DIGITAL CONTROL IN SMPCs AND FPGA-BASED PROTOTYPING

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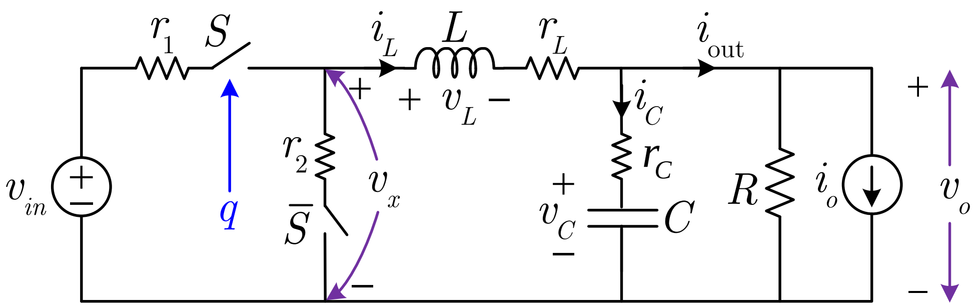
Module 04: Modeling Techniques and Mode Validation using MATLAB

Lecture 39: Discrete-Time Transfer Functions and Closed Loop Block Diagrams

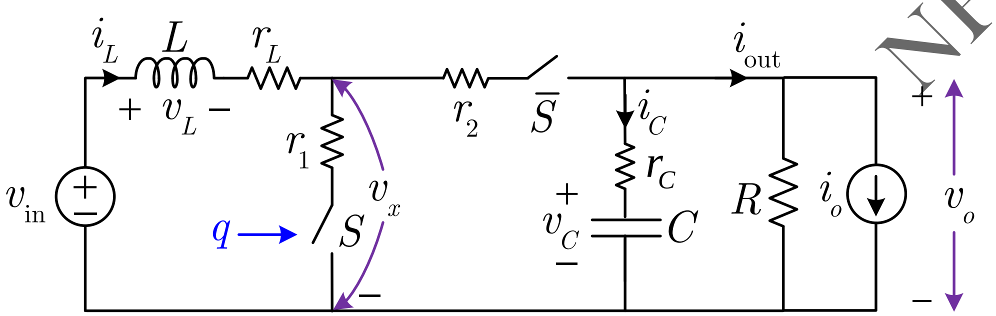
CONCEPTS COVERED

- Step-by-step derivations of various discrete-time transfer functions
- Discrete-time control-to-current and control-to-output TFs
- Loop transfer function under digital voltage mode control
- Loop transfer function under digital current mode control

State-Space Modeling of DC-DC Converters



Synchronous buck converter



Synchronous boost converter

State variables:

$x_1 = i_L$ inductor current

$x_2 = v_C$ capacitor voltage

$$x = \begin{bmatrix} i_L \\ v_C \end{bmatrix}$$

Input variables:

$u_1 = v_{in}$ input voltage

$u_2 = i_o$ sink current

$$u = \begin{bmatrix} v_{in} \\ i_o \end{bmatrix}$$

State-Space Modeling of DC-DC Converters (contd…)

- State-space model

$$\dot{x} = A_q x + B_q u$$

Buck converter matrices

Vin

load

$$A_q = \begin{bmatrix} -\dfrac{(r_e + \alpha r_C)}{L} & -\dfrac{\alpha}{L} \\ \dfrac{\alpha}{C} & -\dfrac{\alpha}{RC} \end{bmatrix} \qquad B_q = \begin{bmatrix} \dfrac{q}{L} & \dfrac{\alpha r_C}{L} \\ 0 & -\dfrac{\alpha}{C} \end{bmatrix}$$

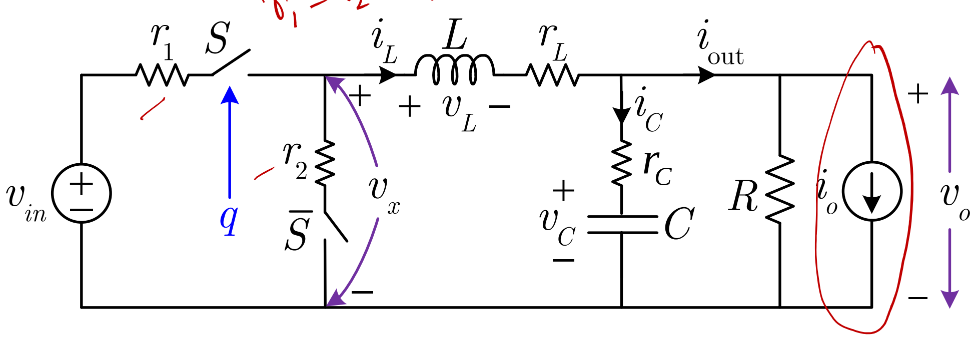
where $\quad r_e = r_1 + r_L \quad$ and $\quad \alpha = \dfrac{R}{R + r_C}$

