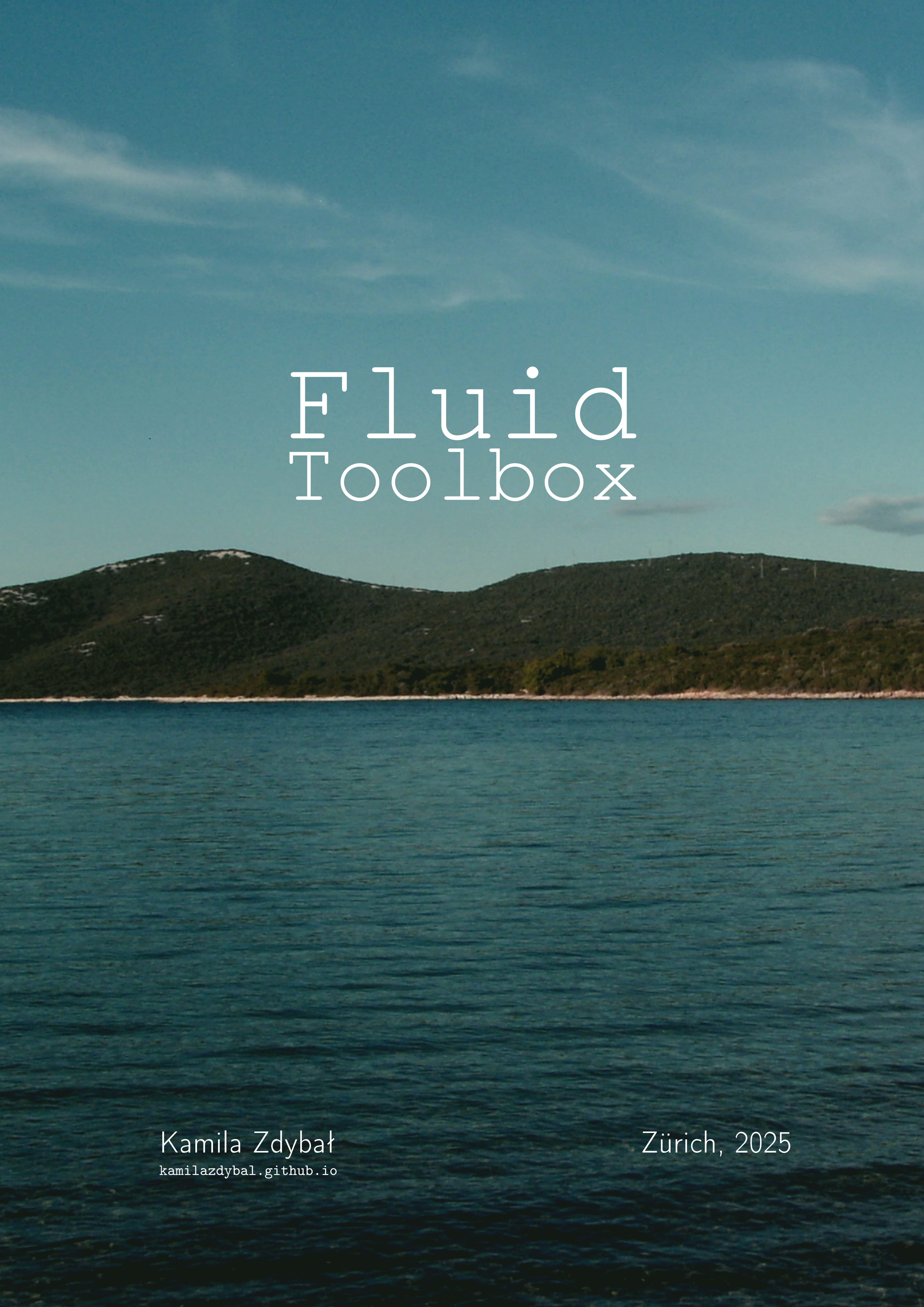

Fluid Toolbox

Kamila Zdybał
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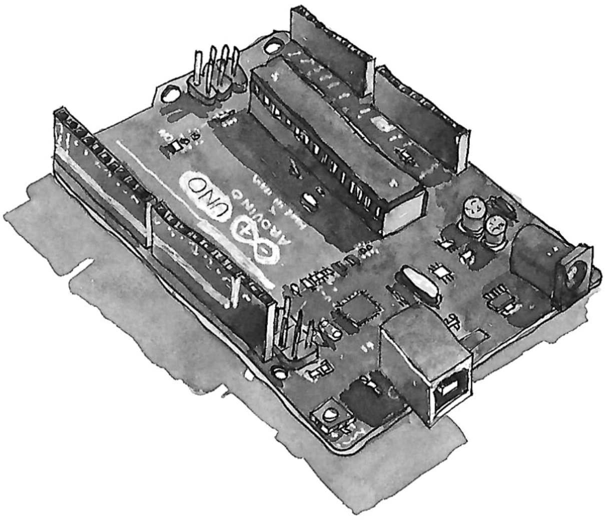
Zürich, 2025



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You can contact me with any suggestions, corrections, or thoughts:
`kamilazdybal at gmail dot com`

`Fluid Toolbox`
`version 1.1`
Typeset with ♥ for LaTeX

PREFACE

"Use it well."
— Prof. Albus Dumbledore

`Fluid Toolbox` is a collection of human-readable, pseudo-random study notes that inspires you to think deeper about various fluid dynamics concepts. It is meant to be used complimentary to the regular textbook since it may provide additional insights, but it will not substitute the thoroughness of a standard course in the subject. I believe that working side-by-side with the course it can become a useful toolbox of concepts that are ready-to-understand and ready-to-use.

Why is this text created?

I had a goal of collecting in one place the most important fluid dynamics concepts, as well as some prerequisites to studying fluid dynamics. Much of the knowledge presented here comes from my search for understanding that I was often missing when reading textbooks, and which was difficult to find in a way that would be engaging, illustrative, and would make intuitive sense to me. Some of the understanding presented here comes from my personal explorations and thinking on the subject, some of it comes from sources that I found very helpful – I will make references to those as we go. I have hopes that this document will become a helpful resource that you have been looking for. I would also like to challenge you to ponder more deeply about the concepts presented here and to seek the beauty in studying fluid motion!

Some of my thoughts on learning

