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CAAM 28200

HW3

2/18/22

1. horizontal nullcline is when $\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so x_2 is constant but x_1 is changing.

This is when $\frac{d}{dt} x_2 = 0 = r_2 x_2 (1 - a_{21} x_1 - a_{22} x_2)$

4 nullclines

$x_2 = 0$ (extinction)

$$\frac{1 - a_{21} x_1}{a_{22}} = a_{22} x_2$$

$$x_2 = \frac{1}{a_{22}} - \frac{a_{21}}{a_{22}} x_1$$

vertical nullcline when $\frac{d}{dt} x_1 = 0 = r_1 x_1 (1 - a_{11} x_1 - a_{12} x_2)$ or this is when $\frac{d}{dt} x_1 = 0 = r_1 x_1 (1 - a_{11} x_1 - a_{12} x_2)$

4 equilibria

$x_1 = 0$ (extinction)

$$x_2 = \frac{1 - a_{12} x_1}{a_{11}} = \frac{1}{a_{11}}$$

Equilibrium when $\frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Also when $x = \begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}$

such as $x_1 = 0, x_2 = 0$ (extinction)

(one population is extinct)

or when $x_2 = \frac{1 - a_{11} x_1}{a_{12}} < \frac{1 - a_{21} x_1}{a_{22}} \Rightarrow a_{22} - a_{11} a_{22} x_1 = a_{12} - a_{12} a_{21} x_1$

$$a_{22} - a_{12} = (a_{11} a_{22} - a_{12} a_{21}) x_1$$

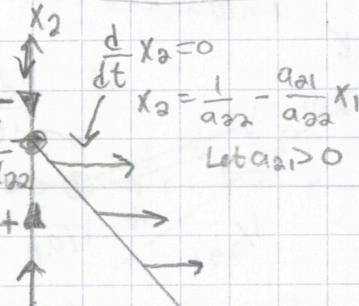
$$x_1 = \frac{a_{22} - a_{12}}{a_{11} a_{22} - a_{12} a_{21}}$$

$$\text{and } x_2 = \frac{1 - a_{11}}{a_{12}} \quad (\text{intersection of 2 lines})$$

If $x_2 = 0$,

$$f = \frac{d}{dt} x_1 = r_1 x_1 (1 - a_{11} x_1)$$

when $x_1 = 1/a_{11}, \frac{dx_1}{dt} = 0$ (carrying capacity)



If $x_1 = 0$

$$\frac{d}{dt} x_2 = r_2 x_2 (1 - a_{22} x_2)$$

if $x_2 = 1/a_{22}, \frac{dx_2}{dt} = 0$

(carrying capacity)

Equilibria at

$$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}$$

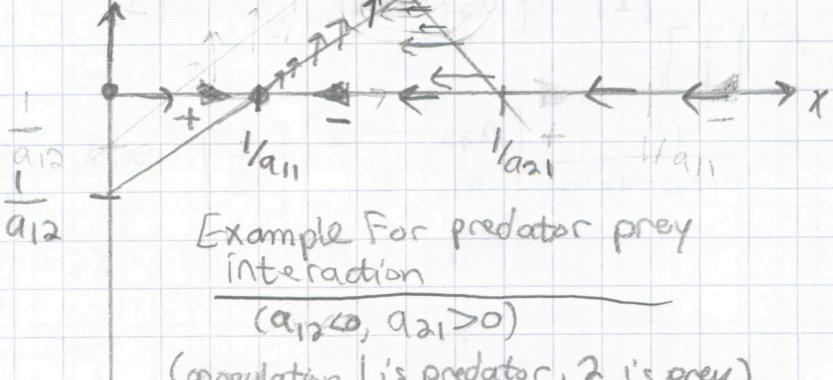
$$x = \begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}$$

and the intersection of 2 lines

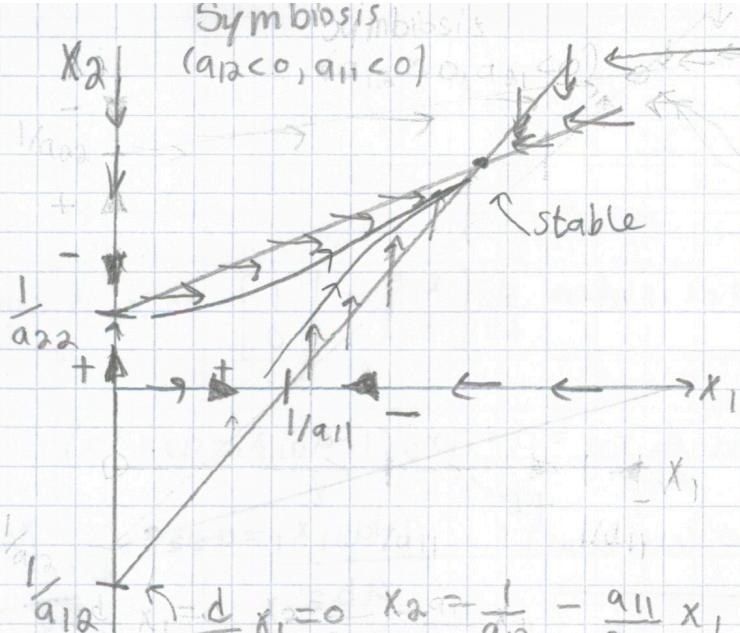
Example For predator prey interaction

$$(a_{12} > 0, a_{21} > 0)$$

(population 1 is predator, 2 is prey)



(1 cont.)



$$\frac{d}{dt}X_2 = 0$$

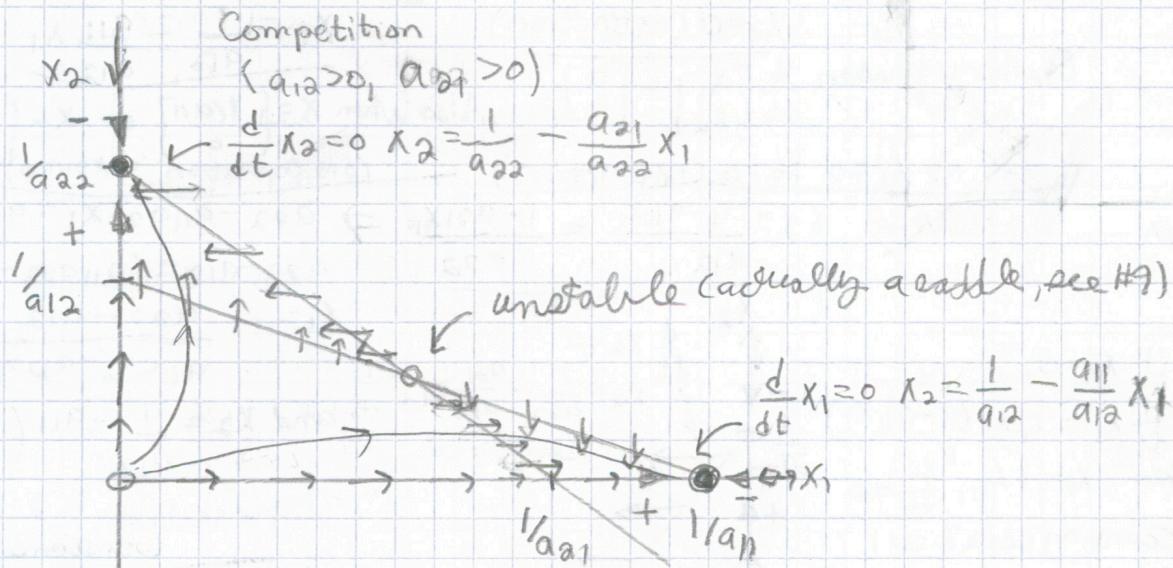
$$X_2 = \frac{1}{a_{22}} - \frac{a_{21}}{a_{22}} X_1$$

