

Kameel Khabaz

CAAM 282

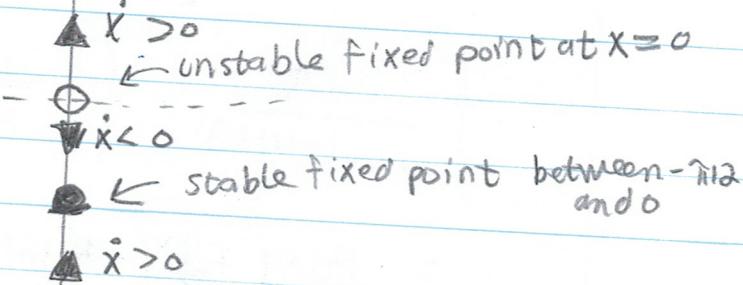
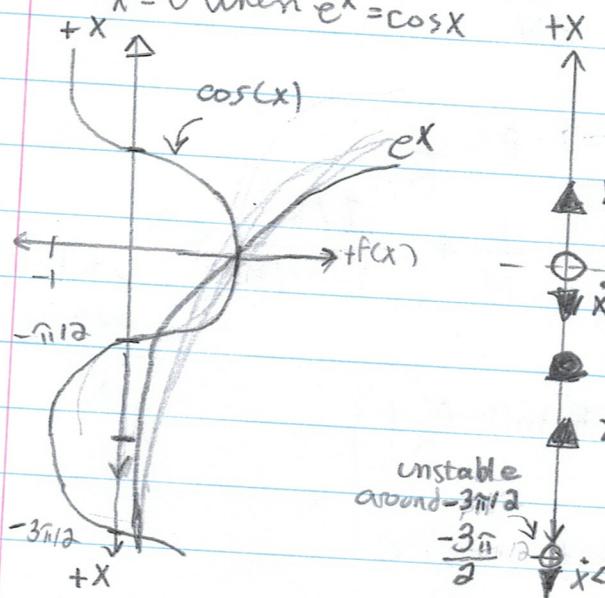
HW1

11/20/22

1. Strogatz 2.2.7:

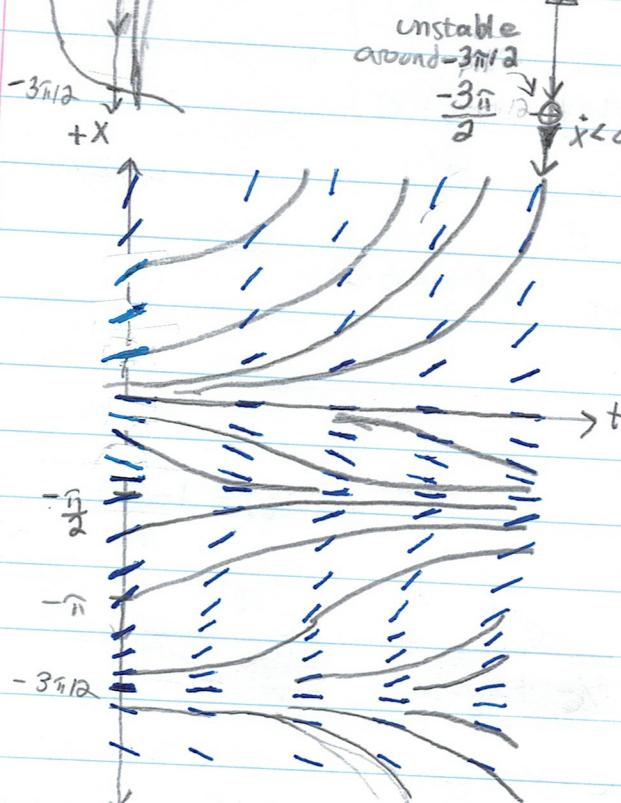
$$\dot{x} = e^x - \cos x \quad \mathcal{N} = [-\frac{3\pi}{2}, \frac{\pi}{2}]$$

$\dot{x} = 0$  when  $e^x = \cos x$



There will be more fixed points  
at smaller  $x$  closer to  $-\frac{3\pi}{2}$ ,  $-\frac{\pi}{2}$ , ...  
 $-\frac{3\pi}{2}, -\frac{5\pi}{2}, -\frac{7\pi}{2}, \dots$  ...

