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CAAM 28200
HW4
2/16/22

Compute Jacobian to linearize the system about

6.3.10: a. $J(v) = \begin{bmatrix} y & x \\ 2x & -1 \end{bmatrix}$ $v = \begin{bmatrix} x \\ y \end{bmatrix}$ At the origin, $J(x) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = A$ $\det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\right) = 0$

$(-\lambda)(-1-\lambda) = 0$

$\lambda = 0, -1$

Since we have one eigenvalue that is <0 and one eigenvalue that is equal to 0, this predicts that we have a line of stable fixed points. Alternatively, this can be predicted from the fact that the trace τ of the Jacobian is <0 and the determinant is $=0$. So linearization predicts that the origin is a non-isolated fixed point.

b. Since one eigenvalue is 0, we know that the fixed point at the origin is not hyperbolic and so the Hartman-Grobman theorem does not apply. Instead, the origin is in fact an isolated fixed point because if I perturb in any direction away from the origin, I will get a $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}$ that is nonzero.

$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x^2 - y \end{bmatrix}$ if $z = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\frac{d}{dt} z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $z = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\frac{d}{dt} z = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

Since $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix}$ is always nonzero slightly away from the origin, the fixed point is in fact isolated

$\frac{d}{dt}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\frac{d}{dt}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\frac{d}{dt}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\frac{d}{dt}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

c. nullclines at $v = \begin{bmatrix} x \\ y \end{bmatrix}$

$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ when $\dot{x} = xy = 1$, $\dot{y} = x^2 - y = 0$, $x^2 = y$

need this to have $\dot{y} = 0$

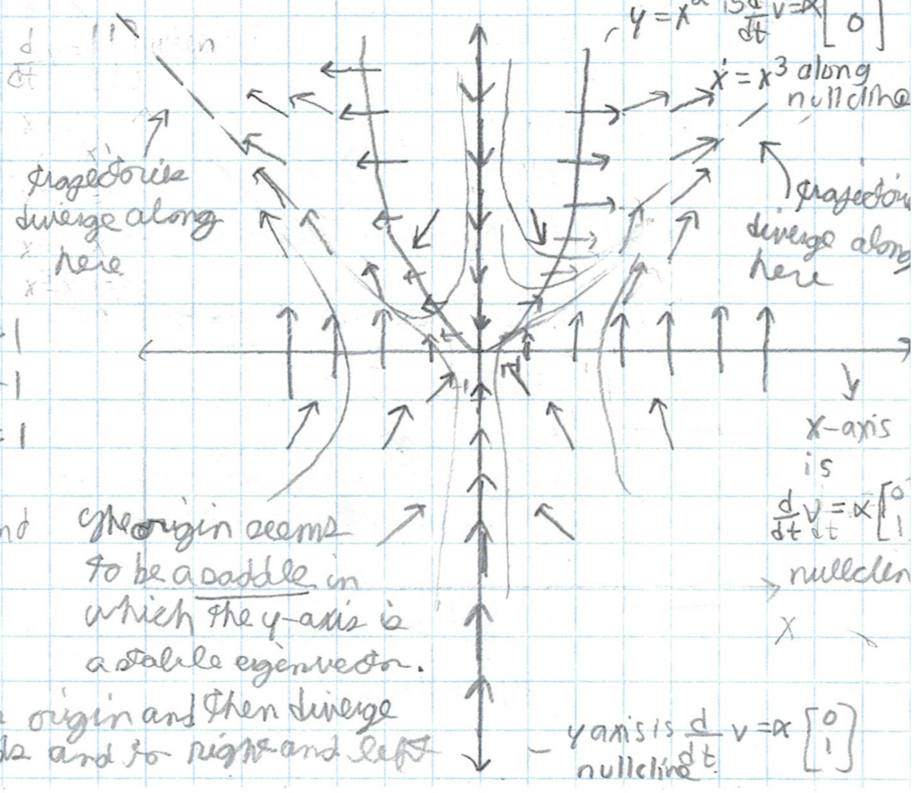
$\dot{x} = y \cdot x = x^2 x = x^3$

$\frac{d}{dt} v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\dot{x} = xy = 0$, $\dot{y} = x^2 - y = 1$

if $x = 0, y = -1$

if $y = 0, x = 1$

so both $\alpha\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}$



The origin seems to be a saddle in which the y-axis is a stable eigenvector.

Trajectories approach the origin and then diverge along the dotted lines upwards and to right and left

6.3.10 (cont.): d. see computer pages.

6.3.13: $\dot{x} = -y - x^3$ linearization predicts center:

$\dot{y} = x$ $\quad v = \begin{bmatrix} x \\ y \end{bmatrix}$ $\quad J(v) = \begin{bmatrix} -3x^2 & -1 \\ 1 & 0 \end{bmatrix}$ $\quad J\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A$

at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\dot{x}=0, \dot{y}=0$ so origin is fixed pt. Let $x_* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}$ $\quad (-\lambda)^2 + 1 = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$

$\lambda = \pm i$ so $Real(\lambda) = 0$

so linearization predicts center.

Consider $V(x,y) = x^2 + y^2$

Then $\frac{dV}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$

$= 2x\dot{x} + 2y\dot{y}$

$= 2x(-y-x^3) + 2y(x)$

$= -2xy - 2x^4 + 2xy$

$\frac{dV}{dt} = -2x^4$

Thus, $\frac{dV}{dt} < 0$ around $x_* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

The minimum of $V(x,y) = 0$ by inspection as $x^2 \geq 0$ and $y^2 \geq 0$ for $x, y \in \mathbb{R}$.

Since $V(x,y)$ is a Lyapunov function there are no closed orbits near x_*. ~~This~~ This is true ($\frac{dV}{dt} < 0$) for all of $\Omega = \mathbb{R}^2$, the neighborhood can be arbitrarily large so x_* must be asymptotically stable and all of Ω must be the basin of attraction of x_*.

Since linearization showed us that the eigenvalues have complex components at the origin, x_* must exhibit oscillatory dynamics so it must be a nonlinear spiral sink

6.4.9. a. if G is constant, we treat it as so

$z = \begin{bmatrix} I \\ C \end{bmatrix}$ $\quad \dot{I} = I - \alpha C = 0 \Rightarrow I = \alpha C$

$\dot{C} = \beta(I - C - G) = 0 \Rightarrow 0 = \cancel{\beta I} - \beta C - \beta G$
$\beta\alpha C$

$\beta(\alpha C - C - G) = 0$

$\alpha C - C - G = 0$

$C(\alpha - 1) = G \therefore C = \frac{G}{\alpha - 1}$ $\quad I = \frac{\alpha G}{\alpha - 1}$

At these values of C and I, $\dot{C} = 0$ and $\dot{I} = 0$ so this is a fixed point (equilibrium) of the system.

This equation is linear so we can calculate

$J(z) = \begin{bmatrix} 1 & -\alpha \\ \beta & -\beta \end{bmatrix} = A$

$Tr(A) = 1 - \beta$

$det(A) = -\beta + \alpha\beta = \beta(\alpha - 1)$

note $\alpha > 1$ in the problem and $\beta \geq 1$

$\therefore det(A)$ is always $> 0 \therefore$ no saddles

if $\beta = 1$, $Tr(A) = 1 - \beta = 1 - 1 = 0$ $\therefore$ complex λ's

$det(A) = \alpha - 1$

$\therefore$ equilibrium is a center so economy just oscillates, purely imaginary eigenvalue

if $\beta > 1$, then $Tr(A) < 0 \therefore$ source not possible

if $Tr(A)^2 > 4det(A)$, have real λ's

$(1-\beta)^2 > 4\beta(\alpha - 1)$

if $\beta > 1$,

so if $(1-\beta)^2 > 4\beta(\alpha-1)$, fixed point is a sink (stable node)

if $(1-\beta)^2 < 4\beta(\alpha - 1)$, fixed point is a spiral sink (stable spiral)

if $(1-\beta)^2 = 4\beta(\alpha-1)$ we only get one eigenvalue that is negative so we have a stable star

6.4.9 (cont.). for $\dot{I} = I - \alpha C = 0$ $\quad k > 0$

$I = \alpha C$

$\dot{C} = \beta(I - C - G_0 - kI) = 0$

$\beta(\alpha C - C - G_0 - k\alpha C) = 0$

$C(\alpha - 1 - k\alpha) = G_0$

$$C = \frac{G_0}{\alpha(1-k)-1}$$

So since $\alpha > 1$, need $1 - k > 0$ $\therefore k < 1$
if $k > 1$, then the equilibrium ceases to exist for Eq. to be in 1st nonnegative quadrant,

$\therefore k_c = 1$

$\therefore$ if $k > k_c$, then Eq. is in ~~a~~ nonrealistic quadrant (not 1st quadrant)

so to have economically sensible equilibrium, need

$\alpha(1-k) - 1 > 0$

$\therefore \alpha(1-k) > 1$

$\therefore 1 - k > \frac{1}{\alpha}$ $\quad \therefore k < \underbrace{1 - \frac{1}{\alpha}}$

$\alpha > 1$

so $1 - 1/\alpha$ must be ≤ 1,

so $k < 1$

Let's get ~~eigenvectors~~ eigenvalues:

$$A = \begin{bmatrix} 1 & -\alpha \\ \beta(1-k) & -\beta \end{bmatrix}$$

The model is linear in I and C, so we can't get limit cycles and only possible long-term behavior is for economy to diverge to infinity when Eq. is not in 1st quadrant

$\lambda = \frac{1}{2}\mathrm{tr}(A) \pm \sqrt{\frac{1}{4}\mathrm{Tr}(A)^2 - \det(A)}$

$\lambda = \frac{1}{2}(1-\beta) \pm \sqrt{\frac{1}{4}(1-\beta)^2 - (-\beta + \alpha\beta(1-k))}$

$\mathrm{Tr}(A) = 1 - \beta$ $\quad$ if $\beta > 1$, then $\mathrm{Tr}(A) < 0$

$\det(A) = (1-\beta) + \alpha(\beta(1-k)) = (1-\beta) + \alpha\beta(1-k) = 1 + \beta(\alpha(1-k) - 1)$

So, if $k > k_c = 1$, then 2nd term is negative in $\det(A)$.
if $\beta > 1$, 1st term in $\det(A)$ is negative
$\therefore \det(A) < 0$
$\therefore$ we have a saddle. This means that long term behavior of economy is to diverge, so I and C keep increasing since they cannot be negative.

