Design and Analysis of Algorithms

CAIRO UNIVERSITY

Lecture 09: Flow Networks

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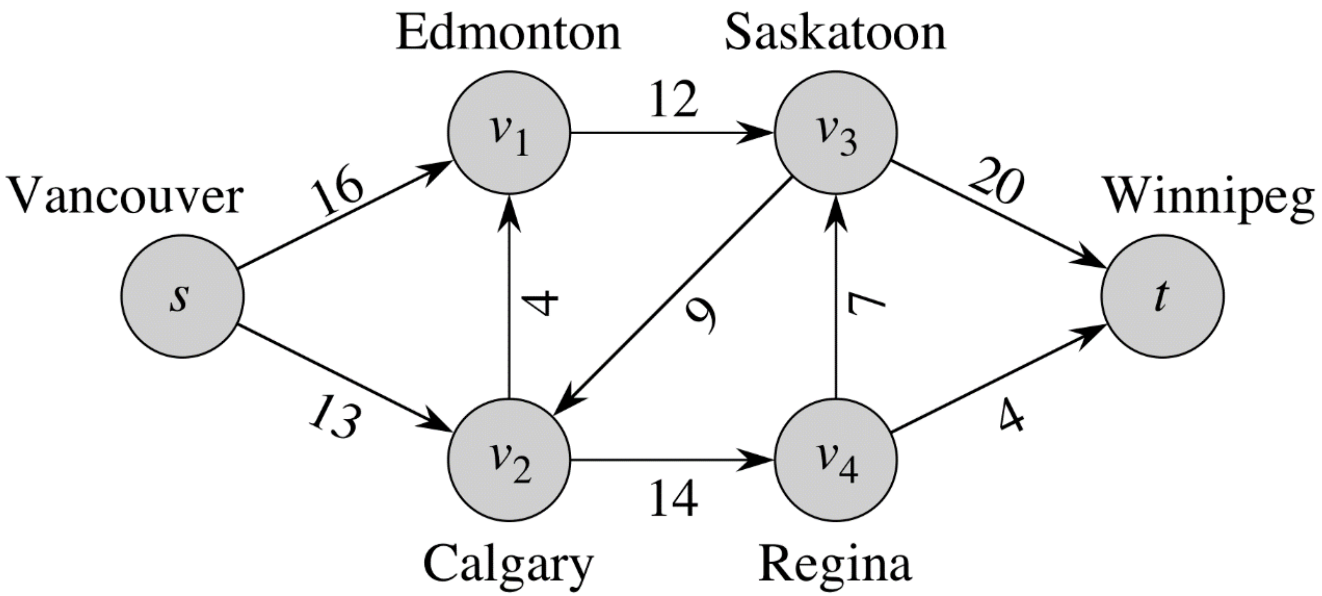
Agenda

- What is a flow network?
- Ford-Fulkerson method
- Residual network
- Augmenting paths
- Min-cut
- Edmonds-Karp algorithm

