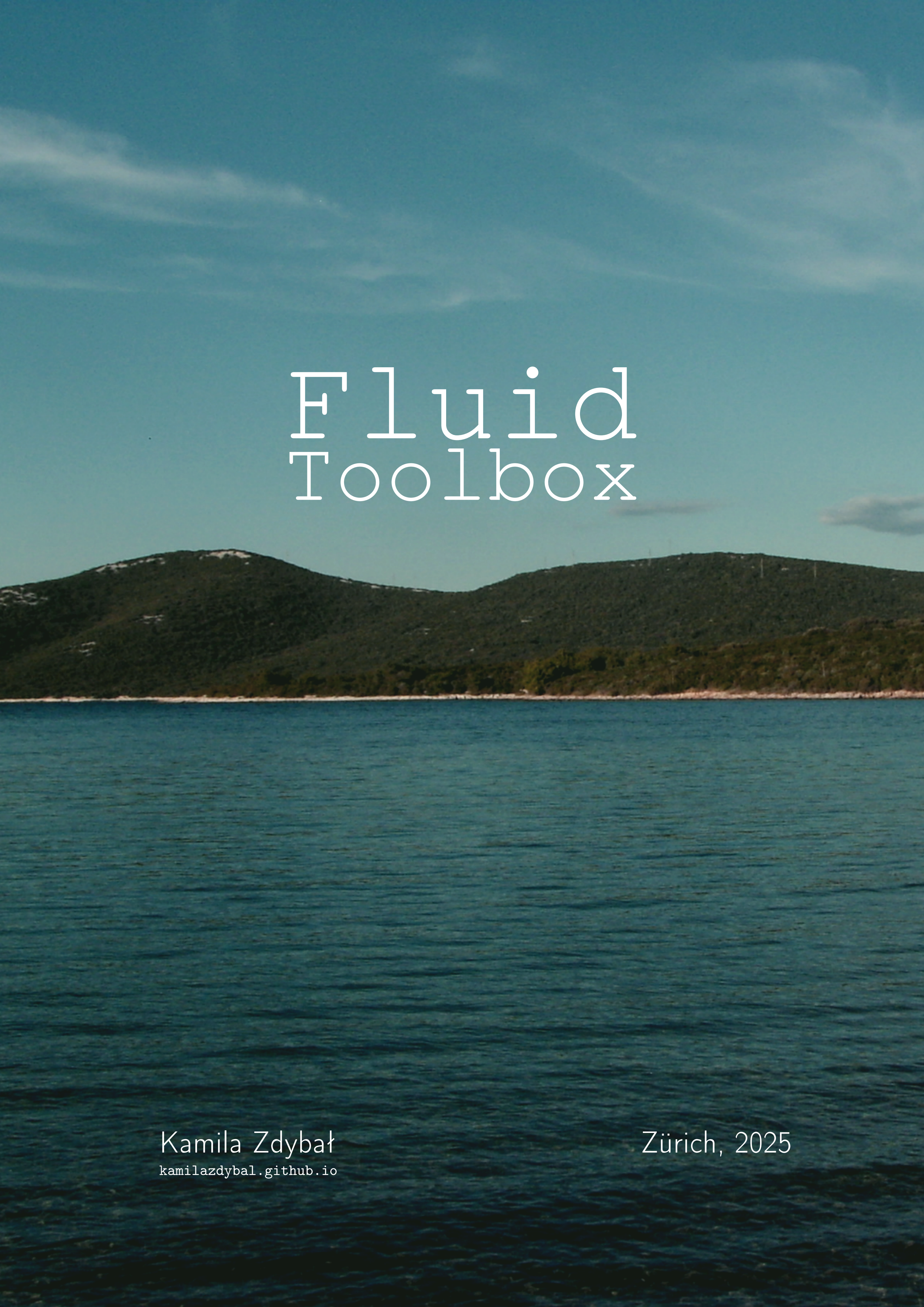
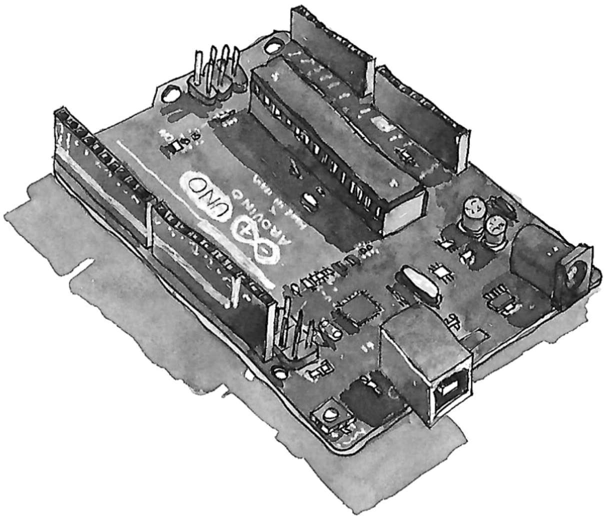