6.5.6.

a. $\dot{x} = -kxy$ $\qquad \dot{x} = 0 = -kxy$ $\therefore x = 0$ or $y = 0$

$\dot{y} = kxy - ly$ $\qquad \dot{y} = 0 = kxy - ly$ $\therefore y(kx - l) = 0$

$y = 0$ or $kx - l = 0$

$x = \frac{l}{k}$

if $y = 0$, both $\dot{x} = 0$ and $\dot{y} = 0$
so we have a line of fixed points

$$J\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -ky & -kx \\ ky & kx - l \end{bmatrix}$$

pick a point on x axis so $y = 0$

$x \neq 0$ $(x > 0)$

$$J\left(\begin{bmatrix} l/k \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -kx \\ 0 & kx - l \end{bmatrix} = A$$

$\det(A) = 0$

$\mathrm{tr}(A) = kx - l$

so if $x > l/k$, the line of fixed points on the x axis is unstable. If $x < l/k$ the line of fixed points on the x axis is stable

if $x = l/k$ the fixed pt is neither stable nor unstable as $\mathrm{tr}(A) = 0$

b. nullclines when

$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ or $\quad$ $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \alpha\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

x is constant

line

$\therefore x = l/k$

$kxy - ly = 0$

$\therefore x = l/k$

y is a free parameter

$\dot{x} = -kxy = 0$

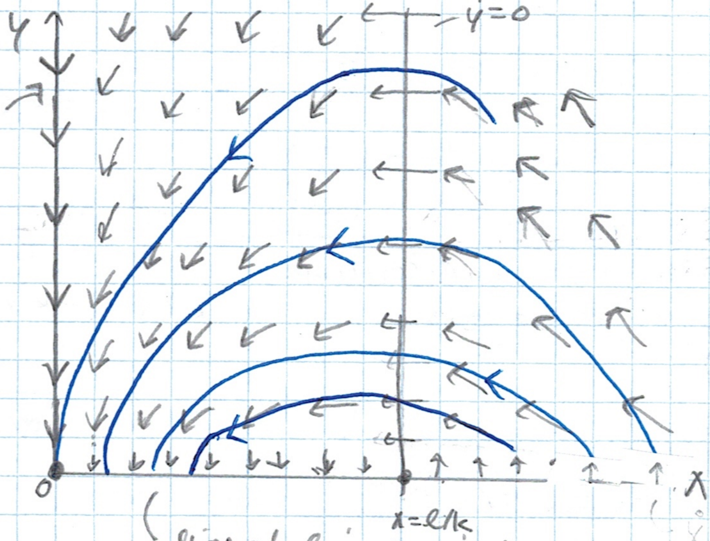
Line $x = 0$

Line $y = 0$ $\quad$ - if $y = 0$ $\dot{y}$ also $= 0$

6.5.6 (cont.):

$k, l > 0$

b.



trajectories must start at $y_0 > 0$ (as $y=0$ is fixed point).

line of fixed points in which stable if $x < l/k$, unstable if $x > l/k$, neither if $x = l/k$

c. $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = (kxy - ly)(-kxy)^{-1}$

$$\frac{dy}{dx} = \frac{kxy - ly}{-kxy} = -1 + \frac{l}{kx}$$

$$\int dy = \int (-1 + l/(kx))\, dx$$

$$y = -x + \frac{l}{k}\ln(x) + C \quad C \text{ is a constant}$$

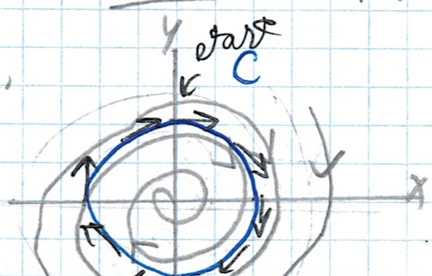
$\therefore E(x,y) = C = y + x - \frac{l}{k}\ln(x)$ so this quantity is conserved for system (C is constant on trajectories)

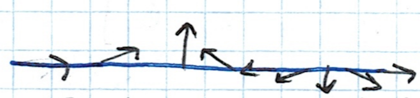
d. I added flow trajectories to my plot in part b. As $t \to \infty$ $y \to 0$ asymptotically, so the epidemic slowly dies out, but y never reaches exactly 0 in finite time

e. We saw that $y=0$ is unstable if $x > l/k$ so for epidemic need $x > l/k$. Epidemic occurs if $x > l/k$

6.8.1:

a.

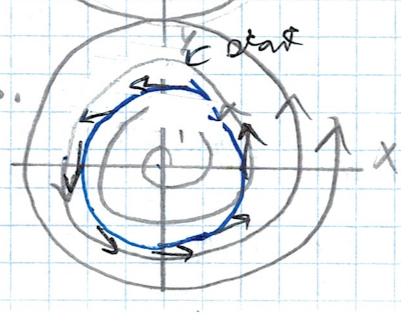




Going counterclockwise

Index: $I_C = +1$

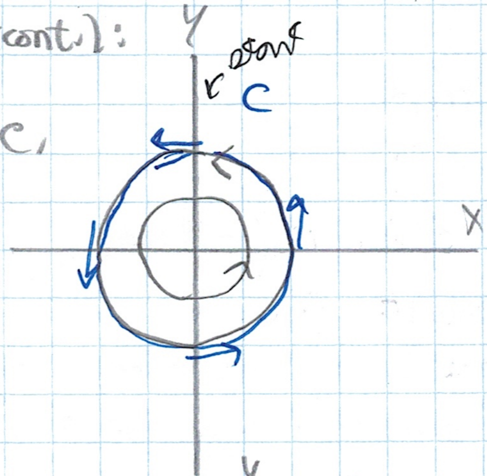
b.

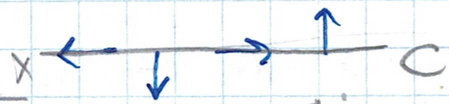


Index counterclockwise so $I_C = +1$

6.8.1 (cont.):

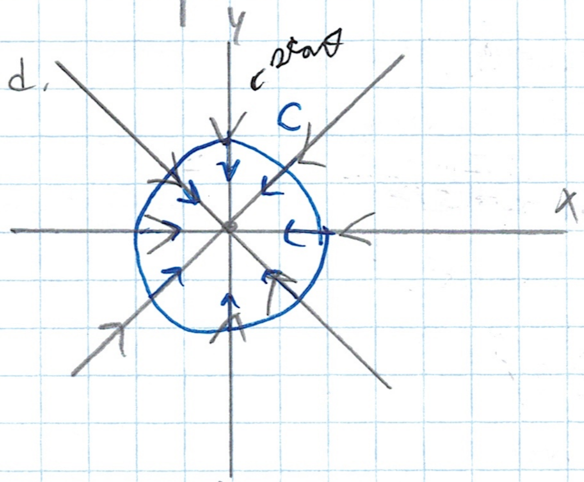
c.

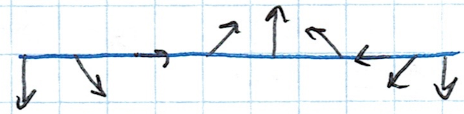




1 rotation counterclockwise
so $I_C = +1$

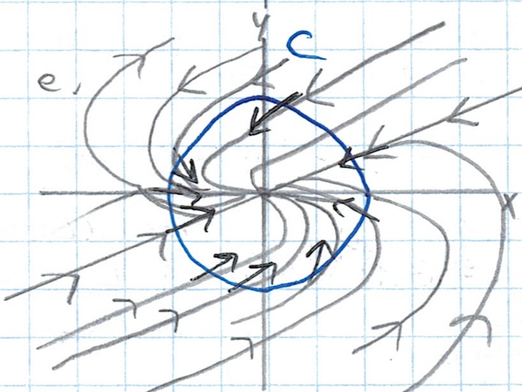
d.

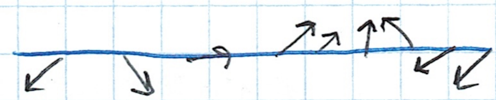




1 rotation counterclockwise
so $I_C = +1$

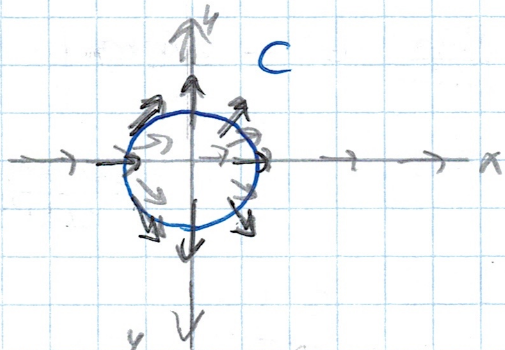
e.

