NYU Tandon Bridge

Homework 8

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02/19/2021

Question 7

a) **Exercise 6.1.5**

b. What is the probability that the hand is a three of a kind?

Step 1: Calculate the total number of 5-card hands

There are 52 cards in a deck.
We need to choose 5 cards.

So, the total number of hands is $C(52,5)$.

Step 2: Calculate the number of 5-card hands with a 3 of a kind

We know that:
- There are 13 ranks
- There are 4 suits

We need to choose:
- 3 cards of the same rank
- 2 cards of different ranks

3 cards of the same rank: $C(13,\ 1) \cdot C(4,\ 3)$. (pick a rank and choose 3 cards in the rank)

2 cards of different ranks: $C(12,\ 2) \cdot C(4,\ 1)^2$. (pick two ranks and for each rank choose 1 of 4 suits)

So, the total number of 3 of a kind 5 card hands is $C(13,\ 1) \cdot C(4,\ 3) \cdot C(12,\ 2) \cdot C(4,\ 1)^2$

Step 3: Calculate the the probability of of a three-of-a-kind

$$p(three - of - a - kind) = \frac{\#\ three{-}of{-}a{-}kinds}{\#\ 5{-}card\ hands}$$

$$\boxed{\begin{aligned} p(three - of - a - kind) &= \frac{C(13,1)\cdot C(4,3)\cdot C(12,2)\cdot C(4,1)^2}{C(52,5)} \\ p(three - of - a - kind) &\approx 0.02112 \end{aligned}}$$

c. What is the probability that all 5 cards have the same suit?

Step 1: Calculate the total number of 5-card hands

There are 52 cards in a deck.
We need to choose 5 cards.

So, the total number of hands is $C(52,5)$.

Step 2: Calculate the number of 5-card hands with 5 cards of the same suit

We know that:
- There are 13 ranks
- There are 4 suits

We need to choose:
- 1 suit
- 5 ranks from the suit

Choose one of four suits: $C(4,\ 1)$
Choose 5 cards from that suit: $C(13,\ 5)$

So, the total number of hands with 5 cards of the same suit is $C(4,\ 1)\cdot C(13,\ 5)$

Step 3: Calculate the the probability of of 5 cards with same suit

$$p(flush + straight\,flush) = \frac{\#\,flushes\ (including\ straight\,flushes)}{\#\ 5{-}card\ hands}$$

$$p(flush + straight\,flush) = \frac{C(4,1)\cdot C(13,5)}{C(52,5)}$$
$$p(flush + straight\,flush) \approx 0.00198$$

d. What is the probability that the hand is a two of a kind

Step 1: Calculate the total number of 5-card hands

There are 52 cards in a deck.
We need to choose 5 cards.

So, the total number of hands is $C(52, 5)$.

Step 2: Calculate the number of 5-card hands with exactly one pair

We know that:

- There are 13 ranks
- There are 4 suits

We need to choose:

- 2 cards of the same rank
- 3 cards of different ranks

Choose two cards of same rank: $C(13, 1) \cdot C(4,\ 2)$ (choose a rank and choose two suits for that rank)
Choose three cards of different ranks: $C(12,\ 3) \cdot C(4,\ 1) \cdot C(4,\ 1) \cdot C(4,\ 1)$ (choose 3 ranks and for each card choose 1 of 4 suits)

So, the total number of hands with exactly one pair is $C(13,\ 1) \cdot C(4,\ 2) \cdot C(12,\ 3) \cdot C(4,\ 1)^3$

Step 3: Calculate the the probability of of 5 cards with same suit

$$p(pair) = \frac{\#\ 5\text{−}card\ hands\ with\ one\ pair}{\#\ 5\text{−}card\ hands}$$

$$p(pair) = \frac{C(13, 1) \cdot C(4, 2) \cdot C(12, 3) \cdot C(4, 1)^3}{C(52, 5)}$$

$$p(pair) \approx 0.422569$$

b) Exercise 6.2.4

a. The hand has at least one club.

Step 1: Calculate the total number of 5-card hands

There are 52 cards in a deck.
We need to choose 5 cards.

So, the total number of hands is $C(52, 5)$.

Step 2: Calculate the number of 5-card hands with no clubs

We know that:
- There are 4 suits
- Each suit has 13 ranks

We need to choose:
- Any 5 cards from three suits of possibilities

$C(3 * 13,\ 5) = C(39,\ 5)$

So, the total number of hands with no clubs is $C(39,\ 5)$

Step 3: Calculate the the probability of of 5 cards with no clubs

$$p(no\ clubs) = \frac{\#\ hands\ with\ not\ clubs}{\#\ 5\text{−}card\ hands}$$
$$p(no\ clubs) = \frac{C(39,5)}{C(52,5)}$$

Step 4: Calculate the the probability of of 5 cards with at least one club

We know that: Hands with at least one club is the complement of hands with no clubs.
So: $p(\overline{no\ clubs}) = 1 - p(no\ clubs)$
$p(1^{+}club) = 1 - p(no\ clubs)$

$$\boxed{\begin{aligned} p(1^{+}club) &= 1 - \frac{C(39,\ 5)}{C(52,\ 5)} \\ p(1^{+}club) &\approx 0.7784 \end{aligned}}$$

b. The hand has at least two cards with the same rank.

