NYU Tandon Bridge Winter 2021 Homework 11

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Question 5

Part a.

Theorem: $For\ any\ n \geq 1,\ n^3 + 2n\ is\ divisible\ by\ 3.$

Proof by induction on the value of n.

I) Base case:

- $n = 1 :\ n^3 + 2n = 1^3 + 2 \cdot 1 = 3.$ Divisible by 3.

II) Inductive step:

- Inductive hypothesis: $For\ k \geq 1,\ k^3 + 2k\ is\ divisible\ by\ 3.$
- Show that for $For\ k + 1,\ (k+1)^3 + 2(k+1)\ is\ divisible\ by\ 3.$

$(k+1)^3 + 2(k+1)$	
$k^3 + 3 \cdot k^2 \cdot 1 + 3 \cdot k \cdot 1^2 + 1^3 + 2(k+1)$	Distribute cube
$k^3 + 3k^2 + 3k + 1 + 2k + 2$	Distribute factor of 2
$k^3 + 2k + 3k^2 + 3k + 3$	Rearrange elements and sum them up
$k^3 + 2k + 3(k^2 + k + 1)$	Factor by 3
$3m + 3(k^2 + k + 1)$	By inductive hypothesis, $k^3 + 2k$ is divisible by 3. So we can express it as $3m$ for some integer m
$3(m + k^2 + k + 1)$	Factor by 3.

Since m and k are integers, $(m + k^2 + k + 1)$ is also an integer. Let's set $g = (m + k^2 + k + 1)$

As such $(k+1)^3 + 2(k+1)$ can be written as $3g$ where g is an integer.

Any integer expressed in this way is divisible by 3.

$\boxed{\text{Therefore, } (k+1)^3 + 2(k+1) \text{ is divisible by 3. } \blacksquare}$

Part b.

Theorem: $For\ any\ n \geq 1,\ n\ can\ be\ written\ as\ a\ product\ of\ prime\ numbers.$

Proof by strong induction on the value of n.

I) Base case:

- $n = 1 : 1\ is\ already\ a\ prime\ number,\ so\ it\ is\ a\ product\ of\ prime\ numbers.$

II) Inductive step:

- Inductive hypothesis:
  $For\ k \geq 1,\ all\ numbers\ j\ from\ 1\ through\ k\ can\ be\ expressed\ as\ a\ product\ of\ primes.$
- Show that k+1 can be written as a product of prime factors.

$k + 1$	
$(k+1) \cdot (\frac{2}{2})$	Multiply by $\frac{2}{2} = 1$.
$2 \cdot (\lceil \frac{k+1}{2} \rceil)$	Factor a 2. Take the ceiling to be sure to get an integer.
$2 \cdot (prime_1 \cdot prime_2 \cdot \ldots prime_m)$	Since $(\lceil \frac{k+1}{2} \rceil)$ is between 1 and k, by the inductive hypothesis, we know that it can be written as a product of primes. i.e. $prime_1$ * $prime_2$ ... * $prime_m$ for some positive integer m.

So, $k + 1$ can be written as $2 \cdot prime_1 \cdot prime_2 \cdot \ldots prime_m$

Since 2 is itself a prime number, we conclude that $k + 1$ can be written as a product of primes. ∎

Question 6

Problem 7.4.1

Part a.

$\sum_{j=1}^{3} j^2$	$\frac{n(n+1)(2n+1)}{6}$
$1^2 + 2^2 + 3^2$	$\frac{3(3+1)(2(3)+1)}{6}$
$1 + 4 + 9$	$\frac{3\cdot4\cdot7}{6}$
14	$\frac{84}{6}$
	14

$\sum_{j=1}^{3} j^2 = \frac{3(3+1)(2(3)+1)}{6} = 14$. Therefore, $P(3)$ is true.

Part b.

To get $P(k)$, we replace n by k everywhere in the equation.

$$P(k) = \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$$

Part c.

To get $P(k+1)$, we replace n by k+1 everywhere in the equation.

$$P(k+1) = \sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

Part d.

The inductive proof is for every positive integer.

1 is the smallest positive integer.

Therefore, P(1) must be proven in the base case.

Part e.

We must prove that $P(k+1)$ **must be true**. This is equivalent to saying $\sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$

Part f.

The inductive hypothesis is that $for\ k \geq 1,\ P(k)\ is\ true.$ This is equivalent to saying $For\ k \geq 1,\ \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}.$

Part g.

Define $P(n)$ to be the assertion that: $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

Theorem: $For\ n \geq 1,\ P(n)\ is\ true.$

Proof by induction on the value of n.

I) Base case:

- The base case is when $n = 1$

$\sum_{j=1}^{1} j^2$	$\frac{n(n+1)(2n+1)}{6}$
1^2	$\frac{1(1+1)(2(1)+1)}{6}$
1	$\frac{1*2*3}{6}$
	1

$\sum_{j=1}^{1} j^2 = \frac{1(1+1)(2(1)+1)}{6} = 1.$ Therefore, $P(1)$ is true and the theorem holds for the base case.

II) Inductive step:

- Inductive hypothesis: $for\ k \geq 1,\ P(k)\ is\ true,\ i.e.\ For\ k \geq 1,\ \sum_{j=1}^{k} j^2 = \frac{k(k+1)(2k+1)}{6}$.
- Show that $for\ k+1,\ P(k+1)\ is\ true,\ i.e.\ \sum_{j=1}^{k+1} j^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$.