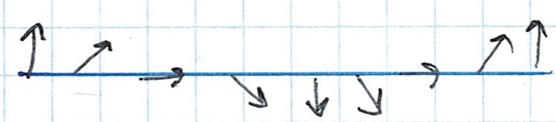




1 rotation counterclockwise
∴ $I_C = +1$

6.8.2:
$\dot{x} = x^2$
$\dot{y} = y$

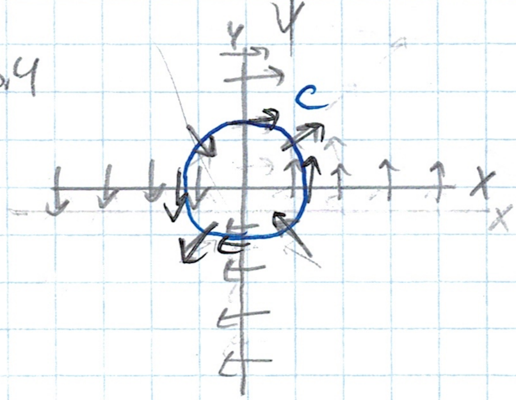


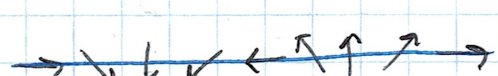


0 rotations counterclockwise
∴ $\boxed{I_C = 0}$

6.8.4

$\dot{x} = y^3$
$\dot{y} = x$





1 rotation clockwise
∴ $\boxed{I_C = -1}$

Conservative Systems:

1. $E = MLg - gML\cos(\theta(t)) + \frac{1}{2}ML^2\dot{\theta}(t)^2$

$$\frac{dE(\theta(t),\dot{\theta}(t))}{dt} = gML\sin(\theta(t))\cdot\dot{\theta}(t) + ML^{\cancel{2}}\dot{\theta}(t)\cdot\cancel{\dot{\theta}(t)}\left(-\frac{g}{\cancel{L}}\sin(\theta(t))\right)$$

$$= MLg\sin(\theta)\cdot\dot{\theta} - MLg\sin(\theta)\cdot\dot{\theta} = 0$$

$$\therefore \frac{dE}{dt} = 0 \quad \text{(constant } E \text{ along trajectories)}$$

Since $E(\theta(t),\dot{\theta}(t))$ is nonconstant on open sets but is constant on trajectories (since we just found $\frac{d}{dt}E(\theta(t),\dot{\theta}(t))$, this means that any trajectory $\theta(t)$ must move along inside level set E such that $E(\theta(t),\dot{\theta}(t)) = G(\theta(0),\dot{\theta}(0))$. This means that sources and sinks cannot exist.

2. See computer pages

3. see computer pages

Two-Body Problem

1.

$$L = \left(\frac{r(t)}{M+m}\right)^2 \left(\overset{Mm(M+m)}{\cancel{mM^2+Mm^2}}\right)\dot{\theta}(t) = \frac{r(t)^2}{(M+m)^{\cancel{2}}}\cdot Mm\cancel{(M+m)}\cdot\dot{\theta}(t)$$

$$L = \frac{Mm}{M+m}r(t)^2\dot{\theta}(t) \checkmark$$

Since L is constant, all trajectories are confined to the same L (L must be constant throughout an orbit trajectory). So, we can just rearrange the equation such that we write $\dot{\theta}(t)$ in terms of the other variables. $\Rightarrow \dot{\theta}(t) = \left(\frac{M+m}{Mm}\right)\frac{L}{r(t)^2}$ ✓

This equation must also be conserved over trajectories.

2.

$$\frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = \frac{M+m}{Mm}\left[\frac{Mm}{M+m}r\dot{\theta}^2 - \frac{GMm}{r(t)^2}\right]$$

$$= r\dot{\theta}^2 - \frac{G(M+m)}{r(t)^2} = \cancel{r}\left(\frac{M+m}{Mm}\right)^2\cdot\frac{L^2}{r(t)^{\cancel{4}3}} - \frac{G(M+m)}{r(t)^2}$$

$$\boxed{\frac{d^2}{dt^2}r(t) = \left(\frac{M+m}{Mm}\right)^2\frac{L^2}{r(t)^3} - \frac{G(M+m)}{r(t)^2} = \frac{d}{dt}\dot{r}(t)}$$

$$\boxed{\frac{d}{dt}\begin{bmatrix} r \\ \frac{d}{dt}r \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} r \\ \dot{r} \end{bmatrix} = \begin{bmatrix} \dot{r}(t) \\ \left(\frac{M+m}{Mm}\right)^2\frac{L^2}{r(t)^3} - \frac{G(M+m)}{r(t)^2} \end{bmatrix}}$$

Two Body Problem (cont.):

1. Mechanical Energy $E(r,\dot r) = \frac{1}{2} m\left[\left(\frac{M}{M+m} r\dot\theta\right)^2 + \left(\frac{M}{M+m}\dot r\right)^2\right] + \frac{1}{2} M\left[\left(\frac{m}{M+m} r\dot\theta\right)^2 + \left(\frac{m}{M+m}\dot r\right)^2\right] - \frac{GMm}{r}$

$$= \frac{1}{2} m\left(\frac{\cancel{M}}{\cancel{M+m}}\cancel{r}\cdot\frac{\cancel{M+m}}{\cancel{M}m}\frac{L}{r(t)^{\cancel{2}}}\right)^2 + \frac{1}{2} m\left(\frac{M}{M+m}\dot r\right)^2 + \frac{1}{2} M\left(\frac{\cancel{m}}{\cancel{M+m}}\cancel{r}\cdot\frac{\cancel{M+m}}{M\cancel{m}}\frac{L}{r^{\cancel{2}}}\right)^2$$

$$+ \frac{1}{2} M\left(\frac{m}{M+m}\dot r\right)^2 - \frac{GMm}{r}$$

$$= \frac{\cancel{m}}{2}\cdot\frac{L^2}{m^{\cancel{2}}r^2} + \frac{m}{2}\cdot\frac{M^2\dot r^2}{(M+m)^2} + \frac{\cancel{M}}{2}\cdot\frac{L^2}{M^{\cancel{2}}r^2} + \frac{M}{2}\frac{m^2\dot r^2}{(M+m)^2} - \frac{GMm}{r}$$

$$= \frac{1}{2}\left(\frac{L^2}{mr^2} + \frac{L^2}{Mr^2} + \frac{M^2m\dot r^2 + Mm^2\dot r^2}{(M+m)^2}\right) - \frac{GMm}{r}$$

$$= \frac{1}{2}\left(\frac{M+m}{Mm}\cdot\frac{L^2}{r^2} + \frac{Mm(\cancel{M+m})\dot r^2}{(M+m)^{\cancel{2}}}\right) - \frac{GMm}{r}$$

$$E(r,\dot r) = \frac{1}{2}\left(\frac{M+m}{Mm}\cdot\frac{L^2}{r^2} + \frac{Mm}{M+m}\cdot\dot r^2\right) - \frac{GMm}{r} \checkmark$$

2. $\frac{d}{dt}E(r(t),\dot r(t)) = \frac{1}{\cancel{2}}\cdot\cancel{(-2)}\,\frac{M+m}{Mm}L^2 r^{-3}\dot r(t) + \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot\frac{Mm}{M+m}\cdot\dot r\cdot\cancel{\frac{d}{dt}\dot r(t)}$

$$\left[\left(\frac{M+m}{Mm}\right)^2\frac{L^2}{r^3} - \frac{G(M+m)}{r^2}\right] \cancel{-(-1)}\,GMm r^{-2}\dot r(t)$$

$$= \cancel{-\frac{M+m}{Mm}\frac{L^2}{r^3}\cdot\dot r} + \cancel{\frac{Mm}{M+m}\dot r\cdot\left(\frac{M+m}{Mm}\right)^2\frac{L^2}{r^3}} - \cancel{\frac{Mm}{M+m}\cdot\dot r\cdot G(M+m)\cdot\frac{1}{r^2}} + \cancel{\frac{GMm}{r^2}\cdot\dot r}$$

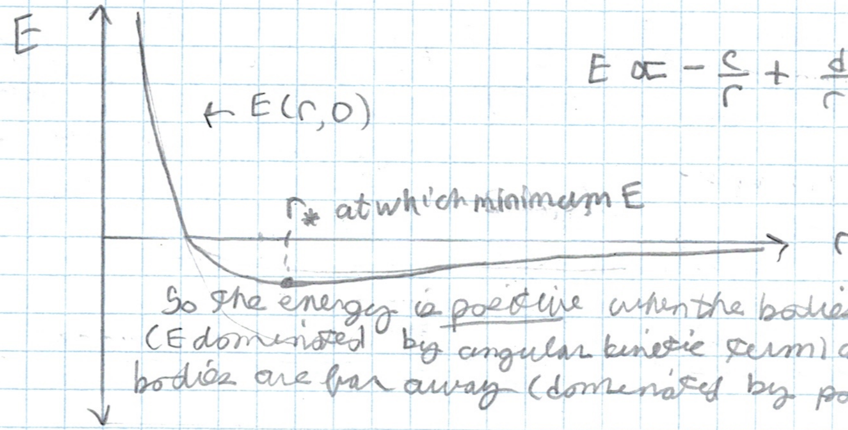
$= 0 \checkmark$ $\quad \frac{dE}{dt} = 0$ so energy is conserved.

Two-Body (cont'd):

Gravitational Potential Term — Angular kinetic term

3. If $\dot{r}=0$, $E(r,0) = -\frac{GMm}{r} + \frac{1}{2}\left[\frac{(M+m)L^2}{Mmr^2} + 0\right] = -\frac{GMm}{r} + \frac{(M+m)L^2}{2Mmr^2}$

Thus the energy function is dominated by the angular kinetic term for fixed $L>0$ as it is proportional to $\frac{1}{r^2}$. This also means that at large r, this term will become small and the gravitational potential term will dominate



$E \propto -\frac{c}{r} + \frac{d}{r^2}$ (c, d are real-valued constants)

So the energy is positive when the bodies are very close (E dominated by angular kinetic term) and negative when bodies are far away (dominated by potential term).