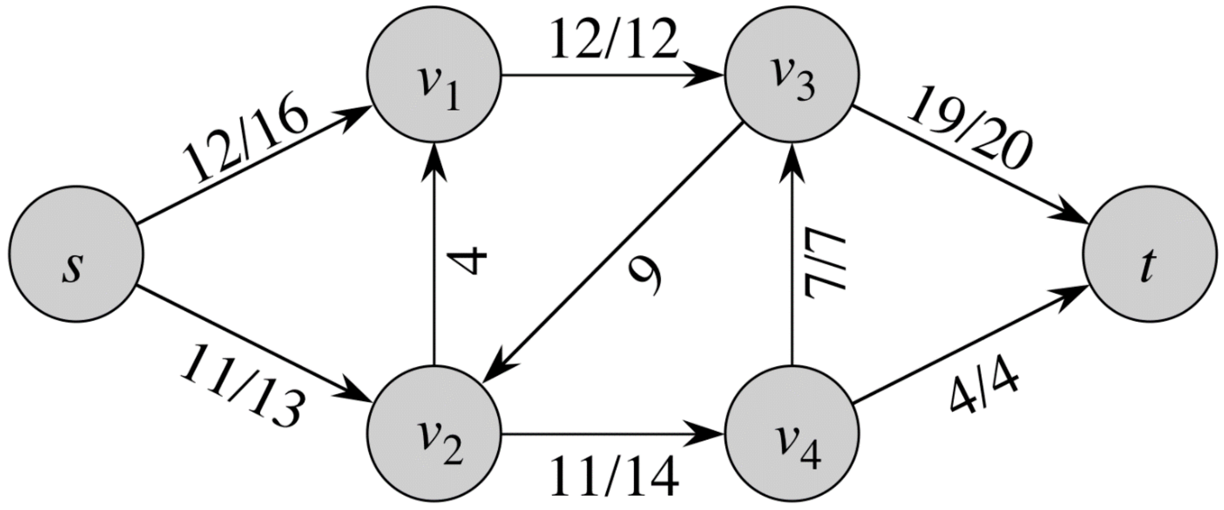
Real life application

- Find max water rate flowing from source (Vancouver) to sink (Winnipeg) in the shown pipe network based on the shown capacities.

- Can you guess an upper-bound for what we can get from just looking?
- Iteratively how do you come up with the best solution?
- Between iterations, how to simplify and prepare the network for the next iteration?

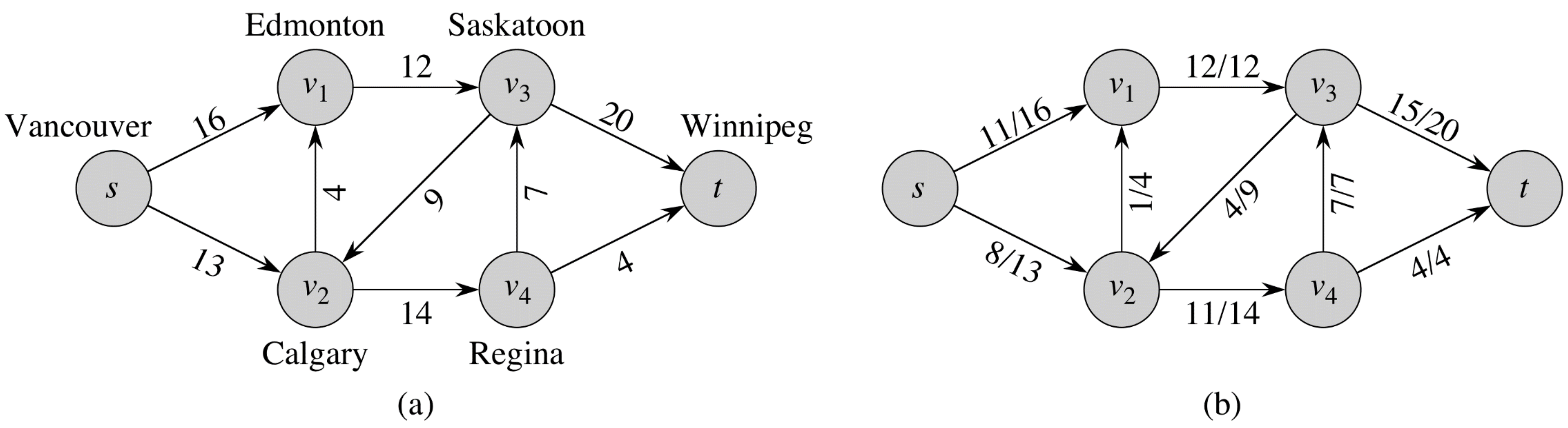


(a)



What is a flow network?

- ***Flow network*** is a directed graph $G(V, E)$
- Each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. No self-loops.
- If there is $(u, v) \in E$, then there is no edge (v, u) in reverse direction, and $c(v, u) = 0$.
- Typically, there is source s and sink t.



What is a flow?

