An International Journal
computers & mathematics with applications

PERGAMON Computers and Mathematics with Applications 39 (2000) 29–38
www.elsevier.nl/locate/camwa

de Rham Diagram for *hp* Finite Element Spaces

L. DEMKOWICZ, P. MONK, L. VARDAPETYAN AND W. RACHOWICZ
Texas Institute for Computational and Applied Mathematics
The University of Texas at Austin
Austin, TX 78712, U.S.A.

(*Received and accepted October 1999*)

Abstract—We prove that the *hp* finite elements for **H**(curl) spaces, introduced in [1], fit into a general de Rham diagram involving *hp* approximations. The corresponding interpolation operators generalize the notion of *hp* interpolation introduced in [2] and are different from the classical operators of Nedelec and Raviart-Thomas.

Keywords—*hp* finite elements, **H**(curl), **H**(div) spaces.

1. INTRODUCTION

The so-called de Rham diagram relating standard H^1-conforming elements with $\mathbf{H}(\text{curl})$-conforming elements of Nedelec and $\mathbf{H}(\text{div})$-conforming elements of Raviart-Thomas, is a standard tool in proving various properties of mixed finite element approximations; see, e.g., [3]. In particular, Bossavit [4] was the first to notice the connection between the good behavior of edge elements and the commuting properties of the diagram. The de Rham diagram relates the interpolation operators of Nedelec and Raviart-Thomas with differential operators ∇, $\nabla\times$, $\nabla\circ$. Unfortunately, these classical definitions *do not* generalize to elements with variable order of approximation. In this paper, we introduce a sequence of finite element (FE) spaces defined on a tetrahedron, cube or prism, and define corresponding interpolation operators in such a way that de Rham diagram commutes in exactly the same way as for elements of uniform order of approximation.

In the presentation, we restrict ourselves to purely algebraic aspects of the construction and delay discussion of approximation properties of the introduced operators (in terms of both h and p-extensions) to a future work.

2. THE TETRAHEDRAL ELEMENT

2.1. H^1-Conforming Element

Let K be the standard, master tetrahedron,

$$K = \left\{\mathbf{x} \in \mathbb{R}^3 : 0 < x_1 < 1,\, 0 < x_1 + x_2 < 1,\, 0 < x_1 + x_2 + x_3 < 1\right\} \tag{2.1}$$

The work has been supported by Air Force under Contract F49620-98-1-0255.

0898-1221/00/$ - see front matter Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX
PII: S0898-1221(00)00062-6

with faces $S = S_i$, $i = 1, \dots, 4$, edges $e = e_i$, $i = 1, \dots, 6$, and vertices $\mathbf{a} = \mathbf{a}_i$, $i = 1, \dots, 4$. The *hp* finite element space of shape functions V_{hp} is defined as a subspace of space of polynomials of order p whose restrictions to element faces S reduce to polynomials of possibly *lower* order p_S, and the corresponding restrictions over edges e to polynomials of order p_e,

$$V_{hp} = \{q \in \mathcal{P}^p : q|_S \in \mathcal{P}^{p_S}\, \forall S,\ q|_e \in \mathcal{P}^{p_e}\, \forall e\}. \tag{2.2}$$

We assume that

$$p_S \le p, \tag{2.3}$$

for every face S, and

$$p_e \le p_S, \tag{2.4}$$

for every edge e and face S adjacent to edge e. In practice, these conditions are satisfied by implementing the *minimum rule* when constructing FE meshes [5,6], i.e., assigning a specific order to each element, and then setting the order of approximation for element sides to the minimum order of the corresponding adjacent elements and, similarly, setting the order for each edge to the minimum order over all elements adjacent to the edge.

Let q be now a sufficiently regular function defined over tetrahedron K. The corresponding *hp*-interpolant $q_{hp} = \Pi_{hp} q$ is now defined as a function from the finite element space V_{hp} that satisfies the following conditions:

- q_{hp} interpolates q at the vertex nodes,

$$q_{hp}(\mathbf{a}) = q(\mathbf{a}), \qquad \text{for every vertex } \mathbf{a}, \tag{2.5}$$

- over each edge e, restriction of $q - q_{hp}$ is orthogonal in the H_0^1 scalar product to polynomials of order p_e, vanishing at the vertices,

$$\int_e \frac{\partial}{\partial s}(q - q_{hp}) \frac{\partial \phi}{\partial s}\, de = 0, \qquad \forall \phi \in \mathcal{P}_0^{p_e}, \tag{2.6}$$

- over each side S, restriction of $q - q_{hp}$ is orthogonal in the H_0^1 scalar product to polynomials of order p_S, vanishing along the boundary of the side,

$$\int_S \boldsymbol{\nabla}_S (q - q_{hp}) \boldsymbol{\nabla}_S \phi\, dS = 0, \qquad \forall \phi \in \mathcal{P}_0^{p_S}, \tag{2.7}$$

- over the whole element, difference $q - q_{hp}$ is orthogonal in the H_0^1 scalar product to polynomials of order p, vanishing over the element boundary,

$$\int_K \boldsymbol{\nabla} (q - q_{hp}) \boldsymbol{\nabla} \phi\, dK = 0, \qquad \forall \phi \in \mathcal{P}_0^{p}. \tag{2.8}$$

Determination of the interpolant can be viewed as a successive solution of approximate Dirichlet problems, first solved over edges, then over sides and, finally, over the whole element. In each of these steps, the solution from the previous step is used to implement Dirichlet boundary conditions.

