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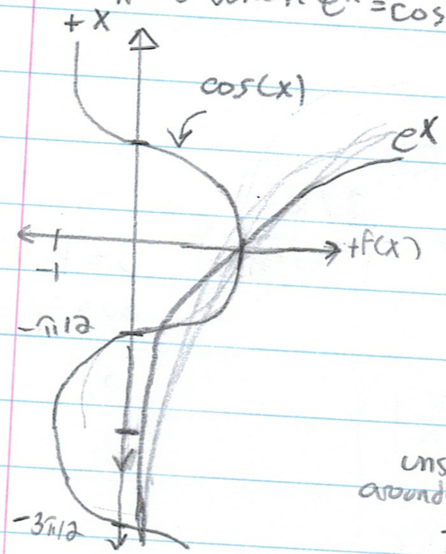
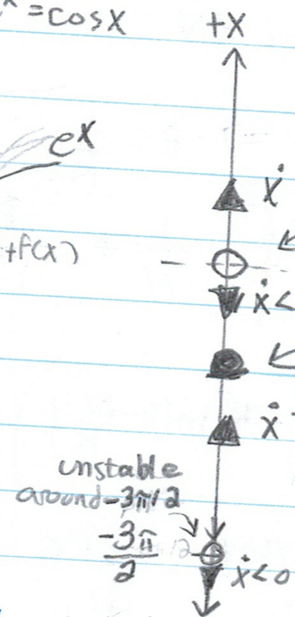
CAAM 282
HW1
1/20/22

1. Strogatz 2.2.7:

$\dot{x} = e^{x} - \cos x \quad \Omega = [-\frac{3\pi}{2}, \pi]$

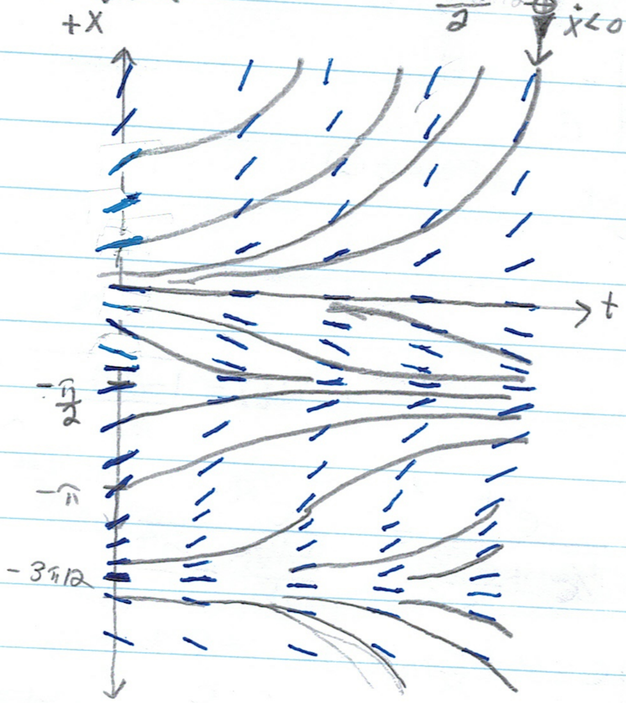
$\dot{x} = 0$ when $e^{x} = \cos x$





There will be more fixed points at smaller x close to $-\frac{3\pi}{2}, -\frac{5\pi}{2}, -\frac{7\pi}{2}, \ldots$

These fixed points alternate in stability



2: 2.3.1(a):

a, $\frac{dN}{dt} = \dot{N} = rN(1-N/K)$

$$\int \frac{dN}{N(1-N/K)} = \int r\,dt = rt + C$$

$$\frac{1}{N(1-N/K)} = \frac{A}{N} + \frac{B}{1-N/K}$$

$$1 = A(1-N/K) + BN = A - AN/K + BN$$

$\underline{1 = A}$ $\quad 0 = -\frac{AN}{K} + BN$

$\Downarrow$

$$BN = \frac{N}{K} \Rightarrow B = \frac{1}{K}$$

$$\therefore \int \frac{dN}{N(1-N/K)} = \int \frac{1}{N} + \frac{1/K}{1-N/K}\,dN$$

$u = 1-N/K$
$du = -\frac{1}{K}dN$
$dN = -K\,du$

$$= \ln|N| + \frac{1}{\cancel{K}}\left(-\cancel{K}\ln\left|1-\frac{N}{K}\right|\right)$$

$$\Rightarrow \ln|N| - \ln|1-N/K| = rt + C$$

$$e^{\ln\left|\frac{N}{1-N/K}\right|} = e^{rt+C}$$

$$\frac{N}{1-N/K} = Ce^{rt} \quad \text{(different C)}$$

$$N = Ce^{rt} - \frac{Ce^{rt}N}{K}$$

$$N\left(1+\frac{Ce^{rt}}{K}\right) = Ce^{rt}$$

$$N = \frac{\cancel{C}e^{rt}/\cancel{C}}{\frac{1}{C}+\frac{\cancel{C}e^{rt}}{K}/\cancel{C}} = \frac{e^{rt}}{\frac{1}{C}+\frac{e^{rt}}{K}} = \frac{e^{rt}}{\frac{1-N_0/K}{N_0}+\frac{e^{rt}}{K}}$$

if $t=0$, $N_0 = \frac{C}{1+C/K} \Rightarrow C = N_0 + \frac{C}{K}N_0 \Rightarrow C(1-N_0/K) = N_0$

$\Rightarrow C = N_0/(1-N_0/K)$

$$\boxed{N(t) = \frac{e^{rt}}{\frac{1-N_0/K}{N_0}+\frac{e^{rt}}{K}}}$$

3; 2.3.4

a. show max of $f = \frac{\dot{N}}{N}$ at intermediate N

$$f = \frac{\dot{N}}{N} = r - a(N-b)^2$$

Find when $f = \frac{\dot{N}}{N}$ is at its max (so $\frac{df}{dN} = 0$)

$\frac{df}{dN} = -2a(N-b) = 0$ when $N = b$

When $N=0$, $f = \frac{\dot{N}}{N} = r$ This is the maximum effective growth rate

