Fluid Toolbox

with Python

Science Docs
PDFs for explorers and experimenters

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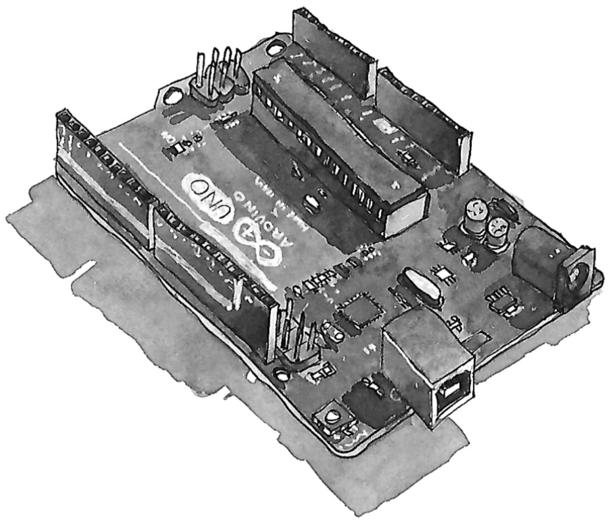

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`Fluid Toolbox`
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CONTENTS

Preface

"Use it well."
— Prof. Albus Dumbledore

Fluid Toolbox with Python is a collection of human-readable, pseudo-random study notes. It contains descriptions and explanations of fluid mechanics concepts, presented in a way that resembles the record of my understanding of them. It is meant to be used complimentary to the regular textbook since it may provide additional insight but it will not substitute thoroughness of a standard course in the subject. I believe that working side by side with a course it can become indeed a useful toolbox of concepts that are ready-to-use and ready-to-understand.

These notes are under constant development as I learn more and understand more.

Why is this text created?

I have a goal of laying the ground for the most important fluid mechanics concept and collecting them in once place. Much of the knowledge presented here comes from my searching of understanding that I was often missing when reading textbooks, and which was difficult to find in a way that would be pleasant and intuitive. Much of the understanding presented here comes from my personal explorations and thinking on the subject and therefore I have hopes that Fluid Toolbox will become a helpful resource that you have been looking for, similarly as it would have been helpful when I was looking for similar set of notes.

Structure of the text

Each concept is presented as a separate chapter. At some point, each concept will be as well supported by a Python script.

A note of inspiration and motivation

Acknowledgements

This PDF is created during my PhD at Université libre de Bruxelles.

I am forever grateful to all people I met in my life

who supported my passion for fluid dynamics.

Bruxelles, 2018

Personal code of conduct

There are certain directives that I would like to state clear at the very beginning to give you an honest overview of what you will encounter later in this text. This is done so that you have a freedom to escape if you see that it won't work for you. And this is done to convince the rest of you that this is what you want to read.

1. I dislike the avoidance to use "I" or "we" in scientific materials. Science is done by humans and textbooks are written by humans, so it feels unnatural to avoid speaking about what "we have done" or what "I would like to explain". When the author is writing his biography trying to sound like someone else wrote about him, that sounds artificial to me. However, I still respect the materials that make the aforementioned avoidance. Otherwise our papers wouldn't be published if we didn't stick to the convention?

2. If I am using difficult wording, it's mostly for fun.

3. Learning is an interactive process and we don't have to limit our thinking to what is said in the text. So I will sometimes ask a brain-teasing question or challenge you to *pause and ponder*.

4. Very likely, you will not get a long term profit from just reading the textbook. You can only see if you've learned the skill by trying it out so I recommend using the method of *active recall* when studying fluid dynamics concepts (and really any concept). One of the approaches (which I often used to check my understanding when writing this text) is called *the Feynman technique*. Combined with *spaced repetition* it can prove to be a useful tool for build your long-lasting understanding of fluid dynamics.

CHAPTER 1

CHANGES

Ch-ch-ch-ch-changes
Turn and face the strange
Ch-ch-changes
There's gonna have to be a different man
— David Bowie

What does it mean for a quantity to change in time and in space?

