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HW3
2/18/22

1. horizontal nullcline is when $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ so x_2 is constant but x_1 is changing.

This is when $\frac{d}{dt} x_2 = 0 = r_2 x_2 (1 - a_{21} x_1 - a_{22} x_2)$

4 nullclines

$x_2 = 0$ (extinction)

$\frac{1 - a_{21} x_1}{a_{22}} = \cancel{a_{22}}\, x_2$ $\quad x_2 = \frac{1}{a_{22}} - \frac{a_{21}}{a_{22}} x_1$

vertical nullcline when $\frac{d}{dt} x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ so x_1 is constant and x_2 is changing,

So this is when $\frac{d}{dt} x_1 = 0 = r_1 x_1 (1 - a_{11} x_1 - a_{12} x_2)$

$x_1 = 0$ (extinction)

$x_2 = \frac{1}{a_{12}} - \frac{a_{11}}{a_{12}} x_1$ $\quad x_1 = \frac{1 - a_{12} x_2}{a_{11}}$

Equilibrium when $\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, Also when $x = \begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}$ or $x = \begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}$

Such as $x_1 = 0, x_2 = 0$ (extinction) (one population is extinct)

or when $x_2 = \frac{1 - a_{11} x_1}{a_{12}} = \frac{1 - a_{21} x_1}{a_{22}} \rightarrow a_{22} - a_{11} a_{22} x_1 = a_{12} - a_{12} a_{21} x_1$

$a_{22} - a_{12} = (a_{11} a_{22} - a_{12} a_{21}) x_1$

$x_1 = \frac{a_{22} - a_{12}}{a_{11} a_{22} - a_{12} a_{21}}$

and $x_2 = \frac{1}{a_{12}} - a_{11}(\quad)$

(intersection of 2 lines)

if $x_2 = 0$,

$f = \frac{d}{dt} x_1 = r_1 x_1 (1 - a_{11} x_1)$

when $x_1 = 1/a_{11}$ $\frac{dx_1}{dt} = 0$

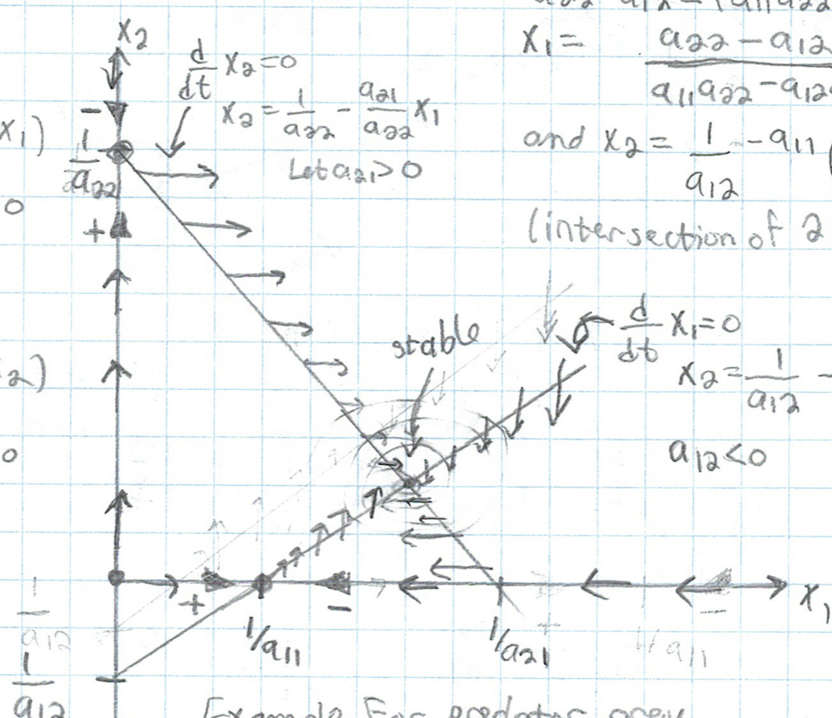
(carrying capacity)

if $x_1 = 0$

$\frac{d}{dt} x_2 = r_2 x_2 (1 - a_{22} x_2)$

if $x_2 = 1/a_{22}$ $\frac{dx_2}{dt} = 0$

(carrying capacity)



Example for predator prey interaction

$(a_{12} < 0, a_{21} > 0)$

(population 1 is predator, 2 is prey)

Equilibria at

$x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

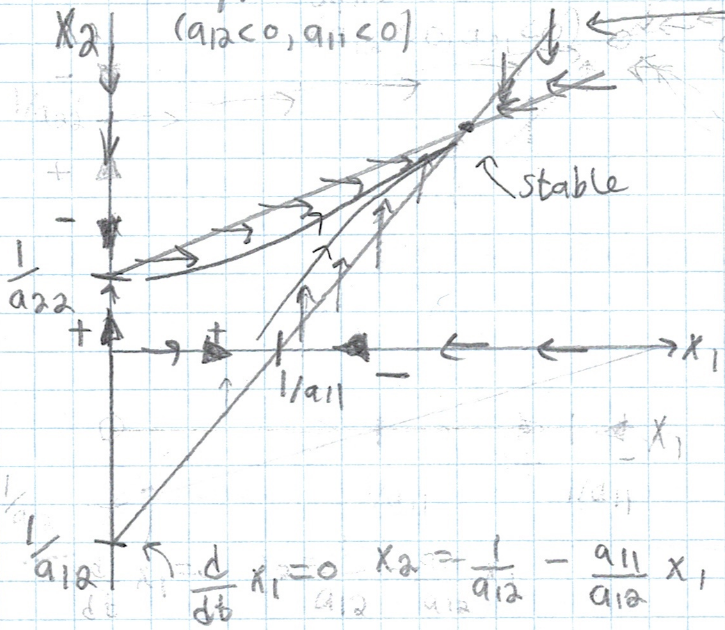
$x = \begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}$

$x = \begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}$

and the intersection of 2 lines

1; (1 cont.)

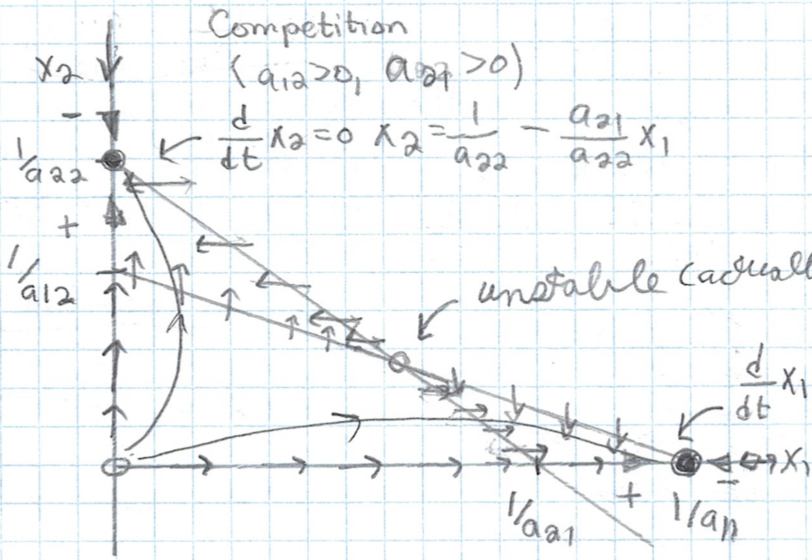
Symbiosis ($a_{12}<0, a_{11}<0$)



Equilibrium here if $\left|\frac{a_{11}}{a_{12}}\right|<\left|\frac{a_{21}}{a_{22}}\right|$ then there would be no equilibrium here

Here $\left|\frac{a_{11}}{a_{12}}\right|>\left|\frac{a_{21}}{a_{22}}\right|$ (so population 2 can support more individuals)

Competition ($a_{12}>0, a_{21}>0$)