$\therefore$ max effective growth rate at intermediate N

if $N > b$ or $N < b$, $\frac{\dot{N}}{N} < r$

b. fixed points when $f = \dot{N} = 0$

$$\dot{N} = rN - aN(N-b)^2 = 0$$
$$\dot{N} = N(r - a(N-b)^2) = 0$$
$N = 0$ $\quad r = a(N-b)^2$

$$N = b \pm \sqrt{\frac{r}{a}}$$

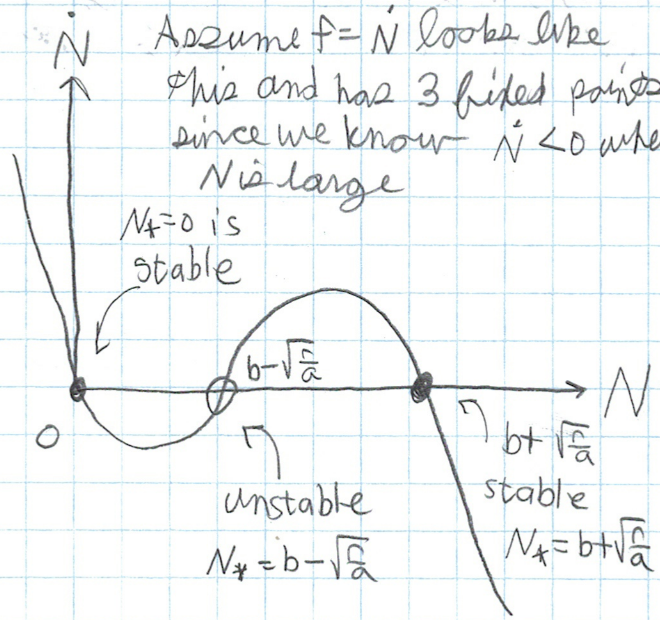
fixed points are $N_* = 0,\ b - \sqrt{\frac{r}{a}},\ b + \sqrt{\frac{r}{a}}$

$$f' = \frac{d}{dN}\dot{N} = r - a[(N-b)^2 + 2N(N-b)]$$
$$= r - a(N-b)^2 + 2aN(N-b)$$

$$f'(0) = r - ab^2$$

if $r < ab^2$, $N_* = 0$ is stable
if $r > ab^2$, $N_* = 0$ is unstable

Assume $f = \dot{N}$ looks like this and has 3 fixed points since we know $\dot{N} < 0$ when N is large



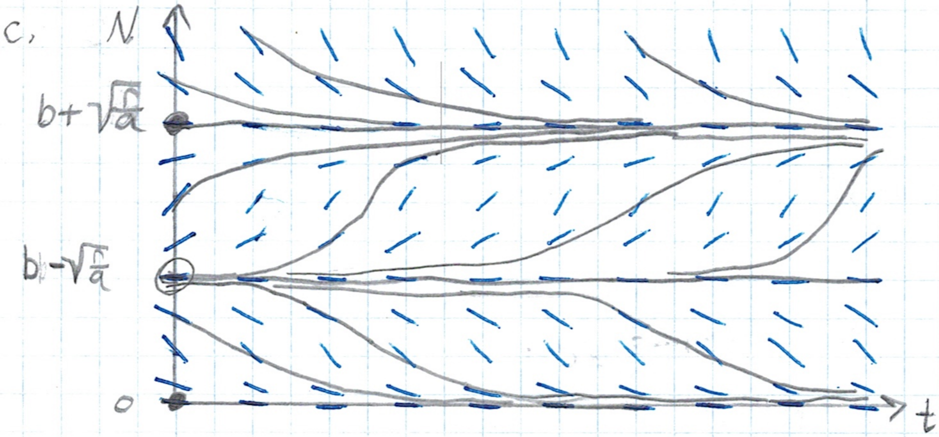
$$f'(b - \sqrt{\tfrac{r}{a}}) = 2a(b - \sqrt{\tfrac{r}{a}})(\sqrt{\tfrac{r}{a}})$$

$$f'(b + \sqrt{\tfrac{r}{a}}) = -2a(b - \sqrt{\tfrac{r}{a}})(-\sqrt{\tfrac{r}{a}})$$

Stability depends on parameter set

2,3,4 (cont.):

c.



The trajectories are unique

d. The difference is that the logistic equation has only 2 fixed points, while this has 3 fixed points. This means when N_0 is small (small initial population), even as small as 1 individual, the logistic model predicts that the population increases to the carrying capacity. However, this Allee model predicts that the population decreases back to 0. In this case, the Allee model is more realistic than the logistic model because individuals must be able to mate for the population to increase.

Otherwise, the models are qualitatively similar. Both have $\dot{N} < 0$ when $N >$ the largest fixed point (which makes sense).

The allee model will also have higher growth rates at populations closer to the carrying capacity (the highest fixed point) than the logistic model since the sigmoid-shaped trajectories are shifted upwards.

The fixed point at $N=0$ is stable for both models, and the fixed point at the highest N is also stable. For the allee model, there is a third fixed point at intermediate N (which is unstable).

4

2.3.6.

a. $\dot{x} = 0 = s(1-x)x^a - (1-s)x(1-x)^a$ $\Rightarrow x=0$ is fixed point

$s(1-x)x^a = (1-s)x(1-x)^a$ $x=1$ is fixed point

$$0 = s\cancel{(1-x)}\cancel{x}x^{a-1} - (1-s)\cancel{x(1-x)}(1-x)^{a-1}$$

$$= \underbrace{(1-x)x}_{x=1,0}\underbrace{[sx^{a-1} - (1-s)(1-x)^{a-1}]}$$

$$sx^{a-1} = (1-s)(1-x)^{a-1}$$

$$\left(\frac{s}{1-s}\right)^{1/a-1} = \frac{(1-x)^{a-1}}{x^{a-1}} = \left(\frac{1-x}{x}\right)^{a-1}$$

$$\frac{1-x}{x} = \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}} \Rightarrow 1-x = x\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}$$

$$1 = x\left(1 + \left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}\right)$$

$$x = \frac{1}{1+\left(\frac{s}{1-s}\right)^{\frac{1}{a-1}}}$$

$$x = \frac{1}{1+\left(\frac{s}{1-s}\right)^{1/(a-1)}}$$ ← 3rd fixed point

This fixed point is between 0 and 1

b. $\ddot{x} = s(-1)x^a + s(1-x)ax^{a-1} - (1-s)\left[(1-x)^a + ax(1-x)^{a-1}\right]$