4. From our sketch, we saw that $E(r,0)$ has one minimum at a $E<0$. It occurs when

$$\frac{d}{dr}E(r,0)=0$$

$$\frac{d}{dr}E = \cancel{-}(-1)\,GMm\,r^{-2} + \frac{1}{\cancel{2}}\cdot(-\cancel{2})\frac{(M+m)}{Mm}L^2 r^{-3}$$

$$= \frac{GMm}{r^2} - \frac{M+m}{Mm}\frac{L^2}{r^3} = 0$$

$$= \frac{1}{r^2}\left(GMm - \frac{M+m}{Mm}\frac{L^2}{r}\right) = 0$$

$$\hookrightarrow r\,\cancel{GMm} = \frac{\frac{M+m}{Mm}\frac{L^2}{\cancel{r}}}{GMm} \qquad \therefore r = \frac{(M+m)L^2}{GM^2m^2}$$

This is where $E(r,0)$ is minimized

$$\frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = 0 \quad \text{when} \quad \left(\frac{M+m}{Mm}\right)^2\frac{L^2}{r(t)^3} = \frac{G(M+m)}{r^2} = 0$$

$$\frac{1}{r^2}\left(\left(\frac{M+m}{Mm}\right)^2\frac{L^2}{r} - G(M+m)\right) = 0$$

$$\downarrow$$

$$r\,\cancel{G(M+m)} = \left(\frac{\cancel{M+m}}{Mm}\right)^{\cancel{2}}\frac{L^2}{\cancel{r}}\cdot\frac{1}{G\cancel{(M+m)}}$$

$$r = \frac{(M+m)L^2}{GM^2m^2}$$

So both $E(r,0)$ and $\dot{r}(t)$ are minimized at the same distance $r = \frac{(M+m)L^2}{GM^2m^2}$. If two mass start at this r with $\dot{r}=0$ then r will stay at this value throughout the orbit so the two masses must follow a circular orbit. Physically, this distance corresponds to the distance at which the gravitational force balances the centrifugal "force" for a given angular momentum

since $\frac{d}{dt}\dot{r}=0$

5. If $\dot{r} \neq 0$ then $E(r,\dot{r}) = -\frac{GMm}{r} + \frac{1}{2}\frac{M+m}{Mm}\frac{L^2}{r^2} + \frac{Mm}{2(M+m)}\dot{r}^2$
$\downarrow$

The planets are moving with a nonzero radial velocity, so they are getting closer together or further apart. An equilibrium (in the r-$\dot{r}$ plane) can only occur when $\dot{r}=0$ as if $\dot{r} \neq 0$ then r is increasing or decreasing, so we just need to show that $\dot{r}$ cannot go to 0 and stay at 0 as t increases. We know that if $\dot{r}=0$, then the equilibrium at which $\dot{r}$ stays at 0 is for $r = \frac{(M+m)L^2}{GM^2m^2}$ (see #4). So we have $\dot{r}=0$, $r = \frac{(M+m)L^2}{GM^2m^2}$ for all time as a valid trajectory. Since our governing equation is continuously differentiable, if $\dot{r}$ starts at a nonzero value, then the trajectory cannot merge to the $\dot{r}=0$, $r = \frac{(M+m)L^2}{GM^2m^2}$ rest trajectory as trajectories cannot merge due to the uniqueness THM.

~~So~~ if $\dot{r} \neq 0$ then the system can't be at rest and the only possible equilibrium occurs along $\dot{r}=0$.

We already solved for the radius of the equilibrium in #4 and found that $\dot{r}=0$, $\frac{d}{dt}\dot{r}=0$ at $r = \frac{(M+m)L^2}{GM^2m^2}$

6. $E(r,\dot{r}) = \underbrace{-\frac{GMm}{r} + \frac{1}{2}\frac{M+m}{Mm}\frac{L^2}{r^2}} + \frac{1}{2}\frac{Mm}{M+m}\dot{r}^2$

if r is constant, this is a constant (call it c)

$\therefore$ with r fixed, energy function $\therefore E(\dot{r}) = c + \left(\frac{1}{2}\frac{Mm}{M+m}\right)\dot{r}^2$ is parabolic and increasing in $|\dot{r}|$.

$\therefore$ For a given r, the ~~minimum~~ only global of $E(\dot{r})_{r=r_*}$ is when $\dot{r}=0$. And we showed in #4 that for $\dot{r}=0$, the global minimum of $E(r)$ is when $r_* = \frac{(M+m)L^2}{GM^2m^2}$. Since $E(\dot{r})_{r=r_*}$ and $E(r)_{\dot{r}=0}$ both increase as one leaves the equilibrium, the energy function has a unique global minimizer at positive $r_* = \frac{(M+m)L^2}{GM^2m^2}$ and $\dot{r}=0$. Since the system conserves energy, solution trajectories must remain on level sets of the energy function. Since $E(\dot{r})_{r=r_*}$ and $E(r)_{\dot{r}=0}$ increase as one goes away from the equilibrium, so the energy function is approximately parabolic near the equilibrium, the equilibrium cannot be a saddle and must be a center. Again, since energy is conserved, we know that attracting/repelling equilibria cannot exist and equilibria must be saddles or centers. Since $E(r,\dot{r})$ has a global minimum at $r=r_*$, $\dot{r}=0$ and increases away from it, so the equilibrium must be a center.

and equilibria can only be saddles or centers

7. See computer pages

8. See computer pages

Strogatz Ch. 7:

7.2.5: a. If this is a gradient system then $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\partial V/dx \\ -\partial V/dy \end{bmatrix}$

$$\frac{d}{dt}x = \frac{-\partial V}{\partial x} = f(x,y) \qquad \therefore \frac{-\partial^2 V}{\partial x \partial y} = \frac{\partial f}{\partial y}$$

$$\frac{d}{dt}y = \frac{-\partial V}{\partial y} = g(x,y) \qquad \frac{-\partial^2 V}{\partial y \partial x} = \frac{-\partial^2 V}{\partial x \partial y} = \frac{\partial g}{\partial x}$$

$$\therefore \frac{\partial f}{\partial y} = \frac{-\partial^2 V}{\partial x \partial y} = \frac{\partial g}{\partial x}$$

$$\therefore \frac{\partial f}{\partial y} = \frac{\partial g}{\partial x} \checkmark$$

b. $\vec{F} = f(x,y)\hat{i} + g(x,y)\hat{j}$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ f & g & 0 \end{vmatrix} = \left(\frac{\partial}{\partial x}g - \frac{\partial}{\partial y}f\right)\hat{k}$$

To be conservative (path independent), the curl of $\vec{F}$ must be zero. By Stokes theorem, $\oint_C \vec{F} \cdot d\vec{r} = \iint (\nabla \times \vec{F}) \cdot \vec{n}\, d\sigma$ so for the circulation to be 0 (~~so the~~ potential is conserved when one returns to the same position), then the curl must be 0.

$$\therefore \frac{\partial}{\partial x}g - \frac{\partial}{\partial y}f = 0 \qquad \therefore \frac{\partial g}{\partial x} = \frac{\partial f}{\partial y}$$ we have the same condition

7.2.9 a. $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} \overset{?}{=} \begin{bmatrix} y + x^2 y \\ -x + 2xy \end{bmatrix}$ Let $\dot{x} = -\frac{\partial U}{\partial x}$ $\therefore \int \partial V = \int -\dot{x}\, \partial x$

$$V = -xy - \frac{x^3 y}{3}$$

$$\partial V = \int -\dot{y}\, \partial y = -\int -x + 2xy\, \partial y$$

$$V = xy - \frac{xy^2}{2}$$

The 2 equations for V are not the same so the system is not a gradient system

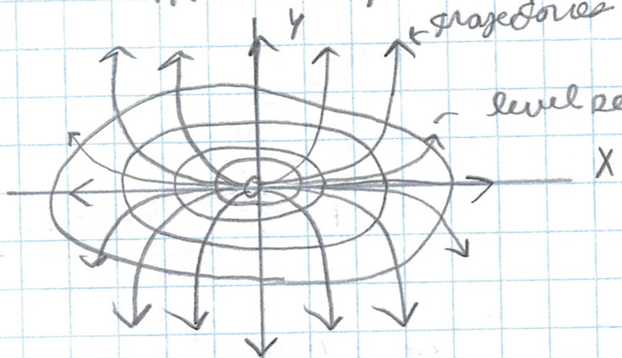
7.2.9 (cont.) $\dot{x} = 2x$ $\dot{y} = 8y$ Let $V(x,y) = -x^2 - 4y^2$

Then $-\frac{\partial V}{\partial x} = 2x = \dot{x}$

$\therefore$ Is a gradient system

$V(x,y) = -x^2 - 4y^2$

$-\frac{\partial V}{\partial y} = -8y = \dot{y}$



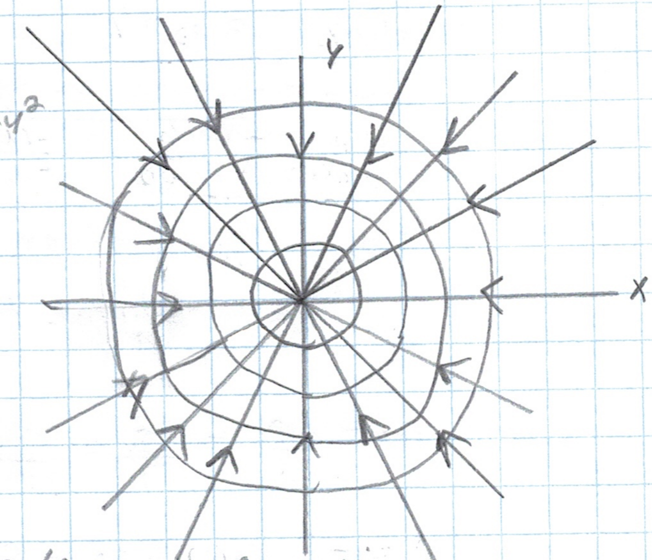
c. $\dot{x} = -2xe^{x^2+y^2}$ $\dot{y} = -2ye^{x^2+y^2}$

