TIME OPTIMAL CONTROL OF 3D CRANE*

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Abstract. A mathematical model of the 3D crane with three control inputs is given. The time-optimal control problem is solved with the use of the variable control parameterization method. The implementation in a real laboratory model is presented.

Keywords. Crane, time-optimal control, nonlinear system.

1. INTRODUCTION

The 3D crane model is a complex nonlinear dynamical system. It consists of a cart, and a payload hanging on a lift-line. Contrary to typical setting [1,3,6,7], movements in three dimensions are considered [4,5]. The cart moves in a horizontal plane: it can be shifted along a rail, which also can be shifted in a perpendicular direction. The length of the lift-line is independently controlled. This leads to a tenth-order system of state equations with three controls. The time-optimal problem is approached by means of the Maximum Principle. The optimal control is determined by the recently developed *method of variable control parameterization* [9,2], generalized to the case of many inputs. Results of practical experiments are presented.

2. 3D CRANE MODEL

The schematic diagram of the crane is given in Fig. 1. There are five measured quantities: x_w denotes the distance of the cart from the center of the rail; y_w denotes the distance of the rail with the cart from the center of the construction frame; R denotes the length of the lift-line; α denotes the angle between the x axis and the lift-line; β denotes the angle between the negative direction on the z axis and the projection of the lift-line onto the yz plane.

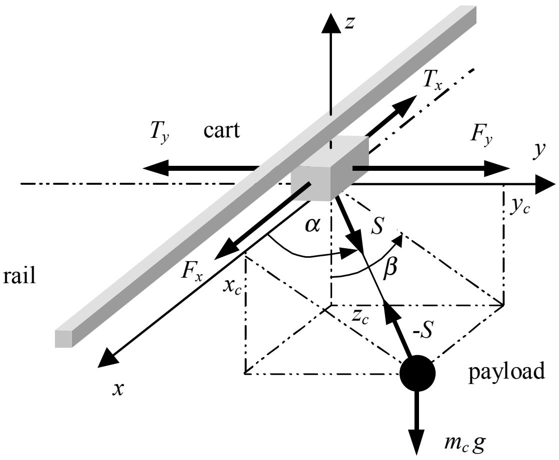

Fig. 1. Forces acting in the 3D crane system

Denote also:

m_c	-	mass of the payload
m_w	-	mass of the cart
m_s	-	mass of the moving rail
x_c, y_c, z_c	-	coordinates of the payload
S	-	reaction force in the lift-line acting on the cart
F_x	-	force driving the cart
F_y	-	force driving the rail with cart
F_R	-	force controlling the length of the lift-line
T_x, T_y, T_R	-	friction forces.

An important element in the construction of mathematical model is the appropriate choice of the

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system of coordinates. The Cartesian system, although simple in interpretation and determining the position in space in a unique way in both directions, is not convenient for the description of the dynamics of rotational motion. The spherical system has therefore been adopted. The position of the payload is described by the angles α and β, shown in Fig. 1. A drawback of the spherical system of coordinates is that for every point on the x-axis, the corresponding value of β is not uniquely determined. However, the points on the x-axis are not attainable in real crane systems.

The following symbols are used in the sequel

$$\mu_1 = \frac{m_c}{m_w}, \qquad \mu_2 = \frac{m_c}{m_w + m_s}$$
$$u_1 = \frac{F_x}{m_w}, \qquad u_2 = \frac{F_y}{m_w + m_s}, \qquad u_3 = \frac{F_R}{m_c}$$
$$T_1 = \frac{T_x}{m_w}, \qquad T_2 = \frac{T_y}{m_w + m_s}, \qquad T_3 = \frac{T_R}{m_c}$$
$$N_1 = u_1 - T_1, \quad N_2 = u_2 - T_2, \quad N_3 = u_3 - T_3 .$$

The position of the payload is described by the equalities

$$\begin{aligned} x_c &= x_w + R\cos\alpha \\ y_c &= y_w + R\sin\alpha\sin\beta \\ z_c &= -R\sin\alpha\cos\beta. \end{aligned}$$

The dynamics of the crane is given by the equations (Fig. 1)

$$\begin{aligned} m_c\ddot{x}_c &= -S_x \\ m_c\ddot{y}_c &= -S_y \\ m_c\ddot{z}_c &= -S_z - m_c g \\ m_w\ddot{x}_w &= F_x - T_x + S_x \\ (m_w + m_s)\ddot{y}_w &= F_y - T_y + S_y , \end{aligned}$$

where S_x , S_y and S_z are the components of the force S

$$\begin{aligned} S_x &= S\cos\alpha \\ S_y &= S\sin\alpha\sin\beta \\ S_z &= -S\sin\alpha\cos\beta. \end{aligned}$$