[For details, refer to Lecture~33 of this course]

State-Space Matrices of a Synchronous Buck Converter

$r_1 = r_2 = r$

$r_e = r + r_L$



Synchronous buck converter

$$A_1 = A_2 = A = \begin{bmatrix} -\dfrac{(r_e + \alpha r_C)}{L} & -\dfrac{\alpha}{L} \\ \dfrac{\alpha}{C} & -\dfrac{\alpha}{RC} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \dfrac{1}{L} & \dfrac{\alpha r_C}{L} \\ 0 & -\dfrac{\alpha}{C} \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & \dfrac{\alpha r_C}{L} \\ 0 & -\dfrac{\alpha}{C} \end{bmatrix}$$

$$B_1 = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

in absence of current sink

Discrete –Time Large Signal and Small Signal Model

$C_{eq} = [\alpha r_c \quad \alpha]$

$\alpha = \frac{R}{R + r_c}$

Large-Signal Discrete-Time Model

$$x_{n+1} = e^{AT} x_n + e^{A(T - t_{on} - t_s)} (e^{A t_{on}} - I) A^{-1} B_1 v_{in}$$

$$v_o[n] = C_{eq} x_n$$

$B_1 = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$

Small-Signal Discrete-Time Model

$$\tilde{x}_{n+1} = A_{eq} \tilde{x}_n + B_{eq} \tilde{d} \qquad \tilde{v}_o[n] = C_{eq} \tilde{x}_n$$

where $A_{eq} = e^{AT}$ and $B_{eq} = e^{A(T - t_d)} B_1 V_{in} T$

[For details, refer to Lecture~38 of this course]

Small-Signal Transfer Functions

$$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d} \quad \textbf{and} \quad \tilde{v}_o[n] = C_{eq}\tilde{x}_n$$

Apply Z-transformation

$$\Rightarrow \boxed{\tilde{x}(z)\times zI = A_{eq}\tilde{x}(z) + B_{eq}\tilde{d}(z)} \quad \textbf{and} \quad \tilde{v}_o(z) = C_{eq}\tilde{x}(z)$$

$$\Rightarrow \tilde{x}(z) = \left(zI - A_{eq}\right)^{-1} B_{eq}\tilde{d}(z)$$

$$\Rightarrow \frac{\overset{2\times 1}{\tilde{x}(z)}}{\underset{1\times 1}{\tilde{d}(z)}} = \boxed{\left(zI - A_{eq}\right)^{-1} B_{eq}} \; 2\times 1$$

Small-Signal Transfer Functions

$$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d}$$

$$\frac{\tilde{x}(z)}{\tilde{d}(z)} = \boxed{\left(zI - A_{eq}\right)^{-1} B_{eq}}$$

$$\tilde{i}_L(z) = [1\ \ 0]\tilde{x}(z)$$

$$\Rightarrow \boxed{G_{id}(z) = \frac{\tilde{i}_L(z)}{\tilde{d}(z)} = [1\ \ 0]\left(zI - A_{eq}\right)^{-1} B_{eq}}$$

Control-to-current transfer function

$$\tilde{x}(z) = \begin{bmatrix} \tilde{i}_L(z) \\ \tilde{v}_c(z) \end{bmatrix}$$

$$\tilde{i}_L(z) = \begin{bmatrix} 1 & 0 \end{bmatrix} \tilde{x}(z)$$

Small-Signal Transfer Functions

$$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d}$$

$$\tilde{v}_o[n] = C_{eq}\tilde{x}_n$$

$\tilde{v}_o(z) = C_{eq}\tilde{x}(z)$

$\tilde{x}(z) = \underbrace{(zI - A_{eq})^{-1} B_{eq}}\, \tilde{d}(z)$

$$\frac{\tilde{x}(z)}{\tilde{d}(z)} = \left(zI - A_{eq}\right)^{-1} B_{eq}$$

$\frac{\tilde{i}_L(z)}{\tilde{d}(z)} = G_{ic}(z)$

$\frac{\tilde{v}_o(z)}{\tilde{d}(z)} = G_{vd}(z)$

$$\Rightarrow \boxed{G_{vd}(z) = \frac{\tilde{v}_o(z)}{\tilde{d}(z)} = C_{eq}\left(zI - A_{eq}\right)^{-1} B_{eq}}$$