These fixed points alternate  
in stability



Q: 2, 3.1(a):

$$a, \frac{dN}{dt} = \dot{N} = rN(1 - N/K)$$

$$\int \frac{dN}{N(1 - N/K)} = \int r dt = rt + C$$

$$\frac{1}{N(1 - N/K)} = \frac{A}{N} + \frac{B}{1 - N/K}$$

$$1 = A(1 - N/K) + BN = A - AN/K + BN$$

$$\underline{1 = A} \quad 0 = -\frac{AN}{K} + BN$$

$$\Downarrow$$

$$BN = \frac{AN}{K} \Rightarrow B = \frac{A}{K}$$

$$\therefore \int \frac{dN}{N(1 - N/K)} = \int \frac{1}{N} + \frac{1/K}{1 - N/K} dN \quad u = 1 - N/K \\ du = -\frac{1}{K} dN \\ dN = -K du$$

$$= \ln|N| + \frac{1}{K} \left( -K \ln|1 - \frac{N}{K}| \right)$$

$$\Rightarrow \ln|N| - \ln|1 - \frac{N}{K}| = rt + C$$

$$e^{\ln|N|} \left| \frac{N}{1 - N/K} \right| = e^{rt + C}$$

$$\frac{N}{1 - N/K} = Ce^{rt} \quad (\text{different } C)$$

$$N = Ce^{rt} + \frac{Ce^{rt}}{K} N$$

$$N \left( 1 + \frac{Ce^{rt}}{K} \right) = Ce^{rt}$$

$$N = \frac{Ce^{rt}/e}{1 + Ce^{rt}/e} = \frac{Ce^{rt}}{e + Ce^{rt}} = \frac{Ce^{rt}}{\frac{e}{Ce^{rt}} + Ce^{rt}} = \frac{Ce^{rt}}{\frac{1}{C} + Ce^{rt}} = \frac{Ce^{rt}}{\frac{1 - N_0/K}{N_0} + Ce^{rt}}$$

$$\text{if } t=0, N_0 = \frac{C}{1 + C/K} \Rightarrow C = N_0 + \frac{C}{K} N_0 \Rightarrow C(1 - N_0/K) = N_0 \Rightarrow C = N_0 / (1 - N_0/K)$$

$$N(t) = \frac{Ce^{rt}}{\frac{1 - N_0/K}{N_0} + Ce^{rt}}$$

3: 2, 3, 4

a. Show max of  $f = \frac{N}{N}$  at intermediate  $N$

$$f = \frac{\dot{N}}{N} = r - a(N-b)^2$$

Find when  $f = \frac{\dot{N}}{N}$  is at its max (so  $\frac{df}{dN} = 0$ )

$$\frac{df}{dN} = -2a(N-b) = 0 \text{ when } N=b$$

When  $N=0$ ,  $f = \frac{\dot{N}}{N} = r$ . This is the maximum effective growth rate

$\therefore$  max effective growth rate at intermediate  $N$

$$\text{if } N > b \text{ or } N < b, \frac{\dot{N}}{N} < r$$

b. Fixed points when  $f = \dot{N} = 0$

$$\begin{aligned}\dot{N} &= rN - aN(N-b)^2 = 0 \\ \dot{N} &= N(r - a(N-b)^2) = 0 \\ N &= 0 \quad r = a(N-b)^2\end{aligned}$$

$$\begin{aligned}f' &= \frac{d}{dN} \dot{N} = r - a[(N-b)^2 + 2N(N-b)] \\ &= r - a(N-b)^2 + 2aN(N-b)\end{aligned}$$

