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VARIABLE CONTROL PARAMETERIZATION FOR TIME-OPTIMAL PROBLEMS[1]

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Abstract: A gradient matching method for computing bang-bang time-optimal controls is described. A horizon-depending auxiliary functional, parameterized by the horizon weighting factor, is minimized for an appropriately chosen sequence of the parameter values. Every iteration of quasi-Newton search for optimal switching times and horizon is preceded by a possible generation of new switchings and followed by a reduction of coinciding ones. The generations result from an analysis of the gradient of the auxiliary functional w.r.t. control. The algorithm normally converges to a local minimum in control space, and the auxiliary functional decreases monotonously during each stage of optimization with constant weighting parameter. An example of time-optimal control of a pendulum-cart system is solved.

Keywords: Time-optimal control, Bang-bang control, Dynamic programming, Gradient methods.

1. INTRODUCTION

The persistent interest in time-optimal control is well motivated by applications in space, aviation and other engineering domains. The efficiency of many industrial processes, especially the robotized ones, depends strongly on the execution time of repetitive operations. The total time of task accomplishment is often critical in competitive or game-like situations. There are well known difficulties resulting from great sensitivity of time-optimal solutions in real control systems and, on the other hand, from the theoretical and computational complexity. In non-convex problems, occurrence of a large number of local optima may be a serious obstacle. In recent years, novel computational algorithms based on well-established results of optimal control theory together with the growing computational power made it possible to overcome the difficulties and solve many time-optimal control problems in real-time (Pesch, 1994; Turnau, *et al*., 1999).

Currently prevailing approaches to numerical solution of time-optimal problems are divided into *direct* and *indirect* (Stryk and Bulirsch, 1992). In the direct approach the original infinite-dimensional optimization problem is approximated by a finite-dimensional one solved by typical nonlinear programming methods, like SQP (Kraft, 1994; Stryk, 1993; Barclay, *et al.*, 1998). Direct methods feature larger areas of convergence, however their overall convergence rates are rather slow. In the indirect approach the adjoint equations are combined with the original state equations to form a *multi-point boundary value problem*. This may be efficiently solved using shooting methods (Bulirsch, 1971; Lastman, 1978). Especially good convergence properties are attributed to multiple-shooting algorithms (Buchauer, *et al*., 1994), but still the necessity to define the proper control structure and initialize the adjoint variables in a sufficient vicinity of the optimal values remains a serious limitation. Therefore in practice both types of methods are used in conjunction: the control structure and/or adjoint variables estimates are obtained by the direct approach and then this solution is refined with an indirect method (Shen and Tsiotras, 1999). Another remedy

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often used to enlarge the area of convergence of indirect methods is to parameterize the problem and employ homotopy (continuation) strategies. Some approaches rely on collocation type formulation which is well suited for boundary value problems (Hargraves and Paris, 1987; Stryk, 1993).

This paper describes a method of solving time-optimal problems which follows the results in (Sirisena, 1974; Korytowski, *et al*., 1998, Szymkat, *et al*., 1999). A horizon-depending auxiliary functional, parameterized by the horizon weighting factor, is minimized for an appropriately chosen sequence of the parameter values. The algorithm exploits information derived from the shape of the gradient of the auxiliary criterion in control space to get indications on how to construct a series of control reparameterizations leading to the fulfillment of the necessary conditions of optimality (*gradient matching*). The general idea is to keep the dimension of the decision space small throughout the optimization, contrary to typical direct methods - but still preserve the large area of convergence. In the proposed approach every iteration of quasi-Newton search for optimal switching times and horizon is preceded by a possible generation of new switchings and followed by a reduction of those on active constraints, including coinciding ones. An important feature is that the optimization is monotonous during every stage with constant value of the horizon weighting parameter in the auxiliary criterion.

In previous versions of the method the search for optimal horizon was based on a continuation strategy combined with optimal horizon estimates. That approach forced relatively small horizon increments because of the requirement of continuity and possibly conservative horizon estimates. In the present version, instead of the hierarchical strategy the horizon becomes a decision variable in the gradient optimization part of the algorithm.