For $x=0$,

$$\ddot{x}(0) = \cancel{s(-1)(0)^a}_0 + \cancel{s(1-0)a(0)^{a-1}}_0 - (1-s)[\cancel{(1-0)^a}^1 + 0]$$

$$= -\cancel{s}(1-s) = s-1 \quad \text{since } 0 \le s \le 1,$$

$\ddot{x}(0) = s-1 < 0 \quad \therefore x=0$ is stable fixed point

for $x=1$

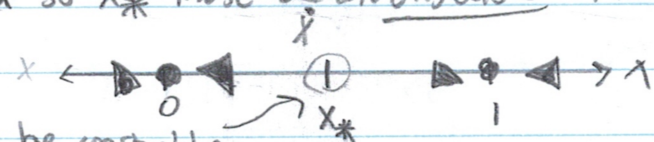
$$\ddot{x}(1) = s(-1)\cancel{1}^a + \cancel{s(0)a1^{a-1}} - \cancel{(1-s)[0^a + a1(0)^{a-1}]}$$

$= -s < 0 \therefore x=1$ is stable fixed point small number

4 (cont'd), c. Since $x=0$ is stable fixed point, $\dot{x}(0+\epsilon) < 0$

Since $x=1$ is stable fixed point, $\dot{x}(1-\epsilon) > 0$

since we know $\dot{x}$ must be <0 for x just greater than 0 and $\dot{x}$ is >0 for x just below 1, and that there is a 3rd fixed point $0 < x_* < 1$, then $\dot{x} < 0$ below x_* and $\dot{x} > 0$ above x_* so x_* must be an unstable fixed point.



must be unstable

5 (2.4.7): $f = \dot{x} = ax - x^3 = x(a - x^2)$

$f = \dot{x} = 0$ when $x = 0, \pm\sqrt{a}$ (3 fixed points)

$\frac{d}{dx} f = a - 3x^2$

$\frac{df}{dx}(0) = a$ if $a > 0$, $f(0)$ is unstable fixed pt

if $a < 0$, $f(0)$ is stable fixedpt (and the only fixed pt)

if $a = 0$, see below

$\frac{df}{dx}(\sqrt{a}) = a - 3a = -2a$ if $a > 0$ $f(\sqrt{a})$ is stable fixedpt

if $a < 0$, $\sqrt{a}$ is not real, so no fixed pt here.

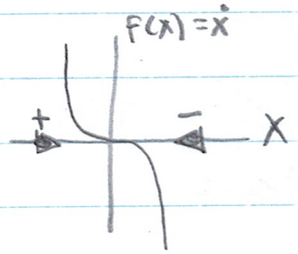
if $a = 0$ see below

$\frac{df}{dx}(-\sqrt{a}) = a - 3a = -2a$ if $a > 0$, $f(-\sqrt{a})$ is stable fixedpt

if $a < 0$, $\sqrt{a}$ is not real, so no fixed pt here.

if $a = 0$ see below

if $a = 0$, $f(x) = -x^3$



if $a = 0$, there is only one fixed point at $x = 0$. This is a stable fixed point since $\dot{x} > 0$ when $x < 0$ and $\dot{x} < 0$ when $x > 0$.

1.6. 2,4,9;

a. $\dot{x} = -x^3 = \frac{dx}{dt}$

$-\int x^{-3} \frac{dx}{\cancel{x^3}} = \int + dt$

$\frac{\cancel{-}\, x^{-2}}{\cancel{-}2} = t + C$

$\frac{x^{-2}}{2} = t + C$

$\frac{1}{x^2}\ \cancel{x^{-2}} = 2t + C$ (different C)

$x^2 = \frac{1}{2t+C}$

$x = \pm\sqrt{\frac{1}{2t+C}}$

Decay as $t \to \infty$ is proportional to $\sqrt{\frac{1}{2t}}$

This decay is much <u>slower</u> than exponential

$\lim_{t\to\infty} x^2 = \lim_{t\to\infty} \frac{1}{2t+C} = \frac{1}{\infty} = 0$

$x^2 \to 0$ as $t \to \infty$

$\therefore x \to 0$ as $t \to \infty$

b. See computer pages

We see on the plot that the decay for $\dot{x} = -x^3$ is much slower than $\dot{x} = -x$ after a certain time point (when they cross)

7(2.5.2): For $\dot{x} = 1+x^2$, $\displaystyle\int \frac{dx}{1+x^2} = \int dt$

$\Rightarrow \tan^{-1} x = t + c$

as $t+c \to \frac{\pi}{2}$, $\tan\left(\frac{\pi}{2}\right) \to +\infty$ $\therefore$ $x \to +\infty$ as $t+c \to \frac{\pi}{2}$

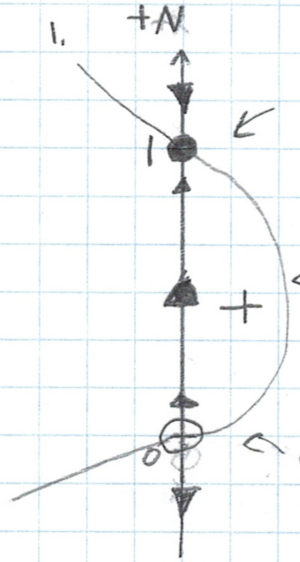
$\therefore$ solution for $\dot{x} = 1+x^2$ diverges to $+\infty$ in finite time

We know that $1+x^{10} > 1+x^2$ for $x>1$, which means that the solution for $\dot{x} = 1+x^{10}$ will diverge faster than the solution for $\dot{x} = 1+x^2$, so since we know $\dot{x} = 1+x^2$ diverges in finite time, $\dot{x} = 1+x^{10}$ must also diverge to $+\infty$ in finite time.

Part 1:

8. $N_* = 0, 1$

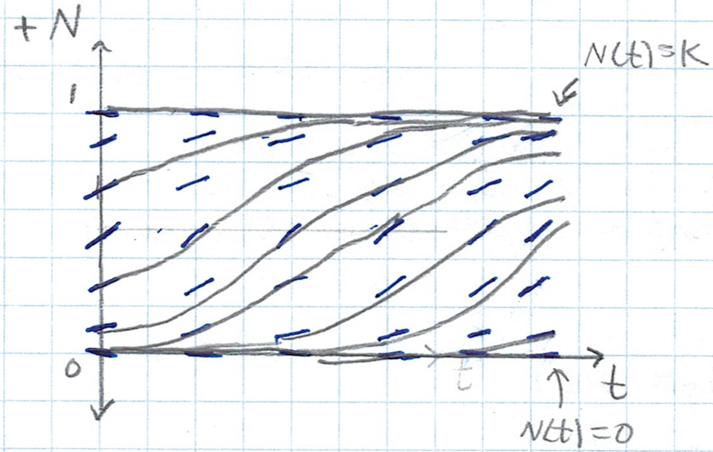
1.



stable fixed point at $N=1$

$\leftarrow f(x) = \frac{d}{dt} N(t) = rN(t)(1-N(t))$

unstable fixed point at $N=0$



We see that if $0<N<1$, then $\frac{d}{dt}N(t) > 0$ and that $N=1$ is a stable fixed point. This means that if a trajectory starts from $N(0)>0$ (positive population), then the population trajectory must increase and approach $N=1$ as t increases ($t\to\infty$). This must happen monotonically.

