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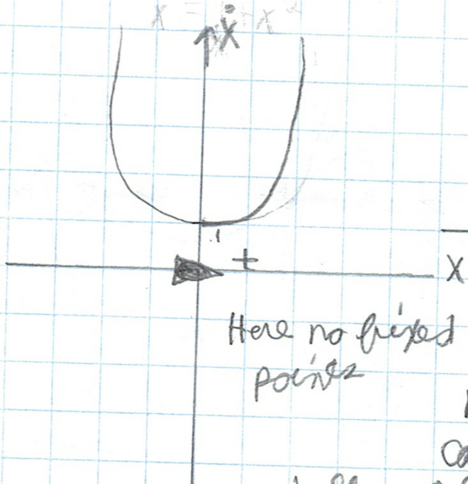
CAAM 28200 HW2
2/1/22

$b^2-4ac = r^2-4=0$
$r=\pm 2$

1: 3.1.1 $\dot{x} = 1+rx+x^2 = x^2+rx+1$

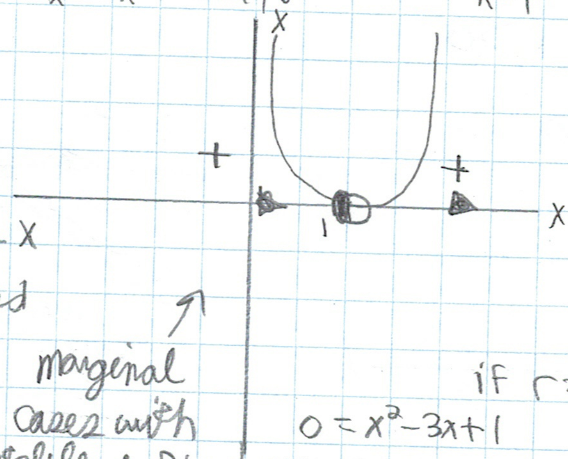
if $r=0$ $\dot{x}=1+x^2$



Here no fixed points

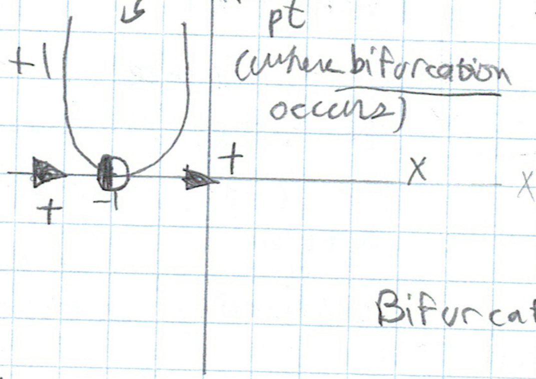
if $r=-2$
$\dot{x}=x^2-2x+1$

$x^2-2x+1=0$
$(x-1)^2=0$
$x=1$

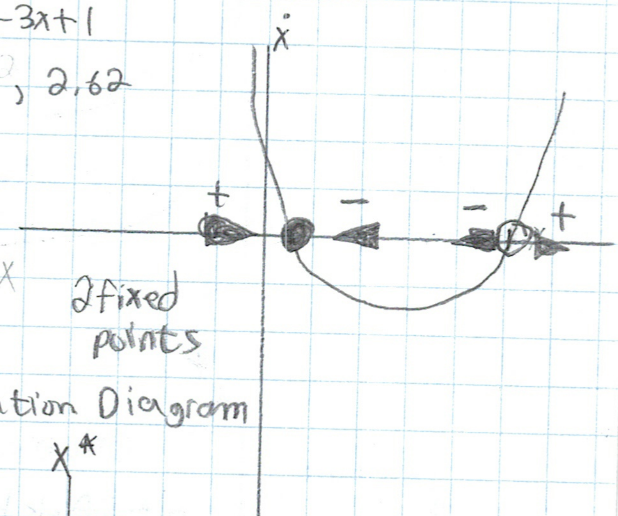


marginal cases with half-stable fixed pt (where bifurcation occurs)

if $r=+2$
$\dot{x}=x^2+2x+1$
$x^2+2x+1=0$
$(x+1)^2=0$
$x=-1$

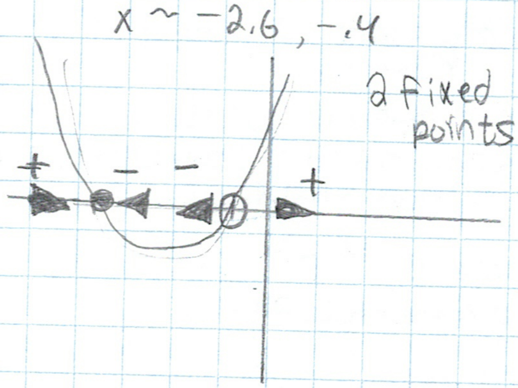


if $r=-3$ $\dot{x}=1-3x+x^2$
$0=x^2-3x+1$
x^* 0.38, 2.62



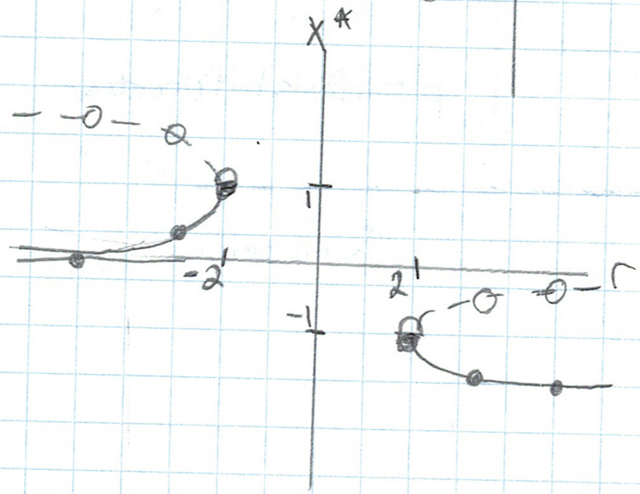
2 fixed points

if $r=+3$
$\dot{x}=x^2+3x+1$
$x\sim -2.6, -.4$

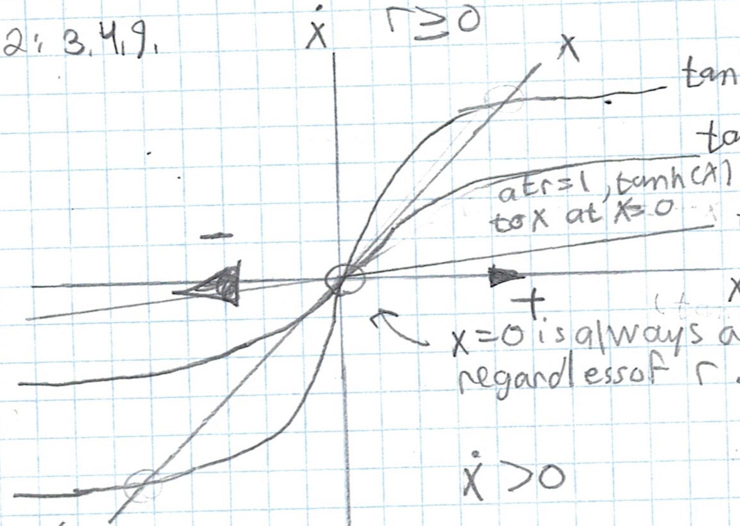


2 fixed points

Bifurcation Diagram



2: 3.4.9.

