NPTEL ONLINE CERTIFICATION COURSES

DIGITAL CONTROL IN SMPCs AND FPGA-BASED PROTOTYPING

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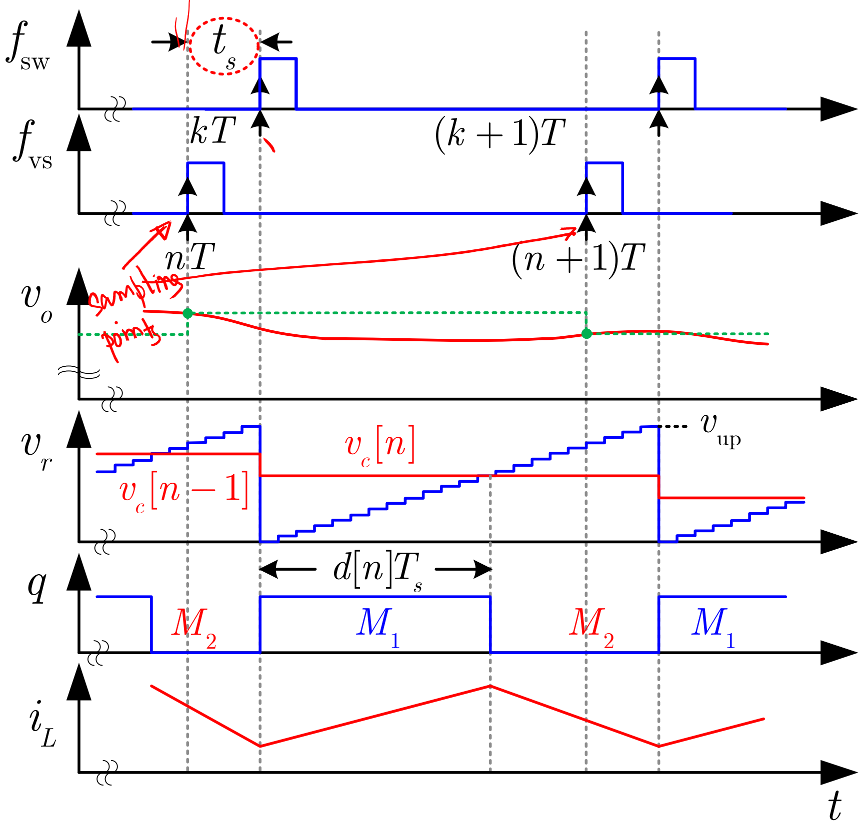
Module 04: Modeling Techniques and Mode Validation using MATLAB

Lecture 38: Derivation of Discrete-Time Small-Signal Models - II

CONCEPTS COVERED

- Discrete-time large-signal modeling
- Formulation of Jacobian matrix
- Discrete-time small-signal modeling

Discrete-Time Large Signal Model under Interval-2 Sampling



$$x(t)\Big|_{t=nT} = x_n \qquad x(t)\Big|_{t=(n+1)T} = x_{n+1}$$

Large-Signal Discrete-Time Model

$$x_{n+1} = e^{AT}x_n + e^{A(T-t_{on}-t_s)}(e^{At_{on}} - I)A^{-1}B_1 v_{in}$$

CCM buck converter with $A_1 = A_2 = A$ under resistive load

Discrete-Time Small Signal Model

Consider perturbations in $x_n = x_{ss} + \tilde{x}_n$ and $t_{on} = T_{on} + \tilde{t}_{on}$

$\underset{2\times1}{x_n} = \begin{bmatrix} x_{1n} \\ x_{2n} \end{bmatrix}$ $\quad \underset{2\times1}{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ $\quad \dfrac{\partial f}{\partial x_n} = \underset{2\times2}{\begin{bmatrix} \dfrac{\partial f_1}{\partial x_{1n}} & \dfrac{\partial f_1}{\partial x_{2n}} \\ \dfrac{\partial f_2}{\partial x_{1n}} & \dfrac{\partial f_2}{\partial x_{2n}} \end{bmatrix}}$