Equilibrium here
if $|\frac{a_{11}}{a_{12}}| < |\frac{a_{21}}{a_{22}}|$

then there would be only
equilibrium here

$$\frac{d}{dt}X_1 = 0 \quad X_2 = \frac{1}{a_{12}} - \frac{a_{11}}{a_{12}} X_1$$

Here $|\frac{a_{11}}{a_{12}}| > |\frac{a_{21}}{a_{22}}|$ (so population 2 can support more individuals)



2. Equilibria are $X = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}$, $X = \begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}$ and when lines intersect
(coexistence equilibrium)

$$AX = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X = A^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_1 = \frac{-a_{12} + a_{22}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$X_2 = \frac{-a_{21} + a_{11}}{a_{11}a_{22} - a_{12}a_{21}}$$

Last equilibrium

3. If the diagonal entries define the carrying capacities as the 2 equilibria with extinction of only one population there are

$$X = \begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix} \quad \text{and} \quad X = \begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}$$

In these cases there are no interactions between the 2 populations

4. need $\frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{12}a_{21}} > 0$ and $\frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{12}a_{21}} > 0$.

if $\det(A) > 0$, we need $a_{22} > a_{12}$ and $a_{11} > a_{21}$
 $a_{22} > 0$ and $a_{11} > 0$, or either $a_{12} > 0$ and $|a_{21}| < a_{22}$ or $a_{12} < 0$
 $a_{21} > 0$ and $|a_{21}| < a_{11}$ or $a_{21} < 0$.

So coexistence is viable always if $a_{21} < 0$ and $a_{11} < 0$

- If a_{21} and $a_{11} > 0$ (or either is > 0) then they must be smaller in magnitude than the corresponding diagonal term, meaning that the population must have a weaker suppressive effect on the other population than on itself for coexistence to be viable, and thus equilibrium

With the nullclines, if $a_{12} < 0$ and $a_{11} < 0$ the intersection must be in the 1st quadrant $(+x_1, +x_2)$. we see this in the symbiosis graph. This is because both nullclines must intersect the exclusion equilibria and have positive slope, so they intersect each other at $x_1 > 0, x_2 > 0$.

- If $a_{12} < 0$ and $a_{21} > 0$ (predator-prey, as in my graph), then if a_{21} is too large, the $\frac{\partial x_2}{\partial t} = 0$ nullcline will be too steep and intersect the $\frac{\partial x_1}{\partial t} = 0$ other nullcline at $x_2 < 0$, as the 2 populations can't interact, so coexistence would not be viable. The case is analogous if a_{12} is too large when $a_{12} > 0, a_{21} < 0$.
- If $a_{12} > 0, a_{21} > 0$ (competition), if $a_{21} > a_{11}$ (as in my graph) then the coexistence equilibrium is unstable so any perturbation will cause the system to move away from the coexistence equilibrium

5. $J(X) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix} \quad f_1 = \frac{dx_1}{dt} = r_1 x_1 - r_1 a_{11} x_1^2 - r_1 a_{12} x_1 x_2$

$\therefore J(X) = \begin{bmatrix} r_1 - 2r_1 a_{11} x_1 - r_1 a_{12} x_2 \\ -r_1 a_{12} x_2 \end{bmatrix} \quad \begin{aligned} \partial_{x_1} f_1 &= r_1 - 2r_1 a_{11} x_1 - r_1 a_{12} x_2 \\ \partial_{x_2} f_1 &= -r_1 a_{12} x_1 \end{aligned}$

$f_2 = \frac{dx_2}{dt} = r_2 x_2 - r_2 a_{21} x_1 x_2 - r_2 a_{22} x_2^2$

$\begin{aligned} \partial_{x_1} f_2 &= r_2 - r_2 a_{21} x_1 - 2r_2 a_{22} x_2 \\ \partial_{x_2} f_2 &= -r_2 a_{21} x_2 \end{aligned}$

expressive
effect of pop1
on pop2.

See next page for more

5(contd).

$$J(x) = \begin{bmatrix} r_1 - r_1 a_{11}x_1 - r_1 a_{12}x_2 & 0 \\ 0 & r_2 - r_2 a_{21}x_1 - r_2 a_{22}x_2 \end{bmatrix} + \begin{bmatrix} -r_1 a_{11}x_1 & -r_1 a_{12}x_1 \\ -r_2 a_{21}x_2 & -r_2 a_{22}x_2 \end{bmatrix}$$

$$\therefore J(x) = \begin{bmatrix} r_1(1-a_{11}x_1-a_{12}x_2) & 0 \\ 0 & r_2(1-a_{21}x_1-a_{22}x_2) \end{bmatrix} + \begin{bmatrix} -r_1 a_{11}x_1 & -r_1 a_{12}x_1 \\ -r_2 a_{21}x_2 & -r_2 a_{22}x_2 \end{bmatrix}$$

$$J\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \quad J\left(\begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} r_1 - r_1 a_{11} \cancel{1/a_{11}} & -r_1 \cancel{a_{11}} \cancel{1/a_{11}} & -r_1 a_{12} \\ 0 & r_2 - r_2 a_{21} \cancel{1/a_{11}} & 0 \end{bmatrix}$$

$$J\left(\begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}\right) = \begin{bmatrix} r_1 - r_1 a_{12}/a_{22} & 0 \\ -r_2 a_{21}/a_{22} & -r_2 \end{bmatrix} = \begin{bmatrix} -r_1 & -r_1 a_{12}/a_{11} \\ 0 & r_2 - \frac{r_2 a_{21}}{a_{11}} \end{bmatrix}$$

In these cases the Jacobian is triangular

At the coexistence equilibrium

$$J(x) = \begin{bmatrix} r_1 \left[\frac{(a_{11}a_{22} - a_{21}a_{12})}{a_{11}a_{22} - a_{21}a_{12}} - \frac{a_{12}(a_{11} - a_{21})}{a_{11}a_{22} - a_{21}a_{12}} \right] & \frac{a_{11}a_{12}}{a_{11}a_{22} - a_{21}a_{12}} & -\frac{a_{12}a_{11}}{a_{11}a_{22} - a_{21}a_{12}} \\ 0 & 0 & 0 \\ \frac{(a_{11}a_{22} - a_{21}a_{12}) - a_{21}a_{22} + a_{22}a_{12} - a_{22}a_{11} + a_{22}a_{12}}{a_{11}a_{22} - a_{21}a_{12}} & & \end{bmatrix}$$

+ (2nd matrix)

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -r_1 a_{11}x_1 & -r_1 a_{12}x_1 \\ -r_2 a_{21}x_2 & -r_2 a_{22}x_2 \end{bmatrix}$$

Do the 1st matrix vanishes
at coexistence eq.

$$\therefore J(x) = \begin{bmatrix} -r_1 a_{11} \left(\frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \right) & -r_1 a_{12} \left(\frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{21}a_{12}} \right) \\ -r_2 a_{21} \left(\frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \right) & -r_2 a_{22} \left(\frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \right) \end{bmatrix}$$

