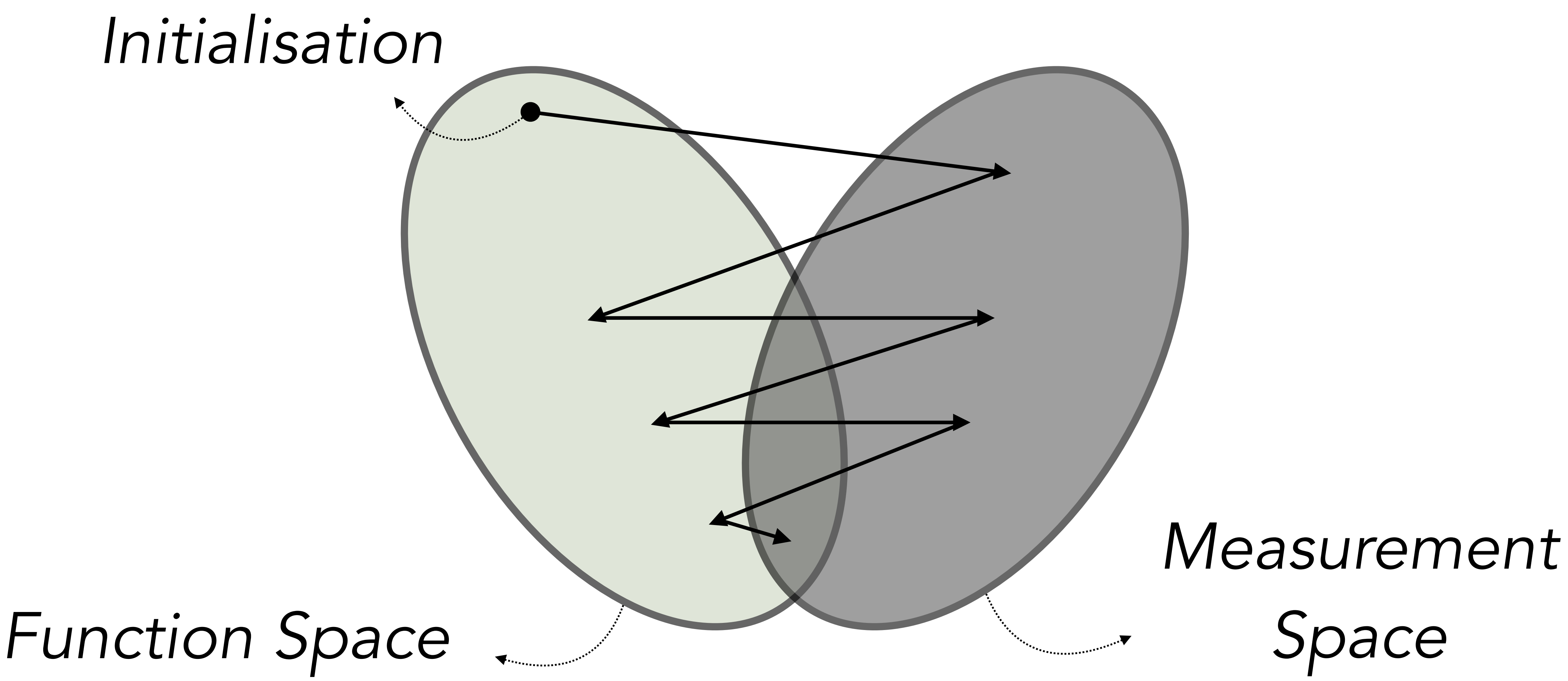
Method of Alternating Projections

- The iterations $x_{\ell+1} = x_1 + (\mathrm{Id} - \Pi\mathcal{V})x_\ell$ converge to Dy,
- y can be obtained using an integrator.



