NYU Tandon Bridge

Homework 3

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01/29/2021

Question 7

a) Exercise 3.1.1

a. All multiples of 3 are elements in A. 3*9 = 27. So $27 \in A$ is **True**

b. A perfect square can be written $y=x^2$ for an integer x. $\sqrt{27} \simeq 5.196$ which is not an integer. So $27 \in B$ is **False.**

c. A perfect square can be written $y=x^2$ for an integer x. $\sqrt{100} \simeq 10$. So $100 \in B$ is **True.**

d. $\{3, 6, 9\} \subsetneq \{4, 5, 9, 10\}$ so E is not a subset of C. Conversely,

$\{4, 5, 9, 10\} \subsetneq \{3, 6, 9\}$ so C is not a subset of E. So, $E \subseteq C$ or $C \subseteq E$ is **False**

e. 3 * 1= 3, 3 * 2 = 6, 3 * 3 = 9, so 3, 6, and 9 are all integer multiples of 3. As such,

$\{ 3, 6, 9 \} \subseteq \{ x \in Z : \ x \text{ is an integer multiple of } 3 \}$ so $E \subseteq A$ is **True.**

f. $27 \in A \text{ but } 27 \notin E$. So not all elements of A are in E. Therefore, $A \subset E$ is **False**.

g. The set {3, 6, 9} is not a multiple of 3 and therefore not an element in A (even though the elements inside the set are multiples of 3). So, $E \in A$ is **False.**

b) Exercise 3.1.2

a. 15 is an element not a set. So, $15 \subset A$ is **False**.

b. Since 3 * 5 = 15, we get $15 \in A$. Also, 3 * 1 = 3, so $3 \in A$ but $3 \notin \{15\}$. As such, all elements in {15} are also elements in A, but not all elements in A are in {15}. Therefore, $\{15\} \subset A$ is **True.**

c. All elements in $\varnothing$ (none), are also in A. However, A contains elements (e.g. 3) that are not in $\varnothing$. Therefore, $\varnothing \subset A$ is **True**.

d. Since the criteria for being an element in A and A are the same (because they're the same set), all elements in A are also in A. Therefore, $A \subseteq A$ is **True**.

e. $\varnothing$ is an empty set, not an element. So, $\varnothing \in B$ is **False**.

c) Exercise 3.1.5

b. **$\{x \in Z^{+}$: x is an integer multiple of 3$\}$**. The set is **infinite**.
Can also be defined as $\{x \in N$: x is a positive integer multiple of 3$\}$.
If we want to write it without English, we can also say $\{3x: x \in Z^{+}\}$.

d. **$\{x \in N$: x is an integer multiple of 10 and $x \leq 1000\}$** and the cardinality is **101**.
If we want to write it without English, we can also say: $\{10x: x \in N, 0 \leq x \leq 100\}$.

d) Exercise 3.2.1

a. 2 is the fourth element in the X roster notation. So, $2 \in X$ is **True.**

b. 2 is the fourth element in the X roster notation. So $\{2\}$ is a subset of X. Therefore, $\{2\} \subseteq X$ is **True**.

c. $\{2\}$ does not appear in the roster as an element in X. So, $\{2\} \in X$ is **False**.

d. 3 does not appear in the roster as an element in X. So, $3 \in X$ is **False**.

e. $\{1, 2\}$ is the third element in the X roster notation. So, $\{1, 2\} \in X$ is **True**.

f. 1 and 2 are both elements in X. So, the subset containing only 1 and 2 must be a subset of X. Therefore, $\{1, 2\} \subseteq X$ is **True**.

g. 2 and 4 are both elements in X. So, the subset containing only 2 and 4 must be a subset of X. Therefore, $\{2, 4\} \subseteq X$ is **True**.

h. $\{2, 4\}$ does not appear as an element in the roster for X. So, $\{2, 4\} \in X$ is **False**.

i. 3 is not an element in X. So, $\{2, 3\} \subseteq X$ is **False**.

j. $\{2, 3\}$ does not appear as an element in the roster for X. So, $\{2, 3\} \in X$ is **False**.

k. There are 6 elements in X. Therefore, $|X| = 7$ is **False**.

Question 8

Exercise 3.2.4

b. Let $A = \{1, 2, 3\}$. What is $\{X \in P(A): 2 \in X\}$?

Step 1: Determine P(A)

P(A) is a set of all the subsets of A.