In practice, the *hp* space is represented as a span of shape functions grouped into four categories:

- vertex nodes, linear, shape functions,
- mid-edge nodes shape functions that, for a particular edge e, span polynomials $\mathcal{P}_0^{p_e}$ vanishing at the end-points, and vanish over all other edges,
- mid-side nodes shape functions that, for a particular side S, span polynomials $\mathcal{P}_0^{p_S}$ vanishing along the boundary of the side, and vanish over all other sides,
- middle node shape functions that span polynomials $\mathcal{P}_0^p$ vanishing over the element boundary.

We refer to [5,6] and the literature therein for numerous examples of constructions of such functions.

The *hp*-interpolant is then determined as a sum of four contributions,

$$q_{hp} = q^v_{hp} + q^e_{hp} + q^s_{hp} + q^m_{hp}, \tag{2.9}$$

where

- q^v_{hp} is a linear function interpolating q at vertices a,
- over each edge e, q^e_{hp} is the H^1_0-projection of $q - q^v_{hp}$ onto the space of mid-edge node shape functions,
- over each side S, q^s_{hp} is the H^1_0-projection of $q - q^v_{hp} - q^e_{hp}$ onto the space of mid-side node shape functions,
- over the whole element, q^m_{hp} is the H^1_0-projection of $q - q^v_{hp} - q^e_{hp} - q^s_{hp}$ onto the space of middle node shape functions.

It is the use of particular shape functions that makes each of the contributions $q^v_{hp}, \dots, q^m_{hp}$ uniquely defined.

REMARK 1. The described construction holds for $p \geq 4$. For $p = 3$, the middle node contribution is missing, for $p = 2$, both mid-side and middle nodes contributions vanish and, finally, for $p = 1$, the approximation reduces to the standard linear element. ∎

2.2. H(curl)-Conforming Element

The *hp* finite element space of shape functions $\mathbf{W}_{hp}$ is defined as a subspace of space of vector-valued polynomials of order p such that the *tangential components* of their restrictions to element faces S reduce to polynomials of possibly *lower* order p_S, and the *tangential components* of their restrictions over edges e reduce to polynomials of order p_e,

$$\mathbf{W}_{hp} = \{\mathbf{E} \in \mathcal{P}^p : \mathbf{n} \times \mathbf{E}|_S \in \mathcal{P}^{p_S}\, \forall S,\ (\mathbf{E}|_e)_t \in \mathcal{P}^{p_e}\, \forall e\}. \tag{2.10}$$

Here, $\mathbf{n}$ stands for the outward normal unit vector, and $\mathbf{E}_t$ denotes the tangential component of field $\mathbf{E}$ along edge e. As previously, we assume that

$$p_S \leq p, \tag{2.11}$$

for every face S, and

$$p_e \leq p_S, \tag{2.12}$$

for every edge e and face S adjacent to edge e.

Let now $V_{h,p+1}$ be a FE space of scalar-valued shape functions defined in the previous subsection, i.e., we assume that each order of approximation p, p_S, p_e has been raised by one. It is easy to see that gradient operator ∇ maps space $V_{h,p+1}$ into the space $\mathbf{W}_{hp}$ just defined.[1]

We define now the *hp*-interpolation operator for space $\mathbf{H}(\text{curl})$. Let $\mathbf{E}$ be a sufficiently regular field defined over tetrahedron K. The corresponding *hp*-interpolant $\mathbf{E}_{hp} = \Pi^{\text{curl}}_{hp}\mathbf{E}$ is now defined as a function from the finite element space $\mathbf{W}_{hp}$ that satisfies the following conditions:

- over each edge e, the tangential component of restriction of $\mathbf{E} - \mathbf{E}_{hp}$ is orthogonal in the L^2 scalar product to polynomials of order p_e,

$$\int_e (\mathbf{E} - \mathbf{E}_{hp})_t\, \phi\, de = 0, \qquad \forall \phi \in \mathcal{P}^{p_e}, \tag{2.13}$$

[1]The compatibility condition: $q \in V_{h,p+1}$ iff $\nabla q \in \mathbf{W}_{hp}$ was the basis for the construction of the *hp* element for Maxwell's equations introduced in [1].

