NYU Tandon Bridge

Homework 5

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Question 3

a) Exercise 4.1.3

b.

The function in question is $f : R \rightarrow R, f(x) = 1 / (x^2 - 4)$.

If we set x = -2 or x = 2, the denominator becomes 0 and the resulting *f(x)* is not a real number.

For x = -2:

$$f(x) = 1 / ((-2)^2 - 4)$$

$$f(x) = 1 / (4 - 4)$$

$$f(x) = 1 / 0$$

For x = 2:

$$f(x) = 1 / (2^2 - 4)$$

$$f(x) = 1 / (4 - 4)$$

$$f(x) = 1 / 0$$

Therefore, the function is not well-defined from **R** to **R.**

c.

The function in question is $f : R \to R,\ f(x) = \sqrt{x^2}$.

This function operates in two steps:

1. Squares the real number x.
2. Takes the square root of the result of step 1.

The only limiting factor we need to consider is that the number inside of the radical must be non-negative.

Regardless of whether x is negative, zero, or positive, x^2 is always non-negative.

As such, the number inside the radical is always non-negative.

For any non-negative real number, the square root function is well defined and returns a non-negative real number.

Therefore, regardless of the real value of x, f(x) will map it to a non-negative real number.

We conclude that $f(x) = \sqrt{x^2}$ is well defined from **R** to **R**. The range is all non-negative real numbers, i.e. $[0, \infty)$.

b) Exercise 4.1.5

b.

The function in question is $f: \{2, 3, 4, 5\} \rightarrow Z, f(x) = x^2$.

Since the domain is small, by exhaustion:

$f(2) = 2^2 = 4$

$f(3) = 3^2 = 9$

$f(4) = 4^2 = 16$

$f(5) = 5^2 = 25$

Therefore, the range of the function f is $\{4, 9, 16, 25\}$.

d.

The function in question is $f: \{0,1\}^5 \rightarrow Z. For\ x \in \{0,1\}5,\ f(x) = number\ of\ 1's\ in\ x.$

In string notation, the numbers in $\{0,1\}^5$ range from 00000 to 11111.

Hence, the number of 1's in x can go from 0 to 5 with every integer value in between. The order of the zeros and ones does not matter.

For example:

f(00000) = 0

f(00001) = 1

f(00011) = 2

f(00111) = 3

f(01111) = 4

f(11111) = 5

Therefore, the range of the function *f* is $\{0,\ 1,\ 2,\ 3,\ 4,\ 5\}$.

h.

The function in question is $f:\ \{1,\ 2,\ 3\} \times \{1,\ 2,\ 3\} \rightarrow Z \times Z,\ where\, f(x,y)\ =\ (y,\ x)$.

This function reverses the order of the first and second items in the ordered pair (x, y).

Given that the ordered pair is created by crossing the same set {1, 2, 3} with itself, the resulting set of ordered pairs in {1, 2, 3} x {1, 2, 3} is perfectly symmetrical.

As such, reversing the order of all pairs will create the same set of pairs as {1, 2, 3} x {1, 2, 3}.

By exhaustion:

f((1, 1)) = (1, 1)

f((1, 2)) = (2, 1)

f((1, 3)) = (3, 1)

f((2, 1)) = (1, 2)

f((2, 2)) = (2, 2)

f((2, 3)) = (3, 2)

f((3, 1)) = (1, 3)

f((3, 2)) = (2, 3)

f((3, 3)) = (3, 3)

Therefore, the range of *f* is {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}.

i.

The function in question is $f:\ \{1,\ 2,\ 3\} \times \{1,\ 2,\ 3\} \to Z \times Z,\ where\, f(x, y)\ =\ (x,\ y + 1).$

This function takes the ordered pair (x, y) and adds 1 to the second item y in the ordered pair.

Given that the ordered pair is created by crossing the same set {1, 2, 3} with itself, the resulting set of ordered pairs in {1, 2, 3} x {1, 2, 3} is:
{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)}

The output of *f* similar to this set but with every second item incremented by 1.