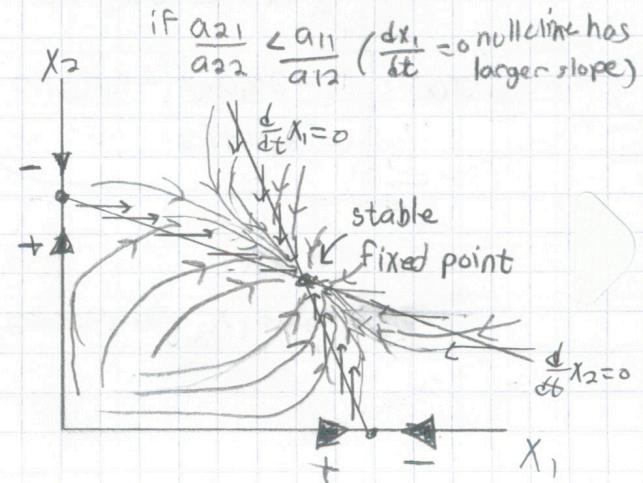
at coexistence equilibrium

4 (cont), if $a_{12} > 0$ and $a_{21} > 0$ (competition), then we can get a stable equilibrium if the slope of the $\dot{x}_2 = 0$ nullcline is less than the slope of the $\dot{x}_1 = 0$ nullcline

$$0 = \frac{dx_1}{dt}, \text{nullcline} \Rightarrow \frac{a_{21}}{a_{22}} < \frac{a_{11}}{a_{12}}$$

$$\Rightarrow a_{11}a_{12} < a_{22}a_{11}$$

So the coexistence eq. is stable if each population has a smaller suppressive effect on the other population than on itself ($a_{12}a_{21} < a_{11}a_{22}$)



6. $J\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$ eigenvalues are $(r_1 - \lambda)(r_2 - \lambda) = 0$
 $\lambda = r_1, r_2$

so eigenvalues are r_1, r_2 since $r_1 > 0$ and $r_2 > 0$
 this equilibrium is always unstable (source),

7.

$$J\left(\begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -r_1 & -r_1 a_{12}/a_{11} \\ 0 & r_2 - r_2 a_{21}/a_{11} \end{bmatrix} = A \quad \det(A - \lambda I) = 0$$

$$(-r_1 - \lambda)(r_2 - r_2 a_{21}/a_{11} - \lambda) = 0$$

We showed in #4 that if
 $a_{21} < a_{11}$ and $\det(A) > 0$, then
 coexistence is viable. Here we

showed that for this extinction
 equilibrium to be a saddle,

we need λ_2 to be positive which

requires that $a_{21} < a_{11}$, (the same requirement
 as before). So if $\det(A) > 0$ and coexistence

is viable, then $a_{22} < a_{11}$, then the equilibrium
 is a saddle. If $\det(A) > 0$ and the equilibrium is a saddle, the coexistence

is viable.

$$J\left(\begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}\right) = \begin{bmatrix} r_1 - r_1 a_{12}/a_{22} & 0 \\ -r_2 a_{21}/a_{22} & -r_2 \end{bmatrix} = A \quad \det(A - \lambda I) = 0$$

$$(r_1 - r_1 a_{12}/a_{22} - \lambda)(-r_2 - \lambda) = 0$$

We see that for this equilibrium to be a saddle,
 we need $a_{12} < a_{22}$. In #4, this was
 the other requirement for coexistence
 to be viable if $\det(A) > 0$.

$$\lambda_1 = r_1 - r_1 a_{12}/a_{22} \quad \lambda_2 = -r_2$$

$$\lambda_1 = r_1(1 - a_{12}/a_{22}) \quad (r_1 > 0 \text{ so } \lambda_1 > 0)$$

$$\text{if } a_{12}/a_{22} < 1, \text{ then } \lambda_1 > 0$$

$$(a_{12} < a_{22})$$

So this means if $a_{12} < a_{11}$ and $a_{12} < a_{22}$,
 then both exclusion equilibria are saddle

and if $\det(A) > 0$, coexistence is then viable. And if $\det(A) > 0$ and coexistence
 is viable, then $a_{22} < a_{11}$ and $a_{21} < a_{11}$, so both exclusion equilibria must be
 saddle.

Now we find eigenvectors for stable perturbations:

For $\lambda_1 = -r_1$ at $x = [1/a_{11}, 0]$,

so stable perturbations occur $V = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

when only the population of the species that
 doesn't go extinct is changed. The extinction

$$\begin{bmatrix} -r_1 & -r_1 a_{12}/a_{11} \\ 0 & r_2 - r_2 a_{21}/a_{11} \end{bmatrix}$$

$$-(-r_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -r_1 a_{12}/a_{11} \\ 0 & r_2 + r_2 a_{21}/a_{11} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$\therefore V_2 = 0$ V_1 can be anything

7 (cont'd). The exclusion equilibrium is stable along the direction with the extinct species remaining extinct.

$$\text{For } \lambda = -r_2 \text{ for } x = [0 \ 1/a_{22}], (A - \lambda I)v = \begin{bmatrix} r_2 + r_1 - \gamma a_{12}/a_{22} & 0 \\ -\gamma a_{21}/a_{22} & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Similarly, the direction of stable perturbation is with the extinct species remaining extinct. If only the non-extinct population is perturbed, then the exclusion

$$\therefore v_1 = 0 \quad v_2 \text{ can be anything } v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

stable equilibrium is stable, so the stable manifold is a curve that is tangent to the eigenvector.

Since the stable manifold is when the extinct species remains extinct, in this case, the stable manifold is a line so the stable eigenvector is the stable manifold.

8. In #5 I wrote down the Jacobian at the coexistence equilibrium. If we plug $r_1 = r_2 = 1$ and $a_{11} = a_{22} = 1/10$,

$$J(x_*) = \begin{bmatrix} -1/10 \left(\frac{1/10 - a_{12}}{1/100 - a_{21}a_{12}} \right) & -a_{12} \left(\frac{1/10 - a_{12}}{1/100 - a_{21}a_{12}} \right) \\ -a_{21} \left(\frac{1/10 - a_{21}}{1/100 - a_{21}a_{12}} \right) & -1/10 \left(\frac{1/10 - a_{21}}{1/100 - a_{21}a_{12}} \right) \end{bmatrix}$$

Let $a_{12} = a_{21} = -1/30$ competition

$$J(x_*) = \begin{bmatrix} -1/10 \left(\frac{1/10 + 1/30}{1/100 - 1/900} \right) & 1/30 \left(\frac{1/10 + 1/30}{1/100 - 1/900} \right) \\ 1/30 \left(\frac{1/10 + 1/30}{1/100 - 1/900} \right) & -1/10 \left(\frac{1/10 + 1/30}{1/100 - 1/900} \right) \end{bmatrix} = \begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$$

$$\det(J(x_*) - \lambda I) = 0 \quad J(x_*) - \lambda I = \begin{bmatrix} -3/2 - \lambda & 1/2 \\ 1/2 & -3/2 - \lambda \end{bmatrix}$$

$$\det(J(x_*) - \lambda I) = (-3/2 - \lambda)^2 - (1/2)^2 = 0$$

$$\lambda^2 + 3\lambda + 9/4 - 1/4 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda + 2)(\lambda + 1) = 0$$

$$\lambda = -2, -1$$

Both eigenvalues are negative so coexistence is stable in this case.

Let $a_{12} = a_{21} = 1/30$ competition

$$J(x_*) = \begin{bmatrix} -1/10 \left(\frac{1/10 - 1/30}{1/100 - 1/900} \right) & -1/30 \left(\frac{1/10 - 1/30}{1/100 - 1/900} \right) \\ -1/30 \left(\frac{1/10 - 1/30}{1/100 - 1/900} \right) & -1/10 \left(\frac{1/10 - 1/30}{1/100 - 1/900} \right) \end{bmatrix} = \begin{bmatrix} -3/4 & -1/4 \\ -1/4 & -3/4 \end{bmatrix}$$

$$\det(J(x_*) - \lambda I) = \det \left(\begin{bmatrix} -3/4 - \lambda & -1/4 \\ -1/4 & -3/4 - \lambda \end{bmatrix} \right) = 0$$

$$(-3/4 - \lambda)^2 - (-1/4)^2 = 0$$

$$\lambda^2 + \frac{3}{2}\lambda + \frac{9}{16} - \frac{1}{16} = 0 \quad \lambda = -1, -1/2$$

Both eigenvalues are negative, so coexistence is still stable. However, the eigenvalues are now less negative, so coexistence is less stable.

