Fourier Series
Laboratory II

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1 Aim of the exercise

The purpose of exercise was to experimentally familiarize with the Fourier series - an operation that allows to represent any real periodic signal by the sum of sinusoidal signals. We measured the signals generated by the independent function generator with a vector spectrum analyzer.

2 What is Fourier Series

A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines. Fourier series make use of the orthogonal relationships of the sine and cosine functions. The computation and study of Fourier series is known as harmonic analysis and is extremely useful to break up an arbitrary periodic function into a set of simple terms that can be plugged in, solved individually, and then recombined to obtain the solution to the original problem or an approximation to it to whatever accuracy is desired or practical

3 Course of measurements

During our measurements we skipped first part of exercise and measured all signals in configuration for second part (Fig. 1).

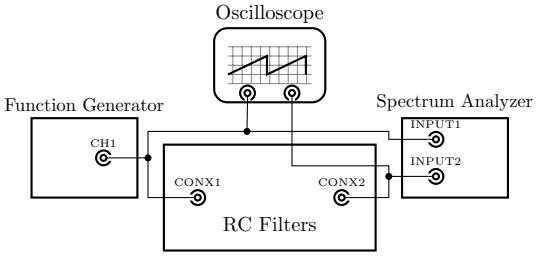


Figure 1: Measurements configuration

After connecting Function Generator, Spectrum Analyzer, Oscilloscope and board with RC filters according to above configuration (Fig. 1) we set Function Generator to Amplitude to $V_{pp} = 5$V and Frequency 1 kHz. With everything set up we proceeded with exercise and recording output of Oscilloscope and Spectrum Analyzer for signals:

- Sin wave
- Square wave 50% duty cycle
- Square wave 25% duty cycle
- Triangle wave 50% symmetry ratio
- Triangle wave 40% symmetry ratio

For Square wave 50% duty cycle and Triangle wave 40% symmetry ratio we also recorded Response for both RC Circuits (Fig. 2)

(a) Circuit A (b) Circuit B

Figure 2: Measured RC Circuits

4 Method for calculating Fourier Series coefficients

All calculations are made using Matlab with Signal Processing Toolbox, source code can be found in Appendix A

First step in our calculation is generating signals that we measured during exercise. Pure signals were obtained with built-in functions of Signal Processing Toolbox and Filtered signals were evaluated using NI Multisim SPICE software.

Next step is finding first 10 coefficients of Fourier series for a signal which we calculated by taking inner product of signal with basis vector. In our calculations we used $\sin n\omega t$, $\cos n\omega t$ and 1 as our basis vectors.

$$a_0 = \langle s(t), 1 \rangle \tag{1}$$

$$a_n = \langle s(t), \cos n\omega t \rangle \tag{2}$$

$$b_n = \langle s(t), \sin n\omega t \rangle \tag{3}$$

During exercise we recorded Fourier series of a signal in Amplitude-Phase form so we need to convert a_n and b_n coefficients into amplitude A_n and phase φ_n

$$A_n = \sqrt{a_n^2 + b_n^2} \tag{4}$$

$$\varphi_n = \arg(a_n - jb_n) \tag{5}$$

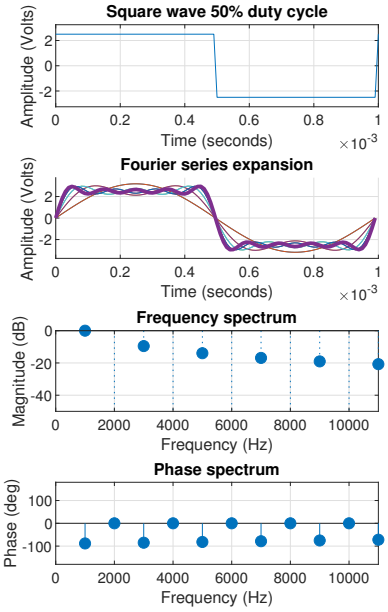


Figure 3: example plot

Knowing first 10 coefficients we can plot approximation of a signal using one of following formulas. Example of approximation of square wave is on the second plot in Figure 3.

