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HW5
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$$7.3.1 \quad a. \quad \begin{aligned} \dot{x} &= x - y - x(x^2 + 5y^2) = x - y - x^3 - 5xy^2 \\ \dot{y} &= x + y - y(x^2 + y^2) = x + y - x^2y - y^3 \end{aligned}$$

$\text{At } (0,0), \dot{x} = \dot{y} = 0$
 $\therefore x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a fixed pt

$$J(A) = \begin{bmatrix} 1-3x^2-5y^2 & -1-10xy \\ 1-2xy & -x^2-3y^2 \end{bmatrix}$$

$$J\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{tr}(J) = 2 \quad \text{Det}(J) = 2$$

$$\det(J) = 2 > \frac{1}{4}(\text{tr}(J))^2 = \frac{1}{4}4 = 1$$

\therefore complex eigenvalues

\therefore spiral source since $\text{tr}(A) > 0$, $\det(A) > 0$, and $\det > \frac{1}{4}\text{tr}^2$

b.

$$\begin{aligned} r^2 &= x^2 + y^2 & x = r\cos\theta & y = r\sin\theta \\ \dot{r} &= x\dot{x} + y\dot{y} = x\cancel{x} - \cancel{xy} - \cancel{xx(x^2 + 5y^2)} + \cancel{xy} + \cancel{yy} - \cancel{yy(x^2 + y^2)} \\ && x^2 & -x^4 - 5x^2y^2 & y^2 & -x^2y^2 - y^4 \end{aligned}$$

$$\begin{aligned} \dot{r} &= (x^2 + y^2) - (x^4 + y^4) - 6x^2y^2 \\ &\quad -x^4 - \cancel{6x^2y^2} - y^4 - \cancel{4x^2y^2} \\ &\quad - (x^4 + \cancel{x^2y^2} + y^4) \\ &\quad - (x^2 + y^2)^2 \end{aligned}$$

$$\begin{aligned} &= r^2 - r^4 - 4r^2\cos^2\theta \cdot r^2\sin^2\theta \\ &\quad - r^4 \cdot \frac{4\cos^2\sin^2}{(2\cos\theta\sin\theta)^2} \end{aligned}$$

$$\begin{aligned} \dot{r} &= r^2 - r^4 - r^4(\sin 2\theta)^2 \\ \dot{r} &= r - r^3 - r^3(\sin 2\theta)^2 \end{aligned}$$

\curvearrowleft angular dependence

$$\therefore \dot{r} = r - r^3 - r^3 \sin^2(2\theta) \quad \text{long circular}$$

$$\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta$$

$$\begin{aligned} \dot{\theta} &= \frac{xy - yx}{r^2} \\ &= \frac{x^2 + xy - xy - y^2}{r^2} \\ &= \frac{-x^2 + y^2 + xy}{r^2} \end{aligned}$$

$$= \frac{x^2 - xy^3 + y^2 + 5xy^3}{r^2}$$

$$= \frac{(x^2 + y^2) + 4xy^3}{r^2}$$

$$= \frac{r^2 + 4r\cos\theta r^3\sin^3\theta}{r^2}$$

$$\dot{\theta} = 1 + 4r^2\cos\theta\sin^3\theta$$

$$c. \quad \dot{r} = r(1 - r^2 - r^2\sin^2(2\theta)) > 0$$

\downarrow

$$1 - r^2 - r^2\sin^2(2\theta) > 0$$

$$1 - r^2(1 + \underbrace{\sin^2(2\theta)}_{\text{max value is } 2}) > 0$$

max value is 2

$$\therefore 1 - 2r^2 > 0$$

$$2r^2 \leq 1 \Rightarrow r^2 \leq \frac{1}{2} \quad r \leq \frac{1}{\sqrt{2}}$$

$\therefore r = \frac{1}{\sqrt{2}}$ circle of maximum radius such that all trajectories have $\dot{r} > 0$ (outward component)

7.3.1 (cont'd);

$$\dot{r} = r(1 - r^2 - r^2 \sin^2 2\theta) < 0$$

\downarrow

$$1 - r^2(1 + \sin^2 2\theta) < 0$$

min is 1

$$1 - r^2 < 0$$

$$r^2 > 1$$

$$r > 1$$

$r_1 = 1$ minimum radius such that all trajectories have radially inward component ($\dot{r} < 0$)

e. Define a region $R = \{(r, \theta) \text{ s.t. } \frac{1}{r} = r, 1 < r < r_2 = 1\}$, R is closed and bounded. r and $\dot{\theta}$ are continuously $\sqrt{r^2}$ differentiable in R . R does not contain equilibria because $\dot{r} = 0$ when $r = 0$ or

R has trajectories confined in R because we know all trajectories have a radially outward component at r_1 and a radially inward component at r_2 , so if $x(t)$ enters R it stays confined in R for all future times

- R contains a closed orbit C and any trajectory confined in R converges to C as $t \rightarrow \infty$ or equals C .

Since C is isolated, it's an attracting limit cycle.

$$\begin{aligned} \dot{r} = 0 \text{ when } r = 0 \text{ or} \\ 1 - r^2 - r^2 \sin^2(2\theta) = 0 \\ 1 = r^2(1 + \sin^2(2\theta)) \\ r^2 = \frac{1}{1 + \sin^2(2\theta)} \\ \text{and } \dot{\theta} = 0 \text{ when } 4r^2 \cos \sin^3 = -1 \\ r^2 = \frac{-1}{4 \cos \sin^3 \theta} \end{aligned}$$

for $\dot{r} = 0$ and $\dot{\theta} = 0$ need

$$\begin{aligned} r^2 = \frac{1}{1 + \sin^2(2\theta)} = \frac{-1}{4 \cos \sin^3 \theta} \\ (2 \sin^2 \theta) \cdot \frac{-2 \cos \sin^3 \theta}{\sin(2\theta)} = -1 - \sin^2(2\theta) \end{aligned}$$

$$\begin{aligned} 0 = 1 = \sin 2\theta (\frac{2 \sin \theta \cos \theta}{\sin 2\theta + 2 \sin^2 \theta}) (5 \sin) \\ -1 = \sin 2\theta \cdot \sin \theta \cdot (2 \cos \theta + 2 \sin \theta) \\ -1 = 2 \cos \theta \sin^2 \theta \cdot 2(\cos \theta + \sin \theta) \end{aligned}$$

Graphically, I determined that the minimum of the function for $0 < \theta < 2\pi$ is ≈ -0.72 which is > -1 . \therefore never have function = -1 \therefore never $\dot{r} = 0$ and $\dot{\theta} = 0$ if $r \neq 0$.

7.3.2: See computer pages

$$\begin{aligned} 7.3.12, a, \dot{E}_1 = \dot{P} + \dot{R} + \dot{S} = P(aR - S) - P(a-1)(C) \\ + R(aS - P) - R(a-1)(C) \\ a(PR + RS + SP) - (PS + PR + SR) + S(CP - R) - S(a-1)(C) \\ = (a-1)(C) \quad \leftarrow \quad \boxed{\begin{array}{l} P(aS - P) - R(a-1)(C) \\ S(CP - R) - S(a-1)(C) \\ -(P+R+S)(a-1)(C) \\ - E_1(a-1)(C) \end{array}} \\ = \dot{E}_1 = (a-1)(C) - E_1(a-1)(C) \\ = (1-E_1)(a-1)C \\ \dot{E}_1 = (1-E_1)(a-1)(PR + RS + PS) \quad \checkmark \end{aligned}$$

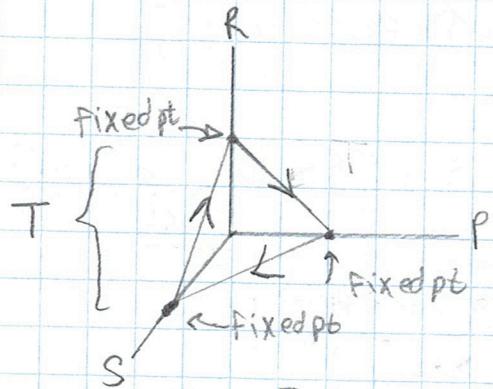
$\therefore E_1, \dot{E}_1$ is an invariant trajectory. In 3D space, the equation $P+R+S=1$ is a plane.

b, the subset T with the added condition $P, R, S \geq 0$ is a triangle.

$T = \{ \text{set } P+R+S=1 \text{ and } P, R, S \geq 0 \}$

7.3.12 (cont.)

- c. Fixed points when $P=0, R=0, S=1$ \leftarrow 3 fixed points when $P=R=S=0$
- $P=1, R=0, S=0$
- $P=0, R=1, S=0$



$\checkmark \dot{S}=0$
 if $S=0$, we are restricted to $P+R=1$, so
 there must be a trajectory along that line.
 Similarly, there are trajectories along the
 lines $P+S=1$ (for $R=0$) and $R+S=1$ (for $P=0$).
 $\checkmark \dot{R}=0$ $\checkmark \dot{P}=0$.

