STAT/CAAM 28200 - Assignment 4 - Due Friday February 25

This is an individual assignment. You may collaborate with others but are expected to give each problem an honest attempt on your own first, and must submit your own work, in your own words. Show all relevant work and cite all sources used. If you get stuck, come to office hours. Write your name on your submission.

Submit assignments on gradescope (linked from canvas). You may take a picture of written work and upload the images, scan your work and upload a pdf, or, if you are ambitious, tex it. Please submit one collated file.

All Strogatz problems are from Strogatz Second Edition, the one with the black cover.

Strogatz Problems from Chapter 6: Please complete:

6.3.10, 6.3.13 (Hint: try finding a Lyapunov function.), 6.4.9 parts a and b, 6.5.6, 6.8.1, 6.8.2, 6.8.4

Conservative Systems and Energy Functions: This problem is all about mechanical systems that satisfy conservation constraints. Our goal is to use the conservation constraints to reduce the dimensionality of the problem, then visualize the level sets of the conserved quantity. Recall that trajectories are restricted to move within these level sets. Note that there is *much* more to be said about conservative systems than what we have done in class. Indeed, Hamiltonian mechanics recasts all of Newtonian mechanics using conservation constraints, and Noether's theorem beautifully demonstrates how conservation laws arise from symmetries. We will content ourselves here with the visuals.

As a warm up, let's start with a familiar problem, the undamped pendulum. Consider a pendulum with mass M, hanging on a massless rod with length L, that moves without friction or drag. Let $\theta(t)$ represent the angle the pendulum is lifted counterclockwise away from its rest position (hanging straight down). Then recall that the pendulum obeys the equation:

$$\frac{d^2}{dt^2}\theta(t) = -\frac{g}{L}\sin(\theta) \tag{1}$$

where g is the acceleration due to gravity. This is a second order equation, so, as usual, we study it in the phase plane.

Suppose that, at some time t, the pendulum is at angle $\theta(t)$ moving with angular vecolity $\frac{d}{dt}\theta(t) = \dot{\theta}(t)$. Then the bob at the end of the pendulum is moving with tangential velocity $v(t) = L\dot{\theta}(t)$. The height of the bob above its rest position is $h(t) = L(1 - \cos(\theta(t)))$. Then, combining potential and kinetic energy, the total mechanical energy of the pendulum at time t is:

$$E(\theta(t), \dot{\theta}(t)) = Mgh(t) + \frac{1}{2}Mv(t)^2 = ML\left(g(1 - \cos(\theta(t)) + \frac{1}{2}L\dot{\theta}(t)^2\right). \tag{2}$$

1. Show that the total mechanical energy E is a conserved quantity. What does this imply about the existence of sinks or sources?
2. Make a 3D surface plot showing the the energy function over the phase plane using $M = 1$, $L = 1$ and $g = 1$ (θ should range from $-\pi$ to π, and you may put an upper and lower bound on $\dot{\theta}$. Make sure to pick the bound large enough so that we see all characteristic behavior.) Color your surface based on the energy function.
3. Add level contours to your plot, and view it from above. Mark the location of any equilibria and mark them as saddles or centers.

Now let's try to visualize a "new" example. Here I've put new in quotes since we'll actually look at an extremely classical problem - the famed two body problem.

The two body problem asks: how do two massive bodies with masses M and m move subject only to the gravitational forces between them?[1]

[1] To be distinguished from the other famous two body problem involving careers and romantic partners.

This is a classical problem with deep historical roots. Ancient astronomers long struggled to explain planetary motion and the motion of the stars. Those attempts sparked the early development of much of mathematics and science, inspired Copernicus, and Kepler, and ultimately Isaac Newton, who invented calculus and Newtonian mechanics and proposed his universal law of gravitation - the first truly successful quantitative theories of nature - in part to explain the motion of two bodies in orbit. Newton solved the problem using geometric arguments in his groundbreaking *Principia*. The problem was later studied by a who's who list of important mathematicians including Leonhard Euler, and physicists including Richard Feynman. It is a mainstay in most physics classes, and we would be remiss to skip it since it is the first and most influential example of a dynamical model. It is, almost without exaggeration, the dynamical model responsible for ushering in the modern age of quantitative science, and is one of the earliest examples of a dynamical model whose success elevated its principals from models to scientific laws.