By exhaustion:

f((1, 1)) = (1, 2)

f((1, 2)) = (1, 3)

f((1, 3)) = (1, 4)

f((2, 1)) = (2, 2)

f((2, 2)) = (2, 3)

f((2, 3)) = (2, 4)

f((3, 1)) = (3, 2)

f((3, 2)) = (3, 3)

f((3, 3)) = (3, 4)

Therefore, the range of *f* is {(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)}

1.

The function in question is $f : P(\{1, 2, 3\}) \to P(\{1, 2, 3\}), for\ X \subseteq A, f(X) = X - \{1\}.$

This function takes in X from the power set of {1, 2, 3} and removes the element 1 from each X . The net effect is that we get the elements of the power set that do not have 1 in them.

Step 1: Determine P(A)

P(A) = {Ø, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

Determine $P(A) - \{1\}$ for all A's:

By exhaustion:

Ø - {1} = Ø

{1} - {1} = Ø

{2} - {1} = {2}

{3} - {1} = {3}

{1, 2} - {1} = {2}

{1, 3} - {1} = {3}

{2, 3} - {1} = {2, 3}

{1, 2, 3} - {1} = {2, 3}

Therefore, the range of *f* is {Ø, {2}, {3}, {2, 3}}

Question 4

Part I

a. Exercise 4.2.2

c.

The function in question is $h: Z \rightarrow Z,\ h(x) = x^3$.

Step 1: Determine if one-to-one.

We know that each number has a unique cube. As such, the function is one-to one.

Step 2: Determine if onto.

There are some values in **Z** that cannot be reached by taking the cube of an integer.

For example, the value y = 5 is a value in the target that cannot be reached by *h*.

$h(1) = 1^3 = 1$

$h(2) = 2^3 = 8$

Since the output of h increases as x increases, h(x) = 5 must be in between h(1) and h(2).

However, the value needed to produce 5 is $x \approx 1.71$, which is not an integer.

Therefore, the function $h: Z \rightarrow Z,\ h(x) = x^3$ is one-to-one but not onto.

g.

The function in question is $f: Z \times Z \to Z \times Z, f(x, y) = (x + 1, 2y)$.

The function outputs an ordered pair. Let's break the output up into each item in the ordered pair:

- The first item is created by taking the integer x and adding 1 to it.
- The second item is created by doubling the integer y.

Step 1: Determine if one-to-one.

- We know that $a = x + 1$ creates a unique integer a for every integer x.
- We know that $b = 2y$ creates a unique integer b for every integer y.

→ As such, the function *f* is one-to-one.

Step 2: Determine if onto.

- Every integer $a = x + 1$ can be reached by an integer $x = a - 1$.
- However, because the second item $b = 2y$ is created by doubling each integer y. b can never be an odd integer.

  If we take b = 1, the solution to $1 = 2y$ is $y = 0.5$ which is not in the domain **Z.**

→ As such, the function *f* is not onto.

Therefore, the function $f: Z \times Z \to Z \times Z, f(x, y) = (x + 1, 2y)$ is one-to-one but not onto.

k.

The function in question is $f: Z^+ \times Z^+ \rightarrow Z^+, f(x, y) = 2^x + y$.

The function takes in an ordered pair of positive integers x and y and outputs a positive integer created by taking 2 to the power of x and adding it to y.

Step 1: Determine if one-to-one.

- If we take $(x_1, y_1) = (2, 2)$. We get $f(2, 2) = 2^2 + 2 = 6$
- If we take $(x_2, y_2) = (1, 4)$. We get $f(1, 4) = 2^1 + 4 = 6$

→ As such, given that different values of x and y produce the same result, the function *f* is not one-to-one.

Step 2: Determine if onto.

- If we take a = 1. There is no combination of positive integers x and y such that

  $2^x + y = a$.

  Since x and y are in Z^+, their smallest values are each 1.