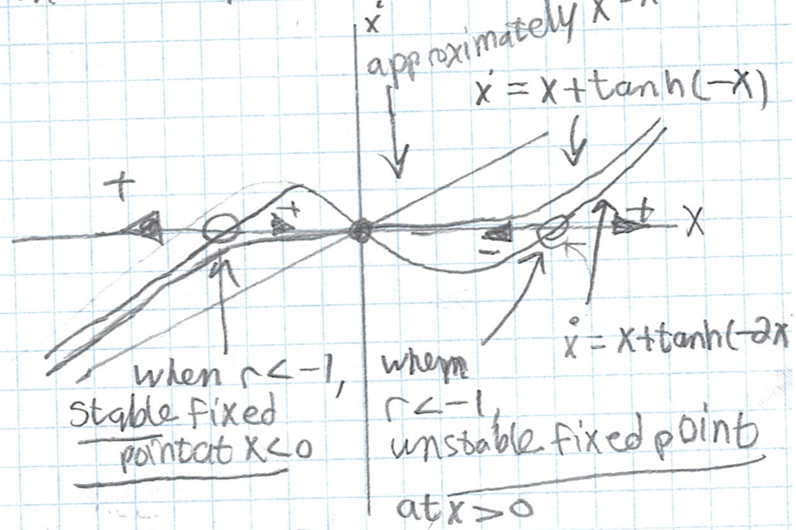


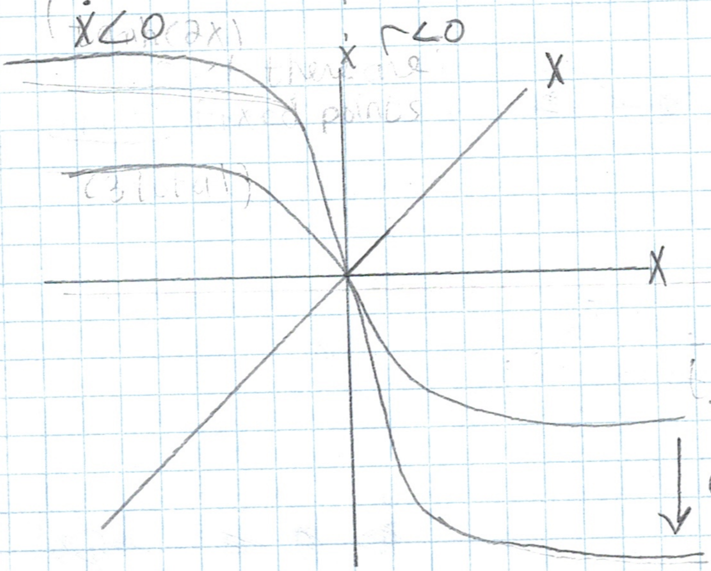
Fixed points are when $x = \tanh(x)$

since $\dot{x} = x + \tanh(rx)$

$x = 0$ is always a fixed point,

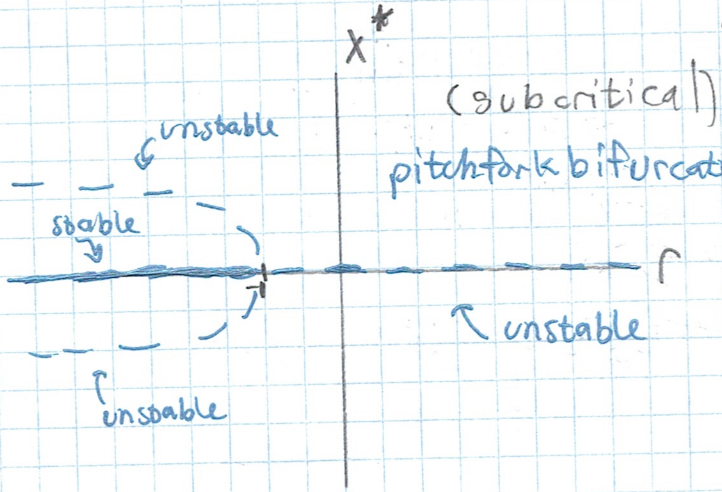
$x=0$ is always a fixed point regardless of r. The $x=0$ fixed point is unstable





∴ bifurcate at $r = -1$

as $|r|$ increases, $\tanh(rx)$ becomes more steep. So at a certain value of r $x + \tanh(rx)$ will intersect the x axis at $x<0$ and $x>0$ (as well as $x=0$). So bifurcation is from 1 to 3 fixed points



$f(x=0) = \dot{x}(0) = (0) + \tanh(-1 \cdot 0) = 0$, $r=-1$

So $x=0$ is a marginal case (since $\dot{x}=0$) so we can confirm that there is a bifurcation there at $r=-1$.

Furthermore, if $r<-1$ (Let $r=-2$),

$f = \dot{x} = x + \tanh(-2x)$

$f' = 1 - 2\,\text{sech}^2(-2x)$

for $x=0$,

$f' = 1 - 2\,\text{sech}^2(0) = 1 - 2 = -1$,

So the slope of $\dot{x}$ is negative at $x=0$

Stable equilibrium at $x=0$ when $r<-1$,

For $r < -1$, Since $\dot{x}$ increases to $+\infty$ as $x \to +\infty$ and decreases to $-\infty$ as $x \to -\infty$ and is continuous on $x \in \mathbb{R}$, the function must intersect $x=0$ at 2 other points resulting in 2 more equilibria. So there are 3 equilibria

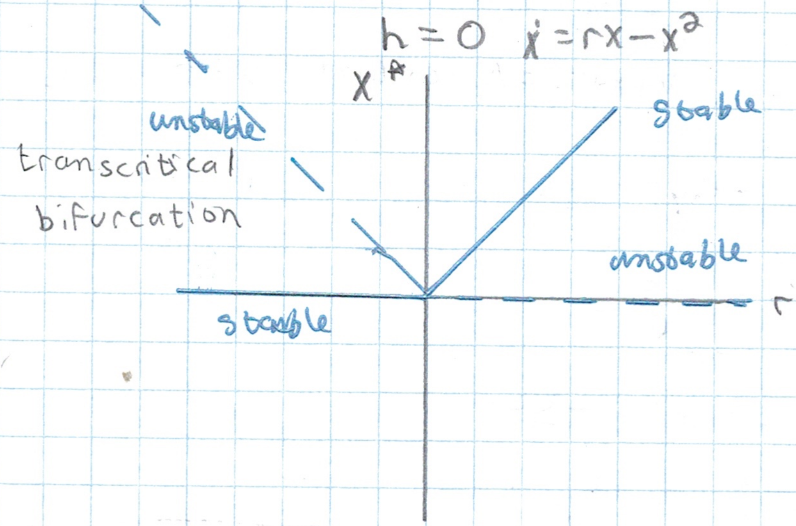
if $r<-1$, there must be 3 equilibria

Since $x_* = 0$ is stable, other 2 equilibria must have + slope so be unstable

3. 3.6.2:

a. The normal form for a transcritical bifurcation is

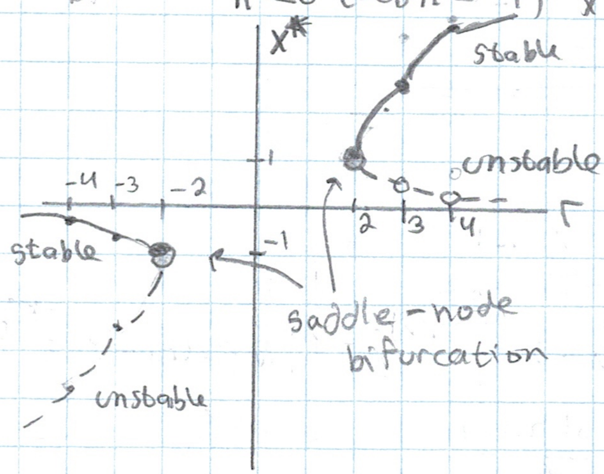
a. $h=0$ $\dot{x}=rx-x^2$ $\frac{d}{dt}x=cx-x^2$



b. $h<0$ (Let $h=-1$) $\dot{x}=-1+rx-x^2$

$b^2-4ac=r^2-4$

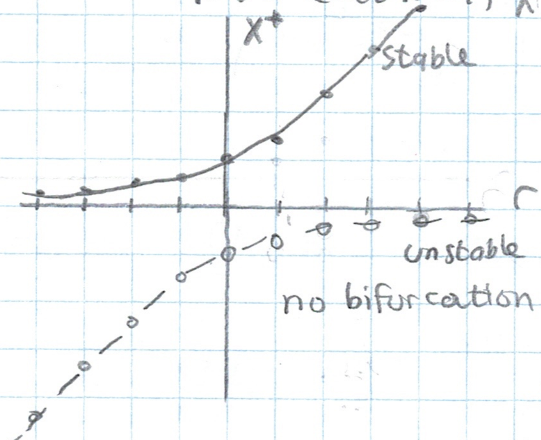
$r=\pm 2$ when only 1 intersection, so we have bifurcations at $r=\pm 2$



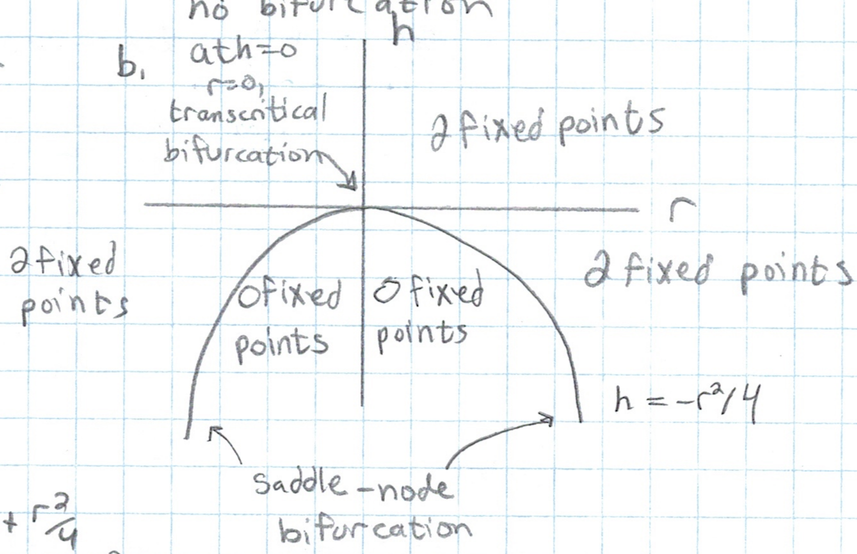
$h>0$ (Let $h=1$) $\dot{x}=1+rx-x^2$

$b^2-4ac=r^2+4>0$ (no real solution for $r=0$)

no parabola with 1 intersection ever, so no bifurcation



b. at $h=0$ $r=0$, transcritical bifurcation



$f=\dot{x}=h+rx-x^2$

$\frac{df}{dx}=r-2x=0$

$x_{max}=r/2$

$f(x_{max})=h+\cancel{\frac{r^2}{2}}\cancel{-\frac{r^2}{4}}+\frac{r^2}{4}=h+\frac{r^2}{4}$

bifurcation when $f(x_{max})=0$ so $h=-\frac{r^2}{4}$

so $\dot{x}=0$ (marginal case)

4. 3.7.6 (a) – (j) use $z(0)=0$

a. $\dot{x} = -kxy$ healthy

sick $\dot{y} = kxy - ly \Rightarrow \dot{y} = -\dot{x} - \dot{z}$

$\dot{z} = ly$ dead

$\int \dot{y} + \dot{x} + \dot{z} = \int 0$

$y + x + z = N$ constant total # people

b. $\dot{z} = ly$

$y = \frac{\dot{z}}{l}$ $\quad \dot{x} = -kxy = -\frac{kx\dot{z}}{l}$

$$\int_{x_0}^{x} \frac{\dot{x}}{x} = \int_0^z -\frac{k}{l}\dot{z}$$

$$\ln|x| - \ln|x_0| = -\frac{kz}{l}$$

$$\ln\left(\frac{x}{x_0}\right) = -\frac{kz}{l}$$

$e \quad e$

$$x = x_0 e^{-kz/l} \Rightarrow \boxed{x(t) = x_0 e^{-kz(t)/l}}$$

c.