Now I show that the 3 trajectories have the correct orientation
 for a heteroclinic cycle:

$$\text{for } P+R=1, \quad \dot{P} = P[aR - (a-1)(PR)] = PR[a - aP + P]$$

$$\dot{P} = PR[a(1-P) + P]$$

$$\begin{cases} P \leq 1 & P > 0 \\ 1-P \geq 0 & \end{cases}$$

$$\therefore \dot{P} \geq 0$$

for $P+S=1$

$$\begin{aligned} \dot{S} &= S[aP - (a-1)(PS)]P \\ &= PS[a - Sa + S] = PS[a(1-S) + S] \\ &\quad \begin{cases} S \leq 1 & 1-S \geq 0 \\ \end{cases} \end{aligned}$$

$$\therefore \dot{S} \geq 0$$

for $R+S=1$

$$\begin{aligned} \dot{R} &= R[aS - (a-1)(RS)]S \\ &= RS[a - Ra + R] \\ &= RS[a(1-R) + R] \\ &\quad \begin{cases} R \leq 1 & 1-R \geq 0 \\ \end{cases} \\ &\therefore \dot{R} \geq 0 \end{aligned}$$

I have now labeled the graph above. As we can see, these 3 orientations result in a cycle graph (heteroclinic cycle).

4. $\dot{P} +$

7.3.12 (cont.)

$$d. \dot{E}_2 = PRS + PRS + PRS$$

$$\dot{E}_2 = PRS$$

$$\dot{E}_1 = P+R+S = 1$$

$$= PRS \left[aR-S + aS-P + aP-R - 3(a-1)(PR+RS+PS) \right]$$

$$= PRS \frac{a(R+S+P) - (R+S+P)}{(a-1)(R+S+P)}$$

$$= PRS \frac{(a-1)}{2} \left[R+S+P - 3(PR+RS+PS) \right] \cdot 2 = [2 - 6(PR+RS+PS)]$$

$$\therefore 1 = \frac{(P+R+S)^2}{a^2} = \frac{1^2}{a^2} = \frac{1}{a^2} = \frac{2-6b}{2}$$

$$\therefore 1 = \frac{P^2+R^2+S^2}{a^2} + \underbrace{a(RS+SP+PR)}_b$$

$$= 2(R^2+P^2+S^2) - 2(PR+RS+PS)$$

$$2(a-2b)$$

$$= 2(1-2b) - 2b$$

$$= 2-6b \quad \therefore (P-R)^2 + (R-S)^2 + (S-P)^2 = 2-6(PR+RS+PS)$$

$$\therefore \dot{E}_2 = \frac{\dot{E}_2(a-1)}{2} \left[(P-R)^2 + (R-S)^2 + (S-P)^2 \right] \quad \checkmark$$

$$e. \text{ if } P=R=S = \frac{1}{3}, \text{ then } \dot{E}_2 = \frac{(a-1)\dot{E}_2}{2} \left[\left(\frac{1}{3}-\frac{1}{3}\right)^2 + \left(\frac{1}{3}-\frac{1}{3}\right)^2 + \left(\frac{1}{3}-\frac{1}{3}\right)^2 \right] = 0$$

As the boundary of T ($P+R+S=1$) we always have one of the three variables equal to 0, so $\dot{E}_2 = PRS = 0$ on boundary

$$\text{of } T. \text{ So, } \dot{E}_2 = \frac{(a-1)}{2} \dot{E}_2[m] = \frac{(a-1)}{2} \cdot 0 \cdot [m] = 0$$

f. Since $\dot{E}_2 = 0$ on the boundary of T , if $a > 1$, then $\dot{E}_2 > 0$

everywhere inside T that is not on the boundary. The maximum of $\dot{E}_2 = PRS$ for $P+R+S=1$ is when the 3 numbers are equal which is at the interior fixed point. So the interior fixed point attracts all trajectories on the interior of T .

g. if $a < 1$, then $\dot{E}_2 < 0$ everywhere not on the boundary of T . The minimum \dot{E}_2 is when one variable is 0, which is the heteroclinic cycle. Do the heteroclinic cycle attract all trajectories in T not at the fixed point?

7.5.6:

$$a. \dot{x} + N(x^2-1) \dot{x} + x = a. \text{ Fixed pt when } \dot{x}=0 \text{ and } x=0$$

$$\therefore \dot{x} = a - x - N(x^2-1)\dot{x}$$

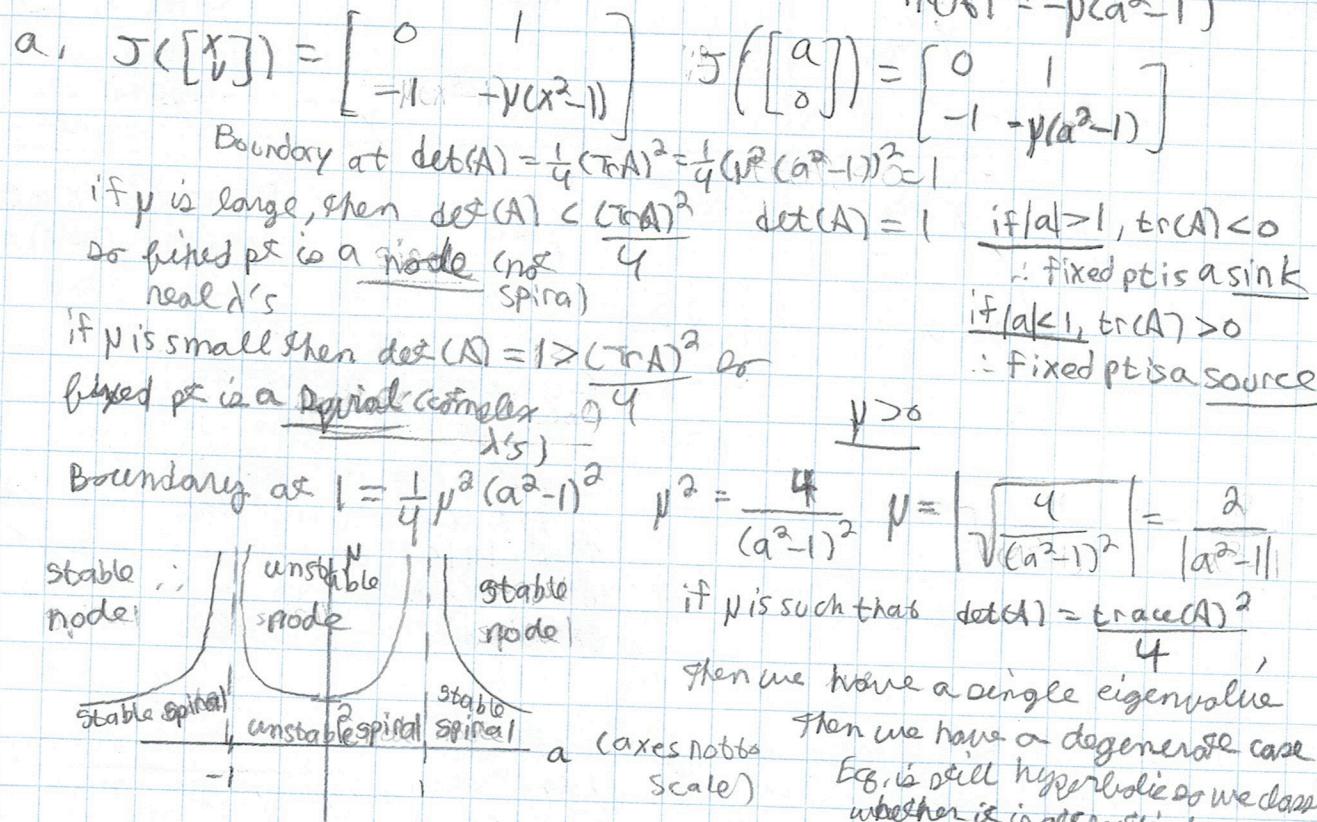
$$\frac{d}{dt} \left[\frac{\dot{x}}{dx/dt} \right] = \left[\frac{dx/dt}{a-x-N(x^2-1)\dot{x}} \right] = \left[\frac{v}{a-x-N(x^2-1)v} \right]$$

$$\text{Let } \dot{x}=v \quad \text{Fixed pt need } v=0 \therefore \frac{d}{dt} v = \dot{x} = a - x - 0 = a - x$$

$$\therefore \text{Fixed pt at } x=a, \dot{x}=0$$

$$a=x$$

7.5.6 (cont.)



$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = a$$

$$0 = a - x - \mu(x^2 - 1)v$$

$$\ddot{x} + \mu(x^2 - 1)\dot{x} = \frac{d}{dt} \left[\dot{x} + \mu \frac{(x^3 - x)}{3} \right] \quad \text{let } F(x) = \frac{x^3}{3} - x$$

$$F = x^2 - 1 \quad \dot{x} = w - \mu F$$

$$\therefore w = \dot{x} + \mu F$$

\therefore our governing equation becomes

$$\frac{d}{dt} \begin{bmatrix} x \\ w \end{bmatrix} = \begin{bmatrix} w - \mu F \\ a - x \end{bmatrix} \quad W = a - x \quad F = \frac{x^3}{3} - x$$

$$\text{Let } y = \frac{w}{\mu} \quad w = \mu y$$

$$\therefore \dot{W} + x = a$$

$$\dot{w} = a - x$$

This coordinate transformation lets us plot in the phase plane.

$$\therefore \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mu(y - F) \\ \frac{1}{\mu}(a - x) \end{bmatrix}$$