- ***Flow*** is a real-valued function $f: V \times V \to \mathbb{R}$ that satisfies:
  - Capacity constraint: For all $u, v \in E$, we require

$$0 \le f(u, v) \le c(u, v)$$

  - For all $u, v \in V - \{s, t\}$, we require that for u:

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

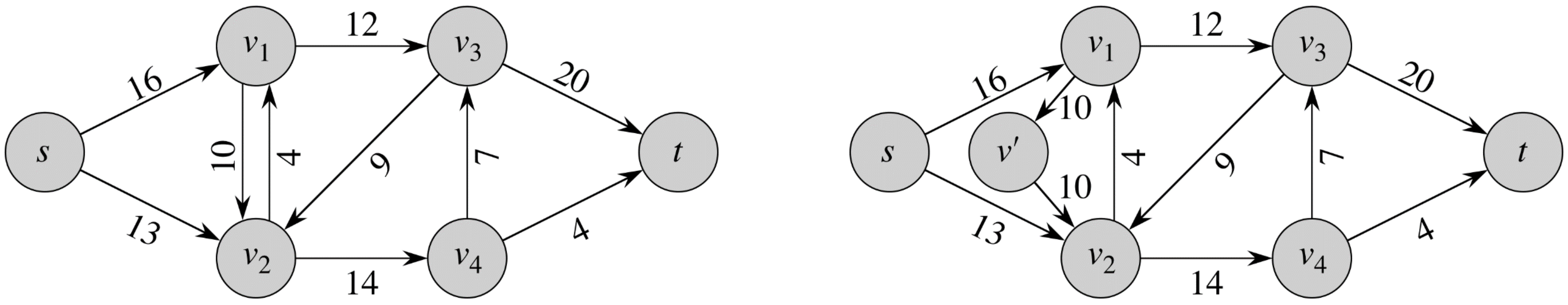
Modeling with antiparallel edges

- Antiparallel edge:

  Edges (v_1, v_2) and (v_2, v_1) are called antiparallel

- Workaround:

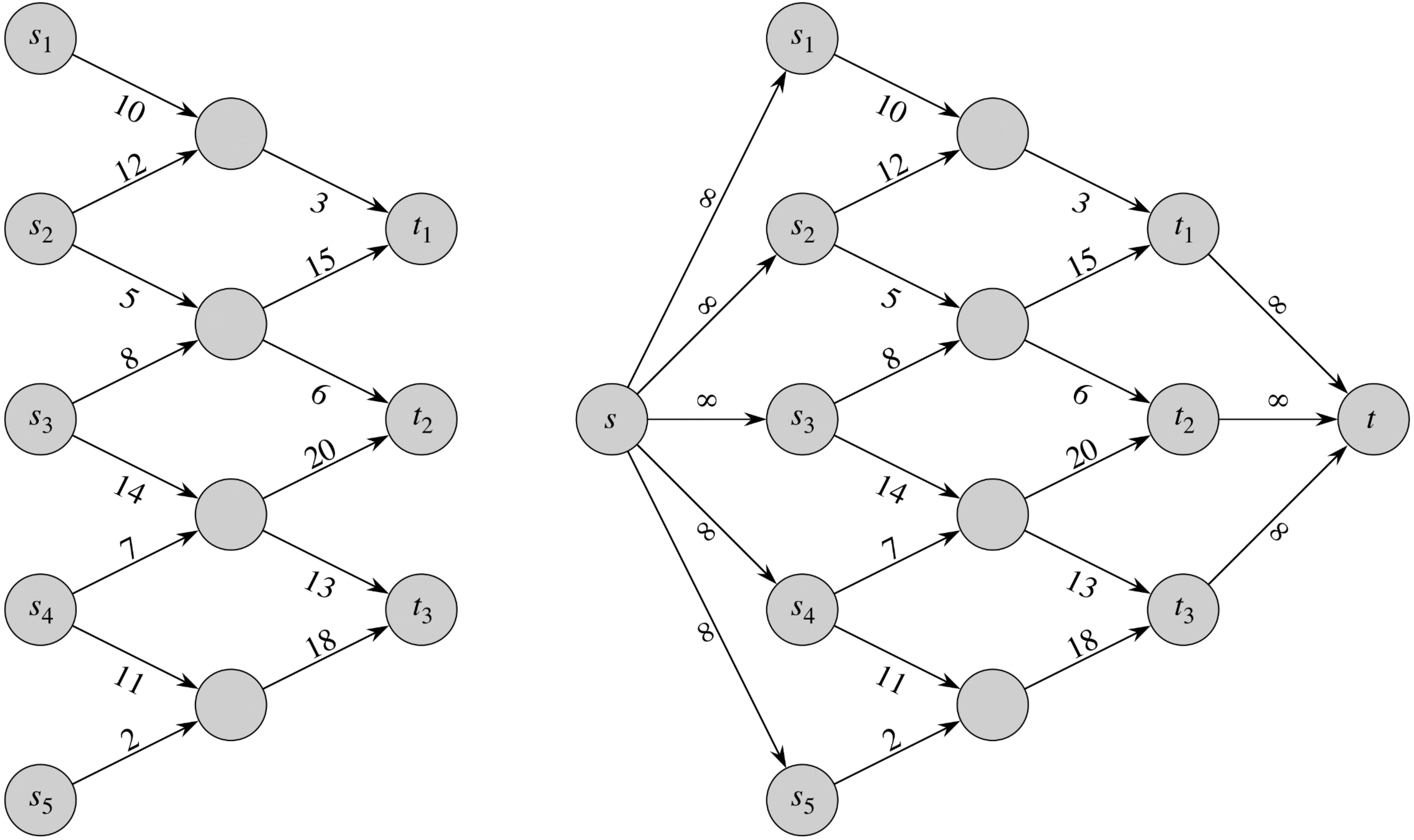
  Split one of the edges using a new vertex



