

Laplace Transform

Laboratory III

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1 Goal of the exercise

2 Laplace Transform

Laplace transform is an integral transform that converts a function of a real variable (time domain) into a function of complex variable (frequency domain). It is a powerful tool for solving differential equations, which turns ODEs into algebraic equations and convolution into multiplication. For function $x(t)$, the Laplace transform is the integral

$$\mathcal{L}[x(t)] = X(s) = \int_0^{\infty} x(t)e^{-st} dt \quad (1)$$

Where $s = \sigma + j\omega$ $\sigma, \omega \in \mathbb{R}$

In order to get solution of differential equation solved in s-domain it is necessary to apply inverse Laplace transform which is given by following complex integral

$$f(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s)e^{-st} ds \quad (2)$$

Integrals like these can be quite difficult to solve, that is why lookup table of [Laplace transforms](#) and [properties](#) can be very handy.

Moving to solving circuits using Laplace transform, there are two approaches, first is constructing differential equation in time domain describing circuit and then solve it using Laplace transform, or second option very similar to solving circuits in steady-state where all components are transformed to s-domain and at the end of calculation transform back to t-domain.

$$R \xrightarrow{\mathcal{L}} R \quad C \xrightarrow{\mathcal{L}} \frac{1}{sC} \quad L \xrightarrow{\mathcal{L}} sL \quad (3)$$

3 Course of measurements

First we tested two circuits: low-pass and high-pass configuration of RC circuit and RLC circuits. After connecting oscilloscope to the wave generator and prototype board, we generated square wave with $v_{pp} = 1V$, $v_{offset} = .5V$, frequency of 100Hz and duty cycle of 50%. Then we read from the oscilloscope Voltage value at times 1τ , 5τ and 10τ (where 10τ is just as fail-safe) and time when voltage reaches 10% and 90% of the highest value.

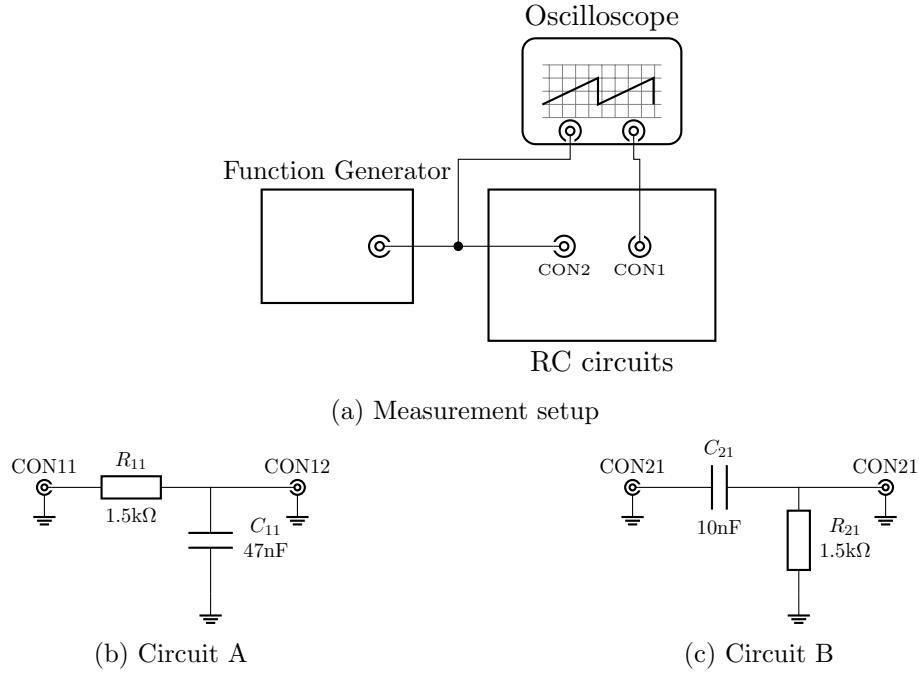


Figure 1: RC Circuits

For RLC circuits we tested two different resistances before RC circuit influence output characteristic. We measured response for 2 cases: Generator out resistance ($50\ \Omega$) + resistance selected by jumper wire. In our case we tested jumper on $1.1\text{k}\Omega$ resistance path and $3.3\text{k}\Omega$. After that we checked response of the circuit:

- if response was sinusoid with decreasing amplitude - resistance was smaller then RC
- if response was exp. decay if R was higher
- if response was aperiodic critical waveform if resistance was equal RC

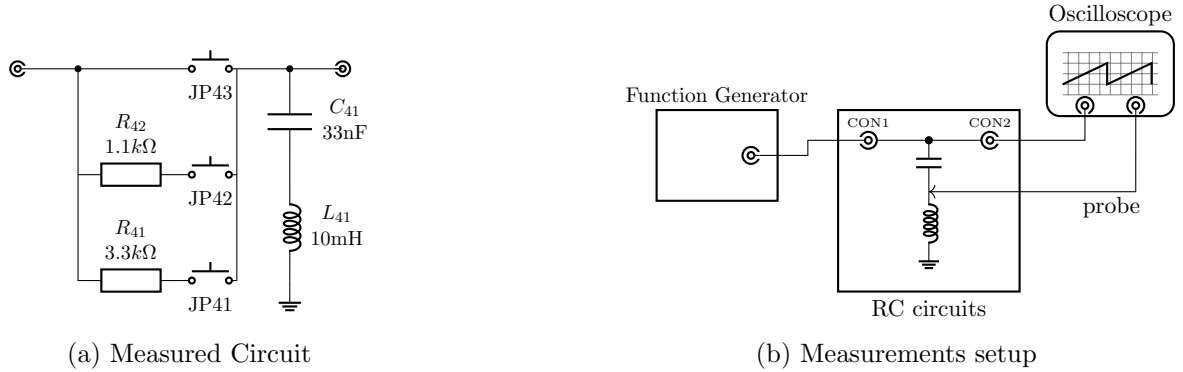


Figure 2: RLC circuit

4 Theoretical calculations

For all calculations we used Matlab with Symbolic Math Toolbox. Source code can be found in Appendix. A

In all three circuit input voltage was square wave with 50% duty cycle which can be described in time domain by

$$v_{in}(t) = V_{offset}\mathbf{1}(t) + V_{pp}\mathbf{1}(t - \frac{T}{2}) - V_{pp}\mathbf{1}(t - T) \quad (4)$$

and in frequency domain by

$$V_{in}(s) = \mathcal{L}[v_{in}(t)] = \frac{V_{offset}}{s} + \frac{e^{-\frac{T}{2}s}}{s} - \frac{e^{-Ts}}{s} \quad (5)$$

4.1 Circuit A

First we need to transform circuit to s-domain according to (3)

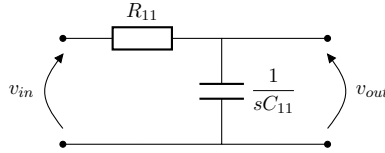
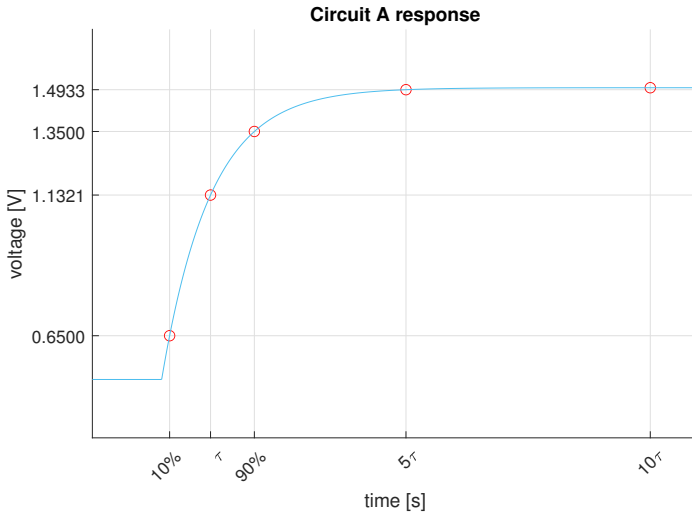


Figure 3: Circuit A schematics

output voltage of circuit A can be described using simple voltage divider

$$V_{out}(s) = V_{in}(s) \frac{\frac{1}{sC_{11}}}{R_{11} + \frac{1}{sC_{11}}} \quad (6)$$

after applying inverse Laplace transform to $v_{out}(s)$ we obtain below plot with marked τ , 5τ , 10τ , 10% and 90% of output voltage.