$\dot{z} = ly$

$x + y + z = N$

$\Rightarrow y = N - x - z$ $\quad \therefore \dot{z} = ly = l(N - x - z)$

$= l\left[N - z - x_0 e^{-kz/l}\right]$

$\therefore \dot{z} = l\left[N - z - x_0 e^{(-kz/l)}\right] = l\left[N - z(t) - x_0 e^{\frac{-kz(t)}{l}}\right]$

d. We wrote $\dot{z}$ entirely in terms of $z(t)$

measure z in units $\frac{l}{k}$ so $z = \frac{l}{k}u$ $\quad u = \frac{k}{l}z$

$$\frac{du}{dt} = \frac{k}{l}\frac{dz}{dt} = \frac{k}{\cancel{l}}\left(N - \left(\frac{l}{k}u\right) - x_0 e^{-\cancel{\frac{k}{l}}\cancel{\frac{l}{k}}u}\right)\cdot \cancel{l}$$

$$\frac{du}{dt} = kN - lu - kx_0 e^{-u}$$

Let $t = c\tau$ so

$$\frac{du}{d\tau} = \frac{du}{dt}\frac{dt}{d\tau} = \left[kN - lu - kx_0 e^{-u}\right]c \quad \text{Let } c = \frac{1}{kx_0}$$

$$\therefore \frac{du}{d\tau} = \left[\frac{\cancel{k}N}{\cancel{k}x_0} - \frac{lu}{kx_0} - \frac{\cancel{kx_0}e^{-u}}{\cancel{kx_0}}\right]$$

$$\frac{du}{d\tau} = \frac{N}{x_0} - \frac{lu}{kx_0} - e^{-u}$$

$$\frac{du}{d\tau} = a - bu - e^{-u} \quad \text{Let } a = \frac{N}{x_0},\ b = \frac{l}{kx_0}$$

So we were able to nondimensionalize the equation by scaling z and t

4 (cont.):
3.7.6(a)-(j)

$z(0)=0$

e. $a = \frac{N}{x_0}$ $a \geq 1$ because $x+y+z=N$ at all times so $x_0+y_0+z_0=N$. These are all non-zero integers (x_0 and N must be positive), so $N \geq x_0$

$\Rightarrow \frac{N}{x_0} \geq 1 \Rightarrow a \geq 1$

$b = \frac{l}{kx_0} > 0$ since $l>0$, $k>0$, and $x_0>0$

so $b>0$

f. Fixed points when $\frac{du}{d\tau}=0$

fixed points at intersection $\therefore a - bu - e^{-u} = 0$

$\hookrightarrow a - bu = e^{-u}$

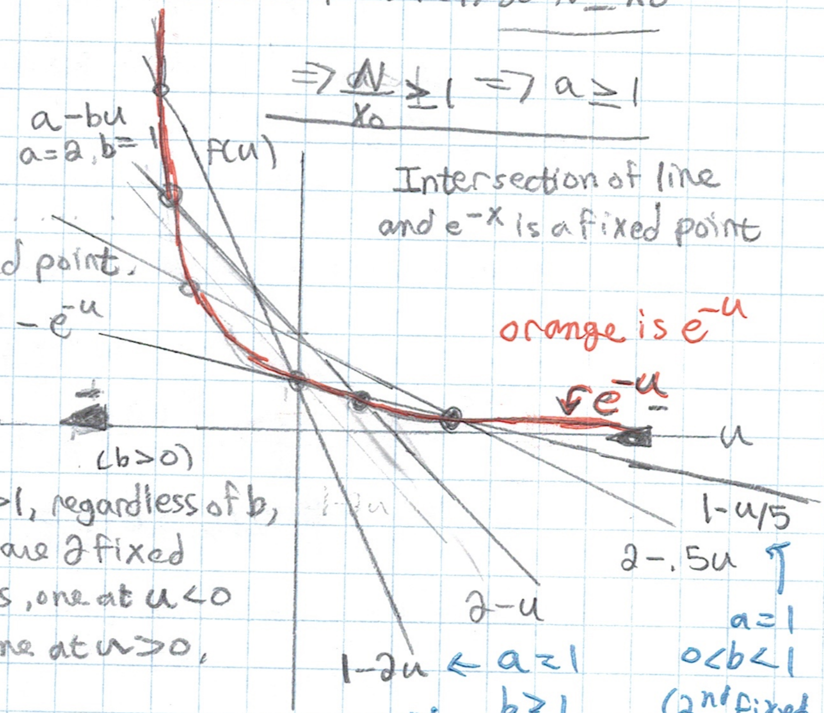
~~if a=0,~~

If $a=1$ then $1-bu=e^{-u}$ when $u=0$. So if $a=1$, $u=0$ is a fixed point. ~~(the only one)~~ if $b \neq 1$, there is a second fixed point (see graph). if $0<b<1$, 2nd fixed point is at $u>0$. if $b>1$, 2nd fixed point is at $u<0$.

if $a>1$, $u=0$ is not a fixed point.

$\frac{du}{d\tau} = [a-bu] - e^{-u}$

if $a>1$, regardless of b, there are 2 fixed points, one at $u<0$ and one at $u>0$.



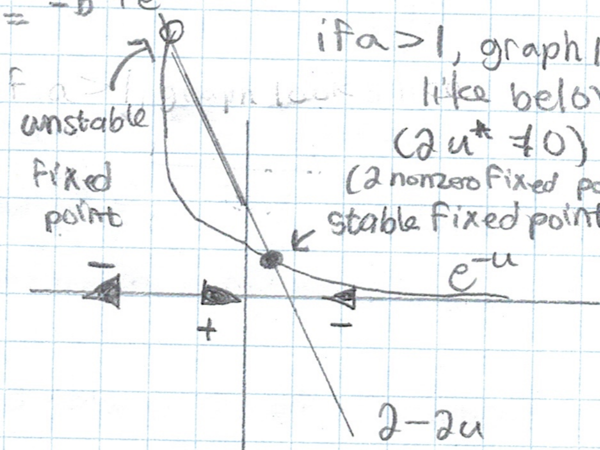
$f = \frac{du}{d\tau} = a - bu - e^{-u}$

$f' = \frac{df}{du} = -b + e^{-u}$

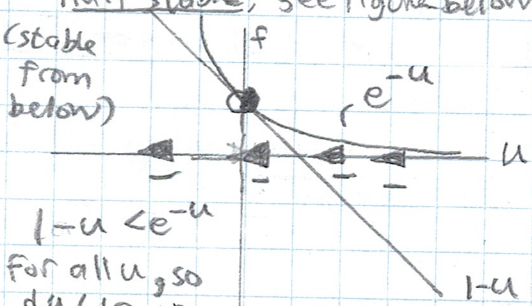
$f'(0) = \frac{df}{du}(0) = -b + e^{0} = 1-b$

- if $b>1$ and $a=1$, $f'(0)<0$ so $u^*=0$ is stable, so $u^*<0$ must be unstable.
- if $b<1$ and $a=1$, $f'(0)>0$ so $u^*=0$ is unstable, so $u^*>0$ must be stable.

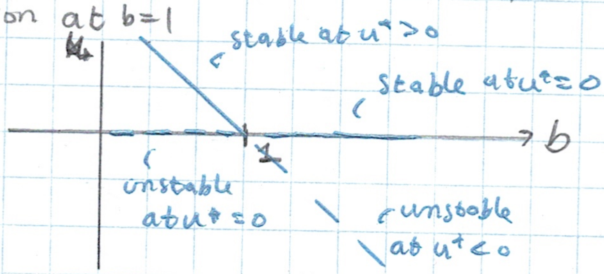
if $a=1$ and $b=1$, line $1-u$ is tangent to e^{-u} ($f'(0)=0$). f is always ≤ 0 for all u, so the fixed pt $u^*=0$ is half stable, see figure below



So if $a>1$, $u^*<0$ is unstable and $u^*>0$ is stable



Notice how if $a=1$ there is a transcritical bifurcation at $b=1$



4 (cont.):
3.7.6(a)-(j)

g. $\dot{y} = kxy - ly$ max $y(t)$ when $\dot{y} = 0$

$0 = kxy - ly = y(kx - l)$

can't be maximum since $y \geq 0$ for all t ← $\therefore y = 0$ or $\underline{kx = l}$ → $x = \frac{l}{k}$

We know from (b) that $x(t) = x_0 e^{-kz/l}$

$\therefore$ When $y(t)$ is maximum, $x(t) = \frac{l}{k}$ $\therefore \frac{l}{k} = x_0 e^{-kz/l}$ at max $y(t)$

$\dot{z}(t)$ is maximum when $\frac{d}{dt}\dot{z}(t) = -l\dot{z} - lx_0\left(-\frac{k}{l}\right)e^{-kz/l}\dot{z} = 0$

$= \left[-l + l\left(+\frac{k}{l}\right)\left(x_0 e^{-kz/l}\right)\right]\dot{z}$

$-l + l\left(\cancel{\frac{k}{l}}\right)\left(\cancel{\frac{l}{k}}\right) = 0$

$\therefore \frac{d}{dt}\dot{z}(t) = 0 \cdot \dot{z} = 0$

So $\dot{z}(t)$ is at max at same time as $y(t)$ is max

From part (d) $u = kz/l$ and

$\dot{u}(t)$ is max when $\frac{d\dot{u}(t)}{dt} = 0$ $\dot{u}(t) = kN - lu - kx_0 e^{-u}$

$\frac{d}{dt}\dot{u}(t) = -l\dot{u} - kx_0(-1)e^{-u}\dot{u} = 0$

$= \dot{u}\left[-l + kx_0 e^{-u}\right] = 0$

$-l + \underbrace{kx_0 e^{-\frac{kz}{l}}} = -l + k \cdot \frac{l}{\cancel{k}} = 0$