Let $V(x,y) = e^{x^2+y^2}$

$-\frac{\partial V}{\partial x} = -2xe^{x^2+y^2}$

$-\frac{\partial V}{\partial y} = -2ye^{x^2+y^2}$

$\therefore$ This is a gradient system



7.2.12 Let $V = x^m + ay^n$ be a Lyapunov function

if $a>0$, (m,n) are even, then $V(x,y) > 0$ for any $x,y \in \mathbb{R}^2$

$\frac{d}{dt}V = mx^{m-1}\dot{x} + any^{n-1}\dot{y} = mx^{m-1}(-x+2y^3-2y^4) + any^{n-1}(-x-y+xy)$

$= -mx^m + \underline{2mx^{m-1}y^3} - 2mx^{m-1}y^4$
$\underline{-anxy^{n-1}} - any^n + \underline{anxy^n}$

Want these to cancel Let $m=2$ $n=4$, $a=1$

$\therefore V = x^2+y^4$ $\frac{dV}{dt} = 2x(-x+2y^3-2y^4) + 4y^3(-x-y+xy)$
$= -2x^2 + \cancel{4xy^3} - \cancel{4xy^4} - \cancel{4xy^3} - 4y^4 + \cancel{4xy^4}$

$\frac{dV}{dt} = -2x^2 - 4y^4$

$\frac{dV}{dt} < 0$ for any $x,y \in \mathbb{R}^2$ $\therefore V(x,y) = x^2+y^4$ is a Lyapunov function

Since we found a Lyapunov function $V(x,y)$ at which $\underline{V(0,0)=0}$, $\underline{V(x,y)>0}$ for $x \in N = \mathbb{R}^2$, $x \neq x_0 = (0,0)$ and $\underline{\frac{d}{dt}V(x,y) < 0}$ then N is a forward invariant set with no closed orbits and $x_* = \langle 0,0 \rangle$ is asymptotically stable with N as its basin of attraction. $\therefore$ no periodic solutions

Three Body Problem:

1.

distance earth to sun $= 148.04 \cdot 10^6$ km

$M_{earth} = M_E = 5.972 \cdot 10^{24}$ kg

$M_{sun} = M_S = 1.989 \cdot 10^{30}$ kg

Let sun be at $r_S = 0$. Then COM is $r_{com} = \dfrac{M_S r_S + M_E r_E}{M_S + M_E}$

$r_E = 148.04 \cdot 10^6$ km

$$= \frac{0 + 5.972 \cdot 10^{24}\,\text{kg} \cdot 148.04 \cdot 10^9\,\text{m}}{5.972 \cdot 10^{24} + 1.989 \cdot 10^{30}\,\text{kg}}$$

$\dfrac{r_{cm}}{r_E} \cdot 100\% = \underline{3 \cdot 10^{-4}\%}$

$r_{cm} = 4.44 \cdot 10^5$ m

so center of mass of system is practically at the sun's $r\ 4 \cdot 10^5$ m center of mass (in fact $r_{cm} \ll$ sun's radius of $696 \cdot 10^6$ m)

$\hookrightarrow$ center of mass is contained inside the sun

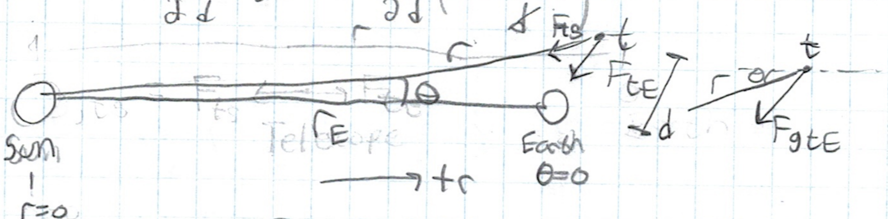
This is because the sun is much more massive than the earth.

$\therefore$ safe to assume that $\underline{r_{ts} = r}$

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} = \frac{2\pi}{3.154 \cdot 10^7\,\text{s}} = \boxed{1.99 \cdot 10^{-7}\,\text{s}^{-1}}$$

2. $F_g(d) = \dfrac{-\partial}{\partial d} V_g(d) = \dfrac{-\partial}{\partial d}\left(-\dfrac{GMm}{d}\right) = -d^{-2}(GMm) = \boxed{-\dfrac{GMm}{d^2} = F_g(d)}$

$\hookrightarrow$ This is the force due to gravity



$F_{gS}(r,\theta) = -\nabla V_S(r,\theta)$

$F_{gE}(r,\theta) = -\nabla V_E(r,\theta)$

$d = \| \vec{r} - \vec{r}_E \|$

$d = \| \langle r\cos\theta - r_E,\ r\sin\theta \rangle \|$

$V_{g,net}(r,\theta) = V_S(r,\theta) + V_E(r,\theta)$

$\therefore F_{g,net} = F_{g,sun} + F_{g,Earth}$

$= -\nabla V_S(r,\theta) - \nabla V_E(r,\theta)$

$F_{g,net} = -\nabla(V_S(r,\theta) + V_E(r,\theta))$

$V_S(r,\theta) = -\dfrac{GM_S m_t}{r}$

$V_E(r,\theta) = -\dfrac{GM_E m_t}{d}$

$\therefore F_{g,net} = -\nabla(V_{g,net}(r,\theta))$

3. Let $V_C(r) = -\frac{1}{2} m_t r^2 \omega^2 \therefore -\partial_r V_c(r) = -\left(\frac{-1}{2}\right) 2 m_t r \omega^2 = m_t r \omega^2 = F_c$

$\therefore \boxed{V_c(r,\theta) = -\dfrac{m_t r^2 \omega^2}{2}}$ is potential function

4. $X(t) = \begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix} \quad \dfrac{d^2}{dt^2} X(t) = \begin{bmatrix} d/dt\ r(t) \\ d/dt\ \theta(t) \end{bmatrix}$

$V(x) = V_S(r,\theta) + V_E(r,\theta) + V_c(r,\theta)$

$V(x) = -\dfrac{GM_S m_t}{r} - \dfrac{GM_E m_t}{d} - \dfrac{m_t r^2 \omega^2}{2}$

$-\nabla V(r,\theta) = -\nabla V_{g,net}(r,\theta) - \nabla V_c(r,\theta)$

$-\nabla V(r,\theta) = F_{g,net}(r,\theta) + F_c(r,\theta) = F_{net}(r,\theta) = m_t a$

$\therefore$ - gradient of total potential is the total force (as gradient is distributive). (gradient of sum is sum of gradients)

4 (cont) We know $F_{net} = ma$

This is potential in terms of the forces (has mass)

$$\Rightarrow a = \frac{F_{net}}{m_t}$$

$$\Rightarrow \frac{d^2}{dt^2}x = \frac{1}{m_t}F_{net}(r,\theta) = \frac{1}{m_t}\left(-\nabla V(r,\theta)\right) \therefore \frac{d^2x}{dt^2} = \frac{-\nabla V(r,\theta)}{m_t}$$

$$=$$

$$\frac{d^2x}{dt^2} = \frac{-\nabla V(r,\theta)}{m_t} = \frac{-\nabla}{\cancel{m_t}}\left(-\frac{GM_S\cancel{m_t}}{r} - \frac{GM_E\cancel{m_t}}{d} - \frac{\cancel{m_t}r^2\omega^2}{2}\right)$$

$$\therefore \frac{d^2x}{dt^2} = -\nabla\left(-\frac{GM_S}{r} - \frac{GM_E}{d} - \frac{r^2\omega^2}{2}\right)$$

$$\therefore \frac{d^2x}{dt^2} = -\nabla V(x) \qquad x = \begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix}$$

$$d = \sqrt{(r\cos\theta - r_E)^2 + (r\sin\theta)^2}$$

This is the net potential in terms of the dynamics, cancel out mass of moving object

5. see computer

Homework 4

CAAM 28200: Dynamical Systems with Applications

Kameel Khabaz

February 27, 2022

Problem 6.3.10

My computer-generated phase portrait is shown below:

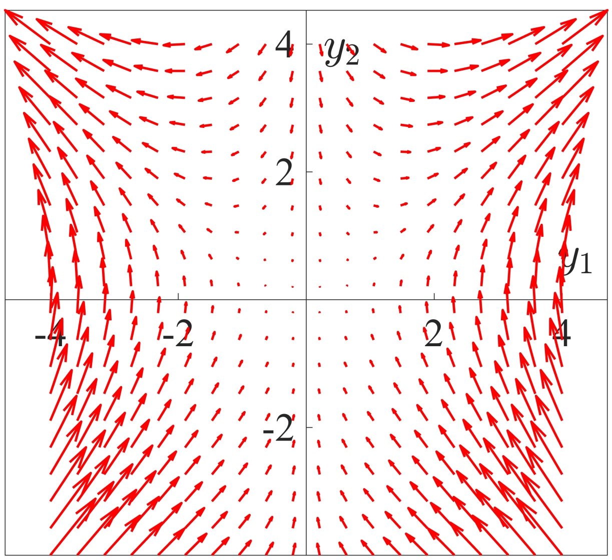


Figure 1: Phase Portrait

Conservative Systems and Energy Functions - Pendulum 2,3

My 3D surface plot showing the total energy function over the phase plane and my contour plot with the equilibria marked are shown below. Here I have bounded the velocity $\dot{\theta}$ by the maximum velocity for a pendulum of $v_{max} = \sqrt{2gh_{max}}$ We clearly see from the energy function and the level contours that the equilibrium is a center.

