High order Ne´de´lec elements with local complete sequence properties

Joachim Scho¨berl and Sabine Zaglmayr

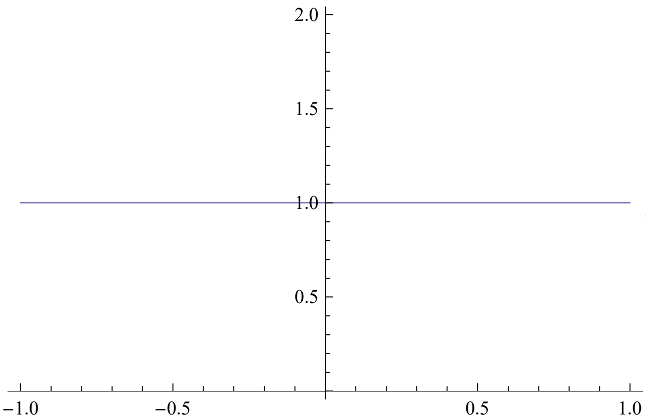
In[1470]:=
```
Needs["VectorAnalysis`"]
SetCoordinates[Cartesian[x, y, z]];
```

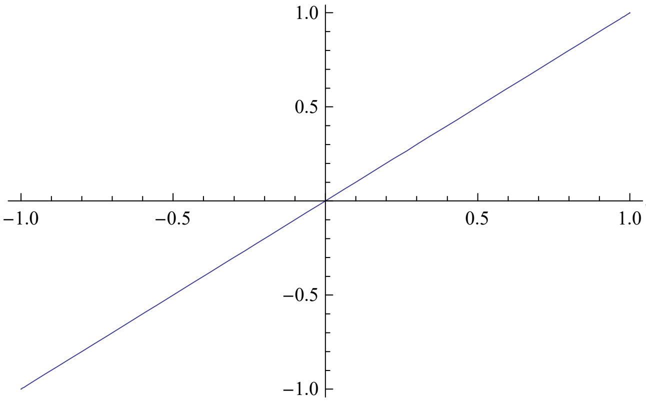
the Legendrepolynomials $l_i(x)$ (i = 0, ... p)

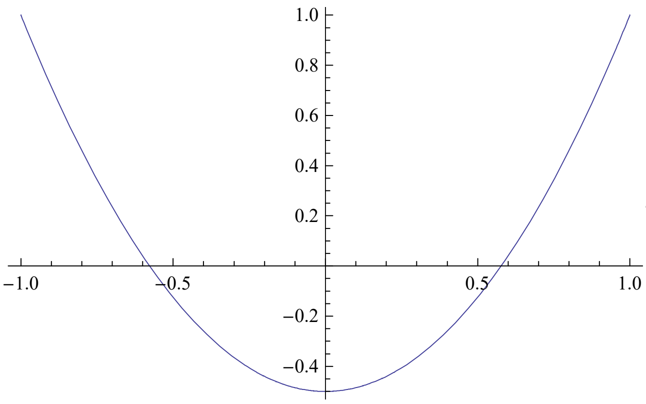
$l_i(x) := P_i(x)$

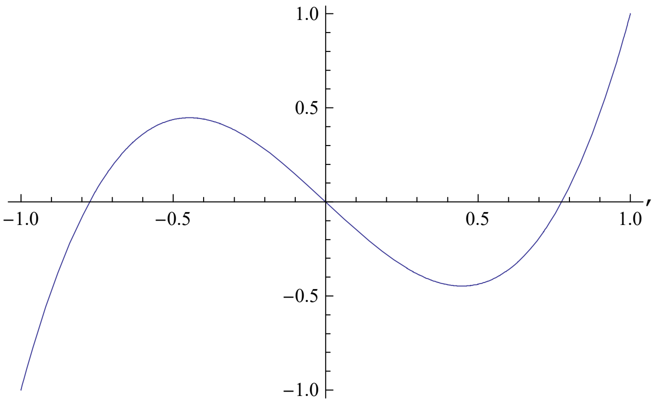
In[1472]:=
```
l(i_, x_) := P_i(x);
```

In[1473]:=
```
Table[Plot[l[i, x], {x, -1, 1}], {i, 0, 5}]
```

Out[1473]= {,

,

,



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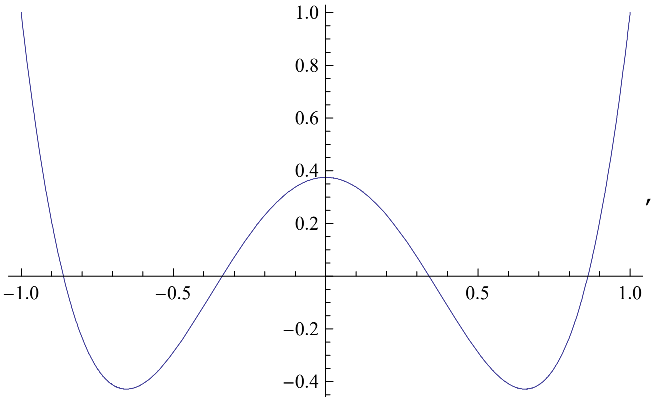

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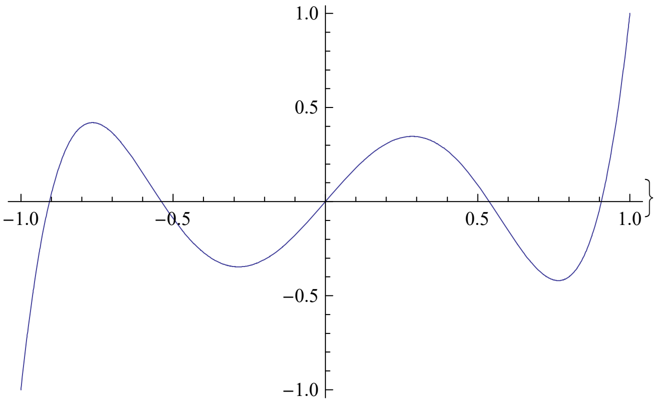


}

In[1470]:= **the integrated Legendre − polynomials** $L_i(i = 2, \ldots, p)$

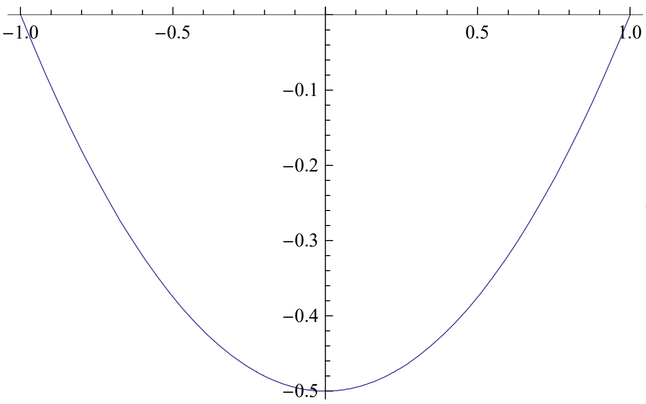
$$L_{i\_}[x\_] := \int_{-1}^{x} P_{i-1}(z)\, dz$$

In[1474]:= $L(i\_, x\_) := \int_{-1}^{x} l(i-1, z)\, dz$

In[1475]:= $\text{For}\left[i = 2,\ i \le 6,\ i{+}{+},\quad L(i, x\_) = \int_{-1}^{x} l(i-1, z)\, dz \quad\right]$

In[1476]:= $\text{Table}[\text{Plot}[L(i, x), \{x, -1, 1\}], \{i, 2, 5\}]$

Out[1476]= {



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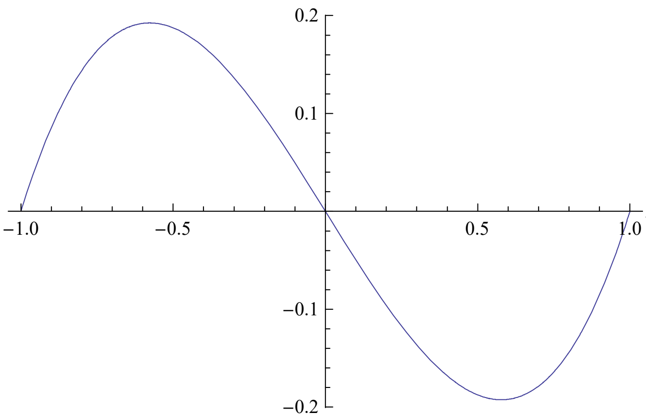

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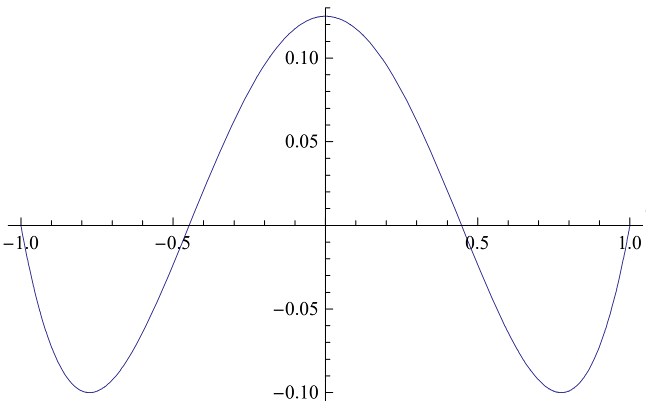

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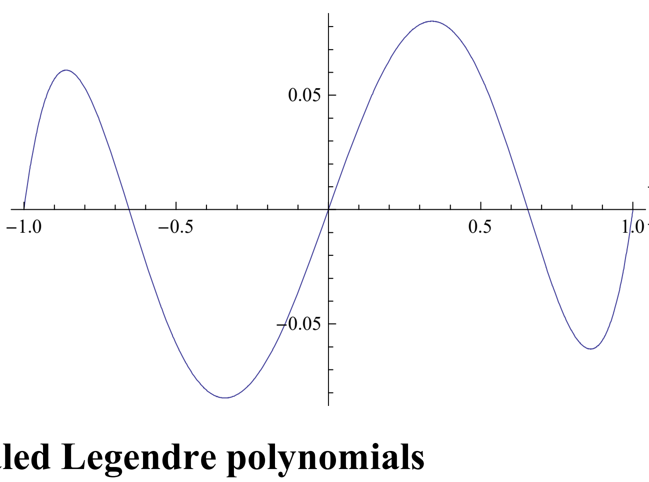