$\therefore \frac{d}{dt}\dot{u}(t) = 0$ $x_0 e^{-kz/l} = \frac{l}{k}$ when $y(t)$ and $\dot{z}(t)$ are at maximum

So $\dot{u}(t)$ is at max at same time as max of both $\dot{z}(t)$ and $y(t)$

h. From (d), $t = c\tau$ so when $t = 0$, $\tau = 0$ $u = \frac{kz}{l}$ so $z(0) = 0 \Rightarrow u(0) = 0$

$\frac{d}{d\tau}\left(\frac{du}{d\tau}\right) = -b\left(\frac{du}{d\tau}\right) + e^{-u}\left(\frac{du}{d\tau}\right) = \left(\frac{du}{d\tau}\right)\left[-b + e^{-u}\right]$

When $t = \tau = 0$, $u = 0$ so $\frac{d}{d\tau}\left(\frac{du}{d\tau}\right) = \frac{du}{d\tau}(-b+1)$

At $t = \tau = 0$, $\frac{du}{d\tau} = a - 1 \geq 0$ since $a \geq 1$ if $b < 1$ → $0 < b < 1$, so $\underline{-b + 1 > 0}$

$\therefore \frac{d}{d\tau}\left(\frac{du}{d\tau}\right)$ is ≥ 0 (so $\dot{u}(t)$ is increasing at $t = 0$)

$\therefore$ max of $\dot{u}(t)$ is when $-b + e^{-u} = 0$

$b = e^{-u}$ so this happens when $e^{-u} < 1$ (and > 0) if $b < 1$

so $u > 0$ so $t_{peak} > 0$, So $\dot{u}(t)$ is max when $t_{peak} > 0$

($\tau_{peak} > 0$ so $t_{peak} > 0$)

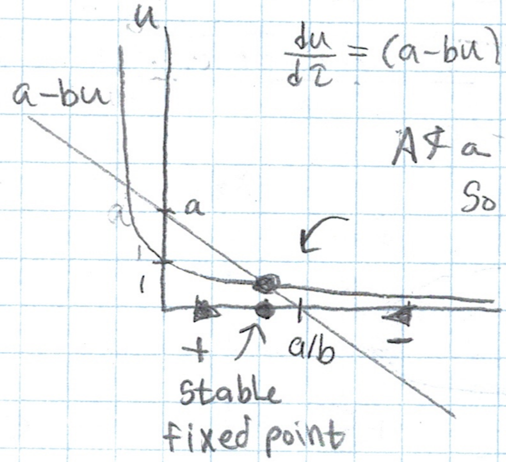
3.7.6 (cont).

h (cont.): $\dot{u}(t)$ eventually decreases to 0 when u converges to a fixed point

$$\frac{du}{d\tau} = a - bu - e^{-u}$$ As u converges to an asymptote, $\dot{u}(t) \to 0$

So if u converges to an equilibrium, then $\dot{u}(t)$ must decrease to 0

$$\frac{du}{d\tau} = (a-bu) - e^{-u}$$



At a $0<u<\frac{a}{b}$ there is a stable fixed point.
So as $t\to\infty$, $u(t)$ must converge to the stable fixed point, so $\dot{u}(t)$ must converge to 0.

i. max of $\dot{u}(t)$ is when $\frac{d}{d\tau}\left(\frac{du}{d\tau}\right) = 0$

$$\frac{d}{d\tau}\left(\frac{du}{d\tau}\right) = -b\left(\frac{du}{d\tau}\right) + e^{-u}\left(\frac{du}{d\tau}\right) = \left(\frac{du}{d\tau}\right)(-b+e^{-u}) = 0$$

$\therefore\ -b+e^{-u}=0$

$e^{-u}=b$

if $b>1$, and $u\geq 0$, then this cannot be true.

In fact, for $b>1$ and $u\geq 0$, $e^{-u}-b<0$
This means that provided $du/d\tau$ is positive (so people are dying), then $\frac{d}{d\tau}\left(\frac{du}{d\tau}\right)$ is decreasing so the epidemic must be on the decline as the rate of deaths are decreasing.
so t_{peak} must be equal to 0.

j. From part (d) we showed that $b = \frac{l}{kx_0}$ $x_0 > 1$ since >1 healthy person at $t=0$

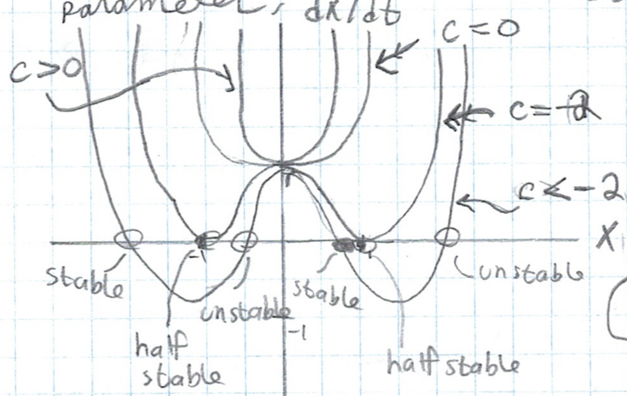
Since $\dot{x} = -kxy$, k is related to the rate at which healthy people get sick (k is a per capita rate, dimensions $(\text{time}\cdot\text{count})^{-1}$)
Since $\dot{z}=ly$, l is related to the rate at which sick people die
(l is a constant death rate, dimensions time^{-1}).

$$b = \frac{1}{x_0}\left(\frac{l}{k}\right)$$ So if $\frac{l}{kx_0}>1$, $b>1$ and no epidemic because the death rate is larger than the rate of people getting sick so the disease can't spread (since people who get it die and can't spread it)

if $\frac{l}{kx_0}<1$, $b<1$ and there is an epidemic because the rate of people getting sick is larger than the death rate, so people can spread the disease.

5,

1. I designed the normal form $\frac{d}{dt}x = x^4 + cx^2 + 1$ in which c is the bifurcation parameter, dx/dt



IF we want 4 roots collide simultaneously define

$$\frac{d}{dt}x = (x-a)(x+a)(x-2a)(x+2a) + ax^4$$

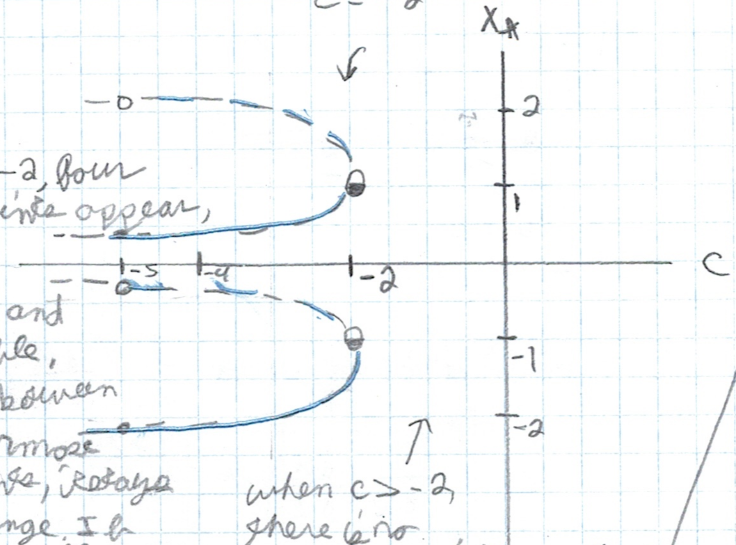
bifurcate at $a=0$

4-node bifurcation

critical parameter value is $c=-2$

2. I will name it a Double saddle node bifurcation

3.



when $c<-2$, four fixed points appear, 2 are unstable and 2 are stable, if $x(0)$ is between the 2 outermost fixed points, it stays in that range. If it is < the smallest fixed point, it converges. If it is > the largest fixed point, it diverges

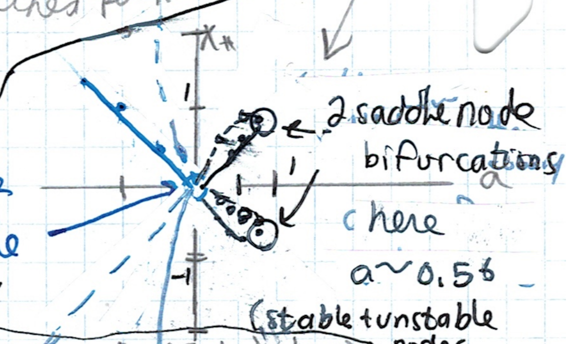
when $c>-2$, there is no fixed point

4. I would expect the bifurcation to occur when the Taylor expansion $f(x|\theta)$ has terms

$f(x_*|\theta_*)=0$, $\partial_x f(x_*|\theta_*)=0$

for fixed point $x_* = \pm 1$

and bifurcation parameter $\theta = -2$. In this case I am at a fixed point and the case is marginal ($f'(x)=0$ at x_*).