- over each side S, the tangential component of restriction $\mathbf{E} - \mathbf{E}_{hp}$ satisfies the following orthogonality conditions:

$$
\begin{aligned}
\int_S \mathrm{curl}_S\,(\mathbf{E} - \mathbf{E}_{hp})\,\mathrm{curl}_S\boldsymbol{\phi}\,dS &= 0, \qquad \forall\boldsymbol{\phi} \in \mathcal{P}_0^{ps}, \\
\int_S (\mathbf{E} - \mathbf{E}_{hp})\,\boldsymbol{\nabla}_S\phi\,dS &= 0, \qquad \forall\phi \in \mathcal{P}_0^{ps+1},
\end{aligned}
\tag{2.14}
$$

  where by $\mathcal{P}_0^{ps}$ we understand the space of vector-valued polynomials of order p whose tangential components vanish along the boundary ∂S, and the surface curl_S is defined as $\mathrm{curl}_S\mathbf{E} = \mathbf{n} \circ (\boldsymbol{\nabla} \times \mathbf{E})$;
- over the whole element K, $\mathbf{E} - \mathbf{E}_{hp}$ satisfies the following orthogonality conditions:

$$
\begin{aligned}
\int_K \boldsymbol{\nabla} \times (\mathbf{E} - \mathbf{E}_{hp})\,\boldsymbol{\nabla} \times \boldsymbol{\phi}\,dK &= 0, \qquad \forall\boldsymbol{\phi} \in \mathcal{P}_0^{p}, \\
\int_K (\mathbf{E} - \mathbf{E}_{hp})\,\boldsymbol{\nabla}\phi\,dK &= 0, \qquad \forall\phi \in \mathcal{P}_0^{p+1},
\end{aligned}
\tag{2.15}
$$

  where, again, by $\mathcal{P}_0^{p}$ we understand the space of vector-valued polynomials of order p whose tangential components vanish along the whole boundary ∂K.

Similarly as in the scalar case, the determination of the interpolant can be viewed as a successive solution of approximate Dirichlet problems for zero-frequency Maxwell's equations, first solved over edges, then over sides and, finally, over the whole element. In each of these steps, the solution from the previous step is used to implement the Dirichlet boundary conditions.

REMARK 2. Problems (2.14) and (2.15) are *not* overdetermined. For instance, problem (2.14) is equivalent to

$$
\begin{aligned}
\int_S \mathrm{curl}_S\,(\mathbf{E} - \mathbf{E}_{hp})\,\mathrm{curl}_S\boldsymbol{\phi}\,dS + \int_S \boldsymbol{\nabla}p\boldsymbol{\phi}\,dS &= 0, \qquad \forall\boldsymbol{\phi} \in \mathcal{P}_0^{ps}, \\
\int_S (\mathbf{E} - \mathbf{E}_{hp})\,\boldsymbol{\nabla}_S\phi\,dS &= 0, \qquad \forall\phi \in \mathcal{P}_0^{ps+1},
\end{aligned}
\tag{2.16}
$$

where $p \in \mathcal{P}_0^{ps+1}$ is a Lagrange multiplier. Substituting in the first equation $\boldsymbol{\phi} = \boldsymbol{\nabla}q$, $q \in \mathcal{P}_0^{ps+1}$, we see quickly that multiplier p is equal to zero. Consequently, the two problems are equivalent to each other. ∎

In practice, the *hp* space is represented as a span of vector-valued shape functions grouped into three categories:

- mid-edge nodes shape functions whose tangential components, for a particular edge e, span polynomials $\mathcal{P}_0^{p_e}$, and vanish over all other edges,
- mid-side nodes shape functions that, for a particular side S, span polynomials $\mathcal{P}_0^{ps}$, and vanish over all other sides,
- middle node shape functions that span polynomials $\mathcal{P}_0^{p}$.

We refer to [7,8] for examples of constructions of such functions.

The *hp*-interpolant is then determined as a sum of three contributions,

$$
\mathbf{E}_{hp} = \mathbf{E}_{hp}^{e} + \mathbf{E}_{hp}^{s} + \mathbf{E}_{hp}^{m}, \tag{2.17}
$$

where

- over each edge e, $(\mathbf{E}_{hp}^{e})_t$ is the L^2-projection of $\mathbf{E}_t$ onto the space of (tangential traces of) mid-edge node shape functions,
- over each side S, $\mathbf{E}_{hp}^{S}$ is the $\mathbf{H}$(curl)-projection of $\mathbf{E} - \mathbf{E}_{hp}^{e}$ onto the space of mid-side node shape functions, with the (weak) constraint on the divergence,
- over the whole element, $\mathbf{E}_{hp}^{m}$ is the $\mathbf{H}$(curl)-projection of $\mathbf{E} - \mathbf{E}_{hp}^{e} - \mathbf{E}_{hp}^{S}$ onto the space of middle node shape functions, with the (weak) constraint on the divergence.

It is again the use of particular shape functions that makes each of the contributions $\mathbf{E}^e_{hp}$, $\mathbf{E}^S_{hp}$, $\mathbf{E}^m_{hp}$ uniquely defined.