There are a few things that I believe will help you in learning a new subject, such as fluid dynamics:

1. I want to take you through the journey of learning as you read these notes. Above all, I will know that it was worth writing this document if you enjoyed the journey of reading it and learning from it!

2. It's not enough to read a textbook or watch a lecture. The real learning and understanding comes when it's only you, your head, and a blank piece of paper. Reading a textbook is an easy task to do, so is watching a lecture.

When you become a spectator in the learning process, it's easy to fool yourself that you understood something. It is only when you have a chance to take action in your learning that you can really use your knowledge and test your deep understanding. Which brings me to...

3. Programming is a great way to roll your sleeves and put things into action.

4. I believe in the quote by Richard Feynman: *Study hard what interests you the most in the most undisciplined, irreverent and original manner possible.* I will sometimes ask a brain-teasing question or challenge you to *pause and ponder*. In your learning, do challenge yourself, and don't let your imagination be limited by simply following pages of any textbook!

This document will be alive for quite some time. I will be coming back to it to add or improve things. You can always access the newest version through my GitHub site: `kamilazdybal.github.io`.

Please feel free to contact me with any suggestions, corrections, or thoughts: `kamilazdybal at gmail dot com`.

CONTENTS

Acknowledgements

I am grateful to all people I encountered in my life

who supported my passion for studying fluid motion.

Among the first ones were Y. Çengel and J. Cimbala

in their inspiring fluid mechanics textbook.