(a) (b)

Modeling with multiple sources/sinks

- Add a new source s and sink t.
- Connect s to all sources with infinite capacity edges.
- Connect all sinks to t with infinite capacity edges.
- Solve for s and t instead.



(a)
(b)

Ford-Fulkerson method

- Method not algorithm because it has several implementations with different complexities
- Greedy, greedy, greedy!!!
- Main ideas
  – Residual networks
  – Augmenting paths
  – Cuts

FORD-FULKERSON-METHOD(G, s, t)

```
1  initialize flow f to 0
2  while there exists an augmenting path p in the residual network G_f
3      augment flow f along p
4  return f
```

Residual network

- Residual network G_f contains the residual capacities from G
- It may contain extra edges to allow for decreasing previously-allocated flows
- Kind of similar to a flow network, except it allows for reversed edges
- Now, why there is no reversed edges in flow networks??

If (u, v) is an edge in E, then the edge capacity should be reduced to $c_f(u, v) = c(u, v) - f(u, v)$.

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \, , \\ f(v,u) & \text{if } (v,u) \in E \, , \\ 0 & \text{otherwise} \, . \end{cases}$$

If (v, u) is an edge in E, then the reversed edge should have capacity $c_f(u, v) = f(v, u)$.

Augmenting paths

- Augmenting path p is a simple path from s to t in the residual network G_f

A path is simple if all the vertices on the path are distinct.

