Laplace Transform
Laboratory III

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1 Goal of the exercise

In this exercise, the primary goal is to validate the accuracy and effectiveness of the transient analysis approach using Laplace transformation. This method allows for the examination and understanding of the behavior of electrical circuits during the transient state. By measuring the voltages in both RC and RLC circuits that are subjected to a unit step input, we can observe and analyze the transient response of the circuits. This experimental verification provides valuable insights into the practical application and usefulness of the Laplace transformation technique in electrical circuit analysis.

2 Laplace Transform

Laplace transform is an integral transform that converts a function of a real variable(time domain) into a function of complex variable(frequency domain). It is powerful tool for solving differential equations, which turns ODEs into algebraic equations and convolution into multiplication. For function $x(t)$, the Laplace transform is the integral

$$\mathcal{L}[x(t)] = X(s) = \int_0^\infty x(t)e^{-st}dt \tag{1}$$

Where $s = \sigma + j\omega \quad \sigma, \omega \in \mathbb{R}$

In order to get solution of differential equation solved in s-domain it is necessary to apply inverse Laplace transform which is given by following complex integral

$$f(t) = \mathcal{L}^{-1}[X(s)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{-st}ds \tag{2}$$

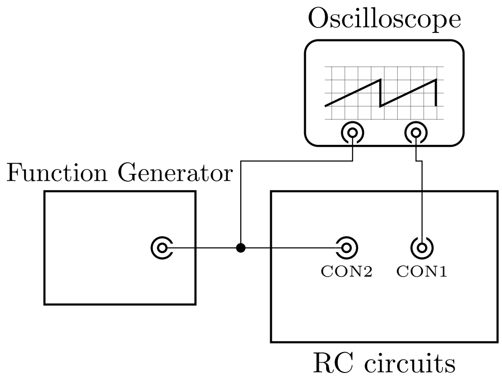
Integrals like these can be quite difficult to solve, that is why lookup table of Laplace transforms and properties can be very handy.

Moving to solving circuits using Laplace transform, there are two approaches, first is constructing differential equation in time domain describing circuit and then solve it using Laplace transform, or second option very similar to solving circuits in steady-state where all components are transformed to s-domain and at the end of calculation transform back to t-domain.

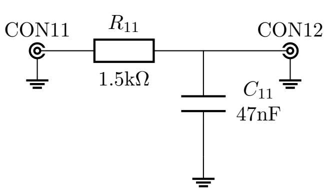
$$R \xrightarrow{\mathcal{L}} R \quad C \xrightarrow{\mathcal{L}} \frac{1}{sC} \quad L \xrightarrow{\mathcal{L}} sL \tag{3}$$

3 Course of measurements

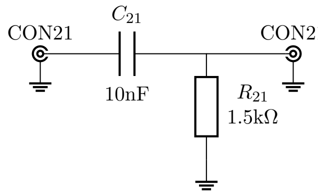
First we tested two circuits: low-pass and high-pass configuration of RC circuit and RLC circuts. After connecting osciloscope to the wave generator and prototype board, we generated square wave with v_{pp} = 1V, v_{offset} = .5V, frequency of 100Hz and duty cycle of 50%. Then we read from the oscilloscope Voltage value at times 1τ, 5τ and 10τ (where 10τ is just as fail-safe) and time when voltage reaches 10% and 90% of the highest value.



(a) Measurement setup



(b) Circuit A

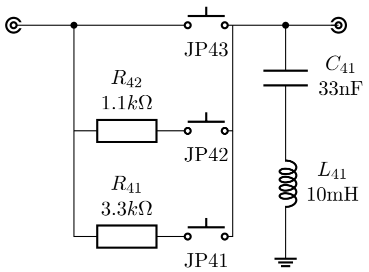


(c) Circuit B

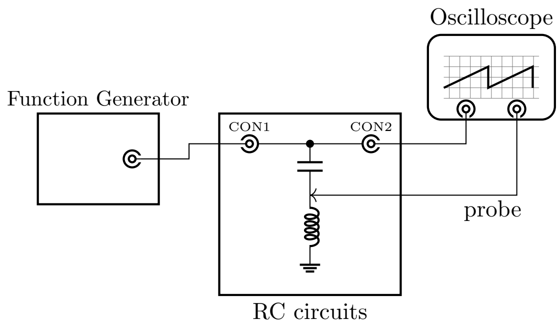
Figure 1: RC Circuits

For RLC circuits we tested two different resistances before RC circuit influence output characteristic. We measured response for 2 cases: Generator out resistance (50 Ω) + resistance selected by jumper wire. In our case we tested jumper on 1.1kΩ resistance path and 3.3kΩ. After that we checked response of the circuit:

- if response was sinusoid with decreasing amplitude - resustance was smaller then RC
- if response was exp. decay if R was higher
- if response was aperiodic critical waveform if resistance was equal RC



(a) Measured Circuit



(b) Measurements setup

Figure 2: RLC circuit

4 Theoretical calculations

For all calculations we used Matlab with Symbolic Math Toolbox. Source code can be found in Appendix. A

In all three circuit input voltage was square wave with 50% duty cycle which can be described in time domain by

$$v_{in}(t) = V_{offset}\mathbf{1}(t) + V_{pp}\mathbf{1}(t - \frac{T}{2}) - V_{pp}\mathbf{1}(t - T) \tag{4}$$

and in frequency domain by

$$V_{in}(s) = \mathcal{L}[v_{in}(t)] = \frac{V_{offset}}{s} + \frac{e^{-\frac{T}{2}s}}{s} - \frac{e^{-Ts}}{s} \tag{5}$$

4.1 Circuit A

First we need to transform circuit to s-domain according to (3)

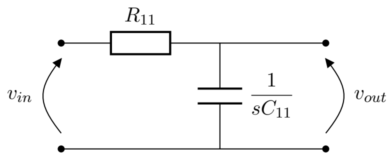
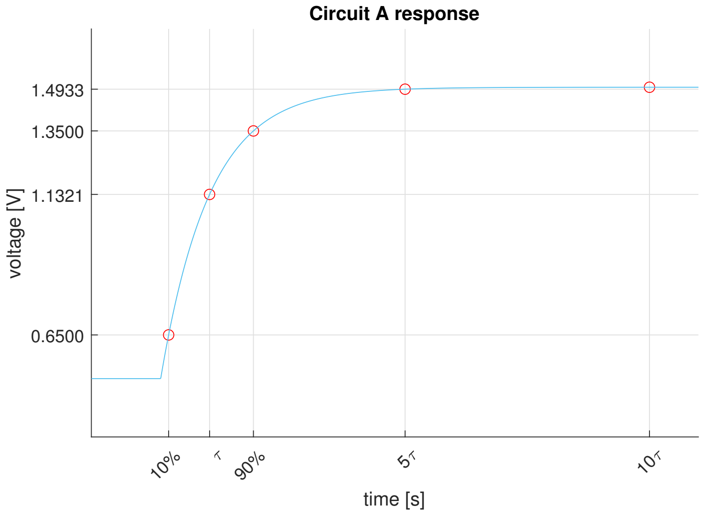


Figure 3: Circuit A schematics

output voltag of circuit A can be described using simple voltage divider

$$V_{out}(s) = V_{in}(s)\frac{\frac{1}{sC_{11}}}{R_{11} + \frac{1}{sC_{11}}} \tag{6}$$

after applying inverse Laplace transform to $v_{out}(s)$ we obtain below plot with marked τ, 5τ, 10τ, 10% and 90% of output voltage.



(a) plot

	time [μs]	voltage [V]
τ	70	1.132
5τ	352	1.493
10τ	705	1.499
10%	11	0.649
90%	133	1.349

(b) table of values

Figure 4: Circuit A output voltage

4.2 Circuit B

steps in circuit B are almost identical to circuit A, first we transformed circuit to s-domain according to (3)

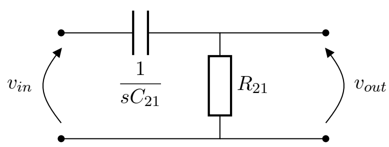
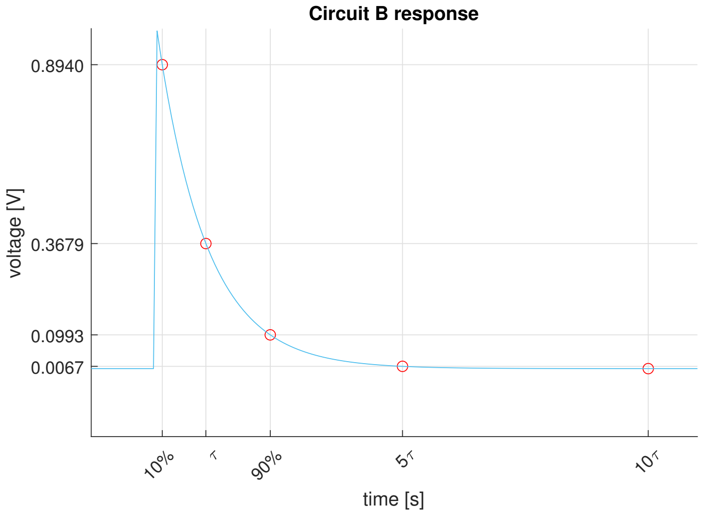