Consider two masses of mass m and M attracted to each other by gravity, and acted on by no other forces. In principal, each body is a point in three dimensional space, with some velocity vector, which again has three entries. So, before reducing the problem, we have a 12 dimensional phase space! Let's try and use symmetries and conservation laws to reduce that dimensionality.

Let $r(t)$ equal the distance between the two masses. We will assume in all that follows that $r(t) > 0$. Imagine drawing a line between the two masses. Somewhere along that line is the center of mass of the system. Then the first mass is a distance $\frac{M}{M+m}r$ from the center of mass, and the second is a distance $\frac{m}{m+M}$ from the center of mass. We will work in a polar frame of reference (coordinate system) centered at the center of mass.

Next, by conservation of momentum (angular and linear), if we fix our coordinate system to follow the center of mass, then the two bodies orbit in some plane. The equations of motion don't change as the center of mass moves since gravitational forces are translationally symmetric with respect to translation of the masses. So, the motion can be reduced to the motion of two bodies in a plane, where the line between the two bodies always passes through the origin, and the relative distances between the bodies and the origin is fixed.

Now we really only have our phase variables to consider. The distance between the two masses, and its rate of change (radial velocity), the angle that the line between the masses forms with respect to some arbitrary reference direction within the plane, and the rate of change in the angle (angular velocity). Let $\theta(t)$ denote the angle at time t. Moreover, gravity is rotationally symmetric, so, if we adopt a rotating frame of reference that rotates with the bodies, then we only have three variables to consider:

1. **Distance between the masses:** $r(t)$. This matters since it controls the size of the force due to gravity pulling the masses together.
2. **Radial Velocity:** $\dot{r}(t) = \frac{d}{dt}r(t)$. This matters since we are dealing with mechanics, so our governing equations follow from Newton's second law, so will be second order.
3. **Angular Velocity:** $\dot{\theta}(t) = \frac{d}{dt}\theta(t)$. This matters since, in a rotating reference frame objects feel a centrifugal force pulling them outwards away from the center of rotaton.[2]

So, although the two body problem has 12 phase variables, we really only need to consider 3. We will cut this down to 2 phase variables using conservation of angular momentum.

Next, the forces. Both objects are attracted towards each other by gravity. The gravitational force is:

$$F_g = -\frac{GmM}{r^2} \tag{3}$$

where G is the universal gravitation constant.

While gravity pulls the bodies together, centrifugal forces pull them apart. The centrifugal force acting on an object in a rotating reference frame is an inertial force associated with the acceleration of the reference frame. The centrifugal force equals the mass o the object times distance of the object from the center of rotation times its angular velocity squared. Therefore, the two masses feel centrifugal forces:

$$\begin{aligned} F_{cm} &= m\frac{M}{M+m}r\dot{\theta}^2 \\ F_{cM} &= M\frac{m}{M+m}r\dot{\theta}^2. \end{aligned} \tag{4}$$

[2]This is, of course, a "fictitious" force that arises from the acceleration of the reference frame.

Notice that the two centrifugal forces are the same, so we can write:

$$F_c = \frac{Mm}{M+m} r\dot{\theta}^2. \tag{5}$$

Both forces act radially, and there are no other forces in the system, so, using Newton's second law, each mass accelerates outward with acceleration:

$$a_m = \frac{1}{m}(F_c - F_g), \quad a_M = \frac{1}{M}(F_c - F_g). \tag{6}$$

Therefore:

$$\frac{d^2}{dt^2} r(t) = \frac{d}{dt}\dot{r}(t) = \left(\frac{1}{m} + \frac{1}{M}\right)(F_c(r,\dot{\theta}) - F_g(r)) = \frac{M+m}{Mm}(F_c(r,\dot{\theta}) - F_g(r)). \tag{7}$$

Thus we arrive at a governing equation for the rate of change in the radial velocity. By definition $\frac{d}{dt} r(t) = \dot{r}(t)$ so we already have a governing equation for the rate of change in the distance between the objects as well. What remains is a governing equation which controls the rate of change in the angular velocity.