Control-to-output transfer function

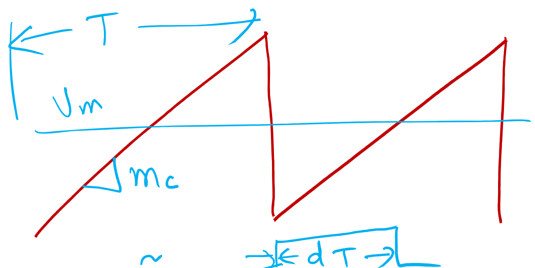
Digital Voltage Mode Control

$$\tilde{x}_{n+1} = A_{\text{eq}}\tilde{x}_n + B_{\text{eq}}\tilde{d}$$

$$\tilde{v}_o[n] = C_{\text{eq}}\tilde{x}_n$$

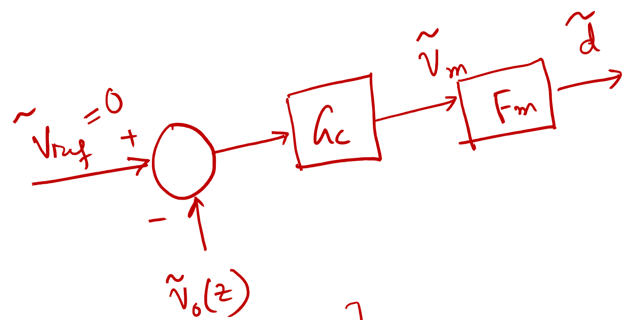
$$\tilde{d} = \frac{\tilde{v}_m}{m_c T} = F_m\tilde{v}_m \qquad F_m = \frac{1}{m_c T}$$

$$\tilde{d} = F_m\tilde{v}_m = -G_c F_m \tilde{v}_o = -G_c F_m C_{\text{eq}}\tilde{x}_n$$



$\tilde{d} = F_m \tilde{v}_m$

$F_m = \frac{1}{m_c T}$



$\tilde{d} = F_m G_c \left[-\tilde{v}_o(z)\right]$

$= -G_c F_m \tilde{v}_o(z)$

$\tilde{v}_o = C_{eq}\tilde{x}(z)$

Digital Voltage Mode Control

$$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d} \quad \textbf{and} \quad \tilde{v}_o[n] = C_{eq}\tilde{x}_n$$

$$\tilde{d} = F_m\tilde{v}_m = -G_cF_m\tilde{v}_o = -G_cF_mC_{eq}\tilde{x}_n$$

Apply Z-transformation

$$zI \times \tilde{x}_n(z) = A_{eq}\tilde{x}_n(z) - G_cF_mB_{eq}C_{eq}\tilde{x}_n(z)$$

