Assignment No. 6: **Multirate Signal Processing**

Reg. No. : 2016-EE-189

Task 1

Statement:

Consider the transfer function of a notch filter,

$$H(z) = \frac{1+a}{2} \frac{\left(z - e^{j\omega_n}\right)\left(z - e^{-j\omega_n}\right)}{\left(z - a\, e^{j\omega_n}\right)\left(z - a\, e^{-j\omega_n}\right)}$$

where ω_n is the frequency to be stopped. This system has two zeros (at ω_n and $-\omega_n$). Sampling rate = F_s.

Question (1):

Read $\mathrm{almostcaught\_high.wav}$, and examine using DTFT to figure out frequency of tone present in the audio.

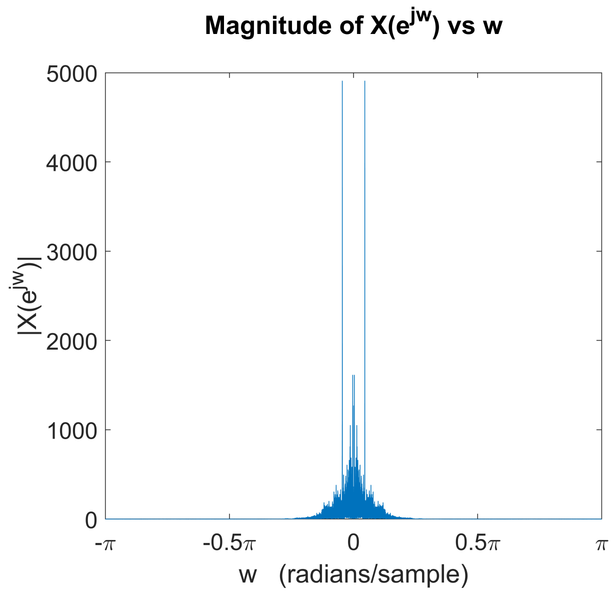
Answer (1):

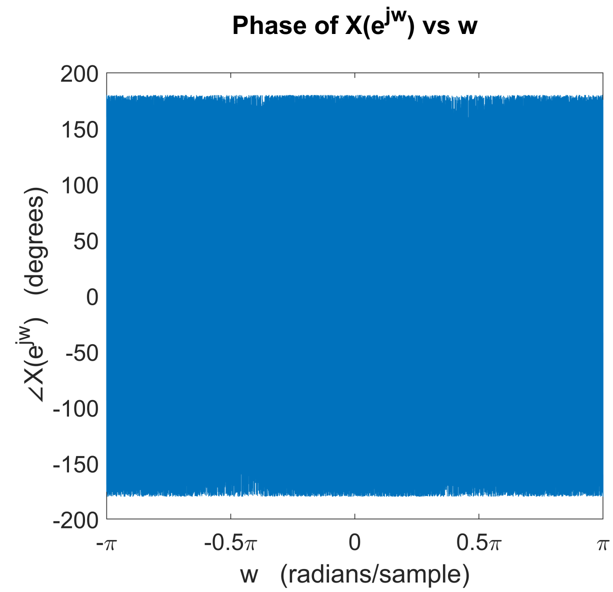
The code, along with the results (inline), to read & compute DTFT of this audio starts below.

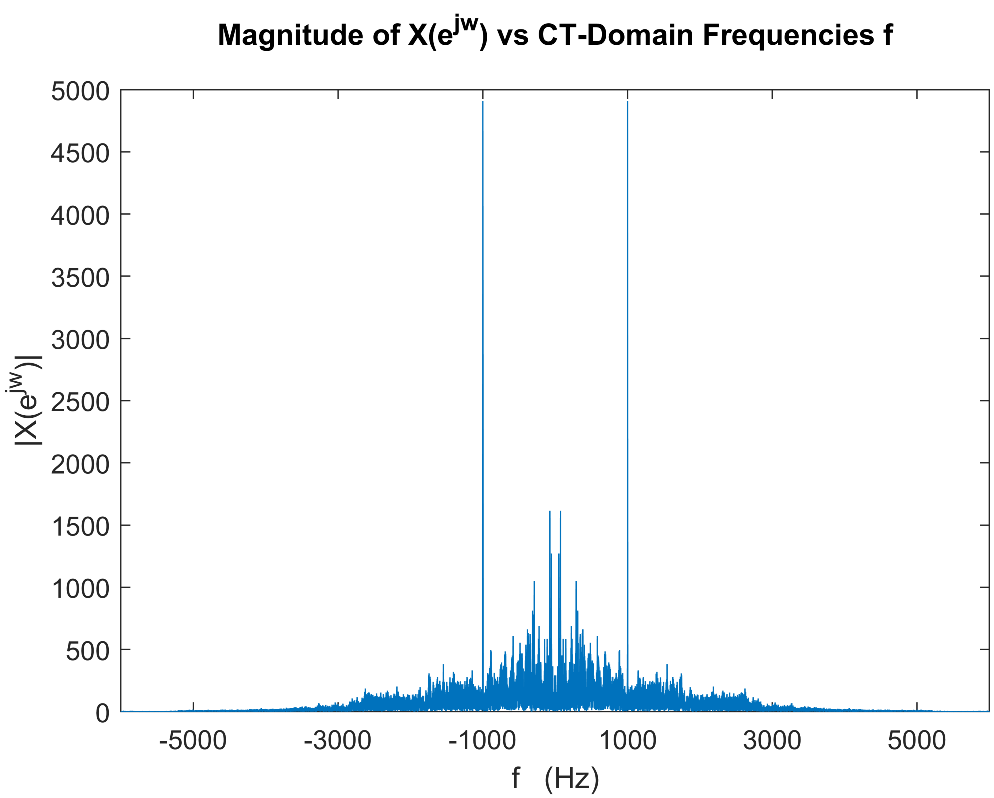
Note: The $\mathrm{dtft\_split}()$, $\mathrm{plotMagPhase}()$, $\mathrm{dnsample}()$, and $\mathrm{sigadd}()$ functions used in different tasks have been defined at end of report.

```
% <task> read and play the provided almostcaught_high.wav file

% read audio data from .wav file (255864×1 double i.e. 1 channel audio)
[x, Fs] = audioread('almostcaught_high.wav');

% get and play the audio x(t) using sound command, using audio's Fs
sound(x, Fs);                                   pause((length(x)-1)/Fs);

% <task> plot DTFT of sound data

% initialize frequency vector (and other vars) for computing DTFT X(w)
w = -pi:2*pi/(length(x)-1):pi;                  % for 1 period [-pi, pi] or DTFT
f = Fs * w/(2*pi);                              % corresponding CT-domain frequencies
x = x.';                                        % convert x[n] to row-vector
n = 0:(length(x)-1);                            % assuming audio starts at n=0

% compute DTFT X(w) using (chunk-wise) matrix-multipication method
if isfile("T1_Xw.mat") load("T1_Xw.mat"); end
```

```
if ~exist('Xw','var')
    Xw = dtft_split(x, n, w, 750);          save("T1_Xw.mat",'Xw');
end

% obtain magnitude and phase (in degrees) of DTFT X(w)
Xw_mag = abs(Xw);       Xw_ang = angle(Xw)*180/pi;

% plot mag. and phase of DTFT X(w) vs w, and mag. of X(w) vs CT-domain frequencies f
plotMagPhase(w, Xw_mag, Xw_ang, 'X', f);
```







The sampling frequency F_s of the given audio sequency is $44100 \text{ samples/sec}$. This means that the range $\omega = [-\pi\ ,\ \pi\ ] \text{ rad/sec}$ maps to the CT frequency-range,

$$f \;=\; \frac{\Omega}{2\pi} \;=\; \left[-\frac{F_s}{2},\ \frac{F_s}{2}\right] \;=\; [-22050,\ 22050]\ \text{Hz}$$

The sharp tone is visible in the DTFT magintude at exactly $f = \pm 1 \text{ kHz}$ (CT) or at $\omega = 0.04535\pi \text{ rad/sec}$ (DT), as,

$$\omega \;=\; \Omega \mathrm{T} \;=\; 2\pi(f)\frac{1}{F_s} \;=\; 2\frac{(1000)}{44100}\pi \;=\; 0.04535\pi \text{ rad/sec}$$

Question (2):

Design a notch filter using mentioned transfer function $H(z)$ and remove the tone.