(a) plot

	time [μ s]	voltage [V]
τ	70	1.132
5τ	352	1.493
10τ	705	1.499
10%	11	0.649
90%	133	1.349

(b) table of values

Figure 4: Circuit A output voltage

4.2 Circuit B

steps in circuit B are almost identical to circuit A, first we transformed circuit to s-domain according to (3)

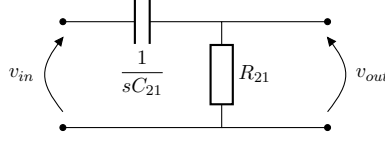
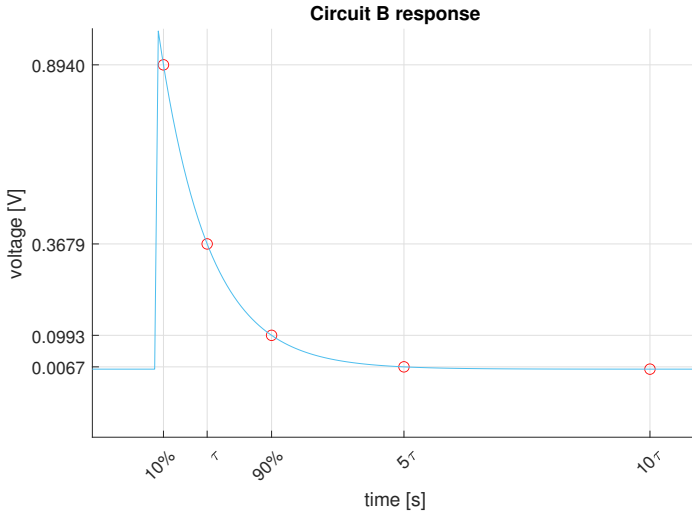


Figure 5: Circuit B schematics

output voltage can be described by voltage divider

$$V_{out}(s) = V_{in}(s) \frac{R_{21}}{R_{21} + \frac{1}{sC_{21}}} \quad (7)$$

after applying inverse Laplace transform to $V_{out}(s)$ we obtain below plot with marked τ , 5τ , 10τ , 10% and 90% of output voltage.



(a) plot

	time [μ s]	voltage [V]
τ	15	0.367
5τ	75	0.007
10τ	150	0.000
10%	2	0.894
90%	35	0.099

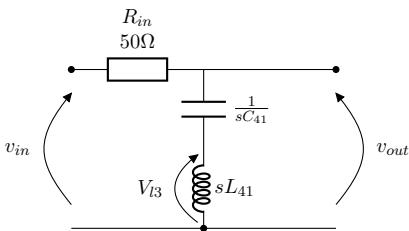
(b) table of values

Figure 6: Circuit B output voltage

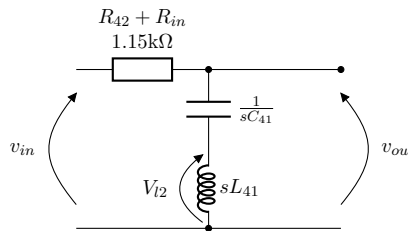
4.3 Circuit C

In third circuit we are also looking for voltage of coil L_{21} .

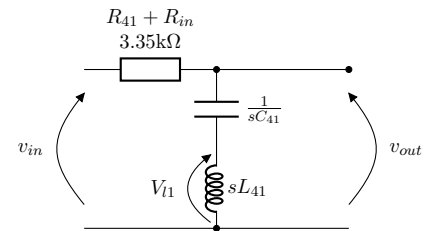
Like in previous 2 circuits we transformed circuit to s-domain according to (3)