Fluid Toolbox

Kamila Zdybał
kamilazdybal.github.io

Zürich, 2025



EX LIBRIS · KAMILA

For more projects similar to this one
visit my personal website: `kamilazdybal.github.io`
or visit me on GitHub: `@kamilazdybal`
You can contact me with any suggestions, corrections, or thoughts:
`kamilazdybal at gmail dot com`

`Fluid Toolbox`
`version 1.1`
Typeset with ♥ for LaTeX

PREFACE

"Use it well."
— Prof. Albus Dumbledore

`Fluid Toolbox` is a collection of human-readable, pseudo-random study notes that inspires you to think a bit deeper about various fluid dynamics concepts. It is meant to be used complimentary to the regular textbook since it may provide additional insights, but it will not substitute the thoroughness of a standard course in the subject. I believe that working side by side with the course it can become a useful toolbox of concepts that are ready-to-understand and ready-to-use.

Why is this text created?

I had a goal of collecting the most important fluid dynamics concept in one place. Much of the knowledge presented here comes from my search for understanding that I was often missing when reading textbooks, and which was difficult to find in a way that would be engaging, illustrative, and would make intuitive sense to me. Some of the understanding presented here comes from my personal explorations and thinking on the subject. I have hopes that it will become a helpful resource that you have been looking for. I would also like to challenge you to ponder more deeply about the concepts presented here and to seek the beauty in studying fluid motion.

Some things I believe will help you learn

There are a few things that I believe will help you in learning a new subject, such as fluid dynamics:

1. I want to take you through the journey of learning as you read these notes. Above all, I will know that it was worth writing this document if you enjoyed the journey of reading it and learning from it!

2. It's not enough to read a textbook or watch a lecture. The real learning and understanding comes when it's only you, your head, and a blank piece of paper. Reading a textbook is an easy task to do, so is watching a lecture.

When you become a spectator in the learning process, it's easy to fool yourself that you understood something. It is only when you have a chance to take action in your learning and recall that you can really use your knowledge and test your deep understanding. Which brings me to...

3. Programming is a great way to roll your sleeves and put things into action.

4. I believe in the quote by Richard Feynman: *Study hard what interests you the most in the most undisciplined, irreverent and original manner possible.* I will sometimes ask a brain-teasing question or challenge you to *pause and ponder*. In your learning, do challenge yourself, and don't let your imagination be limited by simply following pages of any textbook!

This document will be alive for quite some time. I will be coming back to it to add or improve things. You can always access the newest version through my GitHub site: `kamilazdybal.github.io`.

Please feel free to contact me with any suggestions, corrections, or thoughts: `kamilazdybal at gmail dot com`.

CONTENTS

ACKNOWLEDGEMENTS

I am grateful to all people I encountered in my life

who supported my passion for studying fluid motion.

Among the first ones were Y. Çengel and J. Cimbala

in their inspiring fluid mechanics textbook.

Zürich, Switzerland, 2025

Chapter 1

Changes

Chapter 2

Differentiation

CHAPTER 3

MATERIAL DERIVATIVE

3.1 Where space, time, and fluid flow meet

The material derivative describes the *total experienced* change in quantity $\bullet$ as *time goes on* **and** as we *move* across the field of $\bullet$ with fluid velocity, $\vec{V} = \langle u, v, w \rangle$. Hence, the material derivative requires two ingredients; these are visualized in Fig. 3.1. The first ingredient is the field of $\bullet$, which can change spatially and temporally (Fig. 3.1a). The second ingredient is the associated fluid velocity field, $\vec{V}$ (Fig. 3.1b). In this chapter, you can substitute for $\bullet$ any interesting physical quantity that you'd like, such as density, ρ, or temperature, T. Interestingly, this quantity does not need to be a scalar, but can also be a vector or even a tensor.