Figure 5: Circuit B schematics

output voltage can be described by voltage divider

$$V_{out}(s) = V_{in}(s)\frac{R_{21}}{R_{21} + \frac{1}{sC_{21}}} \tag{7}$$

after applying inverse Laplace transform to $V_{out}(s)$ we obtain below plot with marked τ, 5τ, 10τ, 10% and 90% of output voltage.



(a) plot

	time [μs]	voltage [V]
τ	15	0.367
5τ	75	0.007
10τ	150	0.000
10%	2	0.894
90%	35	0.099

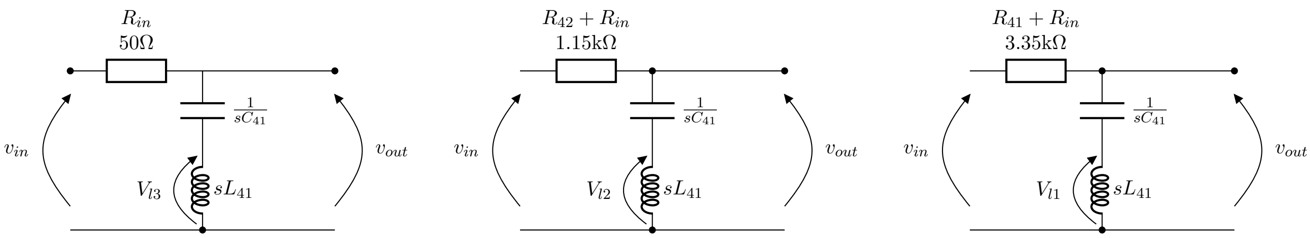
(b) table of values

Figure 6: Circuit B output voltage

4.3 Circuit C

In third circuit we are also looking for voltage of coil L_{21}.

Like in previous 2 circuits we transformed circuit to s-domain according to (3)



(a) JP43 (b) JP42 (c) JP41

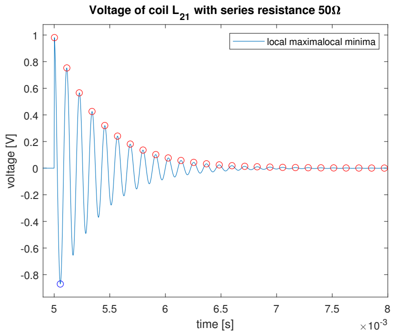
Figure 7: Circuit C schematics

Then solved each circuit for $V_l(s)$ and $V_{out}(s)$

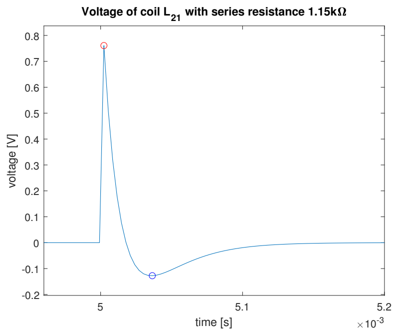
$$V_l(s) = V_{in}(s)\frac{sL_{41}}{R_x + \frac{1}{sC_{21}} + sL_{21}} \tag{8}$$

$$V_{out}(s) = V_{in}(s)\frac{\frac{1}{sC_{41}} + sL_{41}}{R_x + \frac{1}{sC_{41}} + sL_{41}} \tag{9}$$

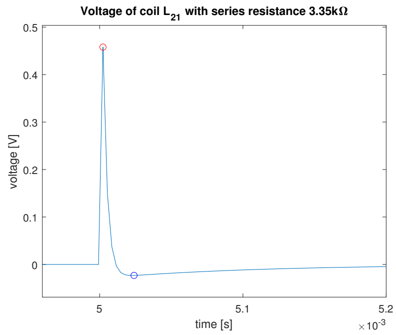
After plugging in value of each component where R_x is series resistance of the circuit, we can transform equation back to time domain and plot results.