Since $\frac{d}{dt}N(t) = 0$ when $N=1$, then a trajectory cannot cross the carrying capacity because trajectories are continuous and $f(x)$ is well-defined (so it can only take on one value at a particular x).

2. $\tau = \lambda t$

$d\tau = \lambda dt$

$\frac{dt}{d\tau} = \frac{1}{\lambda}$

$t(\tau) = \frac{1}{\lambda}\tau$

$\frac{d}{d\tau} N(t(\tau)) = \frac{dN(t(\tau))}{dt} \frac{dt}{d\tau} = \frac{1}{\lambda} rN(t(\tau))(1-N(t(\tau)))$

Let $\lambda = r$

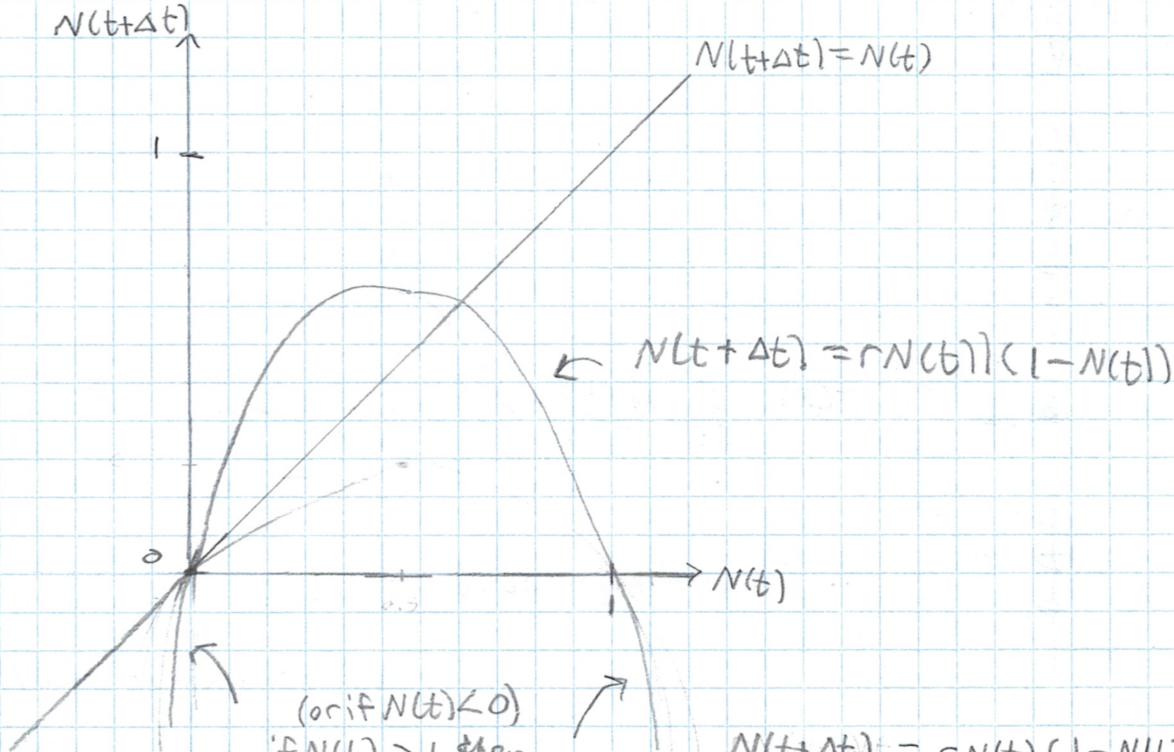
$\therefore \frac{d}{d\tau} N(t(\tau)) = N(t(\tau))(1-N(t(\tau)))$
$= N(1-N) = N(\tau)(1-N(\tau))$

This means that for any logistic equation with any r, we set $\tau = rt$ to change the time units and the differential equation will be independent of r. So regardless of r, if we scale the time units by r, all solutions will be identical.

3. see computer pages

8(cont.).

1.



(or if $N(t)<0$)
if $N(t) > 1$, then
$N(t+\Delta t) = rN(t)(1-N(t))$
is <0, which would indicate a negative population at the next time step. So the logistic map only returns reasonable results for $N(t)$ on $[0,1]$

$N(t+\Delta t) = rN(t)(1-N(t))$
$r=1$

2. fixed point when $N(t+\Delta t) = N(t)$

$$rN(t)(1-N(t)) = N(t)$$
$$rN - rN^2 = N$$
$$0 = rN^2 + (1-r)N = N(rN + 1 - r)$$

So 1 fixed point is $\boxed{N_* = 0}$, another is $rN + 1 - r = 0$, $rN = r-1$, $\boxed{N_* = \frac{r-1}{r}}$

1st fixed point is always at $N=0$

$$N_*(r) = \frac{r-1}{r}$$

if $r>1$, then the second fixed point will be at positive N, if $r<1$, then the 2nd fixed point is <0, which is not realistic.
so, the minimum r for 2 fixed points is r larger than 1 ($r>1$)

3. see computer pages

9. Estimating V_* for flying North to South poles in daylight:

$$V_* \sim \frac{\frac{2\pi R}{2}}{T} \sim \frac{\pi \cdot 6 \cdot 10^6 m}{12 \cancel{hr} \cdot 60 \frac{\cancel{min}}{\cancel{hr}} \cdot 60 \frac{s}{\cancel{min}}} \sim 400 m/s$$

1, $\frac{d}{dt}\phi(t) = 0 = \frac{\omega}{4}\sqrt{4-\sec(\phi(t))^2}$

$\Omega = [-\frac{\pi}{3}, \frac{\pi}{3}]$

$4-\sec^2\phi = 0$

$4 = \sec^2\phi$

$\pm 2 = \sec\phi$

$\phi_* = -\frac{\pi}{3}, +\frac{\pi}{3}$

$\phi_* = (-60°, +60°)$ in degrees latitude

The two equilibrium solutions must exist because as latitude increases, the angular velocity that can be achieved at V_* diverges ($\uparrow$ alot), while the angular velocity of the Earth's rotation is the same. So, at some latitude, the two must be equal, and one has to be able to overtake night by traveling west, at which $\frac{d}{dt}\phi(t) = 0$, for all $t \in (t_0-\varepsilon, t_0+\varepsilon)$, $\varepsilon > 0$

So, equilibrium solutions must exist.

