CMPN302: Algorithms Design and Analysis

Lecture 03: Hashing

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Fall 2017

Goal

- Perform operations (Search, Insert and Delete) as fast as possible

Direct address table

DIRECT-ADDRESS-SEARCH(T, k)

return $T[k]$

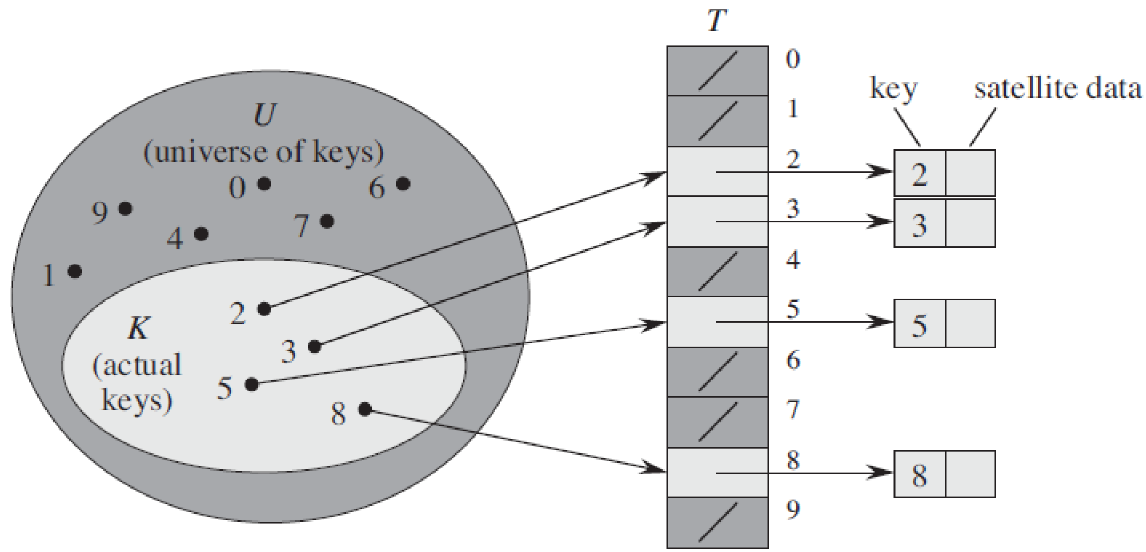
DIRECT-ADDRESS-INSERT(T, x)

$T[x.key] = x$

DIRECT-ADDRESS-DELETE(T, x)

$T[x.key] = \text{NIL}$

Each of these operations takes only $O(1)$ time.

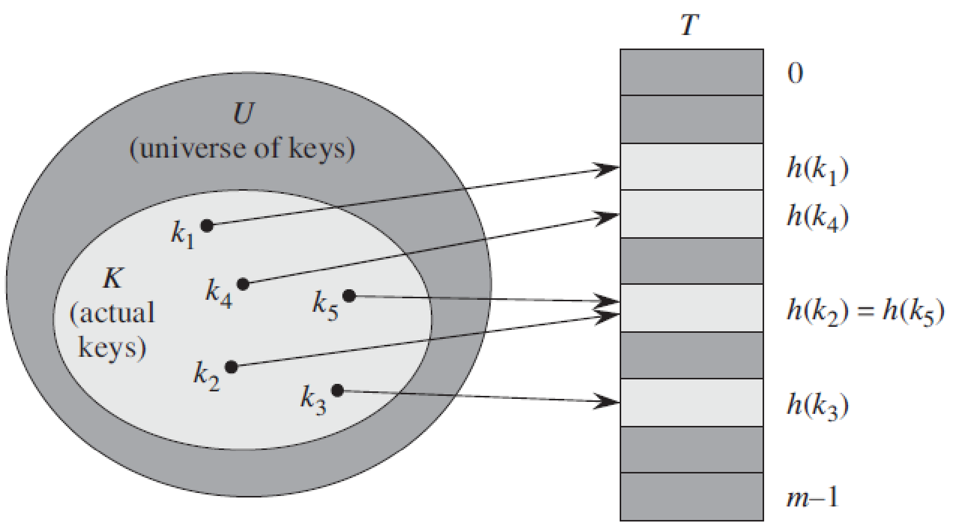


- Operations take $O(1)$ time.
- Data can be stored in table itself without a linked-list.
- Disadv.: If universe U is large.