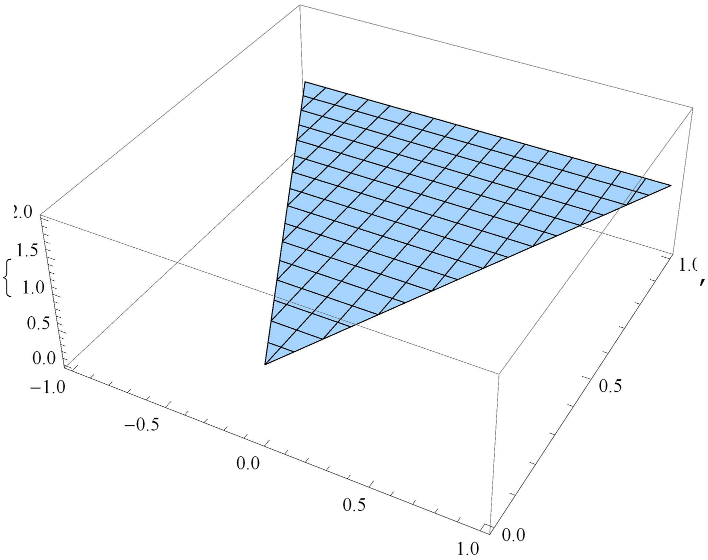
}

the scaled Legendre polynomials

In[1477]:= $\mathrm{lS}(n\_, s\_, t\_) := \mathrm{Simplify}\left[t^n\, l\left(n, \frac{s}{t}\right)\right];$

In[1478]:=
```
Table[Plot3D[lS[i, s, t], {s, -1, 1}, {t, 0, 1},
  RegionFunction → Function[{s, t, z}, 0 > s - t && 0 < s + t]],
 {i,
  0,
  3}]
```

Out[1478]= {



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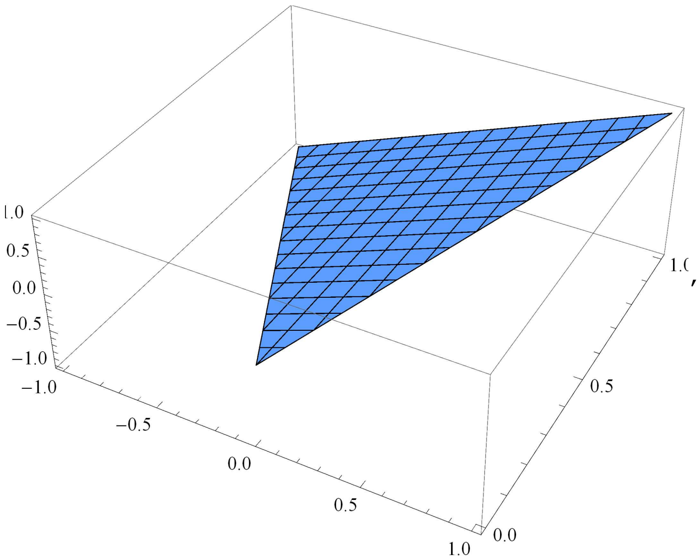

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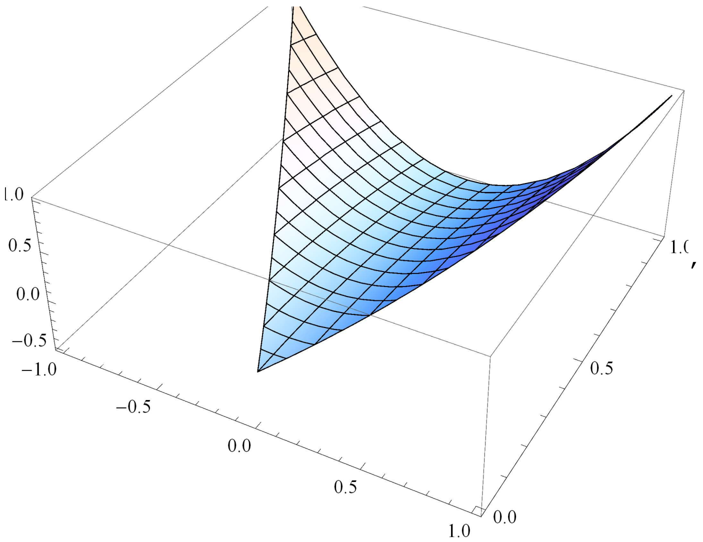
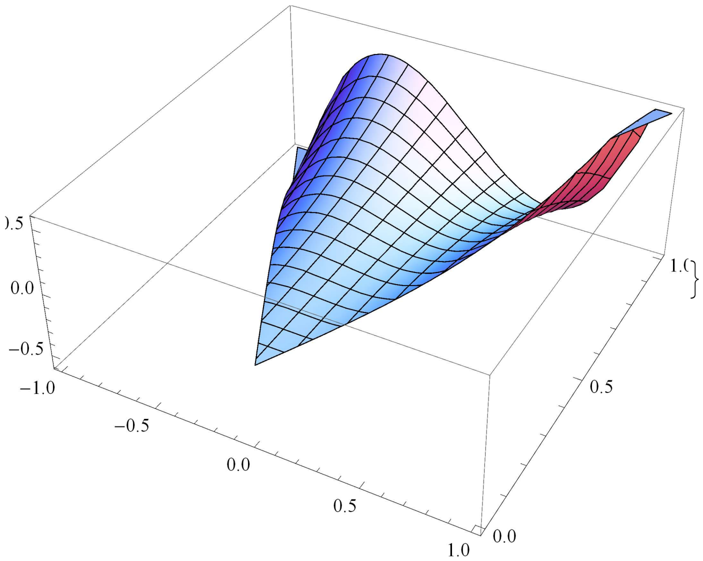


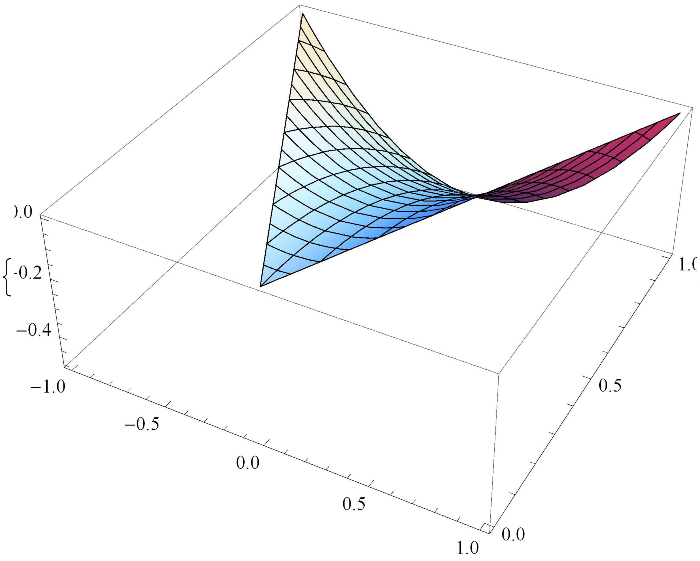


the scaled integrated Legendre polynomials

In[1479]:= **LS(n_, s_, t_) := Simplify**$\left[t^n\, L\left(n,\ \frac{s}{t}\right)\right]$;

In[1480]:=
```
Table[Plot3D[LS[i, s, t], {s, -1, 1}, {t, 0, 1},
  RegionFunction → Function[{s, t, z}, 0 > s - t && 0 < s + t]],
 {i,
  2,
  5}]
```

Out[1480]= {



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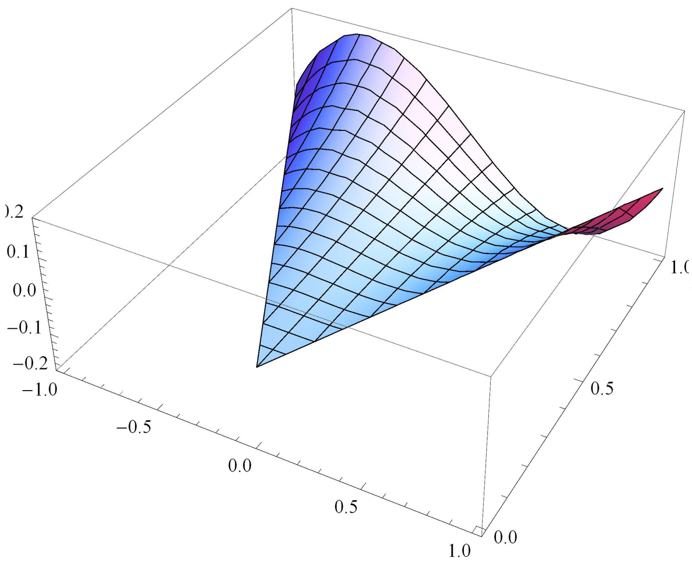