Answer (2):

As found in Q1, the tone is present at $\omega = 0.04535\pi \text{ rad/sec}$. Hence, this will be the frequency to be stopped (ω_n) by $H(z)$:

$$H(z) \;=\; \frac{1+a}{2}\ \frac{(z - e^{j\,0.04535\pi})(z - e^{-j\,0.04535\pi})}{(z - a\,e^{j\,0.04535\pi})(z - a\,e^{-j\,0.04535\pi})}$$

The code, along with the results, to declare this Notch-Filer and use it to remove the tone, starts below.

```
% <task> identify tone (here, at max amplitude) frequency precisely

% get frequency of max amplitude (tone) present in DTFT of audio
wn = abs(w(find(Xw_mag==max(Xw_mag), 1)));
disp(['The tone is at ωn = ', num2str(round(wn/pi, 5)), 'π.']);
```

```
The tone is at ωn = 0.04535π.
```

```
% <task> design Notch Filter corresponding to wn = tone frequency

% define H(z) symbolically
syms z;
a = 0.75;
Hz = ((1+a)/2) * ((z-exp(1i*wn))*(z-exp(-1i*wn))) / ((z-a*exp(1i*wn))*(z-a*exp(-1i*wn)));

% simplify expression to get unfactored form
Hz = vpa(simplify(Hz), 5)
```

```
Hz =
```

$$\frac{9.0865\text{e+}33\, z^2 - 1.7989\text{e+}34\, z + 9.0865\text{e+}33}{1.0385\text{e+}34\, z^2 - 1.5419\text{e+}34\, z + 5.8413\text{e+}33}$$

This result can be expressed as a fraction of polynomials in terms of z^{-1}, as follows:

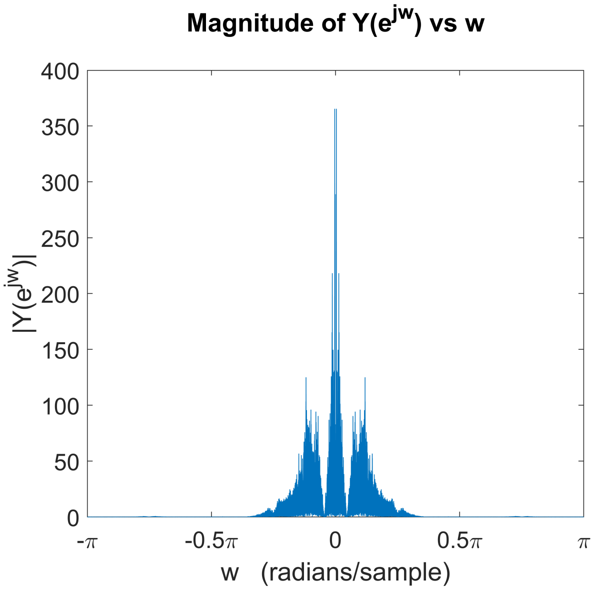
$$\begin{aligned}
H(z) &= \frac{z^{-2}}{z^{-2}} \frac{9.0865 \times 10^{33} z^2 - 1.7989 \times 10^{34} z + 9.0865 \times 10^{33}}{1.0385 \times 10^{34} z^2 - 1.5419 \times 10^{34} z + 5.8413 \times 10^{33}} \\
&= \frac{9.0865 \ - 17.989 \, z^{-1} + 9.0865 \, z^{-2}}{10.385 \ - 15.419 \, z^{-1} + 5.8413 \, z^{-2}} \\
&= \frac{0.875 \ - 1.7323 \, z^{-1} + 0.875 \, z^{-2}}{1 \ - 1.4848 \, z^{-1} + 0.5625 \, z^{-2}} \\
&= \frac{b_0 \, z^0 \ + b_1 \, z^{-1} + b_2 \, z^{-2}}{a_0 \, z^0 + a_1 \, z^{-1} + a_2 \, z^{-2}}
\end{aligned}$$

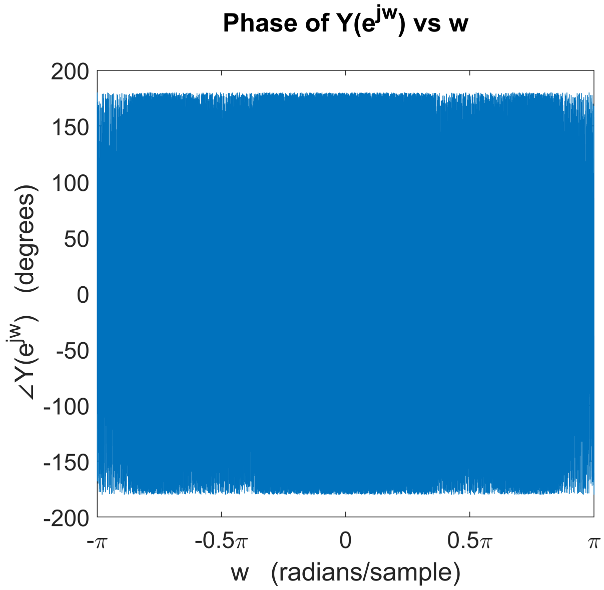
which gives the following CCDE coefficients:

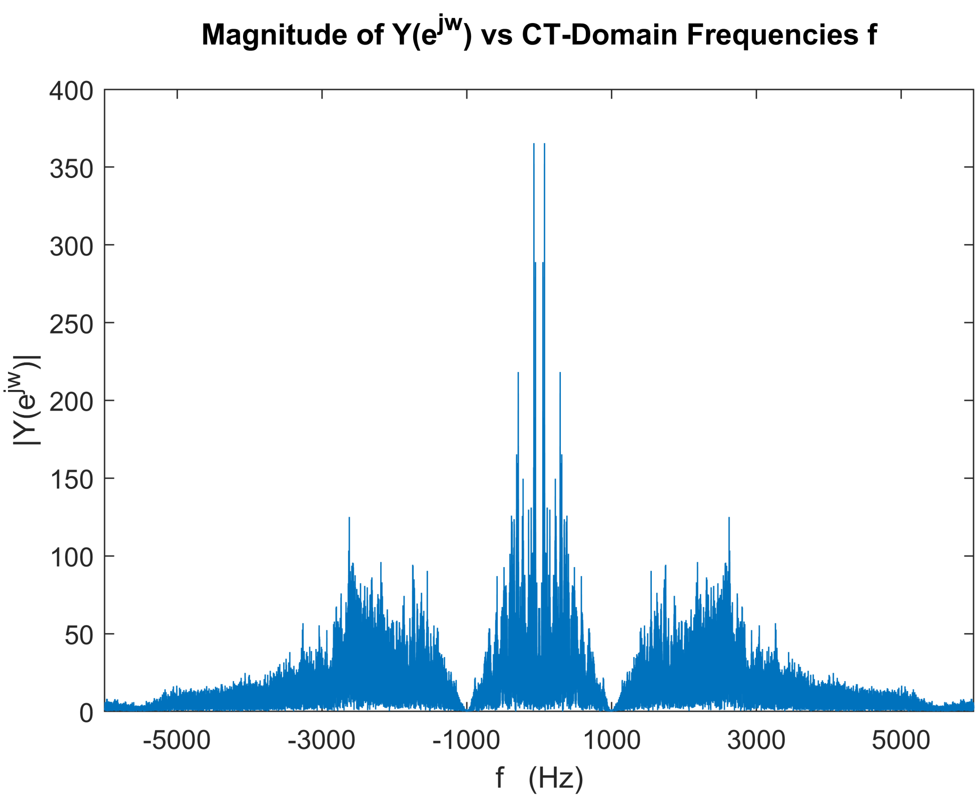
$$\begin{aligned}
b &= \{ 0.875, \ -1.7323, \ 0.875 \} \\
a &= \{ 1, \ -1.4848, \ 0.5625 \}
\end{aligned}$$

```
% define Notch Filter system's CCDE coefficients
b = [0.875 -1.7323 0.875];
a = [1 -1.4848 0.5625];

% <task> remove the tone in x[n] by passing it through Notch Filter

% assuming zero initial conditions (rest)
N_aux = zeros(1, length(a)-1);
M_init = zeros(1, length(b)-1);

% compute system output y[n] for 0 <= n < length(x), with input = sound data x[n],
% using get_system_out()
[y, ny] = get_system_out(b, a, x, N_aux, M_init);
duration = (length(y)-1)/Fs;

% get and play the audio y(t) using sound command, using audio's Fs
sound(y, Fs);
% pause execution while the sound is playing
pause(duration);

% <task> plot DTFT of sound data after passing through system (DTFT of y[n])

% initialize frequency vector (and other vars) for computing DTFT Y(w)
w = -pi:2*pi/(length(y)-1):pi;              % for 1 period [-pi, pi] or DTFT
f = Fs * w/(2*pi);                          % corresponding CT-domain frequencies

% compute DTFT Y(w) using (chunk-wise) matrix-multipication method,
% since this is a large computation, avoid repeating it if already computed
if isfile("T1_Yw.mat") load("T1_Yw.mat"); end
if ~exist('Yw','var')
    Yw = dtft_split(y, ny, w, 750);
    save("T1_Yw.mat",'Xw');
end
```

```
% obtain magnitude and phase of DTFT Y(w)
Yw_mag = abs(Yw);                            % get the magnitude of Y(w)
Yw_ang = angle(Yw)*180/pi;                   % get the phase (angle) of Y(w) in degrees

% plot magnitude and phase of DTFT Y(w) vs w,
% and magnitude of Y(w) vs CT-domain frequencies f
plotMagPhase(w, Yw_mag, Yw_ang, 'Y', f);
```







Note: Compared to the (CT-frequency) plot of $|X(e^{j\omega})|$, this plot of $|Y(e^{j\omega})|$ is zoomed in on the y-axis.

Question (3):

How many multiplications and additions are required per unit time for the filter?

Answer (3):

Given its CCDE coefficients, this Notch Filter system can be represented as,

$$H(z) \;=\; \frac{0.875 - 1.7323\,z^{-1} + 0.875\,z^{-2}}{1 - 1.4848\,z^{-1} + 0.5625\,z^{-2}} \;=\; \frac{Y(z)}{X(z)}$$

which translates to the following CCDE,

$$y[n] - 1.4848y[n-1] + 0.5625y[n-2] \;=\; 0.875x[n] - 1.7323x[n-1] + 0.875x[n-2]$$

$$y[n] \;=\; 1.4848y[n-1] - 0.5625y[n-2] + 0.875x[n] - 1.7323x[n-1] + 0.875x[n-2]$$

Since the output-computing function $\mathrm{get\_system\_out}()$ is essentially using above CCDE representation of the filter to compute output $y[n]$, this CCDE gives us the estimate of multiplications and additions done per sample. Assuming $a_0 = 1$, the multiplication of above R.H.S. with $(1/a_0)$ is not necessary, which leaves us with,

$5\ \mathrm{multiplications}$, and $4\ \mathrm{additions}$, per sample of $y[n]$.