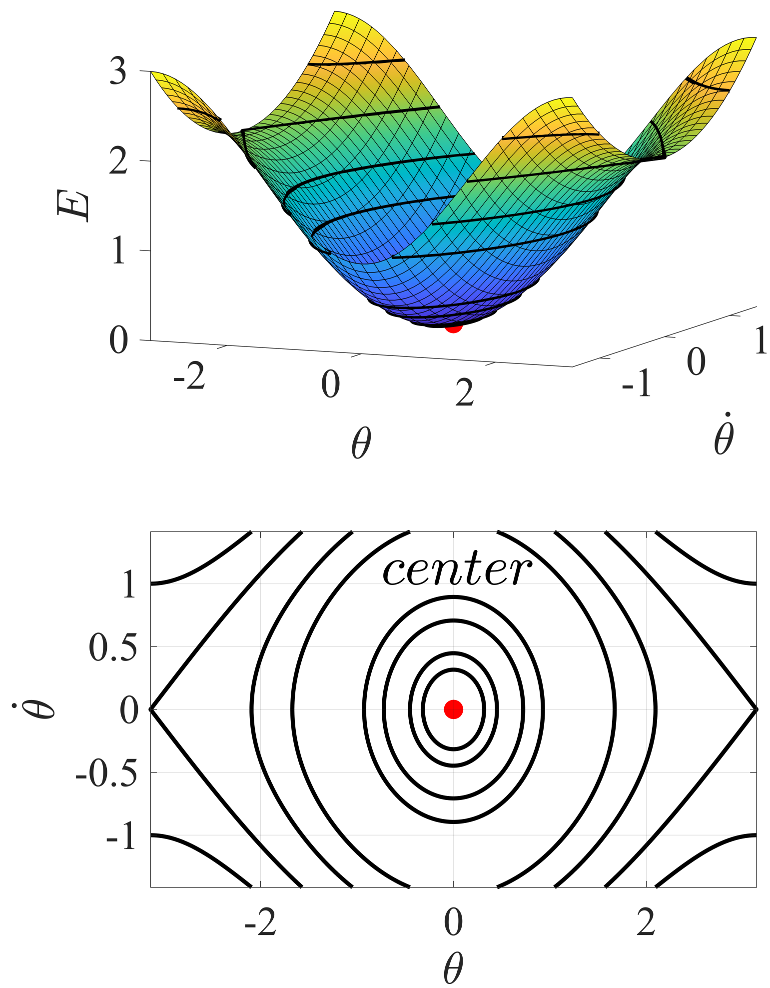


Figure 2: 3D Surface and Level Contour Plots

Two-Body Problem 7

I plotted the energy surface and energy level contours for the given masses and angular momentum. In my plots, I marked the equilibrium with a red dot and added arrows on the level sets to indicate the direction of motion of the system. I also did this with different values of L and m and M. The original plot is shown in Figure 3. In Figure 4, I increased L to $L = 0.2$. We see that as the angular momentum of the system increases, the equilibrium orbital radius must also increase. In Figure 5, I made the masses equal by increasing the smaller mass m to be equal to the larger mass. In Figure 6, I made $M >> m$, in which we can approximate the system as uniform circular motion of the smaller mass m around the larger mass M, which is approximately fixed.

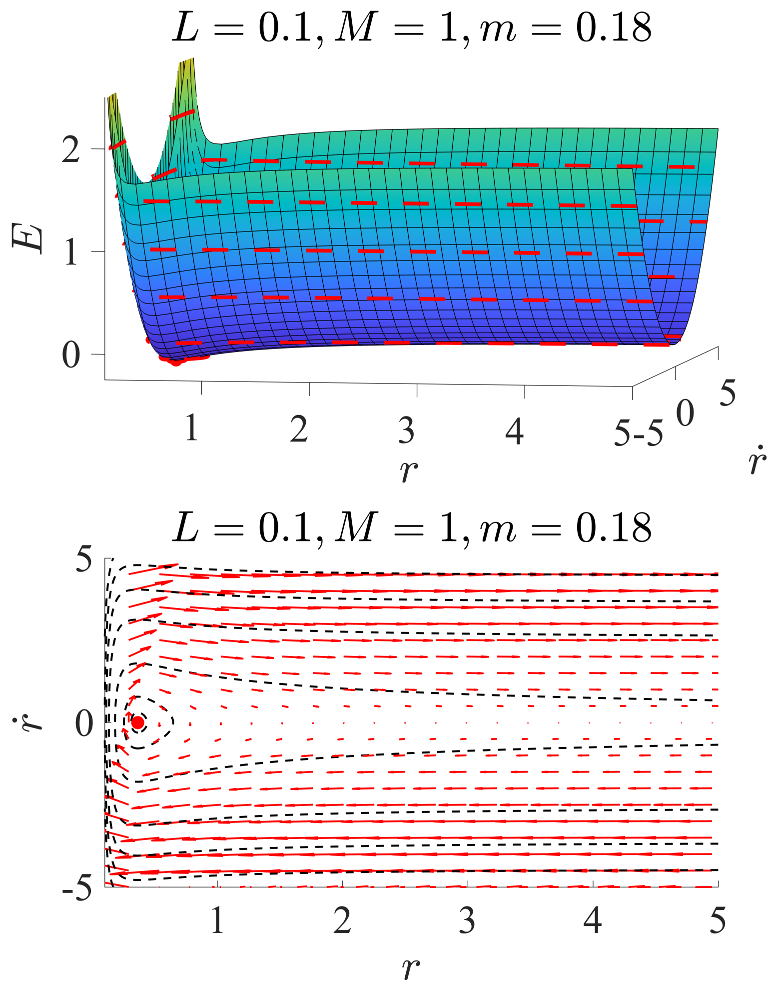


Figure 3: Two-Body Problem

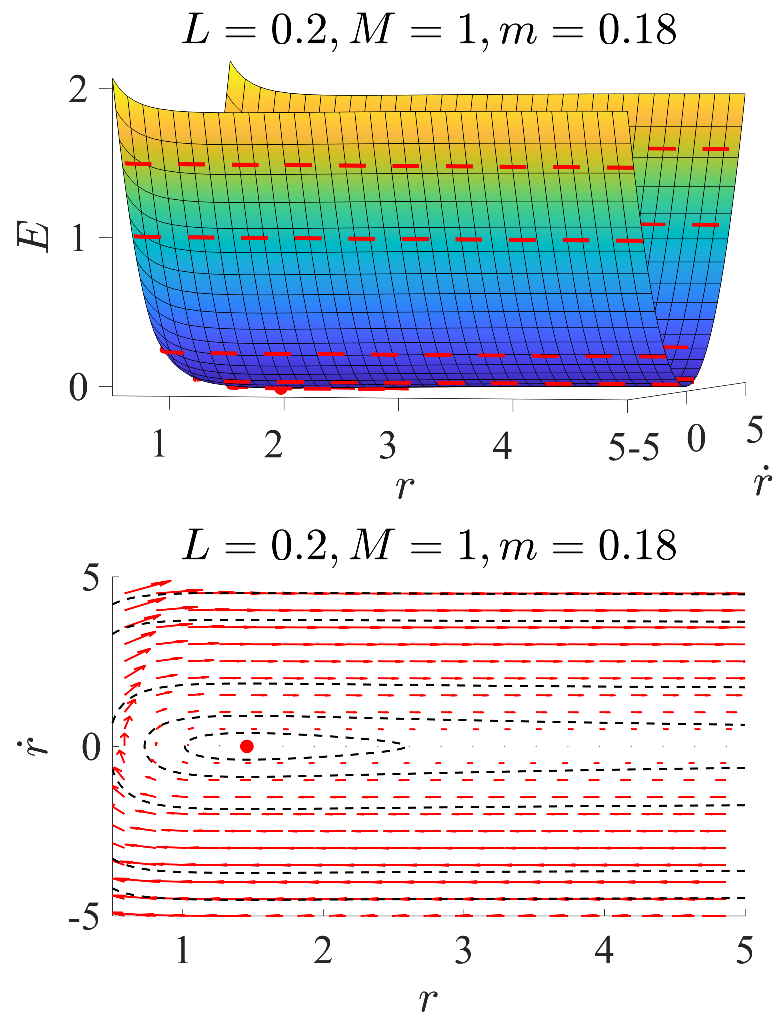


Figure 4: Two-Body Problem $L = 0.2$

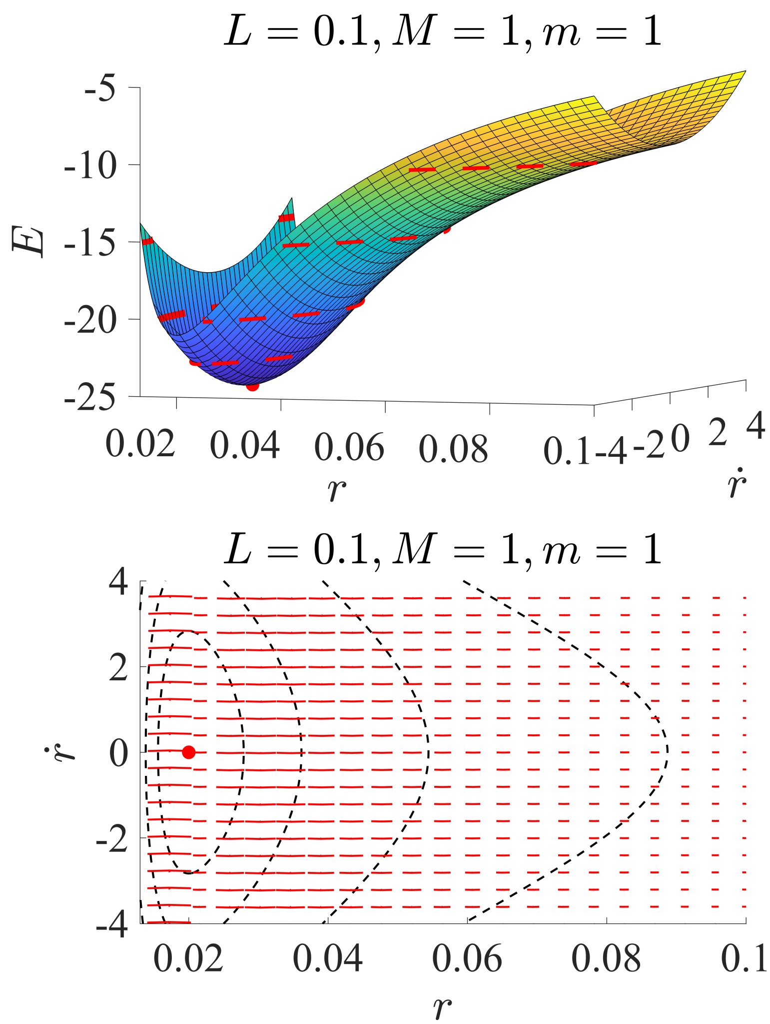
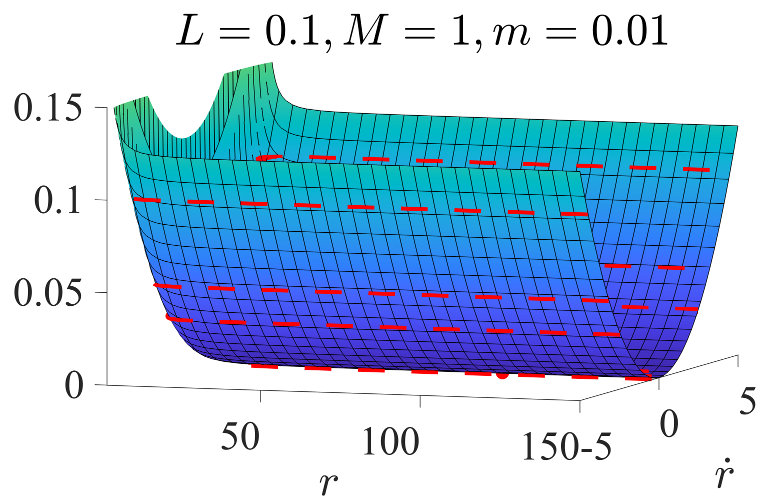