REMARK 3. The described construction holds for $p \geq 3$, with middle node contributions missing for $p = 2$ and the entire approximation reducing to just the mid-edge contributions for $p = 1$. It is also possible to extend the definition to the case of $p = 0$ which leads to the lowest order Nedelec's elements of the second type [9]. The element space of shape functions is then spanned by just six edge shape functions whose tangential components are constant along one particular edge, vanish along the remaining ones, and extend linearly in between. The space does not contain complete linear polynomials. ∎

2.3. H(div)-Conforming Element

Things get simpler now. The *hp* finite element space of shape functions $\mathbf{X}_{hp}$ is defined as a subspace of space of vector-valued polynomials of order p such that the *normal components* of their restrictions to element faces S reduce to polynomials of possibly *lower* order p_S,

$$\mathbf{X}_{hp} = \{\mathbf{u} \in \mathcal{P}^p : \mathbf{n} \circ \mathbf{u}|_S \in \mathcal{P}^{p_S}\ \forall S\}. \tag{2.18}$$

Here $\mathbf{n}$ stands for the outward normal unit vector, and we again assume that

$$p_S \leq p, \tag{2.19}$$

for every face S.

Let now $\mathbf{W}_{h,p+1}$ be an FE space of vector-valued shape functions defined in the previous section, i.e., we assume that each order of approximation p, p_S, p_e has been raised by one. It is easy to see that curl operator $\nabla\times$ maps space $\mathbf{W}_{h,p+1}$ into the space $\mathbf{X}_{hp}$ just defined.

We define now the *hp*-interpolation operator for space $\mathbf{H}(\text{div})$. Let $\mathbf{u}$ be a sufficiently regular field defined over tetrahedron K. The corresponding *hp*-interpolant $\mathbf{u}_{hp} = \Pi^{\text{div}}_{hp}\mathbf{u}$ is now defined as a function from the finite element space $\mathbf{X}_{hp}$ that satisfies the following conditions:

- over each side S, the normal component of restriction $\mathbf{u}-\mathbf{u}_{hp}$ satisfies the L^2 orthogonality condition
$$\int_S \mathbf{n} \circ (\mathbf{u} - \mathbf{u}_{hp})\,\phi\, dS = 0, \qquad \forall\phi \in \mathcal{P}^{p_S}, \tag{2.20}$$
- over the whole element K, $\mathbf{u} - \mathbf{u}_{hp}$ satisfies the following orthogonality conditions:
$$\begin{aligned} \int_K \nabla \circ (\mathbf{E} - \mathbf{E}_{hp})\,\nabla \circ \phi\, dS = 0, &\qquad \forall\phi \in \mathcal{P}^p_0, \\ \int_K (\mathbf{E} - \mathbf{E}_{hp})\,\nabla \times \phi = 0, &\qquad \forall\phi \in \mathcal{P}^{p+1}_0, \end{aligned} \tag{2.21}$$
where by $\mathcal{P}^p_0$ we understand the space of vector-valued polynomials of order p whose normal components vanish along the boundary ∂K, and by $\mathcal{P}^{p+1}_0$ we understand the space of vector-valued polynomials of order $p+1$ whose tangential components vanish along the boundary ∂K.

Similarly as in the previous cases, the determination of the interpolant can be viewed as a successive solution of approximate Dirichlet problems for zero-frequency acoustics equations, first solved over sides and then over the whole element. The solution from the first step is used to implement Dirichlet boundary conditions for the second one.

In practice, the *hp* space is represented as a span of vector-valued shape functions grouped into two categories:

- mid-side nodes shape functions such that their normal components, for a particular side S, span polynomials $\mathcal{P}^{p_S}$, and vanish over all other sides,
- middle node shape functions that span polynomials $\mathcal{P}^p_0$.

The hp-interpolant is then determined as a sum of two contributions,

$$\mathbf{u}_{hp} = \mathbf{u}_{hp}^S + \mathbf{u}_{hp}^m, \tag{2.22}$$

where

- over each side S, $\mathbf{n} \circ \mathbf{u}_{hp}^S$ is the L^2-projection of $\mathbf{n} \circ \mathbf{u}$ onto the space of mid-side node shape functions,
- over the whole element, $\mathbf{u}_{hp}^m$ is the $\mathbf{H}(\mathrm{div})$-projection of $\mathbf{u} - \mathbf{u}_{hp}^S$ onto the space of middle node shape functions, with the (weak) constraint on the curl.

It is again the use of particular shape functions that makes each of the contributions $\mathbf{u}_{hp}^S$, $\mathbf{u}_{hp}^m$ uniquely defined.

