STAT/CAAM 28200 - Assignment 6 - Due Monday March 14

This is an individual assignment. You may collaborate with others but are expected to give each problem an honest attempt on your own first, and must submit your own work, in your own words. Show all relevant work and cite all sources used. If you get stuck, come to office hours. Write your name on your submission.

Submit assignments on gradescope (linked from canvas). You may take a picture of written work and upload the images, scan your work and upload a pdf, or, if you are ambitious, tex it. Please submit one collated file.

All Strogatz problems are from Strogatz Second Edition, the one with the black cover.

Strogatz Questions from Chapter 9: Scans of the exercises in chapters 9 can be found on Canvas under the module for week 8.

Please complete questions:

9.2.1, 9.2.2

Simulation Exercises (Lorenz): The following questions are based on 9.3.9, 9.3.10, and 9.4.1.

Please simulate the Lorenz system for $\sigma = 28$, $b = 8/3$, and r varying from just below the chaos threshold r_H associated with the Hopf bifurcation. Plot the resulting trajectories starting from x_0 near C_+. Illustrate the Hopf bifurcation as r passes r_H. Run your trajectories long enough to clearly illustrate the shape of the strange attractor. Start a second trajectory from an initial condition $y_0 = x_0 + \delta$ near x_0. Plot the relative discrepancy between your two trajectories $\|y(t) - x(t)\|/\|\delta\|$ on a log plot and identify the time region where the discrepancy grows exponentially.

Fit the log growth rate in the log of the relative discrepancy to a line to estimate the Lyapunov exponent. Vary δ to test the robustness of your fit. Based on your fit to the Lyapunov exponent identify how long, starting from $\delta = 10^{-6}$ it will take before the discrepancy is greater than 10^{-1}? Now suppose we reduce δ so that $\delta = 10^{-9}$. How much larger is the new time horizon? (Note: make sure to set a relatively tight error tolerance in your ode solver).

Next, record the sequence of local maxima of the $z(t)$, $\{z_1, z_2, \ldots\}$, then make a scatter plot of z_{n+1} against z_n. Show that the scatter plot is concentrated on a curve thus the sequence of maxima is (approximately) governed by a discrete time map. This is the Lorenz map. Approximate the curve then sketch some solutions to the Lorenz map using a cobweb diagram.

Optional Challenge Problem (The Three Body Problem): Try and simulate three bodies orbiting one another under the influence of gravity. Start two simulations with near to identical initial conditions. Does the three body problem exhibit sensitive dependence on initial conditions?

Maps and Period Doubling: Consider the logistic map:

$$x_{n+1} = r x_n (1 - x_n) \tag{1}$$

and the sine map:

$$x_{n+1} = r \sin(\pi * x_n) \tag{2}$$

for initial $x_0 \in [0, 1]$. For both maps write a code that runs the recursion forward for arbitrary r. Then, slowly increasing r, run a series of long simulations (say a million steps), and generate a scatter plot showing the last 1,000 points visited by the maps at each value of r. Set r to the horizontal coordinate and set the vertical coordinate equal to the x values visited by the recursion. See Strogatz chapter 10 for relevant ranges of r. Compare your plots. What shared features do you observe. *Optional Challenge:* Attempt to identify the values of r where the period doubles. Does the spacing of the period doublings approach a regular pattern?