With $F_s = 44100\ \mathrm{samples/sec}$, new samples of $x[n]$ arrive at a rate of $44100\ \mathrm{samples/sec}$, resulting in,

$5 \times 44100 \;=\; 220500\ \mathrm{multiplications}$, and

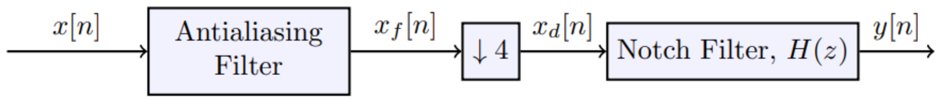
$4 \times 44100 \;=\; 176400\ \mathrm{additions}$ per second.

Next task starts from the next page.

Task 2

Statement:

Now, we'll redo Task 1 but on a decimated version of input signal. Consider the system below:



Question (1):

Design a lowpass FIR filter, using MATLAB's fir1 command, for antialiasing. The length of the filter should be 23.

Answer (1):

The fir1 command has the following syntax,

$$\text{b} = \text{fir1}(n, \omega_n)$$

which uses a Hamming window to design an nth-order lowpass filter when ω_n $(= \omega_{\text{cutoff}}/\pi)$ is a single value. Since we're decimating $x[n]$ by $M = 4$, the AA filter cutoff should be $\pi/M = \pi/4$. The order n would be $(\text{length} - 1) = N - 1$. So we have,

$$\begin{aligned} n &= 23 - 1 = 22 \\ \omega_n &= \frac{\pi/4}{\pi} = \frac{1}{4} \end{aligned}$$

The code, along with the results (inline), to design this antialiasing filter starts below.

```
% <task> design a lowpass FIR filter for antialiasing

M = 4;                                         % decimation factor
AA_N = 23;                                     % AA filter length
AA_n = AA_N-1;                                 % AA filter order
AA_wn = 1/M;                                   % AA filter cutoff/pi

% design low-pass filter for antialiasing using fir(N, wn) command
hlp = fir1(AA_n, AA_wn);
nhlp = 0:(length(hlp)-1);
```

Question (2):

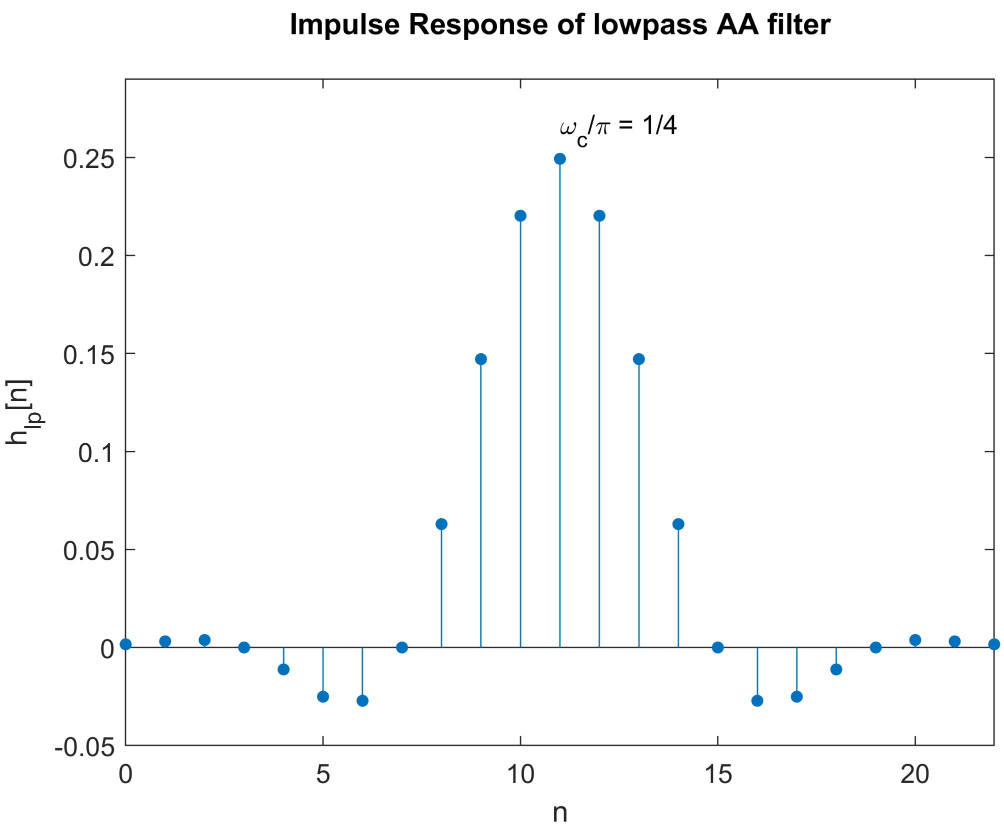
Plot the impulse response as well as the frequency response of the lowpass FIR filter.

Do you recognize the impulse response? Which function does it resemble and why?

Answer (2):

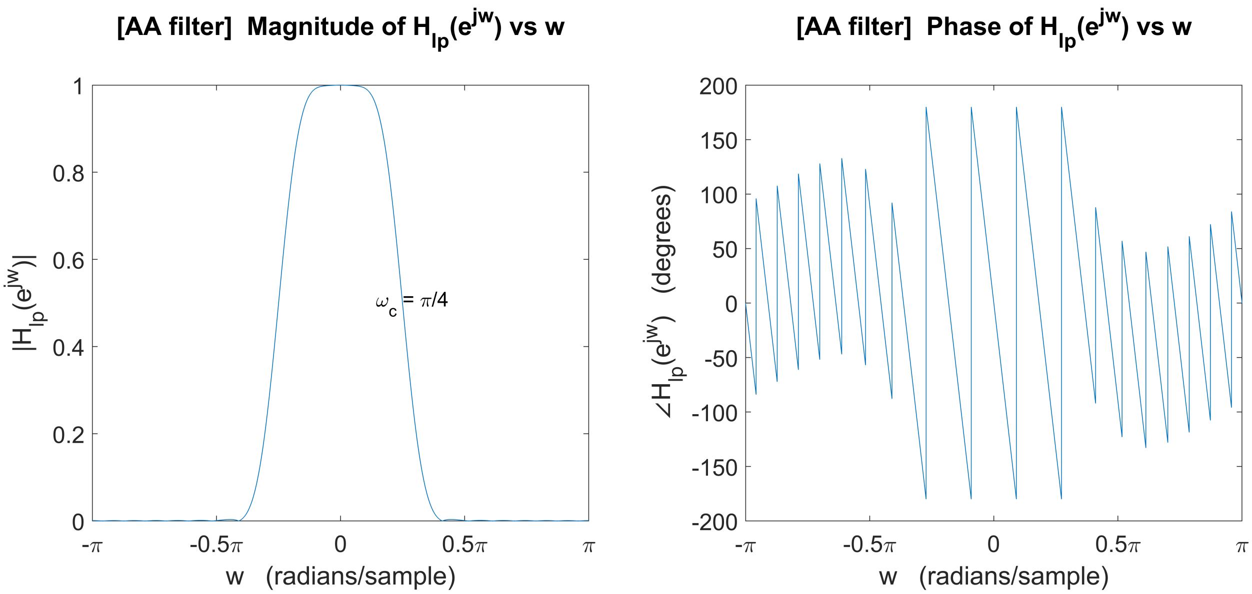
The code to plot the impulse & frequency response of the AA filter starts below.

```
% <task> plot impulse response hlp[n] of the FIR lowpass AA filter

fig = figure;
stem(nhlp, hlp, 'filled', 'MarkerSize', 4);
xlabel('n'); ylabel('h_{lp}[n]');
xlim([min(nhlp) max(nhlp)]);
ylim([-0.05 0.29]);
title({"Impulse Response of lowpass AA filter";''});
text(nhlp(find(hlp==max(hlp), 1)), max(hlp)+0.015, '\omega_c/\pi = 1/4');
```




```
% <task> plot frequency response Hlp(w) of the FIR lowpass AA filter

% initialize frequency vector for computing DTFT Hlp(w)
w = -pi:2*pi/(length(y)-1):pi;              % for 1 period [-pi, pi] or DTFT

% compute DTFT Hlp(w) using matrix-multipication method
Hlpw = hlp * exp(-1i * nhlp' * w);

% obtain magnitude and phase of DTFT Hlp(w)
Hlpw_mag = abs(Hlpw);                       % get the magnitude of Hlp(w)
Hlpw_ang = angle(Hlpw)*180/pi;              % get the phase (angle) of Hlp(w) in degrees
```

```
% plot magnitude and phase of DTFT Hlp(w)
fig = figure; set(fig,'Units','normalized','Position',[0 0 1.4 1]);

% plot magnitude of Hlp(w) vs w
subplot(1, 2, 1);
plot(w, Hlpw_mag);
xlabel('w   (radians/sample)'); ylabel('|H_{lp}(e^{jw})|');
title({'[AA filter]  Magnitude of H_{lp}(e^{jw}) vs w';''}, 'FontSize', 16);
setDTFTradialAxis(1);
text(wn*pi, 0.5, '\omega_c = \pi/4', 'FontSize', 14);
ax = gca; ax.FontSize = 16;

% plot phase of Hlp(w) vs w
subplot(1, 2, 2);
plot(w, Hlpw_ang);
xlabel('w   (radians/sample)'); ylabel('∠H_{lp}(e^{jw})   (degrees)');
title({'[AA filter]  Phase of H_{lp}(e^{jw}) vs w';''}, 'FontSize', 16);
setDTFTradialAxis(1);
ax = gca; ax.FontSize = 16;
```



The impulse response of the lowpass AA filter is a sinc function, given by $\sin(\omega_c n)/\pi n$, which equals ω_c/π for $n = 0$ (but $h_{\text{lp}}[n]$ has been assumed to start at $n = 0$). The Fourier Transform of this sinc function, which gives the Frequency Response of the filter, is,

$$h_{\text{lp}}[n] \;=\; \frac{\sin(\omega_c n)}{\pi\, n} \qquad \Longleftrightarrow \qquad H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

which is a gate (Π) function. This is the typical response of a lowpass filter where frequencies below $\pm\omega_c$ are allowed to pass with gain 1, while higher frequencies are blocked. In our case, $\omega_c = \dfrac{\pi}{M} = \dfrac{\pi}{4}$.