Zürich, Switzerland, 2025

CHAPTER 1

CHANGES

Ch-ch-ch-ch-changes
Turn and face the strange
Ch-ch-changes
There's gonna have to be a different man
— David Bowie

1.1 Derivatives model change

In studying fluid motion, we are inherently interested in **change** in various quantities associated with the fluid. For example, simply because of fluid moving around, its local density or its local temperature might change. In that sense, the main goal of the science of fluid dynamics is to describe that change. We would like to know how flow affects various fluid properties such as density, ρ, pressure, p, temperature, T, or even mixture composition for multicomponent fluids.

Change is mathematically modeled by derivatives. A derivative explains how much one variable, say ϕ_1, changes when we change some other variable, say ϕ_2, and we express this in mathematical terms as

$$\frac{d\phi_1}{d\phi_2},$$

where the letter d stands for *the change of...* and is later followed by the variable that we are speaking of. So really the above ratio means that there is *this much* change in variable ϕ_1 per *this much* change in variable ϕ_2. You can also think of this ratio as *this much* change in variable ϕ_1 per *unit* change in variable ϕ_2.

In fluid dynamics, you will find that we are most interested in two types of change: *change in time* and *change in space*. Since we live in a 3D space with a time arrow, it is justifiable why these two have the biggest popularity, right? Therefore, you will most often encounter dt or dx, dy, dz in the denominator of various forms of derivatives. Let's take pressure, p, as an example. When we write

$$\frac{dp}{dt},$$

you can read this as: there is this much change in p per this much change in t. Similarly, you may encounter expressions like

$$\frac{dp}{dx}, \frac{dp}{dy}, \frac{dp}{dz},$$

which you can read as: change in p per change in x, or y, or z.

There is also another mathematical expression for a derivative and it is

$$\frac{\partial \phi_1}{\partial \phi_2}\,.$$

The operator ∂ (called ″partial“ or ″del“) also stands for *the change of...* but it also gives you a hint that the variable ϕ_1 can change with the change of variables other than ϕ_2. Perhaps it can also change with some ϕ_3 and ϕ_4, even though in this particular ratio from above we are only interested in the change with respect to ϕ_2.

There are what we call *higher-order derivatives*, which can look like this:

$$\frac{\partial^2 \phi_1}{\partial \phi_2^2}\,.$$

What is their meaning? Well, we can also re-write the above as

$$\frac{\partial}{\partial \phi_2}\frac{\partial \phi_1}{\partial \phi_2}\,,$$

and this way it's easier to see that this must have the interpretation of measuring how much the very change in ϕ_1 is changing! In other words, we are describing how the quantity $\frac{\partial \phi_1}{\partial \phi_2}$ changes with the change to ϕ_2.

1.2 Derivatives model various transport processes

A first-order derivative such as, for example,

$$\frac{\partial p}{\partial x}$$

is a model for the *advection* of p. It describes how much change in p are we going to experience along the spatial x-direction. In other words, how much of p is being *pushed* to the adjacent locations on the x-axis.

A second-order derivative,

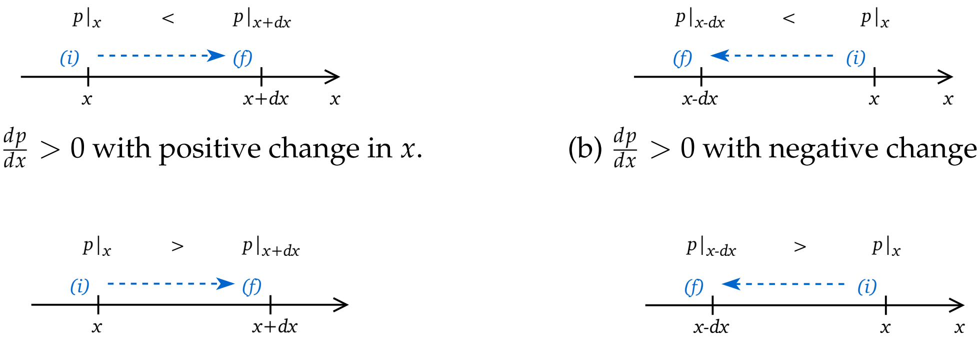
$$\frac{\partial^2 p}{\partial x^2}\,,$$

is a model for the *diffusion* of p. It describes how much change in $\frac{\partial p}{\partial x}$ are we going to experience along the x-direction. While the first-order derivative had the interpretation of how much p is being pushed to the adjacent locations on the x-axis, with the second-order derivative we describe what is the strength of that "pushing" process.