2. PROBLEM FORMULATION

Consider the time-optimal control problem in the system

$$\dot{x} = f(x,u) = f^0(x) + f^1(x)u\,,\ t \in [0,\infty[$$
$$x(0) = x^0 \tag{1}$$

where $x(t) \in \boldsymbol{R}^n$, f^0 and f^1 are continuously differentiable. The target state is denoted by x^f. The set of admissible controls $\boldsymbol{U}$ consists of all right-continuous functions $u : [0,\infty[\to [-u_{\max}, u_{\max}]$. Define an auxiliary criterion depending on the horizon $T \geq 0$

$$S_q(u,T) = \tfrac{1}{2}(x(T) - x^f)^{\mathrm{T}} Q (x(T) - x^f) + qT \tag{2}$$

where q is a nonnegative parameter and $Q = Q^{\mathrm{T}} > 0$. An admissible control u is time-optimal iff $S_0(u,T) = 0$ with minimum T. The adjoint function ψ for problem (1), (2) is determined by

$$\dot{\psi} = -\nabla_x f(x,u)\psi\,,\quad \psi(T) = Q(x^f - x(T))\,. \tag{3}$$

The antigradient of (2) w.r.t. control is given by

$$g(t) = -\nabla_u S_q(u,T)\big|_t = \psi(t)^{\mathrm{T}} f^1(x(t))\,. \tag{4}$$

Define the projection of g onto $\boldsymbol{U}$ at the point u

$$g^U(t) = \begin{cases} 0, & u(t)\,\mathrm{sgn}\, g(t) = u_{\max} \\ g(t), & \text{otherwise.} \end{cases} \tag{5}$$

According to the Maximum Principle, g^U is identically zero on an optimal control.

We confine our considerations to optimal control problems with bang-bang solutions. Singular and Fuller-type solutions can be treated as limit cases. A *bang-bang* control u is fully characterized by a real number $u_0 \in \{-u_{\max}, u_{\max}\}$ and a finite non-decreasing sequence $\tau = (\tau_i)_{i=1}^{i=m} \subset [0,\infty[$

$$u(t) = (-1)^i u_0\,,\quad t \in [\tau_i, \tau_{i+1}[\,,\quad i = 0, 1, ..., m-1$$
$$u(t) = (-1)^m u_0\,,\quad t \in [\tau_m, \infty[\tag{6}$$

where $\tau_0 = 0$. For an empty τ, $m = 0$. The control (6) is also denoted by $u(t;\tau,u_0)$. The restriction of criterion (2) to bang-bang controls defines the functional

$$S'_q(\tau,u_0,T) = S_q(u(\cdot\,;\tau,u_0),T) \tag{7}$$

where $T \in [0,\infty[$, $u_0 \in \{-u_{\max}, u_{\max}\}$, and $\tau = (\tau_i)_{i=1}^{i=m}$ is a finite non-decreasing sequence in $[0,T]$. Let $\tau_{m+1} = T$. The derivative of S'_q w.r.t. τ_i, $i = 1, ..., m$, is as follows (Sirisena, 1978)

$$\nabla_{\tau_i} S'_q(\tau,u_0,T) = 2\eta\, g(\tau_i)\, u(\tau_i;\tau,u_0) \tag{8}$$

where $\eta = +1$ for $\tau_{i-1} \leq \tau_i < \tau_{i+1}$, and $\eta = -1$ for $\tau_{i-1} < \tau_i = \tau_{i+1}$. This derivative is not defined if $\tau_{i-1} = \tau_i = \tau_{i+1}$. The derivative of S'_q w.r.t. horizon is equal to

$$\nabla_T S'_q(\tau,u_0,T) = -\psi(T)^{\mathrm{T}} f(x(T),u(T)) + q$$
$$= h - G(T) \tag{9}$$

where $u(t) = u(t;\tau,u_0)$ and

$$h = q - \psi(T)^{\mathrm{T}} f^0(x(T))\,,\ G(t) = u(t)g(t)\,. \tag{10}$$

3. GENERATIONS AND REDUCTIONS

For non-decreasing sequences $\tau,\gamma \subset \boldsymbol{R}$, $\tau \sim \gamma$ denotes a non-decreasing sequence in $\boldsymbol{R}$ such that the number of its elements equal to any real is the sum of the numbers of such elements in τ and γ. For any $T \in [0,\infty[$ and $u_0 \in \{-u_{\max}, u_{\max}\}$, let τ be a finite, strictly increasing sequence in $]0,T[$. The *switching generation* consists in choosing a finite, non-decreasing sequence γ in $[0,T]$ in such a way that

$$u(t;\tau,u_0) = u(t;\tau \sim \gamma, (-1)^k u_0)\quad \forall\, t \in [0,T[\tag{11}$$

where k is the number of zero elements in γ. The control $u(\cdot\,;\tau \sim \gamma, (-1)^k u_0)$ is the starting point for the next iteration of the optimization procedure, which means that the horizon T and the elements of $\tau \sim \gamma$ become new decision variables.