$$s(t) \approx \frac{a_0}{2} + \sum_{n=1}^{10} (a_n \cos(n\omega t) + b_n \sin(n\omega t)) \tag{6}$$

$$s(t) \approx \frac{A_0}{2} + \sum_{n=1}^{10} (A_n \cos(n\omega t + \varphi_n)) \tag{7}$$

For third and fourth plot in Figure 3 we are calculating Discrete Fourier transform for 10 oscillations of a signal using Fast Fourier transform algorithm from Signal Processing Toolbox and then plotting magnitude and argument of FT in decibel scale on vertical axis.

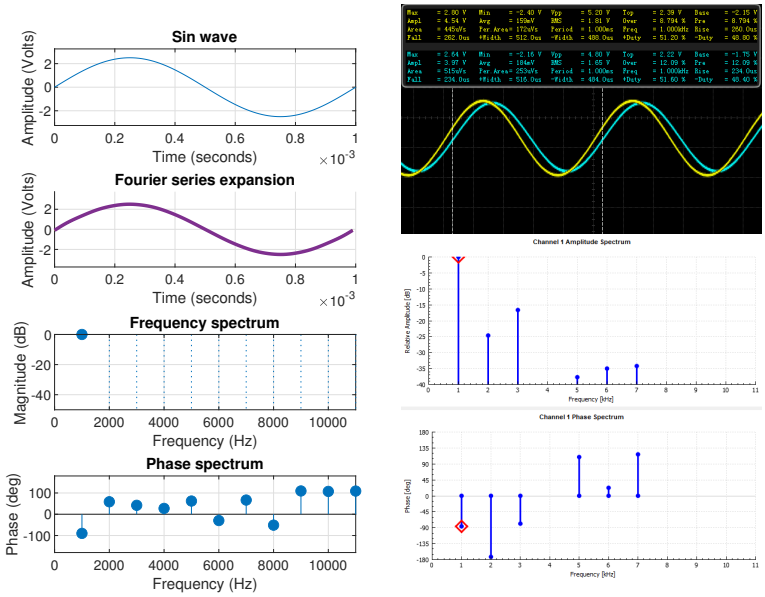
5 Comparison

We are going to be comparing calculated and measured signals, and their Fourier series expansion. During comparison we will determine if the Harmonic analysis is useful in signal analysis and what benefits does it bring.

∗ Yellow signal on oscilloscope is pure signal, Blue is filtered signal

5.1 Pure signals

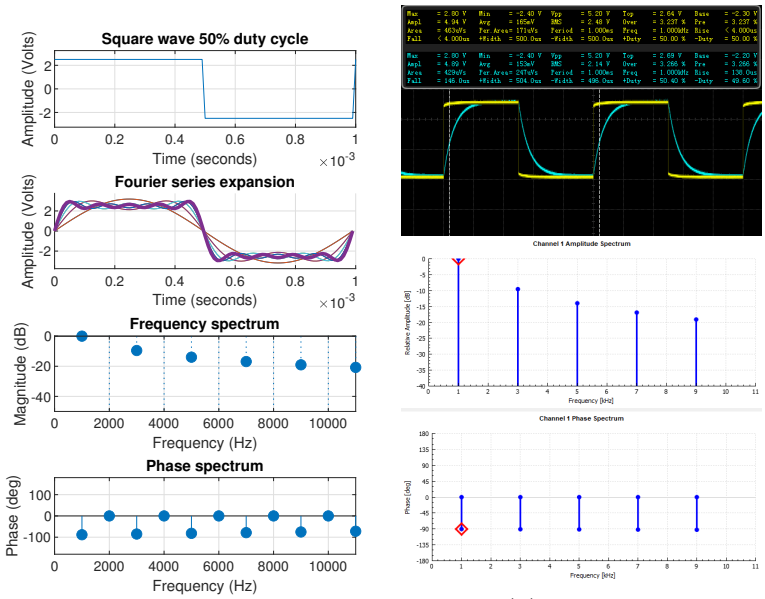
Starting off with first signal, Sinus, we can see that A_n coefficient for 1 kHz is equal to an amplitude of a signal and rest of coefficients are very small almost negligible, phase coefficient φ_n at 1 kHz is approximately -90°therefore according to eq.(7) first element of Fourier series is just $A_1 \sin \omega t$ and the rest of elements are very small, negligible.