1.3 Convention for the sign of a derivative

For the purpose of this demonstration we will look at the derivative $\frac{dp}{dx}$ – change in pressure per change in the x-axis position – which is often encountered in fluid dynamics. We will lay the ground for what does it mean for this derivative to be positive, negative or zero, and why the reasoning makes sense.

Suppose that the initial point is marked with (i) and it is always a point at coordinate x. The point to which we move after one space-step, the final point, is marked with (f) and is either at $x + dx$ or $x - dx$ coordinate, depending on the positive or negative change that we decide to make. The direction of the change on the x-axis is marked with a blue arrow.



(a) $\frac{dp}{dx} > 0$ with positive change in x.

(b) $\frac{dp}{dx} > 0$ with negative change in x.

(c) $\frac{dp}{dx} < 0$ with positive change in x.

(d) $\frac{dp}{dx} < 0$ with negative change in x.

Figure 1.1: Sign of the $\frac{dp}{dx}$ derivative versus directions of change along the x-axis.

We will now find out for all cases which pressure, at (i) or at (f) must be larger. Our aim is to show that situations (a) and (b) in Figure 1.1 must be equivalent – they explain the same physical phenomena. In both of these cases we will show that the pressure is increasing with the increasing x-coordinate, independent of whether we decide to take a step to the right or to the left of our initial point (i).

We will show the analogical result can be said about situations (c) and (d) but in this case the pressure is decreasing with the increasing x-coordinate.

1.3.1 Closing note

The analysis done in this section is often necessary in order to find out whether or not to "put a minus sign" in front of expressions. For instance, as we will show later in the text, such reasoning can help us understand why there is a minus sign in the Euler equation for a fluid element experiencing pressure force: $dp = -\rho v dv$. Oftentimes, the sign of a derivative tells an important information about the nature of the physical phenomena.

CHAPTER 2

DIFFERENTIATION

CHAPTER 3

MATERIAL DERIVATIVE

3.1 Where space, time, and fluid flow meet

The material derivative describes the *total experienced* change in quantity $\bullet$ as *time goes on* **and** as we *move* across the field of $\bullet$ with fluid velocity, $\vec{V} = \langle u, v, w \rangle$. Hence, the material derivative requires two ingredients; these are visualized in Fig. 3.1. The first ingredient is the field of $\bullet$, which can change spatially and temporally (Fig. 3.1a). The second ingredient is the associated fluid velocity field, $\vec{V}$ (Fig. 3.1b). In this chapter, you can substitute for $\bullet$ any interesting physical quantity that you'd like, such as density, ρ, or temperature, T. Interestingly, this quantity does not need to be a scalar, but can also be a vector or even a tensor.