$P(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$

Step 2: Determine which elements are in $\{X \in P(A): 2 \in X\}$

Let us name this set $B = \{X \in P(A): 2 \in X\}$

From the set builder definition, we understand that the set B consists of all elements X in P(A) such that 2 is an element in X. In other words, all elements in P(A) that have a 2 in them.

Therefore, $B = \{\{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}$.

Question 9

a) Exercise 3.3.1

c. A ∩ C = {-3, 0, 1, 4, 17} ∩ {x ∈ Z: x is odd}

$\boxed{A \cap C = \{-3, 1, 17\}}$

d. A ∪ (B ∩ C) = A ∪ ({-12, -5, 1, 4, 6} ∩ {x ∈ Z: x is odd})

A ∪ (B ∩ C) = A ∪ {-5, 1}

A ∪ (B ∩ C) = {-3, 0, 1, 4, 17} ∪ {-5, 1}

$\boxed{A \cup (B \cap C) = \{-5, -3, 0, 1, 4, 17\}}$

e. A ∩ B ∩ C = {-3, 0, 1, 4, 17} ∩ {-12, -5, 1, 4, 6} ∩ C

A ∩ B ∩ C = {1, 4} ∩ C

A ∩ B ∩ C = {1, 4} ∩ {x ∈ Z: x is odd}

$\boxed{A \cap B \cap C = \{1\}}$

b) Exercise 3.3.3

a. $$\bigcap_{i=2}^{5} Ai = A_2 \cap A_3 \cap A_4 \cap A_5$$

$$\bigcap_{i=2}^{5} Ai = \{2^0, 2^1, 2^2\} \cap \{3^0, 3^1, 3^2\} \cap \{4^0, 4^1, 4^2\} \cap \{5^0, 5^1, 5^2\}$$

$$\bigcap_{i=2}^{5} Ai = (\{1,\ 2,\ 4\} \cap \{1,\ 3,\ 9\}) \cap \{1,\ 4,\ 16\} \cap \{1,\ 5,\ 25\}$$

$$\bigcap_{i=2}^{5} Ai = (\{1\} \cap \{1,\ 4,\ 16\}) \cap \{1,\ 5,\ 25\}$$

$$\bigcap_{i=2}^{5} Ai = \{1\} \cap \{1, 5, 25\}$$

$$\boxed{\bigcap_{i=2}^{5} Ai = \{1\}}$$

b. $$\bigcup_{i=2}^{5} Ai = A_2 \cup A_3 \cup A_4 \cup A_5$$

$$\bigcap_{i=2}^{5} Ai = \{2^0, 2^1, 2^2\} \cup \{3^0, 3^1, 3^2\} \cup \{4^0, 4^1, 4^2\} \cup \{5^0, 5^1, 5^2\}$$

$$\bigcup_{i=2}^{5} Ai = (\{1, 2, 4\} \cup \{1, 3, 9\}) \cup \{1, 4, 16\} \cup \{1, 5, 25\}$$

$$\bigcup_{i=2}^{5} Ai = (\{1, 2, 3, 4, 9\} \cup \{1, 4, 16\}) \cup \{1, 5, 25\}$$

$$\bigcup_{i=2}^{5} Ai = \{1, 2, 3, 4, 9, 16\} \cup \{1, 5, 25\}$$

$$\boxed{\bigcup_{i=2}^{5} Ai = \{1, 2, 3, 4, 5, 9, 16, 25\}}$$

e. $\bigcap_{i=1}^{100} Ci = C_1 \cap C_2 \cap \ldots \cap C_{100}$

$$\bigcap_{i=1}^{100} Ci = \{x \in R : -1/1 \le x \le 1/1\} \cap \{x \in R : -1/2 \le x \le 1/2\} \cap \ldots \cap \{x \in R : -1/100 \le x \le 1/100\}$$

We can see that as *i* increases, the set C_i further contracts on itself. C_{100} is a subset of C_{99} which is a subset of C_{98} ... C_2 is a subset of C_1.

Visually this is represented as:

```
     -1                                                                    1
C1   *---------------------------------------------------------------------*

           -1/2                                                    ½
C2          *------------------------------------------------------*
                                        .
                                        .
                                        .
                              -1/100        1/100
C100                            *------------*
```

As such, because we are taking the intersection of all those sets. Only the values in the smallest (innermost) set will be common to all the sets C_i.