2 outer fixed pts diverge, 2 inner ones are ~linear

all 4 nodes collide simultaneously

6. $\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ r & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ $\quad A = \begin{bmatrix} -1 & 1 \\ r & -1 \end{bmatrix}$ $\quad \det\left(\begin{bmatrix} -1-\lambda & 1 \\ r & -1-\lambda \end{bmatrix}\right) = 0$

$(-1-\lambda)^2 - r = 0$

$\sqrt{(-1-\lambda)^2} = \pm\sqrt{r}$

$-1-\lambda = \pm\sqrt{r}$

$\lambda = -1 \pm \sqrt{r}$

$\lambda = -1 \pm \sqrt{r}$

i. if $r<0$, the eigenvalues are complex with negative real part of -1, so the origin is a stable spiral.

ii. If $0<r<1$, the eigenvalues are real and negative so the origin is a stable node.

iii. If $r>1$, one eigenvalue is positive and one is negative, so the origin is a saddle point,

Flow converges to stable fixed pt, diverge from unstable fixed pt

See computer pages for phase portrait

6(cont.). If $r=0$, then $\lambda=-1$ so there is only 1 eigenvalue of algebraic multiplicity 2, but only 1 eigenvector, so it has a geometric multiplicity of 1. In this case, the eigenvector is $A\mathbf{x} = \lambda v$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -1\begin{bmatrix} x \\ y \end{bmatrix}$$

$-x+y=-x \Rightarrow \underline{y=0}$ (x can be anything)

$-y=-y \Rightarrow y$ can be anything $\rightarrow y$ is only 0

so the eigenvector is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

So we have just $\underline{1 \text{ eigenvector}}$, so $\underline{A \text{ is not diagonalizable}}$

$\frac{d}{dt}x = -x+y \quad \dot{y} = 0x - y \Rightarrow \frac{dy}{dt} = -y \Rightarrow \int \frac{dy}{y} = \int -dt$

$\frac{dx}{dt} = -x+y$

$\ln|y| = -t + C$

$e \qquad e$

$\frac{dx}{dt} + x = y$

$y(t) = Ce^{-t} \qquad \underline{y = Ce^{-t}}$

$IR = e^{\int 1 dt} = e^{t}$

The solution for $y(t)$ matches our usual form $Ce^{\lambda t}$

$e^t\left(\frac{dx}{dt} + x\right) = ye^t = C\cancel{e^{-t}}e^t = C$

So $y(t) = Ce^{-t}$

$x(t) = e^{-t}(Ct+D)$ ← The solution for $x(t)$ does not match our usual form because there is an extra $(Ct+D)$ term

C, D are constants

$e^t\frac{dx}{dt} + e^t x = C$

$\int \frac{d}{dt}\left[e^t x\right]dt = \int C \, dt$

$e^t x = Ct + D$

$x = e^{-t}(Ct+D)$

5.1.10: a. $A = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \rightarrow \det\left(\begin{bmatrix} -\lambda & 1 \\ -4 & -\lambda \end{bmatrix}\right) = 0$

$\lambda^2 + 4 = 0$

$\lambda^2 = -4 \quad \lambda = \pm 2i$

real part of $\lambda = 0$, so the origin is $\boxed{\text{Liapunov stable}}$

b. $A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

$\det\left(\begin{bmatrix} -\lambda & 2 \\ 1 & -\lambda \end{bmatrix}\right) = \lambda^2 - 2 = 0$

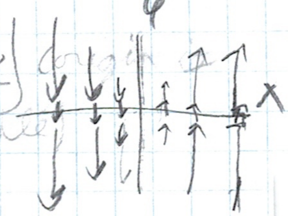
$\lambda^2 = 2$

$\lambda = \pm\sqrt{2}$

one $\lambda > 0$, so the origin is a saddle and thus $\boxed{\text{neither}}$ asymptotically stable nor Liapunov stable

5.1.10 (cont):

c. $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $\quad \det\left(\begin{bmatrix} -\lambda & 0 \\ 1 & -\lambda \end{bmatrix}\right) = 0$ $\quad \boxed{\text{neither}}$



$\lambda^2 - 1 = 0 \quad \lambda = \pm 1$

d. $A = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \det\left(\begin{bmatrix} -\lambda & 0 \\ 0 & -1-\lambda \end{bmatrix}\right) = 0$ $\quad$ Liapunov stable

$-\lambda(-1-\lambda) = 0$

$\lambda + \lambda^2 = 0$

$\lambda(\lambda+1) = 0 \quad \lambda = 0, -1$

e. $A = \begin{bmatrix} -1 & 0 \\ 0 & -5 \end{bmatrix}$ $\quad \det\left(\begin{bmatrix} -1-\lambda & 0 \\ 0 & -5-\lambda \end{bmatrix}\right) = 0$ $\quad$ asymptotically stable

$(-1-\lambda)(-5-\lambda) = 0$

$\lambda = -1, -5$

f. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{bmatrix}\right) = 0$ $\quad \boxed{\text{neither}}$ origin is unstable

$(1-\lambda)^2 = 0$

$\lambda = 1$

5.2.2:

a. $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\quad \det(A - \lambda I) = 0$

$x' = x - y$

$y' = x + y$

$(1-\lambda)(1-\lambda) + 1 = 0$

$1 - 2\lambda + \lambda^2 + 1 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 4\cdot 2 - 4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$\therefore \lambda_1 = 1+i, \ \lambda_2 = 1-i$

for $\lambda_1 = 1+i$,

$(A - \lambda I)v = 0$

$\begin{bmatrix} \cancel{1}-\cancel{1}-i & -1 \\ 1 & \cancel{1}-\cancel{1}-i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-xi - y = 0 \Rightarrow y = -xi$

$x - yi = 0 \Rightarrow x = yi$ Let $y = 1$

$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

for $\lambda_2 = 1-i$

$\begin{bmatrix} \cancel{1}-\cancel{1}+i & -1 \\ 1 & \cancel{1}-\cancel{1}+i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$xi - y = 0 \quad y = xi$ $\quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$x + yi = 0 \quad x = -yi$

Let $y = 1$

So $\lambda_1 = 1+i, \ v_1 = (i, 1)$

$\lambda_2 = 1-i, \ v_2 = (-i, 1)$

The eigenvalues/eigenvectors are complex conjugate

5.2.2 (cont): $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$ we know c_1 and c_2 are complex conjugate

b. $x(t) = c_1 e^{(1+i)t}\begin{bmatrix} i\\ 1\end{bmatrix} + c_2 e^{(1-i)t}\begin{bmatrix} -i\\ 1\end{bmatrix} = e^t\left[c_1 e^{it}\begin{bmatrix} i\\ 1\end{bmatrix} + c_2 e^{-it}\begin{bmatrix} -i\\ 1\end{bmatrix}\right]$

$= e^t\left[c_1(\cos t + i\sin t)\begin{bmatrix} i\\ 1\end{bmatrix} + c_2(\cos t - i\sin t)\begin{bmatrix} -i\\ 1\end{bmatrix}\right]$

$= e^t\left[ic_1\cos t\begin{bmatrix} 1\\ 0\end{bmatrix} + c_1\cos t\begin{bmatrix} 0\\ 1\end{bmatrix} + c_1\sin t\begin{bmatrix} -1\\ 0\end{bmatrix} + ic_1\sin t\begin{bmatrix} 0\\ 1\end{bmatrix} + ic_2\cos t\begin{bmatrix} -1\\ 0\end{bmatrix} + c_2\cos t\begin{bmatrix} 0\\ 1\end{bmatrix}\right.$

$\left. + c_2\sin t\begin{bmatrix} -1\\ 0\end{bmatrix} - ic_2\sin t\begin{bmatrix} 0\\ 1\end{bmatrix}\right]$

$= e^t\left[(c_1+c_2)\left(\cos t\begin{bmatrix} 0\\ 1\end{bmatrix} + \sin t\begin{bmatrix} -1\\ 0\end{bmatrix}\right) + i(c_1-c_2)\left(\sin t\begin{bmatrix} 0\\ 1\end{bmatrix} - \cos t\begin{bmatrix} -1\\ 0\end{bmatrix}\right)\right]$

∴ $x(t)$ c_1 and c_2 must be complex conjugate. Let $c_1 = \frac{a+bi}{2}$, $c_2 = \frac{a-bi}{2}$ ∴ $c_1+c_2 = a$

$i(c_1-c_2) = bi\cdot i = -b$ $\qquad c_1 - c_2 = bi$

$$\boxed{x(t) = e^t\left[a\left(\cos t\begin{bmatrix} 0\\ 1\end{bmatrix} + \sin t\begin{bmatrix} -1\\ 0\end{bmatrix}\right) - b\left(\sin t\begin{bmatrix} 0\\ 1\end{bmatrix} - \cos t\begin{bmatrix} -1\\ 0\end{bmatrix}\right)\right]}$$

10:

1. $v(t) = \frac{dx}{dt}$ $\quad \frac{d}{dt}\left(\frac{dx}{dt}\right) = -\frac{1}{M}\left(kx + b\frac{dx}{dt}\right)$

$\frac{d}{dt}v(t) = -\frac{1}{M}(kx(t) + bv(t))$

∴ $\frac{d}{dt}\begin{bmatrix} x(t)\\ v(t)\end{bmatrix} = \begin{bmatrix} v(t)\\ -\frac{1}{M}(kx(t) + bv(t))\end{bmatrix}$ ← Governing Equation

$Y(t) = \begin{bmatrix} x(t)\\ v(t)\end{bmatrix}$ This is linear since no multiplication of x and v (so we can take linear combinations)

$[k] = kg/s^2$

2. Let $\omega_0^2 = k/M \to (kg/s^2)/kg$ ∴ ω_0^2 has units s^{-2} so ω_0 has units of s^{-1} $\quad [\omega_0] = s^{-1}$

$\mu = b/M \to [bv] = N$ ∴ $[b] = N/(m/s) = kg/s$ ∴ $[\mu] = \frac{(kg/s)}{kg} = s^{-1}$

∴ $\frac{d}{dt}\begin{bmatrix} x(t)\\ v(t)\end{bmatrix} = \begin{bmatrix} v(t)\\ -\omega_0^2 x(t) - \mu v(t)\end{bmatrix}$

$\frac{d^2x}{dt^2} = -\omega_0^2 x - \mu\frac{dx}{dt}$ remeasure t in units u so that $t = \frac{1}{u}\tau$ and $\tau = ut$

∴ $\frac{dx}{dt} = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{1}{u}\frac{dx}{d\tau}$

$\frac{d^2x}{dt^2} = \frac{d^2x}{d\tau^2}\frac{d^2\tau}{dt^2} = \frac{1}{u^2}\frac{d^2x}{d\tau^2}$