  $f(1, 1) = 2^1 + 1 = 3 > a$

  Since the function *f* increases as x and y increase, there is no combination of x and y in the domain that can give us $2^x + y = 1$.

→ As such, given that the function *f* can't produce some values in the target, the function *f* is not onto.

Therefore, the function $f: Z^+ \times Z^+ \rightarrow Z^+, f(x, y) = 2^x + y$ is neither one-to-one nor onto.

b. **Exercise 4.2.4**

b.
The function in question is:

$f:\ \{0,\ 1\}^3 \to \{0,\ 1\}^3$. *The output of f is obtained by taking the input string and replacing the first bit by* 1, *regardless of whether the first bit is a 0 or 1.*

Step 1: Determine if one-to-one.

Since the domain is $\{0,\ 1\}^3$, the possible input values to f are $\{000, 001, 010, 011, 100, 101, 110, 111\}$.

- $f(000)\ =\ 100$
- $f(100)\ =\ 100$

→ As such, given that different input values produce the same result, the function *f* is not one-to-one.

Step 2: Determine if onto.

Intuition: Given that the function replaces the first bit of the string by 1 regardless of the original value, the output can never begin with a 0. So it can't produce the values $\{000, 001, 010, 011\}$ in the target domain.

So, if we set a = 000, no input value to f can produce $f(x)$ = a.

Since the function doesn't touch the last two bits, we only need to try the input strings that match the last two bits of 000. These are 000 and 100.

f(000) = 100

f(100) = 100

→ As such, given that the function *f* can't produce some values in the target, the function *f* is not onto.

Therefore, the function *f* is neither one-to-one nor onto.

c.

The function in question is:

$f: \{0, 1\}^3 \rightarrow \{0, 1\}^3. The\ output\ of\ f\ is\ obtained\ by\ taking\ the\ input\ string\ and\ reversing\ the\ bits.$

<u>Step 1: Determine if one-to-one.</u>

Since the domain is $\{0, 1\}^3$, the possible input values to f are {000, 001, 010, 011, 100, 101, 110, 111}.

By exhaustion:

f(000) = 000

f(001) = 100

f(010) = 010

f(011) = 110

f(100) = 001

f(101) = 101

f(110) = 011

f(111) = 111

f(000) ≠ f(001) ≠ f(010) ≠ f(011) ≠ f(100) ≠ f(101)≠ f(110) ≠ f(111)

→ As such, given that each different value of x maps to a unique result, the function *f* is one-to-one.

<u>Step 2: Determine if onto.</u>

We saw that the range of *f* is {000, 001, 010, 011, 100, 101, 110, 111}. This range is equal to the target $\{0, 1\}^3$.

→ As such, since the function *f* can map inputs to all values in the target, the function *f* is onto.

$\boxed{\text{Therefore, the function } f \text{ is both one-to-one and onto.}}$

d.

The function in question is:

$f: \{0, 1\}^3 \rightarrow \{0, 1\}^4$. *The output of f is obtained by taking the input string and adding an extra copy of the first bit to the end of the string.*

Step 1: Determine if one-to-one.

Since the domain is $\{0, 1\}^3$, the possible input values to f are {000, 001, 010, 011, 100, 101, 110, 111}.

By exhaustion:

- f(000) = 0000
- f(001) = 0010
- f(010) = 0100
- f(011) = 0110
- f(100) = 1001
- f(101) = 1011
- f(110) = 1101
- f(111) = 1111

→ As such, given that each different value of x maps to a unique result, the function *f* is one-to-one.

Step 2: Determine if onto.

We just determined that the range of *f* is {0000, 0010, 0100, 0110, 1001, 1011, 1101, 1111}

The target is {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}.

→ As such, given that the range is not equal to the target, the function *f* is not onto.

Therefore, the function *f* is one-to-one but not onto.

g.

The function in question is:

$Let\ A = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}\ and\ let\ B = \{1\}.\ f:\ P(A) \to P(A).\ For\ X \subseteq A, f(X)\ =\ X - B.$

Step 1: Determine if one-to-one.