2. Equilibria are $x=\begin{bmatrix}0\\0\end{bmatrix}$, $x=\begin{bmatrix}1/a_{11}\\0\end{bmatrix}$ $x=\begin{bmatrix}0\\1/a_{22}\end{bmatrix}$ and when lines intersect (coexistance equilibrium)

$$Ax=\begin{bmatrix}1\\1\end{bmatrix}\to\begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$x=A^{-1}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\frac{1}{a_{11}a_{22}-a_{12}a_{21}}\begin{bmatrix}a_{22}&-a_{12}\\-a_{21}&a_{11}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

so
$$\boxed{x_1=\frac{-a_{12}+a_{22}}{a_{11}a_{22}-a_{21}a_{12}} \qquad x_2=\frac{-a_{21}+a_{11}}{a_{11}a_{22}-a_{21}a_{12}}}$$

Last equilibrium

3. The diagonal entries define the carrying capacities as the 2 equilibria with extinction of only one population are

$$X = \begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}$$

4. need $\dfrac{a_{22}-a_{12}}{a_{11}a_{22}-a_{21}a_{12}} > 0$ and $\dfrac{a_{11}-a_{21}}{a_{11}a_{22}-a_{21}a_{12}} > 0$.

if $\det(A) \geq 0$, we need $a_{22} > a_{12}$ and $a_{11} > a_{21}$

a_{22} and $a_{11} > 0$, so either $a_{12} > 0$ and $|a_{12}| < a_{22}$ or $a_{12} < 0$ and $a_{21} > 0$ and $|a_{21}| < a_{11}$ or $a_{21} < 0$.

So coexistence is viable always if $a_{21} < 0$ and $a_{11} < 0$

- If a_{21} and $a_{11} > 0$ (or either is > 0) then they must be smaller in magnitude than the corresponding diagonal term, meaning that the population must have a weaker suppressive effect on the other population than on itself.

with the nullclines, if $a_{12} < 0$ and $a_{11} < 0$ the intersection and thus equilibrium must be in the 1st quadrant $(+x_1, +x_2)$. we see this in the symbiosis graph. This is because both nullclines must intersect the exclusion equilibria and have positive slope, so they intersect each other at $x_1 > 0$ $x_2 > 0$.

- If $a_{12} < 0$ and $a_{21} > 0$ (predator-prey, as in my graph), then if a_{21} is too large, the $\frac{dx_2}{dt} = 0$ nullcline will be too steep and intersect the other nullcline at $x_2 < 0$, so the 2 populations can't intersect, so coexistance would not be viable.

suppressive effect of pop 1 on pop 2.

The case is analogous if a_{12} is too large when $a_{12} > 0$, $a_{21} < 0$.

- If $a_{12} > 0$, $a_{21} > 0$ (competition) if $a_{21} > a_{11}$ (as in my graph) then the coexistance equilibrium is unstable so any perturbation will cause the system to move away from the coexistance equilibrium

See next page for more

5. $J(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix}$

$f_1 = \frac{dx_1}{dt} = r_1 x_1 - r_1 a_{11} x_1^2 - r_1 a_{12} x_1 x_2$

$\partial_{x_1} f_1 = r_1 - 2 r_1 a_{11} x_1 - r_1 a_{12} x_2$

$\partial_{x_2} f_1 = -r_1 a_{12} x_1$

$f_2 = \frac{dx_2}{dt} = r_2 x_2 - r_2 a_{21} x_1 x_2 - r_2 a_{22} x_2^2$

$\partial_{x_2} f_2 = r_2 - r_2 a_{21} x_1 - 2 r_2 a_{22} x_2$

$\partial_{x_1} f_2 = -r_2 a_{21} x_2$

$$\therefore J(x) = \begin{bmatrix} r_1 - 2r_1 a_{11} x_1 - r_1 a_{12} x_2 & -r_1 a_{12} x_1 \\ -r_2 a_{21} x_2 & r_2 - r_2 a_{21} x_1 - 2 r_2 a_{22} x_2 \end{bmatrix}$$

5 (cont).

$$\text{So, } J(x) = \begin{bmatrix} r_1 - r_1 a_{11} x_1 - r_1 a_{12} x_2 & 0 \\ 0 & r_2 - r_2 a_{21} x_1 - r_2 a_{22} x_2 \end{bmatrix} + \begin{bmatrix} -r_1 a_{11} x_1 & -r_1 a_{12} x_1 \\ -r_2 a_{21} x_2 & -r_2 a_{22} x_2 \end{bmatrix}$$

$$\therefore J(x) = \begin{bmatrix} r_1(1 - a_{11}x_1 - a_{12}x_2) & 0 \\ 0 & r_2(1 - a_{21}x_1 - a_{22}x_2) \end{bmatrix} + \begin{bmatrix} -r_1 a_{11} x_1 & -r_1 a_{12} x_1 \\ -r_2 a_{21} x_2 & -r_2 a_{22} x_2 \end{bmatrix}$$

$$J\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix} \qquad J\left(\begin{bmatrix} 1/a_{11} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} r_1 - r_1 \cancel{a_{11} \cdot 1/a_{11}} - r_1 \cancel{a_{11} \cdot \frac{1}{a_{11}}} & -\frac{r_1 a_{12}}{a_{11}} \\ \cancel{0} & r_2 - \frac{a_{21}}{a_{11}} r_2 \quad 0 \end{bmatrix}$$

$$J\left(\begin{bmatrix} 0 \\ 1/a_{22} \end{bmatrix}\right) = \begin{bmatrix} r_1 - r_1 a_{12}/a_{22} & 0 \\ -r_2 a_{21}/a_{22} & -r_2 \end{bmatrix} \qquad = \begin{bmatrix} -r_1 & -r_1 a_{12}/a_{11} \\ 0 & r_2 - \frac{r_2 a_{21}}{a_{11}} \end{bmatrix}$$

In these cases the Jacobian is triangular

At the coexistance equilibrium

$$J(x) = \begin{bmatrix} r_1 \left[\frac{\cancel{(a_{11}a_{22} - a_{21}a_{12})} \cancel{- a_{11}(a_{22} + a_{12})}^{+a_{11}a_{12}} \cancel{- a_{12}(a_{11} - a_{21})}^{-a_{12}a_{11}}}{a_{11}a_{22} - a_{21}a_{12}} \right] & 0 \\ 0 & \frac{(\cancel{a_{11}a_{22}} - \cancel{a_{21}a_{12}}) - a_{21}\cancel{a_{22}} + \cancel{a_{21}}a_{12} - a_{2\cancel{2}}a_{11} + a_{\cancel{22}}a_{21}}{a_{11}a_{22} - a_{21}a_{12}} \end{bmatrix}$$

$+$ (2nd matrix)

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -r_1 a_{11} x_1 & -r_1 a_{12} x_1 \\ -r_2 a_{21} x_2 & -r_2 a_{22} x_2 \end{bmatrix}$$

So the 1st matrix vanishes at coexistance Eq.

$$\therefore J(x) = \begin{bmatrix} -r_1 a_{11}\left(\frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{21}a_{12}}\right) & -r_1 a_{12}\left(\frac{a_{22} - a_{12}}{a_{11}a_{22} - a_{21}a_{12}}\right) \\ -r_2 a_{21}\left(\frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{21}a_{12}}\right) & -r_2 a_{22}\left(\frac{a_{11} - a_{21}}{a_{11}a_{22} - a_{21}a_{12}}\right) \end{bmatrix}$$