(a) calculated (b) measured

Figure 4: Sin wave

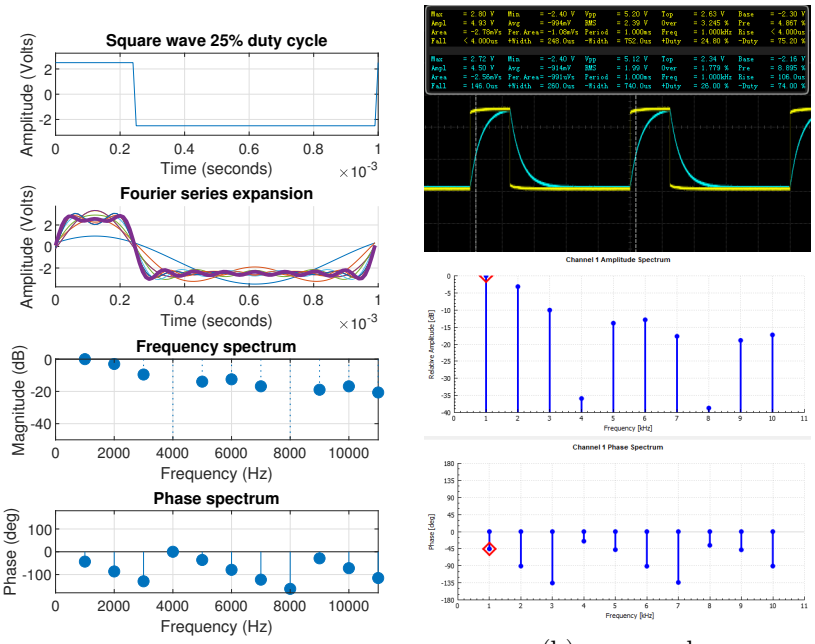
Now moving to more interesting signal than simple sin, square wave, we can see many more components of Fourier series that are not negligible. Starting with first element which is exactly the same as in sin signal we can deduce that square signal have similar shape to sinus and is mostly odd function, but looking further on the rest of coefficients we can see that all non-zero coefficients are shifted by -90°, so the square signal is built entirely from sins and therefore is completely odd.



(a) calculated (b) measured

Figure 5: Square wave 50% duty cycle

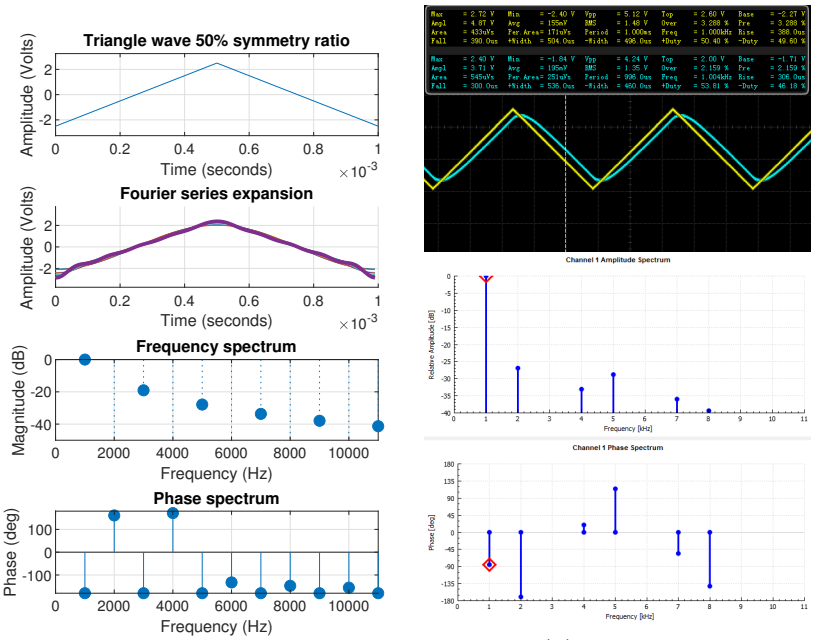
Third signal is also square wave but with duty cycle of 25%. Looking at the most significant coefficients we can see that it is shifted by -45°therefore the signal is neither odd or even, also when looking at both coefficient there is repeating pattern. Phase of a harmonic is changing in order to negate previous, more significant harmonic.