Step 1: Calculate the total number of 5-card hands

There are 52 cards in a deck.
We need to choose 5 cards.

So, the total number of hands is $C(52,5)$.

Step 2: Calculate the number of 5-card hands with no two cards of the same rank

We know that:

- There are 13 ranks
- Each rank has 4 suits

We need to choose:

- 5 different ranks
- Any of 4 suits

Choose 5 ranks: $C(13,\ 5)$
Choose 1 suit from 4 for each of the five cards: $C(4,\ 1)^5$
So, the total number of hands with no two cards of the same rank is $C(13,\ 5) \cdot C(4,1)^5$

Step 3: Calculate the the probability of of 5 cards with no clubs

$$p(no\ same\ rank) = \frac{\#\ hands\ with\ no\ two\ cards\ of\ the\ same\ rank}{\#\ 5{-}card\ hands}$$
$$p(no\ same\ rank) = \frac{C(13,5) \cdot C(4,1)^5}{C(52,5)}$$

Step 4: Calculate the the probability of of 5 cards with at least one club

We know that: Hands with at least two cards of the same rank is the complement of hands with no cards of the same rank.

So: $p(\overline{no\ same\ rank}) = 1 - p(no\ same\ rank)$
$p(1^{+} same\ rank) = 1 - p(no\ same\ rank)$

$$\boxed{\begin{aligned} p(1^{+} same\ rank) &= 1 - \frac{C(13,5) \cdot C(4,1)^5}{C(52,5)} \\ p(1^{+} rank) &\approx 0.4929 \end{aligned}}$$

c. **The hand has exactly one club or exactly one spade.**

Since this asked for "or", I will specify that I solved this problem based on my understanding of the inclusive or.

<u>Step 1: Calculate the total number of 5-card hands</u>

There are 52 cards in a deck.
We need to choose 5 cards.

So, the total number of hands is $C(52,\ 5)$.

<u>Step 2: Calculate the number of 5-card hands with exactly one club</u>

We know that:

- There are 13 club cards

We need to choose:

- 1 of 13 clubs
- 4 cards from other suits

Choose a club: $C(13,\ 1)$
Choose four non-club cards: $C(39,\ 4)$
So, the total number of hands with exactly one club is $C(13,\ 1) \cdot C(39,\ 4)$

<u>Step 3: Calculate the number of 5-card hands with exactly one spade</u>

We know that:

- There are 13 spade cards

We need to choose:

- 1 of 13 spades
- 4 cards from other suits

Choose a spade: $C(13,\ 1)$
Choose four non-spade cards: $C(39,\ 4)$
So, the total number of hands with exactly one spade is $C(13,\ 1) \cdot C(39,\ 4)$

Step 4: Calculate the number of 5-card hands with exactly one club and exactly one spade

We know that:

- There are 13 spade cards
- There are 13 club cards

We need to choose:

- 1 of 13 spades
- 1 of 13 clubs
- 3 cards from other suits

Choose a spade: $C(13,\ 1)$
Choose a club: $C(13,\ 1)$
Choose 3 non-spade cards: $C(26,\ 3)$
So, the total number of hands with exactly one club and exactly one spade is
$C(13,\ 1) \cdot C(13,\ 1) \cdot C(26,\ 3)$

Step 6: Calculate the the probability of 5 card hand with exactly one club or exactly one spade

By definition:

$$p(1\ club\ or\ 1\ spade) = p(1\ club) + p(1\ spade) - p(1\ club\ and\ 1\ spade)$$

$$p(1\ club\ or\ 1\ spade) = \frac{\#\ hands\ 1\ club}{total\ \#\ hands} + \frac{\#\ hands\ 1\ spade}{total\ \#\ hands} - \frac{\#\ hands\ 1\ club\ and\ 1\ spade}{total\ \#\ hands}$$

$$p(1\ club\ or\ 1\ spade) = \frac{\#\ hands\ 1\ club + \#\ hands\ 1\ spade - \#\ hands\ 1\ club\ and\ 1\ spade}{total\ \#\ hands}$$

$$p(1\ club\ or\ 1\ spade) = \frac{2 \cdot C(13,1) \cdot C(39,4) - C(13,1) \cdot C(13,1) \cdot C(26,3)}{C(52,5)}$$

$$p(1^{+}rank) \approx 0.65377$$

d. **The hand has at least one club or at least one spade.**

Since this asked for "or", I will specify that I solved this problem based on my understanding of the inclusive or.

Step 1: Calculate the total number of 5-card hands

There are 52 cards in a deck.
We need to choose 5 cards.

So, the total number of hands is $C(52,\ 5)$.