We shall only use generations constructed in the following way. Denote $\tau=(\tau_i)_{i=1}^{i=m}$, $\tau_0=0$, $\tau_{m+1}=T$, $\tau'=\tau\sim(T)=(\tau'_i)_{i=1}^{i=m+1}$. For simplicity we use abbreviations $\nabla_{.}=\nabla_{.}S'_q(\tau,u_0,T)$, $\nabla'_{.}=\nabla_{.}S'_q(\tau',u_0,T)$.

We say that $a\in[0,T[$ is *efficient* if *a* is different from all elements of τ, it is the smallest point in $[\tau_i,\tau_{i+1}[$ at which *G* attains a minimum in this interval, for some $i=0,1,...,m$, and

$$G(a)<0\,,\qquad G(a)^2>M\left\|\nabla_\tau\right\|^2 \tag{12}$$

where $M\geq 0$ is a predetermined constant.

The sequence γ defining a generation consists only of terms which are efficient or equal to *T*. Assume that $a\leq\tau_m$ is efficient. If $a>0$, γ has exactly two elements equal to *a*. Otherwise, γ has exactly one zero element. Assume that

$$\nabla'_T<\nabla'_{\tau'_{m+1}} \tag{13}$$

and there are no efficient points in $]\tau_m,T[$. If

$$(\nabla'_T)^2+(\nabla'_{\tau'_{m+1}})^2-(\nabla_T)^2>N\,, \tag{14}$$

with $N=M\left\|\nabla_\tau\right\|^2$, then γ has exactly one element equal to *T*. Assume in turn that (13) holds and some $a\in]\tau_m,T[$ is efficient. If (14) holds with $N=G(a)^2$, then γ has exactly one element equal to *T*. Otherwise, γ has exactly two elements equal to *a*. If (13) does not hold and some $a\in]\tau_m,T[$ is efficient, γ has exactly two elements equal to *a*.

Inequalities (13) and (14) are called *horizon efficiency conditions*. Note that from (8) and (9), $\nabla_T=h-G(T)$, $\nabla'_T=h+G(T)$, and $\nabla'_{\tau'_{m+1}}=-2G(T)$. The parameter *M* is used to regulate the frequency of generations.

Reduction of switchings is carried out after every iteration of optimization. For any $T\in[0,\infty[$ and $u_0\in\{-u_{\max},u_{\max}\}$, let τ be a finite, non-decreasing sequence in $[0,T]$ (resulting from optimization). The switching reduction consists in transforming τ into a finite, strictly increasing sequence σ in $]0,T[$ in such a way that

$$u(t;\sigma,(-1)^k u_0)=u(t;\tau,u_0)\quad\forall t\in[0,T[\tag{15}$$

where *k* is the number of zero elements in τ. The control $u(\cdot\,;\sigma,(-1)^k u_0)$ is then subject to subsequent generation tests.

The idea of generation is explained in Figs. 1 and 2. Fig. 1 shows the control (bold solid line) and antigradient (dashed line) at the moment of generation. Note that $G(t)<0$ for $t\in[0,\tau_1[$ and in a subinterval of $[\tau_1,\tau_2[$. The arrows at efficient points indicate the new switching times, generated by $\gamma=(\gamma_1,\gamma_2,\gamma_3)$. The subsequent optimization leads to the situation depicted in Fig. 2, with the new switching times. The switchings τ_1,τ_3,τ_4 which have originated from γ are denoted by arrows. It can be seen that the situation in Fig. 1 corresponds to a stationary point in the space of parameters, but not in the control space. This is the case of *unmatched* control and gradient. On the other hand, Fig. 2 presents the case of complete *matching*. The gradient of S'_q w.r.t. switching times is zero and at the same time the projection of the antigradient of S_q w.r.t. control onto the admissible set vanishes. The necessary condition for an optimum in the control space is thus satisfied.

It should be stressed that in the above algorithm of generation the situation in Fig. 1 is not likely to occur, as the efficiency conditions would normally cause a generation much earlier than at a stationary point in the parameter space. It should be also realized that the transition to the final, fully matched situation, as in Fig. 2, usually requires more generations interleaved with reductions.