Flow networks

Residual networks

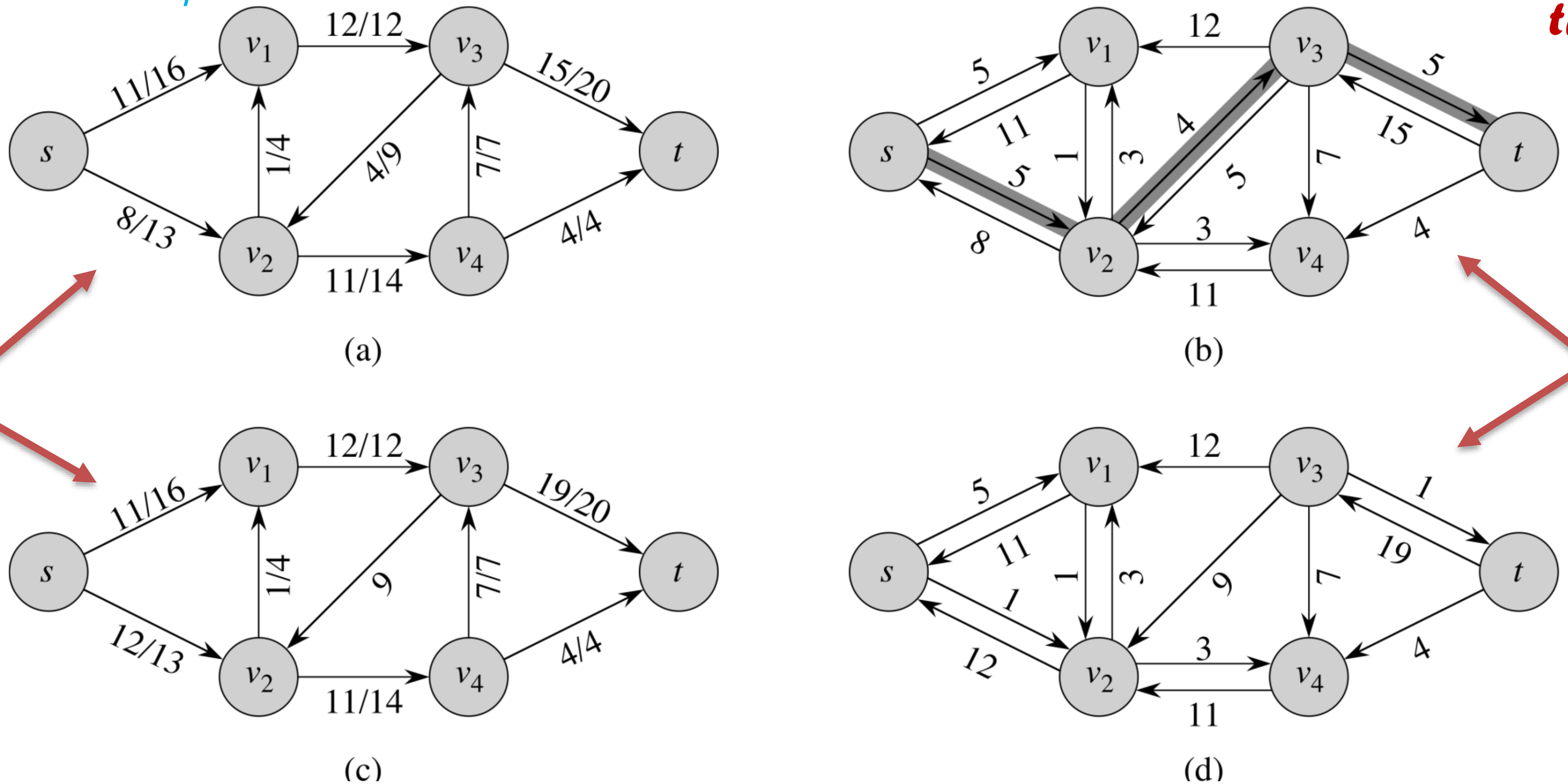


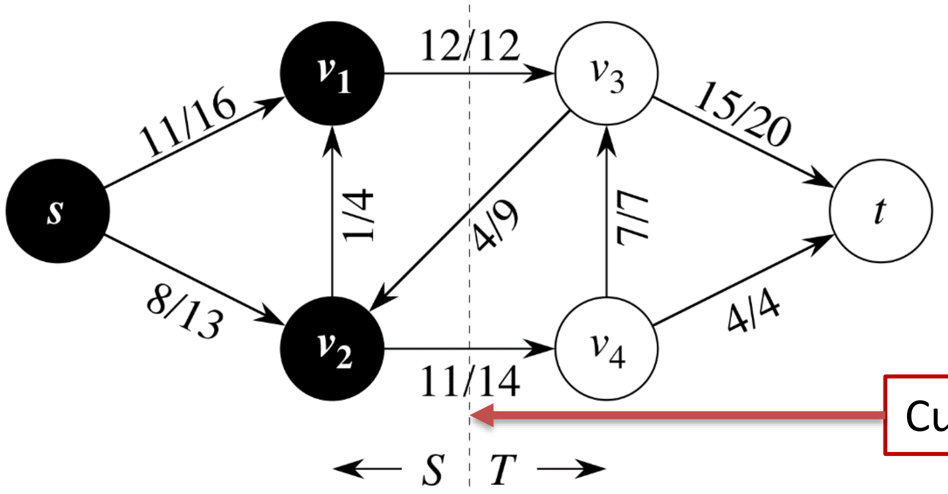
Figure 26.4 **(a)** The flow network G and flow f of Figure 26.1(b). **(b)** The residual network G_f with augmenting path p shaded; its residual capacity is $c_f(p) = c_f(v_2, v_3) = 4$. Edges with residual capacity equal to 0, such as (v_1, v_3), are not shown, a convention we follow in the remainder of this section. **(c)** The flow in G that results from augmenting along path p by its residual capacity 4. Edges carrying no flow, such as (v_3, v_2), are labeled only by their capacity, another convention we follow throughout. **(d)** The residual network induced by the flow in (c).

Cuts of flow networks

- How to know when the algorithm terminates??
- A cut (S, T) of flow network $G(V, E)$ is a partition of V into two sets S and $T = V - S$
- ***Net flow*** $f(S, T)$ across the cut (S, T) is defined as

$$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$

Summation of all flows on edges starting from nodes in S and crossing to nodes in T.
In the shown example, it is 12 + 11.



Cut

Summation of all flows on edges starting from nodes in T and going back to nodes in S.
In the shown example, it is 4.

- All cuts result in net flow = 19 (not optimal)

Cuts of flow networks

- The ***capacity of the cut*** (S, T) is

$$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$

- A ***minimum cut*** of a network is a cut whose capacity is minimum over all cuts of the network

Max-flow min-cut

Theorem 26.6 (Max-flow min-cut theorem)
If f is a flow in a flow network $G = (V, E)$ with source s and sink t, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

Basic Ford-Fulkerson algorithm

```
FORD-FULKERSON(G, s, t)
for each edge (u, v) ∈ G.E
    (u, v).f = 0
while there exists a path p from s to t in the residual network G_f
    c_f(p) = min {c_f(u, v) : (u, v) is in p}
    for each edge (u, v) in p
        if (u, v) ∈ E
            (u, v).f = (u, v).f + c_f(p)
        else (v, u).f = (v, u).f − c_f(p)
```

- Path-finding is performed using either BFS or DFS, thus $O(V+E) = O(E)$
- The while loop is executed $|f^*|$ (if every iteration just adds on unit value), where f^* is the maximum flow
- Overall complexity is $\mathrm{O}(\mathrm{E}\,|f^*|)$