Step 2: Calculate the number of 5-card hands with no clubs and no spades

We know that:

- There are 4 suits, each with 13 cards

We need to choose:

- 5 cards from the diamond and heart suits

Choose 5 heart or diamond cards: $C(52 - 13 * 2,\ 5)$

So, the total number of hands with exactly one club is $C(26,\ 5)$

Step 3: Calculate the probability of 5-card hands with no clubs and no spades

$$p(no\ clubs\ no\ spades)\ = \frac{\#\ hands\ with\ just\ diamonds\ and\ hearts}{\#\ 5{-}card\ hands}$$
$$p(no\ clubs\ no\ spades)\ = \frac{C(26,5)}{C(52,5)}$$

Step 4: Calculate the the probability of 5 card hand with at least one club or one spade

We know that: Hands with at least one club or at least one spade is the complement of hands with no clubs and no spades.

$$p(1^{+}\ club\ or\ 1^{+}\ spade)\ = 1 - p(no\ clubs\ no\ spades)$$

$$p(1^{+}\ club\ or\ 1^{+}\ spade)\ = 1 - \frac{C(26,5)}{C(52,5)}$$
$$p(1^{+}\ club\ or\ 1^{+}\ spade)\ \approx 0.9746$$

Question 8

a) Exercise 6.3.2

a.

<u>Step 1: Calculate p(A)</u>

$\#\ strings\ with\ b\ in\ the\ middle\ = C(1,1) \cdot 6! = 6!$
$\#\ total\ strings = 7!$

$p(A) = \frac{\#\ strings\ with\ b\ in\ the\ middle}{\#\ total\ strings}$

$$\boxed{\begin{aligned} p(A) &= \tfrac{6!}{7!} \\ p(A) &= \tfrac{1}{7} \end{aligned}}$$

<u>Step 2: Calculate p(B)</u>

Since b and c can both occupy any position in the string. We can infer that c can be to the left of b or the right of b with equal probability.

Proof:
Every string where b is to the right of c is coupled 1:1 with a string that has the exact same positions for a,d,e,f,g but with b and c swapped.

Given that c can only be to the right or to the left of b. Those two options must sum to the total number of choices for strings.

Let x denote the number of strings with c to the left or to the right of b (the values are equal).

So $x + x = 7!$
$2x = 7!$
$x = \frac{7!}{2}$

$\#\ strings\ where\ c\ is\ to\ the\ right\ of\ b\ = \frac{7!}{2}$
$\#\ total\ strings = 7!$

$p(B) = \frac{\#\ strings\ with\ c\ to\ the\ right\ of\ b}{\#\ total\ strings}$

$p(B) = \frac{\frac{7!}{2}}{7!}$

$$\boxed{p(B) = \tfrac{1}{2}}$$

Step 3: Calculate p(C)

For the letters "def" to occur together in that order we can treat all 3 as a single unit. We get:

$\#\ strings\ with\ "def"\ = 5!$

$\#\ total\ strings = 7!$

$p(C) = \frac{\#\ strings\ with\ def}{\#\ total\ strings}$

$p(C) = \frac{5!}{7!}$

$\boxed{p(C) = \frac{1}{42}}$

b. What is p(A|C)?

Step 1: Calculate p(A n C)

The criteria are:

- b falls in the middle
- def occurs together in that order

$\#\ strings\ with\ A\ and\ C = C(1,1) \cdot C(2,1) \cdot P(3,3)$

$\#\ strings\ with\ A\ and\ C = 2 \cdot 3!$

$\#\ total\ strings = 7!$

So:

$p(A \cap C) = \frac{\#\ strings\ with\ def\ together\ and\ b\ in\ middle}{\#\ total\ strings}$

$p(A \cap C) = \frac{2 \cdot 3!}{7!}$

Step 2: Calculate p(A|C)

By definition:

$p(A|C) = \frac{p(A \cap C)}{p(C)}$

$p(A|C) = \frac{\frac{2 \cdot 3!}{7!}}{\frac{5!}{7!}}$

$$\boxed{\begin{aligned} p(A|C) &= \frac{2 \cdot 3!}{5!} \\ p(A|C) &= \frac{1}{10} \end{aligned}}$$

c. What is p(B|C)?

Step 1: Calculate p(B n C)

The criteria are:

- c appears to the right of b
- def occurs together in that order

If b is in position 0: (Chooses location of b, then a,c,def,g can occur anywhere after)

- $(\# \ strings \ with \ B \ and \ C)_0 = C(1,1) \cdot 4!$

If b is in position 1: (a/g can come first, then b, then g/a, c, def can occur in any order after)

- $(\# \ strings \ with \ B \ and \ C)_1 = C(2,1) \cdot C(1,1) \cdot 3!$

If b is in position 2: (a, g in any order, then b, then def, c in any order)

- $(\# \ strings \ with \ B \ and \ C)_2 = 2! \cdot C(1,1) \cdot 2!$

If b is in position 3: (def must come first, then b, then a,g,c in any order)