Question (3):

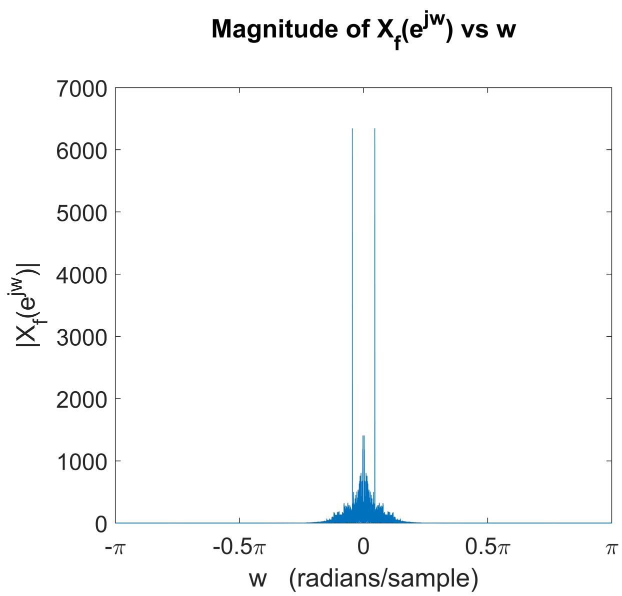
Implement this (decimated) system in MATLAB.

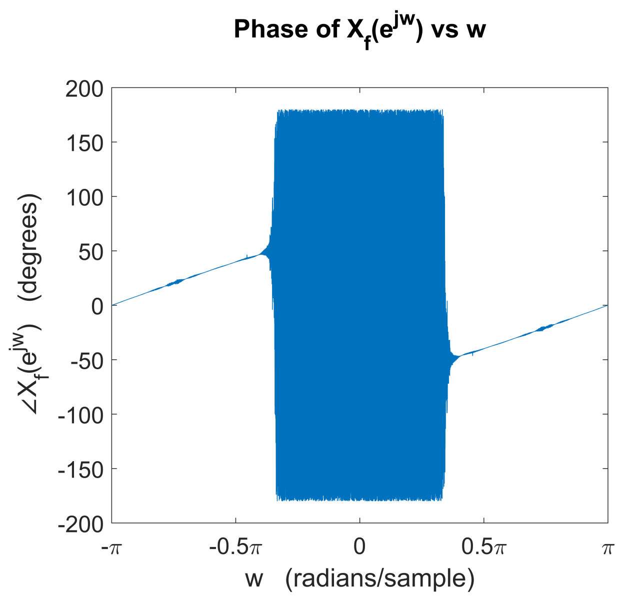
How many additions and multiplications are required for this system per unit time?

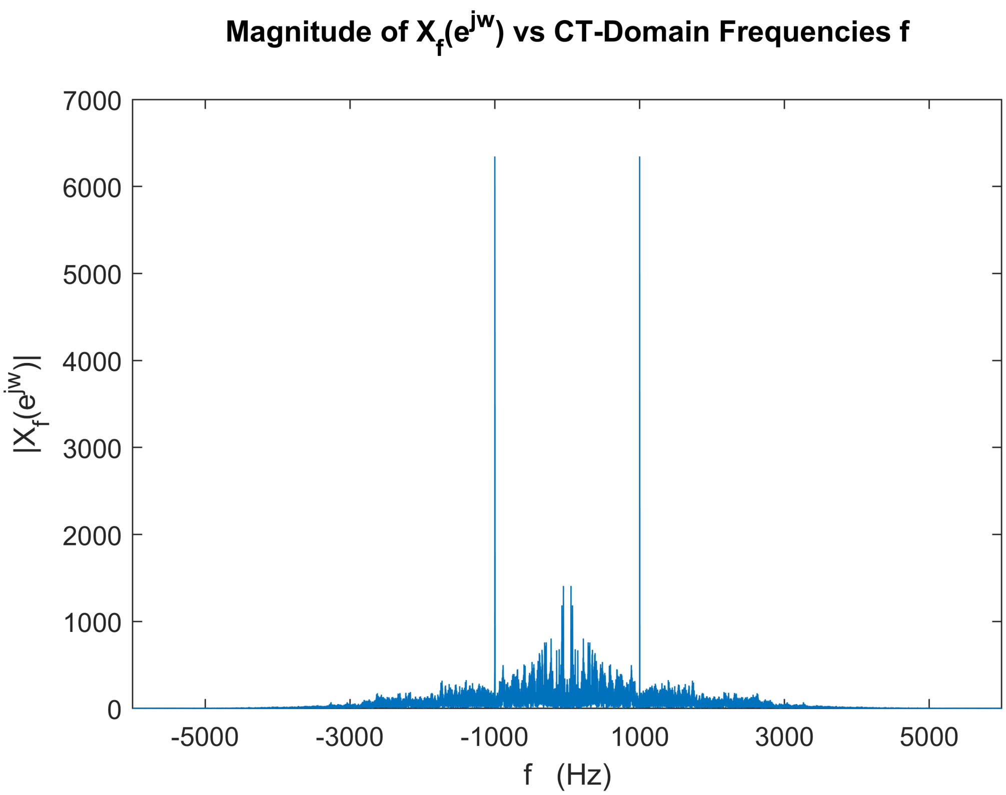
Answer (3):

The code implementing the system with decimated $x[n]$ starts below.

```
% <stage 1> pass x[n] through lowpass AA filter to get xf[n]

% CCDE coefficients for the AA filter
b = hlp;
a = [1];

% assuming zero initial conditions (rest)
N_aux = zeros(1, length(a)-1);
M_init = zeros(1, length(b)-1);

% pass x[n] through hlp[n] to get xf[n],
% using get_system_out()
[xf, nxf] = get_system_out(b, a, x, N_aux, M_init);

% compute and plot DTFT of xf[n]

% initialize frequency vector (and other vars) for computing DTFT X(w)
w = -pi:2*pi/(length(xf)-1):pi;              % for 1 period [-pi, pi] or DTFT
f = Fs * w/(2*pi);                           % corresponding CT-domain frequencies

% compute DTFT Xf(w) using (chunk-wise) matrix-multipication method,
% since this is a large computation, avoid repeating it if already computed
if isfile("T2_Xfw.mat") load("T2_Xfw.mat"); end
if ~exist('Xfw','var')
    Xfw = dtft_split(xf, nxf, w, 750);
    save("T2_Xfw.mat",'Xfw');
end

% obtain magnitude and phase of DTFT Xf(w)
Xfw_mag = abs(Xfw);                          % get the magnitude of Xf(w)
Xfw_ang = angle(Xfw)*180/pi;                 % get the phase (angle) of Xf(w) in degrees

% plot magnitude and phase of DTFT Xf(w) vs w,
% and magnitude of Xf(w) vs CT-domain frequencies f
plotMagPhase(w, Xfw_mag, Xfw_ang, 'X_f', f);
```