Applying Taylor series approximation

t_{on} – scalar $\quad$ Jacobian matrix

$\dfrac{\partial f}{\partial t_{on}} = \underset{2\times1}{\begin{bmatrix} \dfrac{\partial f_1}{\partial t_{on}} \\ \dfrac{\partial f_2}{\partial t_{on}} \end{bmatrix}}$

$$\underset{2\times1}{\tilde{x}_{n+1}} = \underset{2\times2}{\left.\frac{\partial f}{\partial x_n}\right|_{ss}} \underset{2\times1}{\tilde{x}_n} + \underset{2\times1}{\left.\frac{\partial f}{\partial t_{on}}\right|_{ss}} \underset{1\times1}{\tilde{t}_{on}}$$

where $\underset{2\times1}{x_{n+1}} = \underbrace{e^{AT}x_n + e^{A(T-t_{on}-t_s)}(e^{At_{on}} - I)A^{-1}B_1 v_{in}}_{= f(x_n,\, t_{on})} \triangleq \underset{2\times1}{f} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$

[S. Kapat, "An Analytical Approach of Discrete-Time ..." *IEEE APEC*, 2021]

Discrete-Time Small Signal Model (contd..)

$t_{on} \rightarrow$ control input

$$x_{n+1} = \boxed{e^{AT} x_n} + \boxed{e^{A(T-t_{on}-t_s)}(e^{At_{on}} - I)A^{-1}B_1 v_{in}} \triangleq f$$

$\frac{\partial f}{\partial x_n}$

$$x_{n+1} = f_{zi} + f_{zs}$$

where

$$f_{zi} = e^{AT} x_n, \quad f_{zs} = e^{A(T-t_{on}-t_s)}(e^{At_{on}} - I)A^{-1}B_1 v_{in}$$

$$\left.\frac{\partial f}{\partial x_n}\right|_{ss} = \left.\frac{\partial f_{zi}}{\partial x_n}\right|_{ss} = e^{AT}$$

Discrete-Time Small Signal Model (contd..)

$$x_{n+1} = f_{zi} + f_{zs}$$

where

$$f_{zi} = e^{AT} x_n, \quad f_{zs} = e^{A(T-t_{\text{on}}-t_s)}(e^{At_{\text{on}}} - I)A^{-1}B_1 v_{\text{in}}$$

$$\frac{\partial f}{\partial t_{\text{on}}} = \frac{\partial f_{zs}}{\partial t_{\text{on}}} = \frac{\partial}{\partial t_{\text{on}}}\left(\boxed{e^{A(T-t_{\text{on}}-t_s)}(e^{At_{\text{on}}} - I)}A^{-1}B_1 v_{\text{in}}\right)$$

$$\frac{\partial f}{\partial t_{\text{on}}} = \frac{\partial}{\partial t_{\text{on}}}\left((e^{A(T-t_s)}) - (e^{A(T-t_{\text{on}}-t_s)})A^{-1}B_1 v_{\text{in}}\right)$$

$$x_{n+1} = \underbrace{e^{A_2(T-t_{on}-t_s)} \cdot e^{A_1 t_{on}} \cdot e^{A_2 t_s} x_n + \quad (\quad)}_{} $$

$$\partial \qquad \underbrace{\qquad\qquad}_{f}$$

Discrete-Time Small Signal Model (contd..)

Zero state response:

$$f_{\mathrm{zs}} = \left(e^{A(T-t_s)} - e^{A(T-t_{\mathrm{on}}-t_s)}\right)A^{-1}B_1 v_{\mathrm{in}}$$

$$\Rightarrow \frac{\partial f_{\mathrm{zs}}}{\partial t_{\mathrm{on}}} = e^{A(T-t_{\mathrm{on}}-t_s)}B_1 v_{\mathrm{in}}$$

$$\Rightarrow \left.\frac{\partial f_{\mathrm{zs}}}{\partial t_{\mathrm{on}}}\right|_{\mathrm{ss}} = e^{A(T-T_{\mathrm{on}}-t_s)}B_1 V_{\mathrm{in}}$$

[S. Kapat, "An Analytical Approach of Discrete-Time ..." *IEEE APEC*, 2021]