Figure 5: Two-Body Problem Equal Masses



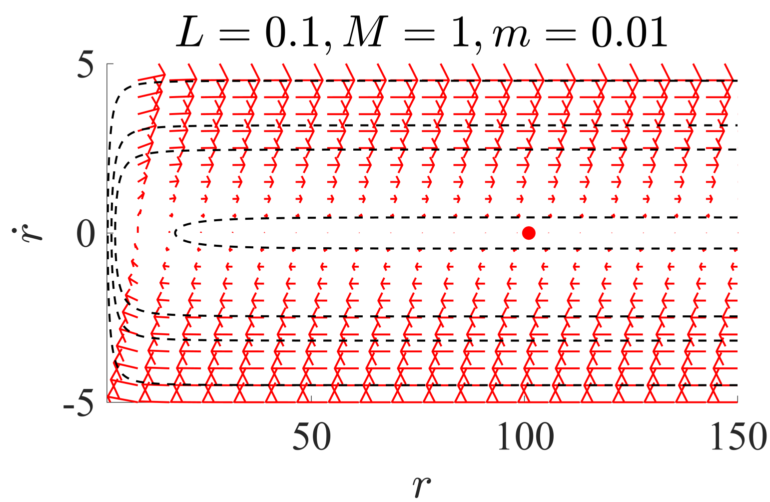


Figure 6: Two-Body Problem $M >> m$

Two-Body Problem 8

I plotted the energy contour in Figure 9, in which the level set for $E = 0$ is the black contour. I also plot a trajectory at negative energy in cyan, and I plot a trajectory at positive energy in magenta. Since kinetic energy must always be positive (if the planets are moving at all), then a negative energy must mean that the potential energy is greater than the kinetic energy. We see that at negative energy, the level sets form elliptical shapes in which the planets get farther and closer together in an elliptical orbit. At the positive energy level set, the planets seem to be just getting closer or farther away from each other, in which $\dot{r}$ stays constant but r increases to infinity as time increases to infinity. In this case, the initial kinetic energy must be greater than the initial potential energy. At the zero energy level set, we see that as r increases, $\dot{r}$ decreases, which means that the radial velocity will become smaller and smaller, so the planets will go farther apart at a slower and slower rate. This corresponds to a system in which at $r = \infty$, the planets have zero potential energy and zero kinetic energy. Then as potential energy becomes more negative as the planets get closer, kinetic energy also increases, and as potential energy increases (less negative as the planets get infinitely far away), kinetic energy decreases to 0.

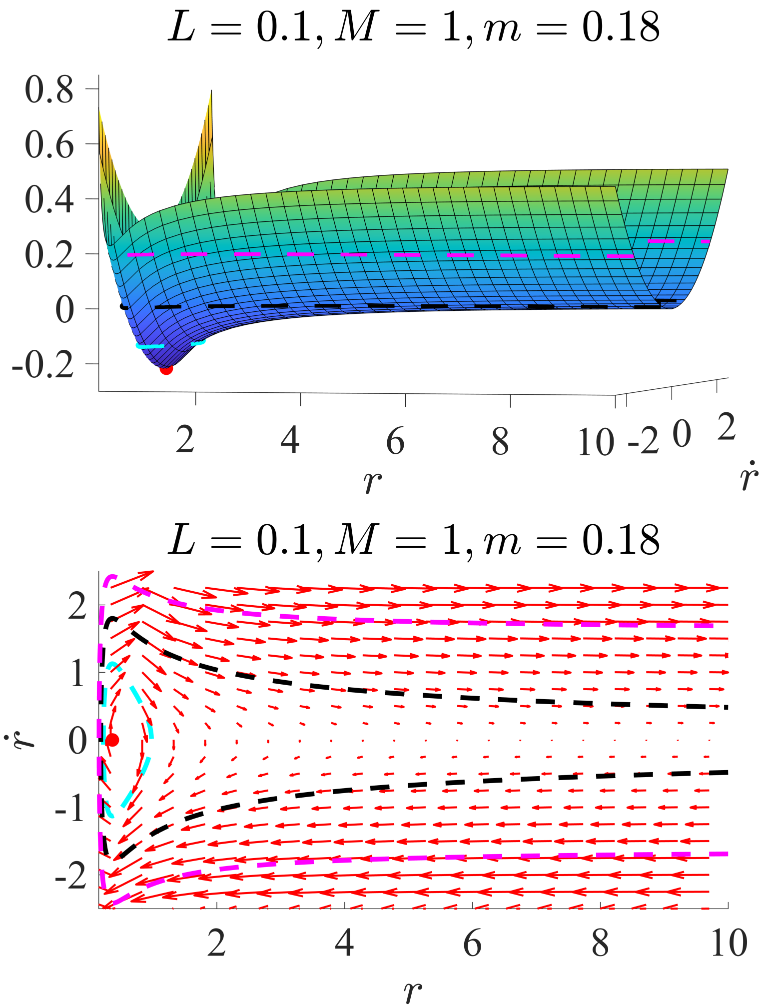


Figure 7: Two-Body Problem With Zero Energy Contour

Three-Body Problem 5

I plotted the potential function using simplified parameters of $M_S = 100$, $M_E = 5$, $r_E = 10$, $G = 1$, $\omega = \sqrt{G(\frac{M_S+M_E}{r_E^3})}$. My plot is shown below:

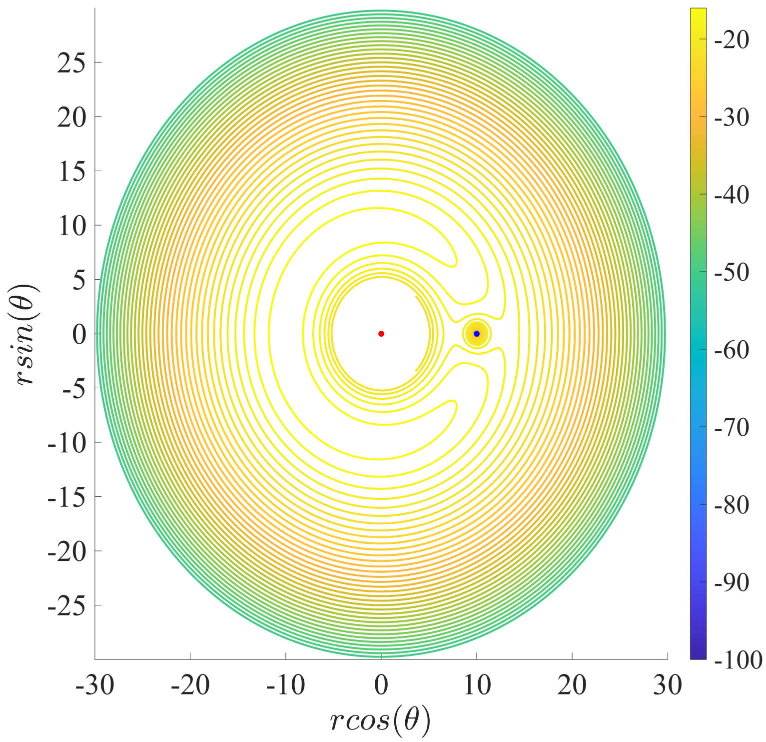


Figure 8: Three-Body Problem with Simplified Parameters

There are supposed to be 5 Lagrange points here. I can clearly see 3 of them, but L4/L5 aren't appearing. Using the physical constants, I am unable to get the Lagrange points to clearly appear because the very large gravitational potential of the sun is masking the effects of the gravitational potential of the centrifugal force.

Since we are plotting the potential, the system moves to minimize the potential energy. Contours depict regions of constant potential. The telescope will move across (perpendicular) to the contours to minimize its potential. We can see this in the governing equation for our system, in which $\frac{d^2}{dt^2}x(t) = -\nabla V(x)$, which means that the telescope will move in the direction that is opposite of the gradient of the potential function. This is the direction down the gradient along which the potential function decreases most quickly.