(8 cont.). This makes sense because the coexistence equilibrium is less stable in the competitive case ($a_{12} = a_{21} = 1/30$) than in the cooperative case ($a_{12} = a_{21} = -1/30$) because when 2 populations are competing, they are more sensitive to competitive exclusion, in which one population excludes the other to extinction. Since the eigenvalues for the competitive case are smaller, this case has a faster rate of convergence, less negative

q. Let $a_{12} = a_{21} = 1/5$

$$J(x_*) = \begin{bmatrix} -1/10 \left(\frac{1/10 - 1/5}{1/100 - 1/25} \right) & -1/5 \left(\frac{1/10 - 1/5}{1/100 - 1/25} \right) \\ -1/5 \left(\frac{1/10 - 1/5}{1/100 - 1/25} \right) & -1/10 \left(\frac{1/10 - 1/5}{1/100 - 1/25} \right) \end{bmatrix} = \begin{bmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{bmatrix}$$

$$\det(J(x_*) - \lambda I) = 0$$

$$(-1/3 - \lambda)^2 - (-2/3)^2 = 0$$

$$\lambda^2 + (2/3)\lambda + 1/9 = 0$$

$$(\lambda + 1)(\lambda - 1/3) = 0$$

$$\lambda = -1, 1/3$$

$$(-1, +1/3)$$

$$\lambda_1 = -1$$

$$(A - \lambda_1 I)v = \begin{bmatrix} -1/3 - (-1) & -2/3 \\ -2/3 & -1/3 - (-1) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

competitive exclusion
extinction of 1 species).

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

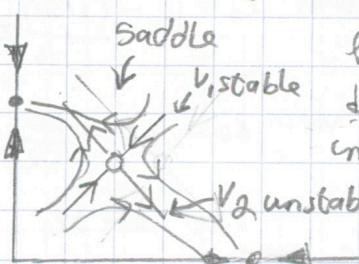
The equilibrium is stable along the direction in which the perturbation in the population is equal for the 2 species (both inc or dec).

$$\lambda_2 = +1/3$$

$$(A - \lambda_2 I)v = \begin{bmatrix} -1/3 + 1/3 & -2/3 \\ -2/3 & -1/3 - 1/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The equilibrium is unstable along the direction in which the perturbation is unequal for the 2 species (one increases and the other decreases), this makes sense as the population that increases will continue to increase and outcompete the other population that decreased due to the perturbation.



\rightarrow due to perturbation

(9 cont'd.) Increasing the competition coefficients decreased the stability of the coexistence state because each population now has a stronger depressive effect on the other population than on itself ($a_{12} = a_{21} > a_{11} = a_{22}$)

10. Case 1: $a_{12} = a_{21} = -1/30$ (symbiosis)

In #8, we had gotten $\lambda = -2 - 1$

$$J(x_0) = A = \begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$$

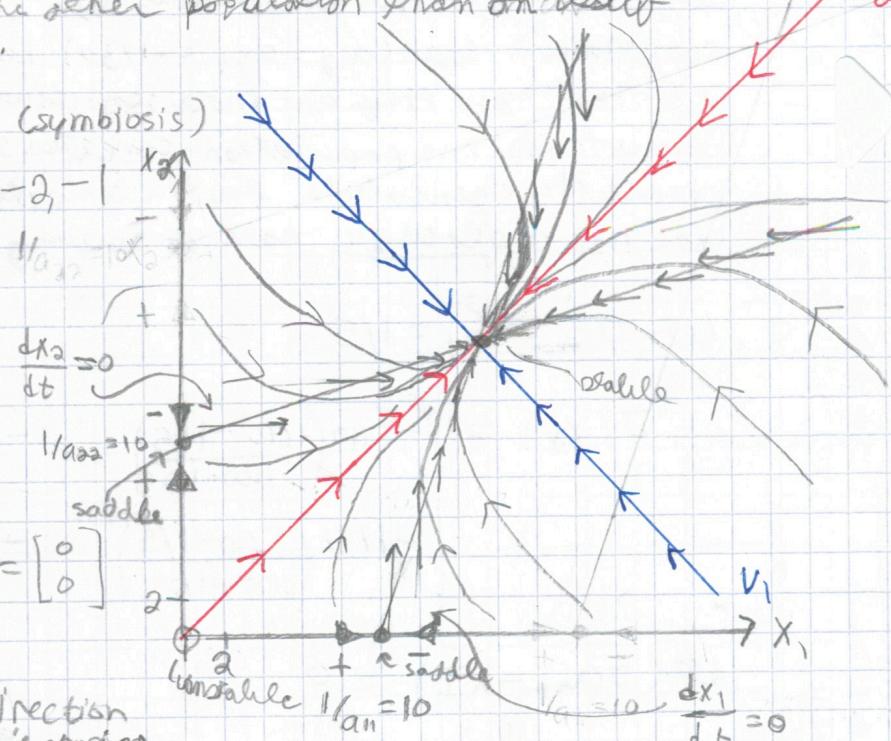
$$(A - \lambda I) V = 0$$

For $\lambda_1 = -2$

$$\begin{bmatrix} -3/2 - (-2) & 1/2 \\ 1/2 & -3/2 - (-2) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$|V_1| > |\lambda_2|$
fast eigendirection
as $t \rightarrow -\infty$, trajectories



For $\lambda_2 = -1$

$$\begin{bmatrix} -3/2 + 1 & 1/2 \\ 1/2 & -3/2 + 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

slow eigendirection
near the fixed point
This is parallel to
approaching trajectories

$$\frac{dx_2}{dt} = 0 \text{ when } x_2 = \frac{1}{a_{22}} - \frac{a_{21}}{a_{22}} x_1$$

$$x_2 = 10 - \frac{(-1/30)}{1/10} x_1 = 10 + \frac{x_1}{3}$$

$$\frac{dx_1}{dt} = 0 \text{ when } x_2 = \frac{1}{a_{12}} - \frac{a_{11}}{a_{12}} x_1$$

$$x_2 = -30 - \frac{1/10}{-1/30} x_1 = -30 + 3x_1$$

case 2: $a_{12} = a_{21} = 1/30$ (competition)