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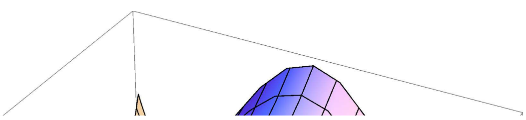
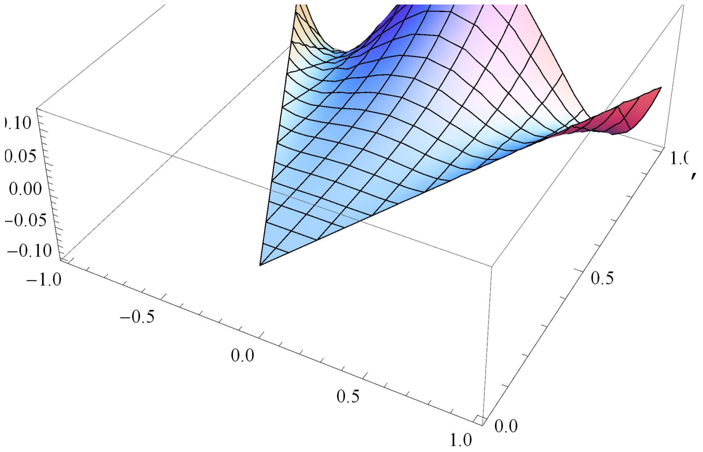
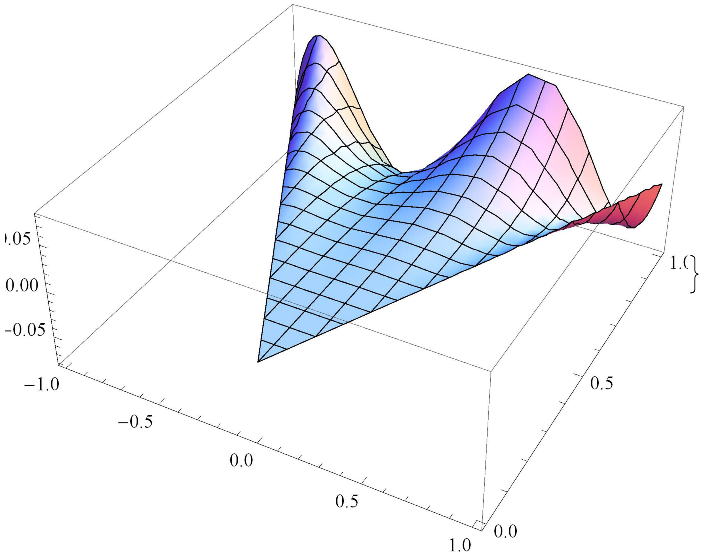






In[1481]:=

the barycentric coordinates

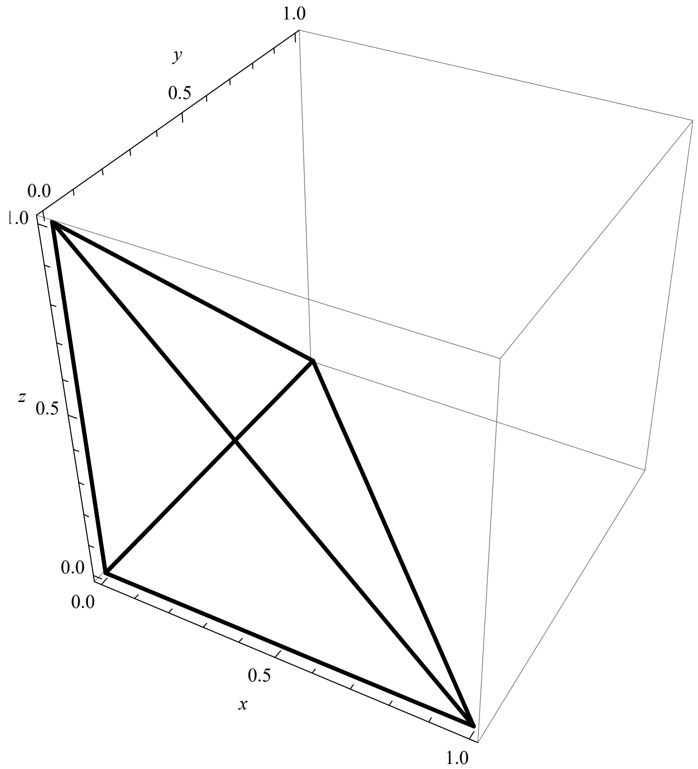
λ_i (i = 1, 2, 3, 4)

In[1482]:=
```
Clear[point, lamda, px, edge, face]
point[0] = {0, 0, 0};  point[1] = {1, 0, 0}; point[2] = {0, 1, 0}; point[3] = {0, 0, 1};
edge[0] = {0, 1}; edge[1] = {0, 2}; edge[2] = {0, 3};
edge[3] = {1, 2}; edge[4] = {1, 3}; edge[5] = {2, 3};
face[0] = {0, 1, 2}; face[1] = {0, 1, 3}; face[2] = {0, 2, 3}; face[3] = {1, 2, 3};

px = {0, 0, 0};
For[i = 0, i < 4, ++i, px += point[i] * lamda[i];]
sum = 0;
For[i = 0, i < 4, ++i, sum += lamda[i]];
px = Append[px, sum];
sol = Solve[px == {x, y, z, 1}, {lamda[0], lamda[1], lamda[2], lamda[3]}];
lamda[0] = lamda[0] /. sol[[1]];
lamda[1] = lamda[1] /. sol[[1]];
lamda[2] = lamda[2] /. sol[[1]];
lamda[3] = lamda[3] /. sol[[1]];
```

In[1496]:=
```
Clear[lines];
lines = {Thick};
For[i = 0, i < 6, i++,
 p1 = edge[i][[1]];
 p2 = edge[i][[2]];
 line = Line[ { point[p1] , point[ p2] } ] ;
 lines = Append[lines, line];
]
Graphics3D[lines, Axes → True,
 AxesLabel → {x, y, z}, Boxed → True, ViewVertical → {0, 0, 1}]
```

Out[1499]=



In[1500]:=
```
G[func_, fac0_] := Module[{ncf, fac = fac0, cf},
   ncf = 16;
   cf = Table[(-1 + 2 / ncf * (i - 1 / 2)) * fac, {i, 1, ncf} ];
   ContourPlot3D[func, {x, 0, 1}, {y, 0, 1}, {z, 0, 1},
    Contours → cf, RegionFunction → Function[{x, y, z}, lamda[0] > 0],
    ColorFunction → ColorData["TemperatureMap"],
    Mesh → None, ContourStyle → Directive[Opacity[0.5]]
     ]
  ]
```

In[1501]:=

Hierarchical triangular H1 - element of order p using Scaled Legendre Polynomials

Hierarchical tetrahedral H^1-element of order p using Scaled Legendre Polynomials

Vertex-based functions

$$\phi_i^V = \lambda_i \qquad \text{for } i = 1, 2, 3, 4$$

Edge-based functions

for edge $E_m = \{e_1, e_2\}, m = 1, \ldots 6$

for $0 \leq i \leq p-2$

$$\phi_i^{E_m} = L_{i+2}^{\mathcal{S}}(\lambda_{e_1} - \lambda_{e_2}, \lambda_{e_1} + \lambda_{e_2})$$

Face-based functions

for face $F_m = \{f_1, f_2, f_3\}$, m=1,...,4

for $i, j \geq 0, i + j \leq p - 3$

$$\begin{aligned}\phi_{ij}^{F_m} &= L_{i+2}^{\mathcal{S}}(\lambda_{f_1} - \lambda_{f_2}, \lambda_{f_1} + \lambda_{f_2}) \\ &\times \lambda_{f_3}\, \ell_j^{\mathcal{S}}(\lambda_{f_3} - \lambda_{f_1} - \lambda_{f_2}, \lambda_1 + \lambda_2 + \lambda_3)\end{aligned}$$

;

Interior-based function

for $0 \leq i + j + k \leq p - 4$

$$\begin{aligned}\phi_{ijk}^{I} &= L_{i+2}^{\mathcal{S}}(\lambda_1 - \lambda_2, \lambda_1 + \lambda_2) \\ &\times \lambda_3\, \ell_j^{\mathcal{S}}(\lambda_3 - \lambda_1 - \lambda_2, \lambda_1 + \lambda_2 + \lambda_3) \\ &\times \lambda_4\, \ell_k(\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3)\end{aligned}$$

Modified by A. Kameari;

Vertex - based functions

In[1504]:=
```
phiV[i_] := lamda[i];
```

Edge - based functions

In[1505]:=
```
phiE[m_, i_] := LS[i + 2,
   lamda[edge[m][[1]]] - lamda[edge[m][[2]]],
   lamda[edge[m][[2]]] + lamda[edge[m][[1]]] ];
```

Face-based functions

```
In[1506]:= u[f_, i_] := LS[i + 2, lamda[face[f][[1]]] - lamda[face[f][[2]]],
              lamda[face[f][[1]]] + lamda[face[f][[2]]] ];
           v[f_, j_] := lamda[face[f][[3]]] *
              lS[j, lamda[face[f][[3]]] - lamda[face[f][[1]]] - lamda[face[f][[2]]],
               lamda[face[f][[1]]] + lamda[face[f][[2]]] + lamda[face[f][[3]]] ] ;
           phiF[f_, i_, j_] := u[f, i] * v[f, j];
```

Interir-based functions

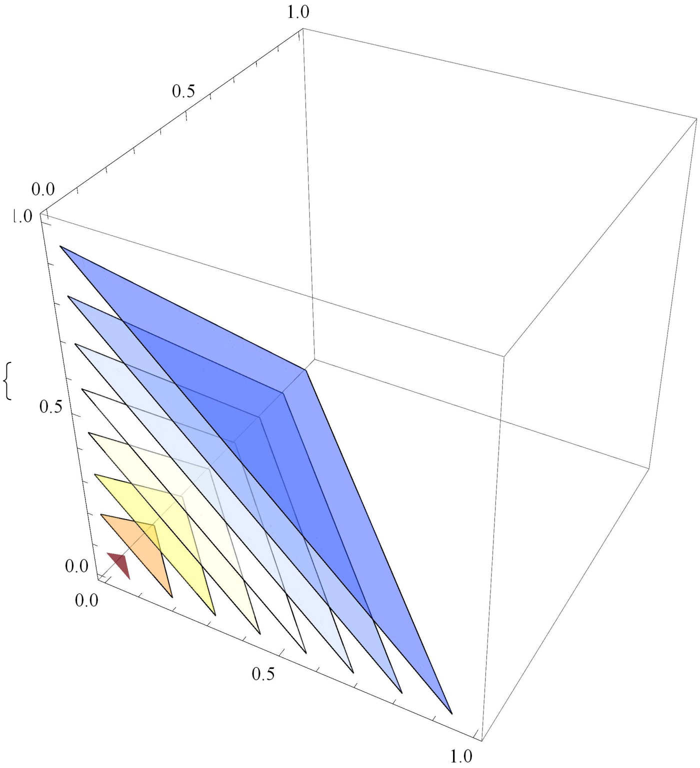
```
In[1509]:= phiI[i_, j_, k_] := LS[i + 2, lamda[0] - lamda[1], lamda[0] + lamda[1] ] * lamda[2] *
              lS[j, lamda[2] - lamda[0] - lamda[1], lamda[0] + lamda[1] + lamda[2] ] *
              lamda[3] * l[k, lamda[3] - lamda[0] - lamda[1] - lamda[2] ];
```

