DIFFERENTIATE-AND-FIRE TIME-ENCODING OF FRI SIGNALS

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Shannon Sampling

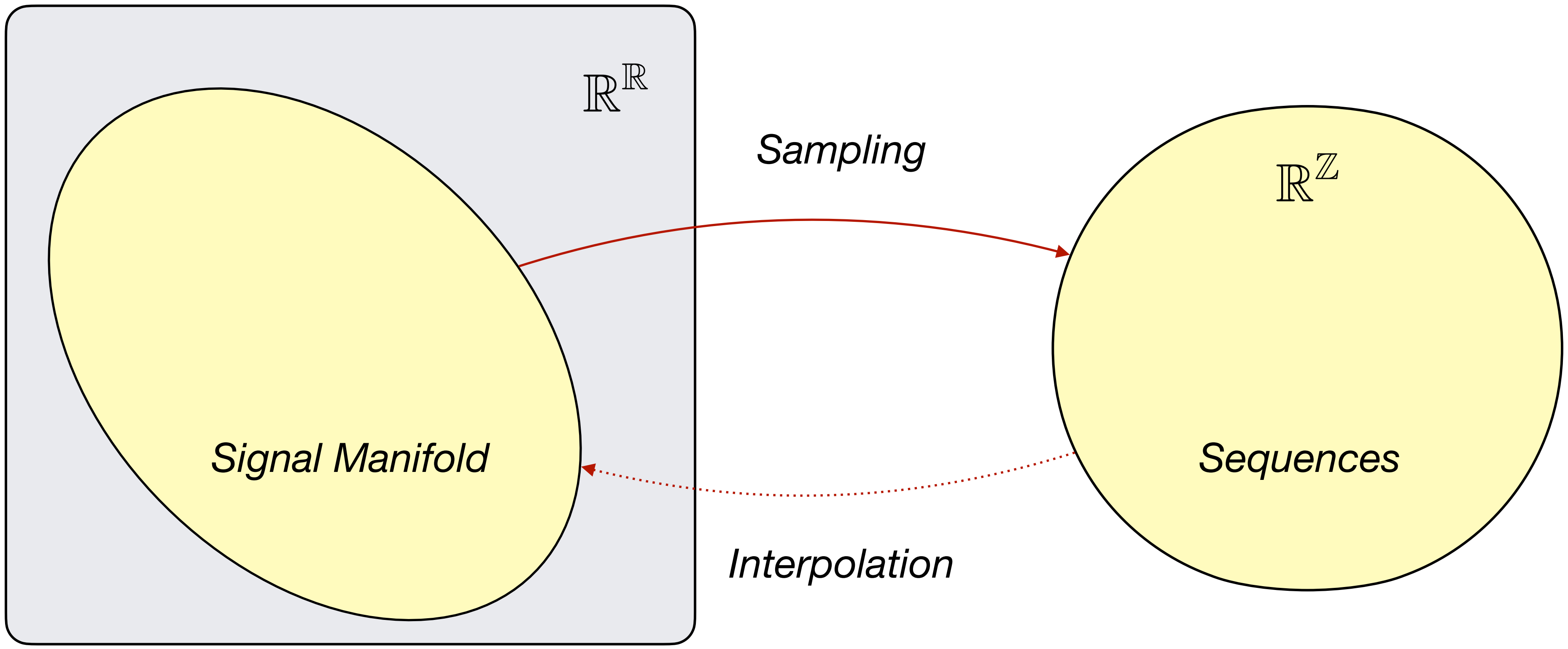

Fig 1: Sampling maps signal manifolds to sequences; interpolation maps sequences to signals

Theorem *(Shannon, 1949)*

If a function f contains no frequencies higher than ξ_N, it is completely described by giving its ordinates at a series of points spaced $T_s = \dfrac{1}{2\xi_N}$ apart.