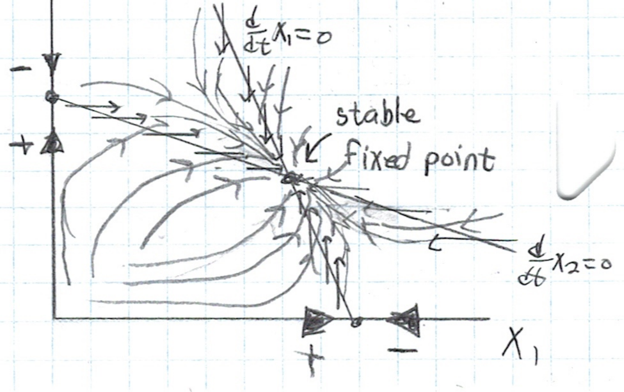
at coexistance equilibrium

4 (cont). if $a_{12} > 0$ and $a_{21} > 0$ (competition), then we can get a stable equilibrium if the slope of the $0 = \frac{dx_2}{dt}$ nullcline is less than the slope of the $0 = \frac{dx_1}{dt}$ nullcline $\Rightarrow \frac{a_{21}}{a_{22}} < \frac{a_{11}}{a_{12}}$

$$\Rightarrow a_{21}a_{12} < a_{22}a_{11}$$

So the coexistance eq. is stable if each population has a smaller suppressive effect on the other population than on itself ($a_{12}a_{21} < a_{11}a_{22}$)

if $\frac{a_{21}}{a_{22}} < \frac{a_{11}}{a_{12}}$ ($\frac{dx_1}{dt} = 0$ nullcline has larger slope)



6. $J\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}$ eigenvalues are $(r_1-\lambda)(r_2-\lambda)=0$

$\lambda = r_1, r_2$

so eigenvalues are r_1, r_2 since $r_1>0$ and $r_2>0$ this equilibrium is always unstable (source).

7.

$$J\left(\begin{bmatrix}1/a_{11}\\0\end{bmatrix}\right) = \begin{bmatrix} -r_1 & -r_1a_{12}/a_{11} \\ 0 & r_2 - r_2a_{21}/a_{11} \end{bmatrix} = A \quad \det(A-\lambda I)=0$$

$$(-r_1-\lambda)\left(r_2 - \frac{r_2a_{21}}{a_{11}} - \lambda\right) = 0$$

$\lambda_1 = -r_1$ ($r_1>0$ so $\lambda_1<0$)

$\lambda_2 = r_2 - \dfrac{r_2a_{21}}{a_{11}}$

$\lambda_2 = r_2(1-a_{21}/a_{11})$

if $a_{21}/a_{11} < 1$ then this eigenvalue is positive ($a_{21}<a_{11}$)

we showed in #4 that if $a_{21}<a_{11}$ and $\det(A)>0$, then coexistance is viable. Here we showed that for this extinction equilibrium to be a saddle, we need λ_2 to be positive which requires that $a_{21}<a_{11}$ (the same requirement as before). So if $\det(A)>0$ ad coexistance is viable, then $a_{22}<a_{11}$, then the equilibrium is a saddle. If $\det(A)>0$ and the equilibrium is a saddle, the coexistance is viable.

we apply similar logic to the other equilibrium:

$$J\left(\begin{bmatrix}0\\1/a_{22}\end{bmatrix}\right) = \begin{bmatrix} r_1 - r_1a_{12}/a_{22} & 0 \\ -r_2a_{21}/a_{22} & -r_2 \end{bmatrix} = A \quad \det(A-\lambda I)=0$$

$$(r_1 - r_1a_{12}/a_{22} - \lambda)(-r_2-\lambda) = 0$$

$\lambda_1 = r_1 - r_1a_{12}/a_{22}$

$\lambda_1 = r_1(1-a_{12}/a_{22})$

$\lambda_2 = -r_2$ ($r_2>0$ so $\lambda_2<0$)

if $a_{12}/a_{22}<1$, then $\lambda_1>0$ ($a_{12}<a_{22}$)

we see that for this equilibrium to be a saddle, we need $a_{12}<a_{22}$. In #4, this was the other requirement for coexistance to be viable if $\det(A)>0$.

So this means if $a_{12}<a_{11}$ and $a_{12}<a_{22}$, then both exclusion equilibria are saddles and if $\det(A)>0$, coexistance is then viable. And if $\det(A)>0$ and coexistance is viable, then $a_{12}<a_{22}$ and $a_{21}<a_{11}$, so both exclusion equilibria must be saddles.

now we find eigenvectors for stable perturbations:

For $\lambda_1 = -r_1$ at $x = [1/a_{11} \;\; 0]$,

$(A-\lambda I)v = 0$

$$\begin{bmatrix} -r_1 & -r_1a_{12}/a_{11} \\ 0 & r_2 - r_2a_{21}/a_{11} \end{bmatrix} - (-r_1)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix} 0 & -r_1a_{12}/a_{11} \\ 0 & r_1+r_2-r_2a_{21}/a_{11} \end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix} = 0$$

$\therefore v_2 = 0$, v_1 can be anything

so stable perturbations occur $V = \begin{bmatrix}1\\0\end{bmatrix}$ when only the population of the species that doesn't go extinct is changed. ~~The exclusion equilibrium~~

7(cont). The exclusion equilibrium is stable along the direction with the extinct species remaining extinct.

For $\lambda = -r_2$ for $x = [0 \ \ 1/a_{22}]$, $(A-\lambda I)v = \begin{bmatrix} r_2+r_1-r_1 a_{12}/a_{22} & 0 \\ -r_2 a_{21}/a_{22} & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$

$\therefore v_1 = 0$ v_2 can be anything $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Similarly, the direction of stable perturbation is with the extinct species remaining extinct. If only the non-extinct population is perturbed, then the exclusion equilibrium is stable, so the stable manifold is a curve that is tangent to the stable eigenvector. Since the stable manifold is when the extinct species remains extinct, in this case, the stable manifold is a line so the stable eigenvector is the stable manifold.

8. In #5 I wrote down the Jacobian at the coexistance equilibrium. If we plug $r_1 = r_2 = 1$ and $a_{11} = a_{22} = 1/10$,

$$J(x_*) = \begin{bmatrix} -1/10\left(\dfrac{1/10 - a_{12}}{1/100 - a_{21}a_{12}}\right) & -a_{12}\left(\dfrac{1/10 - a_{12}}{1/100 - a_{21}a_{12}}\right) \\ -a_{21}\left(\dfrac{1/10 - a_{21}}{1/100 - a_{21}a_{12}}\right) & -1/10\left(\dfrac{1/10 - a_{21}}{1/100 - a_{21}a_{12}}\right) \end{bmatrix}$$

Let $a_{12} = a_{21} = -1/30$ cooperation

$$J(x_*) = \begin{bmatrix} -1/10\left(\dfrac{1/10 + 1/30}{1/100 - 1/900}\right) & 1/30\left(\dfrac{1/10 + 1/30}{1/100 - 1/900}\right) \\ 1/30\left(\dfrac{1/10 + 1/30}{1/100 - 1/900}\right) & -1/10\left(\dfrac{1/10 + 1/30}{1/100 - 1/900}\right) \end{bmatrix} = \begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$$

$\det(J(x_*) - \lambda I) = 0$ $\quad J(x_*) - \lambda I = \begin{bmatrix} -3/2-\lambda & 1/2 \\ 1/2 & -3/2-\lambda \end{bmatrix}$

$\det(J(x_*) - \lambda I) = (-3/2 - \lambda)^2 - (1/2)^2 = 0$

$\lambda^2 + 3\lambda + \cancel{9/4 - 1/4}\,2 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+2)(\lambda+1) = 0$

$\lambda = -2, -1$

Both eigenvalues are negative so coexistence is stable in this case.