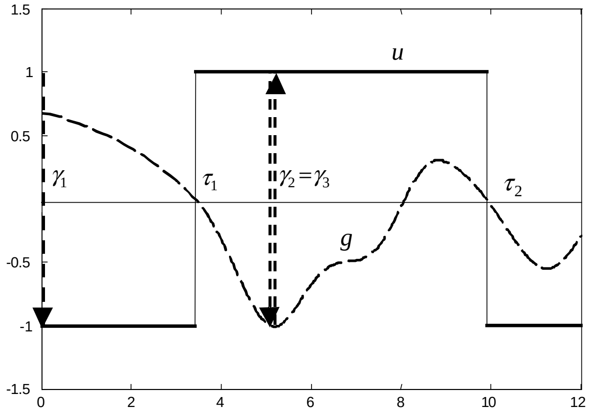


Fig. 1. Example of generation

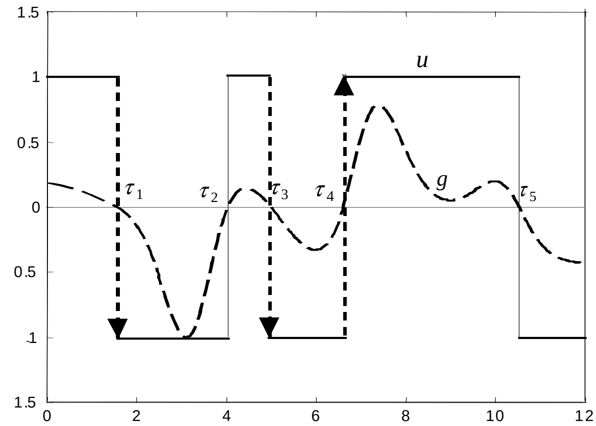


Fig. 2. Completely matched case

4. RULES FOR CHANGING q

Let us first look at an example illustrating the dependence of optimal values of S_q on the horizon *T*, shown in Fig. 3 (note the use of logarithmic scale on the vertical axis). The optima are computed with respect to control for fixed values of *T*. The solid lines on the right represent the optimal values of S_q, going downwards for decreasing values of *q*. The circles denote positions of minima. The plot of S_0*, representing the optimal values of S_0, forms an envelope of those curves. It is clearly seen that the arguments of minima tend to the value of the optimal horizon.

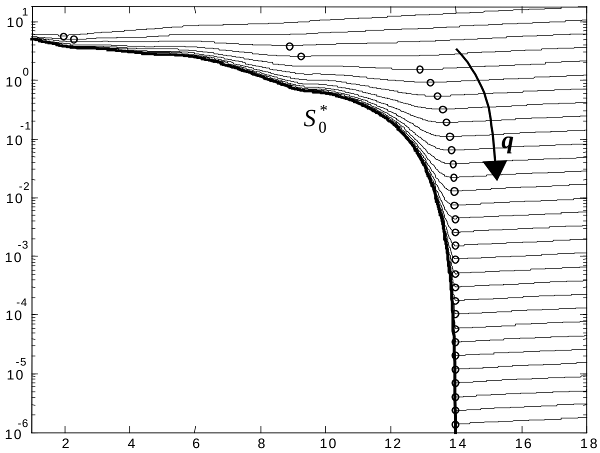


Fig. 3. Optimal values of S_0 and S_q as functions of T

The proposed rules for changing q are justified by the following simple observation. Let

$$\Sigma_q{}^* = \min_{\substack{u\in \boldsymbol{U}\\ T\geq 0}} S_q(u,T), \quad \min_{u\in \boldsymbol{U}} S_q(u,T_q{}^*) = \Sigma_q{}^* . \quad (16)$$

If $q \to 0^+$, then $T_q{}^*$ tends to the time-optimal horizon T^* from the left.

In the algorithm described in Section 5, the asymptotic optimality (with respect to horizon) is assured by stepwise decrease of q. The parameter q is initialized with a moderate value (e.g. 0.05), and kept fixed until the following q *update* conditions are met,

$$\left\|g^U\right\| < \varepsilon_g \text{ and } \left|\nabla_T S_q'\right| < \varepsilon_T \quad (17)$$

where ε_g and ε_T are given thresholds. When this happens, new values: $r_q q$, $r_g \varepsilon_g$ and $r_T \varepsilon_T$ replace q, ε_g, and ε_T, respectively, and are in use until conditions (17) are met again. The positive ratios r_q, r_g and r_T are less than 1. The whole procedure is repeated till the end of the algorithm.

