**STAT/CAAM 28200 - Assignment 2 - Due Wednesday February 9**

This is an individual assignment. You may collaborate with others but are expected to give each problem an honest attempt on your own first, and must submit your own work, in your own words. Show all relevant work and cite all sources used. If you get stuck, come to office hours. Write your name on your submission.

Submit assignments on gradescope (linked from canvas). You may take a picture of written work and upload the images, scan your work and upload a pdf, or, if you are ambitious, tex it. Please submit one collated file.

All Strogatz problems are from Strogatz Second Edition, the one with the black cover.

**Problem 1: Linear Stability Analysis in the Lotka Volterra Model**

We've gone into great detail building the tools for linear stability analysis. Let's run a full example now, starting from a nonlinear system, identifying the equilibria, classifying their stability, and sketching the corresponding phase portraits. This problem will be good practice for your midterm (wink).[1]

Here we will consider a famous example: the Lotka-Volterra equations for interacting populations.

Suppose we have two populations, $x_1(t), x_2(t)$ where $x_1(t)$ is the number of individuals in population 1 at time $t$ and $x_2(t)$ is the number of individuals in population 2 at time $t$. Then let $x(t)$ be the state vector containing the number of individuals in both populations at time $t$.

The Lotka-Volterra model states that:

$$\begin{aligned} \frac{d}{dt}x_1(t) &= r_1 x_1(t)(1 - a_{11}x_1(t) - a_{12}x_2(t)) \\ \frac{d}{dt}x_2(t) &= r_2 x_2(t)(1 - a_{21}x_1(t) - a_{22}x_2(t)) \end{aligned} \tag{1}$$

where $r_1, r_2$ describe the per capita rate of growth of both populations when $x_1$ and $x_2$ are small ($r_1 > 0$ and $r_2 > 0$), the diagonal coefficients $a_{jj}$ describe the competitive/cooperative interactions between individuals of type $j$ and other individuals of type $j$, and the off-diagonal coefficients $a_{12}$ and $a_{21}$ describe competition/cooperation across populations. As a rule $a_{11} > 0$ and $a_{22} > 0$. The coupling coefficients $a_{12}, a_{21}$ can be positive, negative, or zero. If $a_{ij} > 0$ then the populations suppress each other, and if $a_{ij} < 0$ then they help each other. Thus, if $a_{12} < 0$ and $a_{21} < 0$ the two populations are competing. Predator-prey interaction or host-parasite interaction (the parasite benefits from the host, but the host is hurt by the parasite) correspond to setting one coefficient negative and the other positive. If both coefficients are negative then the populations cooperate (think symbiosis).

1. Find all of the nullclines for the Lotka Volterra (LV) equations and sketch some characteristic cases (you don't need to cover all cases here). How many equilibria must the (LV) admit? Where do they occur? Are there any other equilibria?
2. Find all equilibrium $x_*$ for arbitrary $r, a$. (Hint: to find an equilibrium we need to set both rows of equation (1) to zero. What happens if $x_1 = 0$? If $x_2 = 0$? What if both are zero? What if only one is zero? What if neither are zero? In the last case you will need to solve a 2 by 2 linear system.[2]) You should find up to four different equilibria.
3. One of the equilibria corresponds to extinction of both populations, a pair correspond to extinction of one population while the other survives, and the last corresponds to coexistence of the two populations. The carrying capacity of a population is its (nonzero) equilibrium value in absence of the other. How are the diagonal entries $a_{11}$ and $a_{22}$ related to the carrying capacity for both populations (the equilibrium population of $x_1$ when $x_2 = 0$ and visa versa)?
4. A population cannot be negative, so the coexistence equilibrium only makes sense if both entries of $x_*$ are greater than zero. Write down a pair of inequalities involving $a_{22}$, $a_{12}$, $a_{11}$ and $a_{21}$ that ensure that the coexistence equilibrium is viable (is nonnegative). Show that, if $\det(A) > 0$, then coexistence

[1] If you took linear algebra with me you may have seen this problem before.