Since $A = \{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8\}$, $\{2\}$ and $\{1{,}2\}$ are both subsets of A (i.e. elements in P(A)).

- $f(\{2\}) = \{2\} - \{1\} = \{2\}$
- $f(\{1,\ 2\}) = \{1,\ 2\} - \{1\} = \{2\}$

→ As such, given that different input values produce the same result, the function *f* is not one-to-one.

Step 2: Determine if onto.

Since the target $P(A)$ is made up of all possible subsets of A, $\{1,\ 2\} \in P(A)$.

However, since f subtracts $\{1\}$ from all input values, it is impossible to get $\{1, 2\}$ (or any other element of P(A) that contains a 1).

→ As such, given that the function *f* can't produce some values in the target, the function *f* is not onto.

Therefore, the function *f* is neither one-to-one nor onto.

Part II

a. One-to-one, but not onto (over Z → Z^+).

f: Z → Z^+, f(x) = x^2 + 2 if x <= 0, f(x) = x^2 if x > 0

Step 1: Show that this function is one-to-one.

f is one-to-one given that each input produces a different output. The output produced from negative input values is offset from the positive equivalent inputs by 2 and the values do not overlap for any integer x.

- f(-2) = $(-2)^2$ + 2 = 6
- f(2) = 2^2 = 4

→ As such, given that different input values produce unique results, the function *f* is one-to-one.

Step 2: Show that this function is not onto.

Let's consider y = 5:
- y is in the target given that it is a positive integer.
- There is no x such that x^2 = 5 or x^2 + 2 = 5.

→ As such, given that the function *f* can't produce some values in the target, the function *f* is not onto.

Therefore, the function $f : Z \rightarrow Z^+$, $f(x) = x^2 + 2 \; if \; x <= 0, \; f(x) = x^2 \; if \; x > 0$ is one-to-one but not onto.

b. Onto, but not one-to-one over Z → Z⁺.

$f : Z \to Z^+ , f(x) = \sqrt{x^2} + 1$

<u>Step 1: Show that this function is not one-to-one.</u>

- $f(2) = \sqrt{2^2} + 1 = 3$

- $f(-2) = \sqrt{(-2)^2} + 1 = 3$

→ As such, given that different input values map to the same result, the function *f* is not one-to-one.

<u>Step 2: Show that this function is onto.</u>

The function *f* can be split into two parts.

- f(x) = x + 1 for x >= 0
- f(x) = -x + 1 for x < 0

Since the function is symmetric we can consider one portion of the function to determine the range. Let's simplify to f(x) = x + 1.

For all y in $\mathbf{Z}^+$, there exists an x = y - 1 that maps to y when entered into *f*.

→ As such, given that the function *f* maps inputs to all possible values in the target, the function *f* is onto.

Therefore, the function $f : Z \to Z^+ , f(x) = x^2 + 1$ is onto but not one-to-one.

c. **One-to-one and onto over $\mathbf{Z} \rightarrow \mathbf{Z}^{+}$.**

$f: Z \rightarrow Z^{+}, f(x) = -2x + 1 \text{ if } x <= 0, \; f(x) = 2x \text{ if } x > 0$

Step 1: Show that this function is one-to-one.

f is one-to-one given that each input produces a different output. The output produced from negative input values is offset from the positive equivalent inputs by 1 and the values do not overlap for any integer x.

- f(0) = (-2)*(0) + 1 = 1
- f(1) = 2*1 = 2
- f(-1) = (-2)*(-1) + 1 = 3
- f(2) = 2^2 = 4
- f(-2) = (-2)*(-2) + 1 = 5
- ...

→ As such, given that different input values produce unique results, the function *f* is one-to-one.

Step 2: Show that this function is onto.

The function f can be split into two parts.

- f(x) = -2x + 1 for x <= 0
- f(x) = 2x for x > 0

Let's consider each part separately.

- For x <= 0, f(x) = -2x + 1 maps all non-positive integers to all possible positive odd integers.
- For x > 0, f(x) = 2x maps all positive integers to all possible positive even integers.

