STAT/CAAM 28200 - Assignment 1 - Due Friday January 28th

This is an individual assignment. You may collaborate with others but are expected to give each problem an honest attempt on your own first, and must submit your own work, in your own words. Show all relevant work and cite all sources used. If you get stuck, come to office hours. Write your name on your submission.

Submit assignments on gradescope (linked from canvas). You may take a picture of written work and upload the images, scan your work and upload a pdf, or, if you are ambitious, tex it. Please submit one collated file.

Strogatz Questions: The following problems are from Strogatz Second Edition, the one with the black cover. Scans of the exercises in chapters 2 can be found on Canvas under the module for week 2.

Please complete questions:

2.2.7, 2.3.1 (a), 2.3.4, 2.3.6, 2.4.7, 2.4.9, 2.5.2

Question 8: The Logistic Map.

The logistic equation is widely used as a phenomenological/pedagogical model of population growth. It states that a population of size $N(t)$ grows according to the ODE:

$$\frac{d}{dt}N(t) = rN(t)(1 - N(t)) \tag{1}$$

where $N(t)$ is the total number of individuals sclaed by the population's carrying capacity, K. You solved this equation in question 2.3.1.

Not all populations change smoothly in time. Instead, are better modelled by a discrete time difference equation. For example, many plants and animals reproduce on a yearly cycle. Then it makes more sense to use a recursion of the form:

$$\Delta N(t) = N(t + \Delta t) - N(t) = rN(t)(1 - N(t))\Delta t \tag{2}$$

where time t is measured in time steps (reproductive cycles). Our goal will be to compare and constrast the results of these two approaches as r varies.

1. Make a sign chart for the ODE, identify all equilibria, characterize their stability, sketch a slope field, then sketch all solutions. They should match your analytic answers to 2.3.1. Show, without using the explicit solution, that all trajectories starting from a nonnegative population must converge monotonically to the carrying capacity, that is $N(t) \to 1$ as $t \to \infty$ if $N(0) > 0$ monotonically (without ever changing direction). Use these facts to show that no trajectory ever crosses the carrying capacity (all trajectories starting beneath 1 stay beneath 1, all trajectories starting above 1 stay above 1).
2. Find a change of time units $\tau = \lambda t$ for some λ such that:

   $$\frac{d}{d\tau}N(\tau) = N(\tau)(1 - N(\tau)) \tag{3}$$

   for any choice of r. (Hint: λ will depend on r). Use this result to argue that all solutions to the logistic equation are identical up to a change of time units, no matter r.
3. Use a computer program to plot the flow for $r = 1$. These are the normal form solutions. All other solutions are given by rescaling time.

In a sense we have just show that the logistic equation gives entirely boring solutions when we use an ODE. The population always steadily approaches carrying capacity without ever oscillating, and the behavior depends on r in a trivial way. What properties of ODE's ensured this was true?

Let's consider the logistic map now (difference equation).

1. Set up a cobweb plot by plotting the parabola $N(t + \Delta t) = rN(t)(1 - N(t))$ as a function of $N(t)$ and the line $N(t + \Delta t) = N(t)$ on the same axes. Show that the logistic map only returns reasonable results for $N(t)$ on $[0, 1]$ and that if $N(t) > 1$ at some time t then future populations are negative.

2. Show that the logistic map has, at most two fixed points (N_* such that $N(t+\Delta t) = N(t)$ if $N(t) = N_*$). Graphically, fixed points are locations where the parabola in your cobweb plot intersects the line. What is the minimum r such that the logistic map has two fixed points? Solve for the location of the fixed points as a function of r.
3. Complete the following numerical experiment. Carefully draw a cobweb plot for $r = 2, 2.5, 3, 3.25, 3.5$ and 3.57. You may write a computer program to generate these charts. This does not have to be fancy, you can use an excel spreadsheet with the recursion coded as a macro if needed. Run at least 32 steps (you do not need to plot all of them on the cobweb, but plot enough to clearly show the long time behavior). Experiment with changing your initial condition. Make a plot of $N(t)$ for t ranging from 1 to 32. What behaviors do you observe? Does $N(t)$ approach the fixed point for all r? If it does, does it approach the fixed point monotonically? Are there values of r for which the solution appears to approach a steady oscillation? If so, what is the period of the oscillation?

You should see that the logistic map is *much* less well behaved than the logistic equation. In fact, for r large enough, the logistic map is chaotic! For more on the logistic map, and the approach to chaos via period doubling see May's famous 1976 Nature paper *Simple Mathematical Models with Very Complicated Dynamics*. What special properties of ODE's ensures that the logistic equation is well-behaved while the logistic map is not?