$\sum_{j=1}^{k+1} j^2$	
$\sum_{j=1}^{k} j^2 + (k+1)^2$	Summation rule
$\frac{k(k+1)(2k+1)}{6} + (k+1)^2$	By inductive hypothesis
$\frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$	Multiply by 6/6
$\frac{k(k+1)(2k+1) + 6(k+1)^2}{6}$	Combine fractions due to common denominator.
$\frac{k(k+1)(2k+1) + 6(k+1)(k+1)}{6}$	Express $(k+1)^2$ as $(k+1)(k+1)$
$\frac{[k(2k+1) + 6(k+1)](k+1)}{6}$	Factor out (k+1)
$\frac{[2k^2 + k + 6k + 6](k+1)}{6}$	Distribute k and 6.
$\frac{[2k^2 + 4k + 3k + 6](k+1)}{6}$	Rearrange variables
$\frac{[2k(k+2) + 3k + 6](k+1)}{6}$	Factor out (k+2) from $2k^2+4$
$\frac{[2k(k+2) + 3(k+2)](k+1)}{6}$	Factor out (k+2) from 3k+6
$\frac{(k+2)(2k+3)(k+1)}{6}$	Factor out (k+2) from $2k(k+2) + 3(k+2)$
$\frac{(k+2)(2k+2+1)(k+1)}{6}$	Rearrange constants
$\frac{((k+1)+1)(2(k+1)+1)(k+1)}{6}$	Factor out k+1

Therefore, since $\sum_{j=1}^{k+1} j^2 = ((k+1)+1)(2(k+1)+1)(k+1)$. We conclude that P(k+1) is true ∎

Problem 7.4.3

Part c.

Theorem: $For\ n \geq 1, \sum_{j=1}^{n} \frac{1}{j^2} \leq 2 - \frac{1}{n}$

Proof by induction on the value of n.

I) Base case:

- n=1:

$\sum_{j=1}^{n} \frac{1}{j^2}$	$2 - \frac{1}{n}$
$\sum_{j=1}^{1} \frac{1}{j^2}$	$2 - \frac{1}{1}$
$\frac{1}{1^2}$	$2 - 1$
1	1

We get:

$$1 \leq 1$$

$$\sum_{j=1}^{1} \frac{1}{j^2} \leq 2 - \frac{1}{1}$$

Therefore, the theorem holds for the base case.

II) Inductive step:

- Inductive hypothesis: $For\ any\ k \geq 1, \sum_{j=1}^{k} \frac{1}{j^2} \leq 2 - \frac{1}{k}$
- We must show that $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k+1}$

$\sum_{j=1}^{k+1} \frac{1}{j^2} = \sum_{j=1}^{k} \frac{1}{j^2} + \frac{1}{(k+1)^2}$	Summation rule
$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$	By inductive hypothesis
$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)(k+1)}$	Express $(k+1)^2$ as (k+1)(k+1)
$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{k} + \frac{1}{k(k+1)}$	Since: k+1 > k (k+1)(k+1) > k(k+1) $\frac{1}{(k+1)(k+1)} < \frac{1}{k(k+1)}$ So replacing $\frac{1}{(k+1)(k+1)}$ with $\frac{1}{k(k+1)}$ only makes the inequality greater.
$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{k+1}{k(k+1)} + \frac{1}{k(k+1)}$	Multiply by $\frac{k+1}{k+1}$ to get common denominator
$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{(k+1) - 1}{k(k+1)}$	Express as one fraction
$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{k}{k(k+1)}$	Subtract 1's
$\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{(k+1)}$	k's cancel out

Therefore, since $\sum_{j=1}^{k+1} \frac{1}{j^2} \leq 2 - \frac{1}{(k+1)}$, the theorem holds true for k+1 and is proven by induction. ∎

Problem 7.5.1

Part a.

Theorem: $For\ any\ positive\ integer\ n,\ 4\ evenly\ divides\ 3^{2n} - 1.$

Proof by induction on the value of n.

I) Base case:

$n = 1:$

$3^{2n} - 1$
$3^{2\cdot 1} - 1$
$3^{2} - 1$
$9 - 1$
8
$4 \cdot 2$
$4m\ \ (for\ integer\ m = 2)$

Since it can be expressed as 4 times an integer, $3^{2\cdot 1} - 1$ must be divisible by 4.

Therefore, the theorem holds true for the base case n = 1.

II) Inductive step:

- Inductive hypothesis: $For\ any\ k \geq 1,\ 4\ evenly\ divides\ 3^{2k} - 1$
- Show that $for\ k+1,\ 4\ evenly\ divides\ 3^{2(k+1)} - 1$

$3^{2(k+1)} - 1$	
$3^{2k+2} - 1$	Distribute 2 in exponent
$3^2 \cdot 3^{2k} - 1$	Exponent rule
$9 \cdot 3^{2k} - 1$	Multiply out the square
$9 \cdot (4m+1) - 1$	By inductive hypothesis, we know that $3^{2k} - 1 = 4m$ for some integer m So: $3^{2k} = 4m + 1$
$9 \cdot 4m + 9 - 1$	Distribute 9
$9 \cdot 4m + 8$	Subtract 1 from 9
$4 \cdot (9m+2)$	Factor out 4

Since m is an integer, 9m+2 is also an integer.

Therefore, since it can be expressed as 4 times an integer, $3^{2(k+1)} - 1$ must be divisible by 4.

This proves the theorem by induction. ∎