2, For an ODE to admit unique solutions given an initial condition, the function $\frac{d}{dt}x(t)$ must be Lipschitz continuous on the interval $[x_0-\delta, x_0+\delta]$ for $\delta > 0$.

$\forall x_1, x_2 \in [a,b]$, the slope of the secant line from $(x_1, f(x_1))$ to $(x_2, f(x_2))$ must be bounded by a constant K that only depends on f, a, and b for $f(x)$ to be Lipschitz continuous on the interval $[a,b]$ and thus for the ODE to admit unique solutions.

3, See computer page

Homework 1

CAAM 28200: Dynamical Systems with Applications

Kameel Khabaz

January 27, 2022

Problem 2.4.9 (b)

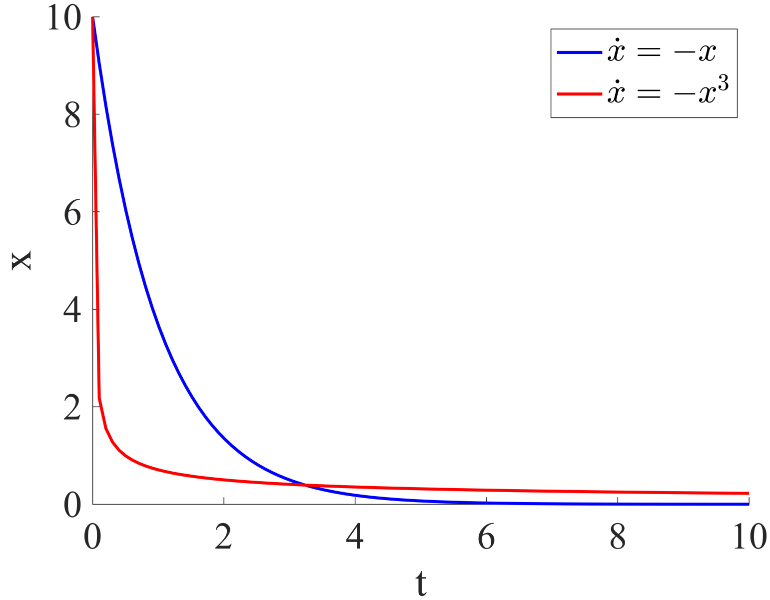


Figure 1: Solution Plots

We see here that the decay for $\dot{x} = -x^3$ is much slower than for $\dot{x} = -x$ as $t \to \infty$. So the system reaches the equilibrium fixed point much more slowly.

Problem 8 Part 1.3

Here I plot the flow for $r = 1$. As we see here, the population always steadily approaches the carrying capacity without oscillations in a simple way because ODEs constrain the geometry of the flow. This is because ODEs are well-defined and finite throughout the entire state space. This means that the trajectories must be continuous and cannot cross. Furthermore, ODEs that are Lipschitz continuous (like $\frac{d}{d\tau}N(\tau) = N(\tau)(1 - N(\tau))$) also must have unique trajectories that cannot merge or branch in finite time.

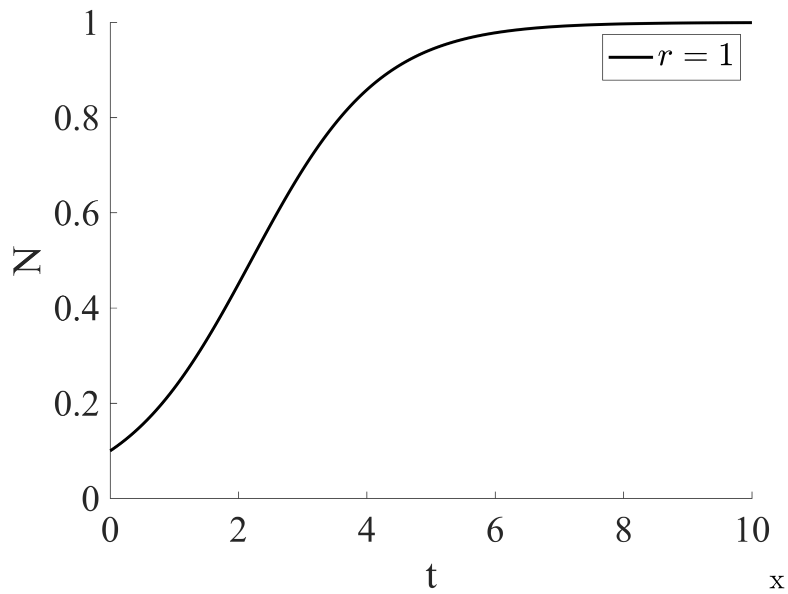


Figure 2: Solution Plots

Problem 8 Part 2.3

Here I plot cobweb plots for two sets of r, one for $r = 2, 2.5, 3$ and one for $r = 3.25, 3.5, 3.57$. I do the plots for two initial conditions, $N_0 = 0.05$ and $N_0 = 0.4$.