Chapter 2

Differentiation

Differentiation is a way to make discrete things continuous.

CHAPTER 3

COMMON FLOW TYPES

Poiseuille, Couette, uniform, etc.

CHAPTER 4

DRAG FORCE

The mathematical description of the drag force begins with making a guess. It is an intuitive guess which answers the question: what physical quantities affect the value of the drag force on an object moving through a fluid? The thought process done on this question gives the following four quantities:

relative fluid-object velocity v fluid density ρ fluid viscosity μ geometry of an object D

We assume that the drag force is a combination of these quantities to some yet unknown powers and we use dimensional analysis to find those powers.

$$F_D = v^a \rho^b \mu^c D^d \tag{4.1}$$

Writing out the units of the above equation we get:

$$\left[\frac{kg \cdot m}{s^2}\right] = \left[\frac{m}{s}\right]^a \cdot \left[\frac{kg}{m^3}\right]^b \cdot \left[\frac{kg}{m \cdot s}\right]^c \cdot \left[m\right]^d$$

Shuffling around a bit:

$$kg \cdot m \cdot s^{-2} = kg^{b+c} \cdot m^{a-3b-c+d} \cdot s^{-a-c}$$

Hence:

$1 = b + c$

$1 = a - 3b - c + d$

$-2 = -a - c$

The simple fact that this set of equations cannot be solved exactly (there is four uknown powers and only three equations) is going to call experiment for help, which you will notice in the next few passages.

Let's then write the set of equations by means of one of the unknowns c:

$a = 2 - c$

$b = 1 - c$

$d = 2 - c$

Subsituting back to the equation 4.1 we get:

$$F_D = v^{2-c} \rho^{1-c} \mu^c D^{2-c} \tag{4.2}$$

Structuring the equation still a bit gives:

$$F_D = v^2 \rho D^2 \cdot \left[\frac{v \rho D}{\mu} \right]^{-c} \tag{4.3}$$

It always feels comfortable to find the Reynolds number in your equation, so there it is:

$$F_D = v^2 \rho D^2 Re^{-c} \tag{4.4}$$

We get finally that the drag force is proportional to some unknown power of the Reynolds number.

In fact, we will not leave it there yet, since there is an interesting last point to say. Instead of the above equation, we will say that the drag force is proportional to some unknown function C_D of the Reynolds number. We also recognise that we may rewrite the quantity $v^2\rho$ as the dynamic pressure $\frac{v^2\rho}{2}$, since multiplying the right hand side by $\frac{1}{2}$ will not spoil the dimensional equality of both sides. The quantity D^2 has got the unit of area, so we exchange it for the quantity $A_\perp$, representing the frontal area of an object. The equation for the drag force becomes:

$$F_D = \frac{v^2 \rho}{2} A_\perp C_D(Re) \tag{4.5}$$

The unknown function $C_D(Re)$ is called the **drag coefficient**. That is where we call for experiment.

Questions:

What would have happened if we wrote the powers in terms of some power other than c?

CHAPTER 5

CIRCULATION

Circulation is defined as:

$$\Gamma = \oint_C \vec{v} \cdot \vec{dl} \tag{5.1}$$

The dot operation gives a scalar which is expressing "how much" in the direction of the other vector is this vector.

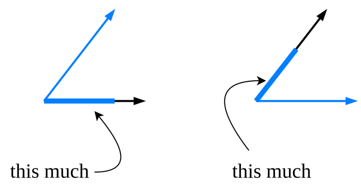

Figure 5.1: Dot product of two vectors.

When they are $\perp$, the dot product is zero.

In the concept of circulation, you ask "how much" at any point on the curve C the velocity vector at that point is in the direction of the curve's geometry. Now, that doesn't yet sound as something to do with "circulating". For the moment, I would think that it's more of an "on-trackness". Something, that in real world would be for instance the measure of how much the vehicle's velocity is in the direction of the road geometry.