(a) calculated (b) measured

Figure 6: Square wave 25% duty cycle

Last of the pure signals is a triangle wave, the signal is very similar to the pure sin wave, even first element is exactly the same as in sin wave, but all the differences are in harmonics with significantly smaller amplitudes that are ”fine tuning” the signal.



(a) calculated (b) measured

Figure 7: Triangle wave 50% symmetry ratio

5.2 RC filters

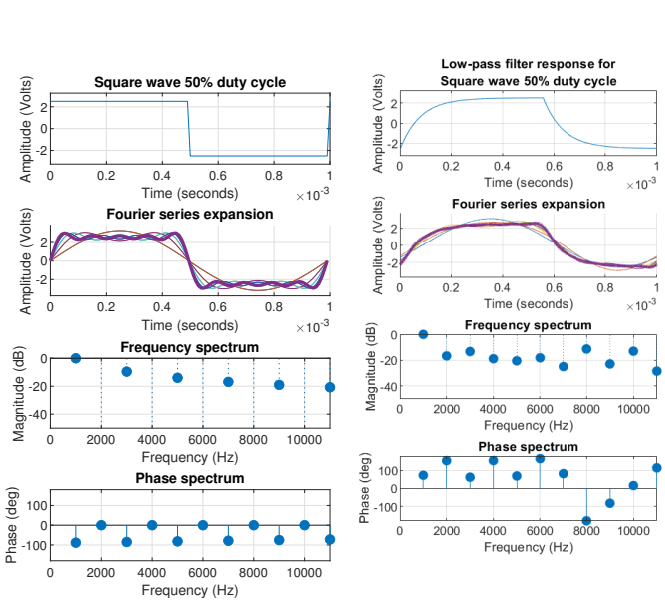
Simple Low-pass and High pass filters can be build with just 2 simple electric components resistor and capacitor, the reason why it is possible is that capacitor is acting differently for low frequencies and high frequencies. From formula for impedance of capacitor $\mathbf{Z} = \frac{1}{j\omega C}$ we can see that if ω is approaching 0, impedance will go to infinity therefore we can treat if like open circuit, but when ω is approaching infinity, impedance will go to 0 so capacitor will act like a wire.

(a) Low-pass filter (b) High-pass filter

Figure 8: RC filters

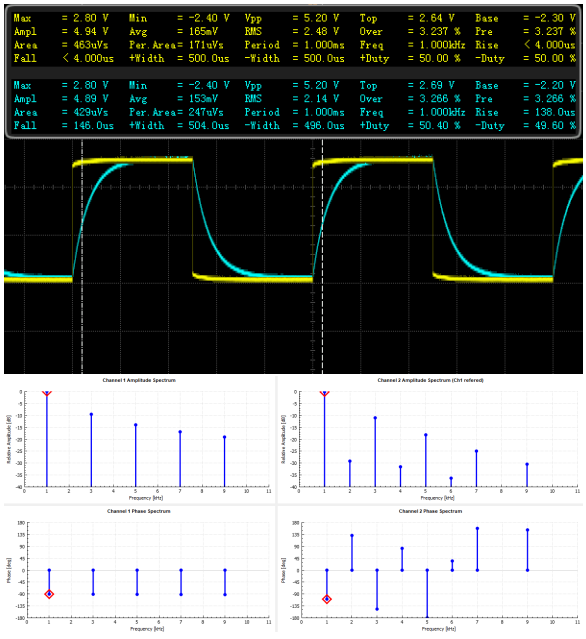
5.2.1 Low-pass filter

First signal that we filtered using low-pass filter and measured was square wave 50% duty cycle. The most notable thing in the comparison of unfiltered and filtered signals is that the amplitudes of harmonics with smaller frequency barely change when those with higher frequencies are changed by several decibels.