Let $a_{12} = a_{21} = 1/30$ competition

$$J(x_*) = \begin{bmatrix} -1/10\left(\dfrac{1/10 - 1/30}{1/100 - 1/900}\right) & -1/30\left(\dfrac{1/10 - 1/30}{1/100 - 1/900}\right) \\ -1/30\left(\dfrac{1/10 - 1/30}{1/100 - 1/900}\right) & -1/10\left(\dfrac{1/10 - 1/30}{1/100 - 1/900}\right) \end{bmatrix} = \begin{bmatrix} -3/4 & -1/4 \\ -1/4 & -3/4 \end{bmatrix}$$

$$\det(J(x_*) - \lambda I) = \det\left(\begin{bmatrix} -3/4-\lambda & -1/4 \\ -1/4 & -3/4-\lambda \end{bmatrix}\right) = 0$$

$(-3/4 - \lambda)^2 - (-1/4)^2 = 0$

$\lambda^2 + \frac{3}{2}\lambda + \frac{9}{16} - \frac{1}{16} = 0$ $\quad \lambda = -1, -1/2$

Both eigenvalues are negative, so coexistance is still stable. However, the eigenvalues are now less negative, so coexistance is less stable. see next page

(8 cont.). This makes sense because the coexistance equilibrium is less stable in the competitive case ($a_{12}=a_{21}=1/30$) than in the cooperative case ($a_{12}=a_{21}=-1/30$) because when 2 populations are competing, they are more sensitive to competitive exclusion, in which one population competes the other out to extinction. Since the eigenvalues for the competitive case are smaller, this case has a <u>slower</u> rate of convergence. ← less negative

9. Let $a_{12}=a_{21}=1/5$

$$J(x_*) = \begin{bmatrix} -1/10\left(\frac{1/10-1/5}{1/100-1/25}\right) & -1/5\left(\frac{1/10-1/5}{1/100-1/25}\right) \\ -1/5\left(\frac{1/10-1/5}{1/100-1/25}\right) & -1/10\left(\frac{1/10-1/5}{1/100-1/25}\right) \end{bmatrix} = \begin{bmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{bmatrix}$$

$\det(J(x_*) - \lambda I) = 0$

$$(-1/3-\lambda)^2 - (-2/3)^2 = 0$$

$$\lambda^2 + (2/3)\lambda + \underbrace{\cancel{1/9 - 4/9}}_{-1/3} = 0$$

$$(\lambda+1)(\lambda-1/3) = 0$$

$$\lambda = -1,\ 1/3$$

$(-1, +1/3)$

Here the eigenvalues are of mixed sign (one > 0 and one < 0). The coexistance equilibrium is now a <u>saddle</u>. So increasing competition coefficients made the eigenstate <u>less</u> stable because there is now an unstable direction. So now if the trajectory approaches from any direction except from the stable eigenvector it will move around the fixed point along the unstable eigenvector to competitive exclusion (extinction of 1 species).

$\lambda_1 = -1$

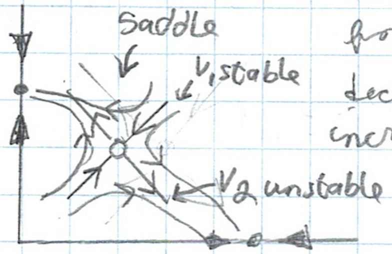
$$(A-\lambda_1 I)v = \begin{bmatrix} \cancel{-1/3-(-1)}\ 2/3 & -2/3 \\ -2/3 & \cancel{-1/3-(-1)}\ 2/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ The equilibrium is <u>stable</u> along the direction in which the perturbation in the population is <u>equal</u> for the 2 species (both inc or dec).

$\lambda_2 = +1/3$

$$(A-\lambda_2 I)v = \begin{bmatrix} \cancel{-1/3-1/3}\ -2/3 & -2/3 \\ -2/3 & \cancel{-1/3-1/3}\ -2/3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ The equilibrium is <u>unstable</u> along the direction in which the perturbation is <u>unequal</u> for the 2 species (one increases and the other decreases), this makes sense as the population that increases due to perturbation will continue to increase and out compete the other population that decreases due to the perturbation.



(9 cont.) Increasing the competition coefficients decreased the stability of the coexistance state because each population now has a stronger suppressive effect on the other population than on itself

10, case 1: $a_{12} = a_{21} = -1/30$ (symbiosis)

In #8, we had gotten $\lambda = -2, -1$

$$A = \begin{bmatrix} -3/2 & 1/2 \\ 1/2 & -3/2 \end{bmatrix}$$

$(A - \lambda I)\,v = 0$

For $\lambda_1 = -2$,

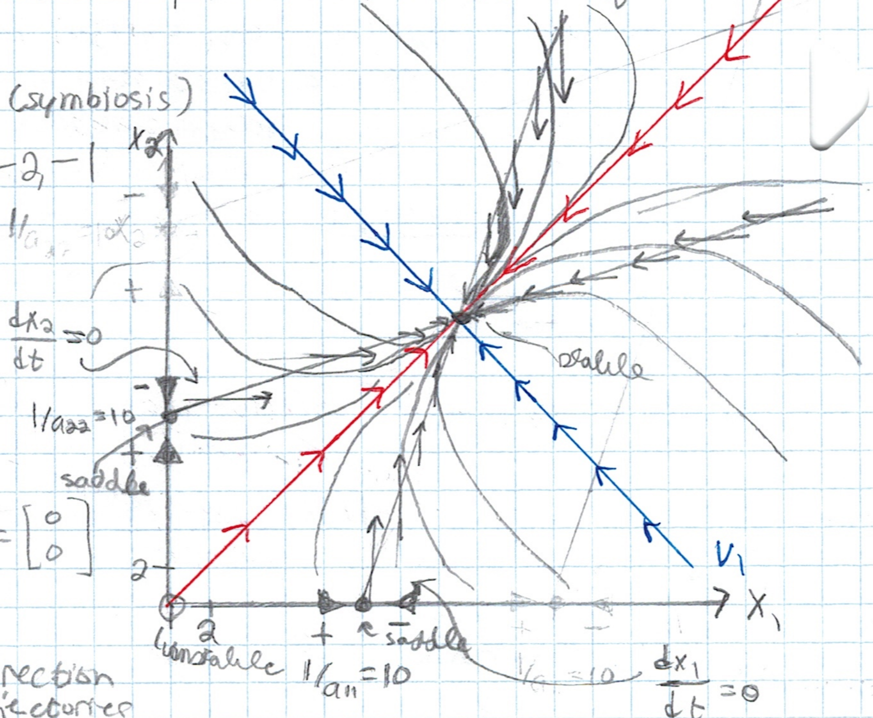
$$\begin{bmatrix} \cancel{-3/2-(-2)}\ 1/2 & 1/2 \\ 1/2 & \cancel{-3/2-(-2)}\ 1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ← $|\lambda_1| > |\lambda_2|$ ∴ fast eigendirection as $t \to -\infty$, trajectories diverge along this direction (∥ to v_1 far from fixed point)

For $\lambda_2 = -1$,

$$\begin{bmatrix} \cancel{-3/2+1}\ -1/2 & 1/2 \\ 1/2 & \cancel{-3/2+1}\ -1/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ← slow eigendirection near the fixed point, trajectories approach from here