It is assumed that the lift-line is always stretched, that is,

$$S_x(x_c - x_w) + S_y(y_c - y_w) + S_z z_c > 0 .$$

As the payload is lifted and lowered with the use of the control force F_R, we have

$$S = F_R - T_R .$$

After appropriate substitutions we obtain

$$\begin{aligned} \ddot{x}_c &= -N_3\cos\alpha \\ \ddot{y}_c &= -N_3\sin\alpha\sin\beta \\ \ddot{z}_c &= N_3\sin\alpha\cos\beta - g \\ \ddot{x}_w &= N_1 + \mu_1 N_3\cos\alpha \\ \ddot{y}_w &= N_2 + \mu_2 N_3\sin\alpha\sin\beta . \end{aligned}$$

We differentiate twice the equations which determine the payload position, with taking into account that the length of the lift-line $R(t)$ varies in time due to the action of the control force F_R. The system of nonlinear equations describing the dynamics of the crane controlled by means of three forces has the form

$$\begin{aligned} \ddot{x}_c &= \ddot{x}_w + (\ddot{R} - R\dot{\alpha}^2)\cos\alpha - (2\dot{R}\dot{\alpha} + R\ddot{\alpha})\sin\alpha \\ \ddot{y}_c &= \ddot{y}_w + (\ddot{R} - R\dot{\alpha}^2 - R\dot{\beta}^2)\sin\alpha\sin\beta + \\ &\quad + 2R\dot{\alpha}\dot{\beta}\cos\alpha\cos\beta + (2\dot{R}\dot{\alpha} + R\ddot{\alpha})\cos\alpha\sin\beta + \\ &\quad + (2\dot{R}\dot{\beta} + R\ddot{\beta})\sin\alpha\cos\beta \\ \ddot{z}_c &= (-\ddot{R} + R\dot{\alpha}^2 + R\dot{\beta}^2)\sin\alpha\cos\beta + \\ &\quad + 2R\dot{\alpha}\dot{\beta}\cos\alpha\sin\beta - (2\dot{R}\dot{\alpha} + R\ddot{\alpha})\cos\alpha\cos\beta + \\ &\quad + (2\dot{R}\dot{\beta} + R\ddot{\beta})\sin\alpha\sin\beta. \end{aligned}$$

Denote

$$\begin{aligned} x_1 &= x_w & x_6 &= \dot{x}_5 = \dot{\alpha} \\ x_2 &= \dot{x}_1 = \dot{x}_w & x_7 &= \beta \\ x_3 &= y_w & x_8 &= \dot{x}_7 = \dot{\beta} \\ x_4 &= \dot{x}_3 = \dot{y}_w & x_9 &= R \\ x_5 &= \alpha & x_{10} &= \dot{x}_9 = \dot{R} \end{aligned}$$

$$\begin{aligned} s_n &\equiv \sin x_n \\ c_n &\equiv \cos x_n \end{aligned}$$

$$\begin{aligned} V_5 &= c_5 s_5 x_8^2 x_9 - 2x_{10}x_6 + g c_5 c_7 \\ V_6 &= 2x_8(c_5 x_6 x_9 + s_5 x_{10}) + g s_7 \\ V_7 &= s_5^2 x_8^2 x_9 + g s_5 c_7 + x_6^2 x_9 . \end{aligned}$$

It is assumed that the friction is proportional to the respective velocity component, and so

$$\begin{aligned} N_1 &= u_1 - k_1\dot{x}_w = u_1 - k_1 x_2 \\ N_2 &= u_2 - k_2\dot{y}_w = u_2 - k_2 x_4 \\ N_3 &= u_3 + k_3\dot{R} = u_3 + k_3 x_{10} . \end{aligned}$$

Finally, we obtain ten state equations describing the dynamics of the crane with varying pendulum length

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = N_1 + \mu_1 c_5 N_3$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = N_2 + \mu_2 s_5 s_7 N_3$$
$$\dot{x}_5 = x_6$$
$$\dot{x}_6 = \left(s_5 N_1 - c_5 s_7 N_2 + (\mu_1 - \mu_2 s_7^2) c_5 s_5 N_3 + V_5\right)/x_9$$
$$\dot{x}_7 = x_8$$
$$\dot{x}_8 = -(c_7 N_2 + \mu_2 s_5 c_7 s_7 N_3 + V_6)/(s_5 x_9)$$
$$\dot{x}_9 = x_{10}$$
$$\dot{x}_{10} = -c_5 N_1 - s_5 s_7 N_2 - (1 + \mu_1 c_5^2 + \mu_2 s_5^2 s_7^2) N_3 + V_7$$

The denominator in the eighth equation includes $\sin x_5$. When the crane operates in its real range, then $\sin x_5 \neq 0$.

