# On-off Threshold Models of Social Contagion

(chaos, zombies, and hipsters)


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Mathematics


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COMPLEX SYSTEMS CENTER

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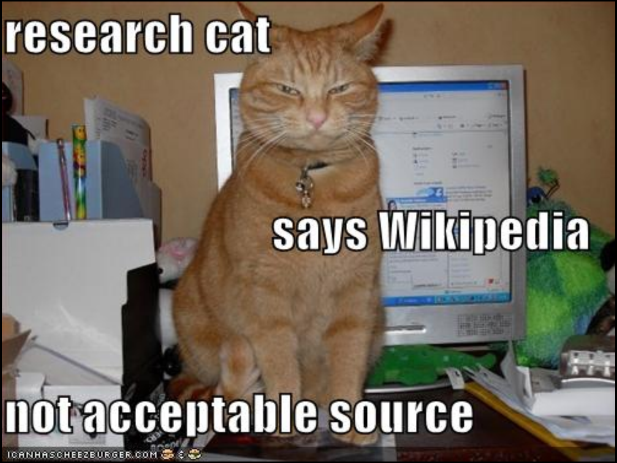
# Outline

You may think

I think


research cat
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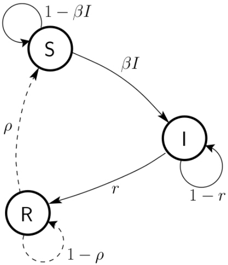
ICANHASCHEEZBURGER.COM

# Classical contagion models

**SIR**

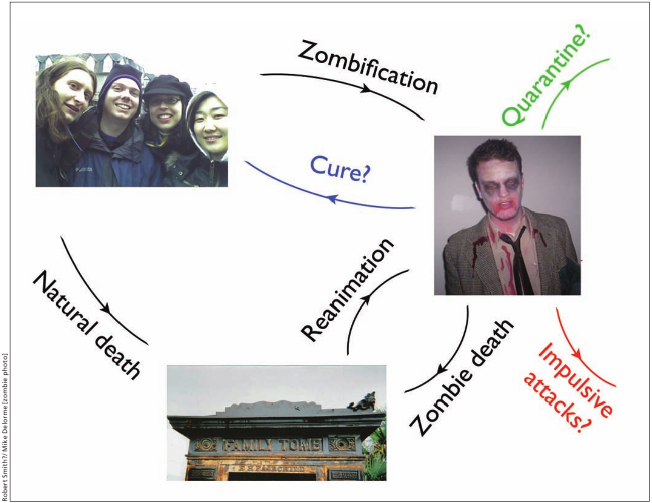
3 possible states:

- Susceptible
- Infected
- Removed

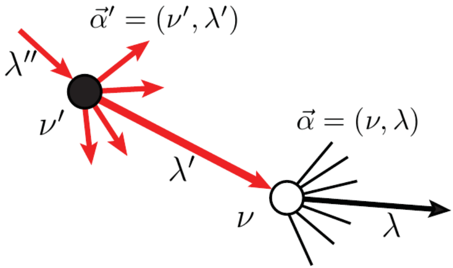



Simplest model (3 variable ODE)
Assumes well-mixed population

# Or SZR


Zombification
Quarantine?
Cure?
Natural death
Reanimation
Zombie death
Impulsive attacks?
FAMILY TOMB
Robert Smith?/ Mike Delorme [zombie photo]

# Networked contagion




## Contagion + Network

- Nodes have states (SIR, 0/1, ↑/↓, etc.)
- State transitions depend on neighbors' states
- Related to: percolation, magnetism, Boolean networks (biology)

# Networks: quick definitions

**A network (graph) is**

a collection of **nodes** and **links**, which connect the nodes.

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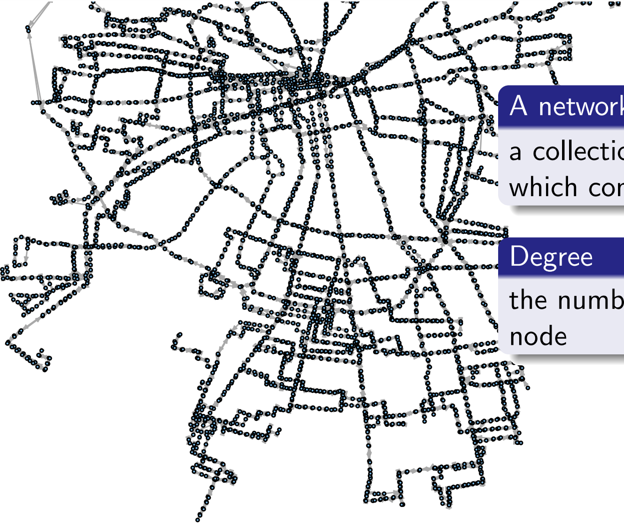