4 Examples

Bandlimited Signals

- Consider bandlimited signals $y \in B([-\pi,\pi]) = \{x \in \mathbb{R}^{\mathbb{R}} \mid \text{supp}(\hat{x}) \subseteq [-\pi,\pi]\}$. The derivative signal $Dy \in B([-\pi,\pi]) \cap V(D\text{sinc})$, i.e.,

$$Dy(t) = \sum_{k\in\mathbb{Z}} y(k)\ D\text{sinc}(t-k) = \sum_{k\in\mathbb{Z}} Dy(k)\ \text{sinc}(t-k).$$

- The samples of the signal and samples of the derivative of the signal satisfy

$$\sum_{k\in\mathbb{Z}} y(k)e^{-\mathrm{j}\omega k} = \sum_{k\in\mathbb{Z}} Dy(k)\frac{1}{\mathrm{j}\omega}e^{-\mathrm{j}\omega k}, \forall\omega \in [-\pi,\pi].$$

- The samples $\{y(k)\}_{k\in\mathbb{Z}}$ can be obtained using the inverse discrete-time Fourier transform as

$$y(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{m\in\mathbb{Z}} Dy(m)\frac{1}{\mathrm{j}\omega}e^{-\mathrm{j}\omega m}e^{\mathrm{j}\omega k}\mathrm{d}\omega = \sum_{m\in\mathbb{Z}} Dy(m)\frac{\text{Si}\left((k-m)\pi\right)}{\pi},$$

where $\text{Si}(t) = \displaystyle\int_0^t \frac{\sin(u)}{u}\mathrm{d}u$ is the Sine integral function.