So, the elements of the set C_{100} will be common to all sets in $\bigcap_{i=1}^{100} Ci$

Therefore, $\bigcap_{i=1}^{100} Ci = \{x \in R : \; -1/100 \le x \le 1/100\}$

f. $\bigcup_{i=1}^{100} Ci = C_1 \cup C_2 \cup \ldots \cup C_{100}$

$$\bigcup_{i=1}^{100} Ci = \{x \in R :- 1/1 \leq x \leq 1/1\} \cap \{x \in R :- 1/2 \leq x \leq 1/2\} \cap \ldots \cap \{x \in R :- 1/100 \leq x \leq 1/100\}$$

We can see that as *i* increases, the set C_i further contracts on itself. C_{100} is a subset of C_{99} which is a subset of C_{98} ... C_2 is a subset of C_1.

Visually this is represented as:

```
     -1                                                                  1
C1    *-----------------------------------------------------------------*

           -1/2                                                ½
C2          *--------------------------------------------------*
                                        .
                                        .
                                        .
                                 -1/100       1/100
C100                               *------------*
```

As such, because we are taking the union of all those sets. Only the values of the largest (outermost) set will contain all the other values of the rest of the sets C_i .

So, the elements of the set C_1 will contain all the other elements in all sets in $\bigcup_{i=1}^{100} Ci$

Therefore, $\bigcup_{i=1}^{100} Ci = \{x \in R : \ -1 \leq x \leq 1\}$

c) Exercise 3.3.4

b. $P(A \cup B) = P(\{a, b\} \cup \{b, c\})$

$P(A \cup B) = P(\{a, b, c\})$

$$\boxed{P(A \cup B) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}}$$

c. $P(A) \cup P(B) = P(\{a, b\}) \cup P(\{b, c\})$

$P(A) \cup P(B) = \{\{\varnothing, \{a\}, \{b\}, \{a, b\}\} \cup P(\{b, c\})$

$P(A) \cup P(B) = \{\{\varnothing, \{a\}, \{b\}, \{a, b\}\} \cup \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$

$$\boxed{P(A) \cup P(B) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}\}}$$

Question 10

a) Exercise 3.5.1

b. In order to be an element in $B \times A \times C$, the element must be an ordered triplet with the first item coming from B = {foam, no-foam}, the second coming from A = {tall, grande, venti}, and the third element coming from C = {non-fat, whole}. We can choose any one element from each set.

Therefore, $(no-foam,\ venti,\ non-fat) \in B \times A \times C$

c. The set $B \times C$ consists of all ordered pairs created with the first element coming from B and the second from C. We get:

$$B \times C = \{(foam,\ non-fat),\ (foam,\ whole),\ (no-foam,\ non-fat),\ (no-foam,\ whole)\}$$

b) Exercise 3.5.3

b. $Z^2 \subseteq R^2$

We know that all elements in **Z** are integers and all elements in **R** are real numbers. We also know that all integers are all real numbers. Therefore, all elements in **Z** are also in **R**. So, we get **Z** $\subseteq$ **R**.

Next, all elements in $\mathbf{Z}^2$ are formed by taking the first element from **Z** and the second element from **Z**. Similarly, all elements in $\mathbf{R}^2$ are formed by taking the first element from **R** and the second element from **R**.

Since, we know that all elements in **Z** are all also elements in **R**. All those integer elements can also be used to create ordered pairs in $\mathbf{R}^2$.

Therefore, since all elements in $\mathbf{Z}^2$ are also in $\mathbf{R}^2$, we conclude that $Z^2 \subseteq R^2$ is **True**.

Note: In fact, for any two sets, A and B, if A $\subseteq$ B, then $A^2 \subseteq B^2$.

c. $\mathbf{Z}^2 \cap \mathbf{Z}^3 = \varnothing$

We know that all elements in $\mathbf{Z}^2$ are formed by taking a first element from **Z** and a second element from **Z**. However, all elements in $\mathbf{Z}^3$ are formed by taking a first element from **Z,** a second element from **Z,** and a third element from **Z.**

As such, all elements in $\mathbf{Z}^2$ are ordered pairs whereas all elements in $\mathbf{Z}^3$ are ordered triplets.

Therefore, there is no overlap between the elements of $\mathbf{Z}^2$ and $\mathbf{Z}^3$.