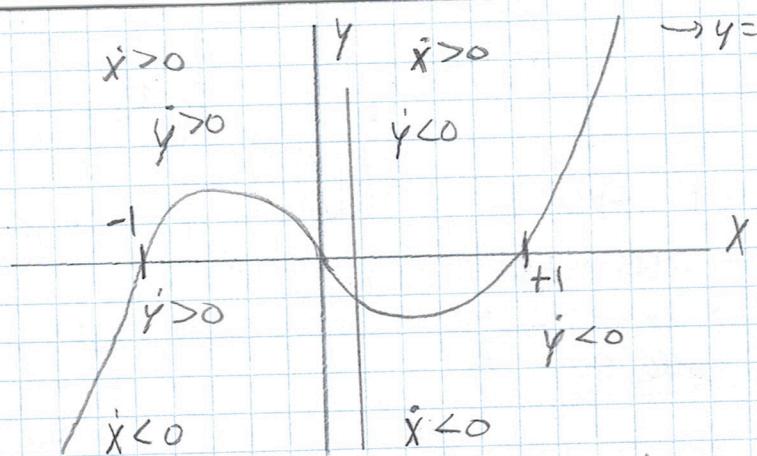
nullclines are now

$$\begin{aligned} \dot{x} = 0 &= \mu(y - F) \quad \therefore y = F = \frac{x^3}{3} - x \\ \dot{y} = 0 &= \frac{1}{\mu}(a - x) \end{aligned}$$

$$\therefore x = a$$

7.5.6 (cont)

b (cont):



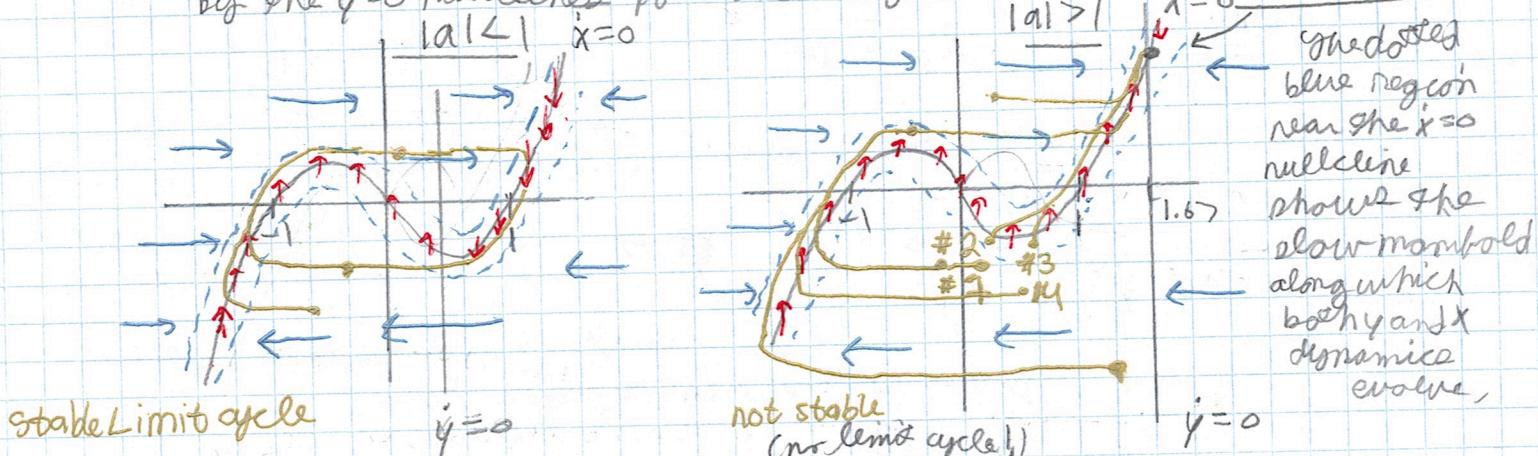
$y = F(x)$ ($\dot{x} = 0$ nullcline)

Other nullcline $\dot{y} = 0$ is at $x = a$ and thus depends on the parameter a .

$$y = F(x) = \frac{x^3}{3} - x \text{ intersects } y = 0 \text{ (has 3 roots) at } x = \pm 1.$$

If we pick a such that $x = a$ this vertical line is in the middle branch of the cubic nullcline, then we know that $|a| < 1$. We showed in part a that if $|a| < 1$, linearization tells us that the fixed point is unstable.

c. If $|a| > 1$, then our system exhibits fine-scale separation, in which x dynamics evolve much faster than y dynamics. This means that horizontal motion occurs much faster than vertical motion. So from an initial condition, the system quickly relaxes to a point on the cubic nullcline at the same y -value (horizontal motion). Then, $\dot{x} \approx 0$ (since we're on the horizontal nullcline) so y dynamics are able to evolve with x relatively constant, whether the trajectory moves up or down determines stability of the limit cycle. This is determined by the $\dot{y} = 0$ nullcline's position (as by a).



So if $|a| < 1$, the trajectories converge to a stable limit cycle.
If $|a| > 1$, the trajectories do not converge to a stable limit cycle. $a_c = 1$

stable limit cycle only if $|a| < a_c$

d. I sketched this in part c in which $a \approx 1.67$. In this case, there is a globally attractive fixed point at the intersection of the 2 nullclines. However, we see that some trajectories travel long distances before converging. For example, trajectories #2 and 3 quickly converge, but trajectories #1 and 4, which start at just a small distance away, diverge and travel long distances before converging back to the fixed pt.

Ch. 8:

$$8.2.2: \quad J(z) = \begin{bmatrix} N+y^2 & -1+2xy \\ -2x & N \end{bmatrix} \quad \begin{array}{l} N=0 \\ z=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}$$

$$z=\begin{bmatrix} x \\ y \end{bmatrix} \quad \therefore J\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = A$$

$$A - \lambda I = \begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} \quad \det(A - \lambda I) = \lambda^2 + 1 = 0$$

$$\lambda^2 = -1$$

$$\therefore \lambda = \pm i$$

8.2.3, See computer pages

$$8.2.4: \quad a. \quad \dot{x} = -y + \mu x + xy^2 \quad r^2 = x^2 + y^2 \quad x = r\cos\theta$$

$$\dot{y} = x + \mu y - x^2 \quad y = r\sin\theta \quad r^2 = x^2 + y^2 \quad \tan\theta = y/x = \text{hyperbolic}$$

$$\begin{aligned} \dot{r} &= -xy + \mu x^2 + x^2 y^2 + xy + \mu y^2 - x^2 y \\ &= \mu(x^2 + y^2) + x^2 y(y-1) \\ \dot{x} &= \mu r^2 + r^2 \cos^2 \theta \sin \theta (r \sin \theta - 1) \\ \dot{r} &= \mu r + r^2 \cos^2 \theta \sin \theta (r \sin \theta - 1) \end{aligned}$$

$$\dot{r} = x\dot{x} + y\dot{y} \quad \dot{\theta} = \frac{\dot{y}x - \dot{x}y}{r^2}$$

$$\therefore \dot{\theta} = \frac{\dot{y}x - \dot{x}y}{r^2 \cos^2 \theta}$$

$$\therefore \dot{\theta} = \frac{\dot{y}x - \dot{x}y}{r^2}$$

$$\begin{aligned} \dot{\theta} &= \frac{\dot{y}x - \dot{x}y}{r^2} = \frac{x^2 + \mu xy - x^3 + y^2 - \mu xy - xy^3}{r^2} \\ &= \frac{x^2 - x^3 + y^2 - xy^3}{r^2} = \cancel{x^2} - r^2 \cos^3 \theta - r \cos \theta \cancel{r^2} \sin^3 \theta \\ \dot{\theta} &= 1 - r \cos^3 \theta - r^2 \cos \theta \sin^3 \theta \end{aligned}$$

$$b. \text{ if } r \ll 1, \text{ then } \dot{\theta} \approx 1 - \frac{r \cos^3 \theta - r^2 \cos \theta \sin^3 \theta}{\text{very small, small}} \approx 1$$

$$\dot{r} = \mu r + r^2 \cos^2 \theta \sin \theta (r \sin \theta - 1)$$

$$\text{Calculate average over one period} \quad \frac{1}{2\pi} \int_0^{2\pi} r^3 \sin^2 \theta \cos^2 \theta d\theta - \int_0^{2\pi} r^2 \sin \theta \cos^2 \theta d\theta$$

$$= \frac{r^3}{2\pi} \cdot \frac{1}{4} \int_0^{2\pi} 4 \sin^2 \theta \cos^3 \theta d\theta - \dots$$

$$I = \frac{r^3}{2\pi} \int_0^{2\pi} \sin \theta \cos^2 \theta d\theta$$

$$\text{Let } u = \cos \theta$$

$$\begin{aligned} &= -r^2 \int_0^{2\pi} \sin \theta u^2 du - \frac{du}{\sin \theta} \\ &+ r^2 \cdot u^3 / 3 = r^2 \cos^3 \theta \int_0^{2\pi} \end{aligned}$$

$$= \frac{r^3}{8\pi} \int_0^{2\pi} (\sin(2\theta))^2 d\theta$$

$$\begin{aligned} &= \frac{r^3}{8\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos(4\theta)) d\theta \\ &= \frac{r^3}{16\pi} \int_0^{2\pi} (1 - \cos(4\theta)) d\theta \end{aligned}$$

$$= \frac{r^3}{16\pi} \int_0^{2\pi} \frac{1}{2} (1 - \cos(4\theta)) d\theta$$

$$= \frac{r^3}{16\pi} \left(\frac{1}{2} \theta - \frac{1}{8} \sin(4\theta) \Big|_0^{2\pi} \right) = \frac{r^3}{16\pi} \left(\frac{2\pi}{2} \right) = \frac{r^3}{8} = 0$$

cont.
8.24(b). So, we found that $\dot{r} = \mu r + \frac{r^3}{8} - 0$ when time averaged around one cycle
 $\therefore \dot{r} \approx \mu r + r^3/8$

c. See computer pages.