∴ $\frac{1}{u^2}\frac{d^2x}{d\tau^2} = -\omega_0^2 x - \mu\frac{1}{u}\frac{dx}{d\tau}$

$\frac{d^2x}{d\tau^2} = -u^2\omega_0^2 x - u\mu\frac{dx}{d\tau}$ Let $u = 1/\omega_0$

∴ $\frac{d^2x}{d\tau^2} = -\frac{1}{\omega_0^2}\omega_0^2 x - \frac{\mu}{\omega_0}\frac{dx}{d\tau}$

∴ $\frac{d^2x}{d\tau^2} = -x - \frac{\mu}{\omega_0}\frac{dx}{d\tau}$ Let $c = \mu/\omega_0$

$[c] = \cancel{s^{-1}/s^{-1}}$

c is dimensionless

$\frac{d^2x}{d\tau^2} = -x - c\frac{dx}{d\tau}$

$\frac{d}{d\tau}\begin{bmatrix} x\\ v\end{bmatrix} = \begin{bmatrix} v\\ -x - cv\end{bmatrix}$

$\frac{d}{d\tau}x$

3. $\frac{d}{d\tau}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} v \\ -x-cv \end{bmatrix}$ $\frac{d}{d\tau}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only if $x, v = 0$ Let $z = \begin{bmatrix} x \\ v \end{bmatrix}$

horizontal nullcline $\frac{d}{d\tau}\begin{bmatrix} x \\ v \end{bmatrix} = \underline{\alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ so need $z = \beta\begin{bmatrix} -c \\ 1 \end{bmatrix}$
constant velocity

$\longleftrightarrow$

vertical nullcline $\frac{d}{d\tau}\begin{bmatrix} x \\ v \end{bmatrix} = \alpha\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so need $v=0$, $-x-cv=1$
constant position $-x=1$ $x=-1$

$\updownarrow$ $\frac{d}{d\tau}z$ is $\parallel$ to $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $z = \beta\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

isocline $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;
increasing position and velocity $\frac{d}{d\tau}\begin{bmatrix} x \\ v \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v=1$, $-x-c=1$, $x=-c-1$ $z = \beta\begin{bmatrix} -c-1 \\ 1 \end{bmatrix}$

isocline $w = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$:
increasing position but decreasing velocity $\frac{d}{d\tau}\begin{bmatrix} x \\ v \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $v=1$ $-x-c=-1$, $x=1-c$ $z=\beta\begin{bmatrix} 1-c \\ 1 \end{bmatrix} = \beta\begin{bmatrix} -c+1 \\ 1 \end{bmatrix}$

The equilibrium at $x=0, v=0$ is a pendulum (oscillator) standing still. If the oscillator is not in this state its position and/or velocity will change. At $x=0, v=0$, neither position nor velocity change (so acceleration and velocity are 0).

The nullcline $\alpha\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is when only the position changes and velocity is constant. This occurs when the position is at $x=(-c)$ the ratio of μ to ω_0, which measures the relative effect of drag to the spring force (or the frequency of the damping to the frequency of the spring).
acceleration is 0
This is analogous to when a spring passes through the rest pt. This is at $z=\beta\begin{bmatrix} -c \\ 1 \end{bmatrix}$ (when $F_x=0$)

The nullcline $\alpha\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is when only the velocity and not the position changes, so the oscillator is at the edge of the path (like a spring at max extension/compression point) and this is momentarily still (velocity is 0, but nonzero acceleration). This is seen since the nullcline is at $z=\beta\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ so velocity is 0 and

The isocline $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is when velocity and position are increasing, which occurs before the constant velocity nullcline at $x=-c-1$ at $z=\beta\begin{bmatrix} -c-1 \\ 1 \end{bmatrix}$

The isocline $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is when position is increasing and velocity is decreasing. This occurs after the constant velocity nullcline at $x=-c+1$ at $z=-\beta\begin{bmatrix} -c+1 \\ 1 \end{bmatrix}$

(cont. 10):

4. $A = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}$ so $\frac{d}{d\tau}\begin{bmatrix} x \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -c \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix}$ trace$(A) = -c$, det$(A) = 1$

$\det(A-\lambda I) = 0 \rightarrow \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -c-\lambda \end{bmatrix}\right) = 0$

$\lambda = \frac{1}{2}(-c) \pm \sqrt{\frac{1}{4}c^2 - 1}$

$-\lambda(-c-\lambda) + 1 = 0$

$c\lambda + \lambda^2 + 1 = 0 \rightarrow \lambda^2 + c\lambda + 1 = 0$

2 methods match

$$\lambda_\pm = \frac{-c \pm \sqrt{c^2-4}}{2}$$

$-\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 - 1} = \lambda_1 = \frac{-c+\sqrt{c^2-4}}{2}$

$-\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 - 1} = \lambda_2 = \frac{-c-\sqrt{c^2-4}}{2}$

if $c \neq \pm 2$, then the 2 eigenvalues are distinct. If $|c|>2$, 2 real eigenvalues. If $|c|<2$, 2 complex eigenvalues

To find the eigenvectors, we need to consider the cases separately.

Note that since $c = \gamma/\omega_0$, it actually needs to be >0 for this to physically make sense.

$0 = (A - \lambda I)v = \begin{bmatrix} \frac{c}{2} \pm \sqrt{\;} & 1 \\ -1 & -\frac{c}{2} \pm \sqrt{\;} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

if $c > 2$ (real eigenvalues), $d = \sqrt{\left(\frac{c}{2}\right)^2 - 1}$ (real number)

so

$\left(\frac{c}{2} \pm d\right)v_1 + v_2 = 0$ and $-v_1 + v_2\left(-\frac{c}{2} \pm d\right) = 0$

For λ_1, $\left(\frac{c}{2} - d\right)v_1 + v_2 = 0 \Rightarrow v_2 + \left(\frac{c}{2} - d\right)\left(-\frac{c}{2} - d\right)v_2 = 0$

$-v_1 + v_2\left(-\frac{c}{2} - d\right) = 0 \Rightarrow v_1 = v_2\left(-\frac{c}{2} - d\right)$

$v_2\left[1 + -\left(\frac{c}{2} - d\right)\left(\frac{c}{2} + d\right)\right] = 0$

$v_2\left[1 - \frac{c^2}{4} + d^2\right] = 0$

$d^2 = \left(\frac{c}{2}\right)^2 - 1$

Let $v_2 = 1$ $v_1 = -\frac{c}{2} - d = -\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 - 1}$

$~~1 - \frac{c^2}{4} + c^2 - 1 = 0~~$

0

$\lambda_1 = -\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 - 1}$ $V_{\lambda_1} = \begin{bmatrix} -\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 - 1} \\ 1 \end{bmatrix}$

so v_2 can be anything

For λ_2, $\left(\frac{c}{2} + d\right)v_1 + v_2 = 0$

$-v_1 + \left(-\frac{c}{2} + d\right)v_2 = 0$

$\therefore v_1 = \left(-\frac{c}{2} + d\right)v_2$

Let $v_2 = 1$ then $v_1 = -\frac{c}{2} + d$

$\therefore v_2 = -v_1\left(\frac{c}{2} + d\right) = -\left(-\frac{c}{2} + d\right)\left(\frac{c}{2} + d\right)v_2$

$V_{\lambda_2} = \begin{bmatrix} -\frac{c}{2} + \sqrt{\left(\frac{c}{2}\right)^2 - 1} \\ 1 \end{bmatrix}$

$v_2 = \underbrace{\left(\frac{c}{2} - d\right)\left(\frac{c}{2} + d\right)}v_2$

$\frac{c^2}{4} - d^2 = \frac{c^2}{4} - \frac{c^2}{4} + 1 = 1$ $\therefore v_2 = v_2$ free parameter

$\lambda_2 = -\frac{c}{2} - \sqrt{\left(\frac{c}{2}\right)^2 - 1}$

(10 cont.):

4 (cont.):

if $c < 2$ (complex eigenvalues), $\lambda_{\pm} = -\frac{c}{2} \pm \boxed{\sqrt{(\frac{c}{2})^2 - 1}}$

$(\frac{c}{2})^2 - 1 < 0$

So $\lambda_{\pm} = -\frac{c}{2} \pm di$

Let $\sqrt{(\frac{c}{2})^2-1} = di$ in which $d = \sqrt{1-(\frac{c}{2})^2}$

$1-(\frac{c}{2})^2 > 0$

$0 = (A-\lambda I)v = \begin{bmatrix} \frac{c}{2} \mp di & 1 \\ -1 & -\frac{c}{2} \mp di \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

For λ_1,

$0 = (\frac{c}{2} - di)v_1 + v_2 \rightarrow 0 = (\frac{c}{2}-di)(-\frac{c}{2}-di)v_2 + v_2 = v_2\left(\underbrace{-(\frac{c}{2}-di)(\frac{c}{2}+di)+1}\right) = 0$

$0 = -v_1 + v_2(-\frac{c}{2} - di) \Rightarrow v_1 = v_2(-\frac{c}{2} - di)$

$-\left(\frac{c^2}{4} + d^2(+1)\right) + 1 = 0$

$\lambda_1 = -\frac{c}{2} + \sqrt{(\frac{c}{2})^2 - 1}$ ∴ v_2 is a free parameter

Let $v_2 = 1$

$1 - \frac{c^2}{4} - d^2 = 0$

$\cancel{1-\frac{c^2}{4}} - \left(\cancel{1-\frac{c^2}{4}}\right) = 0$ ✓

$0 = 0$

∴ $v_1 = 1 \cdot (-\frac{c}{2} - di)$ $\quad v_{\lambda_1} = \begin{bmatrix} -\frac{c}{2} - i\sqrt{1-(\frac{c}{2})^2} \\ 1 \end{bmatrix}$

For λ_2, $0 = (\frac{c}{2} + di)v_1 + v_2 = 0 \rightarrow v_2\left[(\frac{c}{2}+di)(-\frac{c}{2}+di) + 1\right] = 0$