p=1

Vertex - based functions

In[1510]:= `Table[G[phiV[i], 1], {i, 0, 3}]`

Out[1510]= {



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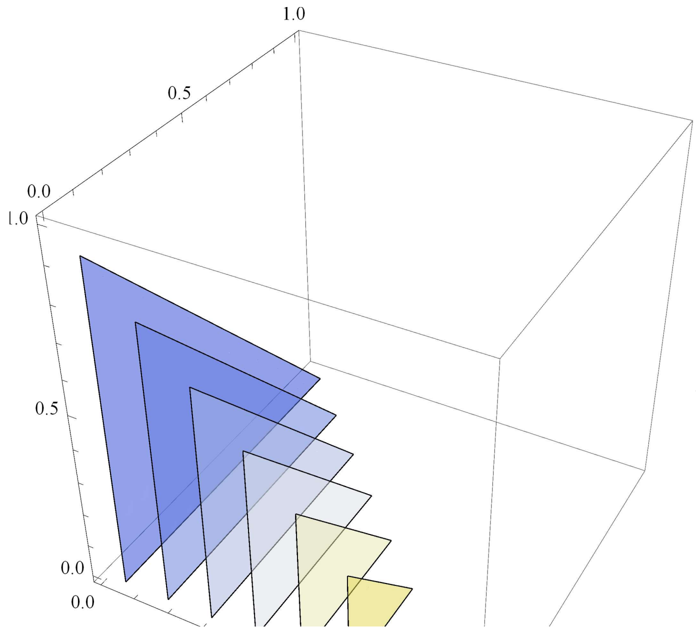

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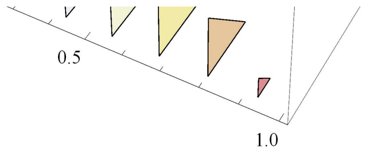
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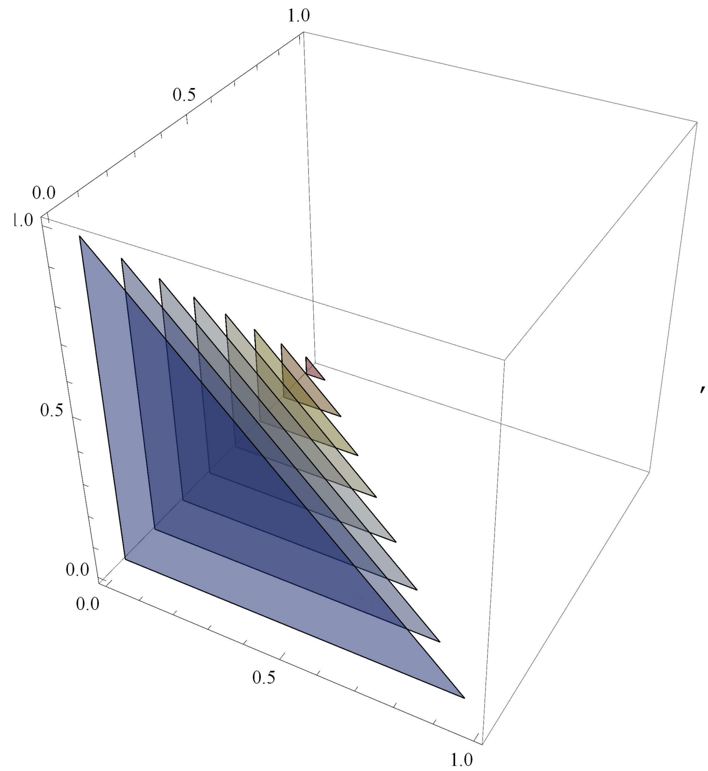
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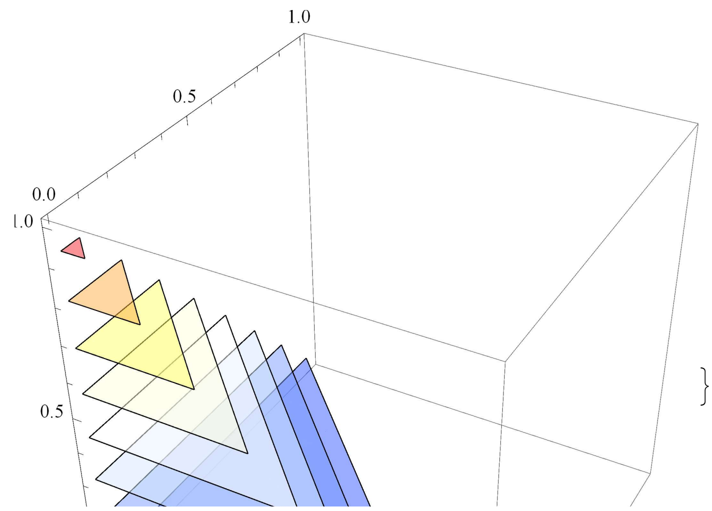
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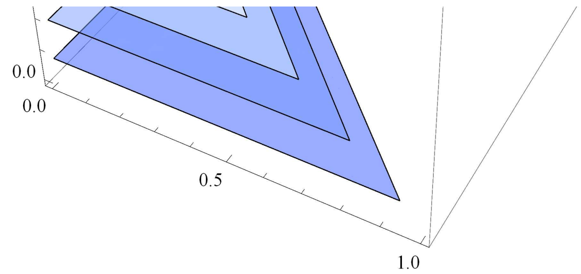
}

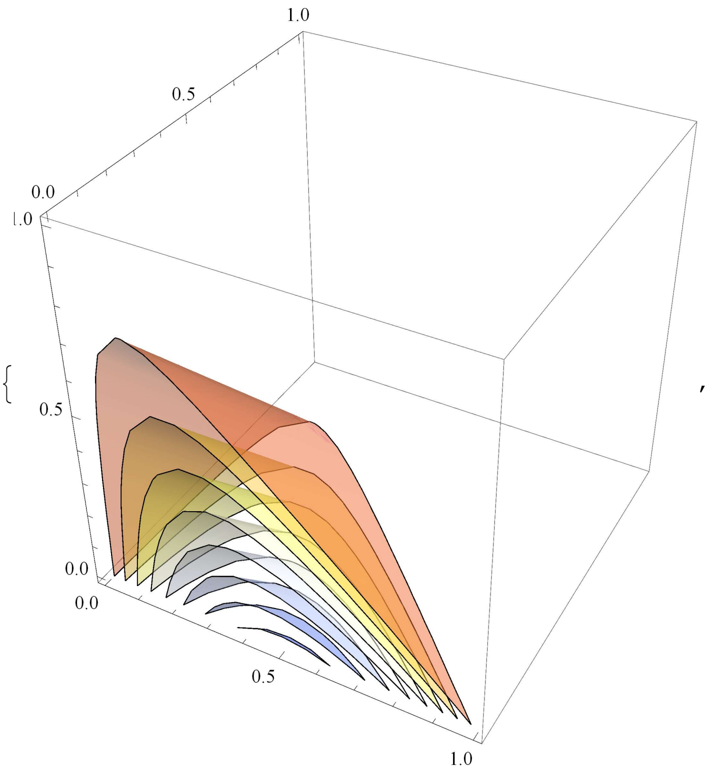


p=2

Edge - based functions

In[1511]:=
```
p = 2;
Table[G[ phiE[e, p - 2], 1 / 2] , {e, 0, 5}]
Factor[Table[phiE[e, p - 2], {e, 0, 5}]]
```

Out[1512]= {



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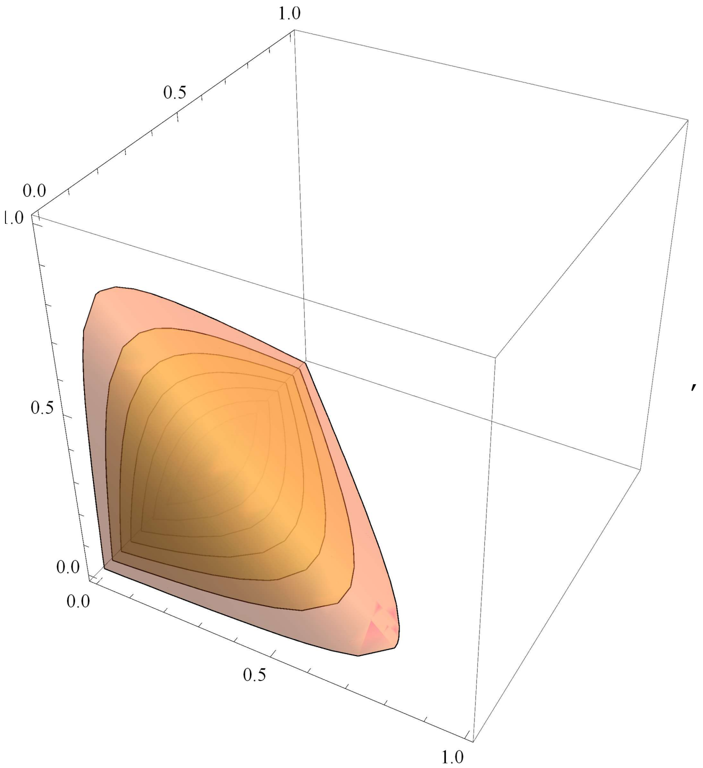