8.2.11:

a. We can linearize: Let $v = \dot{x}$ $\therefore \ddot{x} = \dot{v}$

$$\dot{v} + \mu v + x - x^3 = 0$$

$$\dot{v} = x^3 - x - \mu v$$

$$\frac{d}{dt} \begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ x^3 - x - \mu v \end{bmatrix} \quad \therefore J(\eta) = \begin{bmatrix} 0 & 1 \\ 3x^2 - 1 & -\mu \end{bmatrix} = A \quad J\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ -1 & -\mu \end{bmatrix}$$

$$y = \begin{bmatrix} x \\ v \end{bmatrix} \quad \text{tr}(A) = -\mu$$

$$\det(A) = 1$$

$$1 > \frac{1}{4} \text{tr}(A)^2 = \frac{1}{4} \mu^2$$

when $\mu \approx 0$ so
our fixed point
is a spiral (complex
eigenvalues)

We know that a stable spiral becomes
an unstable spiral when $\text{tr}(A)$ switches
from negative to positive, and vice versa.

\therefore When $\mu > 0$, $\text{tr}(A) < 0 \therefore$ stable spiral

When $\mu < 0$, $\text{tr}(A) > 0 \therefore$ unstable spiral

\therefore when μ decreases through 0, the origin
switches from a stable to unstable spiral

b. See computer pages

8.2.17:

$$\text{When } x_1^* = y_1^* = x_2^* = y_2^* = u$$

$$\text{a. } \dot{x}_1 = -u + F(I - bu - gu)$$

$$\dot{y}_1 = (-u+u)/\tau = 0 \quad \begin{array}{l} \uparrow x_1 \text{ and } x_2 \\ \downarrow \text{equal} \end{array}$$

$$\dot{x}_2 = -u + F(I - bu - gu)$$

$$\dot{y}_2 = (-u+u)/\tau = 0 \quad y_1 = y_2 = 0$$

\therefore This is a fixed point for all chosen parameters

now, we show that the fixed point is unique; $F(I - bu - gu)$

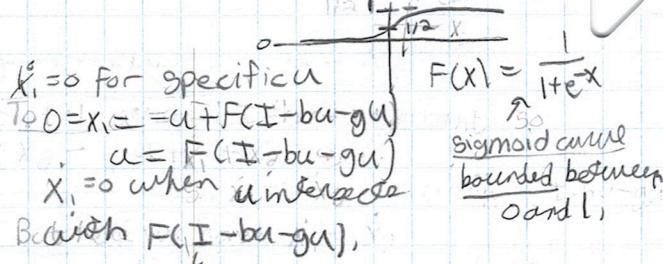
We show that $F(I - u(b+g))$ is monotonically decreasing;

$$F'(I - u(b+g)) = \frac{1}{\tau u} \left(\frac{1}{1 + e^{u(b+g)-I}} \right)' = -\left(1 + e^{u(b+g)-I} \right)^{-2} (b+g) < 0$$

$\therefore F(I - u(b+g))$ is a monotonically decreasing function. u (the line) is monotonically increasing. So there can be at most 1 intersection.

$$\text{So at most 1 s.t. } u = F(I - u(b+g)) \Rightarrow \dot{x}_1 = 0 = \dot{x}_2$$

\therefore at most 1 fixed point



$$F(x) = \frac{1}{1 + e^x}$$

Sigmoid curve
bounded between
0 and 1

For the case where $u < 0$,
 y_1 is also a sigmoid curve that is
reflected / shifted, which must
intersect the line u , i.e. must have u
s.t. $x_1 = 0 = x_2$

$$8.2.17: \quad b. \quad J\left(\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} -1 & F'(I-bx_2-gy_1)(-g) & F'(I-bx_2-gy_1)(-b) & 0 \\ 1/T & -1/T & 0 & 0 \\ F'(I-bx_1-gy_2)(-b) & 0 & F'(-b) & -1 \\ 0 & 0 & 1/T & -1/T \end{bmatrix}$$

At $x_* = \begin{bmatrix} u \\ u \\ u \\ u \end{bmatrix}$ ↓ domain of inside argument (already accounted for chain rule)
 $F'(I-bx_2-gy_1) = F'(I-bx_1-gy_2) = Z$ call it
 $\text{Let } 1/T = d_1$ = $F'(I-bu-gu)=Z$ that
 $= \frac{(1+e^{u(b+g)-I})^{-2}}{d_1^2} > 0$

We know
 $T > 0, b > 0, g > 0$

$$\therefore J(x_*) = \begin{bmatrix} -1 & -gz & -bz & 0 \\ d_1 & -d_1 & 0 & 0 \\ -bz & 0 & -1 & -gz \\ 0 & 0 & d_1 & -d_1 \end{bmatrix} \quad \begin{array}{l} \because F' > 0 \\ \because Z > 0 \\ \because C_2 > 0 \\ \underline{\quad C_3 > 0} \end{array}$$

Let $gz = c_2, I = c_1$ and $c_3 = bz$

$$\therefore J(x_*) = \begin{bmatrix} -c_1 - c_2 & -c_3 & 0 \\ d_1 & -d_1 & 0 \\ -c_3 & 0 & -c_1 - c_2 \\ 0 & 0 & d_1 - d_1 \end{bmatrix}$$

A is constructed, I assume

The stated fact regarding the eigenvalues, that eigenvalues of 9×9 block given by λ 's of 2×2 blocks,

$$c. \quad A = \begin{bmatrix} -c_1 - c_2 \\ d_1 - d_1 \end{bmatrix} \quad B = \begin{bmatrix} -c_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A+B = \begin{bmatrix} -c_1 - c_3 & -c_2 \\ d_1 & -d_1 \end{bmatrix} = C$$

$$\text{tr}(C) = -c_1 - c_3 - d_1$$

$$\det(C) = -d_1(-c_1 - c_3) + c_2d_1$$

$$\det(C) = d_1c_1 + d_2c_2 + d_1c_3$$

$$\text{tr}(C) = -(c_1 + c_3 + d_1)$$

$$\frac{c_1 > 0}{\checkmark} \quad \frac{d_1 > 0 \text{ (timescale} > 0)}{\checkmark}$$

$$c_3 > 0$$

$$\therefore \text{tr}(C) < 0$$

$$\det(C) = d_1(c_1 + c_2 + c_3)$$

$$\therefore c_2 > 0$$

$$\therefore \det(C) > 0$$

$$d. \quad A-B = \begin{bmatrix} c_3 - c_1 & -c_2 \\ d_1 & -d_1 \end{bmatrix}$$

$$\begin{aligned} \det(A-B) &= -d_1(c_3 - c_1) + c_2d_1 \\ &= d_1(c_2 - c_3 + c_1) \\ &= \frac{1}{T}(gz - bz + I) \end{aligned}$$

depending on size of g (if g is small) then $\det(A-B) < 0$,

$$\begin{aligned} \text{tr}(A-B) &= c_3 - c_1 - d_1 \\ &= bz - I - 1/T \end{aligned} \quad \text{need } gz + I < bz : g < \frac{bz - I}{2}$$

e. See computer pages

if T is large then $1/T$ is small, so $\text{tr}(A-B) > 0$,
Large T ($\text{s.t. } \frac{1}{T} < bz - I$) \Rightarrow positive trace

8.7.9; Periodicity is every 2π since $\dot{\theta} = 1$

a. Let r_0 be initial condition on S on $+x$ -axis ($\theta=0, r>0$)

$$\therefore r_1 = P(r_0) \quad \frac{dr}{dt} = r - r^2 \Leftrightarrow \frac{dr}{r-r^2} = dt$$

$$\boxed{\frac{r_{n+1}}{e^{2\pi} r_n + r_{n+1}} = \frac{e^{2\pi} r_n}{r_n + r_{n+1}}} = P(r) \quad \int_0^{2\pi} dt = \int_{r_0}^{r_1} \frac{dr}{r-r^2}$$

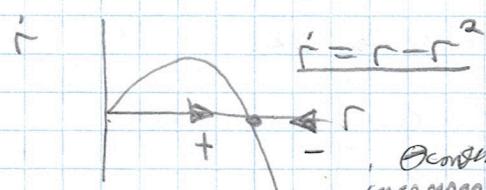
$$2\pi = \int_{r_0}^{r_1} \frac{dr}{r(1-r)} \rightarrow \frac{1}{r(1-r)} = \frac{A}{r} + \frac{B}{1-r}$$

$$1 = A - Ar + Br$$

b. we can analyze r dynamics independently of θ

$$\therefore 2\pi = \int_{r_0}^{r_1} \frac{1}{r} + \frac{1}{1-r} dr$$

$$A = B \quad A = 1 \therefore B = 1$$



oscillatory
increases so have
oscillations

\therefore stable fixed ↓

pt in r_1 , since $\dot{\theta}=1$

in the polar system
we have an isolated closed periodic
orbit that is stable at r_1 ,

which is a stable limit

cycle

c. Let's linearize about $r_* = 1$
and find a map for a perturbation
to determine λ (the multiplier)

$$\therefore \text{let } r = 1 + \delta$$

$$\therefore \dot{r} = (1+\delta) - (1+\delta)^2 = \delta$$

$$= \delta + \delta - 1 - 2\delta = -\delta^2$$

We ignore higher order terms

$$\therefore \dot{\delta} = -\delta^2$$

$$\text{so } \delta(t) = e^{-t} \delta_0$$

When $t = 2\pi$ after 1 period

$$\delta(t) = e^{-2\pi} \delta_0$$

so the characteristic multiplier is $\boxed{e^{-2\pi}} = \lambda$

Since $\lambda < 1$, limit cycle is stable (confirming part b)

$$2\pi = \ln|r_1| - \ln|r_0| = \ln \left| \frac{r_1}{r_0} \right|$$

$$2\pi = 1 \cdot \ln \left| \frac{r_1}{r_0} \right| \ln \left| \frac{r_0}{r_1} \right|$$

$$\ln \left| \frac{r_1}{r_0} \right| = 2\pi + \ln \left| \frac{r_0}{r_1} \right|$$

$$\frac{r_1}{r_0} = e^{2\pi} \cdot \left(\frac{r_0}{r_1} \right)^{-1}$$

$$\therefore r_1 = (1+r_1) C = C - Cr_1$$

$$r_1(1+C) = C$$

$$\therefore r_1 = \frac{C}{1+C} = e^{2\pi} \left(\frac{r_0}{1+r_0} \right)$$

Homework 4

CAAM 28200: Dynamical Systems with Applications

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March 4, 2022

Problem 7.3.2

Using numerical integration, I computed solution trajectories for the governing equation in Exercise 7.3.1 and verified that there is a limit cycle in the trapping region I constructed between $r_1 = \frac{1}{\sqrt{2}}$ and $r_2 = 1$. As we see in the figure, the two blue circles are the inner and outer circles defining the trapping region R . We see that there is a clear limit cycle inside of this trapping region, confirming what we predicted using the Poincarè Bendixson Theorem.