- $(\# \ strings \ with \ B \ and \ C)_3 = C(3,3) \cdot C(1,1) \cdot 3!$

If b is in position 4: (def and one of a/g in any order, then b, then g/a and c in any order)

- $(\# \ strings \ with \ B \ and \ C)_4 = C(3,3) \cdot C(2,1) \cdot 2! \cdot C(1,1) \cdot 2!$

If b is in position 5: (a,def,g in any order, then b, then c)

- $(\# \ strings \ with \ B \ and \ C)_5 = 3! \cdot C(1,1) \cdot C(1,1)$

If b is in position 6: (no way of having c to the right of b if b is last)

- $(\# \ strings \ with \ B \ and \ C)_6 = 0$

$\# \ strings \ with \ B \ and \ C = 4! + 2 \cdot 3! + 8 + 3! + 4 + 3!$
$\# \ strings \ with \ B \ and \ C = 60$
$\# \ total \ strings = 7!$

So:
$p(B \cap C) = \frac{\# \ strings \ with \ def \ together \ and \ b \ in \ middle}{\# \ total \ strings}$
$p(B \cap C) = \frac{60}{7!}$
$p(B \cap C) = \frac{1}{84}$

Alternate explanation:
When def are together, we can consider them as a single "unit". In this case, we end up with 5 elements {a, b, c, def, g}. Even in this case, the same logic about c being to the left or right of b still applies: Every string where c is to the left of b is coupled 1:1 with a string where it is to the left of c simply by swapping b and c (or taking the mirror image). Therefore, the number of strings must be half of the total number of strings that satisfy C, so 5!/2 = 60. Therefore, the probability is $p(B \cap C) = 60/7!$

Step 2: Calculate p(B|C)

By definition:

$$p(B|C) = \frac{p(B \cap C)}{p(C)}$$

$$p(B|C) = \frac{\frac{60}{7!}}{\frac{5!}{7!}}$$

$$\boxed{p(B|C) = \frac{60}{5!}}$$

$$\boxed{p(B|C) = \frac{1}{2}}$$

d. What is p(A|B)?

Step 1: Calculate p(A n B)

The criteria are:

- c appears to the right of b
- b is in the middle

$$\#\ strings\ with\ A\ and\ B = P(5,3) \cdot C(1,1) \cdot 3!$$

$$\#\ strings\ with\ A\ and\ B = 360$$

$$\#\ total\ strings = 7!$$

So:

$$p(B \cap A) = \frac{\#\ strings\ with\ b\ in\ middle\ and\ c\ to\ right\ of\ b}{\#\ total\ strings}$$

$$p(B \cap A) = \frac{360}{7!}$$

$$p(B \cap A) = \frac{1}{14}$$

Step 2: Calculate p(A|B)

By definition:

$$p(A|B) = \frac{p(B \cap C)}{p(C)}$$

$$p(A|B) = \frac{\frac{1}{14}}{\frac{1}{2}}$$

$$\boxed{p(A|B) = \frac{1}{7}}$$

e. Which pairs of events among A, B, and C are independent?

By definition:

- If P(X) = P(X|Y), then the events are independent.

We know that:

- $P(A) \neq P(A|C)$
- $P(B) = P(B|C)$
- $P(A) = P(A|B)$

Therefore:

- A and C are not independent
- B and C are independent
- A and B are independent

b) Exercise 6.3.6

b. The first 5 flips come up heads. The last 5 flips come up tails.

We know that:
- A biased coin is flipped 10 times.
- The probability of heads is 1/3.
- The probability of tails is 2/3.
- The outcomes of the coin flips are mutually independent.

Let's define A to be a single coin flip.

Given that the coin flips are mutually independent, we can say:

$$p(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8 \cap A_9 \cap A_{10}) = p(A_1) \cdot p(A_2) \dots p(A_{10}).$$

$$\boxed{p(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8 \cap A_9 \cap A_{10}) = (\tfrac{1}{3})^5 \cdot (\tfrac{2}{3})^5}$$

c.

We know that:
- A biased coin is flipped 10 times.
- The probability of heads is 1/3.
- The probability of tails is 2/3.
- The outcomes of the coin flips are mutually independent.

Let's define A to be a single coin flip.

Given that the coin flips are mutually independent, we can say:

$$p(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8 \cap A_9 \cap A_{10}) = p(A_1) \cdot p(A_2) \dots p(A_{10}).$$

$$\boxed{p(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8 \cap A_9 \cap A_{10}) = (\tfrac{1}{3})^1 \cdot (\tfrac{2}{3})^9}$$

c) Exercise 6.4.2

a.

You choose a die at random, and roll it six times, getting the values 4, 3, 6, 6, 5, 5. What is the probability that the die you chose is the fair die? The outcomes of the rolls are mutually independent.