So, for all y in $\mathbf{Z}^{+}$, there exists an x that maps to y when entered into *f*.

→ As such, given that the function *f* maps inputs to all possible values in the target, the function *f* is onto.

Therefore, the function $f: Z \rightarrow Z^{+}, f(x) = -2x + 1 \text{ if } x <= 0, \; f(x) = 2x \text{ if } x > 0$ is both one-to-one and onto.

d. Neither one-to-one nor onto over $\mathbf{Z} \rightarrow \mathbf{Z}^{+}$.

$f : Z \rightarrow Z^{+}, f(x) = x^2 + 2$

<u>Step 1: Show that this function is not one-to-one.</u>

- $f(2) = 2^2 + 2 = 4$
- $f(-2) = (-2)^2 + 2 = 4$

→ As such, given that different input values map to the same result, the function *f* is not one-to-one.

<u>Step 2: Show that this function is onto.</u>

Let's consider y = 1.

- y is in the target given that it is a positive integer.
- There is no x such that $x^2 + 2 = 1$.

→ As such, given that the function *f* can't produce some values in the target, the function *f* is not onto.

Therefore, the function $f : Z \rightarrow Z^{+}, f(x) = x^2 + 2$ is neither onto nor one-to-one.

Question 5

a) Exercise 4.3.2

c. We know that a function has a well defined inverse if and only if it is a bijection.

Step 1: Check if *f* is one-to-one

We need to show that $x \neq x' \rightarrow 2x + 3 \neq 2x' + 3$. To prove this statement let's prove the contrapositive. Contrapositive: $2x + 3 = 2x' + 3 \rightarrow x = x'$

- Start by assuming $2x + 3 = 2x' + 3$
- Subtract both sides by 3: $2x = 2x'$
- Divide both sides by 2: $x = x'$

→ We just proved by contrapositive that $x \neq x' \rightarrow 2x + 3 \neq 2x' + 3$. So, f is one-to-one.

Step 2: Check if f is onto

We know that f: **R** → **R,** f(x) = 2x + 3. This means that for every real number y in the target, there exists a real number x in the domain, such that y = 2x + 3.

→ As such, *f* is onto. Let's solve for x to get the inverse.

Step 3: Solve for x in terms of y

We have f(x) = 2x + 3

y = 2x + 3

y - 3 = 2x

$\frac{y-3}{2} = x$

Step 4: Express the inverse in terms of y

Therefore, the inverse function is: $f^{-1} : R \rightarrow R,\ f^{-1}(y) = \frac{y-3}{2}$

d.

The function in question is:

Let A be defined to be the set {1, 2, 3, 4, 5, 6, 7, 8}. f: P(A) → {0, 1, 2, 3, 4, 5, 6, 7, 8}. For X ⊆ A, f(X) = |X|. Recall that for a finite set A, P(A) denotes the power set of A which is the set of all subsets of A.

We know that a function has a well defined inverse if and only if it is a bijection.

Step 1: Check if *f* is one-to-one

f maps the elements in the power set of A to their cardinality.

Since A = {1, 2, 3, 4, 5, 6, 7, 8}, we know that {1} and {2} are both subsets of A (and therefore members of P(A)).

- *f*({1}) = 1
- *f*({2}) = 1

→ Since different input values map to the same output value, *f* is not one-to-one.

Since *f* is not one-to-one, it cannot be a bijection. As such, the function *f* does not have a well-defined inverse f^{-1}.

g.

The function in question is:

f: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f is obtained by taking the input string and reversing the bits.

We know that a function has a well defined inverse if and only if it is a bijection.

Step 1: Check if f is one-to-one

Since the domain is small, we can check by exhaustion:

- f(000) = 000
- f(001) = 100
- f(010) = 010
- f(011) = 110
- f(100) = 001
- f(101) = 101
- f(110) = 011
- f(111) = 111

→ As such, since each input value maps to a unique output, f is one-to-one.