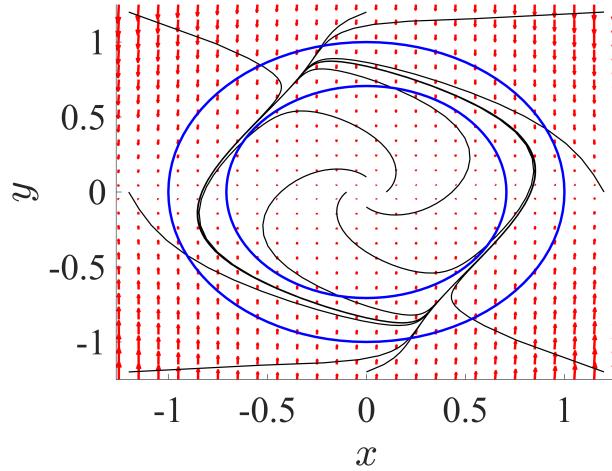


Figure 1: Phase Portrait

Problem 8.2.3

The phase portraits for the system $\dot{x} = -y + \mu x + xy^2$ and $\dot{y} = x + \mu y - x^2$ are shown below. In these portraits, the blue dots are initial conditions. We see here that when $\mu < 0$, the center fixed point at the origin is stable. Furthermore, as μ increases from -0.5 to -0.1 , the initial conditions far from the origin (such as the one at $(-0.9, 0)$) start diverging from the origin. In addition, the spiral trajectories indicate that we may have a limit cycle. So what this implies is that there is a limit cycle that is unstable because trajectories starting inside the cycle converge to the origin

and points starting outside the cycle diverge to infinity. Furthermore, when μ is further increased to -0.05 and to -0.01 , we see that more and more of the trajectories start to diverge, and only trajectories that start very close to the origin still converge to a limit cycle. So the limit cycle is getting smaller and smaller as we approach $\mu = 0$. At the marginal case of $\mu = 0$, we just have an unstable spiral source at the origin. This behavior hints at a subcritical Hopf bifurcation because we have a stable fixed point and an unstable limit cycle that converge.

We confirm this hypothesis when we see that at positive values of μ , the origin becomes unstable and the unstable limit cycle disappears. This agrees with the behavior of a subcritical Hopf bifurcation. We also see that as μ increases, the solutions spiral less and less, and they begin to diverge more quickly. Note that a subcritical Hopf bifurcation also has a stable limit cycle that is stable regardless of μ , but we are not seeing it in the plots. This may be because the stable limit cycle is far from the origin.