$$N = b \pm \sqrt{\frac{r}{a}}$$

$$f'(0) = r - ab^2$$

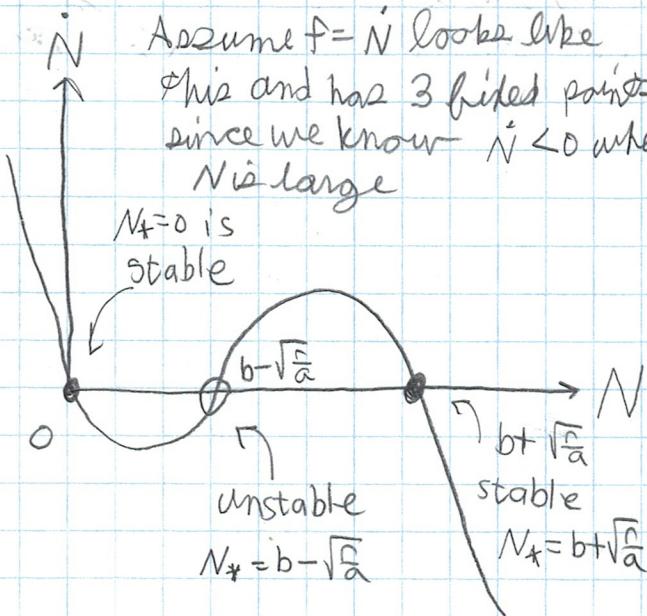
Fixed points are  $N_* = 0, b - \sqrt{\frac{r}{a}}, b + \sqrt{\frac{r}{a}}$

if  $r < ab^2$ ,  $N_* = 0$  is stable  
if  $r > ab^2$ ,  $N_*$  is unstable

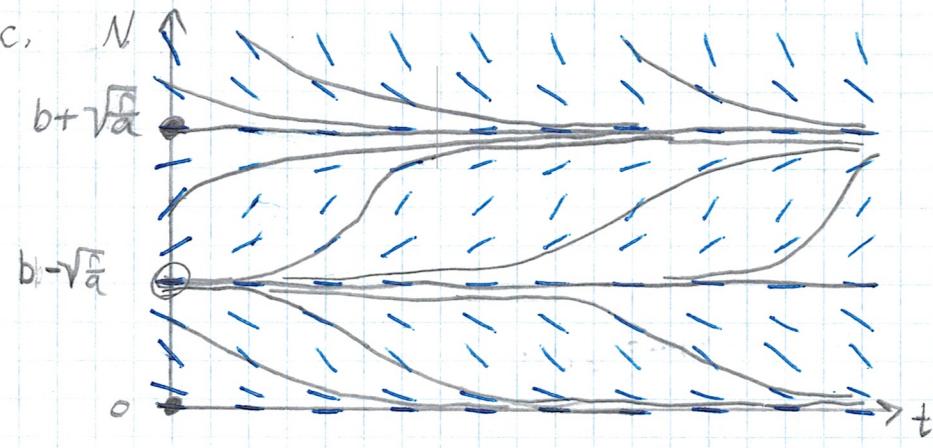
$$f'(b - \sqrt{\frac{r}{a}}) = 2a(b - \sqrt{\frac{r}{a}})(\sqrt{\frac{r}{a}})$$

$$f'(b + \sqrt{\frac{r}{a}}) = -2a(b + \sqrt{\frac{r}{a}})(\sqrt{\frac{r}{a}})$$

Stability depends on parameter  
set



2,3,4 (cont. 1):



The trajectories  
are unique

- d. The difference is that the logistic equation has only 2 fixed points, while this has 3 fixed points. This means when  $N_0$  is small (small initial population), even as small as 1 individual, the logistic model predicts that the population increases to the carrying capacity. However, this Allee model predicts that the population decreases back to 0. In this case, the Allee model is more realistic than the logistic model because individuals must be able to mate for the population to increase.

Otherwise, the models are qualitatively similar. Both have  $N < 0$  when  $N >$  the largest fixed point (which makes sense).

The Allee model will also have higher growth rates at populations closer to the carrying capacity (the highest fixed point) than the logistic model since the sigmoid-shaped trajectories are shifted upwards.

The fixed point at  $N=0$  is stable for both models, and the fixed point at the highest  $N$  is also stable. For the Allee model, there are three fixed points at intermediate  $N$  (which is unstable).