REMARK 4. For $p = 1$, the middle node contribution is missing. Similarly, as for the $\mathbf{H}(\mathrm{curl})$-conforming elements, the definition can be extended to the case $p = 0$ in which case it reduces to the classical Raviart-Thomas element. The element space of shape functions is then the span of just four vector-valued shape functions, each of which has a constant normal component over one particular side, vanishes over the remaining ones, and extends linearly in between. ∎

3. de RHAM DIAGRAM

We recall the de Rham diagram

$$H^1 \xrightarrow{\nabla} \mathbf{H}(\mathrm{curl}) \xrightarrow{\nabla\times} \mathbf{H}(\mathrm{div}) \xrightarrow{\nabla\circ} L^2. \tag{3.1}$$

In the diagram, *the range of each of the operators coincides with the null space of the next operator in the sequence, and the last map is a surjection.* The diagram can be restricted to functions satisfying the homogeneous Dirichlet boundary conditions

$$H_0^1 \xrightarrow{\nabla} \mathbf{H}_0(\mathrm{curl}) \xrightarrow{\nabla\times} \mathbf{H}_0(\mathrm{div}) \xrightarrow{\nabla\circ} L_0^2. \tag{3.2}$$

Here, by H_0^1 we understand the subspace of H^1 consisting of functions vanishing on the boundary, $\mathbf{H}_0(\mathrm{curl})$ contains functions whose tangential components vanish over the boundary and $\mathbf{H}_0(\mathrm{div})$ includes functions whose normal components vanish on the boundary and, finally, L_0^2 stands for L^2-functions with zero average.

We have shown in the previous section that the diagram can be defined on the discrete level using the hp spaces V_{hp}, $\mathbf{W}_{h,p-1}$, $\mathbf{X}_{h,p-2}$, $Y_{h,p-3}$, where space Y_{hp} simply coincides with polynomials of order $p - 3$.

$$V_{hp} \xrightarrow{\nabla} \mathbf{W}_{h,p-1} \xrightarrow{\nabla\times} \mathbf{X}_{h,p-2} \xrightarrow{\nabla\circ} Y_{h,p-3}. \tag{3.3}$$

Diagram (3.3) holds for any $p \geq 3$, i.e., the first space in the sequence starts with at least third-order polynomials. For $p = 2$ or $p = 1$, the sequence involves the zero-order spaces,

$$V_{h2} \xrightarrow{\nabla} \mathbf{W}_{h1} \xrightarrow{\nabla\times} \mathbf{X}_{h0} \xrightarrow{\nabla\circ} Y_{h0} = \mathbb{R} \tag{3.4}$$

and

$$V_{h1} \xrightarrow{\nabla} \mathbf{W}_{h0} \xrightarrow{\nabla\times} \mathbf{X}_{h0} \xrightarrow{\nabla\circ} Y_{h0} = \mathbb{R}. \tag{3.5}$$

REMARK 5. Equivalently, we could use the notation

$$V_{hp} \xrightarrow{\nabla} \mathbf{W}_{hq} \xrightarrow{\nabla\times} \mathbf{X}_{hr} \xrightarrow{\nabla\circ} Y_{hs}, \tag{3.6}$$

where

$$q = \max\{p-1, 0\}, \qquad r = \max\{q-1, 0\}, \qquad s = \max\{r-1, 0\}. \tag{3.7}$$

We emphasize that relations (3.7) hold for the middle, face, and the edge nodes orders of approximation, i.e.,

$$\begin{aligned} &q = \max\{p-1,0\}, \qquad q_S = \max\{p_S - 1, 0\}, \qquad q_e = \max\{p_e - 1, 0\}, \\ &r = \max\{q-1,0\}, \qquad r_S = \max\{q_S - 1, 0\}, \\ &s = \max\{r-1,0\}. \end{aligned} \tag{3.8 ▮}$$

As at the continuous level, we can restrict ourselves to functions satisfying homogeneous boundary conditions

$$V^0_{hp} \xrightarrow{\nabla} \mathbf{W}^0_{h,p-1} \xrightarrow{\nabla\times} \mathbf{X}^0_{h,p-2} \xrightarrow{\nabla\circ} Y^0_{h,p-3}. \tag{3.9}$$

Finally, we will need a 2D equivalent of the last diagram,

$$V^0_{hp} \xrightarrow{\nabla} \mathbf{W}^0_{h,p-1} \xrightarrow{\nabla\times} Y^0_{h,p-2}. \tag{3.10}$$

Obviously, for low p, diagrams (3.9) and (3.10) may terminate with trivial spaces.

REMARK 6. In all the diagrams, the last operators in the sequence are surjective. For instance, let $f \in L^2$ such that $\int f = 0$. We look for $\mathbf{u} \in \mathbf{H}_0(\mathrm{div})$ in the form of

$$\mathbf{u} = \nabla \psi. \tag{3.11}$$

It follows then that ψ must be the solution of the Neumann problem

$$\begin{aligned} &\psi \in H^1, \\ \int_K \nabla\psi \nabla\phi \, dK = \int_K f\phi \, dK, \qquad &\forall \phi \in H^1. \end{aligned} \tag{3.12}$$

Due to the fact that f has a zero average, the problem has a solution. On the discrete level, the proof reduces to lengthy but completely elementary calculation of the dimension of the spaces involved. ▮

We can now formulate our main result.