Representation of Neural Signals

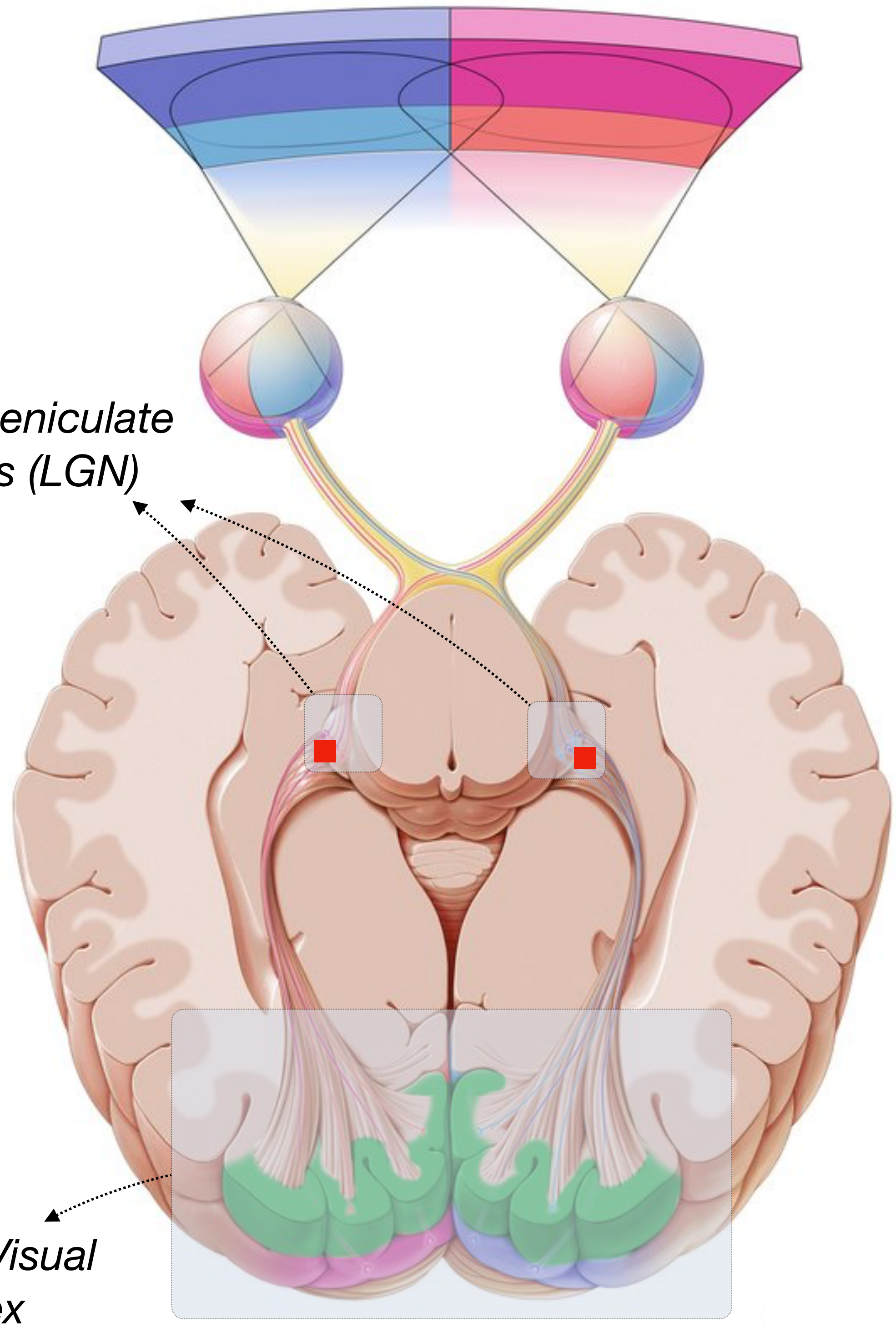

Fig 3: Schematic of the human visual system

- *Parvocellular Pathways* $\rightsquigarrow$ IF-TEM
  - Sustained vision
  - High spatial resolution
  - Colour vision
  - Low contrast vision
- *Magnocellular Pathways* $\rightsquigarrow$ **DIF-TEM**
  - Transient vision
  - Low spatial resolution
  - Monochrome vision
  - High contrast vision