$0 = -v_1 + v_2(-\frac{c}{2} + di) \rightarrow v_1 = v_2(-\frac{c}{2} + di)$

$-\frac{c^2}{4} - d^2 + 1 = 0$

$\lambda_2 = -\frac{c}{2} - \sqrt{(\frac{c}{2})^2 - 1}$ v_2 is free parameter

$1 - \frac{c^2}{4} - (1 - \frac{c^2}{4}) = 0$ ✓

$v_{\lambda_2} = \begin{bmatrix} -\frac{c}{2} + i\sqrt{1-(\frac{c}{2})^2} \\ 1 \end{bmatrix}$

Notice how the eigenvalues are complex conjugates, as expected

if $c = 2$ (repeated eigenvalues) $\lambda_{\pm} = -\frac{c}{2} \pm \sqrt{(\frac{c}{2})^2 - 1} = -\frac{2}{2} \pm \sqrt{\underbrace{(\frac{2}{2})^2 - 1}_{0}}$

$\lambda_{\pm} = -1$

$0 = (A-\lambda I)v = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

∴ one eigenvalue of algebraic multiplicity 2

$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{ii = i + ii} \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ singular system

$v_1 + v_2 = 0$

$v_1 = -v_2$

∴ $v_{\lambda_{\pm}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

only one eigenvector here (geometric multiplicity 1) Since algebraic multiplicity $\neq$ geometric multiplicity, A is not diagonalizable

so not all solns are described by $x(t) = Ce^{\lambda t}v$ (possible

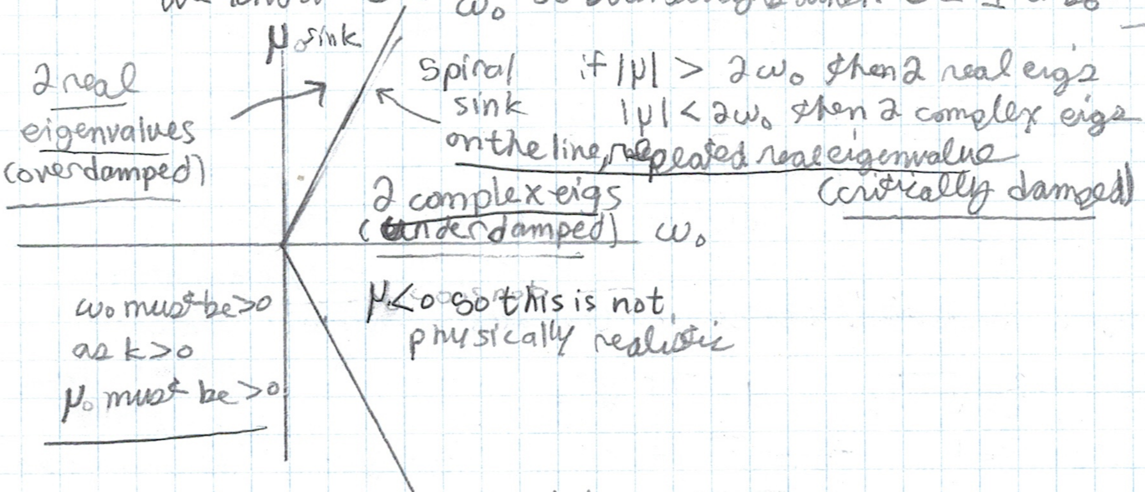
10 (cont.):

5. There are 2 distinct real eigenvalues (overdamped) when $c^2-4>0$

repeated real eigenvalue (critically damped) when $c^2-4=0 \Rightarrow c=\pm 2$

2 distinct complex eigenvalues (underdamped) when $c^2-4<0$

we know $c=\frac{\mu}{\omega_0}$ So boundary is when $c=\pm 2$ so $\mu=\pm 2\omega_0$



6. $x(t)=c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

if $c^2>4$, then λ_1 & λ_2 are real so the general solution is real

so $x(t)=c_1\exp\left(\left[-\frac{c}{2}+\sqrt{\left(\frac{c}{2}\right)^2-1}\,\right]t\right)\begin{bmatrix}-c/2-\sqrt{\left(\frac{c}{2}\right)^2-1}\\1\end{bmatrix}+c_2\exp\left(\left[-\frac{c}{2}-\sqrt{\left(\frac{c}{2}\right)^2-1}\,\right]t\right)\cdot\begin{bmatrix}-c/2+\sqrt{(c/2)^2-1}\\1\end{bmatrix}$

if $c=2$ we only have sol for one trajectory

$\boxed{x(t)=c_1\exp(-t)\begin{bmatrix}1\\-1\end{bmatrix}}$

if $c^2<4$, λ_1 and λ_2 are complex. Let $d=\sqrt{1-\left(\frac{c}{2}\right)^2}$

$x(t)=c_1\exp\left(\left(-\frac{c}{2}+di\right)t\right)\begin{bmatrix}-c/2-di\\1\end{bmatrix}+c_2\exp\left(\left(-\frac{c}{2}-di\right)t\right)\begin{bmatrix}-c/2+di\\1\end{bmatrix}$

$=c_1\exp\left(-\frac{c}{2}t\right)\exp(idt)\left(\begin{bmatrix}-c/2\\1\end{bmatrix}+\begin{bmatrix}-di\\0\end{bmatrix}\right)+c_2\exp\left(-\frac{c}{2}t\right)\exp(-idt)\left(\begin{bmatrix}-c/2\\1\end{bmatrix}+\begin{bmatrix}id\\0\end{bmatrix}\right)$

$\exp(idt)=\cos(dt)+i\sin(dt)$ $\quad\exp(-idt)=\cos(dt)-i\sin(dt)$

$=\exp\left(-\frac{c}{2}t\right)\left[c_1(\cos(dt)+i\sin(dt))\left(\begin{bmatrix}-c/2\\1\end{bmatrix}-i\begin{bmatrix}d\\0\end{bmatrix}\right)+c_2(\cos(dt)-i\sin(dt))\left(\begin{bmatrix}-c/2\\1\end{bmatrix}+i\begin{bmatrix}d\\0\end{bmatrix}\right)\right]$

$=\exp\left(-\frac{c}{2}t\right)\left[(c_1+c_2)\left(\cos(dt)\begin{bmatrix}-c/2\\1\end{bmatrix}+\sin(dt)\begin{bmatrix}d\\0\end{bmatrix}\right)+i(c_1-c_2)\left(\sin(dt)\begin{bmatrix}-c/2\\1\end{bmatrix}-\cos(dt)\begin{bmatrix}d\\0\end{bmatrix}\right)\right]$

Since c_1 and c_2 are complex conjugate, $c_1=\frac{a+bi}{2}$, $c_2=\frac{a-bi}{2}$ $\quad c_1+c_2=a$, $c_1-c_2=bi$

$=\exp\left(-\frac{c}{2}t\right)\left[a\left(\cos(dt)\begin{bmatrix}-c/2\\1\end{bmatrix}+\sin(dt)\begin{bmatrix}d\\0\end{bmatrix}\right)-b\left(\sin(dt)\begin{bmatrix}-c/2\\1\end{bmatrix}-\cos(dt)\begin{bmatrix}d\\0\end{bmatrix}\right)\right]$

$=e^{-ct/2}\left[\cos(dt)\left(a\begin{bmatrix}-c/2\\1\end{bmatrix}+b\begin{bmatrix}d\\0\end{bmatrix}\right)+\sin(dt)\left(a\begin{bmatrix}d\\0\end{bmatrix}-b\begin{bmatrix}-c/2\\1\end{bmatrix}\right)\right]$

$\boxed{\therefore x(t)=e^{-\frac{ct}{2}}\left[\cos\left(t\sqrt{1-\left(\frac{c}{2}\right)^2}\right)\left(a\begin{bmatrix}-c/2\\1\end{bmatrix}+b\begin{bmatrix}\sqrt{1-c^2/4}\\0\end{bmatrix}\right)+\sin\left(t\sqrt{1-\left(\frac{c}{2}\right)^2}\right)\left(a\begin{bmatrix}\sqrt{1-\frac{c^2}{4}}\\0\end{bmatrix}-b\begin{bmatrix}-c/2\\1\end{bmatrix}\right)\right]}$

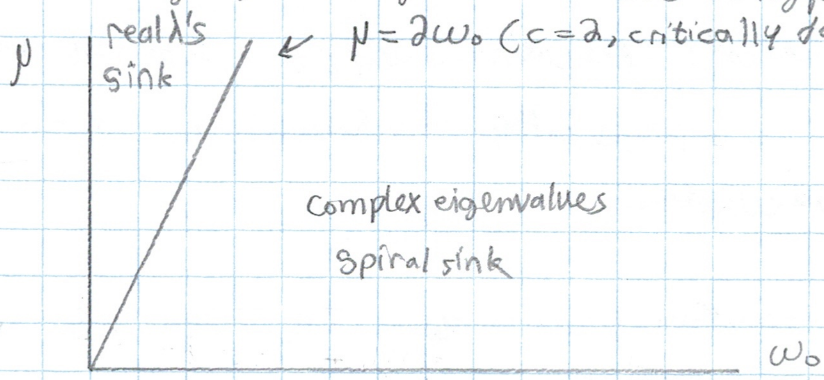
(10 cont.):

7. $$\lambda_\pm = \frac{-c \pm \sqrt{c^2-4}}{2}$$ We had defined $c = \mu/\omega_0$ so
$$\lambda_\pm = \frac{-\mu/\omega_0 \pm \sqrt{\frac{\mu^2}{\omega_0^2}-4}}{2}$$

We know that $\sqrt{\left(\frac{\mu}{\omega_0}\right)^2-4} < \frac{\mu}{\omega_0}$, which means that

$$-\mu/\omega_0 \pm \sqrt{\left(\frac{\mu}{\omega_0}\right)^2-4} < 0$$ regardless of the value of μ and ω_0 (both must be positive for the problem to make physical sense). If the eigenvalues are complex, then the real part is $-\mu/\omega_0$. And we saw that the λ's are also <0 if they are real.

