STAT/CAAM 28200 - Assignment 2 - Due Wednesday February 9

This is an individual assignment. You may collaborate with others but are expected to give each problem an honest attempt on your own first, and must submit your own work, in your own words. Show all relevant work and cite all sources used. If you get stuck, come to office hours. Write your name on your submission.

Submit assignments on gradescope (linked from canvas). You may take a picture of written work and upload the images, scan your work and upload a pdf, or, if you are ambitious, tex it. Please submit one collated file.

All Strogatz problems are from Strogatz Second Edition, the one with the black cover.

Strogatz Questions from Chapter 3: Scans of the exercises in chapters 3 and 5 can be found on Canvas under the module for week 4.

Please complete questions:

3.1.1, 3.4.9, 3.6.2, 3.7.6 (a)-(j) (Note: use $z(0) = 0$)

Question 5: Build-a-Bifurcation.

We've seen that bifurcations can be classified via normal forms recovered by Taylor expanding the governing equation about the bifurcating equilibrium near critical parameter values. Each normal form is a low order polynomial of x with a free parameter (or parameters) introduced in the low order terms. Equilibria occur at roots of the polynomial. So, bifurcations occur when roots split or collide. Different types of bifurcations occur when different root patterns are observed on either side of the bifurcation. Saddle node bifurcations have two roots on on side and zero on the other (two roots collide and annhilate). Transcritical bifurcations see two roots on both sides (two roots pass through one another). Pitchfork bifurcations have three roots on one side, and one on the other (two roots collide at a fixed root and annihilate).

1. Construct your own normal form for a different type of bifurcation. That is, design the simplest polynomial you can where, under some critical change of a parameter, the collection of roots change. For example, you could look for a polynomial where four roots collide simultaneously leaving none behind.
2. Name your bifurcation.
3. Find the critical parameter value, and draw a bifurcation diagram. Describe the qualitative behavior of the flow in the neighborhood of the bifurcation on either side of the critical parameter diagram.
4. *Optional Challenge Question:* Under what conditions on the Taylor expansion of $f(x|\theta)$ in x and parameters θ would you expect your bifurcation to occur?

Question 6: Constructing 2D linear systems: Consider the system of linear differential equations:

$$\frac{d}{dt}x = -x + y\frac{d}{dt}y = rx - y, \tag{1}$$

where r is a real constant. Find a value of r for which the origin is a (i) stable spiral, (ii) stable node, (iii) saddle point. For each of the chosen values of r plot a phase portrait in the (x, y)-phase plane. (An option is to do the latter using Mathematicas StreamPlot function, or the StreamLine command in Matlab.) *Optional Challenge Question:* What happens if $r = 0$? What are the eigenvalues and eigenvectors of the linear system? Is it diagonalizable? Can you find an explicit solution for $y(t)$ and $x(t)$ (Hint: try solving for $y(t)$ first, then use the method of integrating factors to solve for $x(t)$)? Does it match our usual form?

Strogatz Questions from Chapter 5: Scans of the exercises in chapters 3 and 5 can be found on Canvas under the module for week 4.

Please complete questions:

5.1.10, 5.2.2

Question 10: Linear Oscillators. Consider a linear oscillator subject to drag. Let $x(t)$ represent the displacement of the oscillator from its rest position. Let $k > 0$ represent the spring constant, $M > 0$ the mass of the oscillator, and $b \geq 0$ the drag coefficient. Then $x(t)$ is subject to a restoring force, $-kx(t)$ and a drag force $-b\frac{d}{dt}x(t)$. Drag is responsible for slowing the oscillator (dissipation of energy through heat). Applying Newton's Second Law:

$$\frac{d^2}{dt^2}x(t) = -\frac{1}{M}\left(kx(t) + b\frac{d}{dt}x(t)\right). \tag{2}$$

1. Introduce a velocity variable $v(t) = \frac{d}{dt}x(t)$ and convert the 2nd order equation 2 into a system of two 1st order equations in the phase variable $y = [x(t), v(t)]$. Find the governing equation for $y(t)$, and show that it is linear.
2. Introduce the parameters $\omega_0^2 = k/M$ and $b/M = \mu$. Rewrite the system in terms of these parameters. What are their units? Suggest a change of units (either a change of units for time or distance) that would reduce the number of free parameters (eliminat μ or ω_0).
3. Identify all equilibria of the 2 dimension system and solve for all nullclines. Explain the equilibrium and nullclines physically.
4. Find all eigenvalues and eigenvectors associated with the linear system.
5. Under what conditions on ω_0 and μ are there two distinct real eigenvalues (overdamped case), a repeated real eigenvalue (critically damped), two distinct complex eigenvalues (underdamped)? Draw a stability diagram in these two parameters. Mark the over, critically, and under damped cases.
6. Write down the general solution to the oscillator equation in terms of an arbitrary linear combination of terms of the form $\exp(\lambda_j t)v_j$ where λ_j and v_j are eigenvalues and eigenvectors respectively. Separate the real part of the eigenvalues from their imaginary parts, and use Euler's formula ($\exp(i\theta) = \cos(\theta) + i\sin(\theta)$) to write a general form for the solutions in terms of combinations of sines and cosines with exponentially decaying amplitudes.
7. What is the real part of the eigenvalues in terms of ω_0 and μ? Evaluate its sign to show that, in any case, the equilibrium is asymptotically stable and is a hyperbolic sink. Label the regions in your stability diagram where the equilibrium is a sink and where it is a spiral sink. Why, physically, is the equilibrium always a sink? How is the real part of the eigenvalue related to the timescale of decay to the equilibrium?
8. What is the imaginary part of the eigenvalues in terms of ω_0 and μ? How is the imaginary part of the eigenvalue related to the timescale of oscillation (period or frequency)? How does it compare to ω_0? What is ω_0 physically (hint: what choice of b sets the imaginary part equal to ω)?
9. *Optional Challenge Question:* Use the Jordan Normal form for the linear system to find the general form for trajectories in the critically damped case. How does their functional form differ from that seen in the over or underdamped cases?