Question 9: Non-uniqueness. Dynamical systems sometimes arise as solutions to trajectory optimization problems. In trajectory optimization we search for a "best" path, under some chosen constraints. Best is defined by introducing a functional, that is, a function that accepts a trajectory. For example, the total amount of gas used along a driving route or the time elapsed. Mapping services solve trajectory optimization problems when route finding. Best paths are widely used in the physical sciences to study most probable escape paths, rare fluctuations, and noise driven catastrophic transitions. In fact, classical mechanics can be entirely recast in a variational form via Lagrangian mechanics, which replaces Newton's laws with an optimization principle. Here we consider a route finding problem.

Suppose that the Earth was a perfect sphere. Then, on the equinox, exactly one hemisphere of the planet will be lit by daylight, and exactly one other hemisphere will be in night, and the dividing line will run south from the north pole through the south pole and back, along a longitude line. Suppose that the spin of the Earth is much faster than its' orbit around the sun so that the illumination angle remains fixed over the course of a day.

Now suppose that you are given an airplane that can fly at speed v. What is the minimum speed v_* necessary to fly from the North to South Poles staying entirely in daylight? If you can fly that speed then, logistical nightmares aside, you could, in theory, spend an entire year in daylight. But how fast is v_*? Is v_* achievable? Have you ever gone that fast? Take a moment to estimate v_*.

To find v_* we need to find an optimal path. Namely, given a plane that can fly at speed v, what route should it take from the North to South poles in order to minimize the solar time elapsed (perceived change in time as measured by a sundial on the plane) over the course of the flight. Then, given the optimal route for each v, we can solve for v_* by finding the minimum v at which the solar time elapsed is less than 12 hours.

Let R be the radius of the earth (3,958.8 miles), and ω be its angular velocity ($2\pi/24$ in radians per hour). Then the minimum speed is $v_* = \frac{1}{2}\omega R = 518.2$ miles per hour. That is, rather astonishingly, slower than most modern jet airliners. So, if you've flown in any large body jet, you've flown fast enough to evade night (or day if you're some college students) in perpetuity!

For a interactive animated solution go to https://www.mathmouth.com/aroundtheworld. We will primarily be concerned with the resulting dynamical system.

The optimal trajectory satisfies the following differential equation when $v = v_*$:

$$\begin{aligned} \frac{d}{dt}\phi(t) &= \frac{\omega}{4}\sqrt{4 - \sec(\phi(t))^2} \\ \frac{d}{dt}\theta(t) &= \frac{\omega}{4}\sec(\phi(t))^2. \end{aligned} \tag{4}$$

where ϕ is the latitude of the plane, and θ is the longitude of the plane. Notice that the rate of change in the latitude depends only on the current latitude, so we consider the ODE:

$$\frac{d}{dt}\phi(t) = \frac{\omega}{4}\sqrt{4 - \sec(\phi(t))^2} \text{ if } \phi \tag{5}$$

independently. Now this is an ODE in one variable. The state space is $[-2\pi/3, 2\pi/3]$, instead of the usual range of latitudes ($[-\pi/2, \pi/2]$) because the plane can travel faster than daylight moves westward if it is north of $2\pi/3$ or south of $-2\pi/3$ (60 degrees north and south). Outside those latitudes the plane can take any path it likes, and, provided you don't fly into the night, you can always run from sunset fast enough to stay in daylight. Between those latitudes the plane moves westward slower than daylight. At those latitudes, if the plane flies due west, then solar time stands still (the plane moves no closer or farther from the day-night boundary).

1. Show that the differential equation defining latitude admits two equilibrium solutions. Where are they (at what latitude)? Why must they exist?
2. Nevertheless, this dynamical system defines a portion of the optimal trajectory connecting the North and South Poles, so there must be some path starting arbitrarily close to $2\pi/3$ and ending at $-2\pi/3$. Therefore, there are two options. The first, is that the ODE has a unique solution for all initial conditions, in which case trajectories cannot branch or merge at finite times, so the optimal path must be infinitely long, converging to $-2\pi/3$ after flying for infinitely long, and diverging from $2\pi/3$ infinitely far in the past. Alternatively, the ODE does not admit unique solutions, and trajectories can merge and branch at finite times. State the sufficient condition from class for an ODE to admit unique solutions given an initial condition.
3. Plot $f(\phi) = \frac{\omega}{4}\sqrt{4 - \sec(\phi(t))^2}$ and show that the slope $\frac{d}{d\phi}f(\phi)$ diverges at $-2\pi/3$ and $2\pi/3$. What does this imply about the uniqueness of solutions? What does this imply about the existence of finite trajectories travelling at v_* that minimize solar time elapsed?