Picture courtesy: kenhub.com

Time-Encoding of Continuous-Time Signals

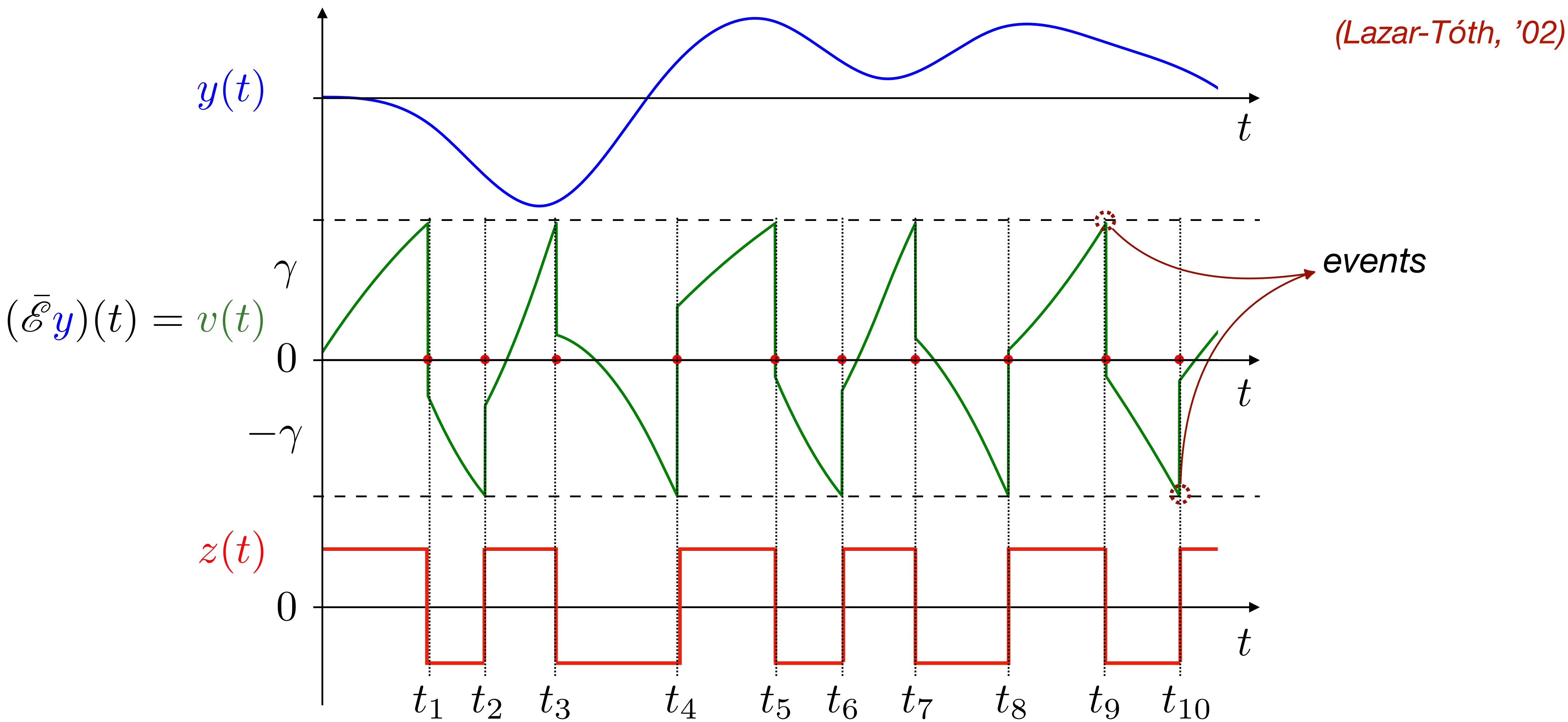

Fig 2: The concept of time-encoding of continuous-time signals

- Mimics peripheral perception of sensory signals
- Low power
- Sparse measurements
- Low redundancy

Time-Encoding Machines

(Kamath-CSS, '21)

Definition

A *time-encoding machine* is a map $\mathscr{T} : \mathbb{R}^{\mathbb{R}} \to \mathbb{R}^{\mathbb{Z}}$ with

- an event operator $\mathscr{E}$ on $\mathbb{R}^{\mathbb{R}}$ and
- references $\{r_n\}_{n\in\mathbb{Z}}$

such that $y \mapsto \mathscr{T} y$ follows

- $\mathscr{T} y = \{t_i \in \mathbb{R} \mid t_i > t_j,\ \forall i > j, i \in \mathbb{Z}\}$ *(ordering)*
- $(\mathscr{E} y)(t_n) = r_n(t_n), \forall t_n \in \mathscr{T} y$ *(characterization)*

- *Sampling density* of $\mathscr{T} y$, $d(\mathscr{T} y) = \sup_{n\in\mathbb{Z}} |t_{n+1} - t_n|$
- $\mathscr{T} y$ is *ϵ-distinct* if $|t_{n+1} - t_n| > \epsilon,\ \forall n \in \mathbb{Z}$