We know that:

- Fair die:
  - P(1-6 | Fair) = ⅙
- Biased die:
  - P(6 | Biased) = 0.25
  - P(1-5 | Biased) = 0.15
- Choosing a die:
  - At random so ½ for both

Since the outcomes of the rolls are mutually independent, we can say:

$$P(A \mid Fair) = P(A_1 \mid Fair) \cdot P(A_2 \mid Fair) \cdot P(A_3 \mid Fair) \cdot P(A_4 \mid Fair) \cdot P(A_5 \mid Fair) \cdot P(A_6 \mid Fair)$$
$$P(A \mid Fair) = P(4 \mid Fair) \cdot P(3 \mid Fair) \cdot P(6 \mid Fair) \cdot P(6 \mid Fair) \cdot P(5 \mid Fair) \cdot P(5 \mid Fair)$$
$$P(A \mid Fair) = \tfrac{1}{6} \cdot \tfrac{1}{6} \cdot \tfrac{1}{6} \cdot \tfrac{1}{6} \cdot \tfrac{1}{6} \cdot \tfrac{1}{6}$$
$$P(A \mid Fair) = (\tfrac{1}{6})^6$$

$$P(A \mid \overline{Fair}) = P(A_1 \mid \overline{Fair}) \cdot P(A_2 \mid \overline{Fair}) \cdot P(A_3 \mid \overline{Fair}) \cdot P(A_4 \mid \overline{Fair}) \cdot P(A_5 \mid \overline{Fair}) \cdot P(A_6 \mid Fair)$$
$$P(A \mid \overline{Fair}) = P(4 \mid \overline{Fair}) \cdot P(3 \mid \overline{Fair}) \cdot P(6 \mid \overline{Fair}) \cdot P(6 \mid \overline{Fair}) \cdot P(5 \mid \overline{Fair}) \cdot P(5 \mid \overline{Fair})$$
$$P(A \mid \overline{Fair}) = 0.15 \cdot 0.15 \cdot 0.25 \cdot 0.25 \cdot 0.15 \cdot 0.15$$
$$P(A \mid \overline{Fair}) = 0.15^4 \cdot 0.25^2$$

By Bayes Theorem:

$$P(Fair \mid Rolls) = \frac{P(Rolls|Fair) \cdot P(Fair)}{P(Rolls|Fair) \cdot P(Fair) + P(Rolls|Not\ Fair) \cdot P(Not\ Fair)}$$

$$\boxed{P(Fair \mid Rolls) = \frac{(\frac{1}{6})^6 \cdot (\frac{1}{2})}{(\frac{1}{6})^6 \cdot (\frac{1}{2}) + 0.15^4 \cdot 0.25^2 \cdot (\frac{1}{2})}}$$
$$\boxed{P(Fair \mid Rolls) \approx 0.4038}$$

Question 9

a) Exercise 6.5.2

a.

We know that:

- There are 4 aces in a deck
- You can have no aces, one ace, two aces, three aces, four aces

$\boxed{\text{So the range of A is } \{0,1,2,3,4\}}$

b.

Step 1: Determine the number of hands with x aces

$\#\ hands\ with\ no\ aces = C(48,5)$

$\#\ hands\ with\ 1\ ace = C(4,1) \cdot C(48,4)$

$\#\ hands\ with\ 2\ aces = C(4,2) \cdot C(48,3)$

$\#\ hands\ with\ 3\ aces = C(4,3) \cdot C(48,2)$

$\#\ hands\ with\ 4\ aces = C(4,4) \cdot C(48,1)$

Step 2: Calculate the probability for each value in the range of A

$p(0\ aces) = \frac{C(48,5)}{C(52,5)}$

$p(1\ ace) = \frac{C(4,1) \cdot C(48,4)}{C(52,5)}$

$p(2\ aces) = \frac{C(4,2) \cdot C(48,3)}{C(52,5)}$

$p(3\ aces) = \frac{C(4,3) \cdot C(48,2)}{C(52,5)}$

$p(4\ aces) = \frac{C(4,4) \cdot C(48,1)}{C(52,5)}$

Step 3: Get the distribution over the random variable A

$(0,\ p(0\ aces)), (1,\ p(1\ ace)), (2,\ p(2\ aces)), (3,\ p(3\ aces)), (4,\ p(4\ aces))$

$$\boxed{(0,\ \tfrac{C(48,5)}{C(52,5)}),\ (1,\ \tfrac{C(4,1) \cdot C(48,4)}{C(52,5)}),\ (2,\ \tfrac{C(4,2) \cdot C(48,3)}{C(52,5)}),\ (3,\ \tfrac{C(4,3) \cdot C(48,2)}{C(52,5)}),\ (4,\ \tfrac{C(4,4) \cdot C(48,1)}{C(52,5)})}$$

b) Exercise 6.6.1

a. Two student council representatives are chosen at random from a group of 7 girls and 3 boys. Let G be the random variable denoting the number of girls chosen. What is E[G]?

<u>Step 1: Find the range of the random variable G</u>

The two student council representatives can be:

- 0 girls and 2 boys
- 1 girl and 1 boy
- 2 girls and 0 boys.