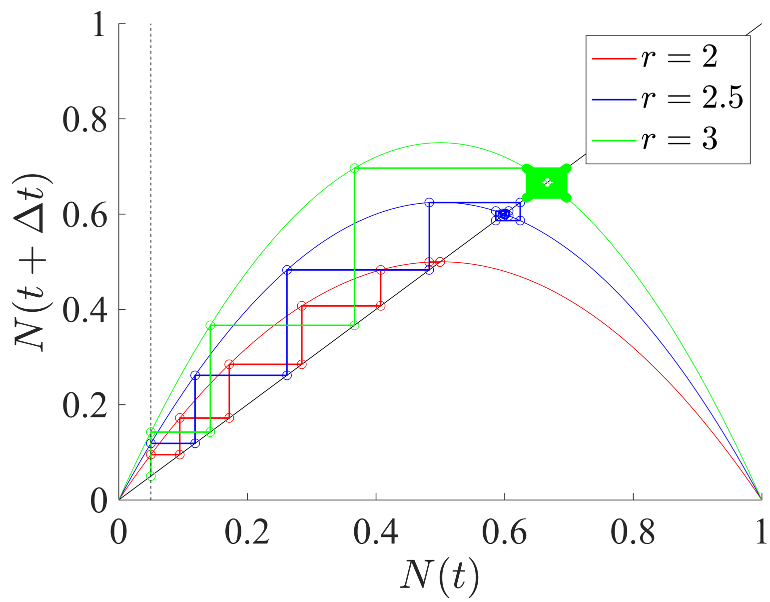


Figure 3: Cobweb Plots $N_0 = 0.05$

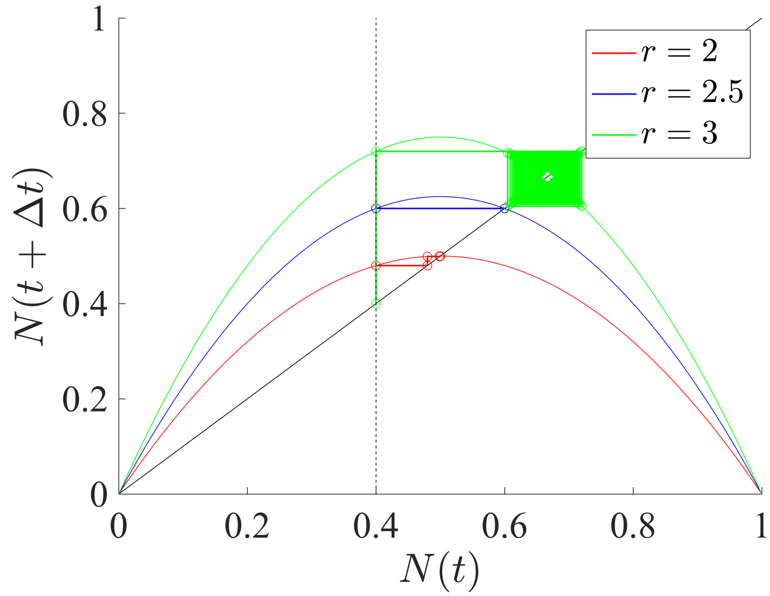


Figure 4: Cobweb Plots $N_0 = 0.4$

We see in the figures above that for $r = 2$ and $r = 2.5$, the logistic map approaches a stable solution for different initial conditions. However, for $r = 2.5$, the approach is not monotonic. When $r = 3$, the solution appears to approach a steady oscillation.

Now I plot cobweb plots for $r = 3.25, 3.5, 3.57$.

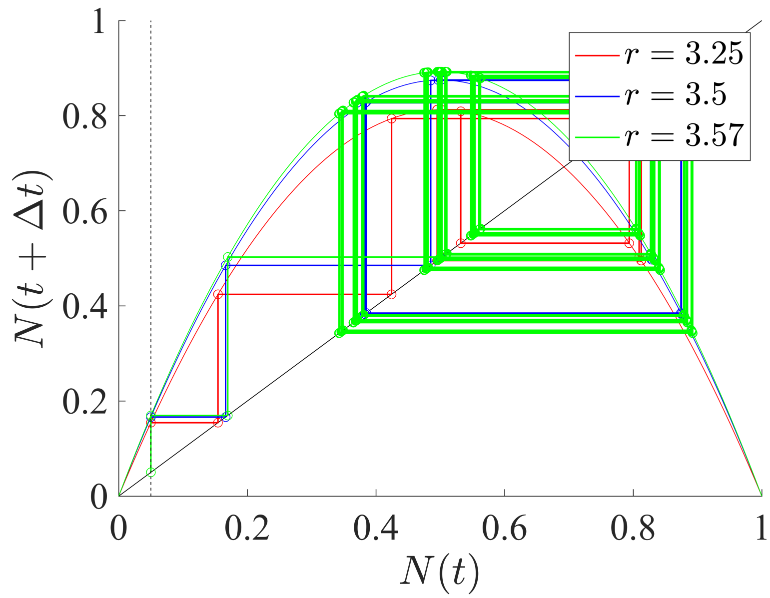


Figure 5: Cobweb Plots $N_0 = 0.05$

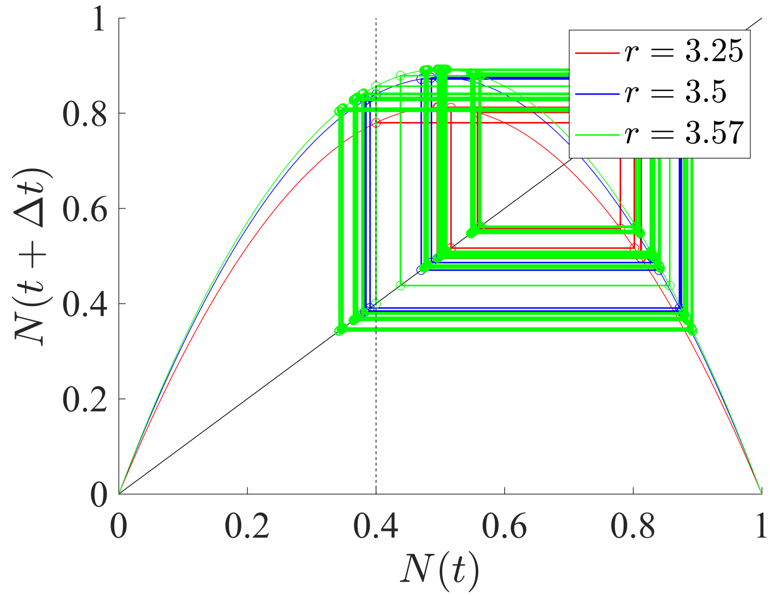


Figure 6: Cobweb Plots $N_0 = 0.4$

We see here that the solutions with $r = 3.25$ and $r = 3.5$ appear to have steady oscillations, while the logistic map with $r = 3.57$ appears to be chaotic. We can confirm these behaviors and approximate the period of the oscillations by plotting $N(t)$ versus t for t ranging from 1 to 32.