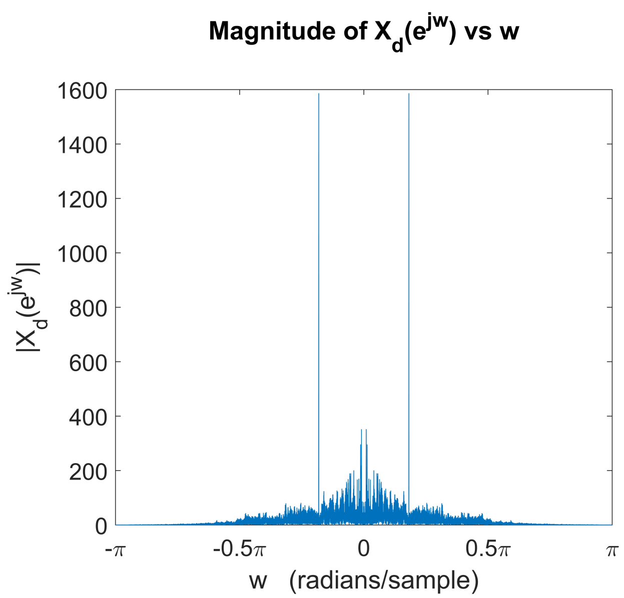


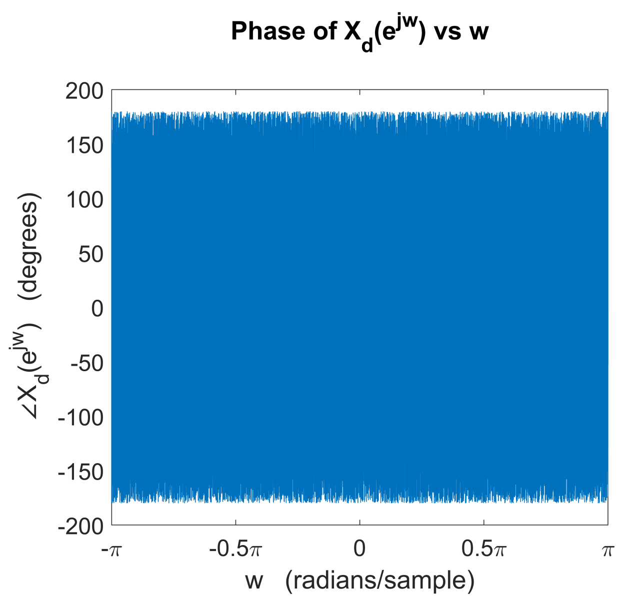
Here, the frequency content higher than,

$$
\begin{aligned}
f &= \pm\frac{\omega_c}{2\pi \mathrm{T}} = \pm\frac{\pi/4}{2\pi(1/44100)} \\
&= \pm 5512.5\ \mathrm{Hz} \\
&\approx \pm 5.5\ \mathrm{kHz}
\end{aligned}
$$

has been removed by the antialiasing filter.

```
% <stage 2> downsample xf[n] by M=4 to get xd[n]

% downsample xf[n] to get xd[n]=xf[nM]
M = 4;                                      % decimation factor
[xd, nxd] = dnsample(xf, nxf, M);

% compute and plot DTFT of xd[n]

% initialize frequency vector (and other vars) for computing DTFT X(w)
w = -pi:2*pi/(length(xd)-1):pi;             % for 1 period [-pi, pi] or DTFT

% compute DTFT Xd(w) using (chunk-wise) matrix-multipication method,
if isfile("T2_Xdw.mat") load("T2_Xdw.mat"); end
if ~exist('Xdw','var')
    Xdw = dtft_split(xd, nxd, w, 750);      save("T2_Xdw.mat",'Xdw');
end

% obtain magnitude and phase (in degrees) of DTFT Xd(w)
Xdw_mag = abs(Xdw);     Xdw_ang = angle(Xdw)*180/pi;

% plot magnitude and phase of DTFT Xd(w) vs w
plotMagPhase(w, Xdw_mag, Xdw_ang, 'X_d');
```





Decimation by 4 is equivalent to multiplying ω axis by 4, shifting by $2\pi i$ ($i = [0, 1, 2, 3]$) to get 4 spectrums, and summing them. As a result, the spectrum plot here, over $[-\pi, \pi]$, has expanded in ω by 4.

Now the tone has shifted to a new DT frequency $\omega = 4(\pm 0.04535\pi) = \pm 0.1814\pi \text{ rad/sec}$. Hence, this will be the new frequency to be stopped (ω_n) by Notch Filter $H(z)$:

$$H(z) = \frac{1+a}{2} \frac{(z - e^{j\,0.1814\pi})(z - e^{-j\,0.1814\pi})}{(z - a\,e^{j\,0.1814\pi})(z - a\,e^{-j\,0.1814\pi})}$$

```
% <stage 3> pass xd[n] through Notch Filter to remove the tone

% identify tone (at max amplitude) freq. precisely from (decimated) audio DTFT Xd(w)
wn = abs(w(find(Xdw_mag==max(Xdw_mag), 1)));
disp(['Now, the tone is at ωn = ', num2str(round(wn/pi, 5)), 'π.']);
```

```
Now, the tone is at ωn = 0.1814π.
```

```
% design Notch Filter corresponding to wn = tone frequency
syms z;
a = 0.75;
Hz = ((1+a)/2) * ((z-exp(1i*wn))*(z-exp(-1i*wn))) / ((z-a*exp(1i*wn))*(z-a*exp(-1i*wn)));
Hz = vpa(simplify(Hz), 5)
```

Hz =

$$\frac{5.6791\text{e+}32\, z^2 - 9.5632\text{e+}32\, z + 5.6791\text{e+}32}{6.4904\text{e+}32\, z^2 - 8.197\text{e+}32\, z + 3.6508\text{e+}32}$$

This result can be expressed as a fraction of polynomials in terms of z^{-1}, as follows:

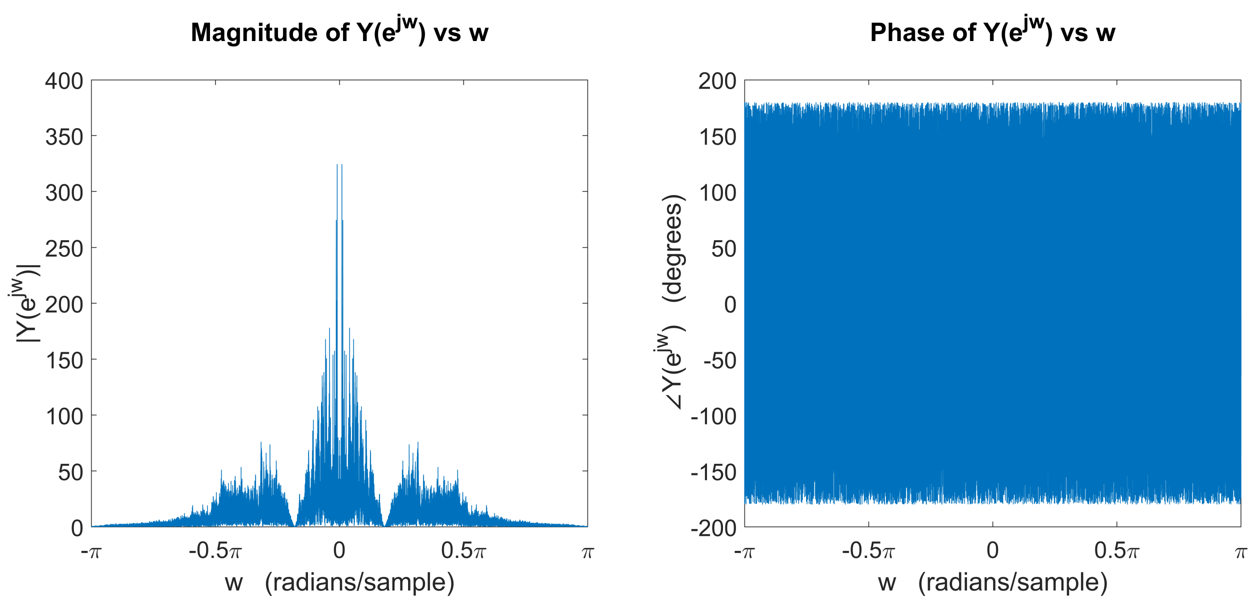
$$\begin{aligned} H(z) &= \frac{z^{-2}}{z^{-2}} \frac{5.6791 \times 10^{32}\, z^2 - 9.5632 \times 10^{32}\, z + 5.6791 \times 10^{32}}{6.4904 \times 10^{32}\, z^2 - 8.197 \times 10^{32}\, z + 3.6508 \times 10^{32}} \\ &= \frac{5.6791\ - 9.5632\, z^{-1} + 5.6791\, z^{-2}}{6.4904\ - 8.197\, z^{-1} + 3.6508\, z^{-2}} \\ &= \frac{0.875\ - 1.4734\, z^{-1} + 0.875\, z^{-2}}{1\ - 1.263\, z^{-1} + 0.5625\, z^{-2}} \\ &= \frac{b_0\, z^0\ + b_1\, z^{-1} + b_2\, z^{-2}}{a_0\, z^0 + a_1\, z^{-1} + a_2\, z^{-2}} \end{aligned}$$

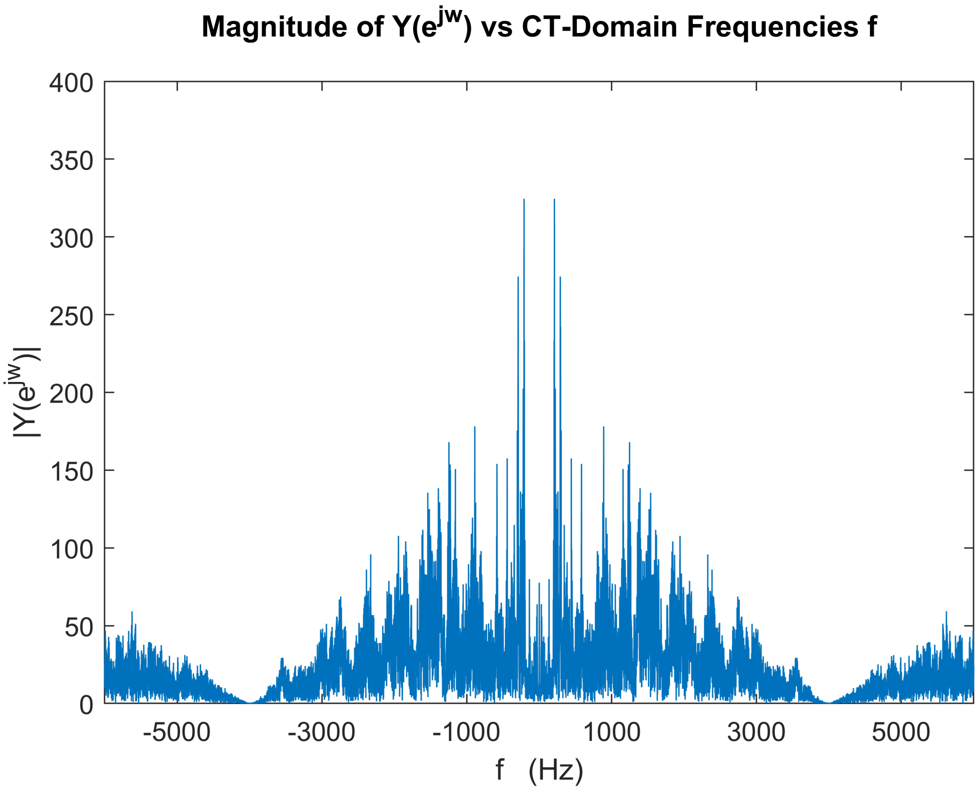
which gives the following CCDE coefficients:

$$\begin{aligned} b &= \{\, 0.875,\ -1.4734,\ 0.875\,\} \\ a &= \{\, 1,\ -1.263,\ 0.5625\,\} \end{aligned}$$

```
% define Notch Filter system's CCDE coefficients
b = [0.875 -1.4734 0.875];
a = [1 -1.263 0.5625];

% assuming zero initial conditions (rest)
N_aux = zeros(1, length(a)-1);
M_init = zeros(1, length(b)-1);

% compute system output y[n], with input = decimated sound data xd[n],
% using get_system_out()
[y, ny] = get_system_out(b, a, xd, N_aux, M_init);
duration = (length(y)-1)/Fs;

% get and play the audio y(t) using sound command, using Fs/M
sound(y, Fs/M);
```

```
% pause execution while the sound is playing
pause(duration);

% plot DTFT of decimated sound data after passing through system (DTFT of y[n])

% initialize frequency vector (and other vars) for computing DTFT Y(w)
w = -pi:2*pi/(length(y)-1):pi;                 % for 1 period [-pi, pi] or DTFT
f = Fs * w/(2*pi);                             % corresponding CT-domain frequencies

% compute DTFT Y(w) using (chunk-wise) matrix-multipication method,
% since this is a large computation, avoid repeating it if already computed
if isfile("T2_Yw_.mat") load("T2_Yw_.mat"); end
if ~exist('Yw_','var')
    Yw_ = dtft_split(y, ny, w, 750);
    save("T2_Yw_.mat",'Yw_');
end

% obtain magnitude and phase of DTFT Y(w)
Yw_mag_ = abs(Yw_);                            % get the magnitude of Y(w)
Yw_ang_ = angle(Yw_)*180/pi;                   % get the phase (angle) of Y(w) in degrees

% plot magnitude and phase of DTFT Y(w) vs w,
% and magnitude of Y(w) vs CT-domain frequencies f
plotMagPhase(w, Yw_mag_, Yw_ang_, 'Y', f, [0 400]);
```





Computations for AA Filter: The length of the decimation filter $h_{\mathrm{lp}}[n]$ is $N = 23$. At $h_{\mathrm{lp}}[n]$ filter stage, the sample arrival rate is $F_s = 44100\ \mathrm{samples/sec}$. Given the AA filter's CCDE coefficients $b = \{b_0, b_1, \ldots, b_{22}\}$ and $a = \{1\}$, it will have a CCDE like,

$$x_f[n] = \sum_{m=0}^{22} b_m\, x[n]$$

which is essentially the form used by $\mathrm{get\_system\_out}()$ to compute $x_f[n]$. This has a total of 23 multiplications, and 22 additions, per sample of $x_f[n]$, giving a total of,

$$23 \times 44100 = 1014300\ \mathrm{multiplications}, \quad \text{and}$$

$$22 \times 44100 = 970200\ \mathrm{additions} \quad \text{per second.}$$

Computations for Notch Filter: As discussed in Task 1 Q3, the Notch Filter requires about 5 multiplications, and 4 additions, per sample of $y[n]$. But after decimation, the sample arrival rate of $x_d[n]$ at the Notch Filter is $F_s/M = 44100/4 = 11025\ \mathrm{samples/sec}$. This leaves us with a total of,

$$5 \times 11025 = 55125\ \mathrm{multiplications}, \quad \text{and}$$

$$4 \times 11025 = 44100\ \mathrm{additions} \quad \text{per second.}$$

which is a reduction by a factor of 4 compared to the undecimated implementation.

Question (4):

Implement an equivalent polyphase-decomposition based decimation system for the given system.

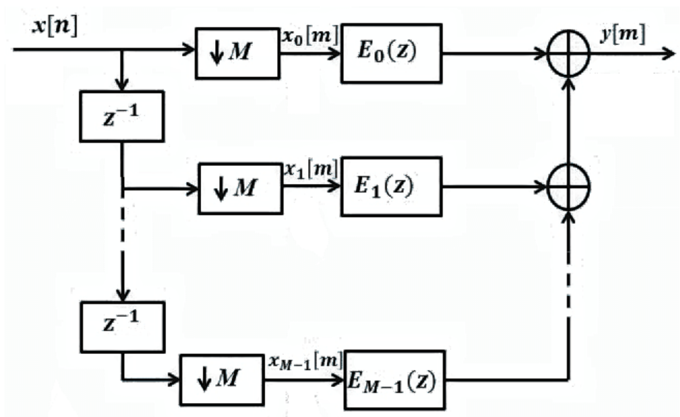
How many additions and multiplications are required per unit time?

Answer (4):

For polyphase implementation of the decimation (AA) filter, it has to be split into M subfilters of the form,

$$e_k[n] = h[\mathrm{nM} + k] = h_k[\mathrm{nM}]$$

where $h[n] = h_{\mathrm{lp}}[n]$ designed previously, and $k = [0, 1, .., M-1]$. This allows to implement the decimation system as follows,



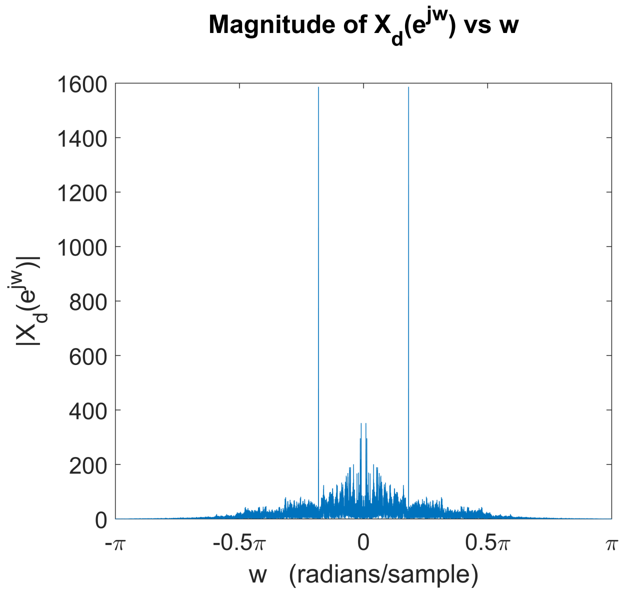
For this, we define the required $e_k[n]$ and downsampled+shifted $x[n]$ sequences, convolve the respective ones, and sum the results to obtain decimated $x_d[n]$. The code implementing this polyphase-decomposition based decimation system starts below.