So the range of G is {0,1,2}

<u>Step 2: Determine the number of outcomes for each value in the range</u>

0 girls = C(7,0)*C(3,2)
1 girl = C(7,1)*C(3,1)
2 girls = C(7,2)*C(3,0)

<u>Step 3: Determine the probability for each value in the range</u>

The total number of choices is C(10,2).

So:

$p(0\ girls) = \frac{C(7,0) \cdot C(3,2)}{C(10,2)}$

$p(1\ girl) = \frac{C(7,1) \cdot C(3,1)}{C(10,2)}$

$p(2\ girls) = \frac{C(7,2) \cdot C(3,0)}{C(10,2)}$

<u>Step 4: Determine the expected value</u>

By definition:

$E[G] = 0 \cdot p(G=0) + 1 \cdot p(G=1) + 2 \cdot p(G=2)$

$E[G] = 0 \cdot p(0\ girls) + 1 \cdot p(1\ girl) + 2 \cdot p(2\ girls)$

$E[G] = 0 \cdot \frac{C(7,0) \cdot C(3,2)}{C(10,2)} + 1 \cdot \frac{C(7,1) \cdot C(3,1)}{C(10,2)} + 2 \cdot \frac{C(7,2) \cdot C(3,0)}{C(10,2)}$

$\boxed{E[G] = \frac{7}{5} = 1.4}$

c) Exercise 6.6.4

a. **A fair die is rolled once. Let X be the random variable that denotes the square of the number that shows up on the die. For example, if the die comes up 5, then X = 25. What is E[X]?**

<u>Step 1: Determine the range of X</u>

The values of X are the squares of the number rolled on the die

$1^2 = 1$

$2^2 = 4$

$3^2 = 9$

$4^2 = 16$

$5^2 = 25$

$6^2 = 36$

So the range of X is $\{1, 4, 9, 16, 25, 36\}$.

<u>Step 2: Determine the probabilities of each element</u>

Each die roll is equally probable in a fair die.

$p(1) = \frac{1}{6}$

$p(4) = \frac{1}{6}$

$p(9) = \frac{1}{6}$

$p(16) = \frac{1}{6}$

$p(25) = \frac{1}{6}$

$p(36) = \frac{1}{6}$

<u>Step 3: Calculate the expected value</u>

Each die roll is equally probable in a fair die.

$E[X] = 1 \cdot p(X = 1) + 4 \cdot p(X = 4) + 9 \cdot p(X = 9) + 16 \cdot p(X = 16) + 25 \cdot p(X = 25)$

$E[X] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6}$

$E[X] = 1 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{1}{6} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{6}$

$\boxed{E[X] = \frac{91}{6}}$

b. What is E[Y]?

Step 1: Determine the range of the random variable Y

In three coin tosses, we can get 0 heads, 1 head, 2 heads, or 3 heads.

We know that Y denotes the square of the number of heads.

- $0^2 = 0$
- $1^2 = 1$
- $2^2 = 4$
- $3^2 = 9$

So the range of Y is {0, 1, 4, 9}.

Step 2: Determine the number of possibilities for each value in the range

- $\#\ choices\ for\ (Y = 0) : \{TTT\}$
- $\#\ choices\ for\ (Y = 1) : \{HTT,\ THT,\ TTH\}$
- $\#\ choices\ for\ (Y = 2) : \{HTH,\ HHT,\ THH\}$
- $\#\ choices\ for\ (Y = 3) : \{HHH\}$

Step 3: Determine the probabilities for each value in the range

$$p(Y = 0) = \frac{\#\ choices\ for\ Y = 0}{total\ \#\ choices}$$
$$p(Y = 0) = \frac{|\{TTT\}|}{|\{TTT, HTT, THT, TTH, HTH, HHT, THH, HHH\}|}$$
$$p(Y = 0) = \frac{1}{8}$$

$$p(Y = 1) = \frac{\#\ choices\ for\ Y = 1}{total\ \#\ choices}$$
$$p(Y = 1) = \frac{|\{HTT, THT, TTH\}|}{|\{TTT, HTT, THT, TTH, HTH, HHT, THH, HHH\}|}$$
$$p(Y = 1) = \frac{3}{8}$$

$$p(Y = 2) = \frac{\#\ choices\ for\ Y = 1}{total\ \#\ choices}$$
$$p(Y = 2) = \frac{|\{HTH, HHT, THH\}|}{|\{TTT, HTT, THT, TTH, HTH, HHT, THH, HHH\}|}$$
$$p(Y = 2) = \frac{3}{8}$$

$$p(Y = 3) = \frac{\#\ choices\ for\ Y = 3}{total\ \#\ choices}$$
$$p(Y = 3) = \frac{|\{HHH\}|}{|\{TTT, HTT, THT, TTH, HTH, HHT, THH, HHH\}|}$$
$$p(Y = 3) = \frac{1}{8}$$

Step 4: Determine the expected value

$$E[Y] = 0 \cdot p(X=0) \ + 1 \cdot p(X=1) \ + 4 \cdot p(X=4) \ + 9 \cdot p(X=9)$$

$$E[Y] = 0 \cdot \tfrac{1}{8} \ + 1 \cdot \tfrac{3}{8} \ + 4 \cdot \tfrac{3}{8} \ + 9 \cdot \tfrac{1}{8}$$

$$\boxed{E[Y] = 3}$$

d) Exercise 6.7.4

a. **A class of 10 students hang up their coats when they arrive at school. Just before recess, the teacher hands one coat selected at random to each child. What is the expected number of children who get his or her own coat?**

Let's define the random variable X to denote the number of people who get the right coat.