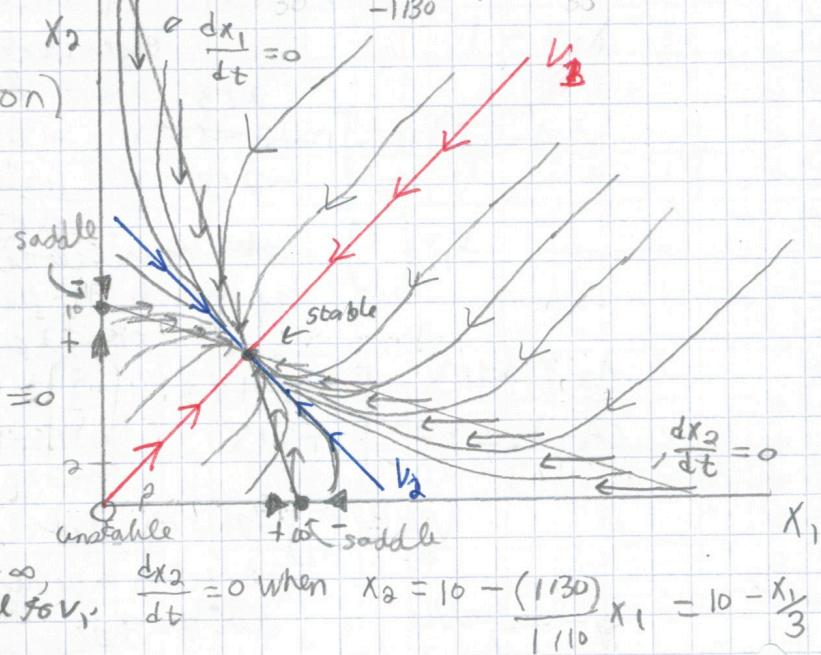
$$J(x_0) = A = \begin{bmatrix} -3/4 & -1/4 \\ -1/4 & -3/4 \end{bmatrix}$$

for $\lambda_1 = -1$,

$$\begin{bmatrix} -3/4 - (-1) & -1/4 \\ -1/4 & -3/4 - (-1) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = 0$$

$$V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

this is the fast
eigendirection. As $t \rightarrow -\infty$,
trajectories are parallel to V_1 .



For $\lambda_2 = -1/2$,

$$\begin{bmatrix} -3/4 - (-1/4) & -1/4 \\ -1/4 & -3/4 - (-1/4) \end{bmatrix} V = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

this is the slow eigendirection, near fixed point,
 V_2 is tangent to approaching trajectories.

$$\frac{dx_1}{dt} = 0 \text{ when } x_2 = 30 - \frac{1/10}{1/30} x_1$$

$$x_2 = 30 - 3x_1$$

10 (cont.).

case 3: $a_{12} = a_{21} = 1/5$ (competition)

$$A = \begin{bmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{bmatrix}$$

$$\lambda = -1, +1/3$$

$$\lambda_1 = 1, v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = +1/3, v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

unstable eigendirection

Trajectories approach from parallel to v_1 , far from the saddle at coexistence, so v_1 is tangent to the stable manifold.

v_2 is tangent to the unstable manifold along which trajectories diverge from the saddle

1. oscillatory solutions when complex λ 's.

Complex λ 's when $\det(A) > \frac{1}{4}\text{tr}(A)^2$ (discriminant < 0)

$$\det(A) = r_1 a_{11} r_2 a_{22} x_{+1} x_{+2} - r_1 r_2 a_{12} a_{21} x_{+1} x_{+2}$$

$$\text{tr}(A) = -r_1 a_{11} x_{+1} - r_2 a_{22} x_{+2}$$

$\det(A) > \frac{1}{4}\text{tr}(A)^2$ for complex λ 's and oscillatory solutions

$$r_1 r_2 x_{+1} x_{+2} (a_{11} a_{22} - a_{12} a_{21}) > \frac{1}{4} (r_1 a_{11} x_{+1} + r_2 a_{22} x_{+2})^2$$

This is the condition

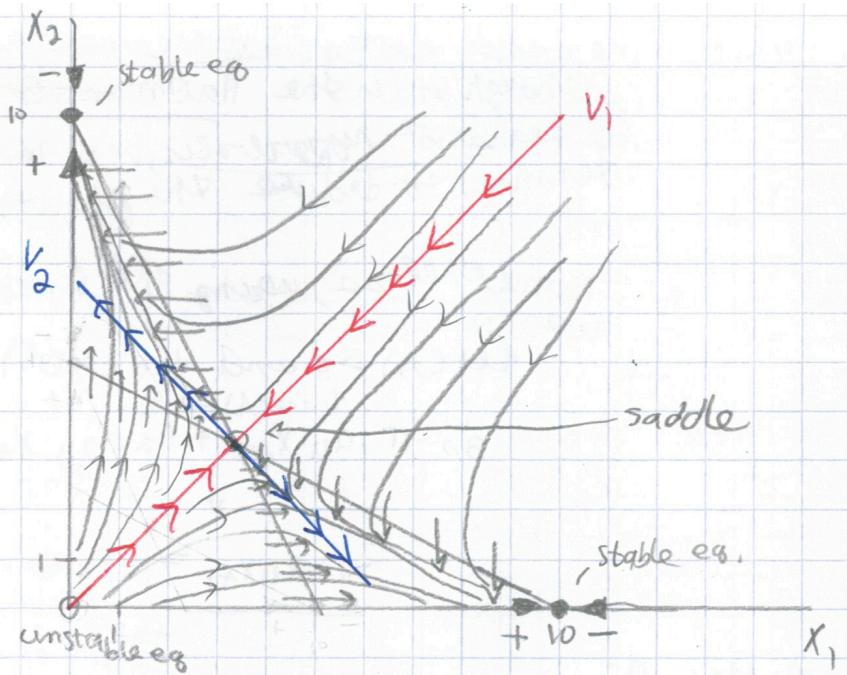
Let's show how predator-prey dynamics can result in oscillations

To simplify, let $r_1 = r_2 = r$, $x_{+1} = x_{+2} = x$, $a_{11} = a_{22} = a$, $a_{12} = -a_{21}$ (one > 0 , one < 0 \Rightarrow predator-prey)

~~$$x^2 (a^2 + a_{12}^2) > \frac{1}{4} (r a x + r a x)^2$$~~

$a^2 + a_{12}^2 > a^2$ This must be always true since $a_{12}^2 > 0$ (a_{12} is real)

Do under these predator-prey conditions we get oscillations? This can also model host-parasite interactions as one population benefits and the other suffers from the interaction.



$$\frac{dx_2}{dt} = 0 \text{ when } x_2 = 10 - \frac{(1/5)}{1/10} x_1$$

$$x_2 = 10 - 2x_1$$

$$\frac{dx_1}{dt} = 0 \text{ when } x_1 = 5 - \frac{1/10}{1/5} x_2$$

$$x_1 = 5 - x_2/2$$

1(cont)- The coexistence Eq is a center when the real part of its 0 and λ is purely imaginary, although under the Hartman-Grobman THM, since the center is not hyperbolic, we really need to consider higher order terms to deduce the flow near the fixed points.

Nonetheless, using the linearization, we get a center when

$a_{11}a_{22} > 0$

$$\det(A) > 0 \text{ and } \text{trace}(A) = 0$$

$$\text{trace}(A) > 0 \text{ so}$$

$$0 < -(\gamma_1 a_{11} X_{+1} + \gamma_2 a_{22} X_{+2}) = 0$$

↓

$$\underline{\gamma_1 a_{11} X_{+1} = -\gamma_2 a_{22} X_{+2}}$$

$$\det(A) > 0 \text{ so } a_{11}a_{22} > a_{12}a_{21}$$

$$\underline{a_{11}a_{22} - a_{12}a_{21} > 0}$$

2 conditions

Problem 2:

$$1. \text{ let } (A - \lambda I) = 0 \rightarrow \det \begin{pmatrix} -\lambda & 1 \\ -r^2 & -2r - \lambda \end{pmatrix} = (-\lambda)(-2r - \lambda) + r^2 = 0$$

| eigenvalue of
alg mult 2

$$2r\lambda + \lambda^2 + r^2 = 0$$

$$\lambda^2 + 2r\lambda + r^2 = 0$$

$$(\lambda + r)^2 = 0$$

$\lambda = -r$, algebraic multiplicity of 2

2. since $r > 0$, $\lambda < 0$, so the equilibrium
is stable

$$3. (A - \lambda I)V = 0$$

$$\begin{bmatrix} -r^2 & 1 \\ -r^2 & -r \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \therefore V = \begin{bmatrix} 1 \\ -r \end{bmatrix}$$

$$v_1 = 1 \quad v_2 = -r$$

There is only one eigenvector
that is linearly independent
so the geometric multiplicity
of $\lambda = -r$ is 1. A