4

2, 3, 6.

$$a. \dot{x} = 0 = s(1-x)x^a - (1-s)x(1-x)^a \Rightarrow x=0 \text{ is fixed point}$$

$$s(1-x)x^a = (1-s)x(1-x)^a \quad x=1 \text{ is fixed point}$$

$$0 = s(1-x)x^{a-1} - (1-s)x(1-x)^{a-1}$$

$$= (1-x)x \left[ sx^{a-1} - (1-s)(1-x)^{a-1} \right]$$

$$x=1, 0$$

$$sx^{a-1} = (1-s)(1-x)^{a-1}$$

$$\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} = \frac{(1-x)^{a-1}}{x^{a-1}} = \left(\frac{1-x}{x}\right)^{a-1}$$

$$\frac{1-x}{x} = \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} \Rightarrow 1-x = x \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}$$

$$1 = x \left(1 + \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}\right)$$

$$x = \frac{1}{1 + \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}}$$

$$x = \frac{1}{1 + \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}} \leftarrow 3^{\text{rd}} \text{ fixed point}$$

This fixed point is between 0 and 1

$$b. \ddot{x} = s(-1)x^a + s(1-x)ax^{a-1} - (1-s)[(1-x)^a + ax(1-x)^{a-1}]$$

For  $x=0$ ,

$$\begin{aligned} \ddot{x}(0) &= \cancel{s(-1)x^a}_0 + \cancel{s(1-x)ax^{a-1}}_0 - (1-s)[\cancel{(1-x)^a}_0 + 0] \\ &= -s(1-s) \neq 0 \text{ since } 0 \leq s \leq 1, \end{aligned}$$

$\ddot{x}(0) = s-1 < 0 \therefore x=0 \text{ is stable fixed point}$

For  $x=1$ ,

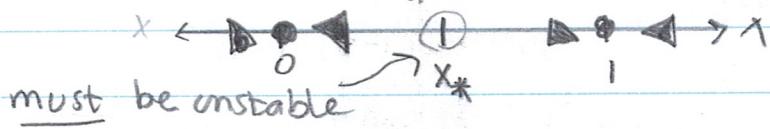
$$\begin{aligned} \ddot{x}(1) &= \cancel{s(-1)x^a}_0 + \cancel{s(1-x)ax^{a-1}}_0 - (1-s)[\cancel{x^a}_0 + a\cancel{x(1-x)^{a-1}}_0] \\ &= -s < 0 \therefore x=1 \text{ is stable fixed point} \end{aligned}$$

small number

4 (cont'd), c. Since  $x=0$  is stable fixed point,  $\dot{x}(0+\epsilon) < 0$

Since  $x=1$  is stable fixed point,  $\dot{x}(1-\epsilon) > 0$

Since we know  $\dot{x}$  must be  $< 0$  for  $x$  just greater than 0 and  $\dot{x}$  is  $> 0$  for  $x$  just below 1, and that there is a 3<sup>rd</sup> fixed point  $0 < x_* < 1$ , then  $\dot{x} < 0$  below  $x_*$  and  $\dot{x} > 0$  above  $x_*$  so  $x_*$  must be an unstable fixed point.



$$5(\partial M 77): f = \dot{x} = ax - x^3 = x(a - x^2)$$

$f = \dot{x} = 0$  when  $x = 0, \pm\sqrt{a}$  (3 Fixed points)

$$\frac{df}{dx} = a - 3x^2$$

$\frac{df}{dx}(0) = a$  if  $a > 0$ ,  $f(0)$  is unstable fixed pt

if  $a < 0$ ,  $f(0)$  is stable fixed pt (and the only fixed pt)  
if  $a = 0$ , see below

$$\frac{df}{dx}(\sqrt{a}) = a - 3a = -2a$$

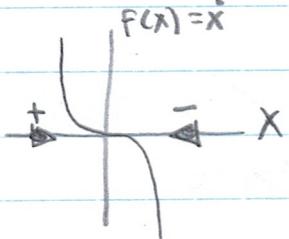
if  $a > 0$ ,  $f(\sqrt{a})$  is stable fixed pt  
if  $a < 0$ ,  $\sqrt{a}$  is not real, so no fixed pt here  
if  $a = 0$ , see below

$$\frac{df}{dx}(-\sqrt{a}) = a - 3a = -2a$$

if  $a > 0$ ,  $f(-\sqrt{a})$  is stable fixed pt  
if  $a < 0$ ,  $-\sqrt{a}$  is not real, so no fixed pt here

$$\text{if } a = 0, f(x) = -x^3$$

if  $a = 0$ , see below



if  $a = 0$ , there is only one fixed point at  $x = 0$

This is a stable fixed point since  $\dot{x} > 0$  when  $x < 0$  and  $\dot{x} < 0$  when  $x > 0$ .