Three-Body Problem 6

There are supposed to be 5 Lagrange points, of which 3 (L1/L2/L3) are unstable saddle points, 2 (L4/L5) are stable local maxima (stability is due to the Coriolis force). In the potential function, there are minima near the earth and sun, as these correspond to the gravitational potential energy

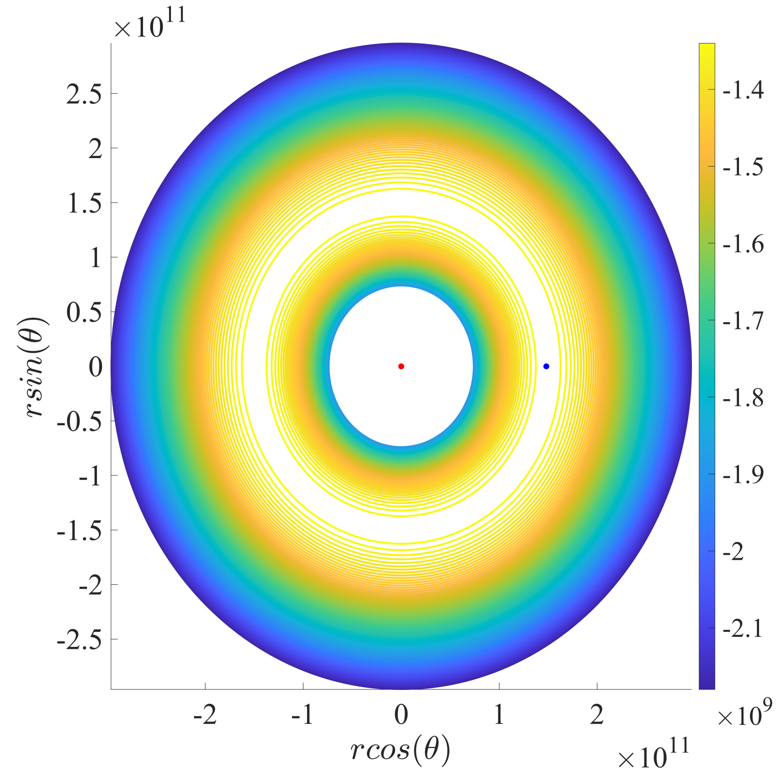


Figure 9: Three-Body Problem with Physics Parameters

terms with a very small radius from either planet.

1 Code

This is all of my code for this homework.

```
% Problem 6.3.10
close all
figure()
Energy = @(Y) [Y(1) * Y(2); Y(1) * Y(1) - Y(2)];
y1 = linspace(-4,4,20);
y2 = linspace(-4,4,20);
[x,y] = meshgrid(y1,y2);
u = zeros(size(x));
v = zeros(size(x));

for i = 1:numel(x)
    Yprime = Energy([x(i); y(i)]);
    u(i) = Yprime(1);
    v(i) = Yprime(2);
end
quiver(x,y,u,v,2,'r','LineWidth',1.5); figure(gcf)
xlabel('$y_1$','Interpreter','Latex')
ylabel('$y_2$','Interpreter','Latex')
set(gca,'FontSize',25,'FontName','times')
axis tight equal;
ax = gca;
ax.XAxisLocation = 'origin';
ax.YAxisLocation = 'origin';
exportgraphics(gcf,'6.3.10.png','Resolution',600)

%% Conservative Systems and Energy Functions: Problem 1.
close all
M = 1;
L = 1;
g = 1;
% Max velocity of pendulum is sqrt(2 * g * hmax)
maxv = sqrt(2 * g * L);
% Define energy grid
Energy = @(x,xdot) M.*L.*(g.*(1-cos(x)) + 0.5 * L * xdot .^2);
angles = linspace(-pi,pi,1000);
vs = linspace(-maxv,maxv,1000);
```

```
[anglesgrid, vsgrid] = meshgrid(angles,vs);
E_grid = Energy(anglesgrid,vsgrid);
xyint = [-pi pi -maxv maxv];

figure()
set(gcf,'Position',[0 0 600 800])
subplot(2,1,1);
hold on
fsurf(Energy, xyint);
scatter3(0,0,Energy(0,0),200,'r','filled')
contour3(anglesgrid, vsgrid, E_grid,[0 .05 .1 .25 .4 1.1 1.5 2 2.5
    3],'k','LineWidth',3);
axis tight
xlabel("$\theta$",'Interpreter','latex')
ylabel("$\dot{\theta}$",'Interpreter','latex')
zlabel("$E$",'Interpreter','latex')
set(gca,'FontSize',30,'FontName','Times')

subplot(2,1,2);
contour3(anglesgrid, vsgrid, E_grid,[0 .05 .1 .25 .4 1.1 1.5 2 2.5
    3],'k','LineWidth',3);
view([0 90])
xlabel("$\theta$",'Interpreter','latex')
ylabel("$\dot{\theta}$",'Interpreter','latex')
hold on
scatter3(0,0,Energy(0,0),200,'r','filled')
text(-.75,1.1,'$center$','Interpreter','latex','FontSize',40)
set(gca,'FontSize',30,'FontName','Times')

%exportgraphics(gcf,'Conservative.2.png','Resolution',600)

%% Two-Body Problem
close all
m = 0.18;
M = 1;
G = 1;
L = 0.1;
r = linspace(0.1,5,1000);
rdot = linspace(-5,5,1000);
```

```
rdot = linspace(-2.5,2.5,1000);
r = linspace(0.15,10,1000);
make_2body_plot(m,M,G,L,r,rdot);

%% Three Body Problem Contour Plot
clear all; close all;
Ms = 1.989 * 10^(30); % kg
Me = 5.972 * 10^(24); % kg
re = 148.04 * 10^9; % m radius of earth's orbit
w = 1.99 * 10^(-7); % s^-1
G = 6.67408 * 10^(-11);

%
Ms = 100;
Me = 5;
re = 10;
G = 1;
w = sqrt(G * (Ms + Me) / (re^3))
%}
V = @(r,theta) - (G .* Ms) ./ r - (G .* Me) ./ sqrt((r.*cos(theta) -
    re).^2 + (r.*sin(theta)).^2) - ((r.^2) .* (w.^2) ./ 2);

rs = linspace(0.5 * re,3*re, 300);
thetas = linspace(0, 2*pi,300);
[r,theta] = meshgrid(rs,thetas);
x = r.* cos(theta);
y = r.* sin(theta);

figure()
xlim([-max(rs) max(rs)])
ylim([min(y(:)) max(y(:))])
hold on
set(gcf,'Position',[0 0 800 800])
contour(x,y,V(r,theta),[-100:1:-2],'LineWidth',2);
%contour(x,y,V(r,theta),10^9 .*[-5:.1:-1.5 -10:1:-2],'LineWidth',2);
scatter(0,0,'r','filled')
scatter(re,0,'b','filled')
colorbar
```

```
set(gca,'FontSize',30,'FontName','Times')
xlabel("$rcos(\theta)$",'Interpreter','latex');
ylabel("$rsin(\theta)$",'Interpreter','latex');

function make_2body_plot(m,M,G,L,r,rdot)
    rstar = (M+m)*L^2 / (G*M*M*m*m);
    figure()
    set(gcf,'Position',[0 0 600 800])
    subplot(2,1,1);
    Energy = @(r,rdot) -G*M*m./r + (1/2) .* ( ((M + m)./(M * m)) .* (L^2 ./ r.^2) ...
                                + ((M.*m)./(M+m)) .* rdot.^2);
    xyint = [min(r) max(r) min(rdot) max(rdot)];
    hold on
    fsurf(Energy, xyint);
    % Make txhe mesh
    [rgrid, rdotgrid] = meshgrid(r,rdot);
    E_grid = Energy(rgrid,rdotgrid);
    levels = [-.05 0 .2 1 1.5];
    contour3(rgrid,rdotgrid,E_grid,[-.15 -.15001],'--c','LineWidth',5);
    contour3(rgrid,rdotgrid,E_grid,[0 0.0001],'--k','LineWidth',5);
    contour3(rgrid,rdotgrid,E_grid,[.2 .20001],'--m','LineWidth',5);

    axis tight
    title("$L = " + L + ", M = " + M + ", m = " + m + "$",'Interpreter','latex')
    xlabel("$r$",'Interpreter','latex')
    ylabel("$\dot{r}$",'Interpreter','latex')
    zlabel("$E$",'Interpreter','latex')
    set(gca,'FontSize',30,'FontName','Times')
    scatter3(rstar,0,Energy(rstar,0),100,'r','filled')
    xlim(xyint(1:2))
    ylim(xyint(3:4))
    zlim([-.3 .85])



    subplot(2,1,2);
```

```
    hold on
    contour3(rgrid,rdotgrid,E_grid,[-.15 -.15001],'--c','LineWidth',3.5);
    contour3(rgrid,rdotgrid,E_grid,[0 0.0001],'--k','LineWidth',3.5);
    contour3(rgrid,rdotgrid,E_grid,[.2 .20001],'--m','LineWidth',3.5);
    view([0 90])

    title("$L = " + L + ", M = " + M + ", m = " + m + "$",'Interpreter','latex')

    % Make arrows on level sets
    [x,y] = meshgrid(downsample(r(20:end),50),downsample(rdot(1:end),50));
    drdot = @(r,rdot) ((M+m)./(M.*m)).^2 * (L^2 ./ (r.^3)) - G.*(M+m) ./ (r.^2);
    drdotdata = drdot(x,y);
    quiver(x,y,y,drdotdata,1.5,'r','LineWidth',1.5);
    xlabel("$r$",'Interpreter','latex')
    ylabel("$\dot{r}$",'Interpreter','latex')
    set(gca,'FontSize',30,'FontName','Times')
    scatter(rstar,0,100,'r','filled')
    xlim(xyint(1:2))
    ylim(xyint(3:4))


    %exportgraphics(gcf,titl + ".png",'Resolution',600);

end
```