$\frac{dx_2}{dt} = 0$ when $x_2 = \frac{1}{a_{22}} - \frac{a_{21}}{a_{22}} x_1$

$x_2 = 10 - \frac{(-1/30)}{1/10} x_1 = 10 + \frac{x_1}{3}$

$\frac{dx_1}{dt} = 0$ when $x_2 = \frac{1}{a_{12}} - \frac{a_{11}}{a_{12}} x_1$

$x_2 = -30 - \frac{1/10}{-1/30} x_1 = -30 + 3x_1$

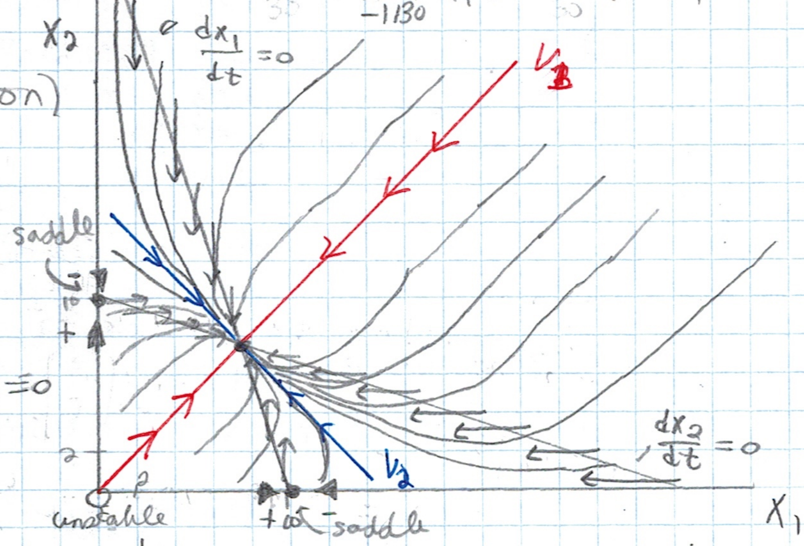
case 2: $a_{12} = a_{21} = 1/30$ (competition)

$$A = \begin{bmatrix} -3/4 & -1/4 \\ -1/4 & -3/4 \end{bmatrix}$$

for $\lambda_1 = -1$,

$$\begin{bmatrix} \cancel{-3/4-(-1)}\ 1/4 & -1/4 \\ -1/4 & \cancel{-3/4-(-1)}\ 1/4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ this is the fast eigendirection. As $t \to -\infty$, trajectories ∥ to V_1.



$\frac{dx_2}{dt} = 0$ when $x_2 = 10 - \frac{(1/30)}{1/10} x_1 = 10 - \frac{x_1}{3}$

For $\lambda_2 = -1/2$,
$$\begin{bmatrix} \cancel{-3/4-(-2/4)}\ -1/4 & -1/4 \\ -1/4 & \cancel{-3/4-(-2/4)}\ -1/4 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\frac{dx_1}{dt} = 0$ when $x_2 = 30 - \frac{1/10}{1/30} x_1$

$x_2 = 30 - 3x_1$

$v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ this is the slow eigendirection, near fixed point, trajectories ∥ to V_2

10 (cont).

case 3: $a_{12} = a_{21} = 1/5$ (competition)

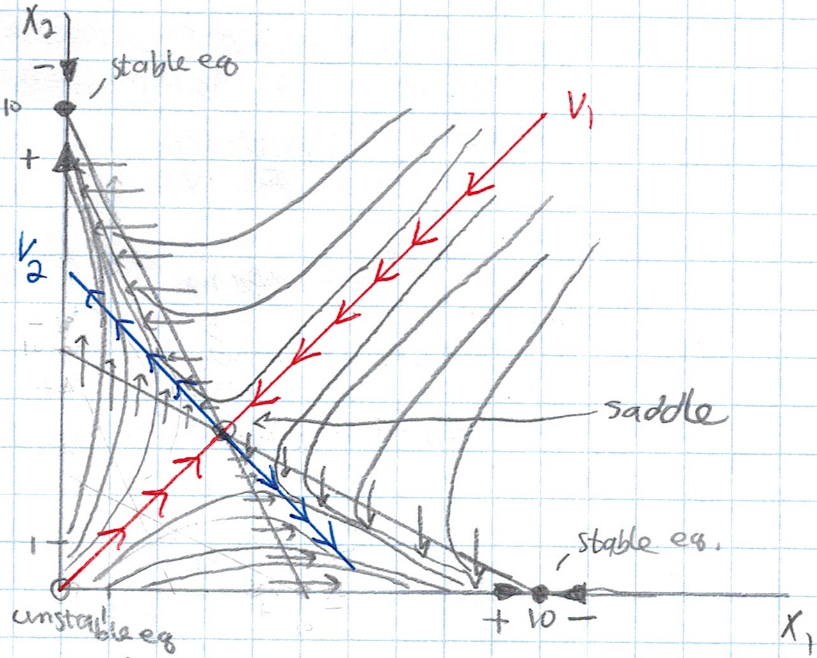
$$A = \begin{bmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{bmatrix}$$

$\lambda = -1, +1/3$

$\lambda_1 = 1,\ v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ stable eigendirection

$\lambda_2 = +1/3,\ v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

unstable eigendirection



$\frac{dx_2}{dt} = 0$ when $x_2 = 10 - \frac{(1/5)}{1/10} x_1$

$x_2 = 10 - 2x_1$

$\frac{dx_1}{dt} = 0$ when $x_2 = 5 - \frac{1/10}{1/5} x_1$

$x_2 = 5 - x_1/2$

1. Oscillatory solutions when complex λ's.

Complex λ's when $\det(A) > \frac{1}{4}\mathrm{tr}(A)^2$ (discriminant < 0)

$\det(A) = r_1 a_{11} r_2 a_{22} x_{*1} x_{*2} - r_1 r_2 a_{12} a_{21} x_{*1} x_{*2}$

$\mathrm{tr}(A) = -r_1 a_{11} x_{*1} - r_2 a_{22} x_{*2}$

$\det(A) > \frac{1}{4}\mathrm{tr}(A)^2$ for complex λ's and oscillatory solutions

$r_1 r_2 x_{*1} x_{*2} (a_{11} a_{22} - a_{12} a_{21}) > \frac{1}{4}(r_1 a_{11} x_{*1} + r_2 a_{22} x_{*2})^2$ This is the condition

Let's show how predator-prey dynamics can result in oscillations

To simplify, let $r_1 = r_2 = r$, $x_{*1} = x_{*2} = x$, $a_{11} = a_{22} = a$, $a_{12} = -a_{21}$ ← (one >0, one <0 ⇒ predator-prey)

$\cancel{r^2} x^2 (a^2 + a_{12}^2) > \frac{1}{4}\overset{2rax}{(\cancel{rax} + rax)}^2 \quad 4\cancel{r^2} a^2 \cancel{x^2}$

$a^2 + a_{12}^2 > a^2$ This must be always true since $a_{12}^2 > 0$ (a_{12} is real)

So under these predator prey conditions we got oscillations. This can also model host parasite interactions as one population benefits and the other suffers from the interactions.

1(cont). The coexistence Eq is a center when the real part of λ is 0 and λ is purely imaginary, although under the Hartman-Grobman THM, since the center is not hyperbolic, we really need to consider higher order terms to deduce the flow near the fixed points.

nonetheless, using the linearization, we get a center when

assuming

$\det(A) > 0$ and $\text{trace}(A) = 0$

$\det(A) > 0$ so $a_{11}a_{22} > a_{12}a_{21}$

trace(A) > 0 so

$\text{so} -(r_1 a_{11} x_{*1} + r_2 a_{22} x_{*2}) = 0$

$(a_{11}a_{22} - a_{12}a_{21} > 0)$

$\downarrow$

2 conditions

$r_1 a_{11} x_{*1} = -r_2 a_{22} x_{*2}$

Problem 2:

1. $\det(A - \lambda I) = 0 \rightarrow \det\left(\begin{bmatrix} -\lambda & 1 \\ -r^2 & -2r-\lambda \end{bmatrix}\right) = (-\lambda)(-2r-\lambda) + r^2 = 0$

$2r\lambda + \lambda^2 + r^2 = 0$

$\lambda^2 + 2r\lambda + r^2 = 0$

1 eigenvalue of alg mult 2

$(\lambda + r)^2 = 0$

$\lambda = -r$, algebraic multiplicity of 2

2. since $r > 0$, $\lambda < 0$ ($\lambda = -r$), so the equilibrium is stable

3. $(A - \lambda I)v = 0$

$\begin{bmatrix} r & 1 \\ -r^2 & -r \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \quad \therefore v = \begin{bmatrix} 1 \\ -r \end{bmatrix}$