The termination conditions have the form

$$S_q < \varepsilon_S^f \text{ or } \left|\nabla_T S_q'\right| < \varepsilon_T^f \text{ and } \left\|g^U\right\| < \varepsilon_g^f \quad (18)$$

where ε_S^f, ε_T^f and ε_g^f are required zero tolerances. In some cases conditions (18) become fulfilled while the horizon is too large. This may happen when the system is driven sufficiently close to the target state and q is small. In order to counteract this phenomenon, the parameter q is appropriately increased. Define $x(\cdot;\tau,u_0)$ as the solution of (1) corresponding to control $u(\cdot;\tau,u_0)$, $D_\tau = \nabla_\tau x(T;\tau,u_0)^{\mathrm{T}}$ and $D_T = \dot{x}(T;\tau,u_0)$. D_τ can be calculated from the solutions of adjoint equations with appropriate final conditions. It is assumed that $\operatorname{rank} D_\tau = n$. Let δ be the minimal norm solution of the equation

$$D_\tau \delta + D_T = 0 . \quad (19)$$

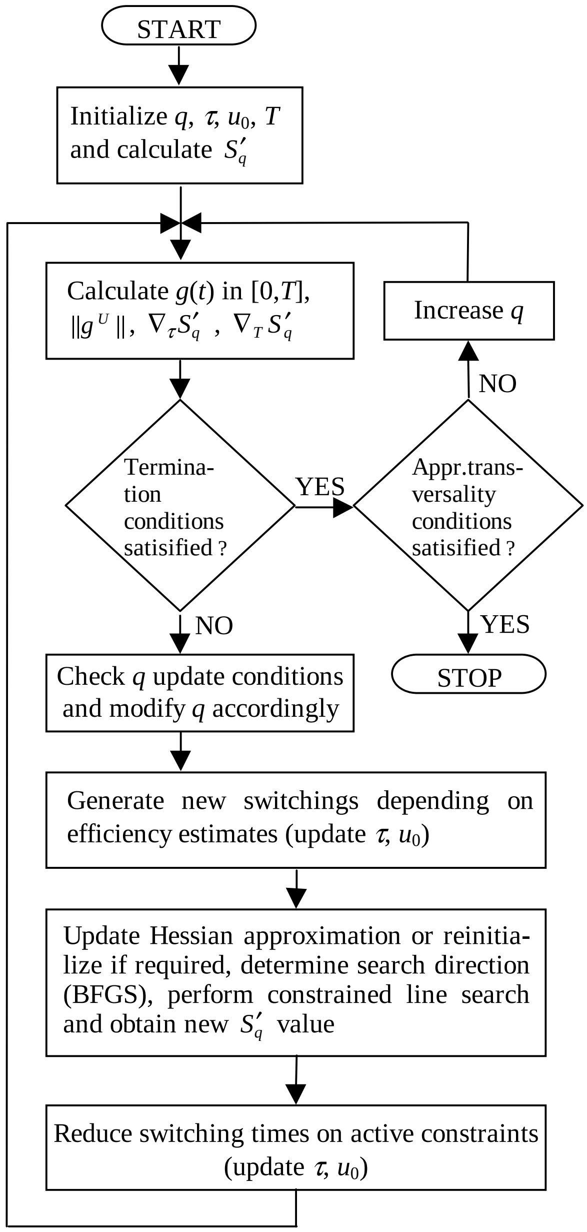
The vector $-\delta$ represents a decomposition of D_T along the columns of D_τ. The linear space spanned by the columns of $[D_\tau\ D_T]$ is tangent to the manifold of constant value of the mapping $(\tau,T) \mapsto x(T;\tau,u_0)$. When D_T approaches a position transversal to $\operatorname{Span}(D_\tau)$, the norm of δ tends to infinity. In practice it is enough to demand that

$$\|\delta\| > \kappa \quad (20)$$

to end the optimization, where $\kappa \gg 0$ is a given constant. Inequality (20) is called the *approximate transversality* condition. If it is not satisfied then q is set to a certain value q_0, sufficiently large to prevent the algorithm from a premature termination.

5. DESCRIPTION OF THE ALGORITHM

The scheme of the numerical method is shown in the following diagram.