THEOREM 1. *The diagram shown below commutes.* ▮

In diagram (3.13), the last interpolation operator, P_{hp}, completing the diagram, is simply the L^2-projection acting from space L^2 onto the space $Y_{hp} = \mathcal{P}^p$.

$$\begin{array}{ccccccc} H^1 & \xrightarrow{\nabla} & \mathbf{H}(\mathrm{curl}) & \xrightarrow{\nabla\times} & \mathbf{H}(\mathrm{div}) & \xrightarrow{\nabla\circ} & L^2 \\ \downarrow \Pi_{hp} & & \downarrow \Pi^{\mathrm{curl}}_{hp} & & \downarrow \Pi^{\mathrm{div}}_{hp} & & \downarrow P_{hp} \\ V_{hp} & \xrightarrow{\nabla} & \mathbf{W}_{h,p-1} & \xrightarrow{\nabla\times} & \mathbf{X}_{h,p-2} & \xrightarrow{\nabla\circ} & Y_{h,p-3} \end{array} \tag{3.13}$$

PROOF.

STEP 1. Let $\mathbf{E} = \nabla q$ and let $\mathbf{E}_{hp}$ and $q_{h,p+1}$ denote the corresponding hp-interpolants of functions $\mathbf{E}$ and q, respectively. We need to show that $\mathbf{E}_{hp} = \nabla(q_{h,p+1})$.

- It follows from the second step of definition of interpolation operator Π_{hp} that

$$\int_e \frac{\partial}{\partial s}(q - q_{h,p+1})\,\phi \, de = 0, \tag{3.14}$$

for all polynomials ϕ of order p with a zero average over the edge. At the same time, for ϕ =const,

$$\int_e \frac{\partial}{\partial s}(q - q_{h,p+1})\,\phi \, de = (q - q_{h,p+1})\,\phi|^{\mathbf{v}_2}_{\mathbf{v}_1} = 0, \tag{3.15}$$

due to the fact that $q_{h,p+1}$ coincides with q at the endpoint vertices $\mathbf{v}_1$, $\mathbf{v}_2$.

- We show now that $\mathbf{n} \times \mathbf{E}_h$ coincides with $\nabla_S q_h$. We know already that $\mathbf{E}_h$ and ∇q_h have the same tangential components along the edges. Thus, over a side S, tangential component $\mathbf{n} \times \mathbf{E}_h$ can be represented as a sum of the tangential component of ∇q_h and a contribution $\mathbf{E}_h^0$ from the space of mid-side node shape functions $\mathcal{P}_0^p$. It follows then from the first orthogonality condition that the $\mathrm{curl}_S \mathbf{E}_h = 0$ and, therefore, the tangential component $\mathbf{n} \times \mathbf{E}_h$ must coincide with the surface gradient of a function from $V_{h,p+1}$. The constraint on the surface divergence coincides then exactly with the definition of operator Π_{hp} and, therefore, the tangential component of $\mathbf{E}_{h,p}$ must coincide with the surface gradient $\nabla_S q_{h,p+1}$.
- We proceed in the same way as in the previous step. We already know that the tangential components of $\mathbf{E}_{hp}$ coincide with the gradient on the boundary. Thus, $\mathbf{E}_{hp}$ must be a sum of the gradient and some contribution from the span of the middle node shape functions $\mathcal{P}_0^p$. The first orthogonality condition assures again that the entire field $\mathbf{E}_h$ must be equal to a gradient, and by the second condition we see that this must be precisely ∇q_h.

STEP 2. Let $\mathbf{u} = \nabla \times \mathbf{E}$ and let $\mathbf{u}_{hp}$ and $\mathbf{E}_{h,p+1}$ denote the corresponding hp-interpolants of functions $\mathbf{u}$ and $\mathbf{E}$, respectively. We need to show that $\mathbf{u}_{hp} = \nabla \times (\mathbf{E}_{h,p+1})$.

- Let $u_n = \mathbf{n} \circ (\nabla \times \mathbf{E}) = \mathrm{curl}_S \mathbf{E}$. In order to conclude that the normal component of u_{hp} coincides with $\mathrm{curl}_S \mathbf{E}_h$, we need to show first the orthogonality condition

$$\int_S \mathrm{curl}_S \left(\mathbf{E} - \mathbf{E}_{h,p+1}\right) \phi \, dS = 0 \tag{3.16}$$

for any test function ϕ from the space of polynomials of order $\mathcal{P}^{p_S}$. But given such a test function ϕ, we can always decompose it into two parts, a constant c, and a second contribution ϕ_0 with zero average. According to diagram (3.10), function ϕ_0 can be identified then with the surface curl of a corresponding mid-side node shape function (space $\mathbf{W}_{h,p+1}^0$). Condition (2.20) follows then directly from condition $(2.14)_1$. Orthogonality with the constants can be proved by integrating by parts and using the edge orthogonality condition (2.13)

$$\int_S \mathrm{curl}_S \left(\mathbf{E} - \mathbf{E}_{h,p+1}\right) c \, dS = \int_{\partial S} \left(\mathbf{E} - \mathbf{E}_{h,p+1}\right)_t c \, ds = 0. \tag{3.17}$$