Hash table

- Reduce table size to m
- Use hash function

$$h : U \rightarrow \{0, 1, \ldots, m-1\}$$



- Problem: collision
- Resolution:
  - Chaining
  - Open addressing

Hash function

- Goal: minimize collisions

- Keys must be integers
  - Convert non-integers to integers, i.e. strings

Division method

$$h(k) = k \ \ mod \ \ m$$

- m should be prime not too close to powers of 2
- Example:
  - $n = 2000$
  - $\alpha = 3$
  - $m = 701$
  - $h(k) = k \ \ mod \ \ 701$

Multiplication method

$$h(k) = \lfloor m(k\,A\,mod\;1) \rfloor, \qquad 0 < A < 1$$

- Extracts the fractional part of $k\,A$ and then multiplies by m (hash table size)
- m better be 2^P
- Assume w, the word size in machine (i.e. 32 bits)
- k fits one word
- Restrict A to $s/2^w$

Multiplication method

$$h(k) = \lfloor m(k\,A \bmod 1) \rfloor, \qquad 0 < A < 1$$

Optimization

- Assume w, the word size in machine (i.e. 32 bits)
- k fits one word
- Restrict A to $s/2^w$
- Compute:

shift right w-p bits

$$ks = r_1 2^w + r_0 \xrightarrow{mod} r_0 \xrightarrow{\times m} q_1 2^p + q_0 \xrightarrow{floor} q_1$$

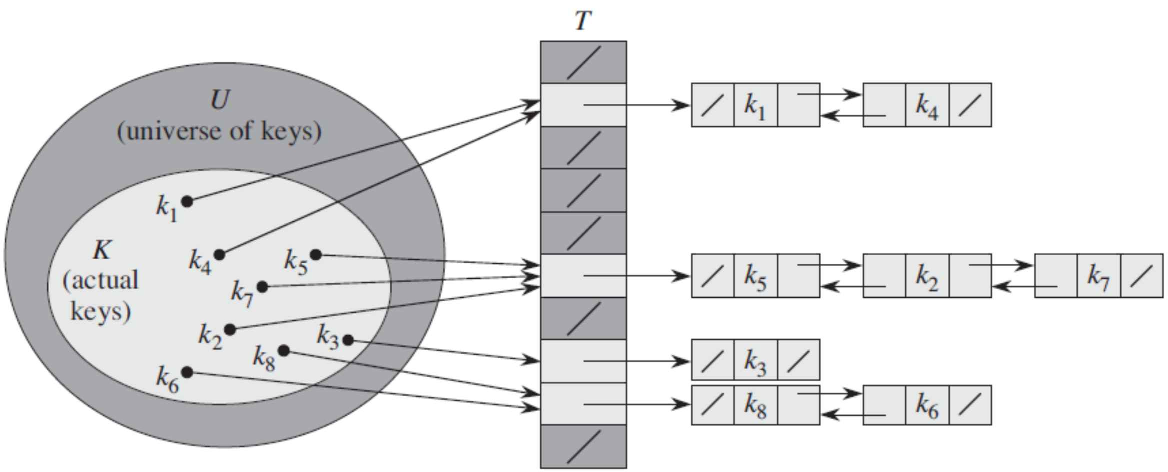
integer fraction integer fraction

Universal hashing

- An adversary can choose the data to *all hash* to the same position if he determines the hashing function
- The problem happens because of determinism!!
- Solution: to avoid determinism, use *randomization*
- At every execution over the *whole dataset*, choose the hash function randomly from a set of hash functions
- This hash function remains the same during the *whole run*, otherwise does not make sense