$v_1 = 1 \quad v_2 = -r$

There is only one eigenvector that is linearly independent, so the geometric multiplicity of $\lambda = -r$ is 1. $\therefore A$

4. since the geometric multiplicity of $\lambda = -r$ is less than the algebraic multiplicity $(1 < 2)$, A is not diagonalizable

1. Let $t = \frac{1}{r}\tau$

$\tau = rt$

$\frac{d\tau}{dt} = r$

$\frac{d^2}{dt^2} x(t) = -\omega_0^2 x(t) - 2\mu \frac{d}{dt} x(t)$

$\frac{d^2}{d\tau^2} x(t) \frac{d\tau^2}{dt^2} = -\omega_0^2 x(t) - 2\mu \frac{d}{d\tau} x(t) \frac{d\tau}{dt}$

$\frac{d}{d\tau}\begin{bmatrix} x \\ \frac{d}{d\tau}x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ \frac{d}{d\tau}x \end{bmatrix}$

$\frac{d^2}{d\tau^2} x(\tau^2) = \frac{-\omega_0^2 x}{r^2} - \frac{2\mu(\tau)}{r} \frac{d}{d\tau} x$

We know that $r = \omega_0 = \mu$

$\therefore \frac{d}{d\tau}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$\frac{d^2}{d\tau^2} x = -\frac{\cancel{r^2}}{\cancel{r^2}} x - \frac{2\cancel{\mu}}{\cancel{r}} \frac{d}{d\tau} x = -x - \frac{d}{d\tau} x$

$\therefore \frac{d}{d\tau} z(\tau) = A z(\tau)$ in which $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

Problem 2 (cont.):

1 (cont.). Switching into this time unit means that we are now measuring time in units of ω_0 and γ, the natural period of the oscillator and the period related to the effect of drag (they are equal in this critically damped case). A unit time interval in τ represents one period ($\Delta\tau = 1$) of the oscillator in its natural frequency $(1/\omega_0)$ or or one period relating to the damping effect $(1/\gamma)$. $1/\gamma$ is the rate of decay of oscillations

2.

$k=0 \quad A^k = A^0 = I$

$$k=1 \quad A^k = A^1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$k=2 \quad A^k = A^2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$A^3 = AA^2 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 2 & 3 \\ -3 & -4 \end{bmatrix}$$

$$\boxed{\therefore A^k = \begin{bmatrix} (-1)^{k-1}(k-1) & k(-1)^{k-1} \\ (-1)^k(k) & (k+1)(-1)^k \end{bmatrix}}$$

See computer pages for script

Proof by induction.

1. Show true for $k=0$ we showed above

$$A' = \begin{bmatrix} (-1)^{a-1}(a-1) & a(-1)^{a-1} \\ (-1)^0(0) & (a+1)(-1)^0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

2. Assume true for k

$$A^k = \begin{bmatrix} (-1)^{k-1}(k-1) & k(-1)^{k-1} \\ (-1)^k(k) & (k+1)(-1)^k \end{bmatrix}$$

3. Show true for $k+1$

$$A^{k+1} = A^1 A^k = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} (-1)^{k-1}(k-1) & k(-1)^{k-1} \\ (-1)^k(k) & (k+1)(-1)^k \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^k(k) & (k+1)(-1)^k \\ -(k-1)(-1)^{k-1} - 2(k)(-1)^k & -k(-1)^{k-1} - 2(k+1)(-1)^k \end{bmatrix} = \begin{bmatrix} (-1)^{(k+1)-1}(k+1-1) & (k+1)(-1)^{(k+1)-1} \\ (k+1)(-1)^{k+1} & (k+1+1)(-1)^{k+1} \end{bmatrix} \checkmark$$

$(k-1)(-1)^k - 2k(-1)^k$
$= (-1)^k[k-1-2k] = (-1)^k(-k-1) = (-1)^{k+1}(k+1)$

$k(-1)^k - 2(k+1)(-1)^k = (-1)^k(\underbrace{k-2k-2}_{-k-2}) = (-1)^{(k+1)}(k+2)$

QED

2 cont. $\therefore$ By induction, pattern must hold for all $k \geq 0$

3. See computer paper

4. $k=0 \therefore A^k = (-1)^0 \begin{bmatrix}1&0\\0&1\end{bmatrix} + (-1)^{0-1}\cdot 0 \cdot \begin{bmatrix}1&1\\-1&-1\end{bmatrix}$

$$A^0 = \begin{bmatrix}1&0\\0&1\end{bmatrix} = I \checkmark$$

$k=1 \therefore A^1 = (-1)^1 \begin{bmatrix}1&0\\0&1\end{bmatrix} + (-1)^{1-1}\cdot 1 \cdot \begin{bmatrix}1&1\\-1&-1\end{bmatrix}$

$$A^1 = \begin{bmatrix}-1&0\\0&-1\end{bmatrix} + \begin{bmatrix}1&1\\-1&-1\end{bmatrix} = \begin{bmatrix}0&0\\-1&-2\end{bmatrix}$$

$$A^1 = A \checkmark$$

$k=2 \therefore A^2 = (-1)^2 \begin{bmatrix}1&0\\0&1\end{bmatrix} + (-1)^{2-1}\cdot 2 \begin{bmatrix}1&1\\-1&-1\end{bmatrix}$

$$A^2 = \begin{bmatrix}1&0\\0&1\end{bmatrix} + -2\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$$

$$A^2 = \begin{bmatrix}-1&-2\\2&3\end{bmatrix} \checkmark$$

$k=3 \therefore A^3 = (-1)^3 \begin{bmatrix}1&0\\0&1\end{bmatrix} + (-1)^{3-1}\cdot 3 \begin{bmatrix}1&1\\-1&-1\end{bmatrix} = \begin{bmatrix}-1&0\\0&-1\end{bmatrix} + \begin{bmatrix}3&3\\-3&-3\end{bmatrix}$

$$A^3 = \begin{bmatrix}2&3\\-3&-4\end{bmatrix} \checkmark$$

So the equation matches what I saw

5. $e^{A\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}(A\tau)^k = \sum_{k=0}^{\infty} \frac{1}{k!}\tau^k \cdot A^k = \sum_{k=0}^{\infty} \frac{1}{k!}\tau^k\left((-1)^k I + (-1)^{k-1}\cdot k \begin{bmatrix}1&1\\-1&-1\end{bmatrix}\right)$

$$e^{A\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\tau^k(-1)^k I + \sum_{k=1}^{\infty} \frac{1}{\cancel{k!}\,(k-1)!}\cancel{\tau^k}(-1)^{k-1}\cancel{k}\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\tau\,\tau^{k-1}$$

$$e^{A\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}(-\tau)^k I + \sum_{k=1}^{\infty} \tau\cdot\frac{1}{(k-1)!}(-1)^{k-1}\tau^{k-1}\begin{bmatrix}1&1\\-1&-1\end{bmatrix} = e^{-\tau}I + \tau\sum_{k=1}^{\infty}\frac{1}{(k-1)!}\cdot(-\tau)^{k-1}\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$$

Let $\ell = k-1$ so we have

$$\tau\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}(-\tau)^{\ell}$$

$$e^{A\tau} = e^{-\tau}\cdot I + e^{-\tau}\cdot\tau\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$$

$$e^{A\tau} = e^{-\tau}\left(\begin{bmatrix}1&0\\0&1\end{bmatrix} + \tau\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\right)$$

The resulting function for non-diagonalizable A differs from the general form when A is diagonalizable because we now have a polynomial term in time in addition to the regular exponential term that we always have. Allowing for polynomial terms in time allows for additional solution behaviors for non-diagonalizable matrices.