(a) JP43



(b) JP42



(c) JP41

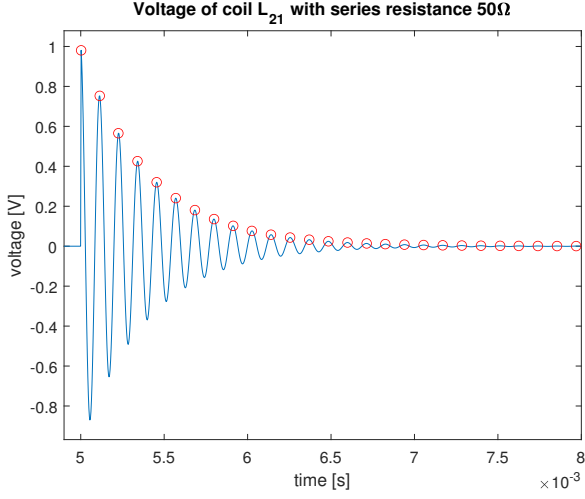
Figure 7: Circuit C schematics

Then solved each circuit for $V_l(s)$ and $V_{out}(s)$

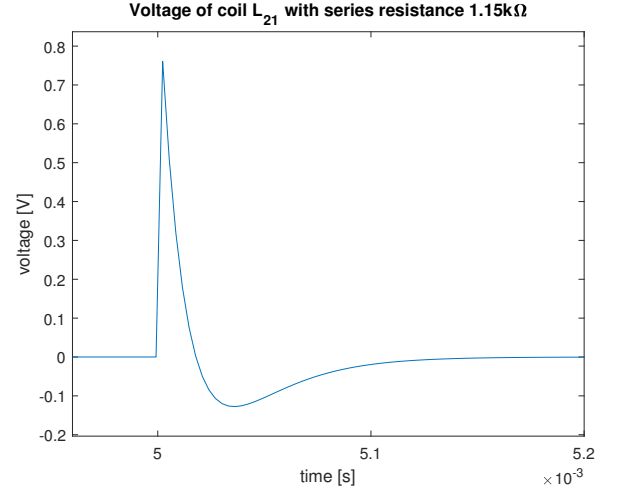
$$V_l(s) = V_{in}(s) \frac{sL_{41}}{R_x + \frac{1}{sC_{21}} + sL_{21}} \quad (8)$$

$$V_{out}(s) = V_{in}(s) \frac{\frac{1}{sC_{41}} + sL_{41}}{R_x + \frac{1}{sC_{41}} + sL_{41}} \quad (9)$$

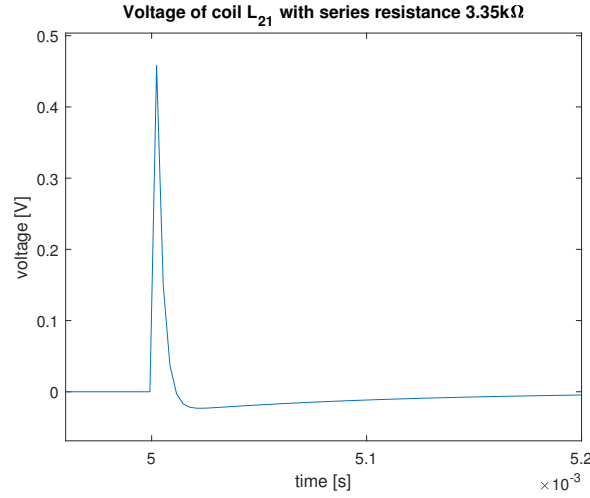
After plugging in value of each component where R_x is series resistance of the circuit, we can transform equation back to time domain and plot results.