- We know already that normal component of u_{hp} coincides with $\mathrm{curl}_S \mathbf{E}_h$ on each side S. Consequently, the first orthogonality condition implies that u_{hp} is divergence-free and, therefore, must coincide with the curl of a function from $\mathbf{W}_{h,p+1}$. But the second orthogonality condition assures that this must be $\nabla \times \mathbf{E}_h$.

STEP 3. Let $q = \nabla \circ \mathbf{u}$ and let q_{hp} and $\mathbf{u}_{h,p+1}$ denote the corresponding L^2-projection of function q and the hp-interpolant of function $\mathbf{u}$, respectively. We need to show that $q_{hp} = \nabla \circ (\mathbf{u}_{h,p+1})$.

In view of the fact that every scalar-valued polynomial with zero-average can be identified as the divergence of a vector-valued function with normal components vanishing on the boundary, compare diagram (3.9), it is sufficient to verify orthogonality condition $(2.21)_1$ for a constant function. But this follows from integration by parts and the side orthogonality condition for interpolant $\mathbf{u}_{hp}$,

$$\int_K \nabla \circ \left(\mathbf{u} - \mathbf{u}_{hp}\right) dK = \int_{\partial K} \mathbf{n} \circ \left(\mathbf{u} - \mathbf{u}_{hp}\right) dS = 0. \qquad 3.18 \ \blacksquare$$

4. GENERALIZATIONS

4.1. Hexahedral and Prismatic Elements

All the presented results generalize in a straightforward manner to the hexahedral and the prismatic element. For definiteness, we define here only the corresponding spaces. Denoting by K the master cube $K = (0,1)^3$, we introduce the following spaces.

- The space of scalar-valued shape functions for the H^1-conforming approximation, consisting of polynomials from space $\mathcal{Q}^p = \mathcal{Q}^{(p^1,p^2,p^3)}$, i.e., of order p^i with respect to x_i such that their restrictions to sides S reduce to polynomials of (possible) lower order $p_S = (p_S^1, p_S^2)$, and their restrictions to edges e reduce to polynomials of (possible) lower order p_e,

$$V_{hp} = \{q \in \mathcal{Q}^p : q|_S \in \mathcal{Q}^{p_S}, \forall S, \ q|_e \in \mathcal{P}^{p_e} \forall e\} . \tag{4.1}$$

  As for the tetrahedral element, we request that orders p_S^i do not exceed the corresponding global orders p^j in the parallel direction, and that the edge orders p_e are not bigger than the corresponding orders p_S^i for adjacent sides in the direction of the edge. The conditions are easily satisfied using again the *minimum rule*.
- The definition of the space of vector-valued shape functions for the $\mathbf{H}$(curl)-conforming approximation is slightly more complicated as different components come from different spaces (see the discussion in [7,8]). Given an order $p = (p^1, p^2, p^3)$, we have the E_1-component coming from space $\mathcal{Q}^{(p^1,p^2+1,p^3+1)}$, the E_2-component coming from space $\mathcal{Q}^{(p^1+1,p^2,p^3+1)}$, and the E_3-component coming from space $\mathcal{Q}^{(p^1+1,p^2+1,p^3)}$. This corresponds to the fact that differentiation of polynomials of the underlying scalar space $V_{h,p+1}$ in direction x_i lowers the order of approximation in that direction only. Restrictions of tangential components of functions $\mathbf{E}$ to faces S reduce to polynomials from space $\mathcal{Q}^{(p_S^1,p_S^2+1)} \times \mathcal{Q}^{(p_S^1+1,p_S^2)}$, and restrictions of tangential components to edges reduce to polynomials of order p_e,

$$\begin{aligned} \mathbf{W}_{hp} = \Big\{ &(E_1, E_2, E_3) \in \mathcal{Q}^{(p^1,p^2+1,p^3+1)} \times \mathcal{Q}^{(p^1+1,p^2,p^3+1)} \times \mathcal{Q}^{(p^1+1,p^2+1,p^3)} \\ &\text{such that } \mathbf{n} \times \mathbf{E}_S \in \mathcal{Q}^{(p_S^1,p_S^2+1)} \times \mathcal{Q}^{(p_S^1+1,p_S^2)} \forall S, E_t|_e \in \mathcal{P}^{p_e} \forall e \Big\} . \end{aligned} \tag{4.2}$$