(a) Plot JP43



(b) Plot JP42



(c) Plot JP41

	voltage [V]	time [μs]
v_{max}	0.98	2
v_{min}	-0.87	55
steady-state		1939
no. of oscillations		18

(d) Table of values JP43

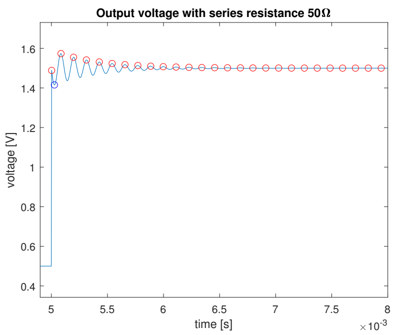
	voltage [V]	time [μs]
v_{max}	0.76	2
v_{min}	-0.127	36
steady-state		117

(e) Table of values JP42

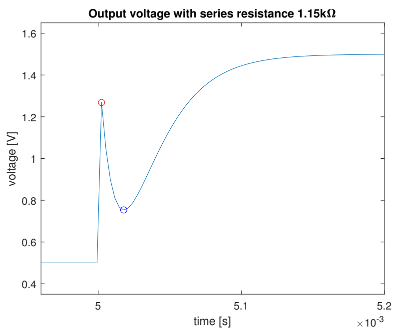
	voltage [V]	time [μs]
v_{max}	0.45	2
v_{min}	-0.023	24
steady-state		117

(f) Table of values JP41

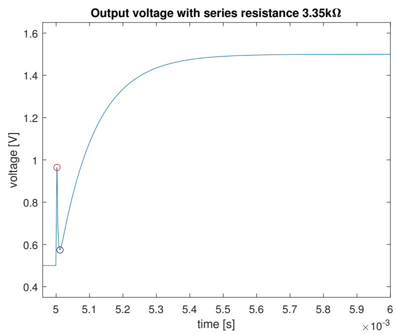
Figure 8: Coil Voltage



(a) Plot JP43



(b) Plot JP42



(c) Plot JP41

	voltage [V]	time [μs]
v_{max}	1.57	86
v_{min}	1.41	27
steady-state		885
no. of oscillations		9

(d) Table of values JP43

	voltage [V]	time [μs]
v_{max}	1.26	2
v_{min}	0.75	17
steady-state		144

(e) Table of values JP42

	voltage [V]	time [μs]
v_{max}	0.96	2
v_{min}	0.57	11
steady-state		501

(f) Table of values JP41

Figure 9: Output voltage

5 Comparison

comparing measurements and calculations results are not very precise in every circuit, most likely by error in measurements(different offset setting) but for some circuits results differ in 3rd decimal place.

	time [μs]	voltage [V]
τ	70	1.132
5τ	352	1.493
10τ	705	1.499
10%	11	0.649
90%	133	1.349

(a) calculated values

	time [μs]	voltage [V]
τ	64	0.632
5τ	225	0.997
10τ	450	1.000
10%	4	0.010
90%	140	0.896

(b) measured values

Figure 10: Circuit A

	time [μs]	voltage [V]
τ	15	0.367
5τ	75	0.007
10τ	150	0.000
10%	2	0.894
90%	35	0.099

(a) calculated values

	time [μs]	voltage [V]
τ	15	0.368
5τ	73	0.008
10τ	145	0.000
10%	1	0.904
90%	35	0.100

(b) measured values

Figure 11: Circuit B

	voltage [V]	time [μs]
v_{max}	0.98	2
v_{min}	-0.87	55
steady-state		1939
no. of oscillations		18

(a) calculated coil voltage

	voltage [V]	time [μs]
v_{max}	1.02	
v_{min}	-0.82	
steady-state		
no. of oscillations		

(b) measured coil voltage

	voltage [V]	time [μs]
v_{max}	1.57	86
v_{min}	1.41	27
steady-state		885
no. of oscillations		9

(c) calculated output voltage

	voltage [V]	time [μs]
v_{max}	1.120	
v_{min}	0.940	
steady-state		1160
no. of oscillations		10