Step 1: Determine the range of the random variable X

In ten people who get their coats, we can get:

- 0 people got the right coat
- 1 people got the right coat
- 2 people got the right coat
- 3 people got the right coat
- 4 people got the right coat
- 5 people got the right coat
- 6 people got the right coat
- 7 people got the right coat
- 8 people got the right coat
- 9 people got the right coat
- 10 people got the right coat

So the range of X is {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}.

Step 2: Calculate the expected value

Since each coat is chosen uniformly at random, the expected value of

$E[X_1] = 1 \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n})$

$E[X_1] = 1 \cdot \frac{1}{n} + 0 \cdot (1 - \frac{1}{n})$

$E[X_1] = 1 \cdot \frac{1}{10} + 0 \cdot (1 - \frac{1}{10})$

$E[X_1] = \frac{1}{10}$

$E[X] = n \cdot E[X_1]$

$E[X] = 10 \cdot \frac{1}{10}$

$\boxed{E[X] = 1}$

Question 10

a) **Exercise 6.8.1**

   a. **What is the probability of exactly 2 defects?**

We know that:

- The defect rate is 1%
- Defects are independent

So we can calculate the probability of 2 errors in 100 Bernoulli trials.

Note: We can choose defect as success or defect as failure, it doesn't matter since k and n-k are both represented as exponents and C(n,k) = C(n,n-k). We'll choose to have defect as a success.
s

By definition, $b(2;\ 100,\ 0.01)\ =\ C(n,k) \cdot p^k \cdot q^{n-k}$

$$b(2;\ 100,\ 0.01)\ = C(100,2) \cdot 0.01^2 \cdot 0.99^{100-2}$$

$$\boxed{b(2;\ 100,\ 0.01)\ = C(100,2) \cdot 0.01^2 \cdot 0.99^{98}}$$

   b. **What is the probability that out of 100 circuit boards made *at least* 2 have defects?**

We know that:

- The probability that out of 100 circuit boards made *at least* 2 have defects is the complement of that out of 100 circuit boards made only 0 or 1 have defects.

<u>Step 1: Calculate the probability that 0 circuit boards have defects</u>

Again, we'll choose to represent defects as successes. So the number of successes is 0.

By definition, $b(0;\ 100,\ 0.01)\ =\ C(n,k) \cdot p^k \cdot q^{n-k}$

$$b(0;\ 100,\ 0.01)\ = C(100,0) \cdot 0.01^0 \cdot 0.99^{100-0}$$

$$b(0;\ 100,\ 0.01)\ = 0.99^{100}$$

<u>Step 2: Calculate the probability that 1 circuit board has defects</u>

By definition, $b(1;\ 100,\ 0.01)\ =\ C(n,k) \cdot p^k \cdot q^{n-k}$

$$b(1;\ 100,\ 0.01)\ = C(100,99) \cdot 0.01^1 \cdot 0.99^{100-1}$$

$$b(1;\ 100,\ 0.01)\ = 100 \cdot 0.01 \cdot 0.99^{99}$$

$$b(1;\ 100,\ 0.01)\ = 0.99^{99}$$

Step 3: Calculate the probability that less than 2 circuit boards has a defect

$$p(defects < 2) = b(0;\ 100,\ 0.01) + b(1;\ 100,\ 0.01)$$
$$p(defects < 2) = 0.99^{100} + 0.99^{99}$$

Step 4: Calculate the probability that at least 2 circuit boards have a defect

$$p(defects \geq 2) = 1 - p(\overline{defects \geq 2})$$
$$p(defects \geq 2) = 1 - p(defects < 2)$$
$$p(defects \geq 2) = 1 - \left(0.99^{100} + 0.99^{99}\right)$$
$$\boxed{\begin{aligned} p(defects \geq 2) &= 1 - 0.99^{100} - 0.99^{99} \\ p(defects \geq 2) &\approx 0.2642 \end{aligned}}$$

c. What is the expected number of circuit boards with defects out of the 100 made?

For a Bernoulli process of independent Bernoulli trials, the expected number of successes n Bernoulli trials with probability of success p is:

$$E[X] = n \cdot p$$
$$E[X] = 100 \cdot 0.01$$
$$\boxed{E[X] = 1}$$

d. Suppose that the circuit boards are made in batches of two.

I - Calculate the probability of at least 2 defects:

We know that:

- Since the boards are made in batches of 2 and that either both or none are defective, the probability that out of 100 circuit boards made *at least* 2 have defects is the complement of the probability that out of 100 circuit boards made 0 have defects.

