Assignment No. 4: **Z-Transform**

Reg. No. : 2016-EE-189

Task 1

Statement:

Let

$$H(z) = \frac{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}{1 + 0.5z^{-1}}$$

where $\omega_0 = 2000\pi/F_s$, and sampling rate $F_s = 11025\ \mathrm{samples/second}$.

Question (1):

Find inverse Z-transform of $H(z)$ on paper.

Find all poles and zeros and draw them in z-plane.

Answer (1):

The work done on paper, i.e. inverse Z-transform and pole-zero plot of $H(z)$, has been attached starting at the next page.

The work done on paper starts from the next page.

$$H(z) = \frac{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}{1 + 0.5 z^{-1}}$$

Inverse Z-transform of H(z) :

since numerator is of higher degree, perform long division,

$$\begin{array}{r|l}
 & 2z^{-1} - 4\cos(\omega_0) - 4 \\
1 + 0.5z^{-1} & 1 - 2\cos(\omega_0) z^{-1} + \cancel{z^{-2}} \\
 & \phantom{1 -{}} +2z^{-1} \qquad + \cancel{z^{-2}} \\
 & \phantom{1 -{}} - \qquad\qquad - \\
\hline
 & 1 + (-2\cos(\omega_0) - 2)\cancel{z^{-1}} \\
 & -4\cos(\omega_0) - 4 + \cancel{(-2\cos(\omega_0) - 2) z^{-1}} \\
 & + \quad + \quad \cancel{-} \\
\hline
 & 4\cos(\omega_0) + 5
\end{array}$$

$$\therefore \quad H(z) = 2z^{-1} - 4\cos(\omega_0) - 4 + \frac{4\cos(\omega_0) + 5}{1 + 0.5z^{-1}} \qquad \text{--- eq. (I)}$$

for $\omega_0 = \frac{2000\pi}{F_s} = \frac{2000\pi}{11025}$ r/s , eq. (I) becomes,

$$H(z) = \underbrace{2z^{-1} - 7.3678}_{C_k} + \frac{\overbrace{8.3678}^{R}}{1 - \underbrace{(-0.5)}_{P} z^{-1}}$$

— eq. (II)

simplifying eq. (I),

$$H(z) = 2z^{-1} - 4\cos(\omega_0) - 5 + 1 + \frac{4\cos(\omega_0) + 5}{1 + 0.5z^{-1}}$$

$$= 2z^{-1} + 1 + (4\cos(\omega_0) + 5)\left(\frac{1}{1+0.5z^{-1}} - 1\right)$$

for which the inverse-z transform can be done easily using z-transform pairs,

$$h[n] = \mathcal{Z}^{-1}\{H(z)\}$$

assuming ROC = $|z| > 0.5$

$$h[n] = 2\,\delta[n-1] + \delta[n] + (4\cos(\omega_0) + 5)\left((-0.5)^n u[n] - \delta[n]\right)$$

— eq. (III)

where it is assumed that the system represented by $H(z)$ is causal, i.e. has the ROC $|z| > 0.5$, in other words that $h[n]$ is a right-sided seq..

for $\omega_o = \frac{2000\pi}{11025}$ r/s, eq. (III) becomes,

$$h[n] = 2\delta[n-1] + \delta[n] + 8.3678\left((-0.5)^n u[n] - \delta[n]\right)$$

$$h[n] = 2\delta[n-1] - 7.3678\,\delta[n] + 8.3678(-0.5)^n u[n]$$

— eq. (IV)

which is the inv-z tr. of $H(z)$ assuming $ROC_h = |z| > 0.5$.

Pole-zero form & plot for $H(z)$ starts at next page.

Pole-Zero form/plot of $H(z)$:

$$H(z) = \frac{1-2\cos(\omega_0) z^{-1} + z^{-2}}{1+0.5z^{-1}} = \frac{z^2 - 2\cos(\omega_0) z + 1}{z(z+0.5)}$$

$$= \frac{z^2 - e^{j\omega_0} z - e^{-j\omega_0} z + 1}{z(z+0.5)}$$

$$= \frac{z(z-e^{j\omega_0}) - e^{-j\omega_0}(z - e^{j\omega_0})}{z(z+0.5)}$$

$$= \frac{(z-e^{j\omega_0})(z-e^{-j\omega_0})}{z(z+0.5)} \quad \begin{matrix} \text{zeros} \\ \text{poles} \end{matrix}$$

so, $H(z)$ has poles at $z=0$ and $z=-0.5$, and zeros at $z=e^{j\omega_0}$ and $z=e^{-j\omega_0}$.