6. even though A is non-diagonalizable, my stability characterization from the eigenvalues was still correct. We see this here because the exponential term always dominates over the polynomial term over long time scales, so the polynomial term ultimately doesn't matter for determining stability. All that matters for determining stability are the eigenvalues, which we found was $-r$ (or -1 after nondimensionalizing). We see this here as we have $e^{-\lambda\tau} = e^{-\tau}$ in the front of our solution, so overall stability is still determined by our eigenvalue

7, see computer pages

Homework 3

CAAM 28200: Dynamical Systems with Applications

Kameel Khabaz

February 16, 2022

Problem 2: Section 2.2

I wrote the script below to show that my pattern of matrix exponentiation holds out to k = 10. The pattern is the following:

$$A^k = \begin{bmatrix} (k-1)(-1)^{k-1} & k(-1)^{k-1} \\ k(-1)^k & (k+1)(-1)^k \end{bmatrix}$$

My code and output are below:

```
% Problem 2  Script for matrix exponential
% Compare multiplication and with pattern
A = [0 1;
    -1 -2];

for k = 0:10
    disp("Exponential A^" + k);
    disp(A^k)
    disp("Exponential A^" + k + " Using Pattern");
    disp(pattern(k))
end

function [A] = pattern(k)
    A = [(-1)^(k-1) * (k-1)  k * (-1)^(k-1);
         (-1)^k * k          (k+1) * (-1)^k];
end
```

```
Exponential A^0
     1     0
     0     1

```

```
Exponential A^0 Using Pattern
      1      0
      0      1

Exponential A^1
      0      1
     -1     -2

Exponential A^1 Using Pattern
      0      1
     -1     -2

Exponential A^2
     -1     -2
      2      3

Exponential A^2 Using Pattern
     -1     -2
      2      3

Exponential A^3
      2      3
     -3     -4

Exponential A^3 Using Pattern
      2      3
     -3     -4

Exponential A^4
     -3     -4
      4      5

Exponential A^4 Using Pattern
     -3     -4
      4      5

Exponential A^5
      4      5
     -5     -6
```

```

Exponential A^5 Using Pattern
     4     5
    -5    -6

Exponential A^6
    -5    -6
     6     7

Exponential A^6 Using Pattern
    -5    -6
     6     7

Exponential A^7
     6     7
    -7    -8

Exponential A^7 Using Pattern
     6     7
    -7    -8

Exponential A^8
    -7    -8
     8     9

Exponential A^8 Using Pattern
    -7    -8
     8     9

Exponential A^9
     8     9
    -9   -10

Exponential A^9 Using Pattern
     8     9
    -9   -10

Exponential A^10
    -9   -10
```

```
    10     11

Exponential A^10 Using Pattern
    -9    -10
    10     11
```

Problem 2: Section 2.3

I now tested plotting solution trajectories for an n-term approximation for the matrix exponential. My code and plots are shown below:

```
%% Crticially Damped Oscillator Matrix Exponential Approximation
A = [0 1;
    -1 -2];

% Solution trajectories are given by y(t) = expo(At) * y(0)
close all
figure();
set(gcf,'Position',[0 0 800 1000])
t = linspace(0,5,100);
z0 = [1; 1];
solns = nan(2,length(t));

subplot(3,2,1);
n = 1
solve_plot(n,A,t,z0);

subplot(3,2,2);
n = 3
solve_plot(n,A,t,z0);

subplot(3,2,3);
n = 5
solve_plot(n,A,t,z0);

subplot(3,2,4);
n = 7
solve_plot(n,A,t,z0);

subplot(3,2,5);
```

```
n = 9
solve_plot(n,A,t,z0);

subplot(3,2,6);
n = 10
solve_plot(n,A,t,z0);
exportgraphics(gcf,"n_approx_trajs.eps")
function solve_plot(n,A,t,z0)
    for i = 1:length(t)
        solns(:,i) = soln_exp_approx(A,t(i),n,z0);
    end
    hold on
    di = min(find(abs(solns(1,:)) > 2))
    xline(t(di),'--r','LineWidth',1.5)
    h1= plot(t,solns(1,:),'Color','k','LineWidth',2);
    h2 = plot(t,solns(2,:),'Color','k','LineWidth',2,'LineStyle','--');
    title("$ n = " + n + " $",'Interpreter','latex')
    set(gca,'FontSize',30,'FontName','times')
    xlabel("$t$",'Interpreter','latex')
    ylabel('$z(\tau)$','Interpreter','latex')
    lgd = legend([h1 h2], "z_1","z_2")
    lgd.Location = "northwest";
    lgd.FontSize = 18;

end

function [soln] = soln_exp_approx(A,t,n,z0)
    expA = [1 0; 0 1];
    for k = 1:n
        expA = expA + (1/factorial(k)) .* (A .*t )^k;
    end
    soln = expA * z0;
end
```

In the plot in Figure 1, I approximated the matrix exponential $\sum_{k=0}^{n} \frac{1}{k!}(At)^k$ for $n = 1$ to $n = 10$ and plotted trajectories starting from $z(0) = [1, 1]$, which corresponds to an initial position and velocity of 1. I then plotted the z_1, or position, variable as a solid line and z_2, or the velocity variable, as a dashed line. I then plotted the time point at which the solution diverges (chosen simply as the position exceeding twice the initial position) as a red dashed vertical line. This figure clearly shows that as n increases, the red line shifts to the right, which means that the n-term matrix exponential approximation converges towards the true trajectory for more time. At early times, as n increases, the approximate trajectory converges more towards the true trajectory (which we know from a critically damped oscillator should be a consistent decrease in both the position and velocity of the oscillator). However, for all of these approximations, regardless of n, the trajectory diverges at later times. This is because an exponential is a power series, and infinitely many terms are needed. Here, we are cutting off that series and including only a small number of terms, so what we get is definitely an approximation. Furthermore, each additional term of the approximation represents a higher-order behavior of the exponential function that contributes a smaller and smaller amount to the overall approximation. Thus, at early times, the function's behavior can be captured with only lower-order terms, but as one goes farther and farther from the $t = 0$ point, higher order terms are needed. We can see this in Figure 2, in which I include 100 terms for the approximation and obtain a much more accurate solution trajectory up to $t = 10$. However, even here, we see that if we increase the time enough, as in the right figure, the approximation will eventually diverge.