(a) Plot JP43

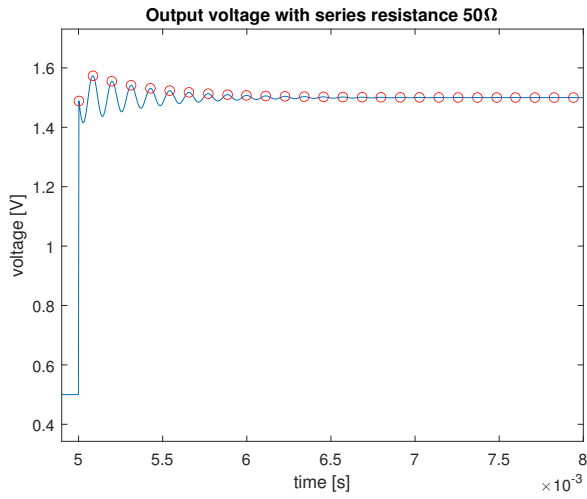


(b) Plot JP42

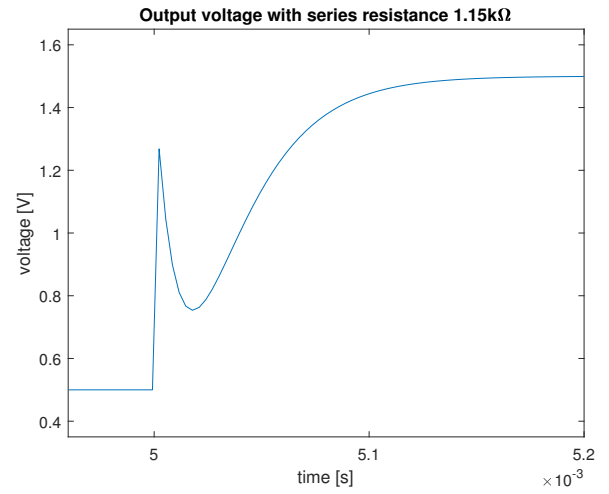


(c) Plot JP41

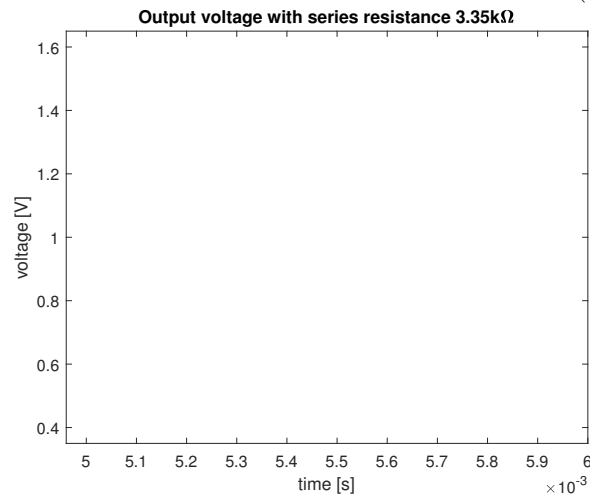
Figure 8: Coil Voltage



(a) Plot JP43



(b) Plot JP42



(c) Plot JP41

Figure 9: Output voltage

5 Comparison

6 Conclusions

A Appendix

[GITHUB repository](#)

```
clear all; close all; clc;
```

```
%% Square wave
```

```
syms t s square
```

```
f = 100;
```

```
T = 1/f;
```

```
w = 2 * pi * f;
```

```
amplitude = 1;
```

```
offset = 0.5;
```

```
square = amplitude * heaviside(t - T/2) - amplitude * heaviside(t - T) + offset * heaviside(t);
```

```
% %% Circuit A
```

```
% C1 = 47e-9;
```

```
% R = 1.5e3;
```

```
% response1(t) = ilaplace(laplace(square) * (1/(s*C1))/(R + 1/(s*C1)));
```

```
% steps = 1000;
```

```
% step = (0.6*T-0.49*T)/steps;
```

```
% t = 0.49*T:step:0.6*T;
```

```
% x = response1(t);
```

```
% tauA = 1.5e3 * 47e-9;
```

```
% tauA1 = tauA;
```

```
% tauA5 = 5*tauA;
```

```
% tauA10 = 10*tauA;
```

```
% percent10 = 0.1*max(x) + offset;
```

```
% percent90 = 0.9*max(x);
```

```
% invresponse = finverse(response1);
```

```
% xpercent10 = double(invresponse(percent10));
```

```
% xpercent90 = double(invresponse(percent90));
```

```
% %% figures
```

```
% fig = figure('Name', 'Circuit A');
```

```
% fig.Position(3:4) = [600 400];
```

```
% grid on;
```

```
% hold on;
```

```
% plot(T/2+tauA1, response1(T/2+tauA1), 'ro')
```

```
% plot(T/2+tauA5, response1(T/2+tauA5), 'ro')
```

```
% plot(T/2+tauA10, response1(T/2+tauA10), 'ro')
```

```
% plot(xpercent10, percent10, 'ro')
```

```
% plot(xpercent90, percent90, 'ro')
```

```
% xticks([xpercent10, T/2+tauA1, xpercent90, T/2+tauA5, T/2+tauA10]);
```

```
% xticklabels({'10%' '\tau' '90%' '5\tau' '10\tau'})
```

```
% yticks(sort(double([response1(T/2+tauA1) percent10 percent90 response1(T/2+tauA5)]),2, "asce
```

```
% plot(t, x)
```

```
% ylim([offset-0.2, amplitude+offset+0.2])
```

```
% xlim([0.49*T, T/2 + 1.1*tauA10])
```

```
% xtickangle(45)
```

```
% title('Circuit A response')
```

```
% xlabel('time [s]')
```

```
% ylabel('voltage [V]')
```

```

% print('img/CircuitA','-depsc')

%% Circuit B
% C2 = 10e-9;
% R = 1.5e3;
% response1(t) = ilaplace(laplace(square) * R/(R + 1/(s*C2)));
% steps = 1000;
% step = (0.6*T-0.49*T)/steps;
% t = 0.49*T:step:0.6*T;
% x = response1(t);
% tauA = 1.5e3 * 10e-9;
% tauA1 = tauA;
% tauA5 = 5*tauA;
% tauA10 = 10*tauA;
% percent10 = double(0.1*max(x))
% percent90 = double(0.9*max(x))
% invresponse = finverse(response1);
% xpercent10 = double(invresponse(percent10))
% xpercent90 = double(invresponse(percent90))
% xpercent90 - T/2
% xpercent10 - T/2
%% figures
% fig = figure('Name', 'Circuit B');
% fig.Position(3:4) = [600 400];
% grid on;
% hold on;
% plot(T/2+tauA1, response1(T/2+tauA1), 'ro')
