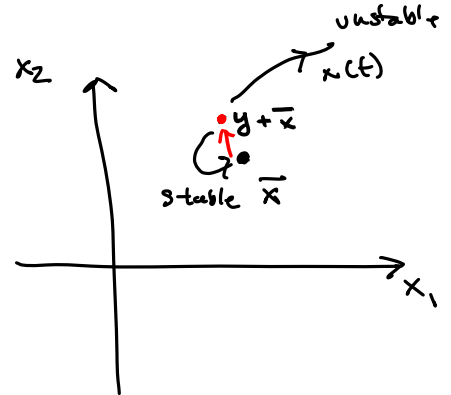


Linearization and stability of equilibria

Illustrate in 2 dimensions:

$$\text{Equilibrium } \bar{x} : f(\bar{x}) = 0$$

CONCEPT:



1) Perturb equilibrium: $\bar{x} \rightarrow \bar{x} + y$

2) Ask: what happens to system starting at $\bar{x} + y$? Does it RETURN to equilibrium \bar{x} ? If so, \bar{x} is stable.

OR, does solution trajectory "MOVE away"? Then \bar{x} is UNSTABLE.

A BIT MORE PRECISELY:

Say \bar{x} is STABLE if, for y near \bar{x} , $x(t) \rightarrow \bar{x}$ as $t \rightarrow \infty$

UNSTABLE if $x(t)$ ^{can} escape a neighborhood U of \bar{x} , no matter how near y is to \bar{x} .

STRATEGY: STUDY DYNAMICS OF SMALL PERTURBATION $y(t)$ AWAY FROM EQUILIB. \bar{x}

$$\frac{dx}{dt} = f(x) \quad ; \quad \text{consider trajectory } x(t) = \bar{x} + y(t) \quad y_i(t) \ll 1 : \text{trajectory is near } \bar{x}$$

$$\frac{d}{dt} (\bar{x} + y(t)) = f(\bar{x} + y(t))$$

$$\frac{d}{dt} \bar{x} + \frac{d}{dt} y(t) = \frac{d}{dt} y(t)$$

$$\rightarrow \boxed{\frac{d}{dt} y(t) = f(\bar{x} + y(t))}$$

Now, get simple expression for $f(\bar{x} + y(t))$

Consider small $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$; $y_1, y_2 \ll 1$

$$\text{In 2-D: } \underline{\underline{f(\bar{x} + y)}} = \begin{pmatrix} f_1(\bar{x}_1 + y_1, \bar{x}_2 + y_2) \\ f_2(\bar{x}_1 + y_1, \bar{x}_2 + y_2) \end{pmatrix} \approx .$$

$$\begin{pmatrix} f_1(\bar{x}_1, \bar{x}_2) \\ f_2(\bar{x}_1, \bar{x}_2) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1} y_1 + \frac{\partial f_1}{\partial x_2} y_2 \\ \frac{\partial f_2}{\partial x_1} y_1 + \frac{\partial f_2}{\partial x_2} y_2 \end{pmatrix} = \underline{\underline{f(\bar{x})}} + Df y, \text{ where}$$

$$Df(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\bar{x}_1, \bar{x}_2) & \frac{\partial f_1}{\partial x_2}(\bar{x}_1, \bar{x}_2) \\ \frac{\partial f_2}{\partial x_1}(\bar{x}_1, \bar{x}_2) & \frac{\partial f_2}{\partial x_2}(\bar{x}_1, \bar{x}_2) \end{bmatrix} \quad \text{JACOBIAN MATRIX.}$$

Since $f(\bar{x}) = 0$ by definition, we have :

$$\boxed{\frac{dy}{dt} = Df y}$$

↑
Fixed matrix

Solutions to this equation:

in terms of EIGENVALUES $\{\lambda_1, \lambda_2\}$ and EIGENVECTORS of $Df(\bar{x})$

Fact: if $Df(\bar{x})$ has two different e.v.s λ_1, λ_2 , write solution to (i) as:

$$y(t) = c_1 e^{\lambda_1 t} \underline{v}_1 + c_2 e^{\lambda_2 t} \underline{v}_2$$

To demonstrate this, just plug:

$$\begin{aligned} \dot{y} &= c_1 \exp(\lambda_1 t) \lambda_1 \underline{v}_1 + c_2 \exp(\lambda_2 t) \lambda_2 \underline{v}_2 \\ &= c_1 \exp(\lambda_1 t) Df(\bar{x}) \underline{v}_1 + c_2 \exp(\lambda_2 t) Df(\bar{x}) \underline{v}_2 \\ &= Df(\bar{x}) y \quad \checkmark \end{aligned}$$

CONCLUDE: (see Sec 5.3.3 of ETA) that

\bar{x} is stable if $\lambda_1 < 0$ and $\lambda_2 < 0$

\bar{x} is UNSTABLE if $\lambda_1 > 0$ or $\lambda_2 > 0$

Unclear if $\lambda_1 = 0$ or $\lambda_2 = 0$.

$$\frac{dx_1}{dt} = -x_1 + x_1^2 = f_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = -x_2 + x_2^2 = f_2(x_1, x_2)$$

$$\bar{x} = (0, 0)$$

$$J(0,0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} (0,0) =$$

$$= \begin{pmatrix} -1+2x_1 & 0 \\ 0 & -1+2x_2 \end{pmatrix} (0,0) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\text{Eigenvalues} = (-1, -1) \rightarrow \underline{\underline{\text{Stable}}}$$