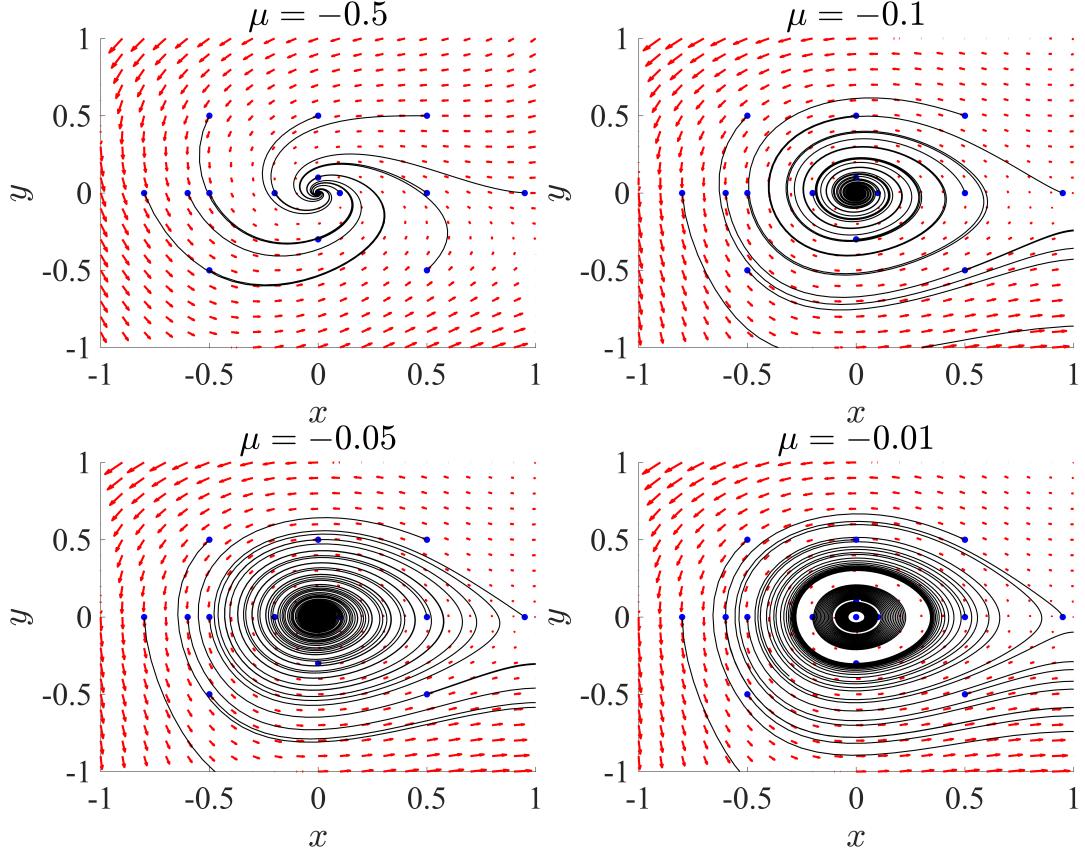


Figure 2: Phase Portraits for $\mu < 0$

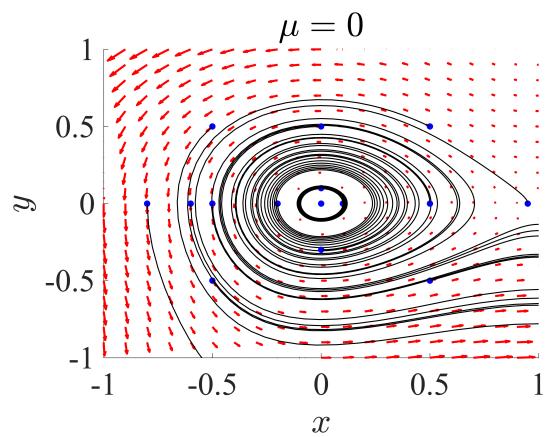


Figure 3: Phase Portrait for $\mu = 0$

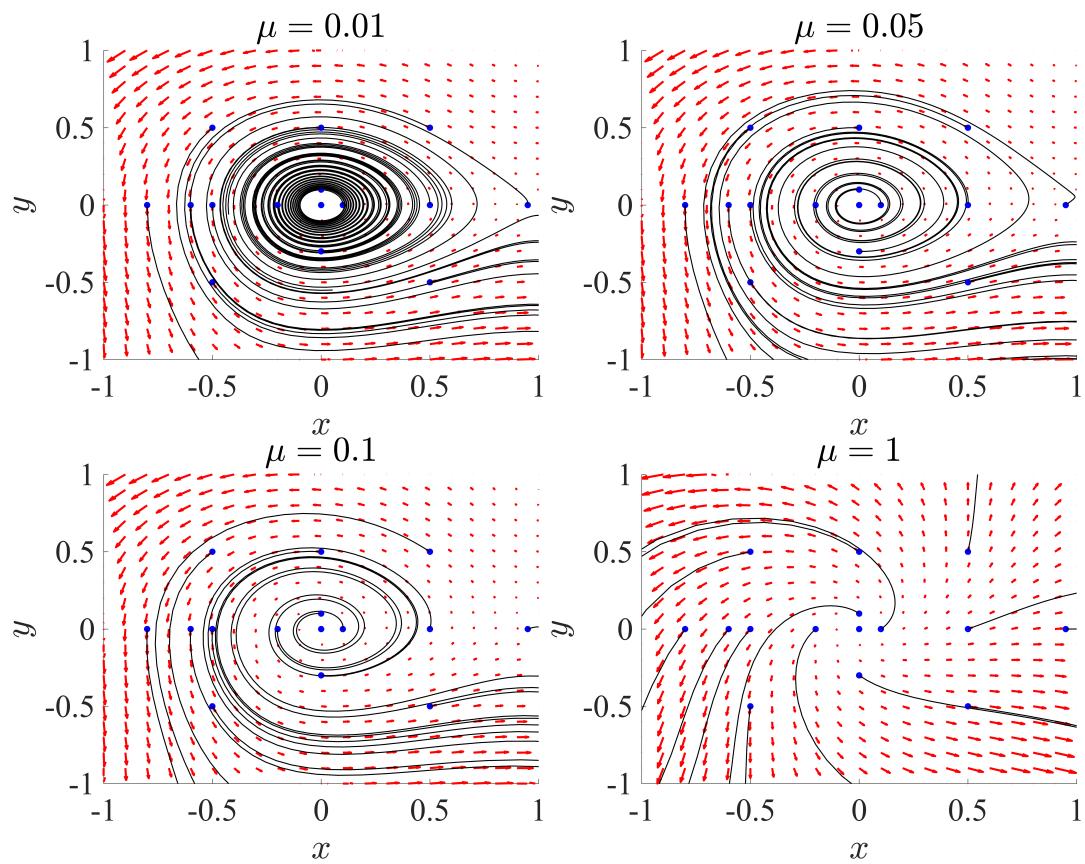


Figure 4: Phase Portrait for $\mu > 0$

Problem 8.2.4.c

Here, I plotted the phase portrait and numerically computed solutions for the approximated polar equation $\dot{\theta} \approx 1$ and $\dot{r} \approx \mu r + \frac{1}{8}r^3$. Initial conditions are shown as blue dots. A circle of radius $r = \sqrt{-8\mu}$ is shown in green. Thus, we see that this circle represents an unstable limit cycle with the predicted radius of $r = \sqrt{-8\mu}$, which confirms our prediction made by finding the radius at which \dot{r} switches signs. I did this in MATLAB and converted between polar and cartesian coordinates as appropriate.

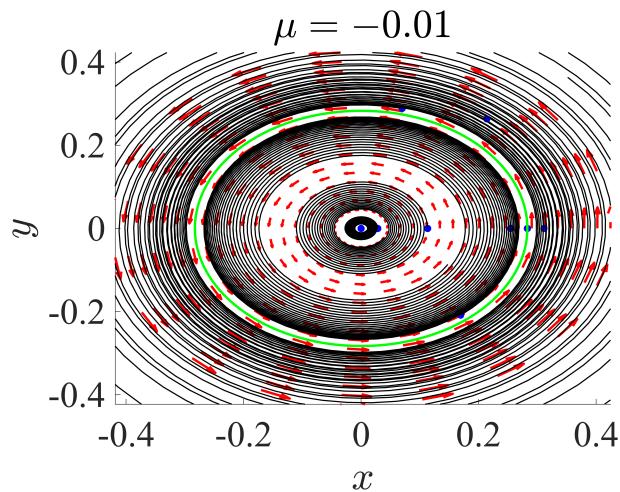


Figure 5: Phase Portrait for Approximated Polar Representation of System

Problem 8.2.11b

I plotted the phase portraits for $\mu > 0$, $\mu = 0$, and $\mu < 0$, shown below. We see here that when $\mu < 0$, the fixed point at the origin is unstable (an unstable spiral), as we previously saw. When $\mu > 0$, the fixed point at the origin is a stable spiral. However, at $\mu = 0$, we have a continuous band of closed orbits surrounding the origin, not limit cycles (which are isolated closed orbit). Thus, we have a degenerate Hopf bifurcation.

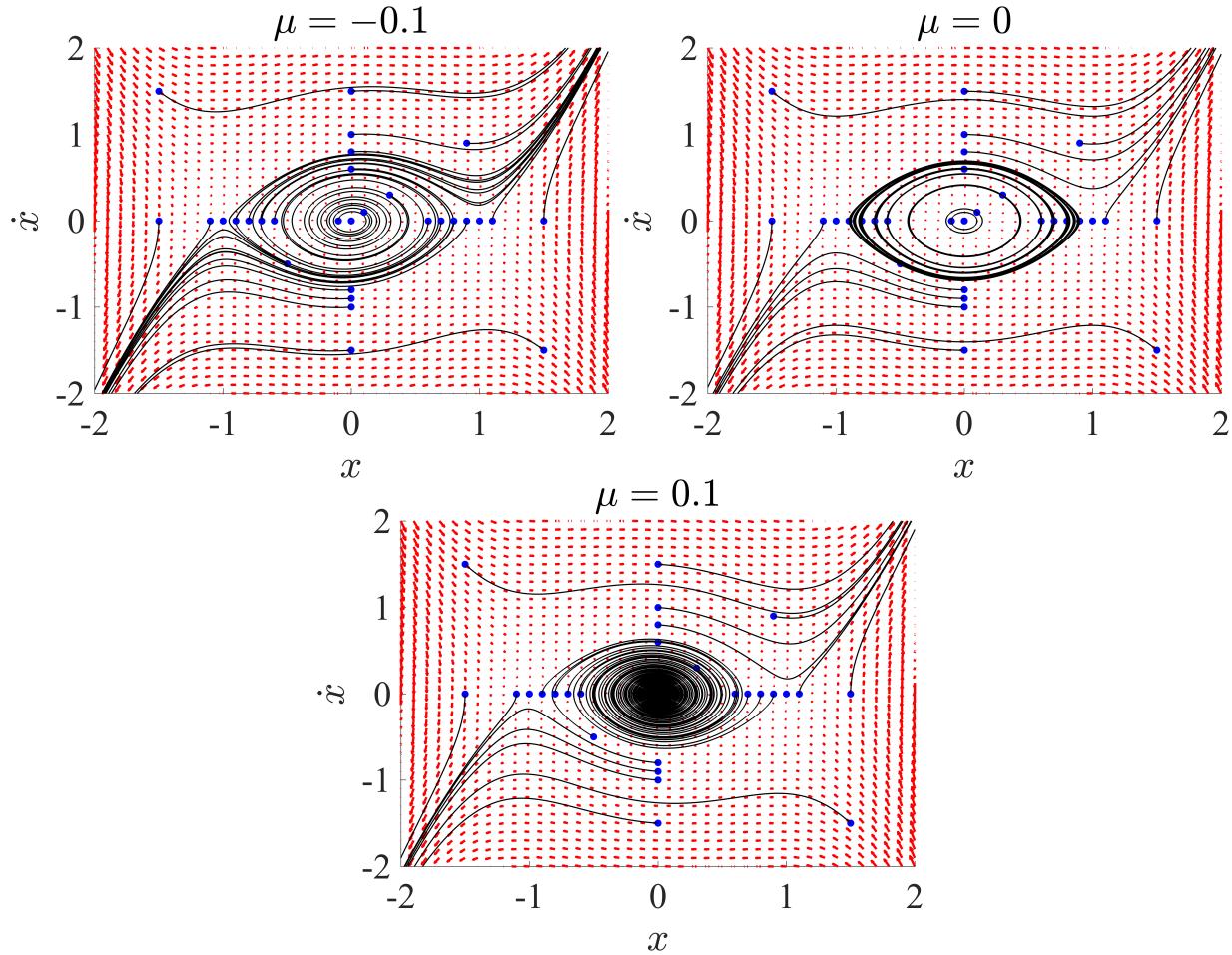


Figure 6: Phase Portraits for damped Duffing Oscillator

Problem 8.2.17e

In this part, I show that the Hopf bifurcation (when the trace of $A - B$ is positive) can be supercritical. To simulate this, note that we have a four-variable system, so we can't plot the dynamics in all four variables. For this reason, I calculate the full ODE and solution trajectories and plot the resulting dynamics of the variables that we are most interested in, which are the activity levels of the two populations of neurons (x_1 and x_2). So when we do this, I plot a phase

plane at constant y_1 and y_2 values. I solve my ODE with y_0 , or the initial state of the system, being at the y_1 and y_2 values of the phase plot. I also start my system at $x_1 \neq x_2$ and $y_1 \neq y_2$. Note that when the trajectories are plotted, we are seeing the projection onto the x_1/x_2 plane.

In the figures, I will also plot the initial conditions of trajectories as blue dots. I also plot the fixed point u^* , which is the point u such that $u = F(I - u(b + g))$ (we determined this in part a of the problem), as a green dot. My results are shown below.