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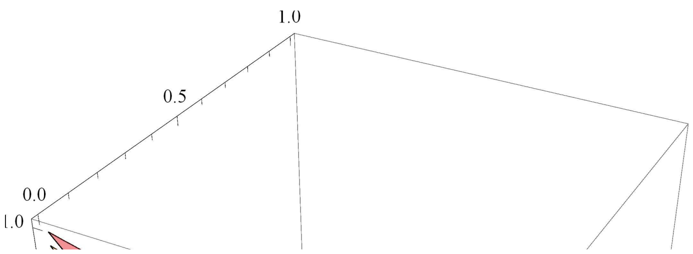
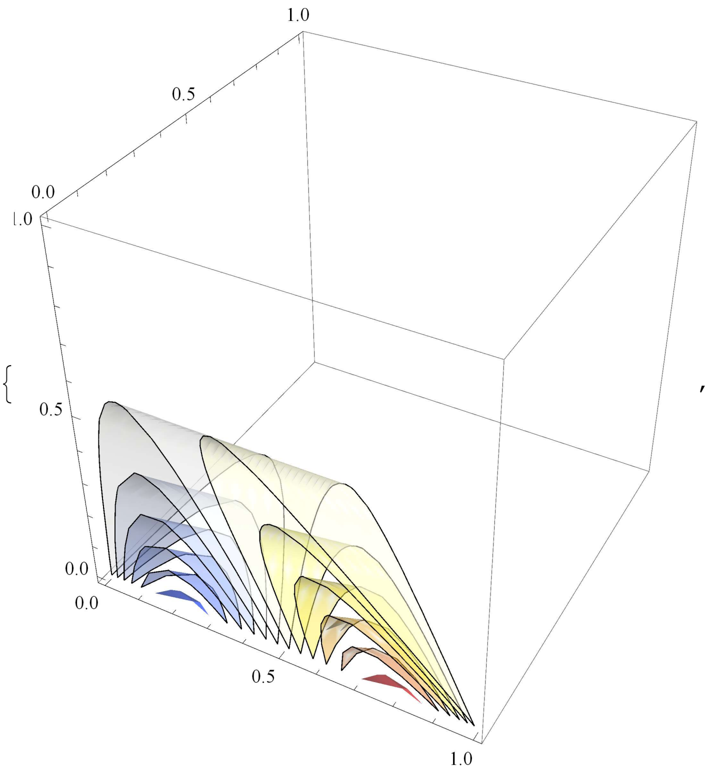


```
Out[1513]= {2 x (-1 + x + y + z), 2 y (-1 + x + y + z), 2 z (-1 + x + y + z), -2 x y, -2 x z, -2 y z}
```

p=3

Edge - based functions

```
In[1514]:= p = 3;
Table[G[phiE[m, p - 2], 1 / 4], {m, 0, 5}]
Factor[Table[phiE[m, p - 2], {m, 0, 5}]]
```

Out[1515]= {

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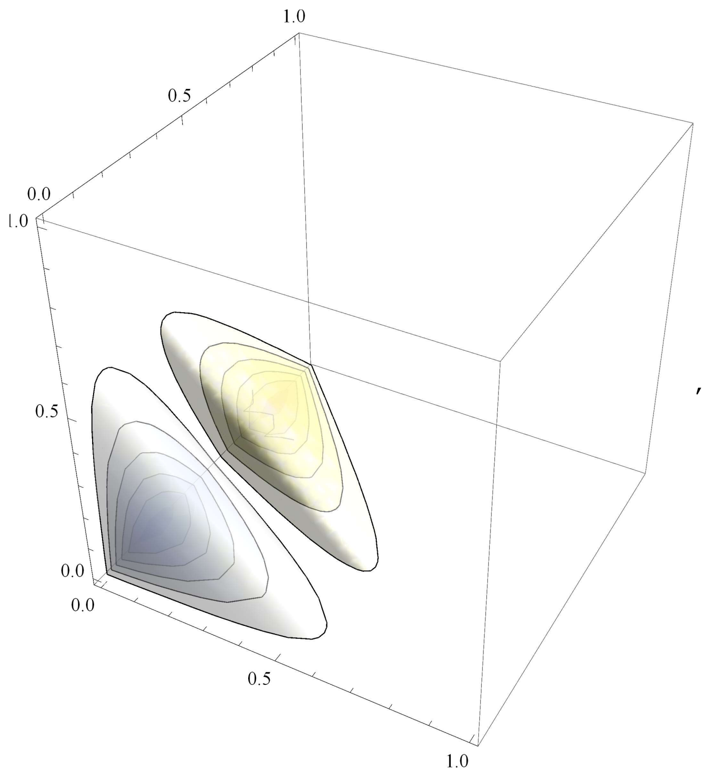

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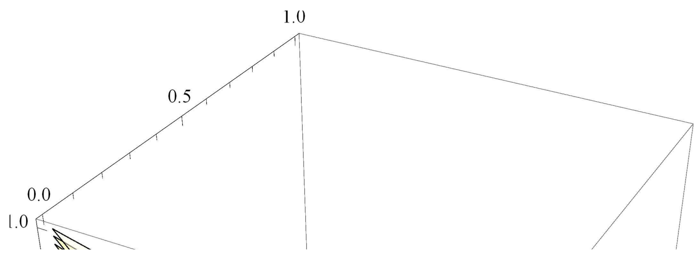
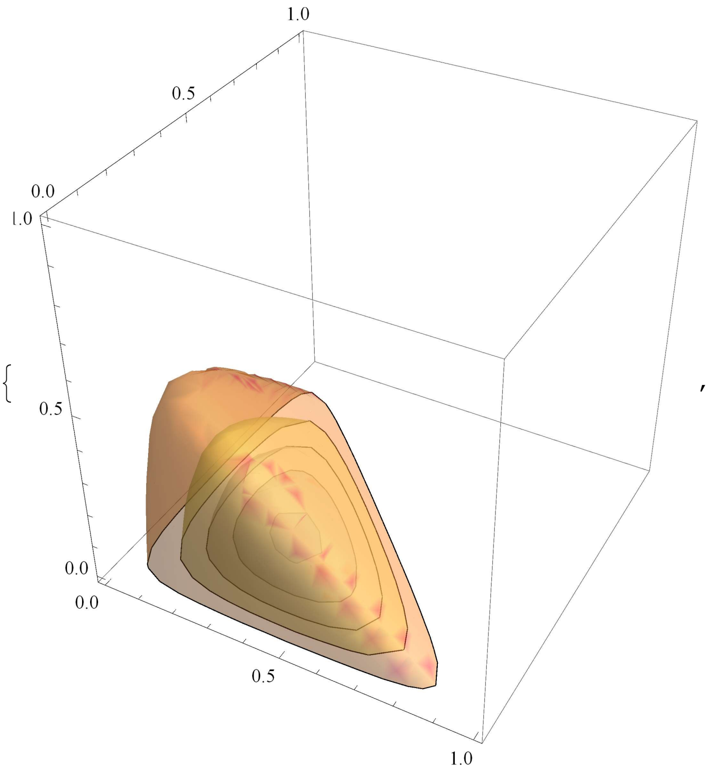


Out[1516]= {-2 x (-1 + x + y + z) (-1 + 2 x + y + z), -2 y (-1 + x + y + z) (-1 + x + 2 y + z),
-2 z (-1 + x + y + z) (-1 + x + y + 2 z), -2 x (x - y) y, -2 x (x - z) z, -2 y (y - z) z}

Face functions

```
In[1517]:= p = 3;
Table[G[phiF[f, i, p - 3 - i], 1 / 8], {i, 0, p - 3}, {f, 0, 3}]
Factor[Table[phiF[f, i, p - 3 - i], {i, 0, p - 3}, {f, 0, 3}]]
```

Out[1518]= {{



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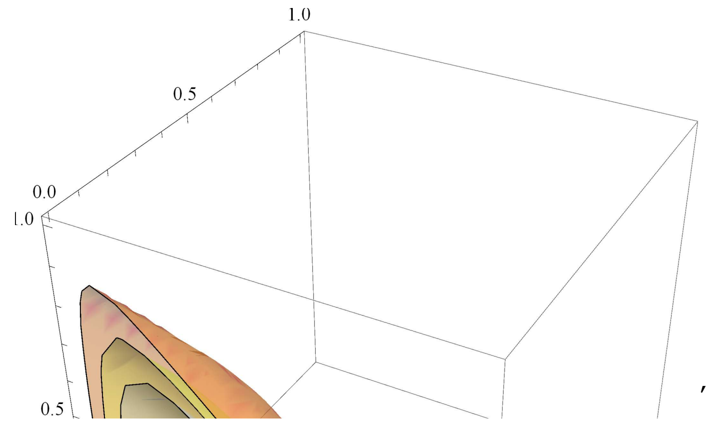

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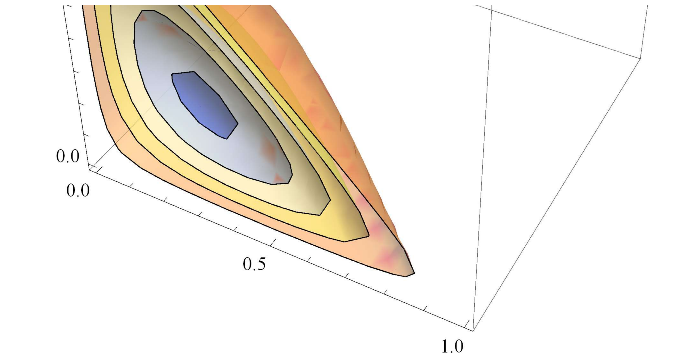
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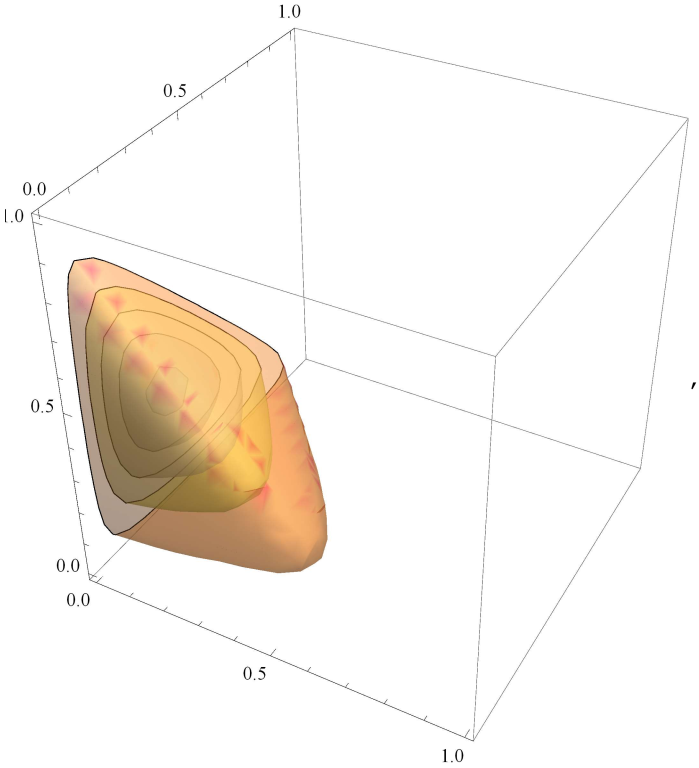
0.5
1.0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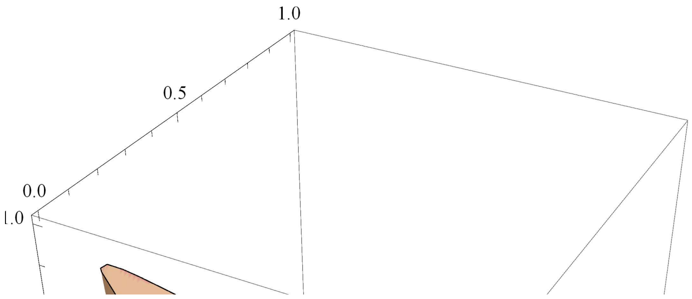
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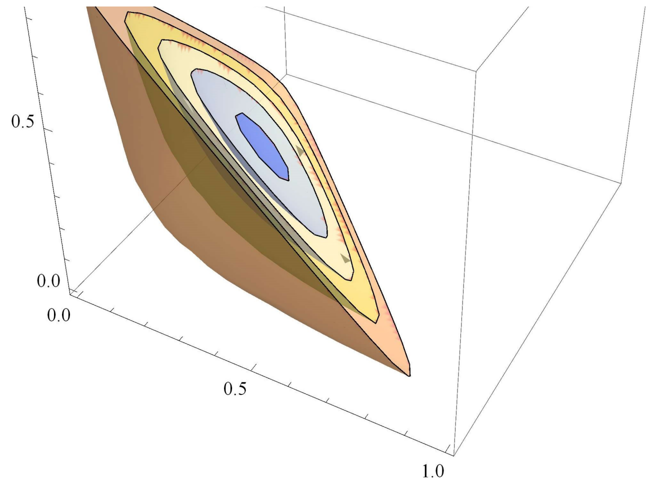