4. since the geometric multiplicity of $\lambda = -r$ is less than the algebraic
multiplicity (1 < 2), A is not diagonalizable

$$1. \text{ let } t = \frac{1}{\tau} z$$

$$z = \tau t$$

$$\frac{dz}{dt} = r$$

$$\frac{d}{dz} \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1-r & \frac{1}{\tau} \end{bmatrix} \begin{bmatrix} x \\ \frac{dx}{dt} \end{bmatrix}$$

$$\therefore \frac{d}{dz} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1-r & \frac{1}{\tau} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\frac{d^2}{dt^2} x(t) = -\omega_0^2 x(t) - 2\mu \frac{1}{\tau} \frac{dx}{dt}$$

$$\frac{d^2}{dz^2} x(t) \frac{d^2}{dt^2} = -\omega_0^2 x(t) - 2\mu \frac{1}{\tau} \frac{d}{dt} x(t) \frac{d^2}{dt^2}$$

$$\frac{d^2}{dz^2} x(z) = -\omega_0^2 x - 2\mu(z) \frac{d}{dz} x \quad \text{we know that } r = \omega_0 = N$$

$$\frac{d^2}{dz^2} x = -\frac{r^2}{\tau^2} x - 2\mu \frac{d}{dz} x = -x - \frac{d}{dz} x$$

$$\therefore \frac{d}{dz} z(z) = Az(z) \text{ in which } A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

FIVE STAR. FIVE STAR. FIVE STAR.

Problem 2 (cont.):

(cont.). Switching into this time unit means that we are now measuring time in units of ω_0 and μ , the natural period of the oscillator and the period related to the effect of drag (they are equal in this critically damped case).

A unit time interval in τ represents one period
 $\tau (\Delta\tau = 1)$

of the oscillator in its natural frequency ($1/\omega_0$) or
or one period relating to the damping effect ($1/\mu$). $1/\mu$
is the rate of decay of oscillations

2,

$$k=0 A^k = A^0 = I$$

$$k=2 A^k = A^2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$k=1 A^k = A^1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}^2$$

$$A^3 = AA^2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

$$\therefore A^k = \begin{bmatrix} (-1)^{k-1}(k-1) & k(-1)^{k-1} \\ (-1)^k(k) & (k+1)(-1)^k \end{bmatrix}$$

See computer page for script

proof by induction.

1. Show true for $k=0$. We showed above

$$A^1 = \begin{bmatrix} (-1)^{0-1}(0-1) & 0(-1)^{0-1} \\ (-1)^0(0) & (0+1)(-1)^0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \checkmark$$

2. Assume true for k

$$A^k = \begin{bmatrix} (-1)^{k-1}(k-1) & k(-1)^{k-1} \\ (-1)^k(k) & (k+1)(-1)^k \end{bmatrix}$$

3. Show true for $k+1$

$$A^{k+1} = A^1 A^k = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} (-1)^{k-1}(k-1) & k(-1)^{k-1} \\ (-1)^k(k) & (k+1)(-1)^k \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^k(k) & (k+1)(-1)^k \\ -(k-1)(-1)^{k-1} & -k(-1)^{k-1} \\ -2(k)(-1)^k & -2(k+1)(-1)^k \end{bmatrix} = \begin{bmatrix} (k+1)^{-1} \\ (-1)(k+1-1) \\ (k+1)(-1)^{k+1} \end{bmatrix} \begin{bmatrix} (k+1)^{-1} \\ (k+1)(-1)^{k+1} \\ (k+1+1)(-1)^{k+1} \end{bmatrix} \quad \checkmark$$

$$(k-1)(-1)^k - 2k(-1)^k \quad k(-1)^k - 2(k+1)(-1)^k = (-1)^k + (-1)^{k+2} = (-1)^{k+1}(k+2) \quad QED$$

2 cont. ∴ By induction, pattern must hold for all $k \geq 0$

3. See computer paper

$$4. \quad k=0 \therefore A^k = (-1)^0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{0-1} \cdot 0 \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\overset{\circ}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$k=1 \therefore A^1 = (-1)^1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{1-1} \cdot 1 \cdot \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\overset{\circ}{A} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix}$$

$$\overset{\circ}{A} = A\overset{\circ}{V}$$

$$k=2 \therefore A^2 = (-1)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{2-1} \cdot 2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\overset{\circ}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + -2 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\overset{\circ}{A} = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \checkmark$$

$$k=3 \therefore A^3 = (-1)^3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-1)^{3-1} \cdot 3 \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ -3 & -3 \end{bmatrix}$$

$$\overset{\circ}{A} = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix} \checkmark$$

Do the equation matches what I have

$$5. \quad e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (At)^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k \cdot A^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k ((-1)^k I + (-1)^{k-1} \cdot k \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix})$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k (-1)^k I + \sum_{k=0}^{\infty} \frac{1}{k!} t^k \frac{(-1)^{k-1}}{(k-1)!} k \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \circ t^{k-1}$$

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} (-t)^k I + \sum_{k=1}^{\infty} t \cdot \frac{1}{(k-1)!} (-1)^{k-1} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = e^{-t} I + t \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \cdot (-t)^{k-1} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

let $\ell = k-1$ so we have

$$t \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \sum_{\ell=0}^{\infty} \frac{1}{(\ell+1)!} (-t)^\ell$$

$$e^{At} = e^{-t} \cdot I + e^{-t} \cdot t \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$e^{At} = e^{-t} \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \right)$$

The resulting function for non-diagonalizable A differs from the general form when A is diagonalizable because we now have a polynomial term in time in addition to the regular exponential term that we always have. Allowing for polynomial terms in time allows for additional solution behaviors for non-diagonalizable matrices.

6. even though A is non-diagonalizable, my stability characterization from the eigenvalues was still correct. we see this here because the exponential term always dominates over the polynomial term over long time scales, so the polynomial term ultimately doesn't matter for determining stability. All that matters for determining stability are the eigenvalues, which we found was $-r$ (or -1 after nondimensionalizing). we see this here as we have $e^{-\lambda t} = e^{-t}$ in the front of our solution, so overall stability is still determined by our eigenvalue

7. See computer pages

Homework 3

CAAM 28200: Dynamical Systems with Applications

Kameel Khabaz

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Problem 2: Section 2.2

I wrote the script below to show that my pattern of matrix exponentiation holds out to $k = 10$.
The pattern is the following:

$$A^k = \begin{bmatrix} (k-1)(-1)^{k-1} & k(-1)^{k-1} \\ k(-1)^k & (k+1)(-1)^k \end{bmatrix}$$