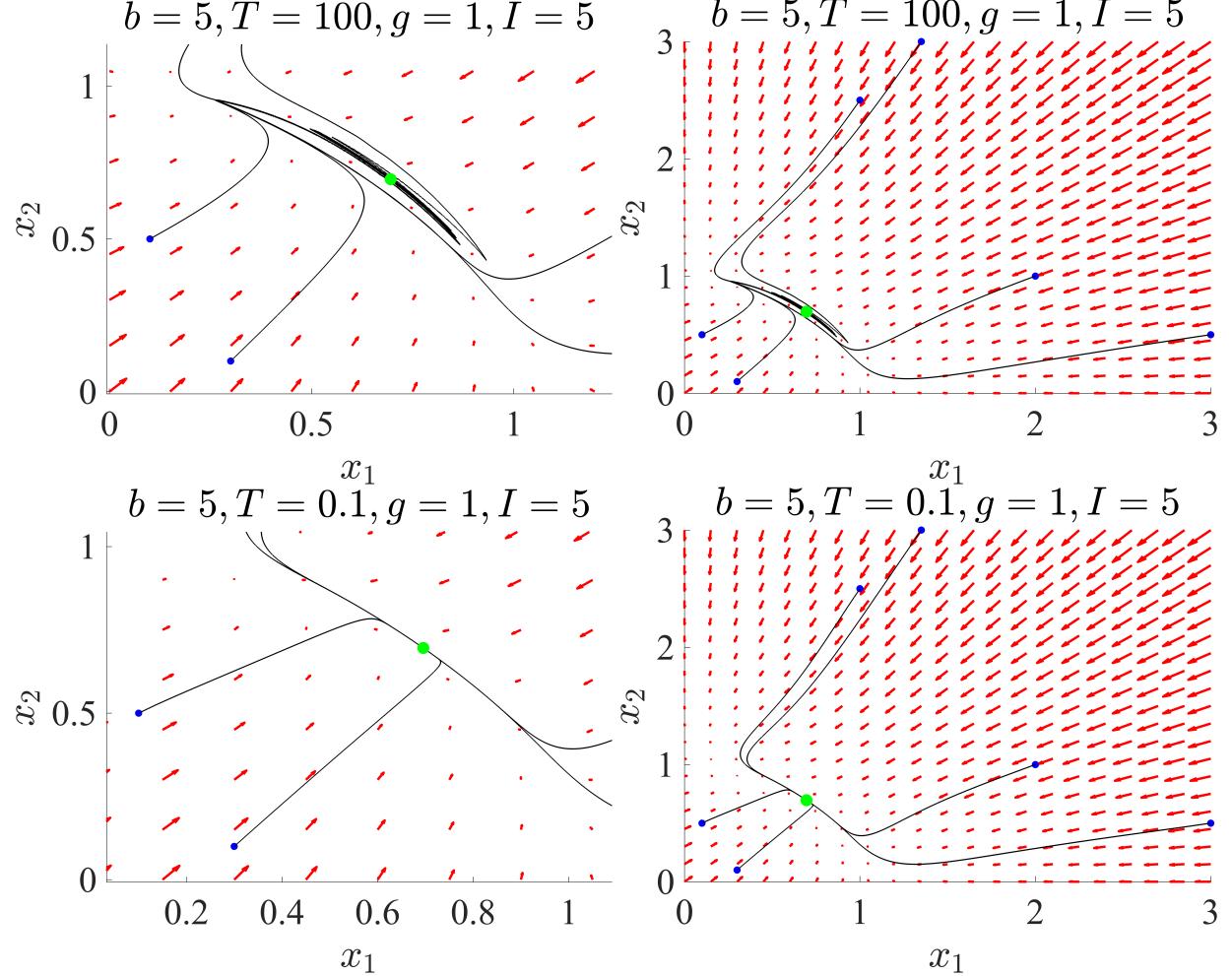


Figure 7: Phase Portraits in x_1/x_2 Plane for Hopf Bifurcation of Binocular Rivalry

We predicted that changing the value of T , which is the time scale of the adaptations in the y -variables building up, will change the sign of the trace of $A - B$ and thus result in a Hopf bifurcation. We see in the figures below that when I set $T = 100$, I seem to have a stable limit cycle around the fixed point (due to the oscillatory pattern) in which we have multiple trajectories spiraling around in that region. Note that the trajectories seem to be on top of each other because we are plotting the projection of the system onto the x_1/x_2 plane. We can confirm this by looking at the dynamics in the y variables, as in the next figure (same exact simulation, just plotting dynamics in the other two variables). Here, we see that when T is large, we have lots of oscillations around the same

region, indicating the stable limit cycle. When T is small, the trajectories seem to spiral at first but then approach a stable fixed point. We also know that this Hopf bifurcation is supercritical because we transition from a stable spiral (at small T) to a stable limit cycle (at large T).

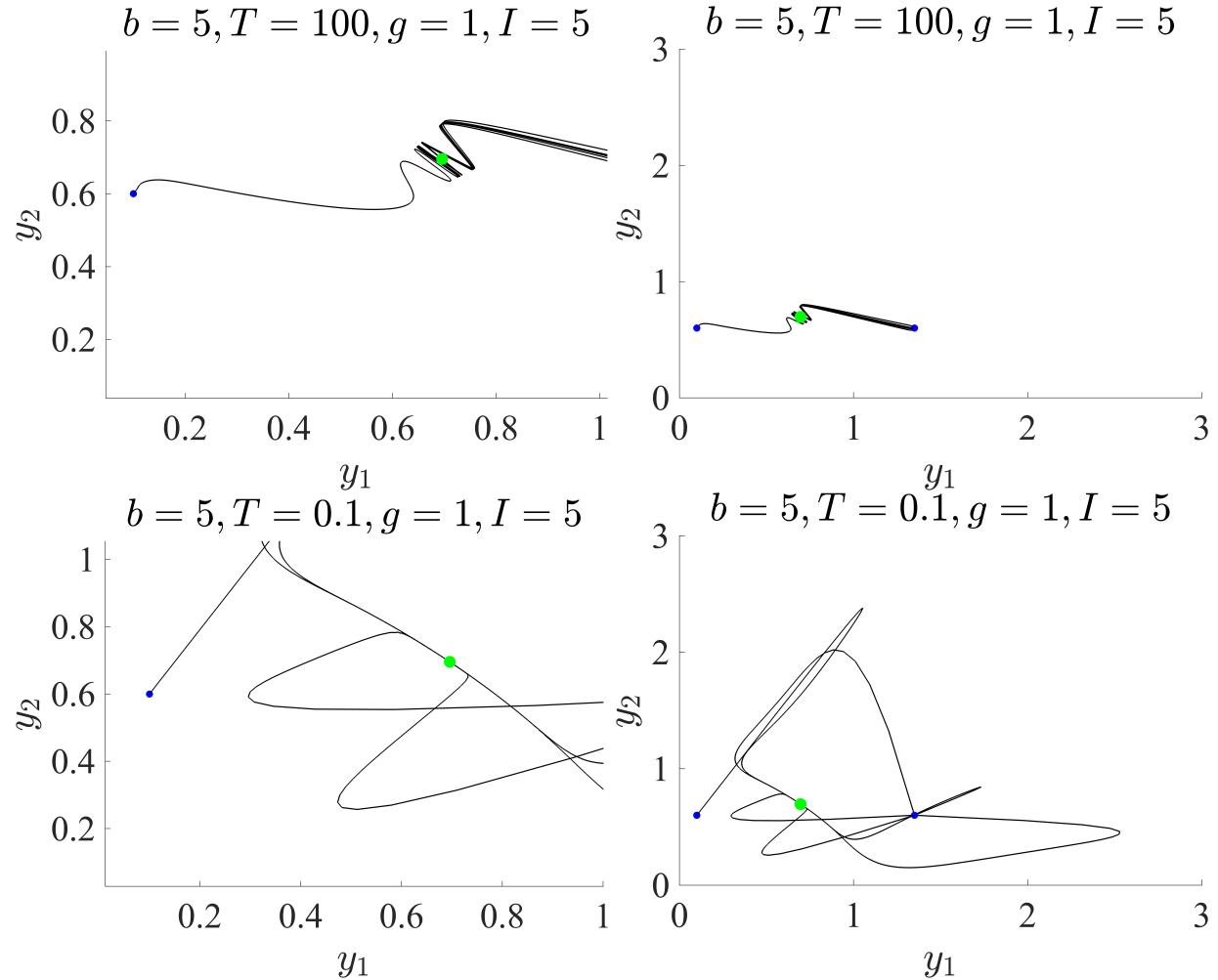


Figure 8: Phase Portraits in y_1/y_2 Plane for Hopf Bifurcation of Binocular Rivalry

Code

My code is shown below.

```

1 % 7.3.2
2
3 [x,y] = meshgrid(-1.25:.1:1.25,-1.25:.1:1.25);
4 xdot = x - y - x * (x.^2 + 5 * y.^2);
5 ydot = x + y - y * (x.^2 + y.^2);
6
7 figure()

```

```

8 hold on
9 quiver(x,y,xdot,ydot,'r','LineWidth',2);
10 f1 = @ode_732;
11 plot_solve([.1 0],f1)
12 plot_solve([- .1 0],f1)
13 plot_solve([0 .1 ],f1)
14 plot_solve([0 -.1 ],f1)
15 plot_solve([1.2 0],f1)
16 plot_solve([1.2 1.2],f1)
17 plot_solve([1.2 -1.2],f1)
18 plot_solve([-1.2 -1.2],f1)
19 plot_solve([-1.2 1.2],f1)
20 plot_solve([-1.2 0],f1)
21 plot_solve([0 1.2],f1)
22 plot_solve([0 -1.2],f1)
23 set(gca,'FontSize',30,'FontName','times')
24 xlabel("$x$",'Interpreter','latex')
25 ylabel("$y$",'Interpreter','latex')
26 rectangle('Position',[-1/sqrt(2) -1/sqrt(2) 2.*1/sqrt(2) 2.*1/sqrt(2)],'Curvature',[1 1],'EdgeColor','b','LineWidth',2)
27 rectangle('Position',[-1 -1 2.*1 2.*1], 'Curvature',[1 1],'EdgeColor','b','LineWidth',2)
28 exportgraphics(gcf,"7.3.2_plot.png","Resolution",600)
29
30 %% 8.2.3
31 close all
32 f2 = @ode_823;
33 mu = 0.01
34 figure()
35 hold on
36 [x,y] = meshgrid(-1:.1:1,-1:.1:1);
37 xdot = -y + (mu.*x) + (x .* y.^2);
38 ydot = x + (mu.*y) - (x.^2);
39 quiver(x,y,xdot,ydot,'r','LineWidth',2);
40 initial_pos = [.5 .5; .1 0; 0 .1; -.2 0; 0 -.3; 0 0; -.5 -.5; .5 0;
                  0 .5; .5 .5; -.5 .5; .5 -.5; -.5 0; .95 0; -.8 0; -.6 0];
41 for i = 1:length(initial_pos)
42     scatter(initial_pos(i,1),initial_pos(i,2),'filled','b');
43     plot_solve(initial_pos(i,:),f2,mu)

