This can be also expressed as

$$\dot{\mu} = \mu \frac{J_1(z)}{zJ_1'(z)} = \mu \frac{1}{\frac{zJ_0(z)}{J_1(z)} - 1} = \cfrac{\mu}{1 - \cfrac{z^2}{4 - \cfrac{z^2}{6 - \cfrac{z^2}{\ddots}}}} \tag{4.27}$$

$$\mathrm{j}\omega\dot{\mu} = \left[0; \tfrac{1}{\mathrm{j}\omega\mu}, \tfrac{4}{\sigma a^2}, \tfrac{6}{\mathrm{j}\omega\mu}, \tfrac{8}{\sigma a^2}, \cdots\right] \tag{4.28}$$

where $z = ka$. The derivation using the equivalent dipole moment is given in [53]. In Fig. 4.6, the profile of the complex permeability is plotted as the frequency where $\sigma = 1.0^7$ Sm^{-1}, $\mu_r = 1.0$. In Fig. 4.7, the profile of the impedance $\mathrm{j}\omega\dot{\mu}$ is plotted as the frequency.

Sphere

Let us consider an isolated sphere applied to a uniform time-harmonic magnetic field $\boldsymbol{H}_0$ as shown in Fig. 4.8. The radius of the sphere, the conductivity, and the permeability are a, σ, and μ, respectively. The coordinate systems (x, y, z) and (r, θ, φ) are introduced where y are defined parallel to $\boldsymbol{H}_0$. We consider only $H_r(r,\theta), H_\theta(r,\theta), E_\varphi(r,\theta)$ as the eddy currents flow in φ-direction. From the MQS approximation of Maxwell's equations, we have

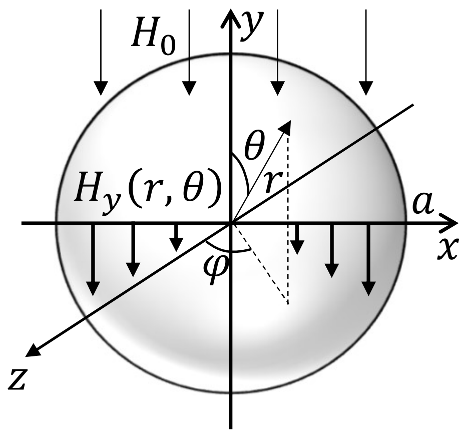

Figure4.8: A sphere immersed in a uniform time-harmonic magnetic field.

$$\frac{1}{r}\left[\frac{\partial}{\partial r}(rH_\theta) - \frac{\partial H_r}{\partial \theta}\right] = \sigma E_\varphi, \tag{4.29}$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\varphi \sin\theta) = -\mathrm{j}\omega\mu H_r, \tag{4.30}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rE_\varphi) = -\mathrm{j}\omega\mu H_\theta \tag{4.31}$$

Eliminating H_r, H_θ, we have

$$\frac{1}{r}\left[\frac{\partial^2}{\partial r^2}(rE_\varphi) + \frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(E_\varphi\sin\theta)\right)\right] + k^2E_\varphi = 0 \tag{4.32}$$

where $k^2 = -\mathrm{j}\omega\sigma\mu$. Applying the variable separation $E_\varphi(r,\theta) = R(r)\Theta(\theta)$, we obtain

$$r\frac{d^2}{dr^2}(rR) + (k^2r^2 - \beta^2)R = 0, \tag{4.33}$$

$$\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{d}{d\theta}(\Theta\sin\theta)\right) + \beta^2\Theta = 0. \tag{4.34}$$

The second equation can be written as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left(\beta^2 - \frac{1}{\sin\theta^2}\right)\Theta = 0. \tag{4.35}$$

Applying the change of variable $t = \cos\theta$, we obtain

$$\frac{d}{dt}\left(\frac{1}{1-t^2}\frac{d\Theta}{dt}\right) + \left(\beta^2 - \frac{1}{1-t^2}\right)\Theta = 0. \tag{4.36}$$

This is the associated Lengendre differential equation. We can find the bounded solutions if and only if $\beta^2 = l(l+1)$ where l is a integer number. The solutions $P_l^1(t), Q_l^1(t)$ are called the l-th order associated Legendre polynomials and l-th order Legendre functions of the second kind.