(a) Pure signal

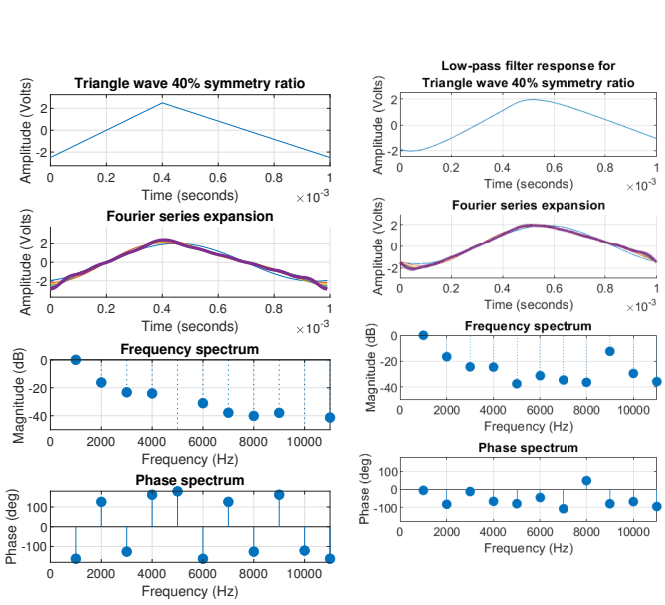
(b) Filtered signal



(c) Measured signal

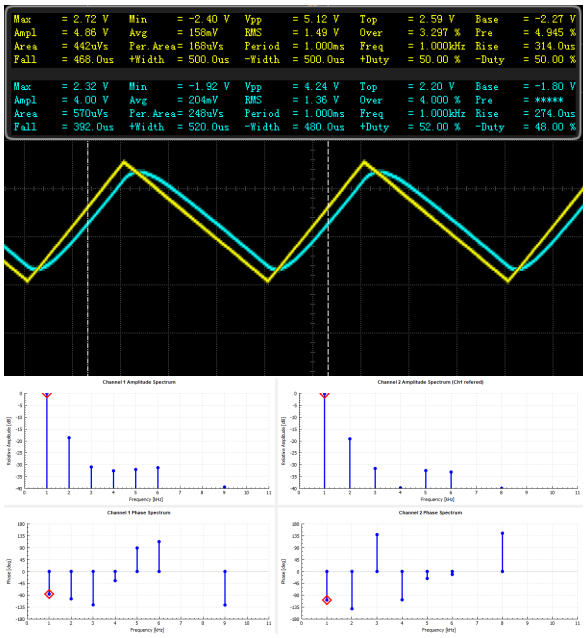
Figure 9: Low-pass filter response for Square wave 50% duty cycle

For filtered triangle wave 40% duty cycle similar to square wave amplitudes of harmonics with higher frequency are decreased by few decibels, and without those higher frequency harmonics which are responsible for "thin tuning". Signal loses sharp shape and retains only rough shape of the signal.