1. We can actually eliminate the angular velocity $\dot{\theta}(t)$ from the problem since angular momentum is conserved. The angular momentum of a orbiting mass is its mass, times its distance from the origin squared, times its angular velocity. We denote the total angular velocity of the system L. Then L is a conserved quantity where:

$$L = \left(m\left(\frac{M}{M+m} r(t)\right)^2 + M\left(\frac{m}{M+m} r(t)\right)^2\right)\dot{\theta}(t). \tag{8}$$

   Show that L reduces to:

$$L = \frac{Mm}{M+m} r(t)\dot{\theta}(t)^2 \tag{9}$$

   then argue that, since L is conserved, all orbit trajectories are confined to the manifold where:

$$\dot{\theta}(t) = \frac{M+m}{Mm}\frac{L}{r(t)^2}. \tag{10}$$

2. So, given the angular momentum L, we can solve for the angular velocity given the distance $r(t)$. This will let us eliminate the angular velocity from the problem. Substitute in for the angular velocity using equation 10 to show that:

$$\frac{d^2}{dt^2} r(t) = \frac{d}{dt}\dot{r}(t) = \left(\frac{M+m}{Mm}\right)^2 \frac{L^2}{r(t)^2} - \frac{G}{M+m}\frac{1}{r(t)^2}. \tag{11}$$

   Convert to the corresponding first order system in the phase plane with coordinates $r, \dot{r}$. We have now succesfully reduced all the way from 12 variables to just 2. This is the power of symmetries and conservation laws.

So far we have only considered conservation of momentum. Mechanical systems, absent any drag forces, also conserve energy. Our goal now is to show that the system governed by equation 11 conserves energy. Total mechanical energy in this system is a combination of the gravitational potential energies of the masses, and their kinetic energy, which can be further divided into radial and tangential (angular) components.

The gravitational potential energy between the masses is:

$$U(r) = -\frac{GMm}{r}. \tag{12}$$

The kinetic energy of each mass can be broken into a radial and a tangential component. These are:

$$\begin{aligned} K_m(r,\dot{r},\dot{\theta}) &= \frac{1}{2}m\left[\left(\frac{M}{M+m} r\dot{\theta}\right)^2 + \left(\frac{M}{M+m}\dot{r}\right)^2\right] \\ K_M(r,\dot{r},\dot{\theta}) &= \frac{1}{2}M\left[\left(\frac{m}{M+m} r\dot{\theta}\right)^2 + \left(\frac{m}{M+m}\dot{r}\right)^2\right] \end{aligned} \tag{13}$$

1. Add up the potential energy, and both kinetic energies, then substitute in for $\dot{\theta}$ in terms of L to show that the total mechanical energy equals:

$$E(r, \dot{r}) = -\frac{GMm}{r} + \frac{1}{2}\left[\frac{M+m}{Mm}\frac{L^2}{r^2} + \frac{1}{2}\frac{Mm}{M+m}\dot{r}^2\right]. \tag{14}$$

2. Show that energy is conserved (check that $\frac{d}{dt}E(t) = 0$.)
3. Show that, along $\dot{r} = 0$ the energy function is dominated by the angular kinetic term for any fixed $L > 0$, and by the gravitational potential term for large r. Sketch the energy function along $\dot{r} = 0$.
4. Show that, along $\dot{r} = 0$, the energy function must have exactly one minimum. Where does it occur? Compare it to the distance r at which $\frac{d^2}{dt^2}r(t) = \frac{d}{dt}\dot{r}(t) = 0$? What does this distance correspond to physically in the problem. If we start the two masses at this radius with $\dot{r} = 0$ what sort of orbit will they follow?
5. Show that, if $\dot{r} \neq 0$ then the system cannot be at rest. Use this constraint to show that the only possible equilibrium occurs along $\dot{r} = 0$. Solve for the possible equilibrium radii. Show there is exactly one such equilibrium.
6. Next, show that, with r fixed, the energy function is parabolic and increasing in $|\dot{r}|$. Use these facts to argue that the energy function should have a unique global minimizer at some positive r and $\dot{r} = 0$. Then all level sets of the energy function must form closed loops around this minimizer in some sufficiently small neighborhood of the minimizer, which we have shown is the unique equilibrium. Trajectories near the equilibrium are constrained to loops about the equilibrium. What type of orbits do they represent? Classify the stability of the equilibrium and state why you know this classification must be true. Do not use linear stability analysis.
7. Make a plot showing the energy surface and energy level contours given masses $m = 0.18$, $M = 1$, gravitational constant $G = 1$ (not the true value, chosen as if we had nondimensionalized), and with angular momentum $L = 0.1$, for r ranging from 0.1 to 5, and $\dot{r}$ ranging from -5 to 5. Mark the equilibrium and some of the closed orbits around the equilibrium. Draw arrows on the level sets to indicate the direction of motion of the system along those level sets. *Optional:* Try playing with the values of L and of the initial masses.
8. *Optional Challenge Problem:* Find the contour at energy level zero. Mark it in a separate color. You should see that it divides to distinctly different types of level sets, and hence trajectories. How do trajectories at negative energy levels behave? What about trajectories at positive energy levels? How do they differ physically? What must be true about the initial kinetic energy of the masses relative to their gravitational potential above and below the zero energy line?