3. OPTIMAL CONTROL PROBLEM AND THE METHOD OF SOLUTION

The state equations of the crane can be written in the vector notation

$$\dot{x}(t) = f(x(t), u(t)) = f^0(x(t)) + f^1(x(t))u(t)$$

where $x(t) \in \boldsymbol{R}^{10}$ is the state vector and $u(t) \in \boldsymbol{R}^3$ is the control vector. The initial state $x(0) = x^0$ is given, together with the target state x^f. The set of admissible controls $\boldsymbol{U}$ consists of all right-continuous functions $u : [0, \infty[\to \boldsymbol{R}^3$ such that

$$u_i(t) \in [u_i^{\min}, u_i^{\max}], \quad i = 1,2,3.$$

Following [9], we describe the main elements of the variable control parameterization method. The only major difference is that in our case the control function is three-dimensional whilst it was scalar in [9]. The time-optimal problem is transformed into a sequence of auxiliary optimization problems. In each of them, the auxiliary criterion depending on the horizon $T \in [0, \infty[$

$$S_q(u,T) = \tfrac{1}{2}(x(T) - x^f)^{\mathrm{T}} Q (x(T) - x^f) + qT$$

is minimized with an appropriately chosen, fixed value of the nonnegative parameter *q*. The matrix *Q* is constant and $Q = Q^{\mathrm{T}} > 0$. Typically, *q* is changed before an exact minimum is achieved, which starts the next problem of the sequence. An admissible control *u* is time-optimal if and only if $S_0(u,T) = 0$ with minimum *T*.

From the Maximum Principle it easily follows that the optimal controls are of bang-bang form, provided the problem is nonsingular. We thus confine our considerations to optimal control problems with bang-bang solutions. A *bang-bang* control *u* is fully characterized by its initial value $u_0 \in \boldsymbol{R}^3$ such that $u_{0i} \in \{u_i^{\min}, u_i^{\max}\}$ for $i = 1,2,3$, and a finite non-decreasing sequence of switching times $\tau_i = (\tau_i^j)_{j=1}^{j=m_i} \subset [0, \infty[$ for each control component u_i. For an empty τ_i, $m_i = 0$. The bang-bang control $u(t)$ is also denoted by $u(t; \tau, u_0)$ where $\tau = (\tau_1, \tau_2, \tau_3)$. The restriction of the auxiliary criterion to bang-bang controls defines the functional

$$S'_q(\tau, u_0, T) = S_q(u(\cdot\,; \tau, u_0), T)$$

with τ being a triple of finite non-decreasing sequences in $[0, T]$.

The discrete decision variables m_i and u_{0i}, $i = 1,2,3$, are determined on the upper level of the minimization algorithm, by the *generation* and *reduction* procedures. On the lower level, the values of τ_i^j and *T* are optimized for fixed m_i and u_{0i} with the use of a gradient (BFGS) method.

The adjoint function ψ for the auxiliary problem is determined by

$$\dot{\psi} = -\nabla_x f(x,u)\psi, \quad \psi(T) = Q(x^f - x(T)).$$

The antigradient of S'_q w.r.t. control is given by

$$g(t) = -\nabla_u S_q(u,T)\Big|_t = \psi(t)^{\mathrm{T}} f^1(x(t)).$$

As in [9], we define g^U, the projection of the antigradient onto the admissible set, and the gradient of S'_q, understood as the vector of derivatives of S'_q w.r.t. τ_i^j, $j = 1,\ldots,m_i$, $i = 1,2,3$, and *T*

$$\nabla S'_q(\tau, u_0, T) = \mathrm{col}(\nabla_\tau S'_q(\tau, u_0, T), \nabla_T S'_q(\tau, u_0, T)).$$

The derivatives with respect to switching times and to the horizon are determined in a similar way as in [8] and [9]

$$\nabla_{\tau_i^j} S'_q(\tau, u_0, T) =$$
$$= \psi(\tau_i^j) f^1(x(\tau_i^j))(u_i(\tau_i^j +) - u_i(\tau_i^j -))$$
$$\nabla_T S'_q(\tau, u_0, T) = (x(T) - x^f)^{\mathrm{T}} Q \dot{x}(T) + q.$$