My code and output are below:

```
1 % Problem 2 Script for matrix exponential
2 % Compare multiplication and with pattern
3 A = [0 1;
4      -1 -2];
5
6 for k = 0:10
7     disp("Exponential A^" + k);
8     disp(A^k)
9     disp("Exponential A^" + k + " Using Pattern");
10    disp(pattern(k))
11 end
12
13 function [A] = pattern(k)
14     A = [(-1)^(k-1) * (k-1)    k * (-1)^(k-1);
15           (-1)^k * k            (k+1) * (-1)^k];
16 end
1
2 Exponential A^0
3      1      0
4      0      1
```

```
5 Exponential A^0 Using Pattern
6      1      0
7      0      1
8
9 Exponential A^1
10     0      1
11     -1     -2
12
13 Exponential A^1 Using Pattern
14     0      1
15     -1     -2
16
17 Exponential A^2
18     -1     -2
19     2      3
20
21 Exponential A^2 Using Pattern
22     -1     -2
23     2      3
24
25 Exponential A^3
26     2      3
27     -3     -4
28
29 Exponential A^3 Using Pattern
30     2      3
31     -3     -4
32
33 Exponential A^4
34     -3     -4
35     4      5
36
37 Exponential A^4 Using Pattern
38     -3     -4
39     4      5
40
41 Exponential A^5
42     4      5
43     -5     -6
```

```
44
45 Exponential A^5 Using Pattern
46      4      5
47      -5     -6
48
49 Exponential A^6
50      -5     -6
51      6      7
52
53 Exponential A^6 Using Pattern
54      -5     -6
55      6      7
56
57 Exponential A^7
58      6      7
59      -7     -8
60
61 Exponential A^7 Using Pattern
62      6      7
63      -7     -8
64
65 Exponential A^8
66      -7     -8
67      8      9
68
69 Exponential A^8 Using Pattern
70      -7     -8
71      8      9
72
73 Exponential A^9
74      8      9
75      -9    -10
76
77 Exponential A^9 Using Pattern
78      8      9
79      -9    -10
80
81 Exponential A^10
82      -9   -10
```

```

83      10      11
84
85 Exponential A^10 Using Pattern
86      -9      -10
87      10      11

```

Problem 2: Section 2.3

I now tested plotting solution trajectories for an n-term approximation for the matrix exponential. My code and plots are shown below:

```

17 %% Critically Damped Oscillator Matrix Exponential Approximation
18 A = [0 1;
19      -1 -2];
20
21 % Solution trajectories are given by y(t) = expo(At) * y(0)
22 close all
23 figure();
24 set(gcf,'Position',[0 0 800 1000])
25 t = linspace(0,5,100);
26 z0 = [1; 1];
27 solns = nan(2,length(t));
28
29 subplot(3,2,1);
30 n = 1
31 solve_plot(n,A,t,z0);
32
33 subplot(3,2,2);
34 n = 3
35 solve_plot(n,A,t,z0);
36
37 subplot(3,2,3);
38 n = 5
39 solve_plot(n,A,t,z0);
40
41 subplot(3,2,4);
42 n = 7
43 solve_plot(n,A,t,z0);
44
45 subplot(3,2,5);

```

```

46 n = 9
47 solve_plot(n,A,t,z0);
48
49 subplot(3,2,6);
50 n = 10
51 solve_plot(n,A,t,z0);
52 exportgraphics(gcf,"n_approx_trajs.eps")
53 function solve_plot(n,A,t,z0)
54     for i = 1:length(t)
55         solns(:,i) = soln_exp_approx(A,t(i),n,z0);
56     end
57     hold on
58     di = min(find(abs(solns(1,:)) > 2))
59     xline(t(di), '--r', 'LineWidth', 1.5)
60     h1= plot(t,solns(1,:), 'Color', 'k', 'LineWidth', 2);
61     h2 = plot(t,solns(2,:), 'Color', 'k', 'LineWidth', 2, 'LineStyle', '--');
62     title("$ n = " + n + " $" , 'Interpreter', 'latex')
63     set(gca, 'FontSize', 30, 'FontName', 'times')
64     xlabel("$t$ ", 'Interpreter', 'latex')
65     ylabel('$z(\tau)$ ', 'Interpreter', 'latex')
66     lgd = legend([h1 h2], "z_1","z_2")
67     lgd.Location = "northwest";
68     lgd.FontSize = 18;
69
70 end
71
72 function [soln] = soln_exp_approx(A,t,n,z0)
73     expA = [1 0; 0 1];
74     for k = 1:n
75         expA = expA + (1/factorial(k)) .* (A .* t )^k;
76     end
77     soln = expA * z0;
78 end

```

In the plot in Figure 1, I approximated the matrix exponential $\sum_{k=0}^n \frac{1}{k!} (At)^k$ for $n = 1$ to $n = 10$ and plotted trajectories starting from $z(0) = [1, 1]$, which corresponds to an initial position and velocity of 1. I then plotted the z_1 , or position, variable as a solid line and z_2 , or the velocity variable, as a dashed line. I then plotted the time point at which the solution diverges (chosen simply as the position exceeding twice the initial position) as a red dashed vertical line. This figure clearly shows

that as n increases, the red line shifts to the right, which means that the n -term matrix exponential approximation converges towards the true trajectory for more time. At early times, as n increases, the approximate trajectory converges more towards the true trajectory (which we know from a critically damped oscillator should be a consistent decrease in both the position and velocity of the oscillator). However, for all of these approximations, regardless of n , the trajectory diverges at later times. This is because an exponential is a power series, and infinitely many terms are needed. Here, we are cutting off that series and including only a small number of terms, so what we get is definitely an approximation. Furthermore, each additional term of the approximation represents a higher-order behavior of the exponential function that contributes a smaller and smaller amount to the overall approximation. Thus, at early times, the function's behavior can be captured with only lower-order terms, but as one goes farther and farther from the $t = 0$ point, higher order terms are needed. We can see this in Figure 2, in which I include 100 terms for the approximation and obtain a much more accurate solution trajectory up to $t = 10$. However, even here, we see that if we increase the time enough, as in the right figure, the approximation will eventually diverge.