(d) measured output voltage

Figure 12: Circuit C JP43

	voltage [V]	time [μs]
v_{max}	0.76	2
v_{min}	-0.127	36
steady-state		117

(a) calculated coil voltage

	voltage [V]	time [μs]
v_{max}	0.920	
v_{min}	-0.100	
steady-state		

(b) measured coil voltage

	voltage [V]	time [μs]
v_{max}	1.26	2
v_{min}	0.75	17
steady-state		144

(c) calculated output voltage

	voltage [V]	time [μs]
v_{max}	0.880	
v_{min}	0.280	
steady-state		

(d) measured output voltage

Figure 13: Circuit C JP42

	voltage [V]	time [μs]
v_{max}	0.96	2
v_{min}	0.57	11
steady-state		501

(a) calculated coil voltage

	voltage [V]	time [μs]
v_{max}	0.721	
v_{min}	0.000	
steady-state		

(b) measured coil voltage

	voltage [V]	time [μs]
v_{max}	0.45	2
v_{min}	-0.023	24
steady-state		117

(c) calculated output voltage

	voltage [V]	time [μs]
v_{max}	0.688	
v_{min}	0.064	
steady-state		

(d) measured output voltage

Figure 14: Circuit C JP41

6 Conclusions

In conclusion Laplace transform in solving circuits is excellent method which gives very accurate results, and works on algebraic level which mean that it can be calculated by a computer as we did in this exercise. Ability to simulate circuits using computer may be not necessary for small circuits but for more complex ones it is unavoidable.