4. VARIABLE PARAMETERIZATION ALGORITHM

The scheme of the numerical algorithm is shown in Fig. 2. In the algorithm, asymptotic optimality (with respect to horizon) is assured by the stepwise decrease of q. The parameter q is initialized with a positive value, and kept fixed until the following *q update* conditions are met

$$\left\|g^U\right\| < \varepsilon_g \quad \text{and} \quad \left|\nabla_T S'_q\right| < \varepsilon_T$$

where ε_g and ε_T are given thresholds. When this happens, new values: $r_q q$, $r_g \varepsilon_g$ and $r_T \varepsilon_T$ replace q, ε_g, and ε_T, respectively, and are in use until the q update conditions are met again. The positive ratios r_q, r_g and r_T are less than 1. The whole procedure is repeated till the end of the algorithm.

The termination conditions have the form

$$S_0 < \varepsilon_S^f \quad \text{or} \quad \left|\nabla_T S'_q\right| < \varepsilon_T^f \quad \text{and} \quad \left\|g^U\right\| < \varepsilon_g^f$$

where ε_S^f, ε_T^f and ε_g^f are required zero tolerances. In some cases the termination conditions become fulfilled while the horizon is too large. This may happen when the system is driven sufficiently close to the target state and q is small. In order to counteract this phenomenon, the *reinitialization* of q is used (see [9]). If the so-called transversality conditions are not satisfied, the parameter q is set to a certain value q_0, sufficiently large to prevent the algorithm from a premature termination.

The *switching generation* consists in choosing three finite, non-decreasing sequences $\gamma_i \subset [0,T]$ of new, additional switching times, and simultaneously replacing u_0 with some u'_0, $u'_{0i} \in \{u_i^{\min}, u_i^{\max}\}$, $i = 1,2,3$. This produces a new bang-bang control $u(\cdot\,;\tau \sim \gamma, u'_0)$ with more switchings. This control is a starting point for the lower level optimization procedure with the elements of $\tau \sim \gamma$ and T as decision variables, where $\tau \sim \gamma$ denotes the triple of monotonically ordered unions of sequences τ_i and γ_i, $i = 1,2,3$. The following control preservation condition is fulfilled

$$u(t;\tau \sim \gamma, u'_0) = u(t;\tau, u_0) \quad \forall t \in [0,T[\,.$$

Precise rules for the generation, based on the analysis of the projection g^U and the gradient $\nabla S'_q(\tau, u_0, T)$, are given in [9].

The *reduction of switchings* is carried out after every iteration of optimization. For any $T \in [0,\infty[$ and $u_0 \in \boldsymbol{R}^3$, $u_{0i} \in \{u_i^{\min}, u_i^{\max}\}$, $i = 1,2,3$, let τ be a triple of finite, non-decreasing sequences of switching times in $[0,T]$ (resulting from optimization). The switching reduction consists in removing all switching times lying on active constraints. This leads to transforming τ into a triple of finite strictly increasing sequences ω in $]0,T[$, and replacing u_0 with u'_0 in such a way that

$$u(t;\omega, u'_0) = u(t;\tau, u_0) \quad \forall t \in [0,T[$$

where $u'_{0i} \in \{u_i^{\min}, u_i^{\max}\}$, $u'_{0i} = u_{0i}$ if the number of zero elements in τ_i is even and $u'_{0i} \neq u_{0i}$ otherwise, for $i = 1,2,3$ (see [9]). The control $u(\cdot\,;\omega, u'_0)$ is then subject to subsequent generation tests.