}}

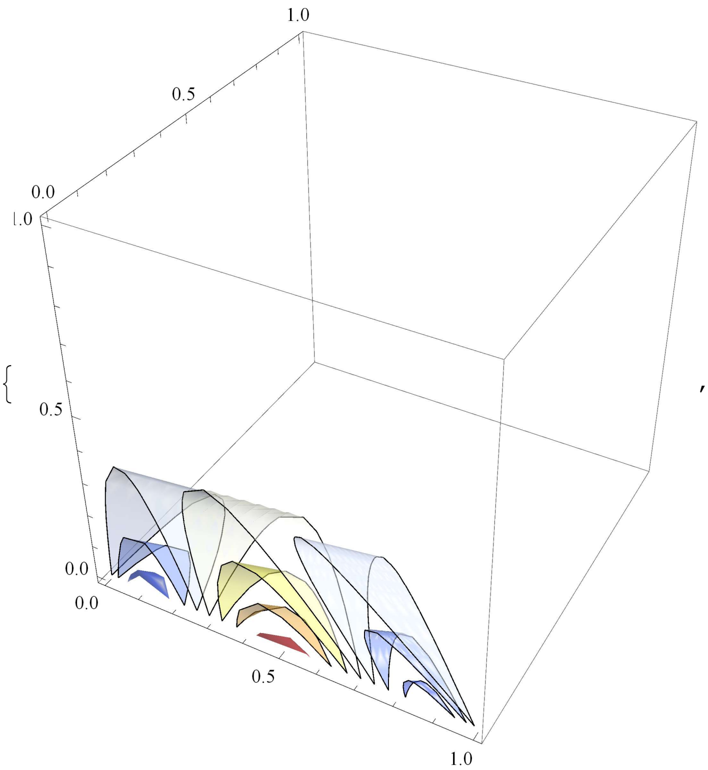
Out[1519]= {{2 x y (-1 + x + y + z), 2 x z (-1 + x + y + z), 2 y z (-1 + x + y + z), -2 x y z}}

p=4

Edge - based functions

In[1520]:=
```
p = 4;
Table[G[phiE[m, p - 2], 1 / 4], {m, 0, 5}]
Factor[Table[phiE[m, p - 2], {m, 0, 5}]]
```

Out[1521]= {



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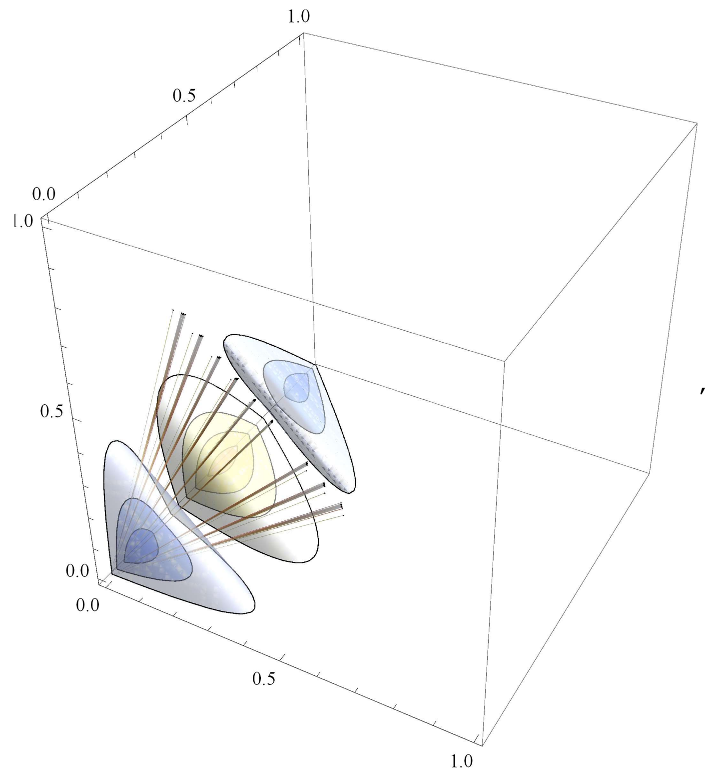

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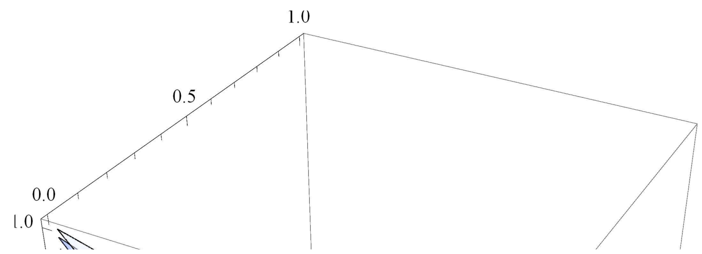
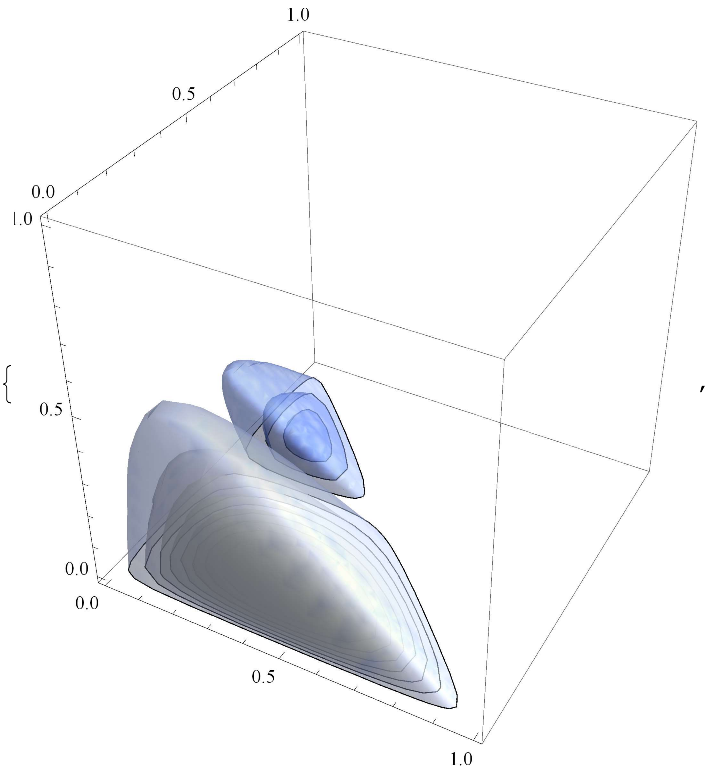


Out[1522]= $\{2\,x\,(-1+x+y+z)\,\left(1-5\,x+5\,x^2-2\,y+5\,x\,y+y^2-2\,z+5\,x\,z+2\,y\,z+z^2\right),$
$2\,y\,(-1+x+y+z)\,\left(1-2\,x+x^2-5\,y+5\,x\,y+5\,y^2-2\,z+2\,x\,z+5\,y\,z+z^2\right),$
$2\,z\,(-1+x+y+z)\,\left(1-2\,x+x^2-2\,y+2\,x\,y+y^2-5\,z+5\,x\,z+5\,y\,z+5\,z^2\right),$
$-2\,x\,y\,\left(x^2-3\,x\,y+y^2\right),\ -2\,x\,z\,\left(x^2-3\,x\,z+z^2\right),\ -2\,y\,z\,\left(y^2-3\,y\,z+z^2\right)\}$

Face functions

In[1523]:=
```
p = 4;
Table[G[phiF[f, i, p - 3 - i], 1 / 32], {i, 0, p - 3}, {f, 0, 3}]
Factor[Table[phiF[f, i, p - 3 - i], {i, 0, p - 3}, {f, 0, 3}]]
```