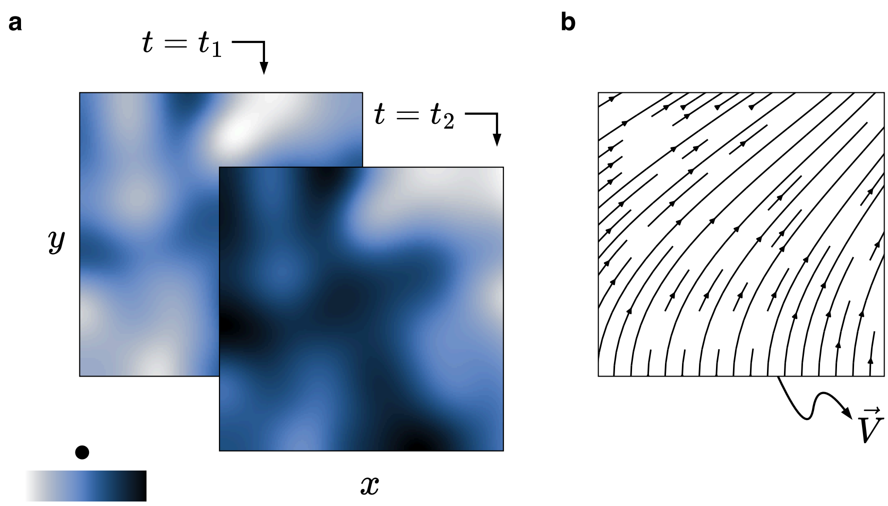


Figure 3.1: Two ingredients needed to compute the material derivative: (**a**) the field of $\bullet$, which can change spatially and temporally, and (**b**) the associated fluid velocity field, $\vec{V}$.

I will start with building a visual intuition for the material derivative. You may consider a 2D field of $\bullet$ that changes in time and space, just like the one presented in Fig. 3.1a. In Fig. 3.2, let's look at the possible reasons for why we might experience change in $\bullet$. In the absence of spatial movement over the (x, y) grid we can only experience change in $\bullet$ if $\bullet$ varies in time. Similarly, in the absence of temporal variation in $\bullet$, we can experience change in $\bullet$ only if we travel along the (x, y) grid and $\bullet$ varies over that grid. With both time and motion present, we experience a superposition of these two effects. That will be our total experienced change in $\bullet$.

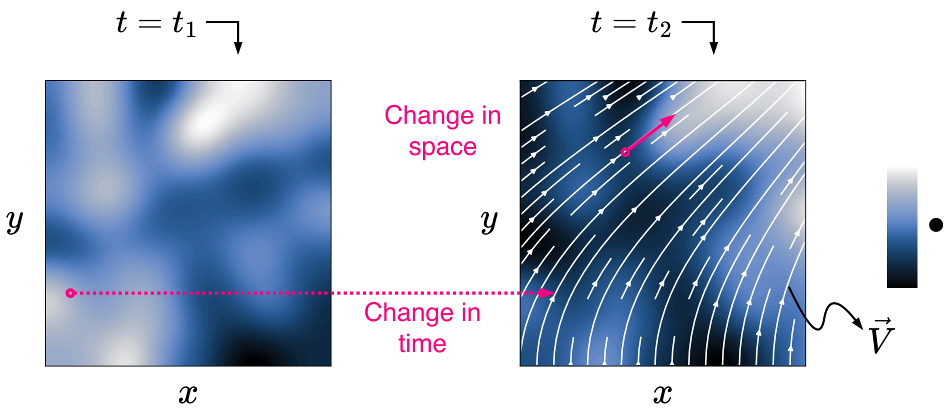


Figure 3.2: A 2D field of some scalar quantity, $\bullet$, that changes in time, t, and space, (x, y). We also consider the associated fluid velocity field, $\vec{V}$. The material derivative is a superposition of two reasons for why $\bullet$ can change.

In mathematical terms, the material derivative, $\frac{D}{Dt}$, is an operator acting on $\bullet$ such that

$$\frac{D\bullet}{Dt} \equiv \frac{\partial \bullet}{\partial t} + \vec{V} \cdot \nabla \bullet \ . \tag{3.1}$$

The superposition that I mentioned before is embedded in the two terms on the right-hand-side of Eq. (3.1). We can now dissect these two terms to better understand why introducing the material derivative is very useful when studying fluid motion.