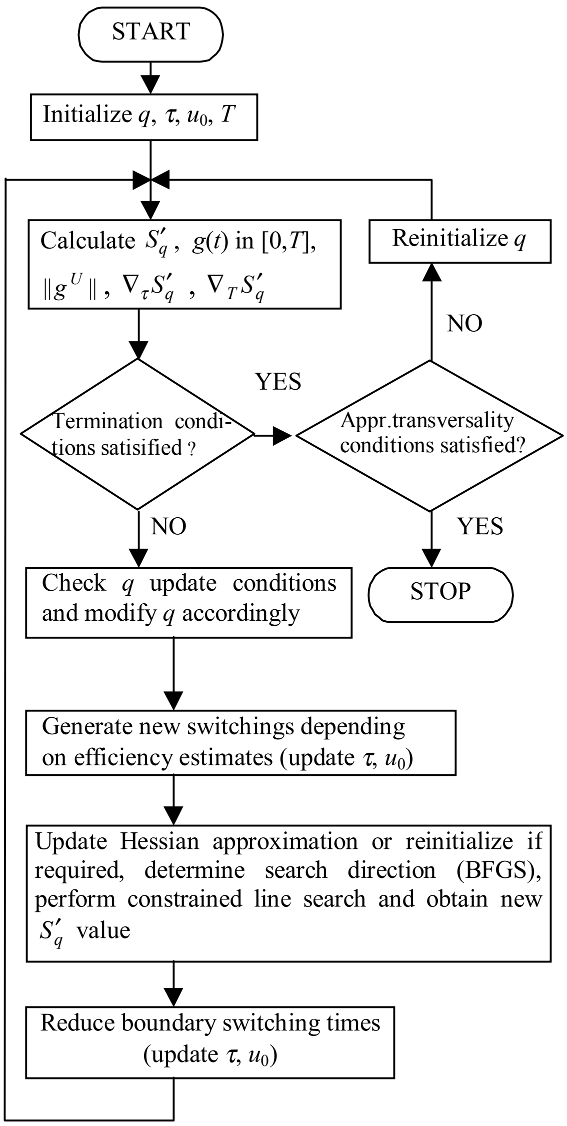

Fig. 2. Variable control parameterization algorithm

5. EXPERIMENTS

5.1 Experiment 1

In the first experiment the time optimal control is only calculated and simulated on a computer. The following parameters are assumed:

$m_c = 1$ kg , $m_w = 2.49$ kg , $m_s = 4.09$ kg ,

$k_1 = 30.12048192771084$ Ns/m,

$k_2 = 11.39817629179331$ Ns/m,

$k_3 = 75$ Ns/m ,

$$x^0 = \mathrm{col}(0, 0.5, 0, 0.5, 0.25\pi, -0.5, 0.25\pi, -0.5, 1, 0)$$

$$x^f = \mathrm{col}(1, 0, 1, 0, 0.5\pi, 0, 0, 0, 1.5, 0) .$$

All components of the control vector take values from the same admissible interval $[-20, 20]$.

The optimal horizon calculated by means of the variable control parameterization method is equal to 5.4 s. The optimal state trajectories are shown in Fig. 2, and the optimal controls are given in Fig. 3, together with the respective antigradient components.

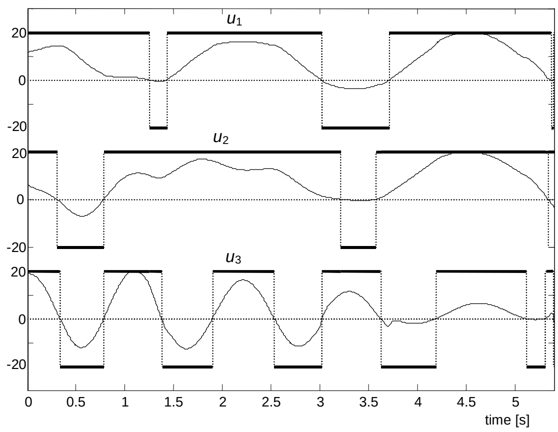

Fig. 3. Time-optimal controls for experiment 1

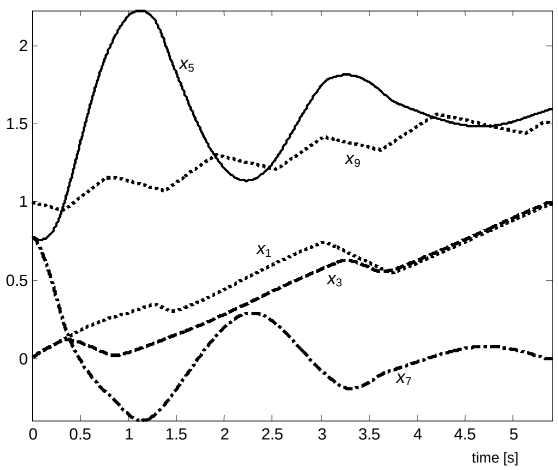

Fig. 4. Time-optimal trajectories for experiment 1

5.2 Experiment 2

In the second experiment the computed optimal control was implemented in a real laboratory model of the 3D crane [5]. A two-stage control problem is considered. In the first stage the payload is lifted, and the cart moves both in the *x* and *y* directions. At the end of the first stage the payload should hang vertically, and the cart should have zero velocity in the *x* direction and a given velocity in the *y* direction. Thus the system is prepared for the payload transportation stage which is omitted in our example. The initial and target states for the first stage are as follows