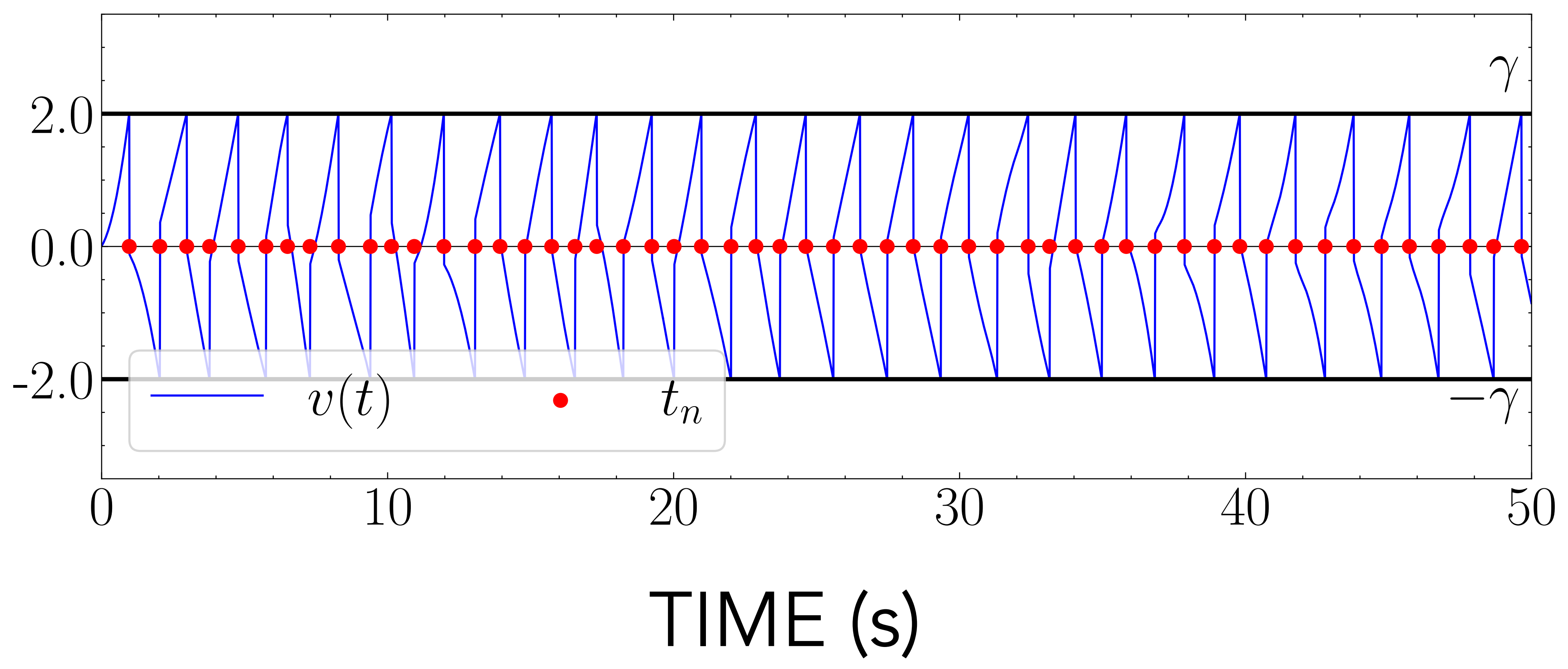
Compactly Supported Generator Kernels

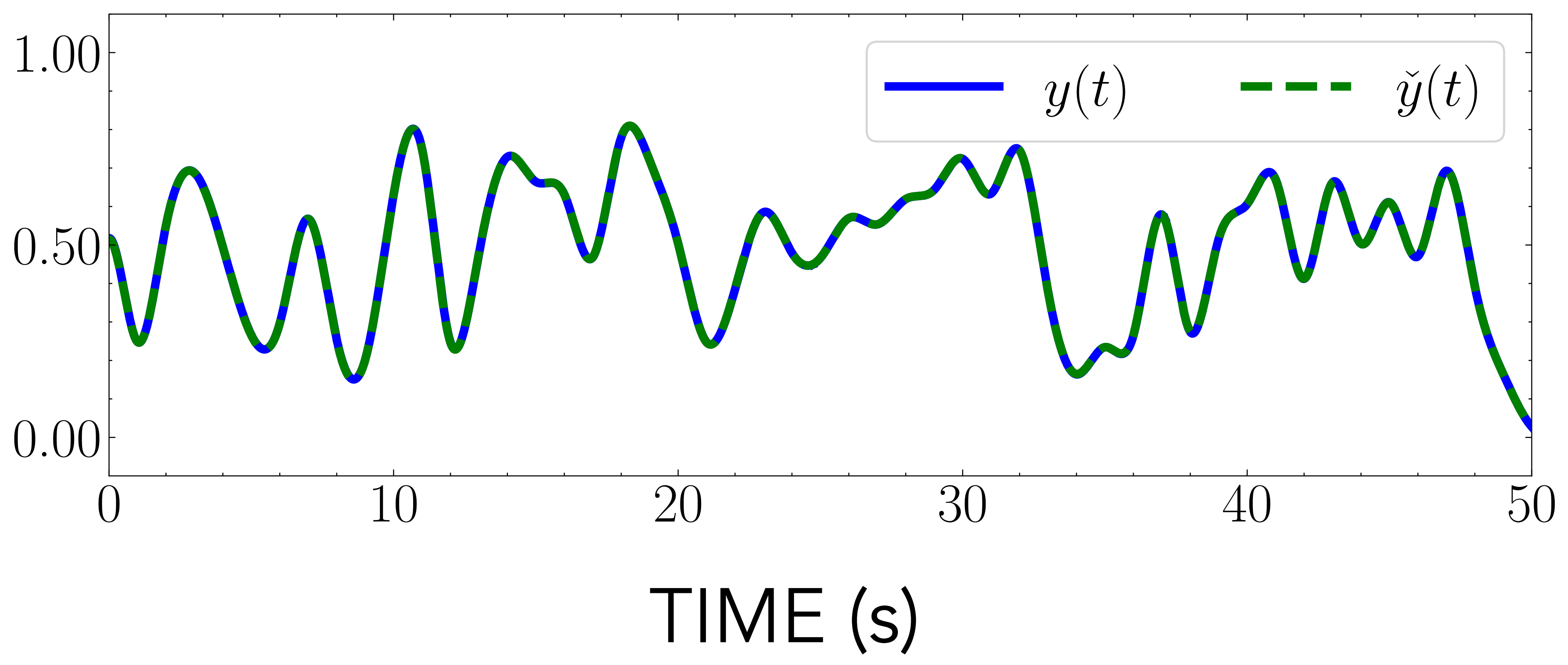
- Consider the signals generated by sequences of the type $\{\cdots, 0, c_0, c_1, \cdots, c_{K-1}, 0, \cdots\}$ and a generator kernel with supp$(\varphi) < K$.
- $L \geq K$ time-instants construct the linear system of equations