(6. 2, 9, 9)

a.  $\dot{x} = -x^3 = \frac{dx}{dt}$

$$-\int x^{-3} \frac{dx}{x^3} = \int dt$$

$$\int \frac{x^{-2}}{2} = t + C$$

$$\frac{x^{-2}}{2} = t + C$$

$$\frac{1}{x^2} = 2t + C \text{ (different } C\text{)}$$

$$x^2 = \frac{1}{2t+C}$$

$$x = \pm \sqrt{\frac{1}{2t+C}}$$

$$\lim_{t \rightarrow \infty} x^2 = \lim_{t \rightarrow \infty} \frac{1}{2t+C} = \frac{1}{\infty} = 0$$

$x^2 \rightarrow 0$  as  $t \rightarrow \infty$

$\therefore x \rightarrow 0$  as  $t \rightarrow \infty$

b. See computer pages

We see on the plot that the decay for  $\dot{x} = -x^3$  is much slower than  $\dot{x} = -x$  after a certain time point (when they cross)

$$7(2,5,2); \text{ for } \dot{x} = 1+x^2, \int \frac{dx}{1+x^2} = \int dt$$

$$\Rightarrow \tan^{-1} x = t + C$$

$$\text{as } t+c \rightarrow \frac{\pi}{2}, \tan\left(\frac{t+c}{2}\right) \rightarrow +\infty \quad \because x \rightarrow +\infty \text{ as } t+c \rightarrow \frac{\pi}{2}$$

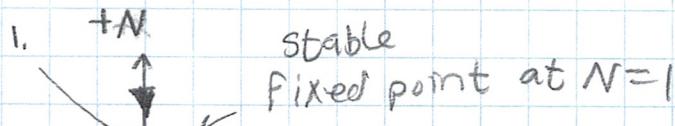
∴ solution for  $\dot{x} = 1+x^2$  diverges in finite time

We know that  $|t+x^{10}| > |t+x^2|$  for  $x \geq 1$ , which means that the solution for  $\dot{x} = |t+x^{10}|$  will diverge faster than the solution for  $\dot{x} = |t+x^2|$ . So since we know  $\dot{x} = |t+x^2|$  diverges infinite time,  $\dot{x} = |t+x^{10}|$  must also diverge in finite time.

to  $\infty$

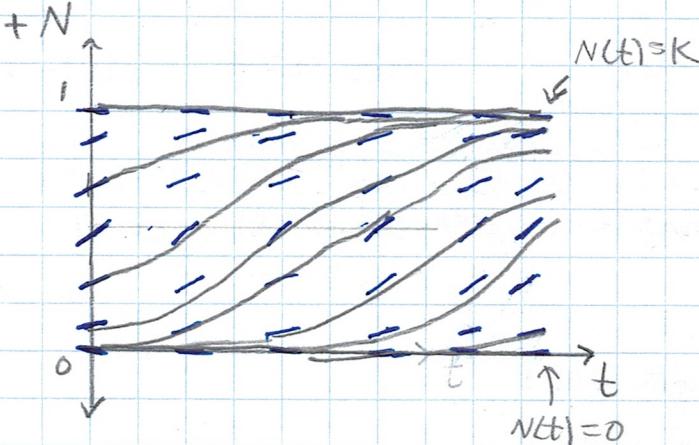
### Part 1:

8.  $N_* = 0, 1$



$$f(x) = \frac{d}{dt} N(t) = rN(t)(1-N(t))$$

unstable fixed point at  $N=0$



Since  $\frac{d}{dt} N(t) = 0$  when  $N=1$ , then a trajectory cannot cross the carrying capacity because trajectories are continuous and  $f(x)$  is well-defined (so it can only take one value at a particular  $x$ ). This must happen monotonically.