$$x^0 = \mathrm{col}(0.23, 0, -0.23, 0, 1.67, 0, 0.1, 0, 1.5, 0)$$

$$x^f = \mathrm{col}(0.23, 0, 0.18, 0.12, 0.5\pi, 0, 0, 0, 1.3, 0) .$$

In the second stage the cart has to be shifted by a given displacement in both axes, and stopped. At the same time the payload has to be lowered and its oscillations damped. The initial state is equal to the final state achieved in the first stage and the target state is equal to

$$x^f = \mathrm{col}(0, 0, 0.58, 0, 0.5\pi, 0, 0, 0, 1.5, 0) .$$

The first two components of the control vector take values from the same admissible interval $[-10, 10]$, and the third component from [–10, 30]. The time-optimal control calculated with the use of the variable parameterization method is shown in Fig. 5. This control was then fed, in an open-loop, to the laboratory model of 3D crane. The obtained trajectories, measured in the real system, are presented in Fig. 6. Fig. 7 presents the differences between the simulated trajectories and measured ones. Note that the relative errors are small despite the lack of feedback.

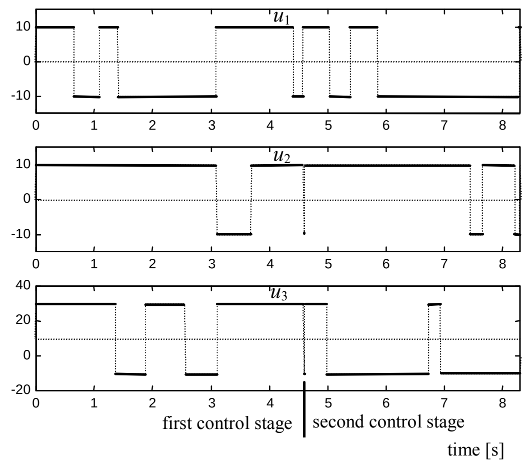

Fig. 5. Time-optimal controls for experiment 2

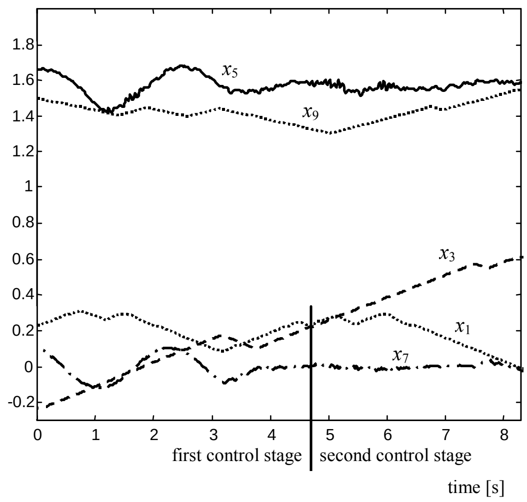

Fig. 6. Time-optimal trajectories for experiment 2, measured in real system

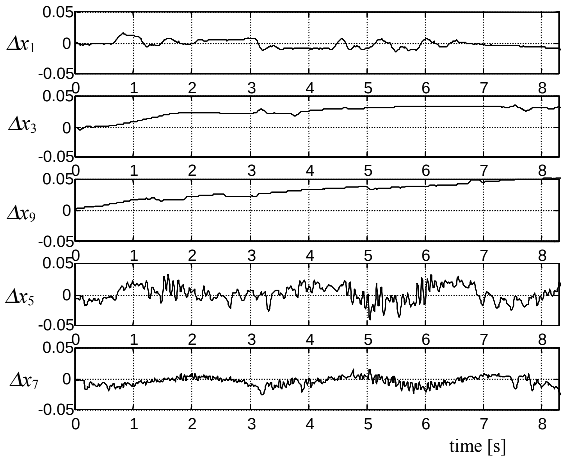

Fig. 7. Differences between simulated time-optimal and measured trajectories (meters for three upper pictures and radians for two bottom ones)

6. CONCLUSIONS

The results show that the derived analytical model of the dynamics of 3D crane can be used for the synthesis of time-optimal control. The variable parameterization method of dynamic optimization proved effective also in an application to higher dimensional nonlinear systems with many controls. The comparison of simulated optimal trajectories with those of the real laboratory model, optimally controlled, indicate the possibility of real-time implementations of time-optimal control.

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