Steady-state conduction in an insulated rod with internal heat production

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This example is based on a Jupyter notebook that can be accessed **here**.

1 Analytic solution

We first derive an expression for the one-dimensional temperature distribution function $T(x)$, resulting from the steady-state heat conduction in a straight rod of length L. We assume that there is an internal heat production, Q_p (W/m^3), at every point in the rod's volume. This may for instance simulate heating up of a wire due to electrical current. The rod is perfectly insulated along its length and it looses heat only through its endpoints (at $x = 0$ and at $x = L$). At steady-state, these endpoints are kept at a fixed temperature, T_0. The material from which the rod is made has thermal conductivity λ $(\frac{W}{mK})$. The system that we will study is presented in Fig. 1a.
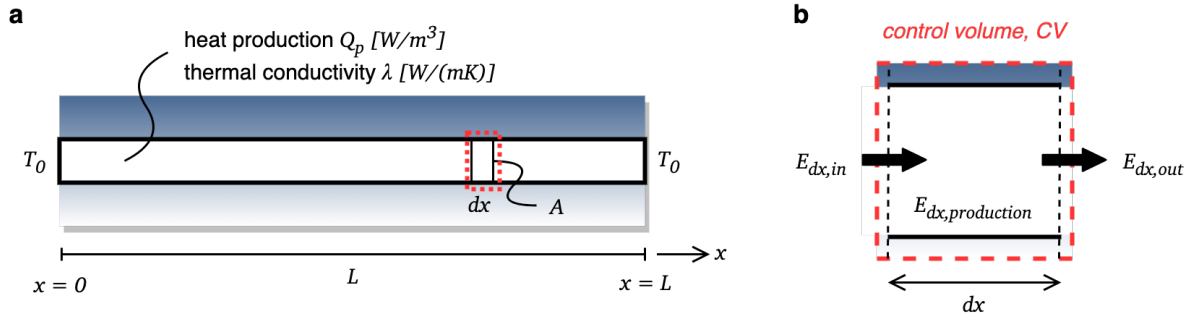

Figure 1: (**a**) Conduction in a lengthwise-insulated rod with internal heat production. (**b**) An infinitesimal control volume (CV) extracted from the rod.

Let's start with writing out the energy balance for an infinitesimal control volume, CV, shown in Fig. 1b. The CV is a slice of length dx extracted from the rod. The energy balance for the CV can be written as

$$\frac{dE_{dx}}{dt} = E_{dx,in} - E_{dx,out} + E_{dx,production}. \tag{1}$$

Note that $E_{dx,in}$, $E_{dx,out}$ and $E_{dx,production}$ are energies per unit time and so have the units of W. The heat flux, ϕ (W), can be modeled using the Fourier's law for one-dimensional heat conduction,

$$\phi = \lambda A \left(-\frac{dT}{dx} \right), \tag{2}$$

where λ is the thermal conductivity and is a property of the material, A is the rod's cross-sectional area and $\frac{dT}{dx}$ is the temperature gradient which can be thought of as the "driving force" for thermal energy transport.

Hence:

$$E_{dx,in} = \lambda A \left(-\frac{dT}{dx} \right)_{x}, \tag{3}$$

$$E_{dx,out} = \lambda A \left(-\frac{dT}{dx} \right)_{x+dx}. \tag{4}$$

The energy per unit time coming from the heat production can be written as Q_p multiplied by the volume of the CV:

$$E_{dx,production} = Q_p A dx. \tag{5}$$

At steady-state, $\frac{dE}{dt} = 0$ and the energy balance becomes

$$\lambda A \left(-\frac{dT}{dx} \right)_{x} - \lambda A \left(-\frac{dT}{dx} \right)_{x+dx} + Q_p A dx = 0. \tag{6}$$

Simplifying the above energy balance we get

$$\frac{\left(\frac{dT}{dx} \right)_{x+dx} - \left(\frac{dT}{dx} \right)_{x}}{dx} = -\frac{Q_p}{\lambda}.$$

It is interesting to note here, that we have lost the dependence on the cross-sectional area of the rod. If we now substitute some function $f(x) = \frac{dT}{dx}$, we notice that we have

$$\frac{f(x+dx) - f(x)}{dx} = -\frac{Q_p}{\lambda}$$

or, in other words,

$$\frac{df(x)}{dx} = -\frac{Q_p}{\lambda}. \tag{7}$$

With the above substitution, the differential equation that we are about to solve becomes

$$\frac{d^2 T}{dx^2} = -\frac{Q_p}{\lambda}. \tag{8}$$

Applying the boundary conditions from both ends of the rod ($T(x = 0) = T_0$ and $T(x = L) = T_0$), the solution to the above differential equation is