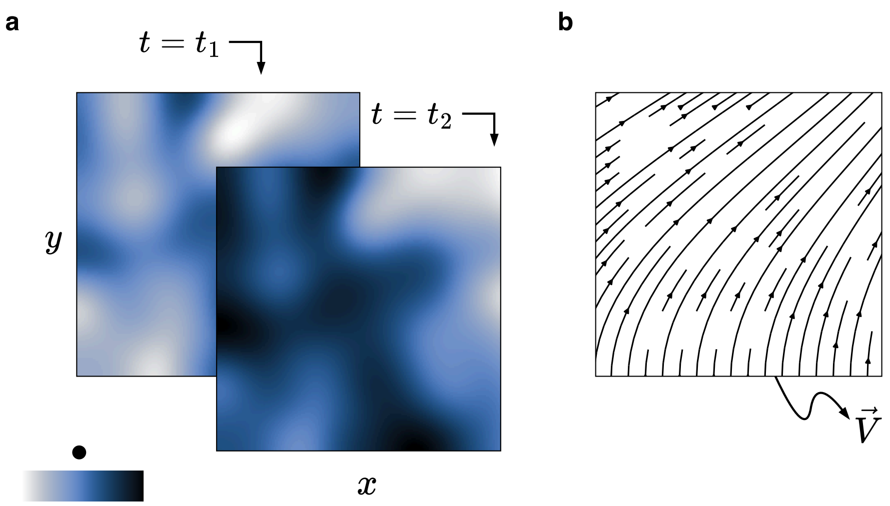


Figure 3.1: Two ingredients needed to compute the material derivative: (**a**) the field of $\bullet$, which can change spatially and temporally, and (**b**) the associated fluid velocity field, $\vec{V}$.

To first start thinking about the material derivative visually, you may consider a 2D field of $\bullet$ that changes in time and space, just like the one presented in Fig. 3.1a. In Fig. 3.2, let's look at the possible reasons for why we might experience change in $\bullet$. In the absence of spatial movement over the (x, y) grid we can only experience change in $\bullet$ if $\bullet$ varies in time. Similarly, in the absence of temporal variation in $\bullet$, we can experience change in $\bullet$ only if we travel along the (x, y) grid and $\bullet$ varies over that grid. With both time and motion present, we experience a superposition of these two effects. That will be our total experienced change in $\bullet$. I will emphasize, however, that in the definition of the material derivative our movement is restricted to one defined by the fluid flow. Hence, we specifically use the flow velocity, $\vec{V}$, and not any other velocity[1].

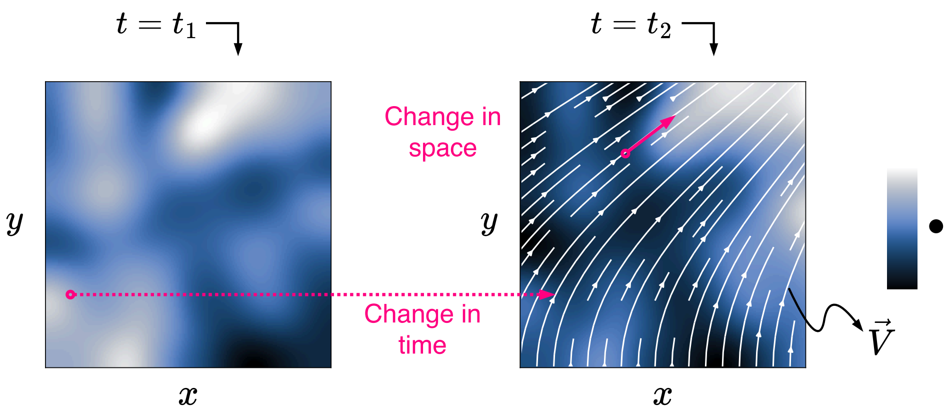


Figure 3.2: A 2D field of some scalar quantity, $\bullet$, that changes in time, t, and space, (x, y). We also consider the associated fluid velocity field, $\vec{V}$. The material derivative is a superposition of two reasons for why $\bullet$ can change.

In mathematical terms, the material derivative, $\frac{D}{Dt}$, is an operator acting on $\bullet$ such that

$$\frac{D\bullet}{Dt} \equiv \frac{\partial\bullet}{\partial t} + \vec{V} \cdot \nabla \bullet \ . \tag{3.1}$$

The superposition that I mentioned before is embedded in the two terms on the right-hand-side of Eq. (3.1). We can now dissect these two terms to better understand why introducing the material derivative is very useful when studying fluid motion.

First, we have $\frac{\partial\bullet}{\partial t}$ which is the plain old[2] partial derivative of $\bullet$ with respect to time. It says that at all possible locations in space, and at any one location, the quantity $\bullet$ can evolve in time. One example of such quantity is temperature. Even if we remain stationary in a specific location, say in a corner of a room, we can still experience change in temperature because our room might be heated (or cooled) and the temperature in our little corner changes in time because of that. The term $\frac{\partial\bullet}{\partial t}$ gives us a recipe for *how* that temperature changes in time in every location of the room.