Our Contributions

Part I Introduction

Part II The Differentiate-and-Fire Time-Encoding Machine

- t-transform
- Sampling sets of DIF-TEM

Part III Time-Encoding of FRI Signals Using DIF-TEM

- Kernel-based Time-Encoding
- Sufficient Conditions for Perfect Recovery

Part IV Simulation Results

Part V Conclusions and Future Work

Differentiate-and-Fire Time-Encoding Machine

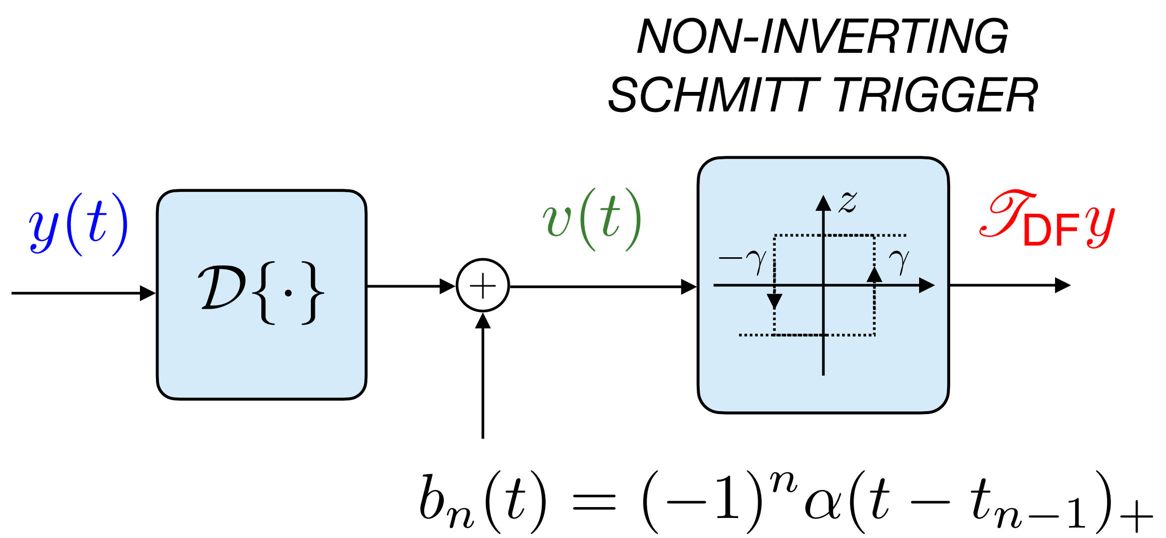

Fig 4: DIF-TEM with a bias signal $b_n(t)$ and threshold γ

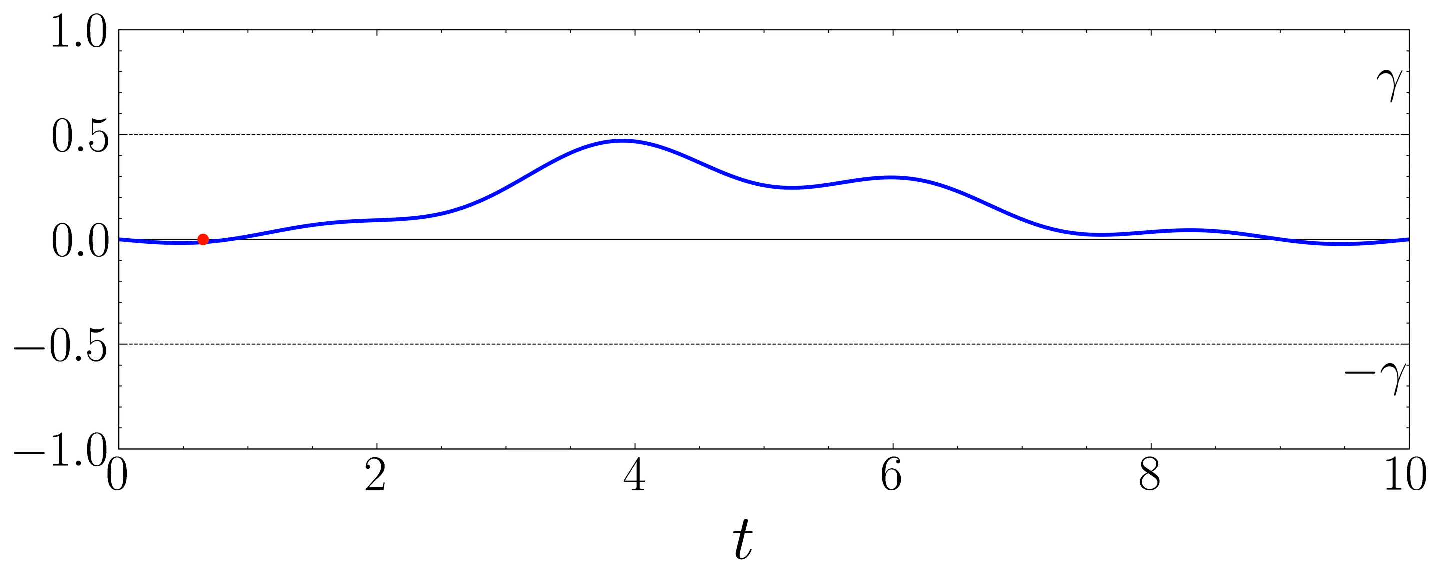

Fig 5: Demonstration of firing in DIF-TEM

Differentiate-and-Fire Time-Encoding Machine

Lemma *(t-transform)*

Let $y \in \mathcal{C}^1(\mathbb{R})$ be the input to the DIF-TEM with parameters $\{\alpha, \gamma\}$. The output of the DIF-TEM $\mathscr{T}_{\mathsf{DF}} y = \{t_n\}_{n \in \mathbb{Z}}$ satisfies:

$$(\mathcal{D}y)(t_n) = (-1)^{n+1} \left(\gamma - \alpha(t_n - t_{n-1})\right), \ \forall t_n \in \mathscr{T}_{\mathsf{DF}} y.$$