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the number of links incident a node

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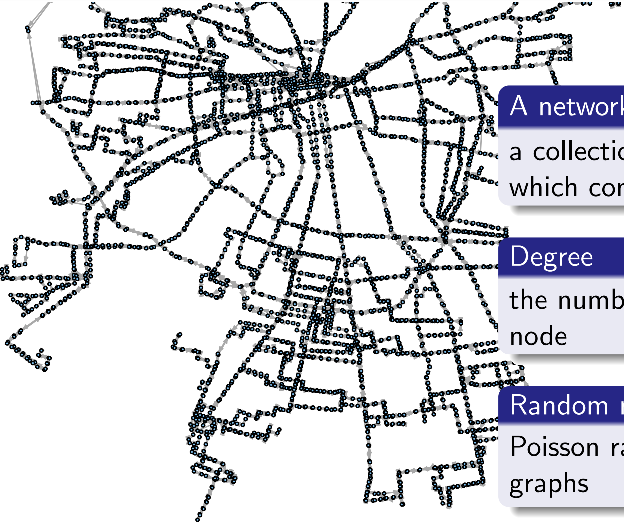

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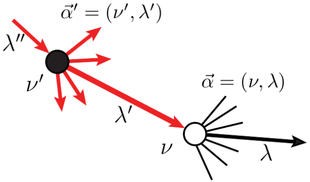
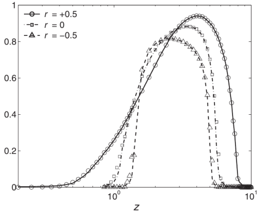
**Random networks**

Poisson random graphs, rewired graphs

# More facts about contagion models




## Typical features:

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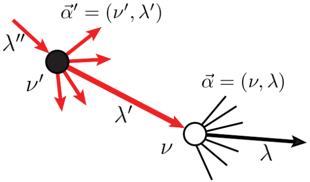


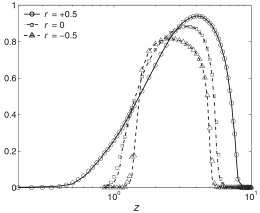


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- **Network structure** or fully mixed

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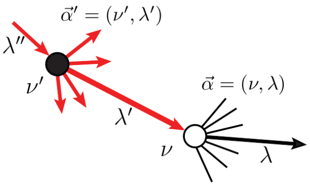


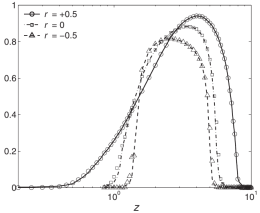


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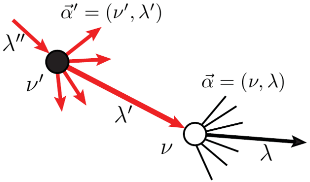


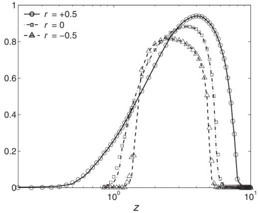


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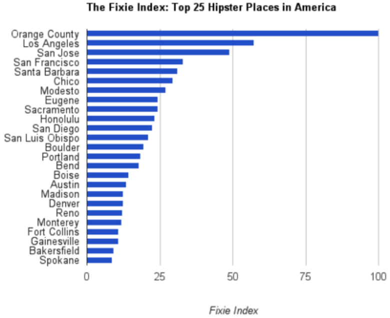
- **Network structure** or fully mixed
- Nodes turn on according to **response function** $f$ and stay on
- **Percolation transition**; varies with network parameters, contagiosity, or initial seed

# Other possible extensions




source: `http://blog.priceonomics.com/post/16013457968/the-fixie-bike-index`

- Social contagion $\rightarrow$ trends, fads, fashion (?)
- Volatility: not something we get from SIR

# Social contagion models

Classical:

- Simmel (1907) "Fashion" limited imitation, class
- Schelling (1971, 1973) segregation
- Granovetter (1978) threshold model
- Granovetter and Soong (1986) off-thresholds, chaos

With networks:

- Watts (2002) thresholds
- Dodds and Watts (2004) universal dynamics
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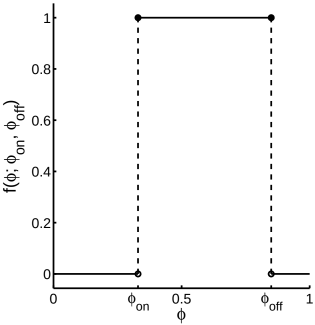
# Our ingredients