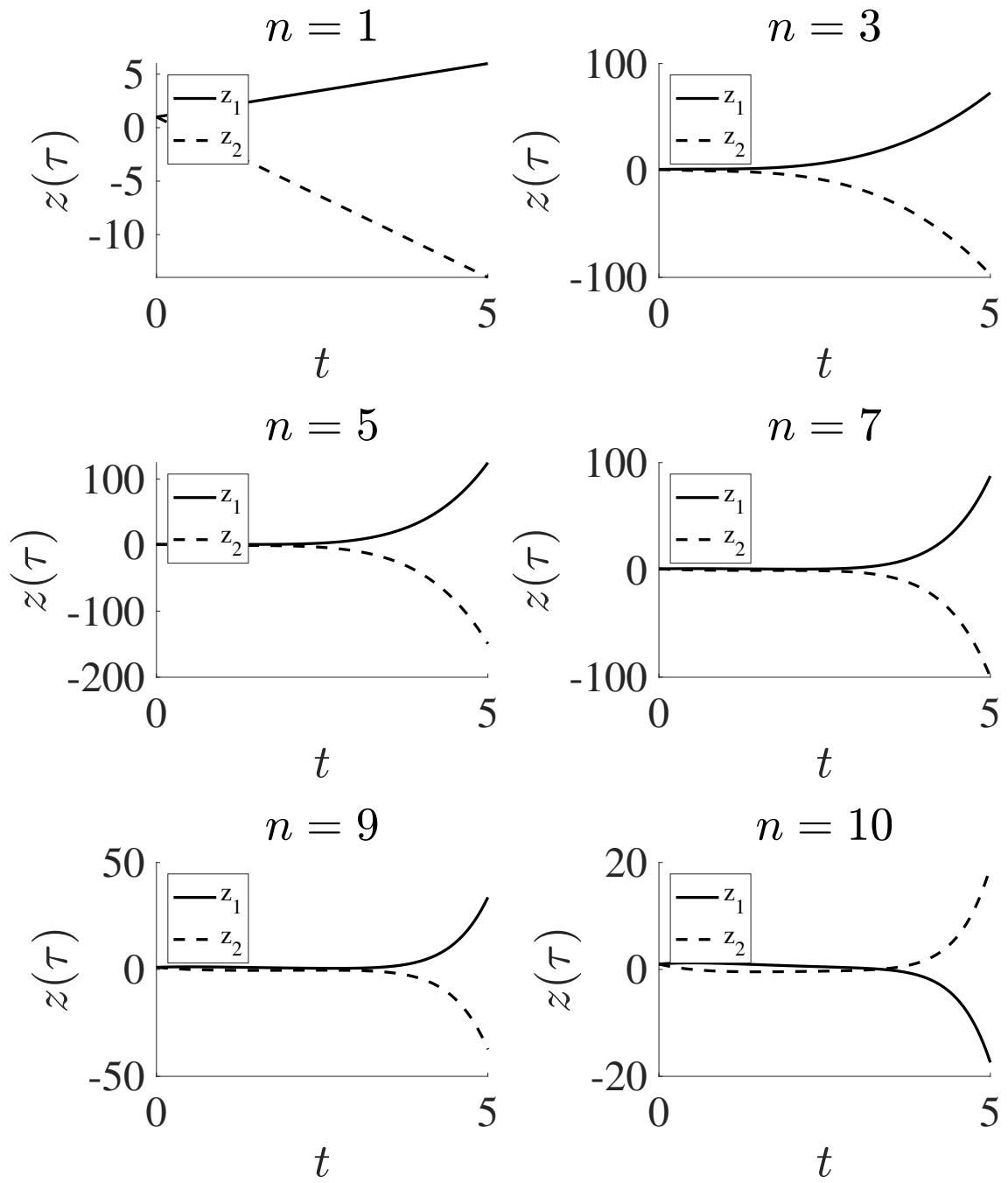


Figure 1: n -term Approximation Solution Trajectories from $z(0) = [1, 1]$

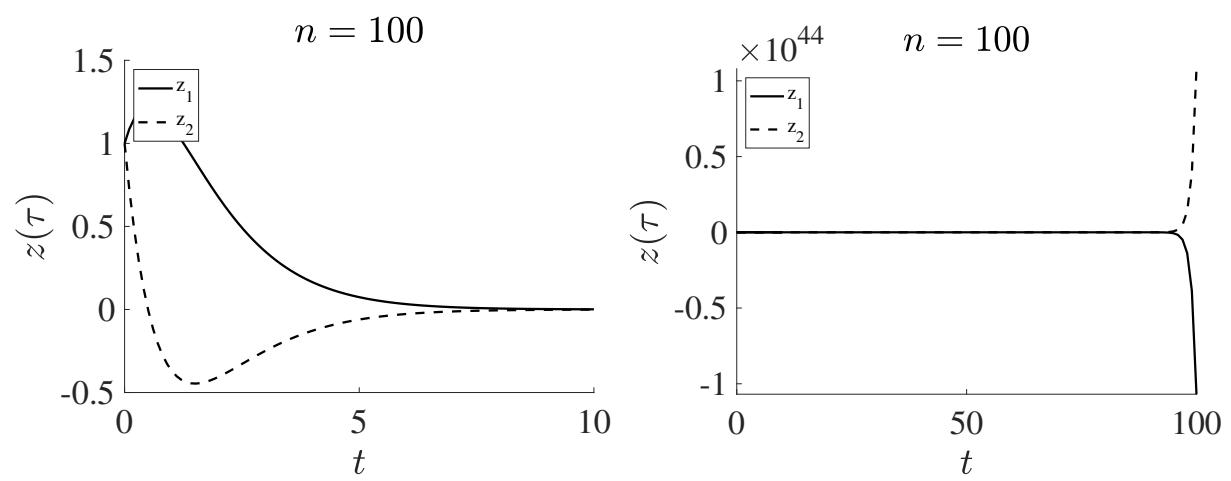


Figure 2: 100-term Approximation Solution Trajectories

Problem 2: Section 2.7

Below, I show my code and plots using the explicit form of the matrix exponential (defined in number 5) to plot the exact solution starting from $z(0) = [1, 1]$.

```
79 %% Problem 2.7 - Solution with Explicit Form of Matrix Exponential
80 figure()
81 t = linspace(0,10,100);
82 explicit_soln_plot(A,t,z0)
83 exportgraphics(gcf,"explicit_form_soln.eps")
84
85 t = linspace(0,100,100);
86 explicit_soln_plot(A,t,z0)
87 exportgraphics(gcf,"explicit_form_soln_lt.eps")
88 function explicit_soln_plot(A,t,z0)
89     solns = nan(2,length(t))
90     for i = 1:length(t)
91         solns(:,i) = exp(-t(i)) * ([1 0; 0 1] + t(i) * [1 1; -1 -1])
92             * z0;
93     end
94     hold on
95     di = min(find(abs(solns(1,:)) > 2))
96     if isinteger(di)
97         xline(t(di), '--r', 'LineWidth', 1.5)
98     end
99     h1= plot(t,solns(1,:), 'Color', 'k', 'LineWidth', 2);
100    h2 = plot(t,solns(2,:), 'Color', 'k', 'LineWidth', 2, 'LineStyle', '--');
101    title("Explicit Form of Matrix Exponential", 'Interpreter', 'latex')
102    set(gca, 'FontSize', 30, 'FontName', 'times')
103    xlabel("$t$", 'Interpreter', 'latex')
104    ylabel('$z(\tau)$', 'Interpreter', 'latex')
105    lgd = legend([h1 h2], "z_1","z_2")
106    lgd.Location = "northeast";
107    lgd.FontSize = 18;
108 end
```

When comparing Figure 2 to Figure 3, we see that using the explicit form for the matrix exponential results in a solution trajectory that matches the expected trajectory even at long time spans. While

using an partial expansion of the matrix exponential means that the excluded higher-order terms result in a solution diverging at large time scales, using the exact form of the exponential includes all higher-order terms and thus results in the correct trajectory. Note how both the approximate solutions (with large n) and the exact solution result in similar behavior at small time spans (such as up to $t = 10$ for $n = 100$).

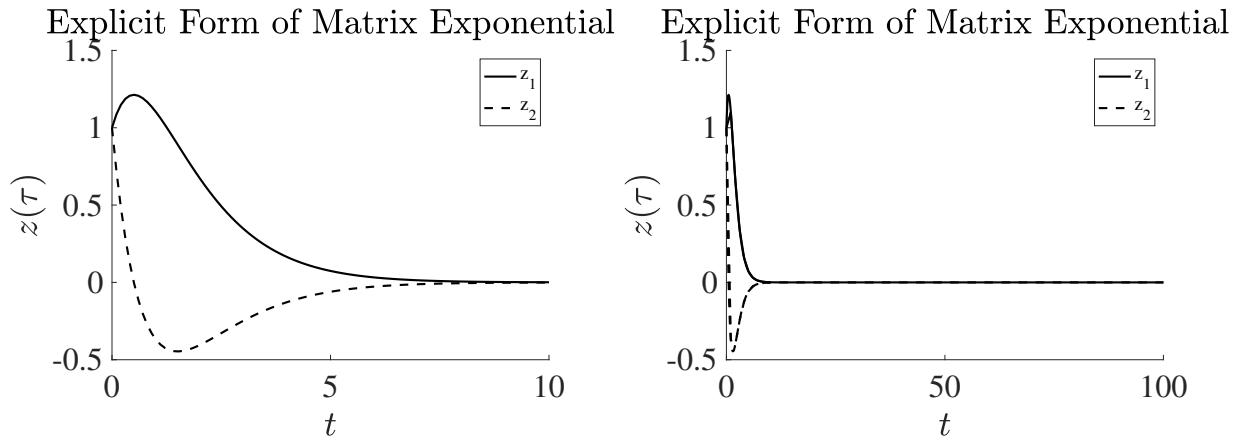


Figure 3: Exact Solution Trajectories