But then you realise an important detail of "$\circ$" on the integral symbol, which means that the curve should be a closed curve - a loop.

When you perform integration, which means summing up every little $\vec{v} \cdot \vec{dl}$ as you go around the loop, you count "how much" at every point on the loop, the velocity vector at these points is in the direction of the loop's geometry (at these points).

If we were to place a small particle at some starting point P on the loop, the circulation would tell us "how much" the velocity field which this particle is subjected to, is tending to move that particle around the loop.

It can be very intuitive when you take a look at these two pictures:

It's no surprise that when the velocity is everywhere perpendicular to the loop's geometry, the circulation around the loop is zero. If you were to place a particle at any point on the loop, such velocity field would act to immediately displace the particle off the loop. Therefore, the particle would have no way of "circulating" around the loop.

On the other extreme is the case when the velocity field is everywhere tangent to the loop's geometry. Anywhere the particle goes on the loop, the velocity at that point would act to keep the particle moving around the loop.

Questions:

1. Why closed loop? Would it have any meaning if we calculated circulation along any general spline?
2. How to chose loops so that the circulation we calculate is of the most meaning to us?
3. What does the zero, positive, negative circulation mean?
4. Can circulation be infinite?

CHAPTER 6

VORTICITY

1. Why is vorticity concept needed? Isn't it something kind of like circulation?

Chapter 7

Stoke's theorem

Chapter 8

Nondimensionalizing

What's with the nondimensionalizing?

Chapter 9

Material derivative

The material derivative is an operator defined as:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{\boldsymbol{V}} \cdot \nabla \tag{9.1}$$

Or, expanding out the terms (in 3D case, where $\vec{\boldsymbol{V}} = \langle u, v, w \rangle$):

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} \tag{9.2}$$

Material derivative is indeed a shorthand for writing a special sum of other operators and it has been created because this set is frequently used in fluid dynamics related equations. Writing it in short as $\frac{D}{Dt}$ simply makes life easier.

CHAPTER 10

CONSERVATION OF MASS

In this chapter we present the derivation of the conservation of mass equation, otherwise known as the **continuity equation**. In principle, it states that mass cannot be lost nor created.

We begin by writing out the overall mass balance inside any control volume CV. The net change of mass inside the control volume is equal to the mass flowing into the CV minus the mass flowing out of the CV. Note here, that when the net change of mass in a CV is not zero (unsteady case), it can only be due to either compression (more mass flowing in than flowing out) or decompression (more mass flowing out than flowing in). The general mass balance is:

$$\text{net change} = \text{flow in} - \text{flow out} \tag{10.1}$$

10.1 Mass flow rate

10.2 Derivation using control volume

We are going to write out the RHS of the equation 10.1 as the difference between mass flow rate in and mass flow rate out in three Cartesian directions.

In the x-direction:

$$\Big(\rho u - \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\Big)dydz - \Big(\rho u + \frac{\partial(\rho u)}{\partial x}\frac{dx}{2}\Big)dydz = -\frac{\partial(\rho u)}{\partial x}dxdydz \tag{10.2}$$

In the y-direction:

$$\Big(\rho v - \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\Big)dxdz - \Big(\rho v + \frac{\partial(\rho v)}{\partial y}\frac{dy}{2}\Big)dxdz = -\frac{\partial(\rho v)}{\partial y}dydxdz \tag{10.3}$$

In the z-direction:

$$\Big(\rho w - \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\Big)dxdy - \Big(\rho w + \frac{\partial(\rho w)}{\partial z}\frac{dz}{2}\Big)dxdy = -\frac{\partial(\rho w)}{\partial z}dzdxdy \tag{10.4}$$