```

```

44 end
45 xlim([min(x(:)) max(x(:))])
46 ylim([min(y(:)) max(y(:))])
47 set(gca,'FontSize',30,'FontName','times')
48 xlabel("$x$",'Interpreter','latex')
49 title("$\mu = " + mu + "$",'Interpreter','latex')
50 ylabel("$y$",'Interpreter','latex')
51 exportgraphics(gcf,"Hopf_823_phase_mu" + mu + ".png",'Resolution',
   ,600)
52
53 %% 8.2.4
54 close all
55 figure()
56 hold on
57 mu = -.01;
58 rlimcycle = sqrt(-8 * mu);
59 rs = linspace(0,1.5*rlimcycle,20);
60 thetas = linspace(0,2*pi,20);
61 [r, theta] = meshgrid(rs,thetas);
62 rdot = mu .* r + (1/8) .* r.^3;
63 thetadot = ones(size(theta));
64 x = r.* cos(theta);
65 y = r.* sin(theta);
66 xdot = rdot .* cos(thetadot) - r .* thetadot .* sin(theta);
67 ydot = rdot .* sin(thetadot) + r .* thetadot .* cos(theta);
68 quiver(x,y,xdot,ydot,'r','LineWidth',2);
69
70 % in r and theta
71 initial_pos = rlimcycle.*[0 0; .1 0; .4 0; .9 0; 1.1 0; 1 0; 1.2 pi;
   0.95 -pi; 1.05 3*pi/2];
72 for i = 1:length(initial_pos)
    [xp,yp] = pol2cart(initial_pos(i,2),initial_pos(i,1));
    scatter(xp,yp,'filled','b');
    plot_solve_polar(initial_pos(i,:),mu)
end
77 set(gca,'FontSize',30,'FontName','times')
78 hold on
79 p = nsidedpoly(1000,'Center',[0 0], 'Radius', rlimcycle);
80 plot(p, 'EdgeColor', 'g','LineWidth',2,'FaceAlpha',0)

```

```

81 xlim([min(x(:)) max(x(:))])
82 ylim([min(y(:)) max(y(:))])
83 xlabel("$x$",'Interpreter','latex')
84 title("$\mu = " + mu + "$",'Interpreter','latex')
85 ylabel("$y$",'Interpreter','latex')
86 exportgraphics(gcf,"Heuristic_824" + mu + ".png",'Resolution',600)
87
88 %% 8.2.11
89
90 close all
91 figure()
92 hold on
93 [x,y] = meshgrid(-2:.1:2, -2:.1:2);
94 mu = 0
95 xdot = y;
96 ydot = x.^3 - x - mu .* y;
97 quiver(x,y,xdot,ydot,'r','LineWidth',2);
98 f1 = @ode_8211;
99 quiver(x,y,xdot,ydot,'r','LineWidth',2);
100 % in r and theta
101 initial_pos = [1.5 0; -1.5 0; 0 -1.5; 0 1.5; 1.5 -1.5; -1.5 1.5; 0
102 0; -.1 0; .1 .1; -.5 -.5; .3 .3; .9 .9; -.6 0; .6 0; 1 0; -1 0; 0
103 1; 0 -1; 0 .8; 0 .6; .7 0; .8 0; .9 0; 1.1 0; 0 -.8; 0 -.9; -.8
104 0; -.7 0; -.9 0; -1.1 0]
105 for i = 1:length(initial_pos)
106 scatter(initial_pos(i,1),initial_pos(i,2),'filled','b');
107 plot_solve(initial_pos(i,:),f1,mu);
108 end
109 set(gca,'FontSize',30,'FontName','times')
110 hold on
111 xlim([min(x(:)) max(x(:))])
112 ylim([min(y(:)) max(y(:))])
113 xlabel("$x$",'Interpreter','latex')
114 title("$\mu = " + mu + "$",'Interpreter','latex')
115 ylabel("$\dot{x}$",'Interpreter','latex')
116 exportgraphics(gcf,"PhasePlot_8211_" + mu + ".png",'Resolution',600)
117
118 %% 8.2.17e
119
120 close all

```

```

117 figure(1)
118 hold on
119 figure(2)
120 hold on
121 tspan = 0:.05:1000;
122 T = 100;
123 b = 5;
124 g = 1;
125 I = 5;
126
127 [x1, y1, x2, y2] = ndgrid([0:.15:3]);
128 F = @(x) 1 ./ (1+exp(-x));
129 x1dot = -x1 + F(I - b.*x2 - g.*y1);
130 x2dot = -x2 + F(I-b.*x1-g.*y2);
131 y1i = 10;
132 y2i = 5;
133 figure(1)
134 quiver(x1(:,y1i,:,:y2i),x2(:,y1i,:,:y2i),x1dot(:,y1i,:,:y2i),x2dot(:,y1i,:,:y2i),'r','LineWidth',2);
135
136 y1o = y1(y1i,y1i,y1i,y1i)
137 y2o = y2(y2i,y2i,y2i,y2i);
138 initial_pos = [1 y1o 2.5 y2o; 2 y1o 1 y2o; y1o .1 3 y2o; 3 y1o .5
    y2o; .1 y1o .5 y2o;.3 y1o .1 y2o];
139 for i = 1:length(initial_pos)
140
141     y0 = initial_pos(i,:);
142     [~,sol] = ode45(@(t,y) ode_8217(t,y,b,g,T,I),tspan,y0);
143     figure(1)
144     scatter(y0(1),y0(3),'filled','b');
145     plot(sol(:,1), sol(:,3),'k','LineWidth',1);
146
147     figure(2)
148     scatter(y0(2),y0(4),'filled','b');
149     plot(sol(:,2), sol(:,4),'k','LineWidth',1);
150 end
151
152 % Solve for fixed point
153 syms u

```

```

154 ustar = solve(F(I - b.*u - g.*u) == u, u);
155
156 figure(1)
157 xlim([min(x1(:)) max(x1(:))])
158 ylim([min(x2(:)) max(x2(:))])
159 set(gca,'FontSize',30,'FontName','times')
160 hold on
161 xlabel("$x_1$",'Interpreter','latex')
162 ylabel("$x_2$",'Interpreter','latex')
163 scatter(ustar,ustar,100,"g",'filled');
164
165 figure(2)
166 xlim([min(y1(:)) max(y1(:))])
167 ylim([min(y2(:)) max(y2(:))])
168 set(gca,'FontSize',30,'FontName','times')
169 hold on
170 xlabel("$y_1$",'Interpreter','latex')
171 ylabel("$y_2$",'Interpreter','latex')
172 scatter(ustar,ustar,100,"g",'filled');
173
174 figure(1)
175 title("$b = " + b + ", T = " + T + ", g = " + g + ", I = " + I + "$",'
    Interpreter','latex')
176 exportgraphics(gcf,"xs_HopfNeurons_b_" + b + "_T" + T + "_g" + g + "
    I_" + I + ".png",'Resolution',600)
177
178 figure(2)
179 title("$b = " + b + ", T = " + T + ", g = " + g + ", I = " + I + "$",'
    Interpreter','latex')
180 exportgraphics(gcf,"ys_HopfNeurons_b_" + b + "_T" + T + "_g" + g + "
    I_" + I + ".png",'Resolution',600)
181
182 function plot_solve(y0, myode, mu)
183     tspan = 0:.01:100;
184     [~,sol] = ode45(@(t,y) myode(t,y,mu),tspan,y0);
185     plot(sol(:,1), sol(:,2),'k','LineWidth',1);
186 end
187 function plot_solve_polar(y0,mu)
188     tspan = 0:.1:100;

```

```

189 % y vector here has r and theta
190 [~,sol] = ode45(@(t,y) ode_polar_824(t,y,mu),tspan,y0);
191 % Convert solution from polar coor to cartesian coor
192 x = sol(:,1).* cos(sol(:,2));
193 y = sol(:,1).* sin(sol(:,2));
194 plot(x,y,'k','LineWidth',1);
195 end
196 function dydt = ode_polar_824(~,yvec,mu)
197 % compute polar derivative
198 r = yvec(1);
199 theta = yvec(2);
200 rdot = mu .* r + (1/8) .* r.^3;
201 thetadot = ones(size(theta));
202 dydt = [rdot; thetadot];
203 end
204 function dydt = ode_732(~,yvec)
205 x = yvec(1);
206 y = yvec(2);
207 dydt = [x - y - x * (x.^2 + 5 * y.^2); x + y - y * (x.^2 + y.^2)];
208 ];
209 end
210 function dydt = ode_823(~,yvec,mu)
211 x = yvec(1);
212 y = yvec(2);
213 dydt = [-y + (mu.*x) + (x .* y.^2); x + (mu.*y) - (x.^2)];
214 end
215 function dydt = ode_8211(~,yvec,mu)
216 x = yvec(1);
217 y = yvec(2);
218 dydt = [y; x.^3 - x - mu .* y];
219 end
220 function dydt = ode_8217(~,yvec,b,g,T,I)
221 x1 = yvec(1);
222 y1 = yvec(2);
223 x2 = yvec(3);
224 y2 = yvec(4);
225 F = @(x) 1 ./ (1+exp(-x));
226 dydt = [-x1 + F(I - b.*x2 - g.*y1);
227 (x1 - y1) ./ T;

```

```
227      -x2 + F(I-b.*x1-g.*y2);  
228      (-y2 + x2) ./ T];  
229 end
```