FFA

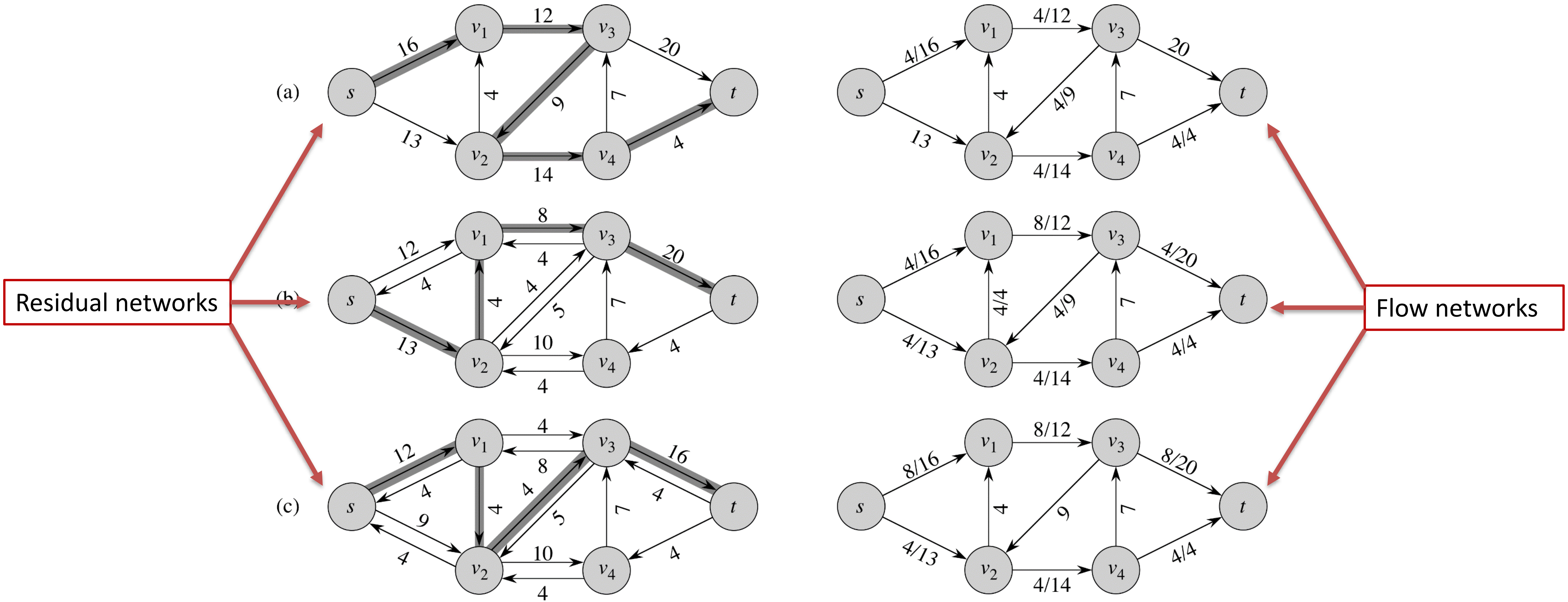
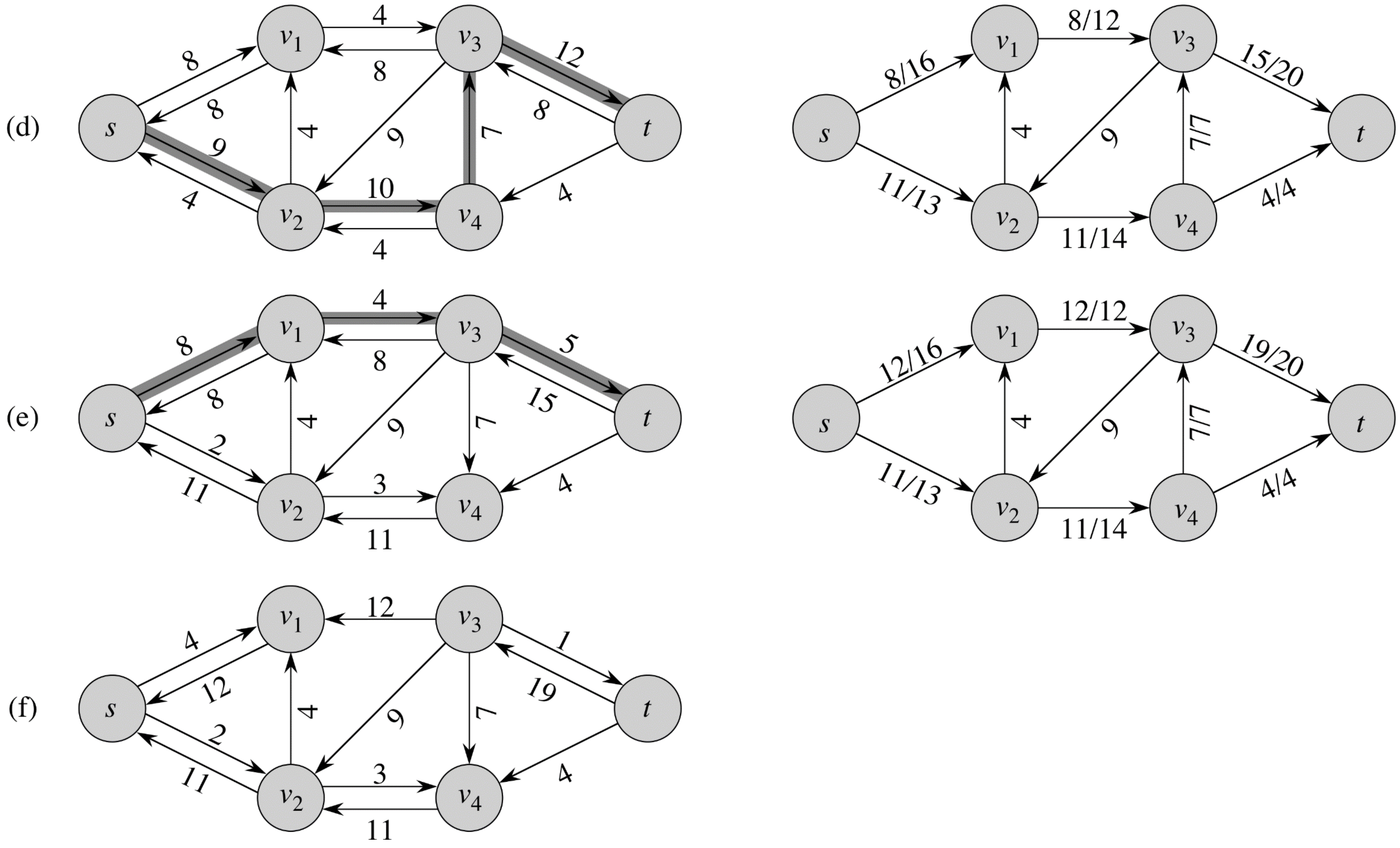