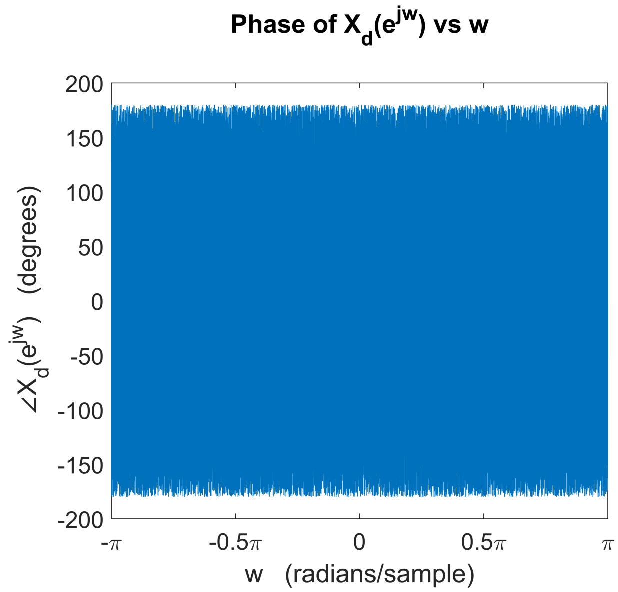
```
% <task> implement polyphase-decomposition based decimation system

xd_nonpolyphase = xd;                   % keep old xd[n] for comparison
M = 4;                                  % decimation factor

xdM = cell(1, M);                       % to hold M (partial) xd[n] sequences
nxdM = cell(1, M);

xd = zeros(1, length(x)/M);             % to hold final decimated sequence
nxd = 0:(length(xd)-1);

% decomposition into M branches
for k=0:(M-1)

    % define ek[n] = hlp[nM+k] (advance by k, scale by M)
    ek = dnsample(hlp, nhlp, M, -k);

    % define xk[n] = x[nM-k] (delay by k, scale by M)
    [xk, nxk] = dnsample(x, n, M, k, 0);
```

```
    % have xk[n] start at n=0
    xk = xk(find(n==0, 1):end);
    nxk = nxk(find(n==0, 1):end);

    % CCDE coefficients for ek (decomposed AA filter)
    b = ek;
    a = [1];

    % assuming zero initial conditions (rest)
    N_aux = zeros(1, length(a)-1);
    M_init = zeros(1, length(b)-1);

    % pass xk[n] through ek[n] using get_system_out()
    [xdM{k+1}, nxdM{k+1}] = get_system_out(b, a, xk, N_aux, M_init);

    % add this partial result to total xd[n]
    [xd, nxd] = sigadd(xd, nxd, xdM{k+1}, nxdM{k+1});

end

% clip result to expected size of decimated sequence
xd = xd(1:length(x)/M);
nxd = nxd(1:length(xd));

% compare result xd[n] with non-polyphase implementation of decimation (Q3)
if isequal(round(xd, 10), round(xd_nonpolyphase, 10))
    disp(['Result of polyphase implmentation of decimation' ...
        ' matches with the non-polyphase implementation in Q3.'])
end
```

```
Result of polyphase implmentation of decimation matches with the non-polyphase implementation in Q3.
```

```
% compute and plot DTFT of xd[n]

% initialize frequency vector (and other vars) for computing DTFT X(w)
w = -pi:2*pi/(length(xd)-1):pi;             % for 1 period [-pi, pi] or DTFT

% compute DTFT Xd(w) using (chunk-wise) matrix-multipication method,
% since this is a large computation, avoid repeating it if already computed
if isfile("T2_Xdw_.mat") load("T2_Xdw_.mat"); end
if ~exist('Xdw_','var')
    Xdw_ = dtft_split(xd, nxd, w, 750);
    save("T2_Xdw_.mat",'Xdw_');
end

% obtain magnitude and phase of DTFT Xd(w)
Xdw_mag_ = abs(Xdw_);                       % get the magnitude of Xd(w)
Xdw_ang_ = angle(Xdw_)*180/pi;              % get the phase (angle) of Xd(w) in degrees

% plot magnitude and phase of DTFT Xd(w) vs w
plotMagPhase(w, Xdw_mag_, Xdw_ang_, 'X_d');
```



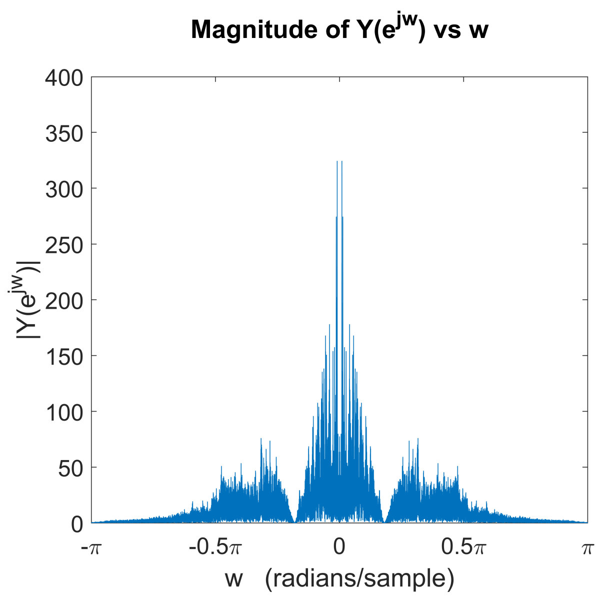


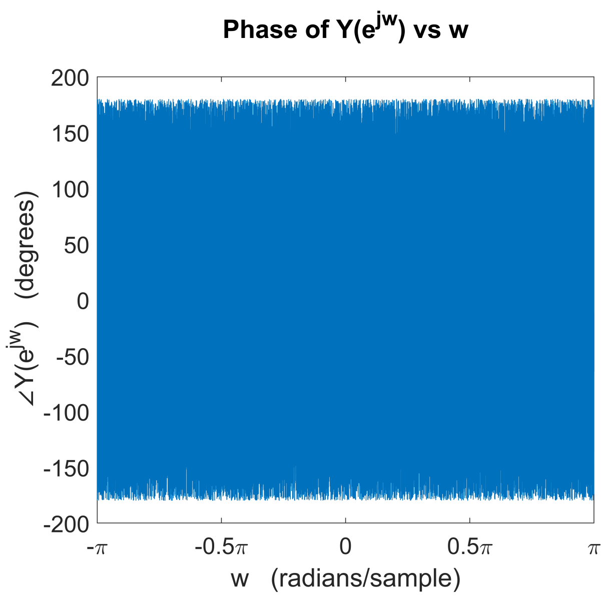
Just as in Q3, after this decimation by $M = 4$, the tone has shifted to DT frequency $\omega = 4(\pm 0.04535\pi)$ $= \pm 0.1814\pi \text{ rad/sec}$. Which, as calculated for Q3, results in following CCDE coefficients for Notch Filter $H(z)$:

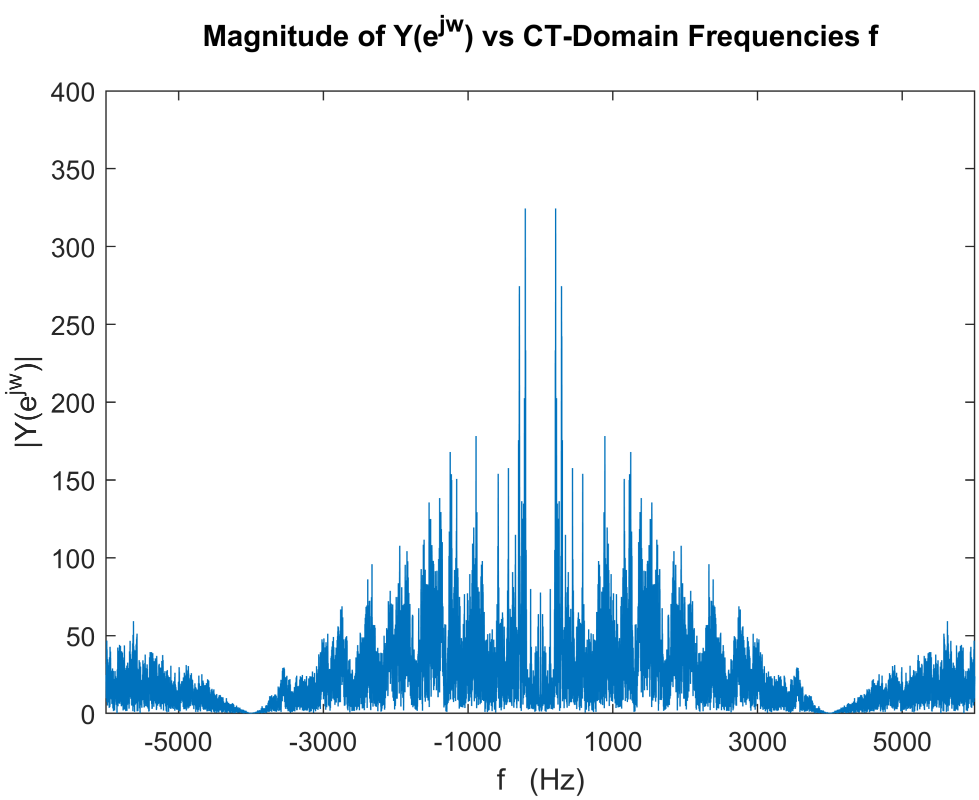
$$b = \{0.875, \ -1.4734, \ 0.875\}$$
$$a = \{1, \ -1.263, \ 0.5625\}$$

```
% define Notch Filter system's CCDE coefficients
b = [0.875 -1.4734 0.875];
a = [1 -1.263 0.5625];

% assuming zero initial conditions (rest)
N_aux = zeros(1, length(a)-1);
M_init = zeros(1, length(b)-1);

% compute output y[n], with input = decimated sound data xd[n], using get_system_out()
[y, ny] = get_system_out(b, a, xd, N_aux, M_init);

% get and play the audio y(t) using sound command, using Fs/M
sound(y, Fs/M);                             pause((length(y)-1)/Fs);

% plot DTFT of decimated sound data after passing through system (DTFT of y[n])

% initialize frequency vector (and other vars) for computing DTFT Y(w)
w = -pi:2*pi/(length(y)-1):pi;              % for 1 period [-pi, pi] or DTFT
f = Fs * w/(2*pi);                          % corresponding CT-domain frequencies

% compute DTFT Y(w) using (chunk-wise) matrix-multipication method
if isfile("T2_Yw__.mat") load("T2_Yw__.mat"); end
if ~exist('Yw__','var')
    Yw__ = dtft_split(y, ny, w, 750);       save("T2_Yw__.mat",'Yw__');
end
```

```
% obtain magnitude and phase of DTFT Y(w)
Yw_mag__ = abs(Yw__);                        % get the magnitude of Y(w)
Yw_ang__ = angle(Yw__)*180/pi;               % get the phase (angle) of Y(w) in degrees

% plot magnitude and phase of DTFT Y(w) vs w,
% and magnitude of Y(w) vs CT-domain frequencies f
plotMagPhase(w, Yw_mag__, Yw_ang__, 'Y', f, [0 400]);
```







Computations for AA Filter: The length of decimation filter $h_{\mathrm{lp}}[n]$ is $N = 23$. But upon decomposition, for a branch k, $e_k[n]$ has the length $N/M = 23/4$, with a decimated version of $x[n]$ arriving to it at a rate of F_s/M. This gives ~ $(N/M)(F_s/M)$ multiplications and $(N/M - 1)(F_s/M)$ additions per second per branch. As there are M branches, we have in total $(N/M)(F_s)$ multiplications and $(N/M - 1)(F_s)$ additions per second, that is,

$$\frac{N}{M} \times F_s = \frac{23}{4} \times 44100 = 23 \times 11025 = 253575 \text{ multiplications}, \quad \text{and}$$

$$\left(\frac{N}{M} - 1\right) \times F_s = \frac{19}{4} \times 44100 = 19 \times 11025 = 209475 \text{ additions} \quad \text{per second.}$$

which is a reduction by a factor of about 4 compared to the non-polyphase implementation.