Out[1524]= {{



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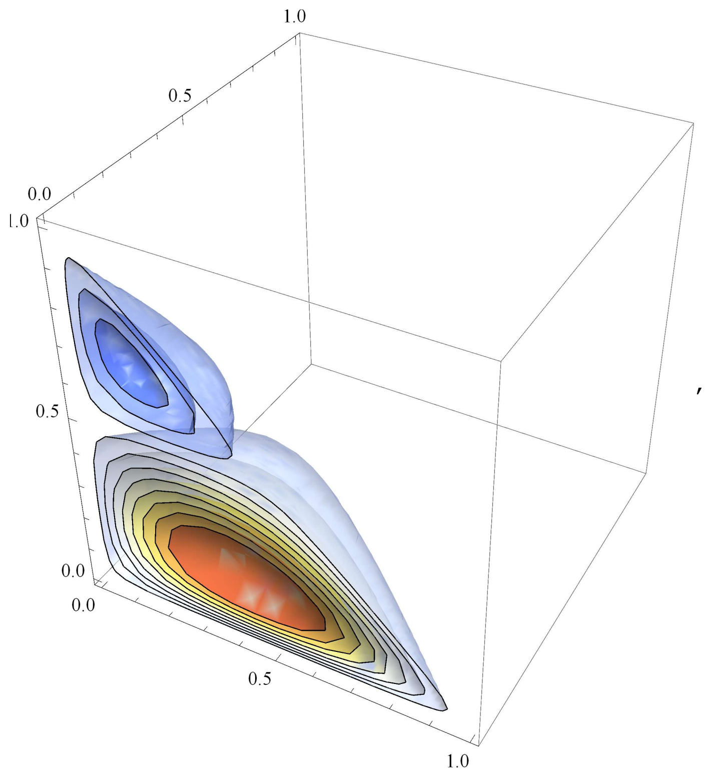

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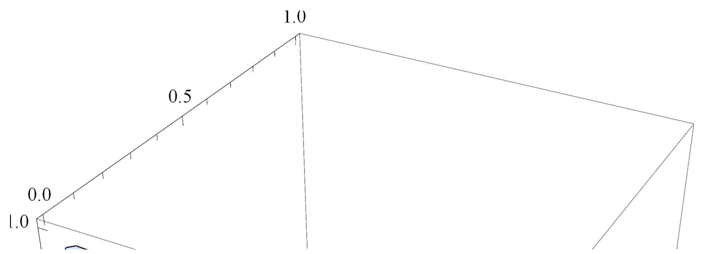
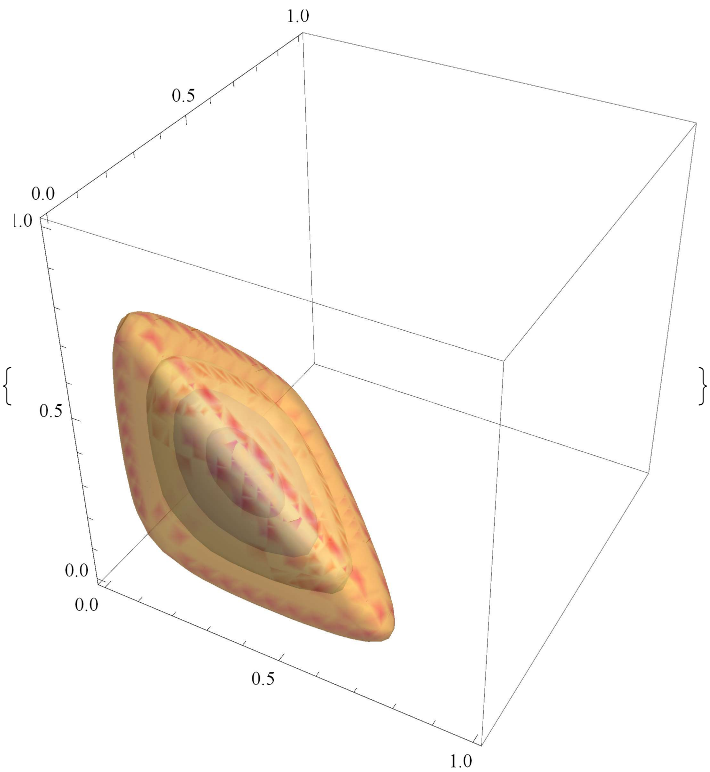


Out[1525]= {{2 x y (-1 + x + y + z) (-1 + 2 y + z), 2 x z (-1 + x + y + z) (-1 + y + 2 z),
 2 y z (-1 + x + y + z) (-1 + x + 2 z), 2 x y (x + y - z) z},
 {-2 x y (-1 + x + y + z) (-1 + 2 x + y + z), -2 x z (-1 + x + y + z) (-1 + 2 x + y + z),
 -2 y z (-1 + x + y + z) (-1 + x + 2 y + z), -2 x (x - y) y z}}

Interior-based functions

In[1526]:=
```
p = 4;
Table[G[phiI[ 0, 0, 0], 1 / 64], {k, 1, p - 3} ]
Factor[Table[phiI[0, 0, 0], {k, 1, p - 3} ]]
```

Out[1527]= {



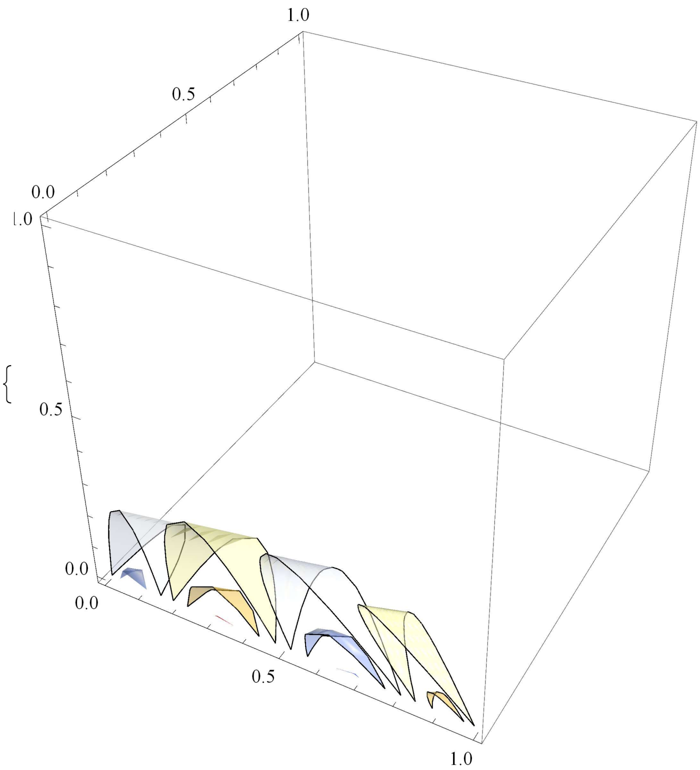
}

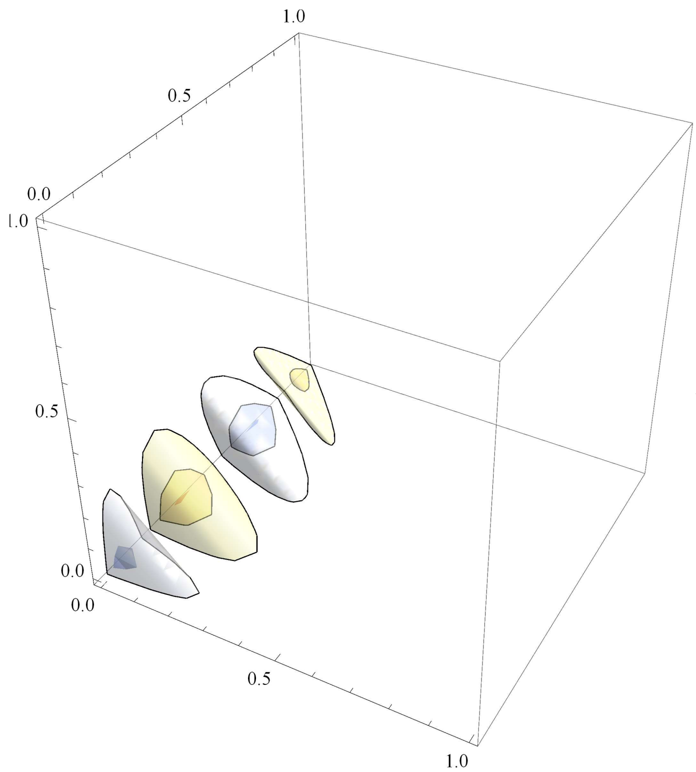
Out[1528]= {2 x y z (-1 + x + y + z)}

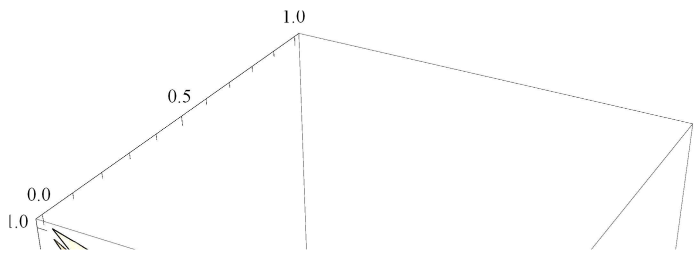
p=5

Edge - based functions

In[1529]:=
```
p = 5;
Table[G[phiE[m, p - 2], 1 / 4], {m, 0, 5}]
Factor[Table[phiE[m, p - 2], {m, 0, 5}]]
```