The net change in time of mass can be written as:

$$\frac{\partial \rho}{dt}dxdydz \tag{10.5}$$

Putting all the terms together into equation 10.1 we obtain:

$$\frac{\partial \rho}{dt}dxdydz = -\frac{\partial(\rho u)}{\partial x}dxdydz - \frac{\partial(\rho v)}{\partial y}dydxdz - \frac{\partial(\rho w)}{\partial z}dzdxdy \tag{10.6}$$

Dividing both sides by $dxdydz$ we get:

$$\frac{\partial \rho}{dt} = -\frac{\partial(\rho u)}{\partial x} - \frac{\partial(\rho v)}{\partial y} - \frac{\partial(\rho w)}{\partial z} \tag{10.7}$$

Lastly, we can observe that the divergence of the quantity $\rho\vec{V}$ is:

$$\nabla(\rho\vec{V}) = \nabla(\rho\langle u, v, w\rangle) = \nabla\langle \rho u, \rho v, \rho w\rangle = \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \tag{10.8}$$

So in the end, we can further write the RHS of the equation 10.7 in a shorter format as:

$$\frac{\partial \rho}{dt} = -\nabla(\rho\vec{V}) \tag{10.9}$$

10.3 Special cases of density function

In the most general case, the density ρ is a function of time and space $\rho = \rho(t, x, y, z)$ and the equation 10.9 is written out for the most general case. Special cases can be defined when certain restrictions are imposed on the density function.

10.3.1 Incompressible flow

When we assume that the density is constant in space, it can be taken out front of the divergence operator and the incompressible continuity equation is:

$$\frac{\partial \rho}{dt} = -\rho\nabla\vec{V} \tag{10.10}$$

The above equation corresponds to the density that is constant throughout the flow field at any moment in time, but can change in time in the entire flow field.

10.3.2 Incompressible, steady flow

The steady flow can be further combined with the incompressible condition, which disables the density to change both in time and space. The stead-state incompressible continuity equation then becomes:

$$0 = \nabla\vec{V} \tag{10.11}$$

or writing the above equation with the use of partial differentiation operator:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{10.12}$$

10.3.3 Compressible, steady flow

First, when the density ρ is only a function of position $\rho = \rho(x, y, z)$ and does not change in time at any point in the flow field, we arrive at the steady state condition where the derivative $\frac{\partial \rho}{dt} = 0$.

The continuity equation then becomes:

$$0 = -\nabla(\rho\vec{V}) \tag{10.13}$$

10.4 Dimension reduction

10.5 Divergence theorem and a different way of looking at the conservation of mass

Chapter 11

Gauss's law

CHAPTER 12

REYNOLDS NUMBER

I this chapter we travel a journey through the many faces of the Reynolds number.

CHAPTER 13

CONSTITUTIVE EQUATIONS

In case you need to understand the constitutive equations for use in the Navier-Stokes equations, you may skip this chapter for now and come back to it once you are in the *Constitutive equations* section inside the Navier-Stokes equations chapter.

CHAPTER 14

THE NAVIER-STOKES EQUATIONS

The Navier-Stokes equations represent the momentum balance on an infinitesimal fluid volume. They state that the sum of forces acting on a fluid volume is equal to the change in momentum for that volume (Newton's second law).

14.1 Incompressible Navier-Stokes

In a three-dimensional, incompressible, unsteady flow we have:

x-direction:

$$\rho g_x - \frac{\partial P}{\partial x} + \mu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\Big) = \rho\Big(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\Big) \tag{14.1}$$

y-direction:

$$\rho g_y - \frac{\partial P}{\partial y} + \mu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\Big) = \rho\Big(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\Big) \tag{14.2}$$

z-direction:

$$\rho g_z - \frac{\partial P}{\partial z} + \mu\Big(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\Big) = \rho\Big(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\Big) \tag{14.3}$$

14.2 Derivation

Sum of the forces is written on the LHS of the above equations. The total change in momentum is written on the RHS of the above equations.