Computations for Notch Filter: The computations for the Notch Filter remain the same as in Q3, which are,

$$5 \times 44100/4 = 55125 \text{ multiplications}, \qquad \text{and} \qquad 4 \times 44100/4 = 44100 \text{ additions}, \qquad \text{per second.}$$

Question (5):

Do you see any gains in number of computations?

Answer (5):

The number of computations for the Notch Filter reduced by a factor of $M = 4$ by using decimation of $x[n]$ before applying the Notch Filter. Furthermore, the computations for the decimation filter also reduced by roughly $M = 4$ upon using polyphase implementation of decimation. A summary is given in folllowing tables.

	Decimation Filter		Notch Filter	
Method	$\times$	$+$	$\times$	$+$
Undecimated	–	–	$(N_n)F_s$	$(N_n - 1)F_s$
Decimated	$(N_d)F_s$	$(N_d - 1)F_s$	$(N_n)F_s/M$	$(N_n - 1)F_s/M$
Decimated - Polyphase	$(N_d/M)F_s$	$(N_d/M - 1)F_s$	$(N_n)F_s/M$	$(N_n - 1)F_s/M$

	Decimation Filter		Notch Filter	
Method	$\times$	$+$	$\times$	$+$
Undecimated	–	–	220500	176400
Decimated	1014300	970200	55125	44100
Decimated - Polyphase	253575	209475	55125	44100

Functions used in the tasks

dtft_split()

Used in Task 1 (Q1, Q2) and Task 2 (Q3, Q4) — to compute DTFT of large sequences by ω-chunk-wise matrix-multiplication method.

Note: *To identify the sharp tone, ω array size should be comparable to that of audio sequence $x[n]$ $(= 255864 \text{ samples})$, i.e. $\Delta\omega \approx 2\pi/(\text{length}(x) - 1)$. Which means, size of matrix $n'\,\omega$, required for $X(\omega) = x\exp(-1i\;n'\,\omega)$ one-line computation, would be $\approx 255864 \times 255864$ doubles $\approx 488\text{ GiB}$ for 64-bit double values. Hence the need to split ω-values into chunks for peice-wise computation of DTFT.*

```
% <function>
% computes DTFT of large sequences by splitting the matrix-multiplication method:
%               Xw = x * exp(-1i * n' * w);
% over chunks of w-values, to be able to hold the n'*w matrix in memory.
%
% <syntax>
% Xw = dtft_split(x, n, w, <wsplit>)
%
% <I/O>
% Xw = output DTFT sequence over range w
% x = DT sequence over range n
% w = DT frequencies array
% wsplit = number to w values to include in one chunk of computation

function Xw = dtft_split(x, n, w, wsplit)

    if ~(nargin==4); wsplit = 750; end
    if wsplit <= 0
        Xw = x * exp(-1i * n' * w);     % one-line matrix-mult. solution
    else
        dtft_split_routine();           % call chunk-wise matrix-mult. method
    end

    function dtft_split_routine
        % compute DTFT X(w) using (chunk-wise) matrix-multipication method
        Xw = zeros(1, length(w));
        wlen = length(w);
        p_b = 1:wsplit:wlen;  p_e = [wsplit:wsplit:wlen, wlen];
        for i   = 1:ceil(wlen/wsplit)
            Xw( p_b(i):p_e(i) ) = x * exp(-1i * n' * w( p_b(i):p_e(i) ));
        end
    end

end
```

dnsample()

Used in Task 2 (Q3, Q4) — to downsample (throw away samples of) a given sequence by factor of M, and to introduce an optional shift before downsampling.

```
% <function>
% downsamples sequence x[n] by a factor M to obtain y[n] = x[nM], adds shift
% before downsampling if optional input 'd' provided, extends sequence to
% include n=r data-point before downsampling if optional input 'r' provided.
%
% <syntax>
% [y, m] = dnsample(x, n, M, <d>, <r>)
%
% <I/O>
% y = output (dilated) sequence = x[nM]
% x = input sequence x[n]
% M = decimation factor
% d = optional delay by 'd' samples before downsampling
% r = n-axis point to include in sequence before downsampling

function [y, m] = dnsample(x, n, M, d, r)

    % M must be a non-zero positive integer
    if (M <= 0)
        error('M must be a non-zero positive number');
    end
    M = round(M, 0);

    % check for shift parameter
    if nargin >= 4
        n = n + d;
    end

    % extend to include n=r, if requested
    if nargin == 5
        if n(end) < r
            n_end = n(end);
            n = [n n_end+1:r];
            x = [x zeros(1, r-n_end)];
        elseif n(1) > r
            n_start = n(1);
            n = [r:n_start-1 n];
            x = [zeros(1, n_start-r) x];
        end
    end

    % mod(a, b): nonzero results are always negative if the divisor is negative
    %            and always positive if the divisor is positive
    diff = mod(n(1), M);
```

```
    % keeping y[0]=x[0.M]=x[0], beginning m value should be
    if diff == 0
        m_b_inx = 1;                        % starts at same value as in n
    else
        m_b = n(1) + M - mod(n(1), M);
        m_b_inx = find(n == m_b, 1);        % index in n of beginning m value
    end

    % decimate x[n] to get y[n]=x[nM]
    m = n(m_b_inx:M:end)/M;
    y = x(m_b_inx:M:end);

end
```

sigadd()

Used in Task 2 (Q4) — to add two sequences of different lengths.

```
% <function>
% implements y[n] = x1[n]+x2[n], for different sized x1 and x2.
%
% <syntax>
% [y, n] = sigadd(x1, n1, x2, n2)
%
% <I/O>
% y = sum sequence over n, which includes n1 and n2
% x1 = first sequence over n1
% x2 = second sequence over n2 (n2 can be different from n1)

function [y, n] = sigadd(x1, n1, x2, n2)

    % n should be from min(n1, n2) to max(n1, n2)
    n = min(min(n1), min(n2)) : max(max(n1), max(n2));

    % for storing x1 and x2 in array spanning length of y
    x1_ = zeros(1, length(n));
    x2_ = x1_;

    % copy x1 into x1_ at its corresponding n values (from n1)
    x1_( find( (n>=min(n1)) & (n<=max(n1)) == 1 ) ) = x1;

    % copy x2 into x2_ at its corresponding n values (from n2)
    x2_( find( (n>=min(n2)) & (n<=max(n2)) == 1 ) ) = x2;

    y = x1_ + x2_;

end
```

plotMagPhase()

Used in Task 1 (Q1, Q2) and Task 2 (Q3, Q4) — to plot given magnitude and phase response against ω, and magnitude against f (CT-domain frequencies) if f-axis provided.

```
% <function>
% plots given magnitude and phase response against given w values, and
% magnitude against f (CT-domain frequencies) if f-axis array provided.
%
% <syntax>
% plotMagPhase(w, Xw_mag, Xw_ang, X, <f>, <mag_ylim>)
%
% <I/O>
% w = DT frequencies array [-pi, pi]
% Xw_mag = DTFT magnitude
% Xw_ang = DTFT phase
% X = name of DTFT for label/title in plot
% f = optional CT frequencies array
% mag_ylim = optional y-axis limit for DTFT magnitude

function plotMagPhase(w, Xw_mag, Xw_ang, X, f, mag_ylim)

    % plot magnitude and phase of DTFT X(w)
    fig = figure;
    set(fig,'Units','normalized','Position',[0 0 1.4 1]);

    % plot magnitude of X(w) vs w
    subplot(1, 2, 1);
    plot(w, Xw_mag);
    xlabel('w   (radians/sample)');
    ylabel(['|',X,'(e^{jw})|']);
    % check and set optional y-axis limit
    if nargin == 6
        ylim(mag_ylim);
    end
    title({['Magnitude of ',X,'(e^{jw}) vs w'];''}, 'FontSize', 16);
    setDTFTradialAxis(1);
    ax = gca;
    ax.FontSize = 16;

    % plot phase of X(w) vs w
    subplot(1, 2, 2);
    plot(w, Xw_ang);
    xlabel('w   (radians/sample)');
    ylabel(['∠',X,'(e^{jw})   (degrees)']);
    title({['Phase of ',X,'(e^{jw}) vs w'];''}, 'FontSize', 16);
    setDTFTradialAxis(1);
    ax = gca;
    ax.FontSize = 16;
```

```
    if nargin >= 5
        % plot magnitude of X(w) vs CT-domain frequencies f
        fig = figure;
        plot(f, Xw_mag);
        xlabel('f   (Hz)'); ylabel(['|',X,'(e^{jw})|']);
        xlim([-6e3 6e3]); xticks(-5e3:2e3:5e3);
        % check and set optional y-axis limit
        if nargin == 6
            ylim(mag_ylim);
        end
        title({'';['Magnitude of ',X,'(e^{jw}) vs CT-Domain Frequencies f'];''});
    end

end
```

```
;
```