We conclude that $\mathbf{Z}^2 \cap \mathbf{Z}^3 = \varnothing$ is **True**.

e. For any three sets, A, B, and C, if $A \subseteq B$, then $A \times C \subseteq B \times C$.

This statement is in the form of a conditional statement. If the premise $A \subseteq B$ is false, then the statement is true regardless of the value of the implication.

So the only case we must consider is when $A \subseteq B$ is true. We need to show that $A \times C \subseteq B \times C$ is also true.

In this case, we know that $A \subseteq B$. Therefore, all elements in A are also elements in B.

The set $A \times C$ is the set of all possible combinations of a first element from A and a second element from C. The set is composed of ordered pairs created from the combination of all elements in A and all elements in C.

Similarly, the set $B \times C$ is the set of all possible combinations of a first element from B and a second element from C. The set is composed of ordered pairs created from the combination of all elements in B and all elements in C.

Since all elements in A are also elements in B. Then those elements can also be used to create ordered pairs for the set $B \times C$. So all elements of $A \times C$ must also be in $B \times C$.

We conclude that if $A \subseteq B$, then $A \times C \subseteq B \times C$ is **True**.

c) Exercise 3.5.6

d. $\{xy\text{: where } x \in \{0\} \cup \{0\}^2 \text{ and } y \in \{1\} \cup \{1\}^2\}$

In this example, we can tell that the set is composed of strings of two elements x and y that are concatenated. Let's get all possible values for x and y and then concatenate them to create the desired set.

We are given that $x \in \{0\} \cup \{0\}^2$.
Therefore: $x \in \{0\} \cup \{00\}$
$x \in \{0, 00\}$

We are given that $y \in \{1\} \cup \{1\}^2$.
Therefore: $y \in \{1\} \cup \{11\}$
$y \in \{1, 11\}$

So all combinations of x and y are:

- 01
- 011
- 001
- 0011

Therefore, we can express the set $\{xy\text{: where } x \in \{0\} \cup \{0\}^2 \text{ and } y \in \{1\} \cup \{1\}^2\}$ in roster notation as **{01, 011, 001, 0011}**.

e. $\{xy\text{: } x \in \{aa, ab\} \text{ and } y \in \{a\} \cup \{a\}^2\}$

In this example, we can tell that the set is composed of strings of two elements x and y that are concatenated. Let's get all possible values for x and y and then concatenate them to create the desired set.

We are given that $x \in \{aa, ab\}$.

We are given that $y \in \{a\} \cup \{a\}^2$.
Therefore: $y \in \{a\} \cup \{aa\}$
$y \in \{a, aa\}$

So all combinations of x and y are:

- aaa
- aaaa
- aba
- abaa

Therefore, we can express the set $\{xy: x \in \{aa, ab\} \text{ and } y \in \{a\} \cup \{a\}^2\}$ in roster notation as **{aaa, aaaa, aba, abaa}**.

d) Exercise 3.5.7

Consider the elements:

- $A = \{a\}$
- $B = \{b, c\}$
- $C = \{a, b, d\}$

c.

$(A \times B) \cup (A \times C) = (\{a\} \times \{b, c\}) \cup (\{a\} \times \{a, b, d\})$

$(A \times B) \cup (A \times C) = \{ab, ac\} \cup (\{a\} \times \{a, b, d\})$

$(A \times B) \cup (A \times C) = \{ab, ac\} \cup \{aa, ab, ad\}$

$(A \times B) \cup (A \times C) = \{aa, ab, ac, ad\}$

f.

$P(A \times B) = P(\{a\} \times \{b, c\})$

$P(A \times B) = P(\{ab, ac\})$

$P(A \times B) = \{\varnothing, \{ab\}, \{ac\}, \{ab, ac\}\}$

g.

$P(A) \times P(B) = P(\{a\}) \times P(\{b, c\})$

$P(A) \times P(B) = \{\varnothing, \{a\}\} \times \{\varnothing, \{b\}, \{c\}, \{b, c\}\}$

$P(A) \times P(B) = \{(\varnothing, \varnothing), (\varnothing, \{b\}), (\varnothing, \{c\}), (\varnothing, \{b, c\}), (\{a\}, \varnothing), (\{a\}, \{b\}), (\{a\}, \{c\}), (\{a\}, \{b, c\})\}$

Question 11

a) Exercise 3.6.2

b.