Step 2: Check if f is onto

We saw that the range of f is {000, 001, 010, 011, 100, 101, 110, 111}. This range is equal to the target $\{0, 1\}^3$.

→ As such, given that the function f can map inputs to all values in the target, the function f is onto. Since f is one-to-one and onto, it has a well-defined inverse.

Step 3: Determine the inverse function

We notice that if we put the result of f(x) back into f, we get the input value x. For example:

- f(110) = 011
- f(011) = 110

This is the case for all inputs and outputs of f. As such, the function f is its own inverse function!

Therefore, the function f is its own inverse function. We can write f^{-1}: $\{0, 1\}^3 \rightarrow \{0, 1\}^3$. The output of f^{-1} is obtained by taking the input string and reversing the bits.

i.

The function in question is: $f: Z \times Z \to Z \times Z,\ f(x,\ y) = (x+5,\ y-2)$

Let's break the function down into 2 components. We will evaluate if each item in the ordered pair is one-to-one and onto.

<u>Step 1: Check if *f* is one-to-one</u>

Item 1	Item 2
We want to establish that $x \neq x' \to x+5 \neq x'+5$	We want to establish that $y \neq y' \to y-2 \neq y'-2$
Taking the contrapositive, we get: $x+5 = x'+5 \to x = x'$	Taking the contrapositive, we get: $y-2 = y'-2 \to y = y'$
Start with the assumption that $x+5 = x'+5$	Start with the assumption that $y-2 = y'-2$
Subtract both sides by 5 $x = x'$	Subtract both sides by 2 $y = y'$

→ We just proved by contrapositive that $x \neq x' \to x+5 \neq x'+5$ and $y \neq y' \to y-2 \neq y'-2$. As such, *f* is one-to-one.

<u>Step 2: Check if *f* is onto</u>

We know that we can break down *f* into

- *a*: **Z** → **Z**, *a*(x) = x + 5
- *b*: **Z** → **Z**, *b*(y) = y - 2.

For every integer m in the target of *a*, there is an integer x in the domain, such that m = x + 5.

For every integer n in the target of *b*, there exists an integer x in the domain, such that n = y + 2.

→ As such, *f* is onto. Let's solve for x and y to get the inverse.

Step 3: Determine the inverse of each item in the pair

Item 1		Item 2	
Equation	Justification	Equation	Justification
m = x + 5	First item in the resulting ordered pair of *f*	n = y - 2	Second item in the resulting ordered pair of *f*
m - 5 = x	Subtract both sides by 5	n + 2 = y	Add 2 to both sides

Step 4: Express the inverse.

Therefore, the inverse function is: $f^{-1} : Z \times Z \rightarrow Z \times Z, f^{-1}(x, y) = (x - 5, y + 2)$

b) Exercise 4.4.8

c.

$f \circ h = f(g(x))$

$f \circ h = f(x^2 + 1)$

$f \circ h = 2(x^2 + 1) + 3$

$\boxed{f \circ h = 2x^2 + 5}$

d.

$h \circ f = g(f(x))$

$h \circ f = g(2x + 3)$

$h \circ f = (2x + 3)^2 + 1$

$\boxed{h \circ f = 4x^2 + 12x + 10}$

c) Exercise 4.4.2

b.

$f \circ h\,(52) = f(h(52))$

$f \circ h\,(52) = f(\lceil \frac{52}{5} \rceil)$

$f \circ h\,(52) = f(11)$

$f \circ h\,(52) = 11^2$

$\boxed{f \circ h\,(52) = 121}$

c.

$g \circ h \circ f\,(4) = g(h(f(4)))$

$g \circ h \circ f\,(4) = g(h(4^2))$

$g \circ h \circ f\,(4) = g(h(16))$

$g \circ h \circ f\,(4) = g(\lceil \frac{16}{5} \rceil)$

$g \circ h \circ f\,(4) = g(4)$

$g \circ h \circ f\,(4) = 2^4$

$\boxed{g \circ h \circ f\,(4) = 16}$

d.