% plot(T/2+tauA5, response1(T/2+tauA5), 'ro')
% plot(T/2+tauA10, response1(T/2+tauA10), 'ro')
% plot(xpercent10, percent10, 'ro')
% plot(xpercent90, percent90, 'ro')
% xticks(sort([xpercent10, T/2+tauA1, xpercent90, T/2+tauA5, T/2+tauA10],2,"ascend"));
% xticklabels({'10%' '\tau' '90%' '5\tau' '10\tau'})
% yticks(sort(double([response1(T/2+tauA1) percent10 percent90 response1(T/2+tauA5)]),2, "ascend"));
% plot(t, x)
% ylim([-0.2, amplitude])
% xlim([0.498*T, T/2 + 1.1*tauA10])
% xtickangle(45)
% title('Circuit B response')
% xlabel('time [s]')
% ylabel('voltage [V]')
% print('img/CircuitB','-depsc')

%% Circuit C
R3 = 50;
R2 = 1.15e3;
R1 = 3.35e3;
C = 33e-9;
L = 10e-3;
response33(t) = ilaplace(laplace(square) * (s*L)/(R3 + 1/(s*C)+s*L));
response32(t) = ilaplace(laplace(square) * (s*L)/(R2 + 1/(s*C)+s*L));
response31(t) = ilaplace(laplace(square) * (s*L)/(R1 + 1/(s*C)+s*L));
response33out(t) = ilaplace(laplace(square) * (s*L + 1/(s*C))/(R3 + 1/(s*C)+s*L));

```



```

response32out(t) = ilaplace(laplace(square) * (s*L + 1/(s*C))/(R2 + 1/(s*C)+s*L));
response31out(t) = ilaplace(laplace(square) * (s*L + 1/(s*C))/(R1 + 1/(s*C)+s*L));
steps = 1000;
step = (0.8*T-0.49*T)/steps;
t = 0.49*T:step:0.8*T;

% R3
x = response33(t);
TF = islocalmax(double(x));
TF(5) = 0;
MF = islocalmin(double(x));
figure('Name','Circuit C3')
plot(t, x, t(TF), x(TF), 'ro')
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.49*T 0.8*T])
title('Voltage of coil L_{21} with series resistance 50\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC3','-depsc')

x = response33out(t);
TF = islocalmax(double(x));
TF(5) = 0;
MF = islocalmin(double(x));
figure('Name','Circuit C3')
plot(t, x, t(TF), x(TF), 'ro')
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.49*T 0.8*T])
title('Output voltage with series resistance 50\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC3out','-depsc')

% R2
x = response32(t);
figure('Name','Circuit C2')
plot(t, x)
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.52*T])
title('Voltage of coil L_{21} with series resistance 1.15k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC2','-depsc')

x = response32out(t);
figure('Name','Circuit C2')
plot(t, x)
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.52*T])
title('Output voltage with series resistance 1.15k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC2out','-depsc')

```

```

% R1
x = response31(t);
figure('Name','Circuit C1')
plot(t, x)
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.52*T])
title('Voltage of coil L_{21} with series resistance 3.35k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC1','-depsc')

x = response31out(t);
figure('Name','Circuit C1')
plot(t, x)
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.6*T])
title('Output voltage with series resistance 3.35k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC1out','-depsc')

```