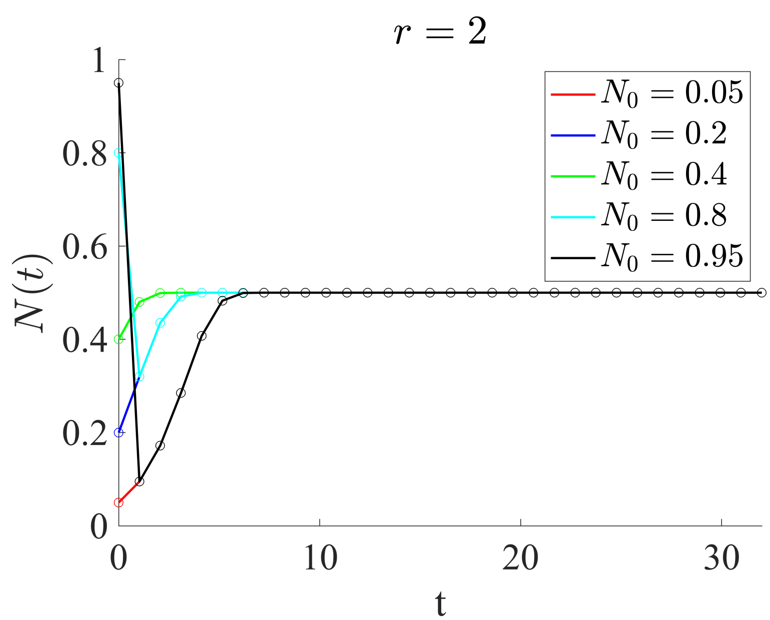


Figure 7: Logistic Map Solution Plot

As expected, for $r = 2$, the solution approaches a steady state regardless of initial condition. Even here, however, the approach for $N_0 = 0.95$ and $N_0 = 0.8$ are not monotonic because N decreases below the stable value before going back up. Thus, $N(t)$ does not approach the fixed point monotonically for some initial conditions.

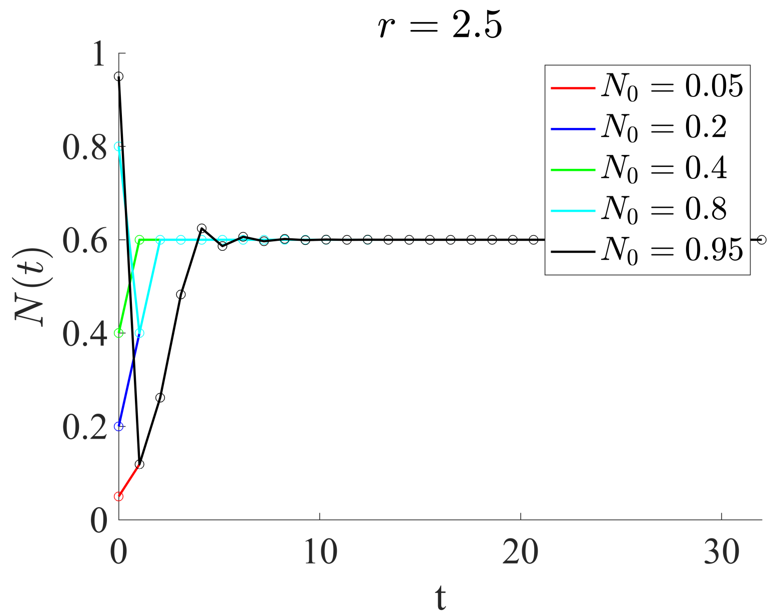


Figure 8: Logistic Map Solution Plot

The solutions similarly approach the fixed point for $r = 2.5$, but the approach is not always monotonic.

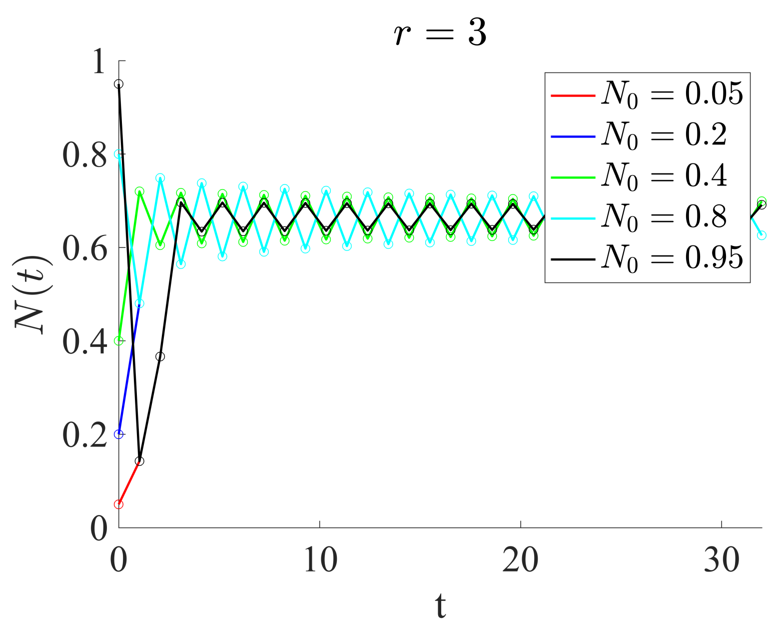


Figure 9: Logistic Map Solution Plot

With $r = 3$, the solutions appear to be in a stable oscillatory cycle of a period of 2 time steps.

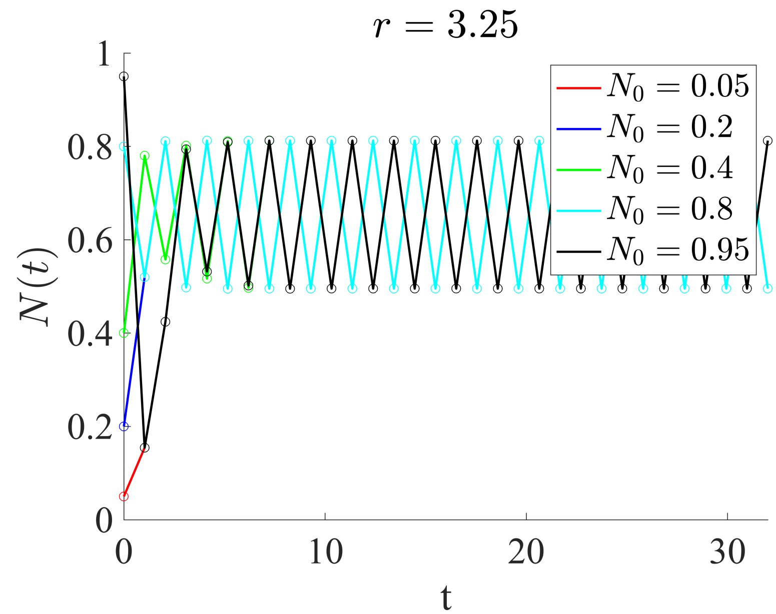


Figure 10: Logistic Map Solution Plot

With $r = 3.25$, the solutions appear to be in a stable oscillatory cycle of a period of 2 time steps.