$h \circ f\,(x) = h(f(x))$

$h \circ f\,(x) = h(x^2)$

$\boxed{h \circ f\,(x) = \lceil \frac{x^2}{5} \rceil}$

d) Exercise 4.4.6

c.

$h \circ f\ (x) = h(f(x))$	
$h \circ f\ (010) = h(f(010))$	
$h \circ f\ (010) = h(110)$	f replaces the first bit of the input by 1. 010 becomes 110.
$h \circ f\ (010) = 111$	h replaces the last bit with a copy of the first bit. 110 becomes 111.

d.
The domain of f i s{000, 001, 010, 011, 100, 101, 110, 111}.

By definition, f replaces the first bit of the input by 1, the outputs are:

- f(000) = 100
- f(001) = 101
- f(010) = 110
- f(011) = 111
- f(100) = 100
- f(101) = 101
- f(110) = 110
- f(111) = 111

Therefore, the range of f is {100, 101, 110, 111}. So the input values that can be entered into h are 100, 101, 110, and 111.

By definition, h replaces the last bit with a copy of the first bit.

- h(100) = 101
- h(101) = 101
- h(110) = 111
- h(111) = 111

Therefore, the range of $h \circ f$ is {101, 111}.

e.
The domain of f i s{000, 001, 010, 011, 100, 101, 110, 111}.

By definition, f replaces the first bit of the input by 1, the outputs are:
- f(000) = 100
- f(001) = 101
- f(010) = 110
- f(011) = 111
- f(100) = 100
- f(101) = 101
- f(110) = 110
- f(111) = 111

Therefore, the range of f is {100, 101, 110, 111}. So the input values that can be entered into g are 100, 101, 110, and 111.

By definition, g reverses the order of the bits.
- g(100) = 001
- g(101) = 101
- g(110) = 011
- g(111) = 111

Therefore, the range of $g \circ f$ is {001, 101, 011, 111}.

e) Exercise 4.4.4

c.

Step 1: Express $g \circ f$

$$g \circ f(x) = g(f(x))$$

As such, the input for g in $g \circ f$ is the result of $f(x)$.

Step 2: Relate f is not one-to-one to $g \circ f$

If f is not one-to-one, the input into g is the same for different values of x.

Let's set the resulting value y = f(x).

Since the input y into g(y) is the same for different values of x, g(y) will have the same result for different values of x.

For example, suppose we have x_1 and x_2 such that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$.

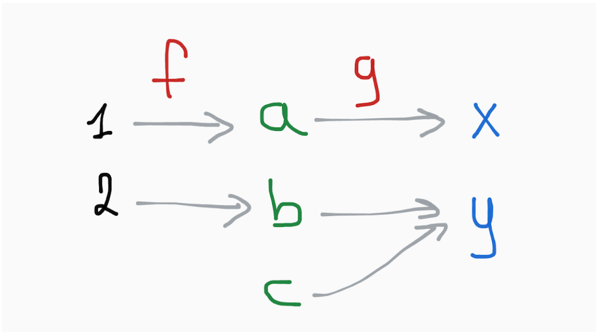
- $g \circ f(x_1) = g(f(x_1)) = g(y)$
- $g \circ f(x_2) = g(f(x_2)) = g(y)$

So $g \circ f(x_1) = g \circ f(x_2)$

As such, if f is not one-to-one, $g \circ f$ will have the same result for different input values.

Therefore, if f is not one-to-one it is not possible that $g \circ f$ is one-to-one.

d.
Consider the example:



<u>Step 1:</u> Show that g is not one-to-one.

In this example, g(b) = g(c) = y

→ As such, g is not one-to-one

<u>Step 2:</u> Show that $g \circ f$ is one-to-one.

- $g \circ f\,(1) = x$
- $g \circ f\,(2) = y$

→ As such, since all input values map to different results, $g \circ f$ is one-to-one.

Therefore, if g is not one-to-one it is possible that $g \circ f$ is one-to-one.