Discrete-Time Small Signal Model (contd..)

$$\tilde{x}_{n+1} = \left.\frac{\partial f}{\partial x_n}\right|_{ss} \tilde{x}_n + \left.\frac{\partial f}{\partial t_{on}}\right|_{ss} \tilde{t}_{on}$$

$$\Rightarrow \tilde{x}_{n+1} = e^{AT}\tilde{x}_n + e^{A(T-T_{on}-t_s)} B_1 V_{in} \tilde{t}_{on}$$

Again On-time $t_{on} = dT$ **and Total delay** $t_d = DT + t_s$

$$\Rightarrow \tilde{t}_{on} = T\tilde{d}$$

$$\Rightarrow \tilde{x}_{n+1} = e^{AT}\tilde{x}_n + e^{A(T-t_d)} B_1 V_{in} T\tilde{d}$$

Discrete-Time Small Signal Model (contd..)

$$\boxed{\tilde{x}_{n+1} = A_{\text{eq}}\tilde{x}_n + B_{\text{eq}}\tilde{d}}$$

where $\boxed{A_{\text{eq}} = e^{AT}}$ **and** $\boxed{B_{\text{eq}} = e^{A(T-t_d)}B_1V_{\text{in}}T}$

Output Voltage $v_o[n] = C_{\text{eq}}x_n$

where $C_{\text{eq}} = \begin{bmatrix}\alpha r_C & \alpha\end{bmatrix}$

$$\Rightarrow \boxed{\tilde{v}_o[n] = C_{\text{eq}}\tilde{x}_n}$$

Discrete-Time Small Signal Analysis

$$\tilde{x}_{n+1} = A_{\text{eq}}\tilde{x}_{n} + B_{\text{eq}}\tilde{d} \quad \textbf{and} \quad \tilde{v}_{o}[n] = C_{\text{eq}}\tilde{x}_{n}$$

Applying Z-transformation

$$\Rightarrow \tilde{x}(z) \times zI = A_{\text{eq}}\tilde{x}(z) + B_{\text{eq}}\tilde{d}(z) \quad \textbf{and} \quad \tilde{v}_{o}(z) = C_{\text{eq}}\tilde{x}(z)$$

$$\Rightarrow \tilde{x}(z) = \left(zI - A_{\text{eq}}\right)^{-1} B_{\text{eq}}\tilde{d}(z)$$

$$\Rightarrow \boxed{\frac{\tilde{v}_{o}(z)}{\tilde{d}(z)} = C_{\text{eq}}\left(zI - A_{\text{eq}}\right)^{-1} B_{\text{eq}}}$$

Control-to-output transfer function

Comparative Analysis : DT SSM under Interval-2 Sampling

Approach-1

$$\tilde{x}_{n+1} = \Phi \tilde{x}_n + \gamma \tilde{d}$$

$$\Phi = e^{AT}$$

$$\gamma = e^{A(T-t_d)} \alpha T = e^{A(T-t_d)} B_1 V_{in} T$$

Approach-2

$$\tilde{x}_{n+1} = A_{eq} \tilde{x}_n + B_{eq} \tilde{d}$$

$$A_{eq} = e^{AT}$$

$$B_{eq} = e^{A(T-t_d)} B_1 V_{in} T$$

Discrete-Time Small-Signal Model Parameters: Buck Converter

$$\tilde{x}_{n+1} = A_{\mathrm{eq}}\tilde{x}_{n} + B_{\mathrm{eq}}\tilde{d}$$

Considering sampling delay t_s

Modulation	A_{eq}	B_{eq}
Trailing-edge	e^{AT}	$e^{A((1-D)T-t_s)}BV_{\mathrm{in}}T$
Leading−edge	e^{AT}	$-e^{A(DT-t_s)}BV_{\mathrm{in}}T$

[S. Kapat, "An Analytical Approach of Discrete-Time ..." *IEEE APEC*, 2021]

CONCLUSION

- Discrete-time large-signal modeling
- Formulation of Jacobian matrix
- Discrete-time small-signal modeling

THANK YOU !