First, we have $\frac{\partial \bullet}{\partial t}$ which is the plain old[1] partial derivative of $\bullet$ with respect to time. It says that at all possible locations in space, and at any one location, the quantity $\bullet$ can evolve in time. One example of such quantity is temperature. Even if we remain stationary in a specific location, say in a corner of a room, we can still experience change in temperature because our room might be heated (or cooled) and the temperature in our little corner changes in time because of that. The term $\frac{\partial \bullet}{\partial t}$ gives us a recipe for *how* that temperature changes in time in every location of the room.

Second, we have $\vec{V} \cdot \nabla \bullet$, that is, a gradient vector, $\nabla \bullet = \langle \frac{\partial \bullet}{\partial x}, \frac{\partial \bullet}{\partial y}, \frac{\partial \bullet}{\partial z} \rangle$, dotted with the fluid velocity vector, $\vec{V}$. The gradient of $\bullet$ is a vector field that describes directions in which $\bullet$ varies. If, and only if, our own spatial movement is aligned (at least to some extent) with the direction of $\bullet$'s gradient, we will experience a change in quantity $\bullet$. Otherwise, if we walk along an isocurve of $\bullet$, we will not experience any change in $\bullet$. The dot product taken between $\vec{V}$ and $\nabla \bullet$ measures the degree of that alignment.

[1]See Chapter 1.

To summarize, the first term on the right-hand-side of Eq. (3.1) describes how we will experience change in $\bullet$ in the absence of our motion through the field of $\bullet$. The second term describes how we will experience additional change in $\bullet$ due to moving around through the field of $\bullet$ but with a very specific velocity, $\vec{V}$. I will emphasize again that in the definition of the material derivative our movement is restricted to one defined by the fluid flow. Hence, we specifically use the flow velocity, $\vec{V}$, and not any other velocity[2]. The material derivative is a neat superposition of these two factors for why $\bullet$ can change. It is also a shorthand for describing change in $\bullet$ in a moving fluid and it has been created because this superposition of effects frequently appears in the governing equations of fluid dynamics. Writing it as $\frac{D\bullet}{Dt}$ simply makes our life easier.

Finally, I would like to present some more ways of writing Eq. (3.1) just to expose you to other possible notations that you might encounter in textbooks. First, some like to write the definition of the material derivative without specifying the placeholder for the physical quantity, $\bullet$, on which it acts:

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{V} \cdot \nabla \,. \tag{3.2}$$

The unspoken assumption here is that the operator $\frac{D}{Dt}$ always acts on *something*, so you can apply this definition to any *something* you like. In this chapter, I chose to explicitly indicate that *something* with the "$\bullet$" symbol. In the most general 3D case, where $\vec{V} = \langle u, v, w \rangle$, we can expand the dot product terms to obtain the following notation:

$$\frac{D\bullet}{Dt} \equiv \frac{\partial\bullet}{\partial t} + u\frac{\partial\bullet}{\partial x} + v\frac{\partial\bullet}{\partial y} + w\frac{\partial\bullet}{\partial z} \,. \tag{3.3}$$

A yet another way of writing the equation above that you may encounter is the following:

$$\frac{D\bullet}{Dt} \equiv \frac{\partial\bullet}{\partial t} + V_i\frac{\partial\bullet}{\partial i} \,. \tag{3.4}$$

This way of writing Eq. (3.3) is using the Einstein notation where it is implied that you should substitute for the dummy index i every possible spatial dimension, *i.e.*, x, y, and z, and, as you substitute, you also sum up all the terms that form for each possible i.

[2]That said, one could, potentially, define a generalization of the material derivative to allow for an arbitrary velocity! Such a new quantity will have a different physical meaning though.

Hungry for more?

You can find a great intuitive description of a material derivative in Chapter 3, §3.5 of the *Transport Phenomena* textbook by Bird, Stewart & Lightfoot [Bird et al., 2002]. They delineate differences between various derivatives on the example of following fish in a river.