We see that if  $0 < N < 1$ ,

then  $\frac{d}{dt} N(t) > 0$  and that

$N=1$  is a stable fixed point, this means that if a trajectory starts from  $N(0) > 0$  (positive population), then the population trajectory must increase and approach  $N=1$  as  $t$  increases ( $t \rightarrow \infty$ ).

2.  $\frac{d\tau}{dt} = \lambda \Rightarrow \frac{d}{d\tau} N(t(\tau)) = \frac{dN(t(\tau))}{dt} \cdot \frac{dt}{d\tau} = \frac{1}{\lambda} rN(t(\tau))(1-N(t(\tau)))$

$$\frac{dt}{d\tau} = \frac{1}{\lambda} \quad dt = \frac{1}{\lambda} d\tau$$

$$t(\tau) = \frac{1}{\lambda} \tau \quad \text{if } \lambda = \text{const}$$

$$\text{Let } \lambda = r$$

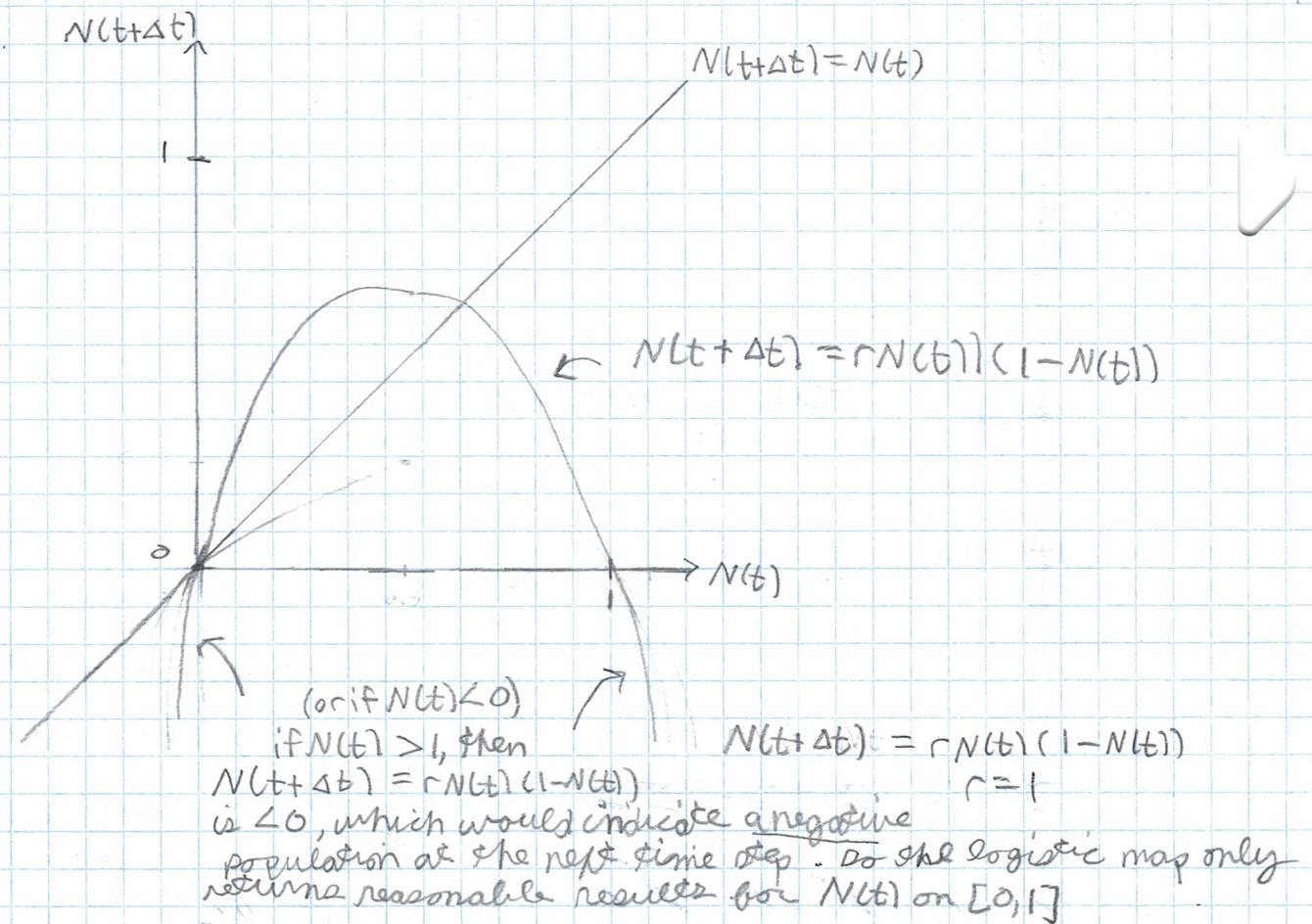
$$\frac{d}{d\tau} N(t(\tau)) = \frac{1}{r} N(t(\tau))(1-N(t(\tau)))$$