Second, we have $\vec{V} \cdot \nabla\bullet$, that is, a gradient vector, $\nabla\bullet = \langle \frac{\partial\bullet}{\partial x}, \frac{\partial\bullet}{\partial y}, \frac{\partial\bullet}{\partial z} \rangle$, dotted with the fluid velocity vector, $\vec{V}$. The gradient of $\bullet$ is a vector field that describes

[1]That said, one could, potentially, define a generalization of the material derivative to allow for an arbitrary velocity! Such a new quantity will have a different physical meaning though.

[2]See Chapter 1.

directions in which $\bullet$ varies. If, and only if, your own movement is aligned (at least to some extent) with the direction of $\bullet$'s gradient, you will experience a change in quantity $\bullet$. Otherwise, if you walk along an isocurve of $\bullet$, you will not experience any change in $\bullet$. The dot product taken between $\vec{V}$ and $\nabla\bullet$ measures the degree of that alignment.

In other words, the first term on the right-hand-side of Eq. (3.1) describes how we will experience change in $\bullet$ in the absence of our motion through the field of $\bullet$. The second term describes how we will experience additional change in $\bullet$ due to moving around through the field of $\bullet$ but with a very specific velocity, $\vec{V}$. The material derivative is a neat superposition of these two factors for why $\bullet$ can change. It is also a shorthand for describing change in $\bullet$ in a moving fluid and it has been created because this superposition of effects frequently appears in the governing equations of fluid dynamics. Writing it in short as $\frac{D}{Dt}$ simply makes our life easier.

Hungry for more?

You can find a great intuitive description of a material derivative in Chapter 3, §3.5 of the *Transport Phenomena* textbook by Bird, Stewart & Lightfoot [Bird et al., 2002]. They delineate differences between various derivatives on the example of following fish in a river.

Finally, I would like to present two more ways of writing Eq. (3.1) just to expose you to other possible notations that you might encounter in textbooks. In the most general 3D case, where $\vec{V} = \langle u, v, w \rangle$, we can expand the dot product terms to obtain:

$$\frac{D\bullet}{Dt} \equiv \frac{\partial\bullet}{\partial t} + u\frac{\partial\bullet}{\partial x} + v\frac{\partial\bullet}{\partial y} + w\frac{\partial\bullet}{\partial z}\,. \tag{3.2}$$

A yet another way of writing the equation above that you might sometimes encounter is the following:

$$\frac{D\bullet}{Dt} \equiv \frac{\partial\bullet}{\partial t} + V_i\frac{\partial\bullet}{\partial i}\,. \tag{3.3}$$

This way of writing Eq. (3.2) is using the Einstein notation where it is implied that you should substitute for the dummy index i every possible spatial dimension, *i.e.*, x, y, and z, and, as you substitute, you also sum up all the terms that form for each possible i.

3.2 Pause and ponder

Let's look at some alternative ways to describe change in both space and time and see why they wouldn't be equally useful as Eq. (3.1)! Suppose I present you with the following quantity:

$$\frac{\partial \bullet}{\partial t} + \frac{\partial \bullet}{\partial x} + \frac{\partial \bullet}{\partial y} + \frac{\partial \bullet}{\partial z} \,. \tag{3.4}$$

How is that quantity different from the definition of the material derivative? In other words, what does the dot product with the velocity vector change in how we described change in space in Eq. (3.2)?

The velocity vector is not our independent motion through the field of $\bullet$. It is our motion when carried by the fluid flow.

This discussion tells us something deeper about the philosophy of describing fluid motion. Material derivative is inherently tied to the continuum assumption in fluid dynamics.

In essence, the material derivative describes our experience change in $\bullet$ because of our motion with the fluid velocity, even though the change in $\bullet$ might happen precisely *due to* fluid motion, or at least be some function of it. Think about the fluid density, ρ, which can change due to local movement of fluid from one location to the next.

CHAPTER 4

DIVERGENCE THEOREMS

CHAPTER 5

COMMON FLOW TYPES

CHAPTER 6

DRAG FORCE

Chapter 7

Circulation

Chapter 8

Vorticity

CHAPTER 9

STOKE'S THEOREM

CHAPTER 10

NONDIMENSIONALIZING

CHAPTER 11

CONSERVATION OF MASS

Chapter 12

Gauss's law

Chapter 13

Reynolds number

CHAPTER 14

CONSTITUTIVE EQUATIONS

Chapter 15

The Navier-Stokes equations

APPENDIX

Ending remarks

The cover photo:
View from the coast of Cres island, Croatia, October 2016.
Photo by: J. Aleksanderek ©

BIBLIOGRAPHY

[Bird et al., 2002] Bird, R. B., Stewart, W., and Lightfoot, E. (2002). Transport phenomena.