3.2 Pause and ponder

Let's look at some alternative ways to describe change in both space and time and see why they wouldn't be equally useful as Eq. (3.1)! Suppose I present you with the following quantity:

$$\frac{\partial \bullet}{\partial t} + \frac{\partial \bullet}{\partial x} + \frac{\partial \bullet}{\partial y} + \frac{\partial \bullet}{\partial z}\,. \tag{3.5}$$

How is that quantity different from the definition of the material derivative? In other words, what does the dot product with the velocity vector change in how we described change in space in Eq. (3.3)?

The velocity vector is not our independent motion through the field of $\bullet$. It is our motion when carried by the fluid flow. In essence, the material derivative describes our experience change in $\bullet$ because of our motion with the fluid velocity, even though the change in $\bullet$ might happen precisely *due to* fluid motion, or at least be some function of it. Think about the fluid density, ρ, which can change due to local movement of fluid from one location to the next.

Chapter 4

Divergence theorems

CHAPTER 5

COMMON FLOW TYPES

CHAPTER 6

DRAG FORCE

CHAPTER 7

CIRCULATION

Circulation is defined as:

$$\Gamma = \oint_C \vec{v} \cdot \vec{dl} \tag{7.1}$$

The dot product, $\vec{v} \cdot \vec{dl}$, returns a scalar which is expressing "how much" in the direction of the other vector is this vector.

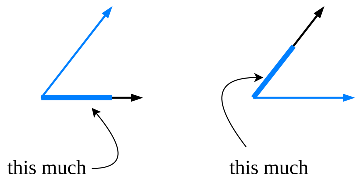


Figure 7.1: Dot product of two vectors.

When they are $\perp$, the dot product is zero.

In the concept of circulation, we ask "how much" at any point on the curve C the velocity vector at that point is in the direction of the curve's geometry. Now, that doesn't yet sound as something to do with "circulating". For the moment, I would think that it's more of an "on-trackness". Something, that in real world would be for instance the measure of how much the vehicle's velocity is in the direction of the road geometry (which we would hope is all of it!).

But to start , notice an important detail of "$\circ$" on the integral symbol, which signals that the curve should be a closed curve – a loop – although not necessarily a perfect circle.

When you perform integration, which means summing up every little $\vec{v} \cdot \vec{dl}$ as you go around the loop, you count "how much" at every point on the loop, the velocity vector at these points is in the direction of the loop's geometry (at these points).

If we were to place a small particle at some starting point P on the loop, the circulation would tell us "how much" the velocity field which this particle is subjected to, is tending to move that particle around the loop.

It can be very intuitive when you take a look at these two pictures:

It's no surprise that when the velocity is everywhere perpendicular to the loop's geometry, the circulation around the loop is zero. If you were to place a particle at any point on the loop, such velocity field would act to immediately displace the particle off the loop. Therefore, the particle would have no way of "circulating" around the loop.

On the other extreme is the case when the velocity field is everywhere tangent to the loop's geometry. Anywhere the particle goes on the loop, the velocity at that point would act to keep the particle moving around the loop.

Questions:

1. Why closed loop? Would it have any meaning if we calculated circulation along any general spline?
2. How to chose loops so that the circulation we calculate is of the most meaning to us?
3. What does the zero, positive, negative circulation mean?
4. Can circulation be infinite?
5. For what velocity field, $\vec{v}$, and the corresponding loop, C, is circulation zero?

CHAPTER 8

VORTICITY

CHAPTER 9

STOKE'S THEOREM

CHAPTER 10

NONDIMENSIONALIZING

Chapter 11

Conservation of mass

CHAPTER 12

GAUSS'S LAW

CHAPTER 13

REYNOLDS NUMBER

Chapter 14

Constitutive equations

CHAPTER 15

THE NAVIER-STOKES EQUATIONS

APPENDIX

Ending remarks

The cover photo:
View from the coast of Cres island, Croatia, October 2016.
Photo by: J. Aleksanderek ©

Bibliography

[Bird et al., 2002] Bird, R. B., Stewart, W., and Lightfoot, E. (2002). Transport phenomena.