  We mention that the approximation generalizes Nedelec's hexahedra of the first kind [10].
- The definition of the $\mathbf{H}$(div)-conforming element is a little bit simpler. The element space of shape functions $\mathbf{X}_{h,p}$ consists of polynomials from space $\mathcal{Q}^{(p^1+1,p^2,p^3)} \times \mathcal{Q}^{(p^1,p^2+1,p^3)} \times \mathcal{Q}^{(p^1,p^2,p^3+1)}$, such that the normal components of their restrictions to element faces S reduce to polynomials of order p_S,

$$\begin{aligned} \mathbf{X}_{hp} = \Big\{ &(u_1, u_2, u_3) \in \mathcal{Q}^{(p^1+1,p^2,p^3)} \times \mathcal{Q}^{(p^1,p^2+1,p^3)} \times \mathcal{Q}^{(p^1,p^2,p^3+1)} \\ &\text{such that } \mathbf{n} \circ \mathbf{u}_S \in \mathcal{Q}^{(p_S^1,p_S^2)} \forall S \Big\} . \end{aligned} \tag{4.3}$$

- Finally, space Y_{hp} coincides simply with space $\mathcal{Q}^{(p^1,p^2,p^3)}$.

The *hp*-interpolations operators are defined in a completely analogous way as for the tetrahedral element and the proof of the fact that de Rham diagram commutes is done in exactly the same way as for the tetrahedral element.

Definitions of *hp* spaces for the prismatic element

$$K = \{(x_1, x_2, x_3) : 0 < x_1 + x_2 < 1, \ 0 < x_3 < 1\} \tag{4.4}$$

are completely analogous and we leave them to the reader.

4.2. Curved Elements

All described properties generalize in the usual way to curved elements provided the transformations are defined in a way consistent with the transformation rules for gradients, curls, and the divergence, comp. [7,8,11]. If a curved element is the image of the corresponding master element through transformation $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$, the corresponding spaces consist of functions defined as follows:

$$\begin{aligned} q(\mathbf{x}) &= \hat{q}(\boldsymbol{\xi}), \\ E^i(\mathbf{x}) &= \hat{E}^j(\boldsymbol{\xi})\frac{\partial \xi^i}{\partial x_j}, \\ u^i(\mathbf{x}) &= \hat{u}^n(\boldsymbol{\xi})\frac{\partial x^i}{\partial \xi_n}J^{-1}, \\ \psi(\mathbf{x}) &= \hat{\psi}(\boldsymbol{\xi})J^{-1}, \end{aligned} \tag{4.5}$$

where $\mathbf{x} = \mathbf{x}(\boldsymbol{\xi})$ and J^{-1} is the inverse Jacobian of the transformation.

Obviously, the generalizations will work for meshes composed of such elements as well.

REFERENCES

1. L. Demkowicz and L. Vardapetyan, Modeling of electromagnetic absorption/scattering problems using *hp*-adaptive finite elements, *Computer Methods in Applied Mechanics and Engineering* **152** (1/2), 103–124, (1998).
2. J.T. Oden, L. Demkowicz, W. Rachowicz and T.A. Westermann, Towards a universal *h-p* adaptive finite element strategy, Part 2. *A posteriori* error estimation, *Comp. Meth. Appl. Meth. Eng.* **77**, 113–180, (1989).
3. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, Berlin, (1991).
4. A. Bossavit, Un nouveau point de vue sur les éléments finis mixtes, *Matapli (Bulletin de la Société de Mathématiques Appliquëes et Industrielles)*, 23–35, (1989).
5. L. Demkowicz, K. Gerdes, C. Schwab, A. Bajer and T. Walsh, HP90: A general and flexible Fortran 90 *hp*-FE code, *Computing and Visualization in Science* 1, 145–163, (1998).
6. L. Demkowicz, T. Walsh, K. Gerdes and A. Bajer, 2D *hp*-adaptive finite element package. Fortran 90 implementation (2Dhp90), TICAM Report 98-14, The University of Texas at Austin, Austin, TX 78712.
7. W. Rachowicz and L. Demkowicz, A two-dimensional *hp*-adaptive finite element package for electromagnetics, TICAM Report 98-15, (July 1998); *Computer Methods in Applied Mechanics and Engineering* (to appear).
8. W. Rachowicz and L. Demkowicz, A three-dimensional *hp*-adaptive finite element package for electromagnetics, (in preparation).
9. J.C. Nedelec, A new family of mixed finite elements in $\mathbb{R}^3$, *Numerische Mathematik* **50**, 57–81, (1986).
10. J.C. Nedelec, Mixed finite elements in $\mathbb{R}^3$, *Numerische Mathematik* **35**, 315–341, (1980).
11. L. Vardapetyan and L. Demkowicz, *hp*-adaptive finite elements in electromagnetics, *Computers Methods in Applied Mechanics and Engineering* **169**, 331–344, (1999).
12. P.A. Raviart and J.M. Thomas, Primal hybrid finite element methods for 2^{nd} order elliptic equations, *Mathematics of Computation* **31** (138), 391–413, (1977).