$$T(x) = -\frac{Q_p}{2\lambda} (x^2 - Lx) + T_0. \tag{9}$$

Below, we draw a graph of the temperature distribution in a copper rod $200m$ long with cross-sectional area of $0.01m^2$. We assume that the thermal conductivity for this rod is $400 \frac{W}{m \cdot K}$. The internal heat production in the entire rod is $20W$ and the steady-state temperature of the endpoints is kept at $T_0 = 0^o C$. The analytic solution is plotted below:
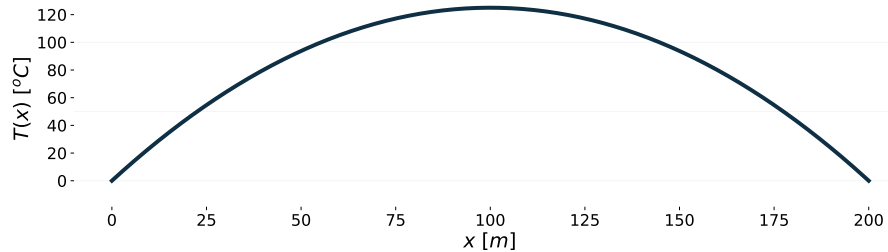

Figure 2: Analytic solution.

Note, that even though the heat flow was assumed to be one-dimensional in this exercise, it does not mean that the geometry of the problem has to be one-dimensional. In fact, we've assumed the rod to be a three-dimensional object having length and a cross-sectional area. Rather, the one-dimensionality of the problem means that it is practical to assume only one of the three directions as an important direction for heat transport. Since the rod was perfectly insulated along its length, the temperature gradients along directions perpendicular to the x-axis are zero.

2 Numerical solution

As a computational example, we are going to numerically solve a second-order linear ODE governing this problem

$$\frac{d^2 T}{dx^2} = -\frac{Q_p}{\lambda}$$

with two Dirichlet boundary conditions: $T(x = 0) = T_0$ and $T(x = L) = T_0$.

2.1 Finite Difference Method (FDM)

Using the second-order central difference approximation we have

$$\frac{T_{i-1} - 2T_i + T_{i+1}}{\Delta x^2} = -\frac{Q_p}{\lambda}, \tag{10}$$

which we can also write as

$$T_{i-1} - 2T_i + T_{i+1} = -\Delta x^2 \frac{Q_p}{\lambda} \tag{11}$$

to have a better view for creating the matrix form of the system of equations. Having selected n points that discretize the domain, we have n algebraic equations to solve. In a matrix form those can be written as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_2 \\ \vdots \\ T_n \end{bmatrix} = \begin{bmatrix} T_0 \\ -\Delta x^2 \frac{Q_p}{\lambda} \\ -\Delta x^2 \frac{Q_p}{\lambda} \\ \vdots \\ T_0 \end{bmatrix} \tag{12}$$

Or in general as: $\mathbf{AT} = \mathbf{R}$. Notice that the matrix $\mathbf{A}$ is a tridiagonal matrix.

The numerical solution using FDM is plotted below on top of the analytic solution using $n = 5$ and then $n = 10$ equations:
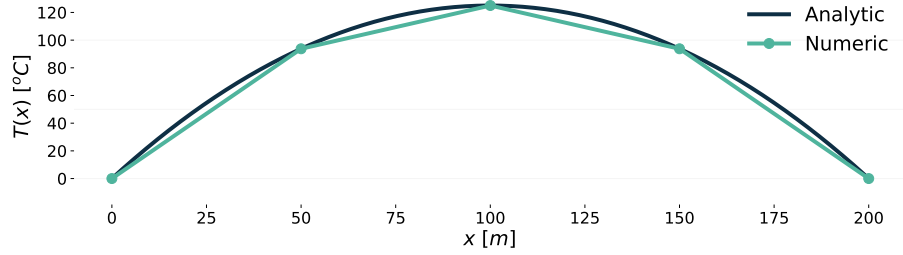

Figure 3: FDM numerical solution using $n = 5$ equations.
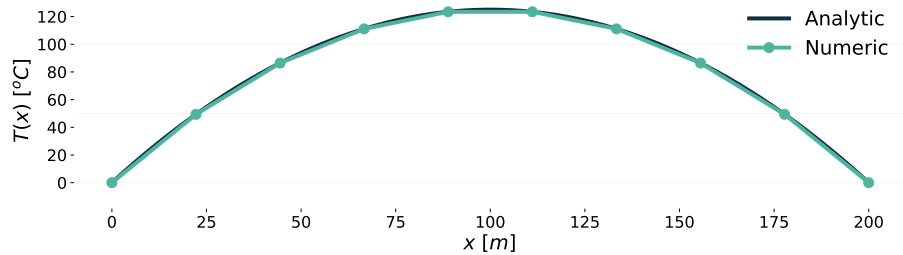

Figure 4: FDM numerical solution using $n = 10$ equations.

3   References

Below is a list of all learning materials that helped me put together this example:

1. Robert Mudde, Peter Hamersma, *The Basics of Transport Phenomena*, Delft University of Technology, available on edX.org as DelftX: TP101x

2. Robert Mudde, Peter Hamersma, *Advanced Transport Phenomena*, Delft University of Technology, available on edX.org as DelftX: TP201x.

3. James C. Sutherland, *Advanced Numerical Methods*, The University of Utah, 2018

4. Joel H. Ferziger, Milovan Perić, Robert L. Street, *Computational Methods for Fluid Dynamics*, 3rd Edition, Springer-Verlag Berlin Heidelberg, 2002.