Since the circuit boards in the batches must both be defective or not, the probability that 2x circuit boards are defective is equivalent to the probability that x batches are defective.

So let's think of the problem in terms of batches.

Step 1: Calculate the probability that 0 batches have defects

By definition, $b(0;\ 50,\ 0.01)\ =\ C(n,k)\cdot p^k\cdot q^{n-k}$

$$b(0;\ 50,\ 0.01)\ = C(50,0)\cdot 0.01^0\cdot 0.99^{50-0}$$
$$b(0;\ 50,\ 0.01)\ = 0.99^{50}$$

Step 2: Calculate the probability that at least 1 batch has a defect

$$p(defected\ batches \geq 1)\ =\ 1\ - p(\overline{defected\ batches \geq 1})$$
$$p(defected\ batches \geq 1)\ =\ 1\ - p(defected\ batches\ <\ 1)$$
$$p(defected\ batches \geq 1)\ =\ 1\ - p(defected\ batches\ =\ 0)$$
$$p(defected\ batches \geq 1)\ =\ 1\ - 0.99^{50}$$
$$p(defected\ batches \geq 1)\ \approx\ 0.395$$

Since the number of defective batches maps 2-to-1 to the number of circuit boards, the probability of 2 defected boards is equivalent to at least 1 defective batch.

Therefore $p(defected\ boards \geq 2)\ =\ 1\ - 0.99^{50}$

$$p(defected\ boards \geq 2)\ \approx\ 0.395$$

II - Calculate the expected number of defective boards:

Step 1: Calculate the expected number of defective batches

$$E[defective\ batches]\ =\ n(batches)\cdot p(defect)$$
$$E[defective\ batches]\ =\ 50\cdot 0.01$$
$$E[defective\ batches]\ =\ 0.5$$

Step 2: Multiply by two to get the expected number of defective circuit boards

$$E[defective\ boards]\ = 2\cdot E[defective\ batches]$$
$$E[defective\ batches]\ =\ 2\cdot 0.5$$
$$\boxed{E[defective\ batches]\ =\ 1}$$

III - Discuss the differences

1. The probability of getting defective circuit boards increased from 0.264 when the boards were made separately to 0.395 when the boards are made in batches of 2.
2. The expected number of defective circuit boards did not change, regardless of whether the boards were made separately or in batches of 2 we expect 1 board to be defective.

b) Exercise 6.8.3

b. What is the probability that you reach an incorrect conclusion if the coin is biased?

We know that:

- If there are less than 4 heads, we conclude that the coin is biased.
- If there are 4 or more heads, we conclude that the coin is fair.

So, in order to reach an incorrect conclusion given that the coin is biased. We would need to consider all situations in which there are 4 or more heads.

Getting 4 or more heads is complementary to getting less than 4 heads, so let's calculate those probabilities:

<u>Step 1: Calculate the probabilities of getting 0, 1, 2, 3, heads given a biased coin.</u>

$$p(0 \mid \overline{fair}) = C(10,0) \cdot 0.3^0 \cdot 0.7^{10} = 0.7^{10}$$
$$p(1 \mid \overline{fair}) = C(10,1) \cdot 0.3^1 \cdot 0.7^9 = 3 \cdot 0.7^9$$
$$p(2 \mid \overline{fair}) = C(10,2) \cdot 0.3^2 \cdot 0.7^8$$
$$p(3 \mid \overline{fair}) = C(10,3) \cdot 0.3^3 \cdot 0.7^7$$

<u>Step 2: Calculate the probability of getting less than 4 heads given a biased coin</u>

$$p(less\ than\ 4 \mid \overline{fair}) = p(0 \mid \overline{fair}) + p(1 \mid \overline{fair}) + p(2 \mid \overline{fair}) + p(3 \mid \overline{fair})$$
$$p(less\ than\ 4 \mid \overline{fair}) = 0.7^{10} + 3 \cdot 0.7^9 + C(10,2) \cdot 0.3^2 \cdot 0.7^8 + C(10,3) \cdot 0.3^3 \cdot 0.7^7$$
$$p(less\ than\ 4 \mid \overline{fair}) = 0.6496$$

<u>Step 3: Calculate the probability of getting 4 or more heads given a biased coin</u>

$$p(4\ or\ more \mid \overline{fair}) = 1 - p(\overline{4\ or\ more} \mid \overline{fair})$$
$$p(4\ or\ more \mid \overline{fair}) = 1 - p(less\ than\ 4 \mid \overline{fair})$$
$$p(4\ or\ more \mid \overline{fair}) = 1 - 0.7^{10} + 3 \cdot 0.7^9 + C(10,2) \cdot 0.3^2 \cdot 0.7^8 + C(10,3) \cdot 0.3^3 \cdot 0.7^7$$
$$p(4\ or\ more \mid \overline{fair}) \approx 0.35$$

Therefore, the probability of reaching an incorrect conclusion given a biased coin is about 0.35.