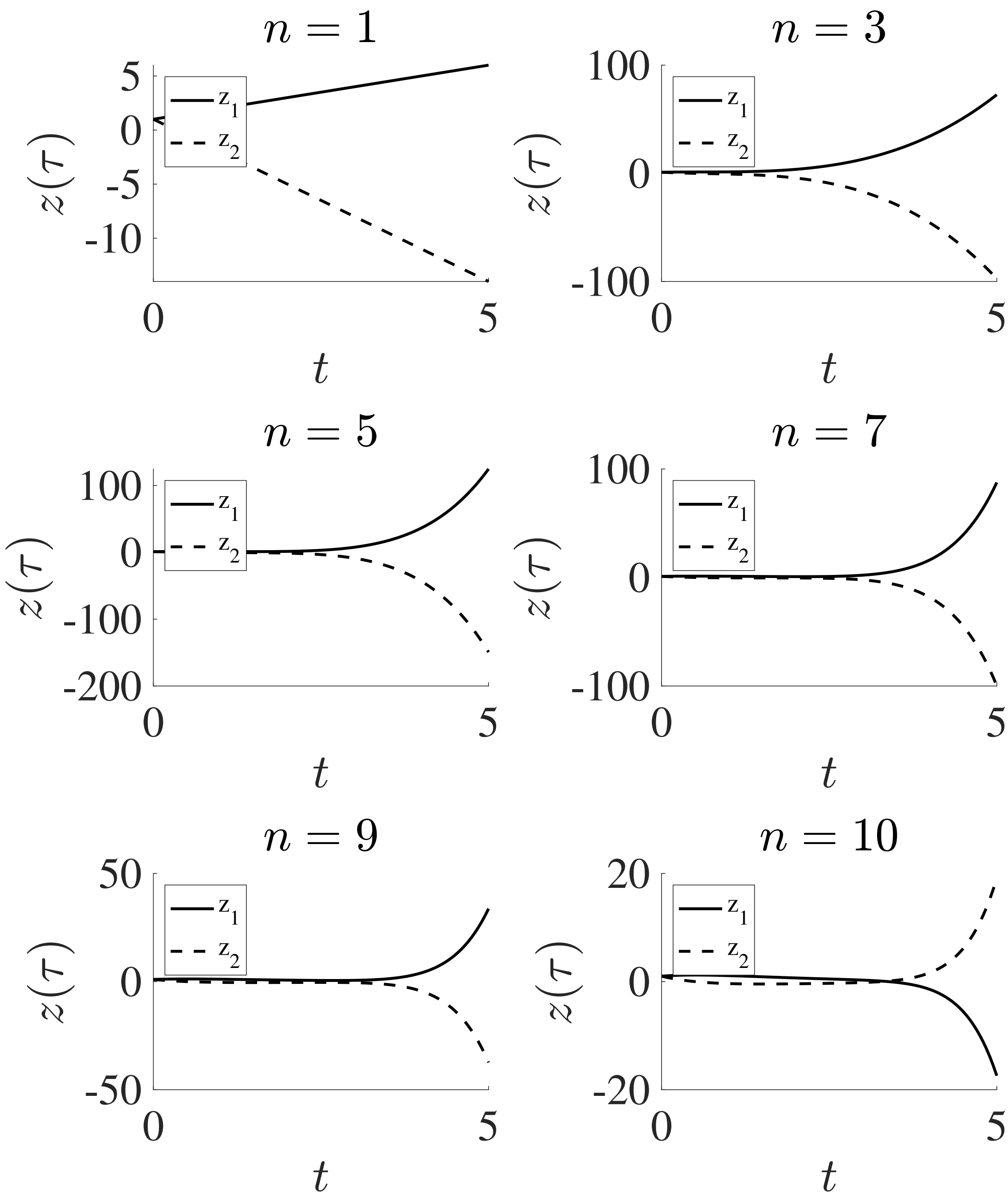


Figure 1: n-term Approximation Solution Trajectories from $z(0) = [1, 1]$

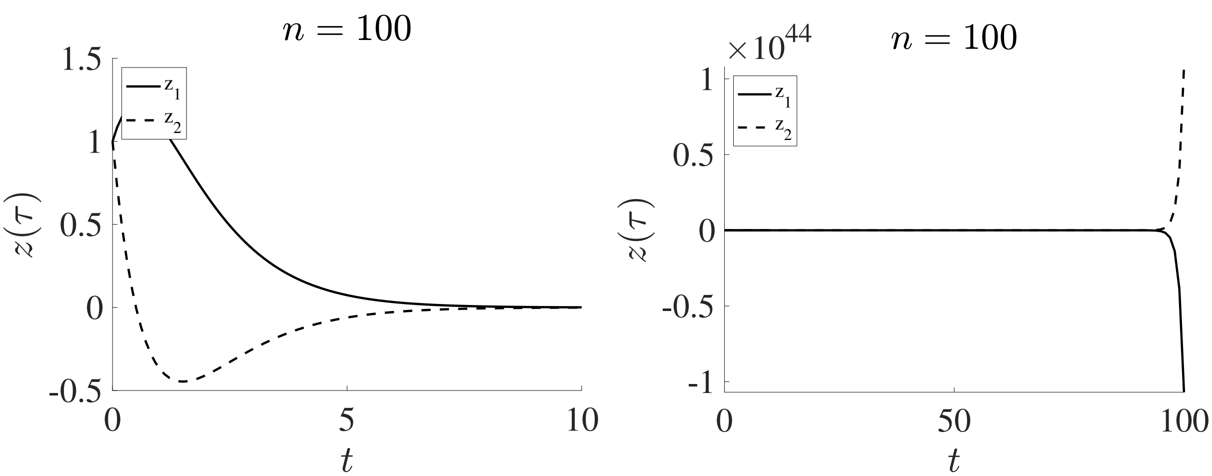


Figure 2: 100-term Approximation Solution Trajectories

Problem 2: Section 2.7

Below, I show my code and plots using the explicit form of the matrix exponential (defined in number 5) to plot the exact solution starting from $z(0) = [1, 1]$.

```
%% Problem 2.7 - Solution with Explicit Form of Matrix Exponential
figure()
t = linspace(0,10,100);
explicit_soln_plot(A,t,z0)
exportgraphics(gcf,"explicit_form_soln.eps")

t = linspace(0,100,100);
explicit_soln_plot(A,t,z0)
exportgraphics(gcf,"explicit_form_soln_lt.eps")
function explicit_soln_plot(A,t,z0)
    solns = nan(2,length(t))
    for i = 1:length(t)
        solns(:,i) = exp(-t(i)) * ([1 0; 0 1] + t(i) * [1 1; -1 -1]) * z0;
    end
    hold on
    di = min(find(abs(solns(1,:)) > 2))
    if isinteger(di)
        xline(t(di),'--r','LineWidth',1.5)
    end
    h1= plot(t,solns(1,:),'Color','k','LineWidth',2);
    h2 = plot(t,solns(2,:),'Color','k','LineWidth',2,'LineStyle','--');
    title("Explicit Form of Matrix Exponential",'Interpreter','latex')
    set(gca,'FontSize',30,'FontName','times')
    xlabel("$t$",'Interpreter','latex')
    ylabel('$z(\tau)$','Interpreter','latex')
    lgd = legend([h1 h2], "z_1","z_2")
    lgd.Location = "northeast";
    lgd.FontSize = 18;

end
```

When comparing Figure 2 to Figure 3, we see that using the explicit form for the matrix exponential results in a solution trajectory that matches the expected trajectory even at long time spans. While using an partial expansion of the matrix exponential means that the excluded higher-order terms result in a solution diverging at large time scales, using the exact form of the exponential includes all higher-order terms and thus results in the correct trajectory. Note how both the approximate solutions (with large n) and the exact solution result in similar behavior at small time spans (such as up to $t = 10$ for $n = 100$).

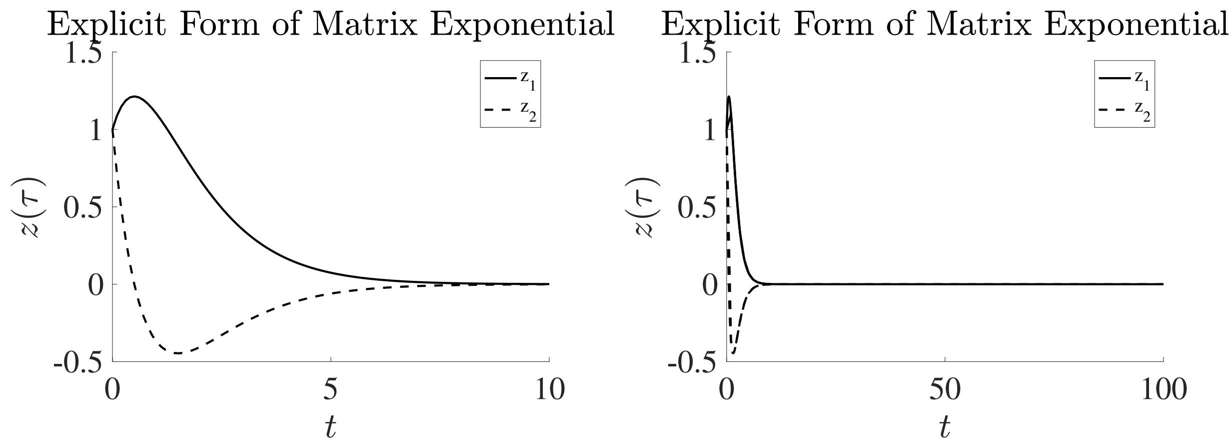


Figure 3: Exact Solution Trajectories