(a) Pure signal

(b) Filtered pure signal

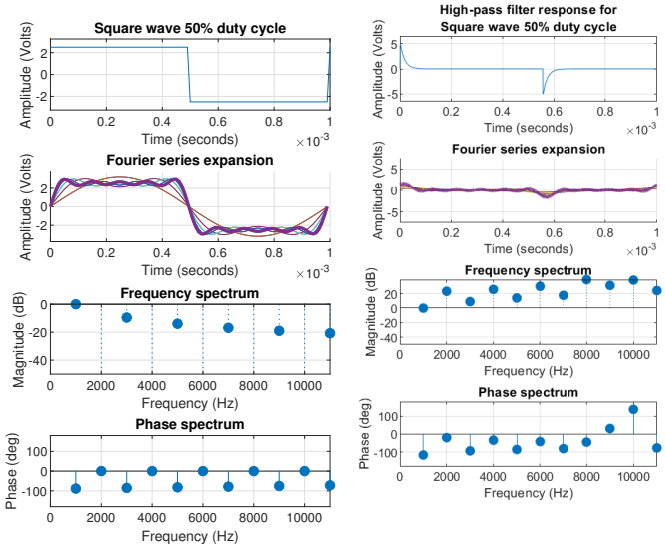


(c) Measured signal

Figure 10: Low-pass filter response for Triangle wave 40% symmetry ratio

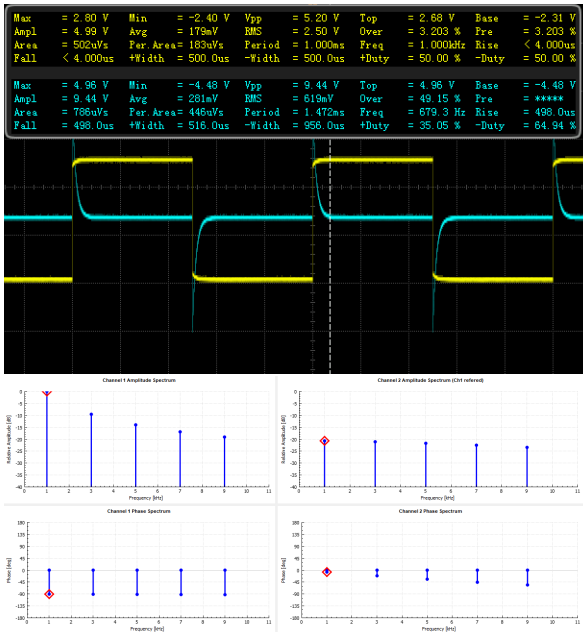
5.2.2 High-pass filter

Moving to High-pass filter we measured the same signals as in Low-pass filter. Starting with the square wave. Looking at amplitudes we can see that only small frequency harmonics are greatly impacted by order of -20dB which is around 0.1 of original amplitude, decrease of the amplitudes of harmonics eases off with increasing frequency.