14.2.1 RHS

In the most general case, the velocity vector $\vec{V} = \langle u, v, w\rangle$ is a function of time and position. We write therefore: $\vec{V}(t,x,y,z) = \langle u(t,x,y,z), v(t,x,y,z), w(t,x,y,z)\rangle$.

The acceleration component is described as the time derivative of the corresponding velocity component. If you were to take the derivative with respect to time of any of the above components of the velocity vector, it would become due to chain rule:

$$a_x = \frac{\partial u(t,x,y,z)}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial t}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} \tag{14.4}$$

Observe then, that $\frac{\partial t}{\partial t} = 1$, $\frac{\partial x}{\partial t} = u$, $\frac{\partial y}{\partial t} = v$, $\frac{\partial z}{\partial t} = w$.

We have therefore for all three components:

$$a_x = \frac{\partial u(t,x,y,z)}{\partial t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{14.5}$$

$$a_y = \frac{\partial v(t,x,y,z)}{\partial t} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \tag{14.6}$$

$$a_z = \frac{\partial w(t,x,y,z)}{\partial t} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \tag{14.7}$$

The terms $\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$, $\frac{\partial w}{\partial t}$ are called **local accelerations**, and the remaining terms where the derivatives are taken with respect to spacial coordinates are called **convective accelerations**[1]. The names *local* and *convective* carry a lot of meaning here, although initially it might be hard to see.

The word *convective* means that it is a quantity that depends on traveling to other regions - in other words - it depends on position. It is going to change due to the fact that the fluid element has traveled to a new place.

The word *local*, on the other hand, suggests that it is a quantity that is intrinsic to the particular fluid element and its change is independent of the particle's position in space. The *local* quantity can change in a fluid element, but solely "in itself" as the time pass (or *locally* and that will happen independent of where the fluid particle travels to).

Notice that this logic fits into how these derivatives look like. The convective terms are all derivatives with respect to position - they account for a change in velocity components due to change in coordinates. The local terms are all derivatives with respect to time - they just change by itself as the time change and position is irrelevant.

The above equations represent the acceleration components. These components are present in the RHS of the Navier-Stokes equations [14.1 - 14.3].

The standard way to write the Newton's second law is that the sum of the forces equals mass multiplied by acceleration. Since the Navier-Stokes equations are written per volume basis, the change in momentum (or the mass multiplied by acceleration) is written per unit of volume, so:

$$\frac{dm}{dV}a_x = \frac{dm}{dV}\frac{\partial u(t,x,y,z)}{\partial t} = \rho\frac{\partial u(t,x,y,z)}{\partial t} \tag{14.8}$$

Multiplying all the acceleration components by the density ρ we are going to obtain the full description for the change in momentum per unit of volume:

$$\rho a_x = \rho\frac{\partial u(t,x,y,z)}{\partial t} = \rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} + \rho v\frac{\partial u}{\partial y} + \rho w\frac{\partial u}{\partial z} \tag{14.9}$$

$$\rho a_y = \rho\frac{\partial v(t,x,y,z)}{\partial t} = \rho\frac{\partial v}{\partial t} + \rho u\frac{\partial v}{\partial x} + \rho v\frac{\partial v}{\partial y} + \rho w\frac{\partial v}{\partial z} \tag{14.10}$$

$$\rho a_z = \rho\frac{\partial w(t,x,y,z)}{\partial t} = \rho\frac{\partial w}{\partial t} + \rho u\frac{\partial w}{\partial x} + \rho v\frac{\partial w}{\partial y} + \rho w\frac{\partial w}{\partial z} \tag{14.11}$$

This concludes the derivation and explanation of the RHS of the Navier-Stokes equations stated at the beginning of this chapter.

[1] See the chapter on Material Derivative for more understanding of the local and convective terms.

14.2.2 LHS

We are now going to crunch the LHS of the Navier-Stokes equations - the sum of the forces acting on a unit volume of fluid. There are two types of forces taken into account here: **body forces** (gravity) and **surface forces** (forces due to pressure difference and dissipative forces due to viscosity).