Successive approximations of the solution are obtained using a quasi-Newton procedure in parameter spaces. In each iteration before the line search, g, $\nabla_\tau S_q'$ and $\nabla_T S_q'$ are evaluated using solutions of the discretized state and adjoint equations with appropriate boundary conditions. Initial control approximations usually do not satisfy necessary optimality conditions. The fulfillment of these conditions is gradually achieved by the use of generations alternating with directional minimizations and reductions.

The strategy of changing q consists in subsequent updates whenever conditions (17) are satisfied (q is then decreased) and possibly, when the termination test (18) is passed (in that case, q is increased). For constant values of q, criterion (2) decreases monotonously. With the properly selected parameters the overall procedure normally converges at least to a local minimum of the auxiliary criterion.

The right choice of the parameter M in the efficiency conditions and the thresholds ε_g and ε_T in the q update conditions, significantly improves performance of the method. For a too large value of M many iterations are spent on approaching a stationary point in the parameter space, although the shape of the gradient w.r.t. control may clearly indicate the need for additional switchings. If M is too small, frequent generations produce non-persistent switchings thus deteriorating numerical effectiveness. If the thresholds ε_g and ε_T are too easy to achieve the generation mechanism may overlook the optimal structure and yield a solution satisfying termination conditions with a too large horizon. When the thresholds are hard to achieve, the algorithm performs some iterations in vain, keeping q unnecessarily fixed and thus holding the progress of horizon optimization.

6. NUMERICAL EXPERIMENTS

The proposed procedure has been implemented in MATLAB and C. In the experiments a 4-th order dynamical system was used (Szymkat, *et al.*, 1999)

$$\dot{x}_1 = x_3, \qquad \dot{x}_2 = x_4$$

$$\dot{x}_3 = \frac{V_1(x,u) + V_2(x)\cos x_2}{D(x)}$$

$$\dot{x}_4 = \frac{V_1(x,u)\cos x_2 + aV_2(x)}{D(x)}$$

$$|u(t)| \le u_{\max}$$

where $V_1(x,u) = u - x_4^2 \sin x_2 - bx_3$, $V_2(x) = \sin x_2 - cx_4$ and $D(x) = a - \cos^2 x_2$. These equations model a scaled laboratory pendulum-cart system. The coefficients are defined as: $a = 11.213554$, $b = 0.425608$, $c = 3.15649 \cdot 10^{-4}$ and the control bound $u_{\max} = 3.935172$. The problem is to determine a control driving the system from the initial state $x^0 = \mathrm{col}(-0.742844, \pi, 0, 0)$ to the final one $x^f = \mathrm{col}(0.445706, 0, 0, 0)$ in a minimum time. The pendulum should be swung up from the down, stable position to the upright unstable equilibrium with a simultaneous shift of the cart. An interesting application of linear programming to time-optimal control of a single pendulum was presented in (Furuta, *et al.*, 1999).

The values of $S_0{*}(T)$ (right-hand y-scale) together with the corresponding values of τ_i (left-hand y-scale) are shown in Fig. 4. The optimal control has seven switching times, $T_0{*} \cong 13.966$ and $u_0{*} = -u_{\max}$.

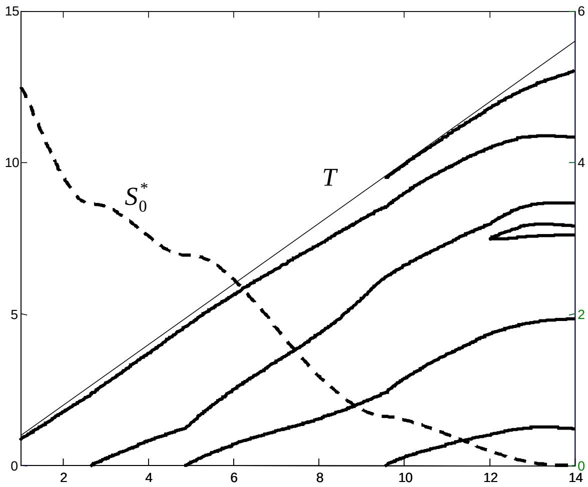


Fig. 4. Values of $S_0{*}$ and optimal switching times

In the first test run the algorithm was started with q = 0.05, T = 11.0, $u_0 = u_{\max}$ and no switchings. Fig. 5 illustrates the progress of computations. The values of $\log_{10} S_0$ (solid line with dots, left-hand y-scale), $\log_{10} q$ (dashed line, left-hand y-scale) and T (solid line, right-hand y-scale) right after subsequent major iterations are shown. The squares represent initial numbers of switching times (left-hand y-scale) for a given iteration.