Out[1530]= {,

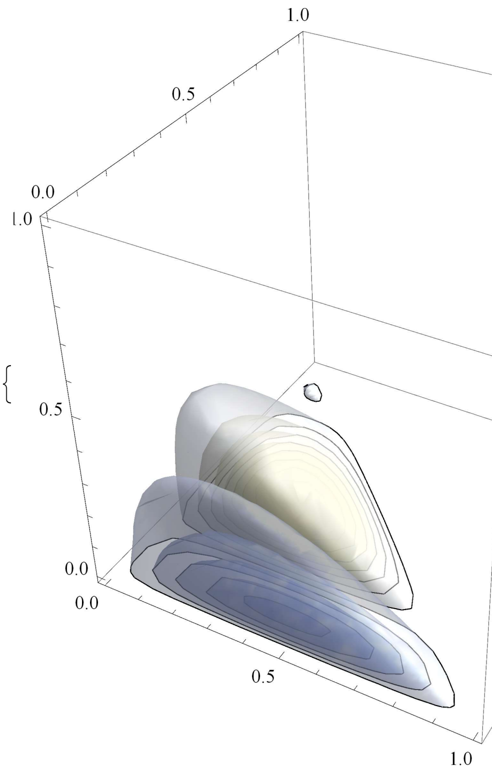
,



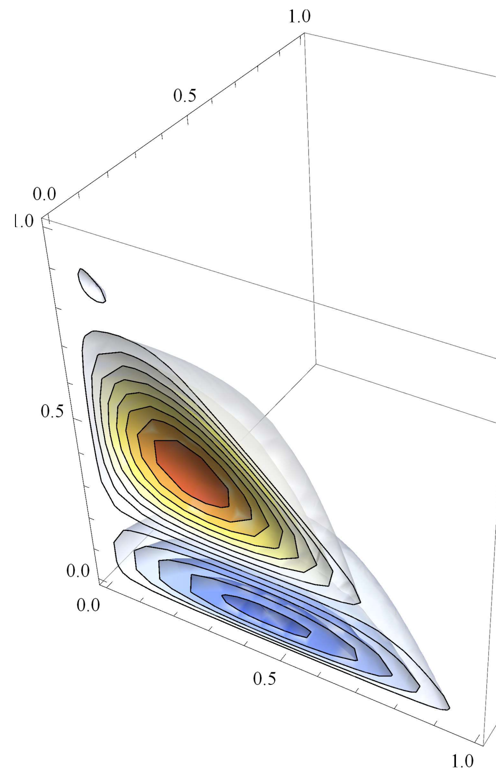
Out[1531]= $\{-2\,x\,(-1+x+y+z)\,(-1+2\,x+y+z)\,\left(1-7\,x+7\,x^2-2\,y+7\,x\,y+y^2-2\,z+7\,x\,z+2\,y\,z+z^2\right),$
$-2\,y\,(-1+x+y+z)\,(-1+x+2\,y+z)\,\left(1-2\,x+x^2-7\,y+7\,x\,y+7\,y^2-2\,z+2\,x\,z+7\,y\,z+z^2\right),$
$-2\,z\,(-1+x+y+z)\,(-1+x+y+2\,z)\,\left(1-2\,x+x^2-2\,y+2\,x\,y+y^2-7\,z+7\,x\,z+7\,y\,z+7\,z^2\right),$
$-2\,x\,(x-y)\,y\,\left(x^2-5\,x\,y+y^2\right),\ -2\,x\,(x-z)\,z\,\left(x^2-5\,x\,z+z^2\right),\ -2\,y\,(y-z)\,z\,\left(y^2-5\,y\,z+z^2\right)\}$

Face functions

In[1532]:=
```
p = 5;
Table[G[phiF[f, i, p - 3 - i], 1 / 32], {i, 0, p - 3}, {f, 0, 3}]
Factor[Table[phiF[f, i, p - 3 - i], {i, 0, p - 3}, {f, 0, 3}]]
```

Out[1533]= {{ 

,



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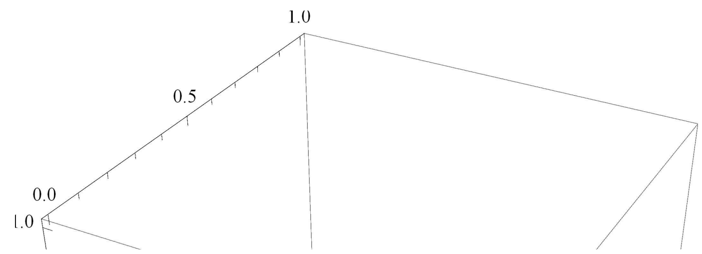


Out[1534]=

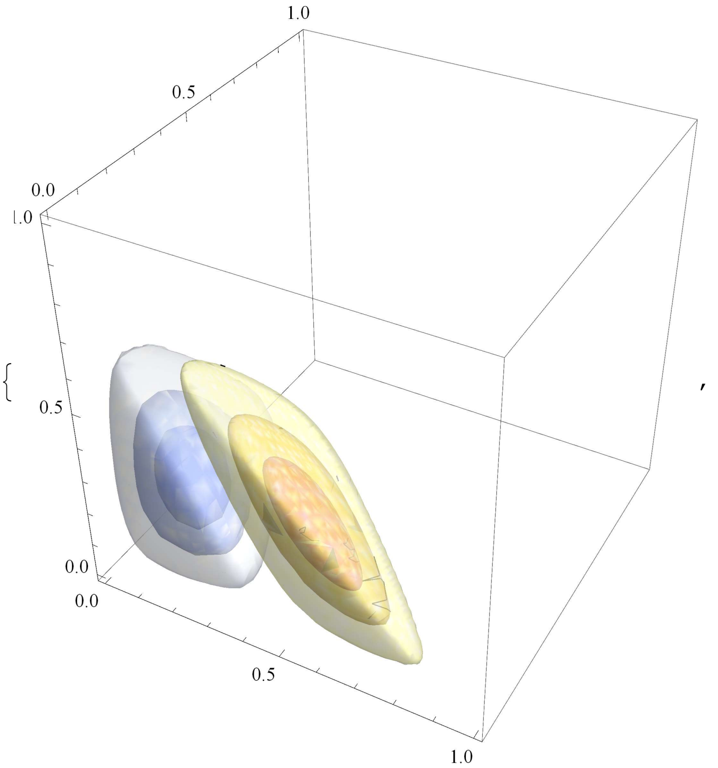
$$\{\{2xy(-1+x+y+z)(1-6y+6y^2-2z+6yz+z^2),\ 2xz(-1+x+y+z)(1-2y+y^2-6z+6yz+6z^2),\ 2yz(-1+x+y+z)(1-2x+x^2-6z+6xz+6z^2),\ -2xyz(x^2+2xy+y^2-4xz-4yz+z^2)\},$$
$$\{-2xy(-1+x+y+z)(-1+2x+y+z)(-1+2y+z),\ -2xz(-1+x+y+z)(-1+2x+y+z)(-1+y+2z),\ -2yz(-1+x+y+z)(-1+x+2y+z)(-1+x+2z),\ 2x(x-y)y(x+y-z)z\},$$
$$\{2xy(-1+x+y+z)(1-5x+5x^2-2y+5xy+y^2-2z+5xz+2yz+z^2),\ 2xz(-1+x+y+z)(1-5x+5x^2-2y+5xy+y^2-2z+5xz+2yz+z^2),\ 2yz(-1+x+y+z)(1-2x+x^2-5y+5xy+5y^2-2z+2xz+5yz+z^2),\ -2xy(x^2-3xy+y^2)z\}\}$$

Interior-based functions

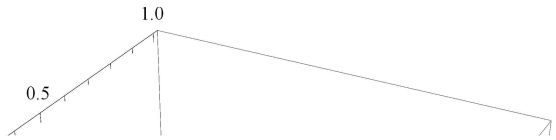
In[1535]:=

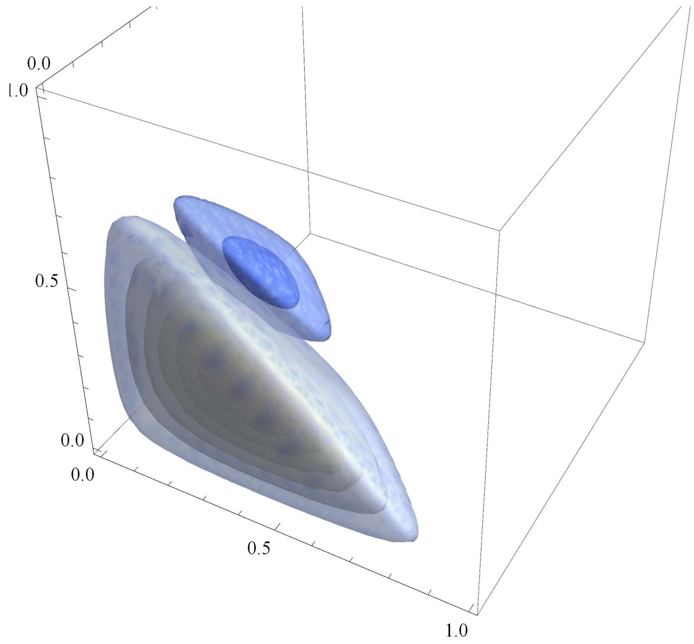
```
p = 5;
{G[phiI[ 1, 0, 0], 1 / 256], G[phiI[ 0, 1, 0], 1 / 256], G[phiI[ 0, 0, 1], 1 / 256] }
Factor[{phiI[1, 0, 0], phiI[0, 1, 0], phiI[0, 0, 1]}]
```

Out[1536]= {



,





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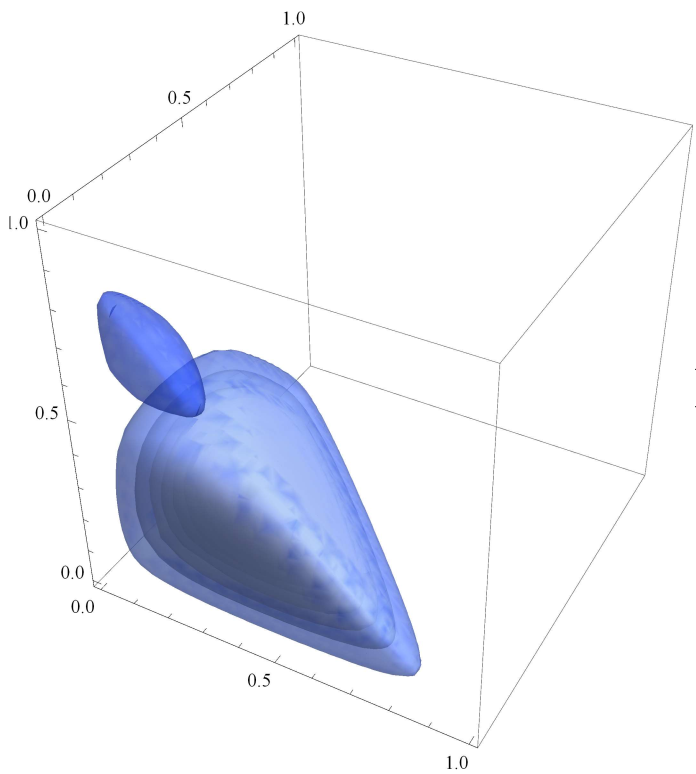

}

Out[1537]= {-2 x y z (-1 + x + y + z) (-1 + 2 x + y + z),
 2 x y z (-1 + x + y + z) (-1 + 2 y + z), 2 x y z (-1 + x + y + z) (-1 + 2 z)}