Gravity forces

The body forces due to gravity are perhaps the most intuitive to figure out. In order to construct the gravity force $F_{g,x}$ on a fluid element, we multiply mass of that element by gravitational acceleration. Inside the Navier-Stokes equations it is not any different. We have the terms: ρg_x, ρg_y, ρg_z, that represent exactly that idea, except they are again written on a per volume basis.

$$\frac{dF_{g,x}}{dV} = \frac{dm}{dV} g_x = \rho g_x \tag{14.12}$$

$$\frac{dF_{g,y}}{dV} = \frac{dm}{dV} g_y = \rho g_y \tag{14.13}$$

$$\frac{dF_{g,z}}{dV} = \frac{dm}{dV} g_z = \rho g_z \tag{14.14}$$

Surface forces

The surface stresses can be represented by the **stress tensor**:

$$\boldsymbol{\sigma}_{IN} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \qquad \boldsymbol{\sigma}_{OUT} = \begin{pmatrix} \sigma_{xx} + \frac{\partial \sigma_{xx}}{\partial x} dx & \sigma_{xy} + \frac{\partial \sigma_{xy}}{\partial x} dx & \sigma_{xz} + \frac{\partial \sigma_{xz}}{\partial x} dx \\ \sigma_{yx} + \frac{\partial \sigma_{yx}}{\partial y} dy & \sigma_{yy} + \frac{\partial \sigma_{yy}}{\partial y} dy & \sigma_{yz} + \frac{\partial \sigma_{yz}}{\partial y} dy \\ \sigma_{zx} + \frac{\partial \sigma_{zx}}{\partial z} dz & \sigma_{zy} + \frac{\partial \sigma_{zy}}{\partial z} dz & \sigma_{zz} + \frac{\partial \sigma_{zz}}{\partial z} dz \end{pmatrix} \tag{14.15}$$

The net force on the fluid element resulting from the surface stresses comes from the difference between stresses on two opposing sides of the fluid element. Therefore we are going to look at the value of:

$$\boldsymbol{\sigma}_{OUT} - \boldsymbol{\sigma}_{IN} = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} dx & \frac{\partial \sigma_{xy}}{\partial x} dx & \frac{\partial \sigma_{xz}}{\partial x} dx \\ \frac{\partial \sigma_{yx}}{\partial y} dy & \frac{\partial \sigma_{yy}}{\partial y} dy & \frac{\partial \sigma_{yz}}{\partial y} dy \\ \frac{\partial \sigma_{zx}}{\partial z} dz & \frac{\partial \sigma_{zy}}{\partial z} dz & \frac{\partial \sigma_{zz}}{\partial z} dz \end{pmatrix} \tag{14.16}$$

Since we are interested in forces resulting from these stresses, we need to multiply the above values by the areas of the corresponding surfaces on which these stresses act.

$$F_\sigma = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} dxdydz & \frac{\partial \sigma_{xy}}{\partial x} dxdxdz & \frac{\partial \sigma_{xz}}{\partial x} dxdxdy \\ \frac{\partial \sigma_{yx}}{\partial y} dydydz & \frac{\partial \sigma_{yy}}{\partial y} dydxdz & \frac{\partial \sigma_{yz}}{\partial y} dzdxdy \\ \frac{\partial \sigma_{zx}}{\partial z} dxdydz & \frac{\partial \sigma_{zy}}{\partial z} dydxdz & \frac{\partial \sigma_{zz}}{\partial z} dzdxdy \end{pmatrix} \tag{14.17}$$

And lastly, we will write the forces per unit of volume:

$$f_\sigma = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} & \frac{\partial \sigma_{xy}}{\partial x} & \frac{\partial \sigma_{xz}}{\partial x} \\ \frac{\partial \sigma_{yx}}{\partial y} & \frac{\partial \sigma_{yy}}{\partial y} & \frac{\partial \sigma_{yz}}{\partial y} \\ \frac{\partial \sigma_{zx}}{\partial z} & \frac{\partial \sigma_{zy}}{\partial z} & \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix} \tag{14.18}$$