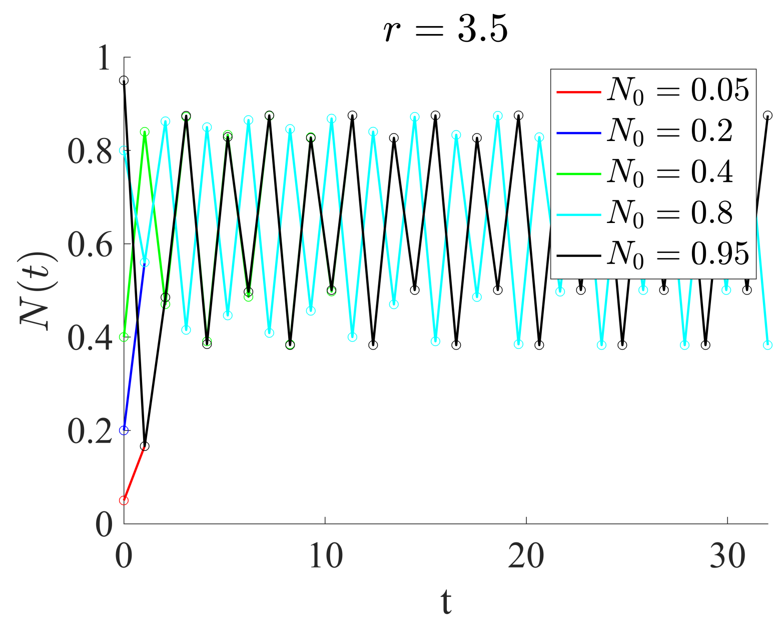


Figure 11: Logistic Map Solution Plot

With $r = 3.5$, the solutions appear to take on more complex oscillatory cycles of a period of 4 time steps.

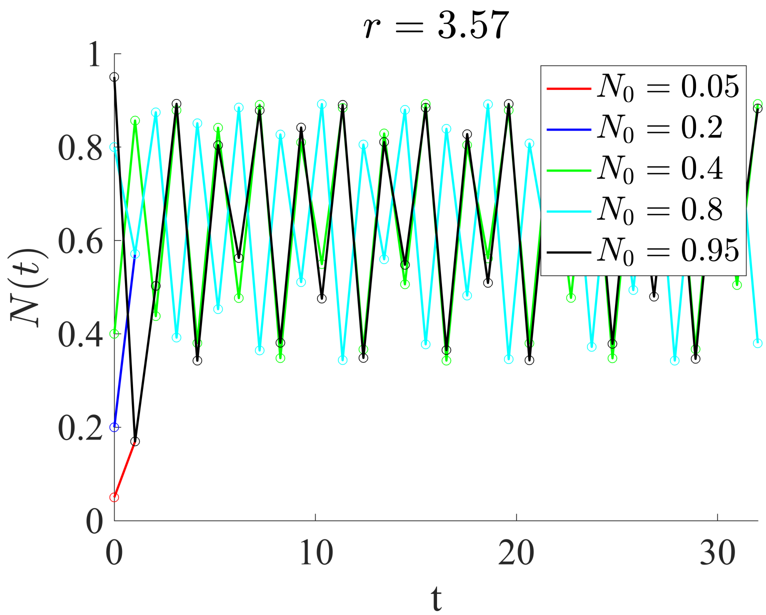


Figure 12: Logistic Map Solution Plot

With $r = 3.57$, the solutions now exhibit chaos and do not appear to have stable oscillations with a finite period. As previously stated, unlike discrete time equations, ODEs constrain the geometry of the flow because they are well-defined and finite throughout the entire state space $\Omega = [0, 1]$. This means that the trajectories must be continuous and cannot cross. Furthermore, ODEs that are Lipschitz continuous also must have unique trajectories that cannot merge or branch in finite time.

Problem 9 Part 3

I plot the function $f(\phi) = \frac{\omega}{4}\sqrt{4 - sec(\phi(t))^2}$ below and plotted the derivative $\frac{d}{d\phi}$. We see in these figures below that the slope $\frac{d}{d\phi}f(\phi)$ diverges at $\phi = \pm\frac{\pi}{3}$, implying that the solution trajectories are not unique. One could stay flying westward at a constant latitude of 60° and choose to follow the optimal trajectory southward at any time. We can easily predict this non-uniqueness because the cusp of the function $f(\phi)$ at $\phi = \pm\frac{\pi}{3}$ tells us that the function is not Lipschitz continuous at those points, violating the uniqueness theorem. This means that there are exist finite trajectories traveling at v_* that minimize the elapsed solar time. Trajectories can start at $\phi = \pm\frac{\pi}{3}$ and branch off at any time.

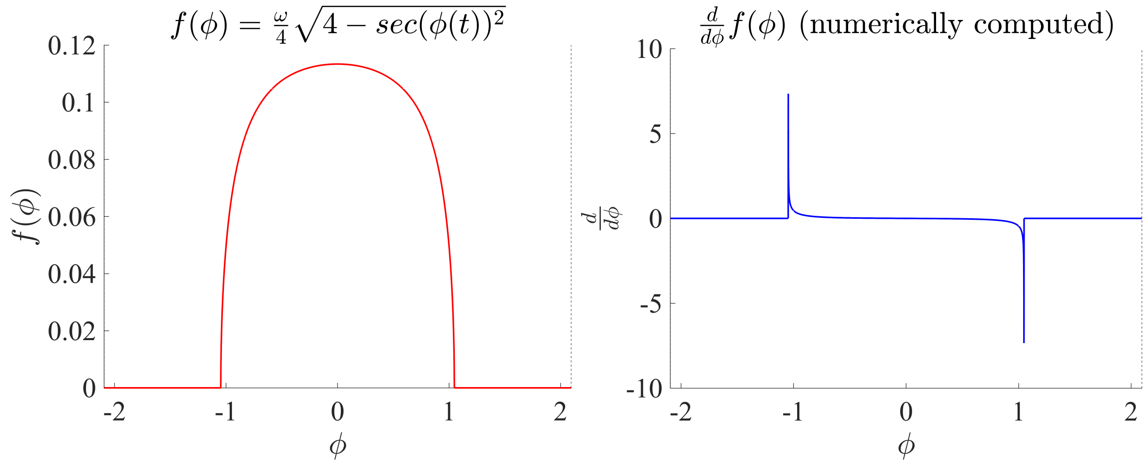


Figure 13: Function Plot