Step	Statement	Law/Identity
1	$(B \cup A) \cap (\overline{B} \cup A)$	
2	$(A \cup B) \cap (\overline{B} \cup A)$	Commutative Law on line 1
3	$(A \cup B) \cap (A \cup \overline{B})$	Commutative Law on line 2
4	$A \cup (B \cap \overline{B})$	Distributive Law on line 3
5	$A \cup \varnothing$	Complement Law on line 4
6	A	Identity Law on line 5

c.

Step	Statement	Law/Identity
1	$\overline{A \cap \overline{B}}$	
2	$\overline{A} \cup \overline{\overline{B}}$	De Morgan's Law on line 1
3	$\overline{A} \cup B$	Double Complement Law on line 2

b) Exercise 3.6.3

b. We can prove this in two ways.

- Proof 1: Counterexample

  If we set A = {1, 2} and B = {1, 2, 3, 4}.

  We get: $A - (B \cap A) = \{1,2\} - (\{1,2\} \cap \{1, 2, 3, 4\})$.

  $A - (B \cap A) = \{1,2\} - \{1,2\}$.

  $A - (B \cap A) = \varnothing$.

  $\boxed{\text{Therefore, since } \varnothing \neq \{1, 2\}\text{, we conclude that } A - (B \cap A) = A \text{ is \textbf{False}.}}$

- Proof 2: Simplify with identity laws then counterexample.

Step	Statement	Law/Identity
1	$A - (B \cap A)$	
2	$A \cap \overline{(B \cap A)}$	Subtraction Law on line 1
3	$A \cap (\overline{B} \cup \overline{A})$	De Morgan's Law on line 2
4	$(A \cap \overline{B}) \cup (A \cap \overline{A})$	Distributive Law on line 3
5	$(A \cap \overline{B}) \cup \varnothing$	Complement Law on line 4
6	$A \cap \overline{B}$	Identity Law on line 5

If we take A = {1,2} and B = $\{x \in N : x \text{ is odd}\}$, then $A \cap \overline{B} = \{2\}$. So, $A \cap \overline{B} \neq A$.

$\boxed{\text{Therefore, } A - (B \cap A) = A \text{ is \textbf{False}.}}$

d. We can prove this in two ways.

- Proof 1: Counterexample.

If we set A = {1,2} and B = {1, 2, 3, 4}.

$(B - A) \cup A = (\{1,2, 3, 4\} - \{1, 2\}) \cup \{1,2\}$

$(B - A) \cup A = \{3, 4\} \cup \{1,2\}$

$(B - A) \cup A = \{1,2, 3, 4\}$

Therefore, since $\{1,2, 3, 4\} \neq \{1, 2\}$, we conclude that $(B - A) \cup A = A$ is **False**.

- Proof 2: Simplify with identity laws.

Step	Statement	Law/Identity
1	$(B - A) \cup A$	
2	$(B \cap \overline{A}) \cup A$	Subtraction Law on line 1
3	$A \cup (B \cap \overline{A})$	Commutative Law on line 2
4	$(A \cup B) \cap (A \cup \overline{A})$	Distributive Law on line 3
5	$(A \cup B) \cap U$	Complement Law on line 4
6	$A \cup B$	Identity Law on line 5

If we set A = {1,2} and B = {1, 2, 3, 4}.

$A \cup B = \{1,2\} \cup \{1, 2, 3, 4\}$

$A \cup B = \{1, 2, 3, 4\}$

Therefore, since $\{1,2, 3, 4\} \neq \{1, 2\}$, we conclude that $(B - A) \cup A = A$ is **False**.

c) Exercise 3.6.4

b.

Step	Statement	Law/Identity
1	$A \cap (B - A)$	
2	$A \cap (B \cap \overline{A})$	Subtraction Law on line 1
3	$(A \cap B) \cap \overline{A}$	Associative law on line 2
4	$(B \cap A) \cap \overline{A}$	Commutative law on line 3
5	$B \cap (A \cap \overline{A})$	Associative law on line 4
6	$B \cap \varnothing$	Complement law on line 5
7	$\varnothing$	Domination law on line 6

c.

Step	Statement	Law/Identity
1	$A \cup (B - A)$	
2	$A \cup (B \cap \overline{A})$	Subtraction Law on line 1
3	$(A \cup B) \cap (A \cup \overline{A})$	Distributive law on line 2
4	$(A \cup B) \cap U$	Complement law on line 3
5	$A \cup B$	Identity law on line 4