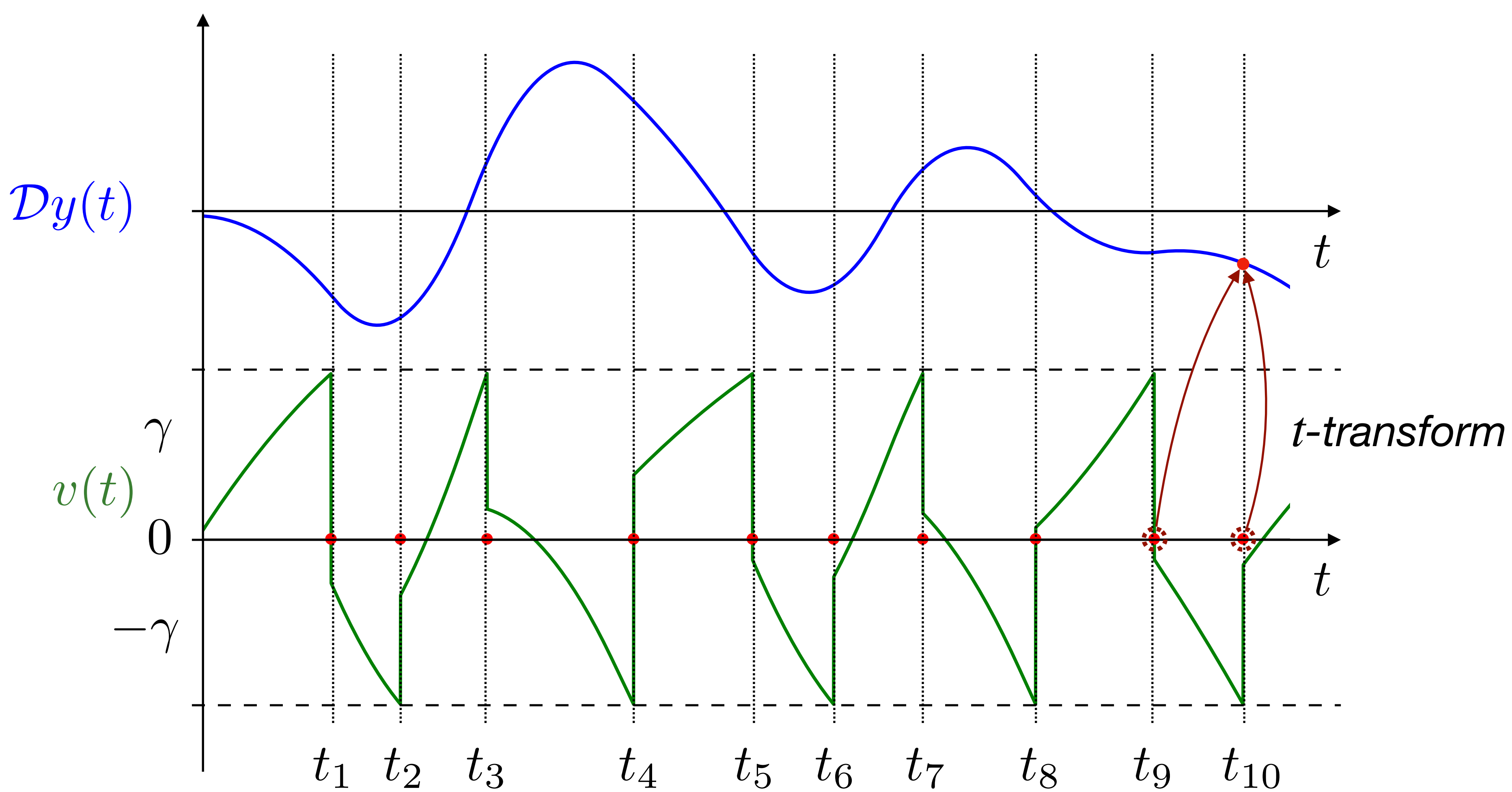

Fig 6: The t-transform gives the samples $\mathcal{D}y(t)$ using the trigger times

Differentiate-and-Fire Time-Encoding Machine

Corollary *(Stability)*

Let $y \in \mathcal{C}^1(\mathbb{R})$ with $\|\mathcal{D}y\|_\infty \leq \beta$ be the input to the DIF-TEM with parameters $\{\alpha, \gamma\}$. The output $\mathscr{T}_{\text{DF}}y = \{t_n\}_{n\in\mathbb{Z}}$ satisfies

$$\frac{\gamma - \beta}{\alpha} \leq t_n - t_{n-1} \leq \frac{\gamma + \beta}{\alpha}.$$

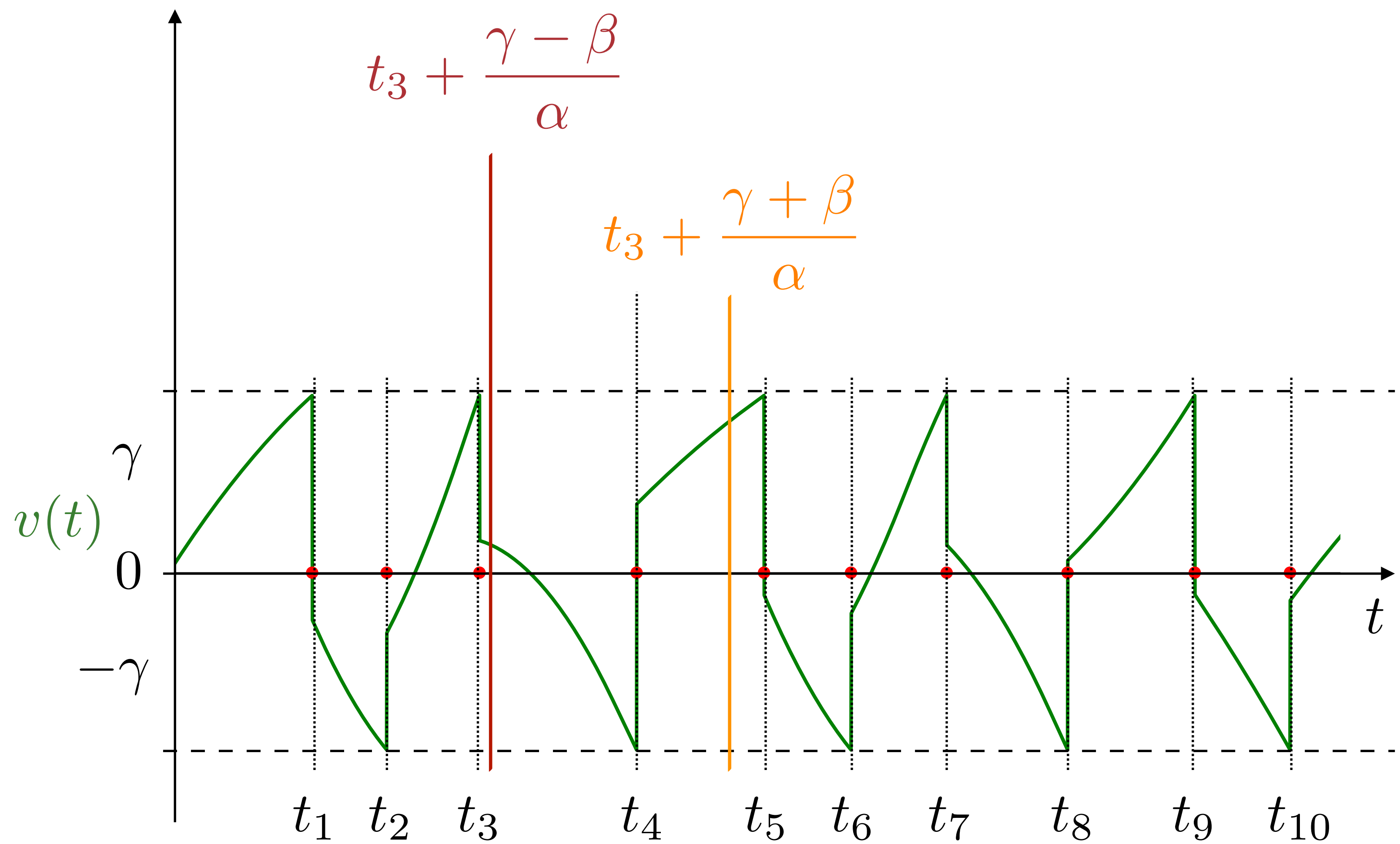

Fig 7: Stability ensures the existence of one trigger time in the interval $\left[t_n + \frac{\gamma-\beta}{\alpha}, t_n + \frac{\gamma+\beta}{\alpha}\right]$