**1. Network structure**

The process takes place on a network

**2. Limited imitation**

node response function:



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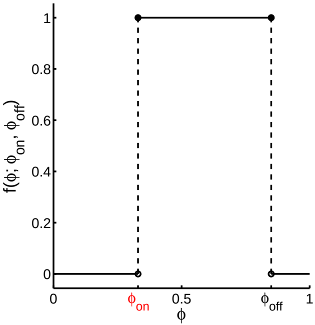

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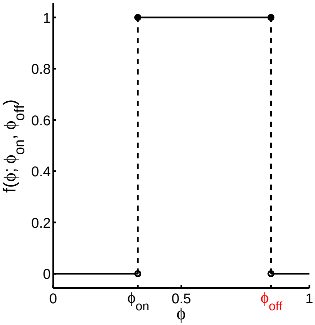

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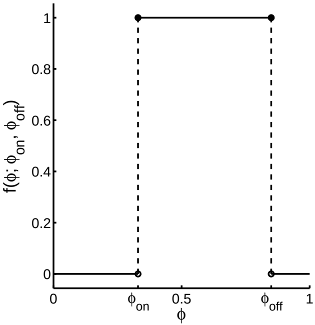
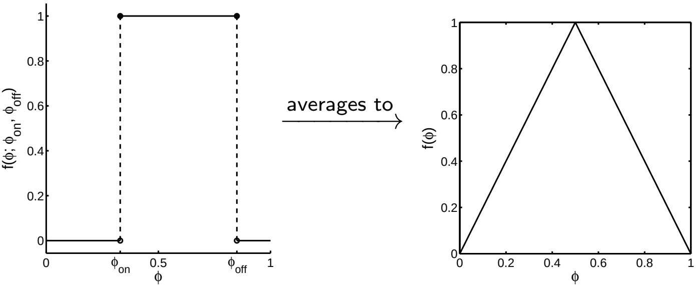


# More about response functions

Distribution of thresholds $P(\phi_{\text{on}}, \phi_{\text{off}})$ characterizes responses




deterministic

probabilistic

# Allowing for randomness

- **D**eterministic or **P**robabilistic node responses
- Similarly, the network can be **F**ixed or **R**ewired

| | **R**ewiring network | **F**ixed network |
|---|---|---|
| **P**robabilistic response | P-R | P-F |
| **D**eterministic response | D-R | D-F |

Also: update probability $\alpha$

# Outline

# Exact dynamics

For all model classes, the dynamics follow:

$$x_i(t+1) = \alpha f_i \left( \frac{\sum_{j=0}^{N} A_{ij} x_j(t)}{\sum_{j=0}^{N} A_{ij}} \right) + (1-\alpha) x_i(t) \tag{1}$$

- $x_i(t)$ = probability node $i$ is "on" at time $t$
- $f_i$ = node $i$ response function ($\phi_{\text{on},i}$ and $\phi_{\text{off},i}$)
- $A_{ij}$ = adjacency matrix (network)
- $\alpha$ = update probability

# Dense limit

For dense Poisson random graphs [$k_{\mathrm{avg}} = \Omega(\log N)$], the system is well-mixed and we recover the 1-d map dynamics:

$$\phi(t+1) = \alpha f(\phi(t)) + (1-\alpha)\phi(t) \tag{2}$$

- $\phi(t)$ = probability any node is "on" at time $t$
- $f = \sum_{i=0}^{N} f_i/N$ = the stochastic response function
- $\alpha$ = the update probability

# Mean field/random mixing

For P-R and D-R (rewired network), we can derive a 1-d map:

$$\rho(t+1) = \alpha\, g(\rho(t)|p_k, f) + (1-\alpha)\rho(t) \tag{3}$$

- $\rho(t)$ = probability that an edge is "on" at time $t$
- $p_k$ = degree distribution with $k_{\mathrm{avg}} = \sum_{k=0}^{\infty} k p_k$
- $f$ = stochastic response function
- $\alpha$ = update probability

# Analysis of mean field

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**Convergence result**

A more connected network implies dynamics closer to $f$.