Since the real part of λ is <0 in any case, the equilibrium must be asymptotically stable and a hyperbolic sink.



The equilibrium must always be a sink because the oscillator is being damped, so the energy must be decreasing to 0. Since energy is decreasing, the amplitude of oscillations must also decrease.

The real part of the solution has the form $x(t) = e^{dt}[\ldots\ldots]$ in which d is the real part of the eigenvalues $(-\mu/\omega_0)$. So the timescale of decay for this exponential decay is just $1/\lambda$, so it is $\frac{\omega_0}{\mu}$ ← if they're complex eigenvalues. So the eigenvalue's real part is the inverse of the timescale of decay. Remember our time is in $\tau = \frac{1}{\omega_0} t$ so we are measuring in units of the natural frequency. This makes sense because if μ increases (stronger damping/drag), the oscillations decay more quickly. If $\mu \to 0$, timescale of decay goes to ∞, so there is no decay in the oscillations (undamped case).

8. When we have 2 complex eigenvalues $(c<2)$, $\sqrt{c^2-4}$,

we saw in part 10.6 that we have a $\cos\left(t\sqrt{1-(c/2)^2}\right)$ term and a

$\sin\left(t\sqrt{1-(c/2)^2}\right)$. So the frequency (in units $\tau = \frac{1}{\omega_0}t$) is $\frac{1}{2\pi}\left(\sqrt{1-(c/2)^2}\right)$

$t = \omega_0 \tau$

This comes from the imaginary part of the eigenvalue, which dictates the timescale of oscillations. So the imaginary part of the eigenvalue is $\sqrt{1-(c/2)^2}\, i$, which factors into the cosine and sine (by Euler's formula) and thus determines the oscillation frequency. In terms of ω_0 and ν, it is ($c = \nu/\omega_0$)

If $b=0$, then $c = \frac{\nu}{\omega_0} = \frac{0}{\omega_0} = 0$ so the frequency is $\frac{1}{2\pi}\left(\sqrt{1-0}\right) = \frac{1}{2\pi}$.

$\nu = b/M = 0$

The frequency is $\frac{1}{2\pi}$ in units of time τ. In the original time units, the frequency is $t = \omega_0\tau = \frac{\omega_0}{2\pi}$ and the period is $\frac{2\pi}{\omega_0}$. This is the frequency/period for a linear oscillator with no damping. So if we set the drag coefficient $b=0$, we remove the damping effect and thus recover the non-damped oscillator timescale (frequency $\omega_0/2\pi$ and period $2\pi/\omega_0$)

So ω_0 is physically the frequency of an undamped oscillator.

$$\sqrt{1-\frac{c^2}{4}} = \sqrt{1-\frac{\nu^2}{4\omega_0^2}}$$ so frequency in units τ is

$$\frac{1}{2\pi}\left(\sqrt{1-\frac{\nu^2}{4\omega_0^2}}\right)$$

Homework 2

CAAM 28200: Dynamical Systems with Applications

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Problem 6

I plotted the phase portraits for $r < 0$, $0 < r < 1$, and $r > 1$ below. We see how the solution is a stable spiral when $r < 0$, a stable node when $0 < r < 1$, and a saddle when $r > 1$.

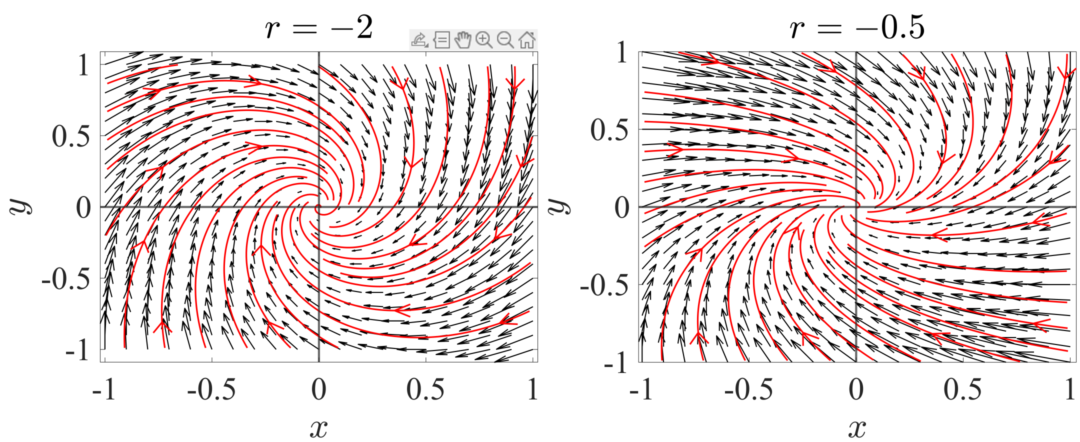

Figure 1: Phase Portraits for $r < 0$

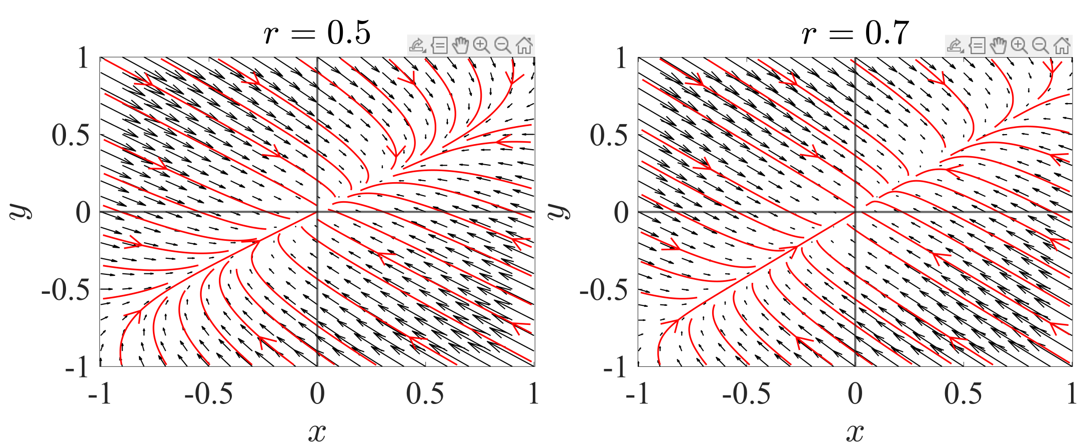

Figure 2: Phase Portraits for $0 < r < 1$

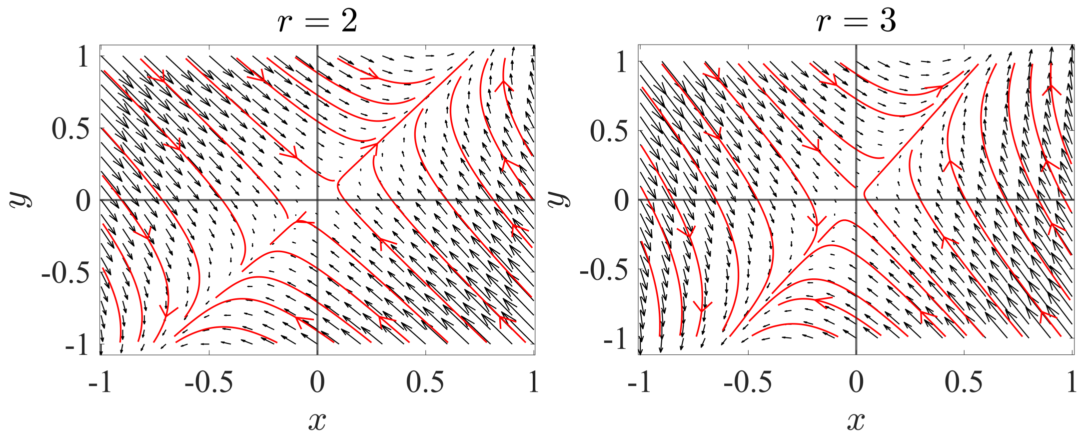

Figure 3: Phase Portraits for $r > 1$

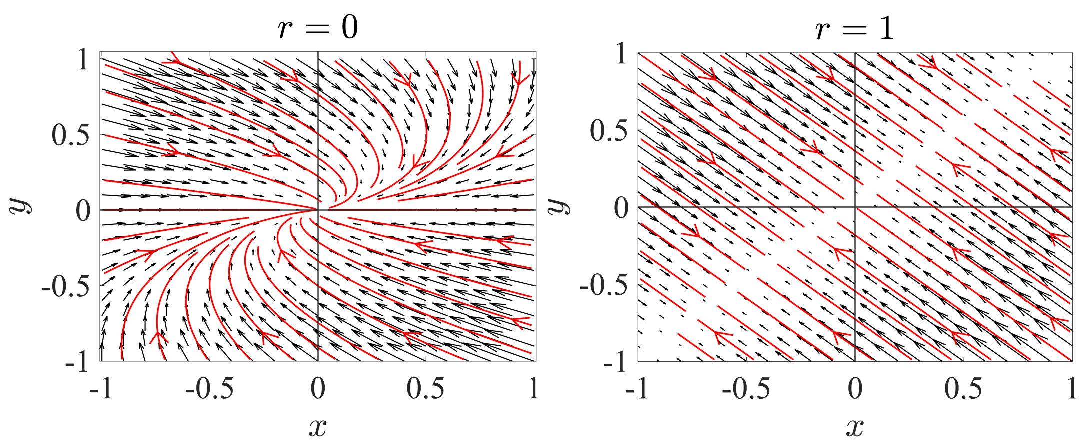

Figure 4: Phase Portraits for Marginal Cases