$$\left|zI_{2\times 2} - A_{eq} + G_cF_mB_{eq}C_{eq}\right| = 0$$

characteristic equation

$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d}$

$zI \times \tilde{x}(z) = A_{eq}\tilde{x}(z) + B_{eq}\tilde{d}(z)$

$\tilde{d}(z) = -G_cF_mC_{eq}\tilde{x}(z)$

$zI\tilde{x}(z) = \left[A_{eq} - G_cF_mC_{eq}\right]\tilde{x}(z)$

$\left(zI - A_{eq} + G_cF_mC_{eq}\right)\tilde{x}(z) = 0$

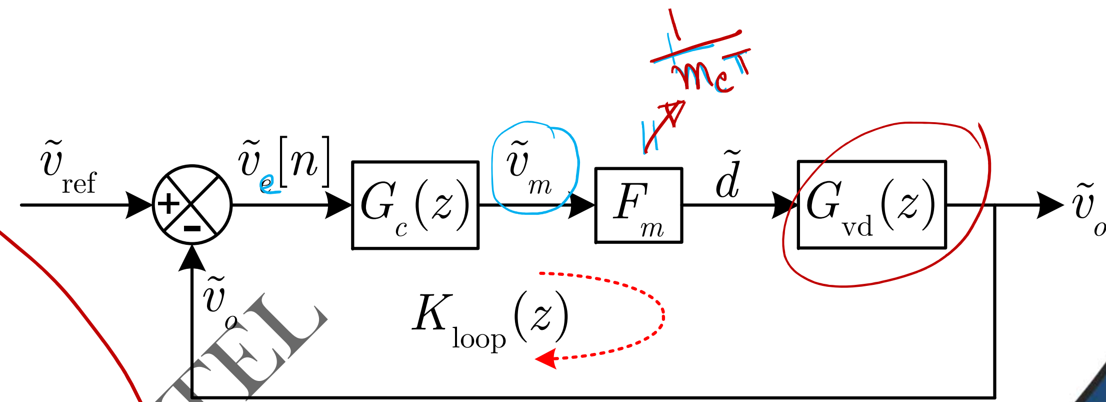
Digital VMC – Loop Transfer Function

$$K_{\text{loop}}(z) = F_m G_c G_{\text{vd}}$$

where $F_m = \dfrac{1}{m_e T}$

$$G_c = K_p + \frac{K_i}{1 - z^{-1}} \quad \textbf{(PI Controller)}$$

$$G_{\text{vd}}(z) = C_{\text{eq}}\left(zI - A_{\text{eq}}\right)^{-1} B_{\text{eq}}$$



Digital PID controller

$$G_c(z) = K_p + K_i \frac{1}{1 - z^{-1}} + K_d(1 - z^{-1})$$

Mixed Signal Peak Current Mode Control

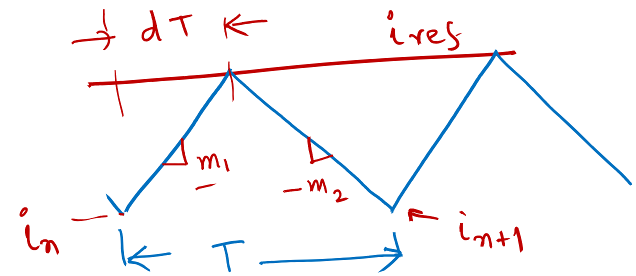
$$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d}$$

$$\tilde{v}_o[n] = C_{eq}\tilde{x}_n$$

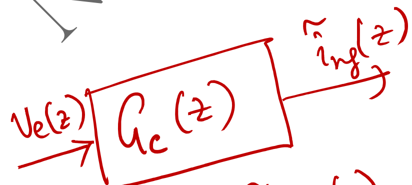
$$\tilde{d} = \frac{\left(\tilde{i}_{ref} - \tilde{i}_L\right)}{m_1 T} = F_m \times \left(\tilde{i}_{ref} - \tilde{i}_L\right) \qquad F_m = \frac{1}{m_1 T}$$

$$i_{ref}[n] = K_p\left(v_{ref} - v_o[n]\right) + u_i[n]$$

$$u_i[n] = K_i\left(v_{ref} - v_o[n]\right) + u_i[n-1]$$

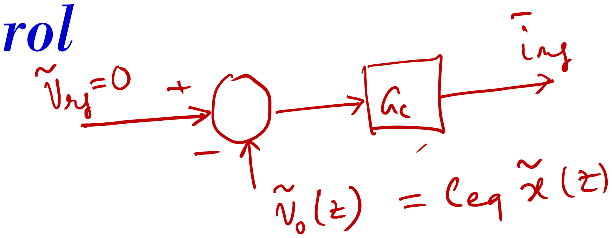


$$dT = \frac{i_{ref} - i_n}{m_1}$$



$$\frac{\tilde{i}_{ref}(z)}{\tilde{v}_e(z)} = G_c(z) = K_p + \left(K_i \times \frac{1}{1-z^{-1}}\right)$$

Mixed Signal Peak Current Mode Control



$$\tilde{x}_{n+1} = A_{\text{eq}}\tilde{x}_n + B_{\text{eq}}\tilde{d} \quad \textbf{and} \quad \tilde{v}_o[n] = C_{\text{eq}}\tilde{x}_n$$