Time-Encoding of FRI Signals

- Consider the T-periodic FRI signal $x \in L^2([0,T[)$ *(Vetterli et al., '02)*

$$x(t) = \sum_{m \in \mathbb{Z}} \sum_{k=0}^{K-1} c_k \varphi(t - \tau_k - mT)$$

  - $\varphi \in L^2(\mathbb{R})$ is a known pulse
  - $\mathbf{c} = [c_0 \; c_1 \; \cdots \; c_{K-1}]^{\mathsf{T}} \in \mathbb{R}^K$ unknown amplitudes
  - $\boldsymbol{\tau} = [\tau_0 \; \tau_1 \; \cdots \; \tau_{K-1}]^{\mathsf{T}} \in \mathbb{R}^K$ unknown shifts

- Rate of innovation is $\dfrac{2K}{T}$

- Fourier-series coefficients of $x \in L^2([0,T[)$

$$\hat{x}_m = \frac{1}{T} \hat{\varphi}(m\omega_0) \sum_{k=0}^{K-1} c_k e^{-\mathrm{j}\omega_0 m \tau_k}; \; \omega_0 = \frac{2\pi}{T}$$

signal reconstruction ≡ parameter estimation

Sum of Weighted Complex Exponentials (SWCE)

Prony's Annihilating Filter

- The *annihilating filter* $\boldsymbol{h} = (h_k)_{k=0}^{K}$ with roots $\vartheta_k = e^{-\mathrm{j}\omega_0 \tau_k}$ *(Prony, 1795)*

$$H(z) = \sum_{k=0}^{K} h_k z^{-k} = h_0 \prod_{k=0}^{K-1} (1 - \vartheta_k z^{-1})$$

- *Annihilation property*

$$(\boldsymbol{h} * \hat{\boldsymbol{x}})_m = \frac{1}{T} \sum_{k'=0}^{K-1} c_{k'} \vartheta_{k'}^{m} \underbrace{\sum_{k=0}^{K} h_k \vartheta_{k'}^{-k}}_{H(\vartheta_{k'})=0} = 0, \; \forall m \in \mathbb{Z}$$

- Prony's method

$$\mathbf{Xh} = \mathbf{0}, \text{ i.e., } \mathbf{h} \in \text{null}(\mathbf{X})$$

  - Roots of $\boldsymbol{h} \rightsquigarrow \boldsymbol{\tau}$

Kernel-Based Time-Encoding

- Recovering $\hat{\mathbf{x}}$ from measurements of $x * g$

$$
\begin{aligned}
y(t) &= (x * g)(t) \\
&= \sum_{m \in \mathbb{Z}} \hat{x}_m \hat{g}(m\omega_0) e^{\mathrm{j}\omega_0 m t}
\end{aligned}
$$

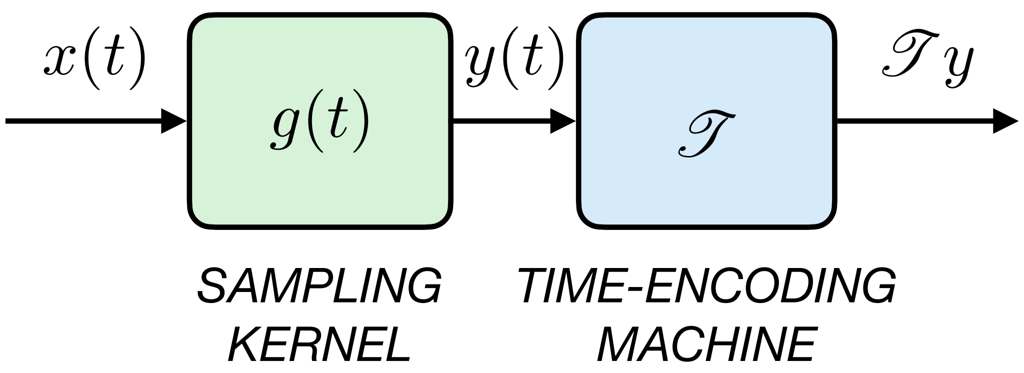

Fig 8: Kernel-based time-encoding of $x(t)$ using $g(t)$ as the sampling kernel

- Let g satisfy *alias-cancellation conditions*:

(Mulleti-CSS, '17)

$$
\hat{g}(m\omega_0) = \begin{cases} g_m \neq 0, & m \in [\![-M, M]\!] \\ 0, & m \notin [\![-M, M]\!] \end{cases}
$$