A Appendix

GITHUB repository

```
clear all; close all; clc;


%% Square wave
syms t s square
f = 100;
T = 1/f;
w = 2 * pi * f;
amplitude = 1;
offset = 0.5;
square = amplitude *heaviside(t - T/2) - amplitude *heaviside(t - T) + offset * heaviside(t);


%% Circuit A
C1 = 47e-9;
R = 1.5e3;
response1(t) = ilaplace(laplace(square) * (1/(s*C1))/(R + 1/(s*C1)));
steps = 1000;
step = (0.6*T-0.49*T)/steps;
t = 0.49*T:step:0.6*T;
x = response1(t);
tauA = 1.5e3 * 47e-9;
tauA1 = tauA;
tauA5 = 5*tauA;
tauA10 = 10*tauA;
percent10 = 0.1*max(x) + offset;
percent90 = 0.9*max(x);
invresponse = finverse(response1);
xpercent10 = double(invresponse(percent10));
xpercent90 = double(invresponse(percent90));
% figures
fig = figure('Name', 'Circuit A');
fig.Position(3:4) = [600 400];
grid on;
hold on;
plot(T/2+tauA1, response1(T/2+tauA1), 'ro')
plot(T/2+tauA5, response1(T/2+tauA5), 'ro')
plot(T/2+tauA10, response1(T/2+tauA10), 'ro')
plot(xpercent10, percent10, 'ro')
plot(xpercent90, percent90, 'ro')
xticks([xpercent10, T/2+tauA1, xpercent90, T/2+tauA5, T/2+tauA10]);
xticklabels({'10%' '\tau' '90%' '5\tau' '10\tau'})
yticks(sort(double([response1(T/2+tauA1) percent10 percent90 response1(T/2+tauA5)]),2, "ascend
plot(t, x)
ylim([offset-0.2, amplitude+offset+0.2])
xlim([0.49*T, T/2 + 1.1*tauA10])
xtickangle(45)
title('Circuit A response')
xlabel('time [s]')
ylabel('voltage [V]')
```

```
print('img/CircuitA','-depsc')

%% Circuit B
C2 = 10e-9;
R = 1.5e3;
response1(t) = ilaplace(laplace(square) * R/(R + 1/(s*C2)));
steps = 1000;
step = (0.6*T-0.49*T)/steps;
t = 0.49*T:step:0.6*T;
x = response1(t);
tauA = 1.5e3 * 10e-9;
tauA1 = tauA;
tauA5 = 5*tauA;
tauA10 = 10*tauA;
percent10 = double(0.1*max(x))
percent90 = double(0.9*max(x))
invresponse = finverse(response1);
xpercent10 = double(invresponse(percent10))
xpercent90 = double(invresponse(percent90))
xpercent90 - T/2
xpercent10 - T/2
% figures
fig = figure('Name', 'Circuit B');
fig.Position(3:4) = [600 400];
grid on;
hold on;
plot(T/2+tauA1, response1(T/2+tauA1), 'ro')
plot(T/2+tauA5, response1(T/2+tauA5), 'ro')
plot(T/2+tauA10, response1(T/2+tauA10), 'ro')
plot(xpercent10, percent10, 'ro')
plot(xpercent90, percent90, 'ro')
xticks(sort([xpercent10, T/2+tauA1, xpercent90, T/2+tauA5, T/2+tauA10],2,"ascend"));
xticklabels({'10%' '\tau' '90%' '5\tau' '10\tau'})
yticks(sort(double([response1(T/2+tauA1) percent10 percent90 response1(T/2+tauA5)]),2, "ascend
plot(t, x)
ylim([-0.2, amplitude])
xlim([0.498*T, T/2 + 1.1*tauA10])
xtickangle(45)
title('Circuit B response')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitB','-depsc')

%% Circuit C
R3 = 50;
R2 = 1.15e3;
R1 = 3.35e3;
C = 33e-9;
L = 10e-3;
response33(t) = ilaplace(laplace(square) * (s*L)/(R3 + 1/(s*C)+s*L));
response32(t) = ilaplace(laplace(square) * (s*L)/(R2 + 1/(s*C)+s*L));
response31(t) = ilaplace(laplace(square) * (s*L)/(R1 + 1/(s*C)+s*L));
response33out(t) = ilaplace(laplace(square) * (s*L + 1/(s*C))/(R3 + 1/(s*C)+s*L));
```

```
response32out(t) = ilaplace(laplace(square) * (s*L + 1/(s*C))/(R2 + 1/(s*C)+s*L));
response31out(t) = ilaplace(laplace(square) * (s*L + 1/(s*C))/(R1 + 1/(s*C)+s*L));
steps = 1000;
step = (0.8*T-0.49*T)/steps;
t = 0.49*T:step:0.8*T;

% R3
x = response33(t);
figure('Name','Circuit C3')
TF = islocalmax(double(x));
TF(5) = 0;
MF = islocalmin(double(x), 'MaxNumExtrema', 1);
plot(t, x, t(TF), x(TF), 'ro', t(MF), x(MF), 'bo')
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.49*T 0.8*T])
title('Voltage of coil L_{21} with series resistance 50\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC3','-depsc')

x = response33out(t);
figure('Name','Circuit C3')
TF = islocalmax(double(x));
TF(5) = 0;
MF = islocalmin(double(x));
plot(t, x, t(TF), x(TF), 'ro', 0.0050271, 1.41531, 'bo')
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.49*T 0.8*T])
title('Output voltage with series resistance 50\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC3out','-depsc')

% R2
x = response32(t);
figure('Name','Circuit C2')
TF = islocalmax(double(x), 'MaxNumExtrema', 10);
TF(5) = 0;
MF = islocalmin(double(x));
plot(t, x, t(TF), x(TF), 'ro', t(MF), x(MF), 'bo')
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.52*T])
title('Voltage of coil L_{21} with series resistance 1.15k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC2','-depsc')

x = response32out(t);
figure('Name','Circuit C2')
TF = islocalmax(double(x));
TF(5) = 0;
MF = islocalmin(double(x));
plot(t, x, t(TF), x(TF), 'ro', t(MF), x(MF), 'bo')
```

```
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.52*T])
title('Output voltage with series resistance 1.15k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC2out','-depsc')

% R1
x = response31(t);
figure('Name','Circuit C1')
TF = islocalmax(double(x));
TF(5) = 0;
MF = islocalmin(double(x), 'MaxNumExtrema', 1);
plot(t, x, t(TF), x(TF), 'ro', t(MF), x(MF), 'bo')
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.52*T])
title('Voltage of coil L_{21} with series resistance 3.35k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC1','-depsc')

x = response31out(t);
figure('Name','Circuit C1')
TF = islocalmax(double(x));
TF(5) = 0;
MF = islocalmin(double(x), 'MaxNumExtrema', 1);
plot(t, x, t(TF), x(TF), 'ro', t(MF), x(MF), 'bo')
ylim([double(min(x)-0.1*max(x)), double(max(x)+0.1*(max(x)))])
xlim([0.496*T 0.6*T])
title('Output voltage with series resistance 3.35k\Omega')
xlabel('time [s]')
ylabel('voltage [V]')
print('img/CircuitC1out','-depsc')
```