$$= N(1-N) = N(\tau)(1-N(\tau))$$

This means that for any logistic equation with any  $r$ , we set  $\tau = rt$  to change the time units and the differential equation will be independent of  $r$ . So regardless of  $r$ , if we scale the time units by  $r$ , all solutions will be identical.

3. see computer pages

8(cont.)



2. fixed point when  $N(t+\Delta t) = N(t)$

$$rN(t)(1-N(t)) = N(t)$$

$$rN - rN^2 = N$$

$$0 = rN^2 + (1-r)N = N(rN + 1 - r)$$

So 1 fixed point is  $N=0$ , another is  $rN + 1 - r = 0$ .  $\frac{dN}{dt} = \frac{r-1}{r}$ . 1st fixed point is always at  $N=0$ .

$$N = \frac{r-1}{r}$$

$$N_{\star}(r) = \frac{r-1}{r}$$

If  $r > 1$ , then the second fixed point will be at positive  $N$ , if  $r < 1$ , then the 2nd fixed point is  $< 0$ , which is not realistic.

so the minimum  $r$  for 2 fixed points is larger than 1 ( $r > 1$ )

3. see computer pages

9. Estimating  $v_*$  for flying North to South poles in daylight:

$$v_* \sim \frac{\frac{2\pi R}{2}}{T} \sim \frac{\frac{\pi \cdot 6 \cdot 10^6 \text{ m}}{12 \text{ hr} \cdot 60 \text{ min} \cdot 60 \text{ s}}}{\text{min}} \sim 40 \text{ m/s}$$

$$1. \frac{d}{dt} \phi(t) = 0 = \frac{\omega}{4} \sqrt{4 - \sec(\phi(t))^2}$$

$$\omega = \left[ -\frac{\pi}{3}, \frac{\pi}{3} \right]$$

$$4 - \sec^2 \phi = 0 \\ 4 = \sec^2 \phi \\ \pm 2 = \sec \phi$$

$$\phi = -\frac{\pi}{3} + \frac{\pi}{3}$$

$$\phi_* = (-60^\circ, +60^\circ) \text{ in degrees}$$

while the angular velocity of the Earth's rotation is the same. So, at some latitude

the two must be equal, and one has to be able to overtake night

by traveling west, at which  $\frac{d}{dt} \phi(t) = 0$ .

So, equilibrium solutions must exist. For all  $t \in (t_0 - \epsilon, t_0 + \epsilon)$ ,  $\epsilon > 0$

2. For an ODE to admit unique solutions given an initial condition, the function  $\frac{dx}{dt}$  must be Lipschitz continuous on the interval  $[x_0 - \delta, x_0 + \delta]$  for  $\delta > 0$ .

$\forall x_1, x_2 \in [a, b]$ , the slope of the secant line from

$(x_1, f(x_1))$  to  $(x_2, f(x_2))$  must be bounded by a constant  $K$  that only depends on  $f$ ,  $a$ , and  $b$  for  $f(x)$  to be Lipschitz continuous on the interval  $[a, b]$  and thus for the ODE to admit unique solutions.

3. See computer page