Since the pressure P is often a variable of interest when it comes to fluids, it is worth decomposing the stress tensor to two terms:

$$\boldsymbol{\sigma}_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} + \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{14.19}$$

The above pressure P is such that for a fluid at rest the stress tensor is simply:

$$\boldsymbol{\sigma}_{ij} = \begin{pmatrix} -P & 0 & 0 \\ 0 & -P & 0 \\ 0 & 0 & -P \end{pmatrix} \tag{14.20}$$

The matrix with τ-terms is called **viscous stress tensor**. In a special case when fluid is inviscid this tensor is zero.

$$\boldsymbol{\tau}_{ij} = \begin{pmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{pmatrix} \tag{14.21}$$

We can now write the forces per unit volume (eq. 14.18) as:

$$f_\sigma = \begin{pmatrix} \frac{\partial(\tau_{xx}-P)}{\partial x} & \frac{\partial \tau_{xy}}{\partial x} & \frac{\partial \tau_{xz}}{\partial x} \\ \frac{\partial \tau_{yx}}{\partial y} & \frac{\partial(\tau_{yy}-P)}{\partial y} & \frac{\partial \tau_{yz}}{\partial y} \\ \frac{\partial \tau_{zx}}{\partial z} & \frac{\partial \tau_{zy}}{\partial z} & \frac{\partial(\tau_{zz}-P)}{\partial z} \end{pmatrix} \tag{14.22}$$

The above formulation is convenient because there *might*[2] exist a known relationship between shear stresses τ_{ij} and velocity gradients (see chapter on *Constitutive equations* for a more intuitive explanation). Adding such relation allows to finally represent the forces coming from stresses by means of desired parameters: pressure, velocity components and viscosity.

14.3 A note about Cauchy's equations

The Navier-Stokes equation can be thought of as a special case of the Cauchy's equation that is strictly to be used for a specific type of fluid: a Newtonian, incompressible fluid. Cauchy's equation states the law of conservation of linear momentum for any type of fluid.

14.4 Assumptions and limitations

[2] I say *might* because we often deal with Newtonian fluids for which this formulation is known. In short, for Newtonian fluids there is a way to express shear stresses as being proportional to velocity gradients (with viscosity being the proportionality constant). It is analogous to the relation given by Newton:

$$\tau \propto \frac{\partial u}{dy} \tag{14.23}$$

However, when dealing with fluids of a more complex nature than Newtonian fluids, this formulation needs to be found accordingly and the remaining derivation of the equations of motion needs to be updated. The Navier-Stokes equations are presented in this chapter for Newtonian fluids.

Questions

1. What would the Navier-Stokes equations reduce to for different flow cases?

APPENDIX

About the author

I am a graduate in civil engineering and a long-time passionate of the fluid business.

My journey through fluid mechanics begun during my first ever visit to a wind tunnel laboratory where my future supervising Professor winked at me when encouraging students to join his research. In the meantime, I worked in a wind engineering company in the UK, where I had a chance to take part in every step of wind tunnel testing of tall buildings, occasionally even gluing some trees to the model. Shortly after finishing my Master studies I was a stagiaire at The von Karman Institute for Fluid Dynamics in Belgium, where apart from falling in love with French language, I studied data decomposition methods and applied them to extract low-rank structures from fluid flow phenomena. I am currently beginning a new scientific journey as a PhD researcher at Université libre de Bruxelles (ULB). My work comprises the development of reduced-order methods for modelling turbulent reactive flows.

Whenever the weather permits I enjoy exploring distant places with my bicycle or while running. I also take pleasure in painting buildings and street signs with watercolours.

The cover photo:
View from the coast of Cres island in Croatia, October 2016.
Photo by: J. Aleksanderek ©

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