[2] You may use the general formula $A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ to solve the linear system with arbitrary coefficients.

is only viable if each population has a stronger suppressive effect on itself than on the other population and is always possible if $a_{12} < 0$ and $a_{21} < 0$ (the two populations cooperate). This scenario would correspond to a symbiosis. Can you explain these constraints using your nullclines?

5. Show that the Jacobian $J(x)$ for the Lotka-Volterra model (1) is:

$$J(x) = \begin{bmatrix} r_1(1 - a_{11}x_1 - a_{12}x_2) & 0 \\ 0 & r_2(1 - a_{21}x_1 - a_{22}x_2) \end{bmatrix} + \begin{bmatrix} -r_1 a_{11} x_1 & -r_1 a_{12} x_1 \\ -r_2 a_{21} x_2 & -r_2 a_{22} x_2 \end{bmatrix}. \tag{2}$$

   Compute the Jacobian at each equilibrium. Show that if $x_1 = 0$ or $x_2 = 0$ then the Jacobian is triangular. Show that at the coexistence equilibrium the first matrix in equation (2) vanishes.

6. Our goal is to analyze the eigenvalues of these matrices. Let's start with the easy equilibria. Show that the extinction equilibrium $x_* = [0, 0]$ is always unstable (is a source) and has eigenvalues $r_1, r_2$. Thus, if we start with small enough populations they always grow.

7. Next consider the equilibria where one population is extinct. We call these exclusion equilibria. Under what conditions are these equilibria saddles? Show that both equilibria are saddles if $\det(A) > 0$ and coexistence is viable, and coexistence is viable if $\det(A) > 0$ and both exclusion equilibria are saddles. Find the eigenvectors associated stable perturbations. How are they associated with the stable manifolds of the exclusion equilibria.

8. Ok, now for the interesting one. To make the problem easier let's assume that $r_1 = r_2 = 1$, $a_{11} = a_{22} = 1/10$. Then consider the following cases $a_{12} = a_{21} = -1/30$. Is coexistence stable in this case? What if $a_{12} = a_{21} = 1/30$? Did coexistence get more or less stable (did the eigenvalues get more of less negative)? *Does your answer make sense (remember that negative coefficients mean cooperation and positive coefficients mean competiton)?

9. Now suppose that $a_{12} = a_{21} = 1/5$. What are the eigenvalues now? What type of equilibrium is the coexistence equilibrium $x_*$? Find the eigenvectors. Along which direction is the system stable or unstable? *Does this answer make sense? *How did increasing the competition coefficients influence the stability of the coexistence state and why?

10. Sketch the flow in each of the three cases described above. Make sure to include all nullclines, equilibria, and make sure that your flow has the correct topological behavior around each equilibrium.

So far the two populations considered either compete with each other or cooperate. If one benefits from the other, while hurting it, then it is possible for the eigenvalues to be complex, and the coexistence state may be a spiral sink, source, or center. This is the case in the famous fox and hare example, in which the fox population benefits by eating the hares. Fox and hare populations are observed to cycle since the eigenvalues at the coexistence equilibrium are complex. A similar model for parasite-host dynamics can show boom bust cycles where a large number of hosts allows the parasite population to grow, leading to death of the hosts, leading to a crash in the host population, and a subsequent crash in the parasite population, which allows the host population to recover.

You have shown that, at the coexistence equilibrium, the Jacobian is:

$$J(x_*) = \begin{bmatrix} -r_1 a_{11} x_{*1} & -r_1 a_{12} x_{*1} \\ -r_2 a_{21} x_{*2} & -r_2 a_{22} x_{*2} \end{bmatrix}. \tag{3}$$

This is just a 2 by 2 matrix. Recall that, given an arbitrary 2 by 2 matrix $J$ with entries $j_{11}, j_{12}, j_{21}, j_{22}$, we can classify the equilibrium by comparing the trace and determinant of the matrix.

1. *Optional Challenge Question:* Identify what conditions on the parameters $r$, $a$, and $x_*$ that would produce oscillatory solutions about the coexistence equilibrium. (You may ignore that $x_*$ depends on $a$ for now). What type of interactions (competitive, cooperative, predator prey, parasite host, etc) are required to observe oscillations? Are there any situations where the coexistence equilibrium is a center?