Perturb and apply Z-transformation

$$\tilde{i}_{\text{ref}}(z) = -G_c(z)\tilde{v}_o(z)$$

$$\tilde{i}_{\text{ref}}(z) = -G_c(z)C_{\text{eq}}\tilde{x}(z)$$

$$G_c(z) = K_p + \frac{K_i}{1 - z^{-1}}$$

Mixed Signal Peak Current Mode Control

$$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d}$$

$$= A_{eq}\tilde{x}_n + B_{eq}F_m\left(\tilde{i}_{ref} - \tilde{i}_L\right)$$

$$\tilde{i}_{ref}(z) = -G_c(z)C_{eq}\tilde{x}(z)$$

$$\boxed{\tilde{i}_L = [1\ \ 0]\tilde{x}}$$

$\tilde{d} = F_m(\tilde{i}_{ref} - \tilde{i}_L)$

$\tilde{x}_{n+1} = A_{eq}\tilde{x}_n + B_{eq}\tilde{d}$

$\tilde{d} = F_m(\tilde{i}_{ref} - \tilde{i}_L)$

$\tilde{i}_L = [1\ \ 0]\tilde{x}_n$

$\tilde{i}_{ref}(z) = -G_c C_{eq}\tilde{x}(z)$

$zI\,\tilde{x}(z) = A_{eq}\tilde{x}(z) + B_{eq}F_m[\tilde{i}_{ref}(z) - \tilde{i}_L(z)]$

$=$

Mixed Signal Peak Current Mode Control

Apply Z-transformation

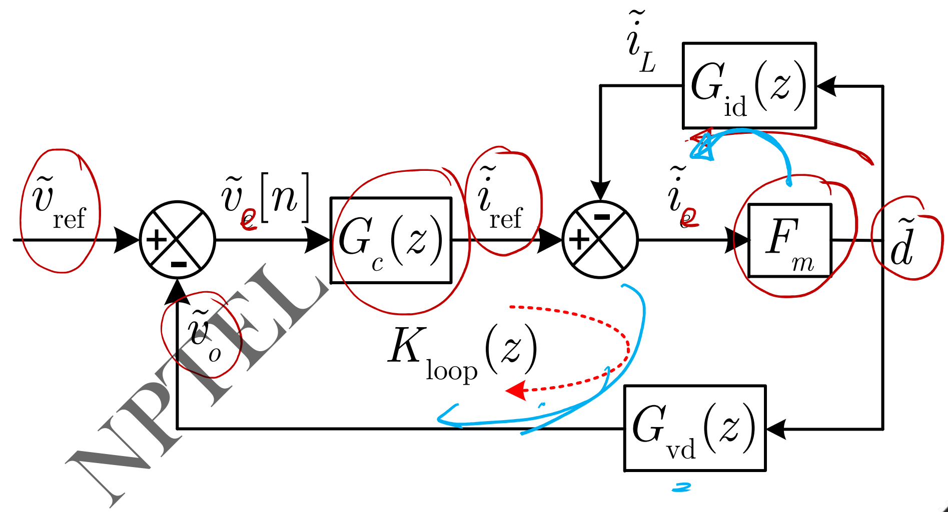
$$zI \times \tilde{x}_n(z) = A_{\text{eq}}\tilde{x}_n(z) - F_m G_c B_{\text{eq}} C_{\text{eq}} \tilde{x}_n(z) - B_{\text{eq}} F_m [1\ \ 0]\,\tilde{x}_n(z)$$

$$\Rightarrow \left| zI_{2\times 2} - A_{\text{eq}} + F_m B_{\text{eq}}[1\ \ 0] + F_m G_c B_{\text{eq}} C_{\text{eq}} \right| = 0$$

Mixed Signal Peak CMC Loop Transfer Function

$\tilde{d} = F_m(\tilde{i}_{ref} - \tilde{i}_L)$

$$K_{\text{loop}}(z) = \frac{F_m G_c G_{\text{vd}}}{1 + F_m G_{\text{id}}}$$



where $F_m = \dfrac{1}{m_1 T}$

$G_c = K_p + \dfrac{K_i}{1 - z^{-1}}$ **(PI Controller)**

$$G_{\text{vd}}(z) = C_{\text{eq}}\left(zI - A_{\text{eq}}\right)^{-1} B_{\text{eq}} \qquad G_{\text{id}}(z) = [1\ \ 0]\left(zI - A_{\text{eq}}\right)^{-1} B_{\text{eq}}$$

CONCLUSION

- Step-by-step derivations of various discrete-time transfer functions
- Discrete-time control-to-current and control-to-output TFs
- Loop transfer function under digital voltage mode control
- Loop transfer function under digital current mode control

THANK YOU !