Then, equation (4.33) can be written as

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left(k^2 - \frac{l(l+1)}{r^2}\right)R = 0. \tag{4.37}$$

This is the spherical Bessel differential equation. The solutions $j_l(kr), n_l(kr)$ are called the spherical Bessel function and spherical Neumann function. Since the spherical Neumann function diverges at the origin, the general solution of our interest can be written as

$$E_\varphi(r,\theta) = \sum_{l=0}^{\infty} j_l(kr)(A_{l1}P_l^1(\cos\theta) + B_{l1}Q_l^1(\cos\theta)). \tag{4.38}$$

From the boundary conditions

$$H_\theta(a,\theta) = H_0\sin\theta, \tag{4.39}$$

$$E_\varphi(r,0) = 0, \tag{4.40}$$

we have

$$E_\varphi(r,\theta) = A_{11}j_1(kr)P_1^1(\cos\theta) = -A_{11}j_1(kr)\sin\theta \tag{4.41}$$

$$H_\theta(r,\theta) = -\frac{A_{11}}{\mathrm{j}\omega\mu r}\frac{d}{dr}(rj_1(kr))\sin\theta \tag{4.42}$$

$$H_r(r,\theta) = \frac{A_{11}}{\mathrm{j}\omega\mu r}j_1(kr)2\cos\theta \tag{4.43}$$

where

$$A_{11} = \frac{\mathrm{j}\omega\mu a H_0}{j_1(ka) + kaj_1'(ka)}. \tag{4.44}$$

The average response can be calculated as

$$\begin{aligned}
\overline{B}_y &= \frac{\mu}{\frac{4\pi a^3}{3}} \int_0^a r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\varphi (H_r \cos\theta - H_\theta \sin\theta) \\
&= \frac{3\mu}{4\pi a^3} \frac{2\pi A_{11}}{\mathrm{j}\omega\mu} \int_0^a r dr \int_0^\pi \left(j_1(kr) 2\cos\theta^2 \sin\theta + \frac{d}{dr}(rj_1(kr)) \sin\theta^3 \right) d\theta \\
&= \frac{3\mu}{4\pi a^3} \frac{2\pi A_{11}}{\mathrm{j}\omega\mu} \frac{4}{3} \int_0^a \left(rj_1(kr) + r\frac{d}{dr}(rj_1(kr)) \right) dr \\
&= \frac{\mu}{a^3} \frac{2A_{11}}{\mathrm{j}\omega\mu} \left[r^2 j_1(kr) \right]_0^a \\
&= \frac{\mu}{a^3} \frac{2A_{11}}{\mathrm{j}\omega\mu} a^2 j_1(ka) \\
&= \frac{2\mu j_1(ka)}{j_1(ka) + kaj_1'(ka)} H_0.
\end{aligned} \tag{4.45}$$

From the definition, we obtain

$$\begin{aligned}
\dot{\mu} &= \frac{\overline{B}_y}{H_0} = \mu \frac{2j_1(z)}{j_1(z) + zj_1'(z)} = \mu \frac{2(z - \tan z)}{(1 - z^2)\tan z - z} \\
&= \cfrac{2\mu}{2 - \cfrac{z^2}{5 - \cfrac{z^2}{7 - \cfrac{z^2}{\ddots}}}}
\end{aligned} \tag{4.46}$$

$$\mathrm{j}\omega\dot{\mu} = \left[0; \tfrac{1}{\mathrm{j}\omega\mu}, \tfrac{10}{\sigma a^2}, \tfrac{7}{2\mathrm{j}\omega\mu}, \tfrac{18}{\sigma a^2}, \tfrac{11}{2\mathrm{j}\omega\mu}, \cdots \right] \tag{4.47}$$

where $z = ka$. This result is consistent with that given in [101]. In Fig. 4.9, the profile of the complex permeability is plotted as the frequency where $\sigma = 1.0^6$ Sm^{-1}, $\mu_r = 10.0$. In Fig. 4.10, the profile of the impedance $\mathrm{j}\omega\dot{\mu}$ is plotted against the frequency.

Once we obtain the complex permeability $\dot{\mu}$, we can use it in FEA. It has been shown the eddy current losses P_{eddy} due to the uniform magnetic field H_0 can correctly be computed by using the complex permeability. The complex power can be written as

$$P = \frac{\mathrm{j}\omega}{2} \int_\Omega \mu |\boldsymbol{H}|^2 d\Omega \tag{4.48}$$

where μ is set to $\dot{\mu}$ in the homogenized material. The real and imaginary parts represent the sum of P_{eddy} and time variation in the stored magnetic energy. Note here that the homogenized materials can be subdivided into the finite elements without considering the skin depth. Thus the number of unknowns can be reduced.