(a) Pure signal

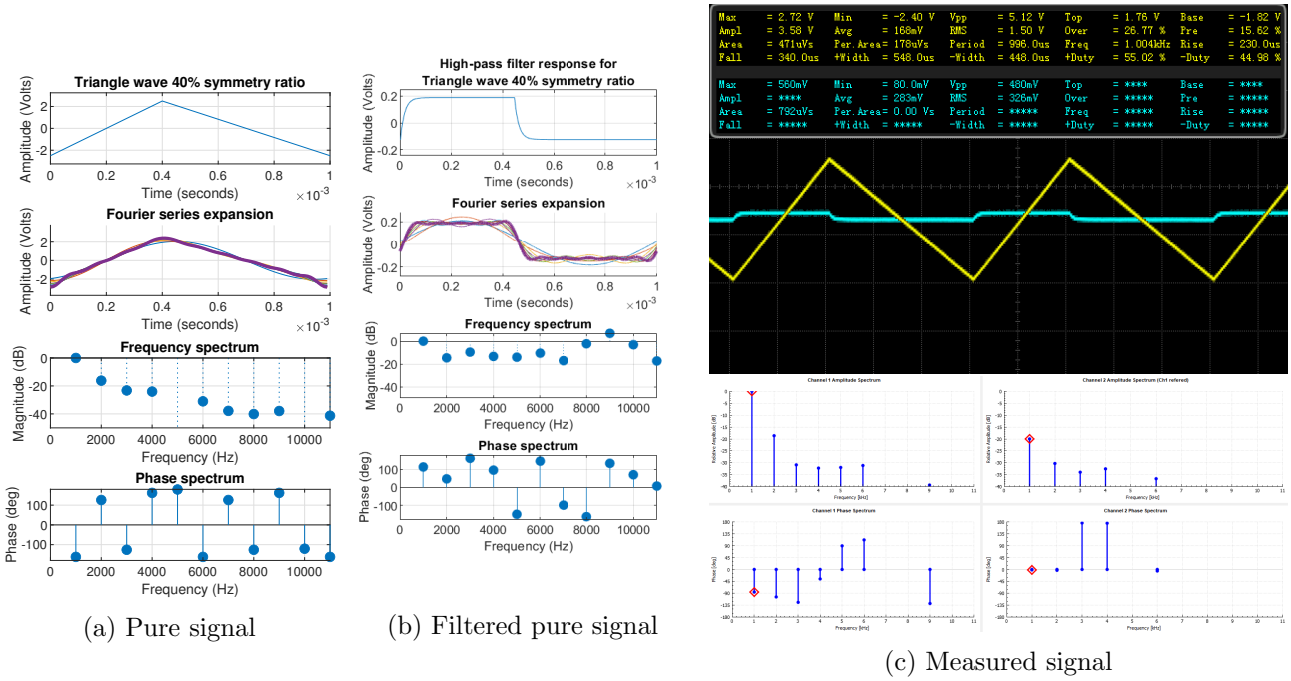
(b) Filtered pure signal



(c) Measured signal

Figure 11: Low-pass filter response for Square wave 50% duty cycle

Moving to last of the signals which is Triangle wave with 40% symmetry ratio. Similar to previous case the biggest difference is in first harmonic with lowest frequency, amplitude is reduced by -20dB another thing is change in phase between filtered and unfiltered first harmonic. Phase of first harmonic change to better match extremes of filtered signal.



(a) Pure signal (b) Filtered pure signal (c) Measured signal

Figure 12: Low-pass filter response for Square wave 50% duty cycle

6 Conclusion

Fourier series is powerful tool in signal analysis. It allows us to split complex signals into much simpler components and point out characteristics of complex periodic signals, as we did in section. 5.1. Splitting signals into simple harmonics also allows us to evaluate effectiveness of for example filters, by comparing Fourier series expansion of filtered and unfiltered signals. Another application of Fourier series which we didn't utilize is splitting the signal into sins and cosines, performing some calculation on simple function and then recombining new sines and cosines to get processed signal.

A Source Code

GITHUB repository

lab2.m

```
clear all; close all; clc;
[Square50_integrator, Square50_differentiatior, Triangle40_integrator, Triangle40_differentiat

fs = 100000;
duration = 0.01;
N = fs * duration;
t = 0:1/fs:duration-1/fs;
f = 1000;
a = 2.119;
phi = 0;

a = 2.5;
phi = 0;

s = a * sin(2*pi*t*f + phi);
tr50 = a * sawtooth(2*pi*t*f + phi, 0.5);
tr40 = a * sawtooth(2*pi*t*f + phi, 0.4);
% tr50 = tr50(size(tr50, 2)/40:size(tr50, 2));

sq50 = a * square(2*pi*t*f + phi, 50);
sq25 = a * square(2*pi*t*f + phi, 25);

Final_Plots(s, 'Sin wave', 'sin');
Final_Plots(tr50, 'Triangle wave 50% symmetry ratio', 'tri50');
Final_Plots(tr40, 'Triangle wave 40% symmetry ratio', 'tri40');
Final_Plots(sq50, 'Square wave 50% duty cycle', 'sqr50');
Final_Plots(sq25, 'Square wave 25% duty cycle', 'sqr25');
Final_Plots(Square50_integrator,['Low-pass filter response for' newline 'Square wave 50% duty
Final_Plots(Square50_differentiatior,['High-pass filter response for' newline 'Square wave 50%
Final_Plots(Triangle40_integrator,['Low-pass filter response for' newline 'Triangle wave 40% s
Final_Plots(Triangle40_differentiatior,['High-pass filter response for' newline 'Triangle wave

close all;
```