Chaining

- Load factor $\alpha = \frac{\mathrm{n}}{\mathrm{m}}$



- Unsuccessful search takes $\Theta(1+\alpha)$
- Can be $\Theta(1)$ if $\alpha = O(1)$, in other words $n = O(m)$

Chaining

- Proof for $\Theta(1+\alpha)$

Let x_i denote the ith element inserted into the table, for $i = 1, 2, \ldots, n$, and let $k_i = x_i.key$. For keys k_i and k_j, we define the indicator random variable $X_{ij} = \mathrm{I}\{h(k_i) = h(k_j)\}$. Under the assumption of simple uniform hashing, we have $\Pr\{h(k_i) = h(k_j)\} = 1/m$, and so by Lemma 5.1, $\mathrm{E}[X_{ij}] = 1/m$. Thus, the expected number of elements examined in a successful search is

Chaining

- Proof for $\Theta(1+\alpha)$

$$
\begin{aligned}
\mathrm{E}\left[\frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}X_{ij}\right)\right]
&= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\mathrm{E}\left[X_{ij}\right]\right) \\
&= \frac{1}{n}\sum_{i=1}^{n}\left(1+\sum_{j=i+1}^{n}\frac{1}{m}\right) \\
&= 1+\frac{1}{nm}\sum_{i=1}^{n}(n-i) \\
&= 1+\frac{1}{nm}\left(\sum_{i=1}^{n}n-\sum_{i=1}^{n}i\right) \\
&= 1+\frac{1}{nm}\left(n^2-\frac{n(n+1)}{2}\right) \\
&= 1+\frac{n-1}{2m} \\
&= 1+\frac{\alpha}{2}-\frac{\alpha}{2n}\,.
\end{aligned}
$$

Open addressing

- Elements occupy hash table itself
- To resolve collisions, instead of inserting in linked list, examine a *next* *empty* position to store the element
- Examining the *next* position is called *probing*

```
HASH-INSERT(T, k)
i = 0
repeat
    j = h(k, i)          ith Next position
    if T[j] == NIL
        T[j] = k
        return j
    else i = i + 1
until i == m
error "hash table overflow"
```

Open addressing

Searching inside hash

- Must search all possible *next* positions

```
HASH-SEARCH(T, k)
i = 0
repeat
    j = h(k, i)
    if T[j] == k
        return j
    i = i + 1
until T[j] == NIL or i == m
return NIL
```

Open addressing

Deletion is tricky

- If we delete an element while there are elements inserted after it, we can't search after deletion because the element is marked as nil
- Solution: mark with a different mark
  - Problem: search time no longer depends on α
- Open addressing is not typically used if keys can be deleted, use chaining instead

Open addressing

Linear probing

$$h(k, i) = (h'(k) + i) \bmod m$$

- Examine consecutive positions ($mod\ m$)
- Problem: *primary clustering*
  - An empty slot preceded by i full slots will be filled with probability $(i+1)/m$

Open addressing

Quadratic probing

$$h(k, i) = (h'(k) + c_1 i + c_2 i^2) \bmod m \;, \tag{11.5}$$

where h' is an auxiliary hash function, c_1 and c_2 are positive auxiliary constants,

for $i = 0, 1, \ldots, m-1$

- Jump by quadratic steps
- Works much better than linear probing
- Problem: if $h(k_1, 0) = h(k_2, 0)$, then $h(k_1, i) = h(k_2, i)$, called *secondary clustering*
  - Still initial position determines the entire sequence

Open addressing

Double hashing

$$h(k, i) = (h_1(k) + i h_2(k)) \bmod m ,$$

where h_1 and h_2 are auxiliary hash functions

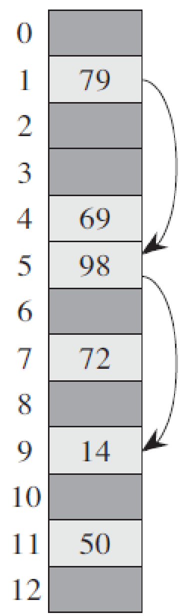


Figure 11.5 Insertion by double hashing. Here we have a hash table of size 13 with $h_1(k) = k \bmod 13$ and $h_2(k) = 1 + (k \bmod 11)$. Since $14 \equiv 1 \pmod{13}$ and $14 \equiv 3 \pmod{11}$, we insert the key 14 into empty slot 9, after examining slots 1 and 5 and finding them to be occupied.