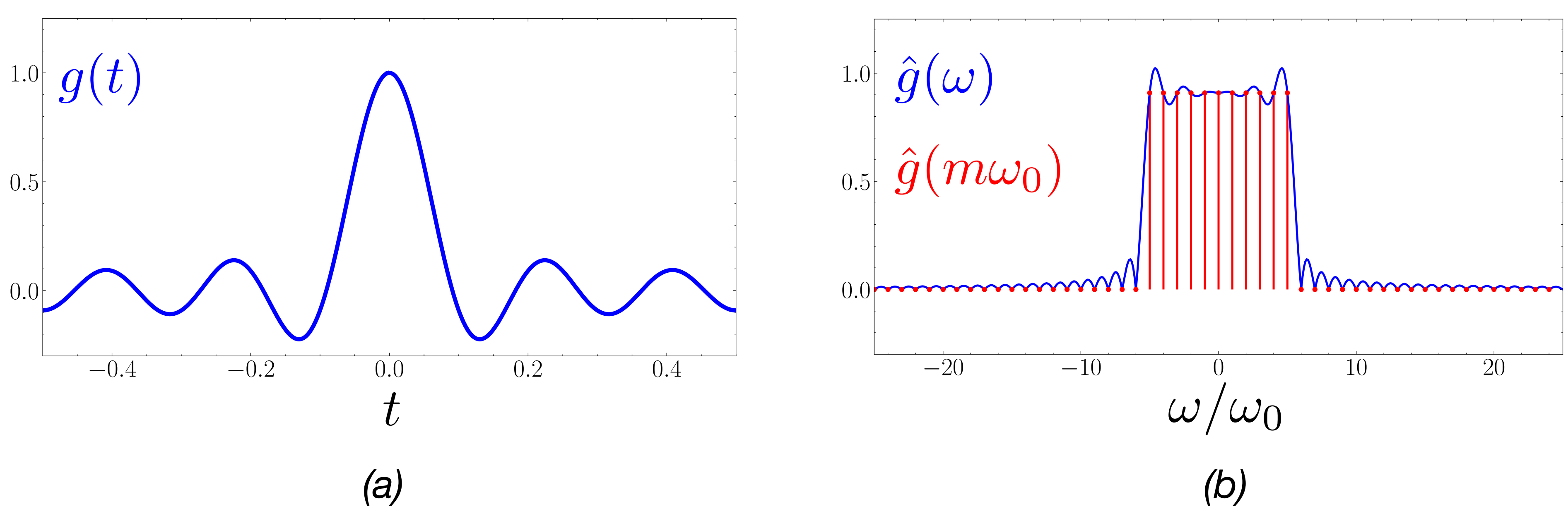

Fig 9: Fourier-domain sum-of-sincs (SoS) kernel: (a) time-domain; (b) Fourier-domain

Sufficient Conditions for Perfect Recovery

- $\mathscr{T}_{\mathsf{DF}} y$ provides crossing-time instants of $\mathcal{D}y$

$$y_n \doteq (\mathcal{D}y)(t_n) = \sum_{m=-M}^{M} \hat{x}_m \cdot (\mathrm{j}\omega_0 m) \cdot e^{\mathrm{j}\omega_0 m t_n}$$

- L measurements in $[0, T[$, $\mathbf{y} = [y_1 \ \cdots \ y_L]^{\mathsf{T}}$

$$\mathbf{y} = \mathbf{G}_{\mathsf{DF}} \mathbf{S} \hat{\mathbf{x}}$$

where $\mathbf{S} = \operatorname{diag}\left((\mathrm{j}\omega_0 m)_{m=-M}^{M}\right) \in \mathbb{C}^{N \times N}$ and

$$\mathbf{G}_{\mathsf{DF}} = \begin{bmatrix} e^{-\mathrm{j}M\omega_0 t_1} & \cdots & e^{-\mathrm{j}\omega_0 t_1} & 1 & e^{\mathrm{j}\omega_0 t_1} & \cdots & e^{\mathrm{j}M\omega_0 t_1} \\ e^{-\mathrm{j}M\omega_0 t_2} & \cdots & e^{-\mathrm{j}\omega_0 t_2} & 1 & e^{\mathrm{j}\omega_0 t_2} & \cdots & e^{\mathrm{j}M\omega_0 t_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ e^{-\mathrm{j}M\omega_0 t_L} & \cdots & e^{-\mathrm{j}\omega_0 t_L} & 1 & e^{\mathrm{j}\omega_0 t_L} & \cdots & e^{\mathrm{j}M\omega_0 t_L} \end{bmatrix}$$

Lemma

The matrix $\mathbf{G}_{\mathsf{DF}}$ is left-invertible when $L \geq 2M + 1$

Sufficient Conditions for Perfect Recovery

- **S** is singular
- $\hat{x}_0$ cannot be recovered
- We solve by setting $\hat{x}_0 = 0$ and $\mathbf{S}_{M+1,M+1} = 1$

- To get $L \geq 2K+1$ time-instants in $[0, T[$

$$d(\mathscr{T}_{\mathsf{DF}} y) < \frac{T}{L}$$

Proposition

Let the T-periodic FRI signal $x(t)$ be observed using a sampling kernel $g(t)$ that satisfies alias-cancellation conditions and a DIF-TEM with parameters $\alpha, \gamma > 0$. The time instants $\{t_n\}_{n=1}^{L} \subset \mathscr{T}_{\mathsf{DF}} \cap [0, T[$ constitute a sufficient representation of x if $L \geq 2K+1$ and the parameters of the DIF-TEM satisfy

$$\frac{\gamma + \|x * \mathcal{D}g\|_\infty}{\alpha} \leq \frac{T}{L}.$$

Simulation Results

- Consider the periodic FRI signal

$$x_1(t) = \sum_{m\in\mathbb{Z}} \sum_{k=0}^{4} c_k \varphi(t - \tau_k - m)$$

  - $\varphi(t) = \beta^{(3)}(10t)$
  - Sampling kernel: SoS
  - $\alpha = 16.4, \gamma = 2$