# Outline

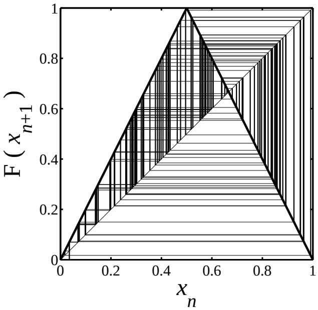
# A specific experiment

## Response functions

$\phi_{\text{on}} \sim U([0, 1/2])$
$\phi_{\text{off}} \sim U([1/2, 1])$
results in tent map $f$ (right)

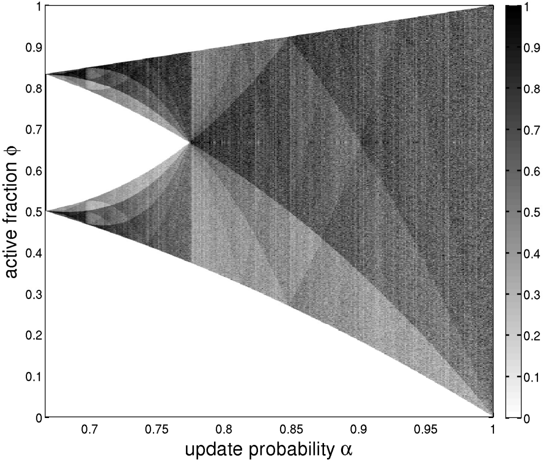
## Networks

Poisson random networks
$N$ large $(10^4, 10^5)$
$k_{\text{avg}}$ tunable parameter



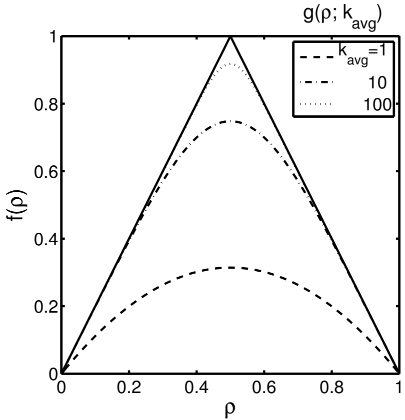
# Dense limit

1-d dense map exhibits chaos for $\alpha > 2/3$

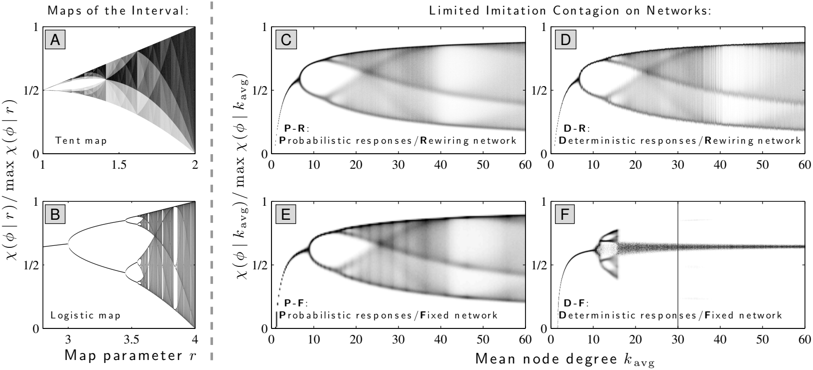


# Confirmation of convergence in mean-field

As expected, $g(\rho; k_{\rm avg}) \nearrow f(\rho)$ as $k_{\rm avg} \to \infty$:

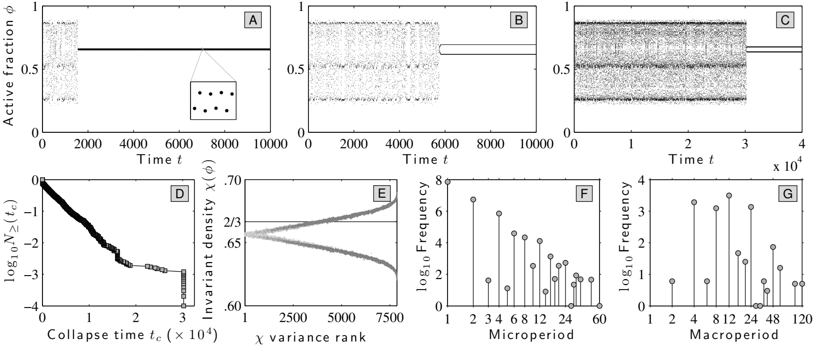


# Random designs show different (crazy) behavior

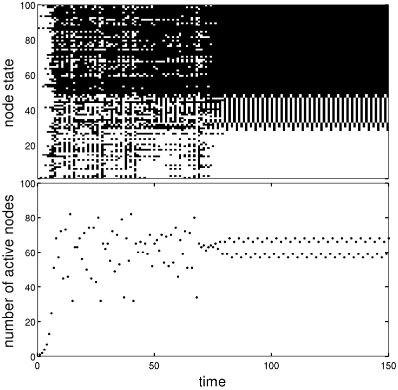


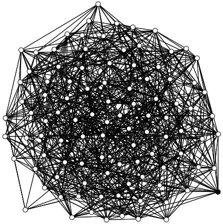

$(\alpha = 1)$

# Deterministic dynamics



# Deterministic dynamics (small network)

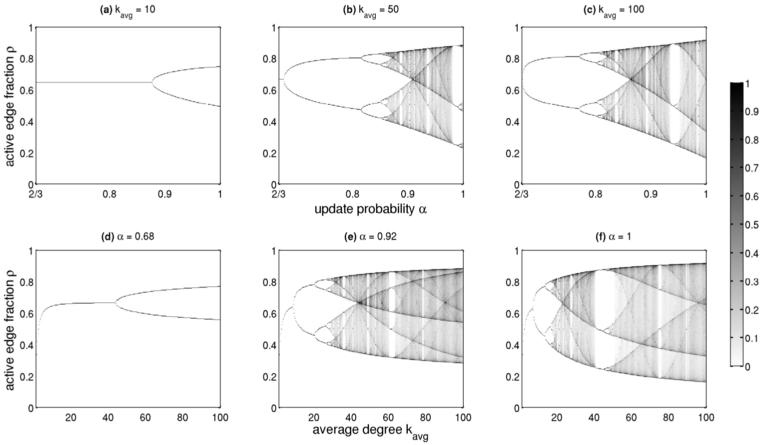


# Movies

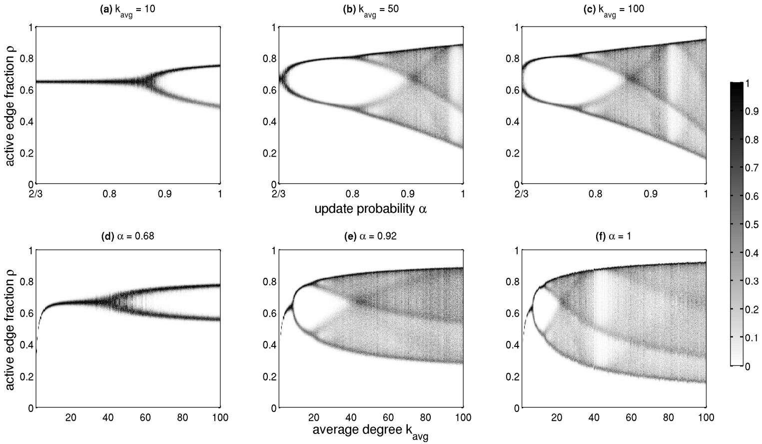
(Show network movies)

# Stochastic dynamics — theory




mean field equations

# Stochastic dynamics — simulations




fully stochastic (P-R)

# 3-d bifurcation structure

(Show movies, Paraview)

# Outline

# Conclusions about the on-off threshold model

- rich dynamical behavior across designs and parameters, results from aversion to conformity

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- rich dynamical behavior across designs and parameters, results from aversion to conformity
- network has “smoothing” effect
- mean field theory works (for P-R, D-R; somewhat for P-F)

# To-do

- D-F dynamics: collapses, individual nodes, transient vs. mean field, transient time, random matrices

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- observational data, experiments?

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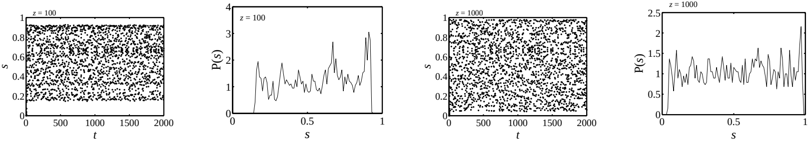
# Invariant densities—stochastic response functions




Trying out higher values of $\langle k \rangle$...

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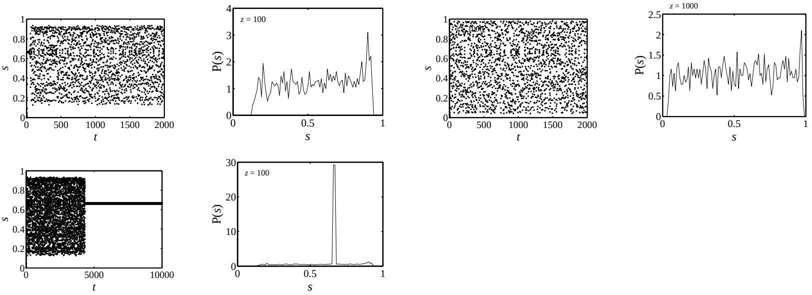


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