$$\mathbf{y} = \mathbf{M}\mathbf{c},$$

where $\mathbf{y} = [Dy(t_1)\ Dy(t_2)\ \cdots\ Dy(t_L)]^\mathsf{T} \in \mathbb{R}^L$, $\mathbf{c} = [c_0\ c_1\ \cdots\ c_{K-1}]^\mathsf{T} \in \mathbb{R}^K$ and $\mathbf{M} \in \mathbb{R}^{L \times K}$ defined by the entries $[M]_{i,j} = D\varphi(t_i - j)$.

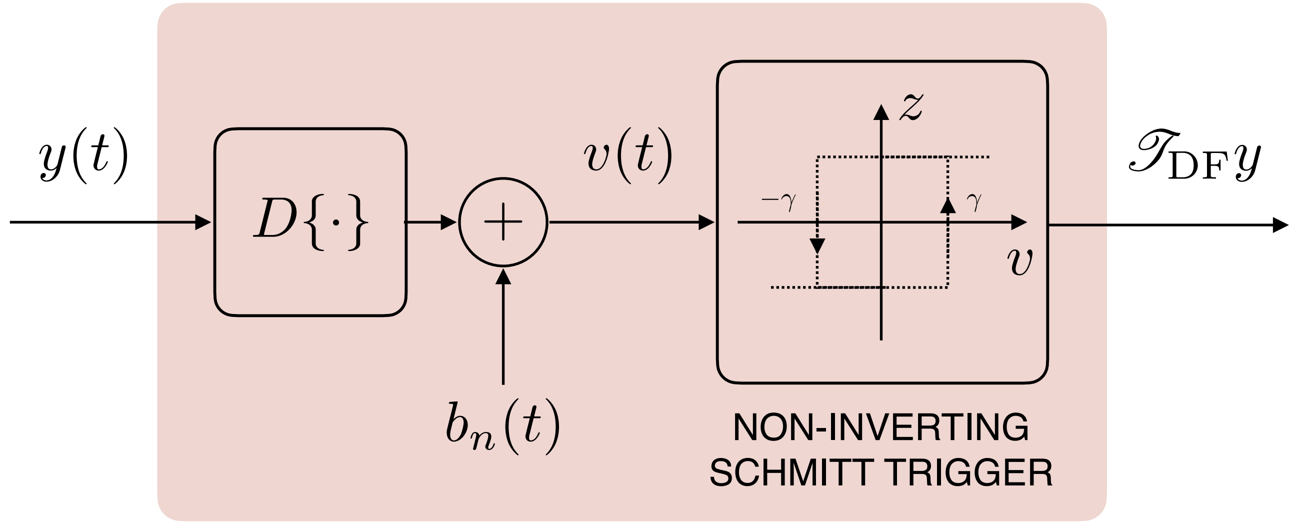
- $\mathbf{M}$ is diagonally dominant and left-invertible, i.e. $\mathbf{c}$ can be uniquely recovered.