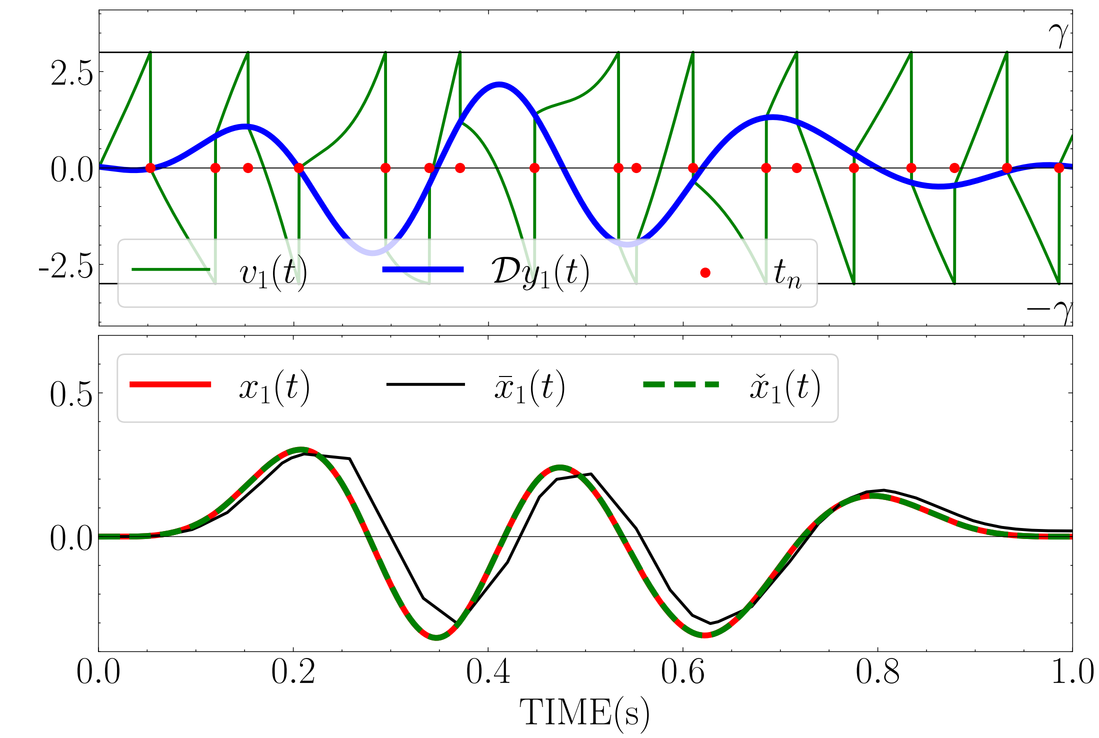

Fig 10: Time-encoding and reconstruction of $x_1(t)$: RSNR$(x_1, \check{x}_1) = 51$ dB; RSNR$(x_1, \bar{x}_1) = 9$ dB

- Consider the periodic FRI signal

$$x_2(t) = \sum_{m\in\mathbb{Z}} \sum_{k=0}^{4} c_k \delta(t - \tau_k - m)$$

  - Sampling kernel: SoS
  - $\alpha = 270, \gamma = 24.4$

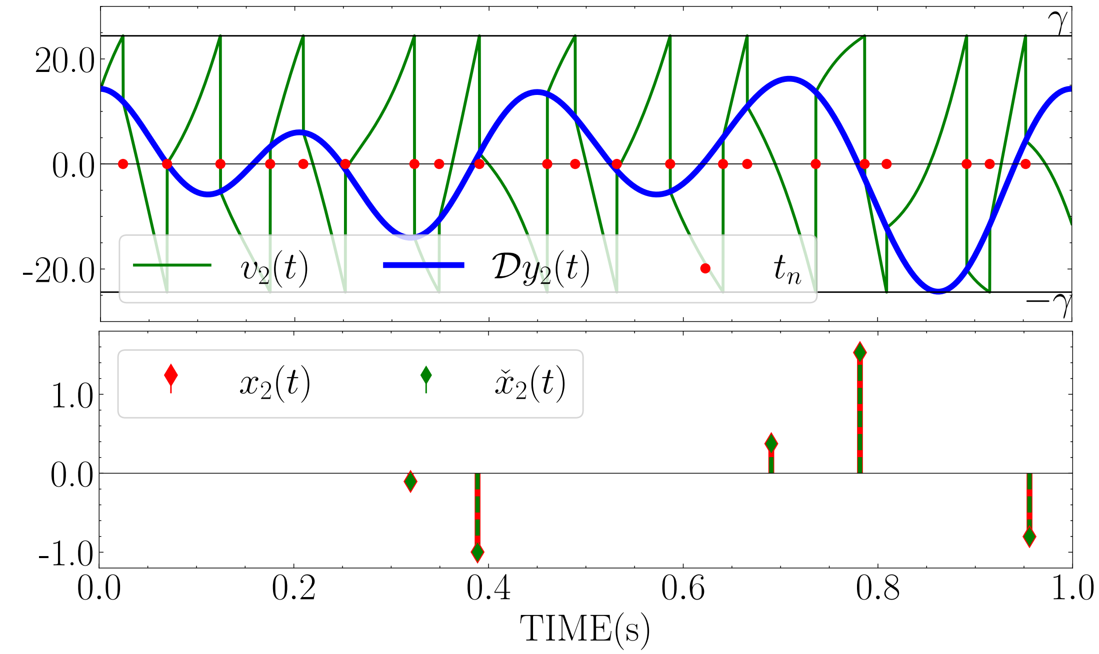

Fig 11: Time-encoding and reconstruction of $x_2(t)$: Perfect reconstruction up to numerical precision

Summary and Conclusions

- We considered differentiate-and-fire time-encoding of FRI signals
- We proposed kernel-based time-encoding and analyzed sampling and reconstruction in the Fourier-domain
- We demonstrated that perfect reconstruction is possible using samples taken at the order of the rate-of-innovation of the signal

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