Strogatz Problems from Chapter 7: Please complete:

7.2.5 a (*optional challenge:* Try part b. Try computing the curl of the vector field defined by $f(x)$, then use Stoke's theorem to argue path independence), 7.2.9, 7.2.12

Lagrange Points, Potential Systems, and the Three Body Problem:

We turn now to the infamous three body problem. What if, instead of considering two bodies in orbit, we considered three?

Even though the two body problem was solved by Newton in the 17th century, the three body problem remains unsolved to this day since it exhibits chaos. In fact, Poincare discovered chaos before we had given chaotic systems the name chaos, when attempting to prove the stability of the solar system!

Here we will focus on a restricted version of the problem. We consider a pair of very large masses, say the Earth and the sun, in a fixed circular orbit, both orbiting their shared center of mass with fixed angular velocity $\dot{\theta} = \omega$. Let M_s be the mass of the sun, and M_E be the mass of the Earth. Then, we introduce a third, *much* lighter body, say a very expensive space telescope. The telesope's mass is orders of magnitude smaller than either of the other two masses, so it doesn't influence their orbits. Instead, their orbits fix a moving potential function that influences the motion of the telescope.

Typically we want to be able to "park" a telescope at a relatively stable location in space relative to the position of the Earth without constantly using precious fuel to adjust the telescope's orbit. For near earth satellites we simply put the satellites in orbit about Earth. If, as was the case for the James Webb Space Telescope, we intend to send a satellite far from earth, then we need to consider its orbit relative to the gravitational pull of the earth and the sun. So, we will try and look for stable locations to "park" our telescope where the forces in the system balance. We will restrict our search to the orbital plane of the earth and the sun. Note that these are also locations in space where smaller orbiting objects (asteroids) will tend to settle. These locations are called the Lagrange points. The James Webb Telescope arrived at one of the Lagrange points earlier this year, where it will stay in a roughly stable orbit using minimal fuel.

As before, we will work in a rotating reference frame centered at the center of mass of the sun and the earth. Then, let r_{ts} and r_{te} denote the distance from the telescope to the sun and earth respectively, and r denote the distance of the telescope from the center of mass of the sun and the earth.

1. Compute the distance of the sun and the earth from their shared center of mass (you may look online for the necessary constants, i.e. mass of the earth and the sun, and the distance from the earth to the sun). Show that the sun is *much* more massive than the earth, so the center of mass of the sun is close to the center of the sun (in fact, show that the center of mass is contained inside the sun). From now on we will assume that the sun is so massive that the center of mass of the system is at the center of the sun. Then $r_{ts} = r$. Compute the angular velocity of the earth around the sun, and call that velocity number ω.

2. The gravitational potential generated by a spherical body with mass M experienced by a mass m at a distance d from the body is:

$$V_g(d) = -\frac{GMm}{d}. \tag{15}$$

   Show that the force due to gravity is recovered by $F_g(d) = -\partial_d V_g(d)$. Write down the force due to gravity pulling on the telescope from the sun and the earth in terms of their two potentials.

3. We are working in a rotating reference frame that is centered at the center of mass of the earth and the sun, rotating at angular velocity ω. Then, the telescope feels a centriugal force:

$$F_c = m_t r \omega^2. \tag{16}$$

   Find a potential function V_c such that $F_c(r) = -\partial_r V_c(r)$.

4. Combine the three potentials to show that the motion of the satellite is governed by the gradient of a net potential. That is show that the governing equation for the telescope takes the form:

$$\frac{d}{dt}x(t) = -\nabla V(x) \tag{17}$$

   where $x(t)$ is the position of the telescope in the orbital plane, and $V(x)$ is some combination of the two gravitational potentials and the centrifugal potential.

5. Plot your potential function using a contour plot. Color the contours by the value of the potential energy function. Does the system move across (perpendicular) to the contours or along them?

6. Identify any local maxima or minima of the potential function (excluding collisions with the sun or the earth). How many equilibrium locations are there for the telescope to rest at? These are the Lagrange points. Are they stable equilibria?