Final Plots.m

```
function [] = Final_Plots(f, name, filename)
frequency = 1000;
duration = 10/frequency;
N = size(f,2);
sampling_frequency = N/duration;
t = 0:1/sampling_frequency:duration-1/sampling_frequency;
fig = figure('Name', name);
fig.Position(3:4) = [300 500];

%% signal
subplot(4,1,1);
plot(t, f);
```

```
xlabel('Time (seconds)');
xlim([0,0.001])
ylabel('Amplitude (Volts)');
ylim([-max(f)-(max(f)*0.3),max(f)+(max(f)*0.3)])
title(join(['', name]));
grid on;

%% fourier series
X = t(1:size(t,2)/10);
% f_short = f(1:size(x,2));
L = 1/frequency;
dx = 2*L/(N/10-1);
x = -L:dx:L;
f_short = f(1:size(x,2));
subplot(4,1,2); hold on;
% plot(x, f_short, 'r*');
A0 = (1/L)*sum(f_short.*ones(size(x)))*dx;
fFS = A0/2;
for k=1:10
    A(k) = (1/L)*sum(f_short.*cos(pi*k*x*frequency))*dx;
    B(k) = (1/L)*sum(f_short.*sin(pi*k*x*frequency))*dx;
    fFS = fFS + A(k)*cos(k*pi*x*frequency) + B(k)*sin(k*pi*x*frequency);
    plot(X, fFS);
    pause(.05);
end
plot(X, fFS,'LineWidth',2);
hold off;

xlabel('Time (seconds)');
xlim([0,0.001])
ylabel('Amplitude (Volts)');
ylim([-max(f)-(max(f)*0.5),max(f)+(max(f)*0.5)])
title('Fourier series expansion');
grid on;

%% fft amplitude
% plot
f_fft = fft(f);
f_oneSide = f_fft(1:N/2);
frequencies = sampling_frequency * (0:N/2-1) / N;
f_magnitude = abs(f_oneSide) / (N/2);
f_phase = angle(f_oneSide) * 180/pi;

freq_of_interest = [1000, 2000, 3000, 4000, 5000, 6000, 7000, 8000, 9000, 10000, 11000];
indices = zeros(size(freq_of_interest));
for i = 1:length(freq_of_interest)
    [~, indices(i)] = min(abs(frequencies - freq_of_interest(i)));
end

% subplot(4,1,3)
% plot(frequencies, db(f_magnitude));
% xlabel('Frequency (Hz)');
% xlim([0, 11000]);
```

```
% ylim([-40, 10]);
% ylabel('Magnitude');
% title('Frequency spectrum');

%stem
subplot(4,1,3)
% stem(frequencies(indices), f_magnitude(indices), 'x'); hold on;
stem(frequencies(indices), 20*log10(f_magnitude(indices)/f_magnitude(11)), 'filled', 'LineStyl
xlabel('Frequency (Hz)');
xlim([0, 11000]);
ylabel('Magnitude (dB)');
ylim([-50,max(20*log10(f_magnitude(indices)/f_magnitude(11)))])
title('Frequency spectrum');
grid on;

%% fft phase
subplot(4,1,4)
stem(frequencies(indices), f_phase(indices), 'filled', 'LineStyle', '-');
xlabel('Frequency (Hz)');
xlim([0, 11000]);
ylabel('Phase (deg)');
ylim([-180, 180]);
title('Phase spectrum');
grid on;

%% saving img
print(join(['img/',filename]), '-depsc');
end
```

Import csv.m

```
function [Vout] = RC_circuit(frequency, N)
figure('Name', 'RC circuit');
Vpp = 5;
frequency = 1000;
T = 1/frequency;
R = 1.5e3;
C1 = 47e-4;
C2 = 10;
dt = T/(N-1);
t = 0:dt:T;
Vout = 0*t;
Vout(N/2+1:N) = Vout(N/2+1:N) + Vpp;
Vout(N/2+1:N) = Vout(N/2+1:N) + (-Vpp)*exp(-(1:N/2)/(R*C1));
Vout(1:N) = Vout(1:N) + (Vpp)*exp(-(1:N)/(R*C1));
Vout(1:N) = Vout(1:N) - Vpp/2;
plot(t, Vout);
pause(.1)
end
```