In early iterations the number of switchings grows gradually from 0 to 7 due to the subsequent generations, but the horizon after initial fluctuations stabilizes at some distance from the optimal value (the S'_q minima with respect to T are shifted from S'_0 minima unless q is close to zero). In the next phase q is decreased, first to $2.5 \cdot 10^{-3}$ (starting from iteration 41), and later to $1.25 \cdot 10^{-4}$ (from iteration 55). This results in T moving rapidly towards $T_0{*}$. After 65 iterations, S'_0 reaches the final value below 10^{-6}. The whole optimization required 89 evaluations of the functional and 65 evaluations of its gradient in total.

In the second experiment the initial values were as follows: $q = 5 \cdot 10^{-5}$, T = 15.5. At the beginning no switchings were assumed and $u_0 = u_{\max}$ was taken. The values of q and T were deliberately chosen to be likely to lead to the situation where the criterion is close to zero and the horizon too large. This happened indeed after 28 iterations. The results of the first phase of this test run are shown in Fig. 6. The checking of the approximate transversality condition made it possible to detect the need for the reinitialization of q. After increasing q to 0.05 the horizon dropped quickly to values below $T_0{*}$. The remaining part of optimization proceeded similarly to the previous case (not shown in Figure).

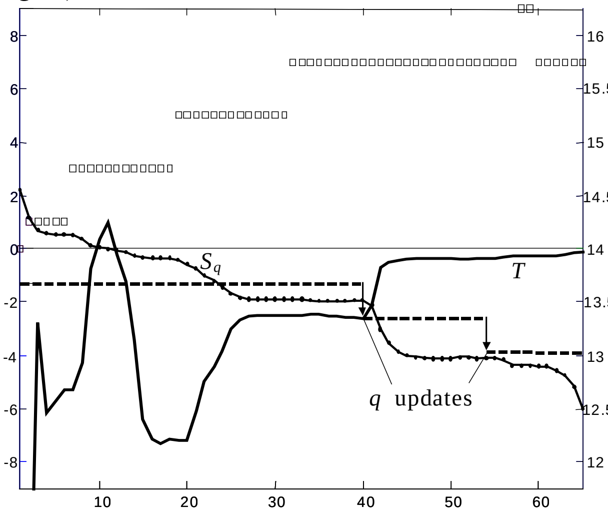


Fig. 5. First test run

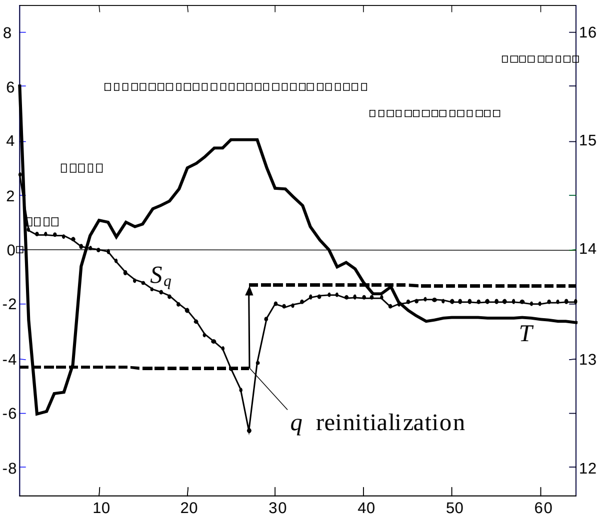


Fig. 6. Initial part of the second run

7. CONCLUSIONS

The algorithm presented in this paper can be started with an arbitrary control structure and successfully converge. Contrary to most direct methods it is possible to keep the dimension of decision space small throughout the optimization, while preserving the large area of convergence (at least for purely bang-bang cases). The rates of convergence are relatively good, compared with methods which explicitly calculate second derivatives. Another advantage is that there is no need to use horizon estimates. On the other hand, the departure from the strict continuation-type procedure in some rare cases may result in stopping at a local minimum (with respect to the horizon) of the auxiliary criterion with a non-zero distance from the target state. The algorithm should be then restarted using a different initial approximation, or the horizon should be increased by forced continuation.

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