**Problem 2: Solving the Critically Damped Oscillator Equation**

On your last homework you considered an damped harmonic oscillator governed by:

$$\frac{d^2}{dt^2}x(t) = -\frac{1}{M}(kx(t) + b\frac{d}{dt}x(t)). \tag{4}$$

After moving into phase space, and renaming parameter groups (see Monday's notes), you should have arrived at a linear system of the form:

$$\frac{d}{dt}y(t) = Ay(t) \tag{5}$$

where:

$$A = \begin{bmatrix} 0 & 1 \\ -\omega_0^2 & -2\mu \end{bmatrix}. \tag{6}$$

On Monday we noticed that something strange happens if $\omega_0 = \mu = r$ where $r$ is some rate (units 1 over time). Then $A$ is:

$$A = \begin{bmatrix} 0 & 1 \\ -r^2 & -2r \end{bmatrix} \tag{7}$$

1. Identify all of the eigenvalues of $A$. How many are there? What are their algebraic multiplicities?
2. Use the eigenvalues to classify the equilibrium and characterize its stability.
3. Identify all linearly independent eigenvectors of $A$. How many are there? What is the geometric multplicity of the eigenvalues?
4. Use your preceding answers to determine if $A$ is diagonalizable.

Ok, so $A$ is not diagonalizable. This is why we call this case *critically* damped. What to do? Well, we can always fall back on the matrix exponential. Let:

$$\exp(At) = e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k. \tag{8}$$

Then, as shown in class:

$$y(t) = e^{At}y(0) \tag{9}$$

given any initial $y(0)$. So, all we need to do is compute the matrix valued function $\exp(At)$. This will be easier if we have fewer variables to track.

1. So, as usual, let's nondimensionalize. We've identified $r$ as a relevant rate parameter. Rescale time to eliminate the dependence on $r$. In particular, define: $t(\tau) = \frac{1}{r}\tau$ and convert to a new phase space variable $z$ where $z_1(\tau) = y_1(t(\tau)) = x(t(\tau))$ and $z_2(\tau) = \frac{d}{d\tau}z_1(\tau) = \frac{d}{d\tau}y_1(t(\tau)) = \frac{d}{d\tau}x(t(\tau))$. Then, show that your new system reads:

   $$\frac{d}{d\tau}z(\tau) = Az(\tau) \tag{10}$$

   where:

   $$A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}. \tag{11}$$

   Explain what switching into this time unit means physically about how we are tracking the motion of the oscillator. That is, what does a unit time interval in $\tau$ represent for the motion of the oscillator?
2. Compute $A^k$ for $k = 0$, $k = 1$, $k = 2$, and $k = 3$. You should start to see a pattern emerging. Write a script and check that your pattern holds out to $k = 10$. *Optional Challenge:* Use induction to prove that the pattern you observed must hold for all $k$.
3. Consider a $n$ term approximation to the matrix exponential $\sum_{k=0}^{n} \frac{1}{k!}(At)^k$. Write a code to plot the resulting estimates for the solution trajectories $z(\tau)$ given $z(0) = [1, 1]$ (start at $x_0 = 1$ with initial velocity 1) for a range of increasing $n$ (you may stop at $n = 10$). The trajectories should converge towards the true trajectory as $n$ increases (at fixed $t$), but will diverge once $t$ is sufficiently large for any $n$. Why?

4. You should have seen that:

$$A^k = (-1)^k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{k-1} k \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \tag{12}$$

   Check that this equation matches the pattern you saw for $k = 0$, $k = 1$, $k = 2$, and $k = 3$.

5. Plug this equation into the definition of the matrix exponential and show that:

$$e^{A\tau} = e^{-\tau} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \tau \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right). \tag{13}$$

   How does the resulting for $z(\tau)$ differ from the general form we saw when $A$ is diagonalizable?

6. Was your stability classification of the equilibrium based on the eigenvalues correct?

7. Use the explicit form for the matrix exponential to plot the exact solution starting from $z(0) = [1, 1]$. How does it compare to your approximate solutions using a partial expansion to approximate the matrix exponential?