Open addressing

- Analysis

Theorem 11.6

Given an open-address hash table with load factor $\alpha = n/m < 1$, the expected number of probes in an unsuccessful search is at most $1/(1-\alpha)$, assuming uniform hashing.

Perfect hashing

- Use two levels of hashing

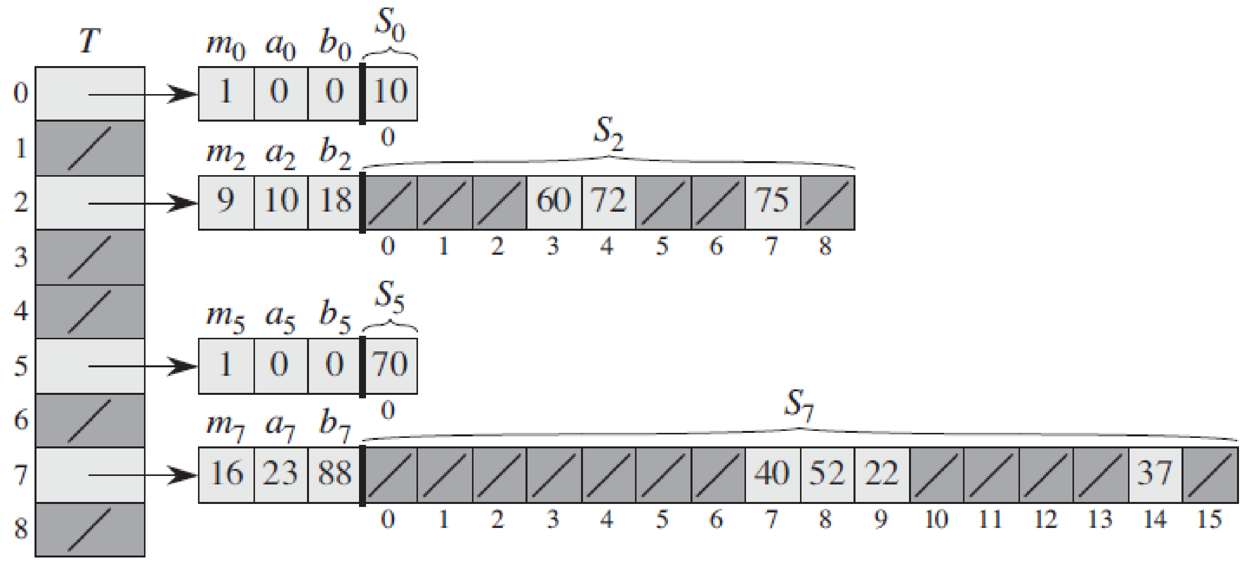


Figure 11.6 Using perfect hashing to store the set $K = \{10, 22, 37, 40, 52, 60, 70, 72, 75\}$. The outer hash function is $h(k) = ((ak + b) \bmod p) \bmod m$, where $a = 3$, $b = 42$, $p = 101$, and $m = 9$. For example, $h(75) = 2$, and so key 75 hashes to slot 2 of table T. A secondary hash table S_j stores all keys hashing to slot j. The size of hash table S_j is $m_j = n_j^2$, and the associated hash function is $h_j(k) = ((a_j k + b_j) \bmod p) \bmod m_j$. Since $h_2(75) = 7$, key 75 is stored in slot 7 of secondary hash table S_2. No collisions occur in any of the secondary hash tables, and so searching takes constant time in the worst case.

Perfect hashing

- Good choice for *static* data
  - Examples:
    - Reserved words in programming language
    - Set of file names in CD-ROM
- $O(1)$ memory access in the worst case
- Universal hashing is used at each level
- Similar to chaining but use a *secondary hash table* instead of *linked lists*
- Careful selection of the secondary hash table can avoid collisions