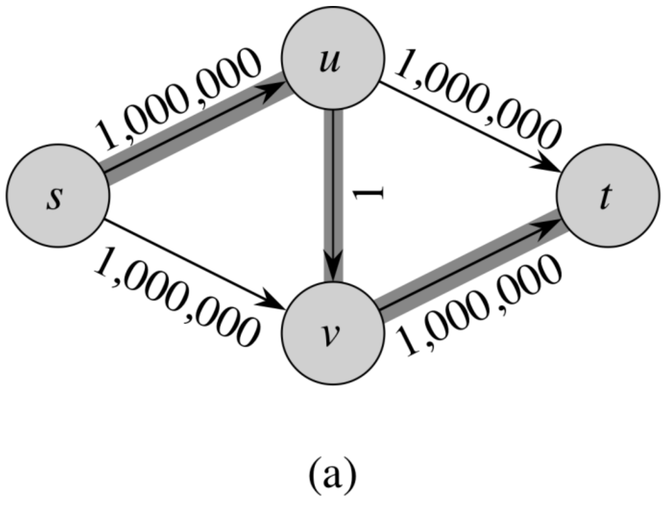
Figure 26.6 The execution of the basic Ford-Fulkerson algorithm. **(a)–(e)** Successive iterations of the **while** loop. The left side of each part shows the residual network G_f from line 3 with a shaded augmenting path p. The right side of each part shows the new flow f that results from augmenting f by f_p. The residual network in (a) is the input network G.

FFA



Worst case for FFA

- $|f^*|$ in this case is 2,000,000



(a) (b) (c)

Edmonds-Karp algorithm

- Uses BFS for augmenting path (shortest path)
- Total number of flow augmentations is $O(VE)$, thus overall complexity $O(VE^2)$
- How does it perform with the worst case for FFA?