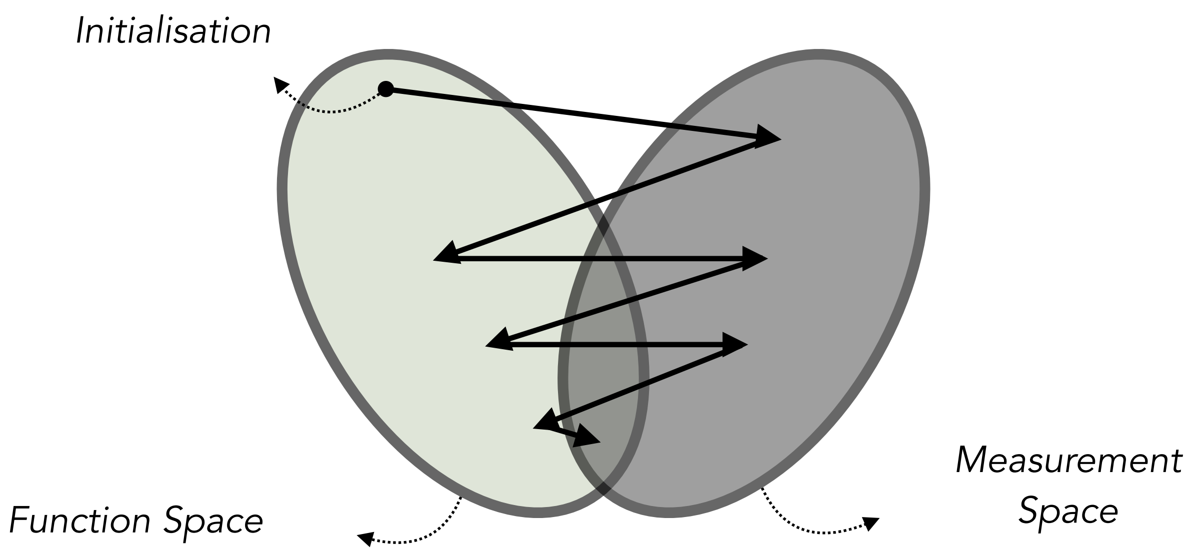


Summary

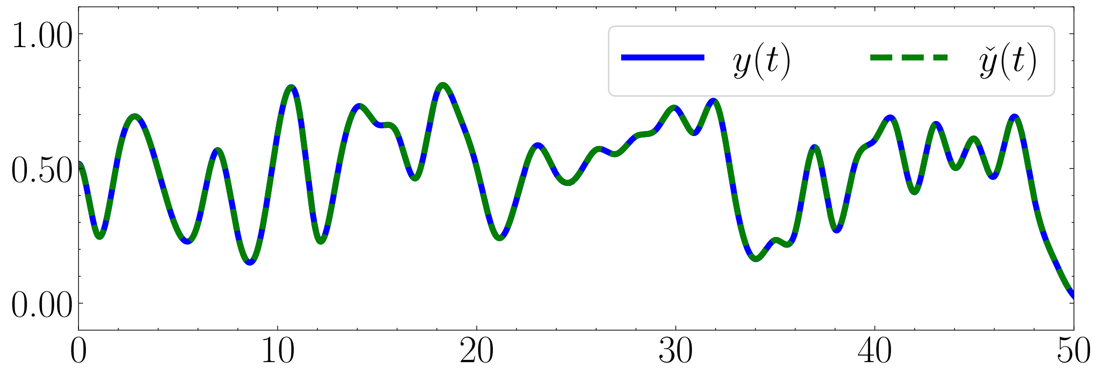
- We proposed a novel differentiate-and-fire time-encoding machine that mimics the dynamic vision sensors.
- We showed, using the addition of a ramp, the sampling density of the time-encoding machine can be bounded.



- We proposed the method of alternating projections to reconstruct signals in shift-invariant spaces from their time-encoded measurements.



- We showed in the particular case where the generator kernel has finite support, reconstruction can be achieved by solving a linear system of equations.



Thank You!