putting $\omega_0 = \frac{2000\pi}{11025}$ r/s in above eq., we get,

$$H(z) = \frac{(z-e^{j0.5699})(z-e^{-j0.5699})}{z(z+0.5)}$$

$$= \frac{(z-0.8420-j0.5396)(z-0.8420+j0.5396)}{z(z+0.5)}$$

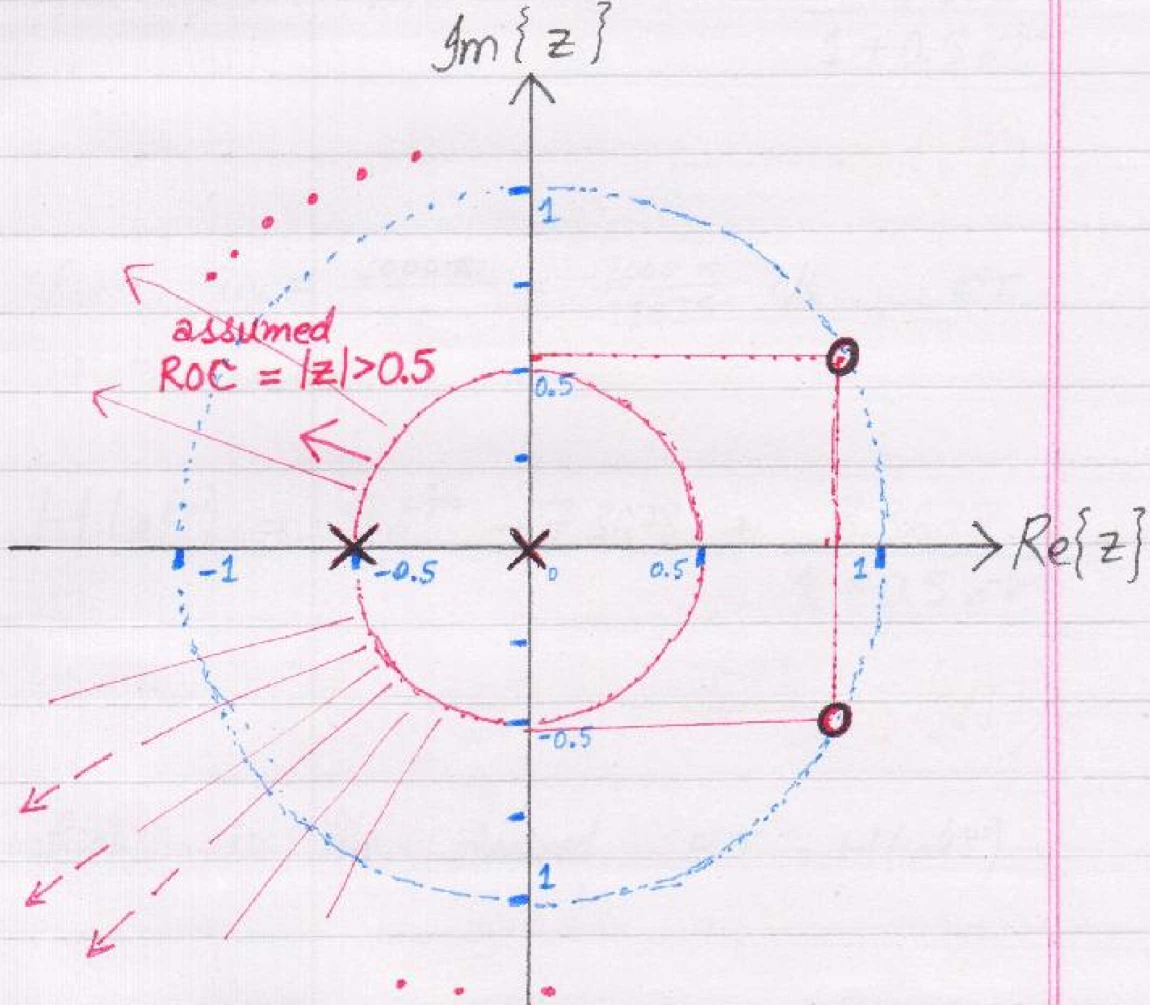
— eq. (V)

from eq. (V), $H(z)$ has,

poles at $\begin{cases} z = 0 \\ z = -0.5 \end{cases}$

zeros at $\begin{cases} z = 0.8420 + j\,0.5396 \\ z = 0.8420 - j\,0.5396 \end{cases}$

So the pole-zero plot of $H(z)$ is given by,



Question (2):

Find inverse Z-transform of $H(z)$ using MATLAB. Plot all poles and zeros in z-plane using MATLAB.

Answer (2):

The code, alongwith the results, to compute inverse Z-tranform of $H(z)$ and plot its pole-zero map starts below.

```
% >>> task <<< compute inverse Z-transform of H(z)

% define sampling frequency Fs and corresponding w0
Fs = 11025;
w0 = 2000*pi/Fs;

% [method 1] using iztrans()

% define H(z) symbolically
syms z;
syms w_0;
Hz = vpa((1-2*cos(w_0)*(z^-1)+(z^-2))/(1+0.5*(z^-1)))
```

Hz =

$$\frac{\frac{1}{z^2} - \frac{2.0\cos(w_0)}{z} + 1.0}{\frac{0.5}{z} + 1.0}$$

```
% compute inverse Z-transform of H(z) using iztrans()
% <syntax> f = iztrans(F, transVar)
h = vpa(iztrans(Hz))
```

h = $(4.0\cos(w_0) + 5.0)\ \left(1.0\ (-0.5)^n - 1.0\,\delta_{n,0}\right) + 2.0\,\delta_{n-1.0,0} + 1.0\,\delta_{n,0}$

Which matches the inverse Z-transform in eq. (III) derived on paper.

```
% substitute w0 value to get numerical sequence h[n]
h = vpa(subs(h, {w_0}, {w0}), 5)
```

h = $8.3678\ (-0.5)^n + 2.0\,\delta_{n-1.0,0} - 7.3678\,\delta_{n,0}$

Which matches the inverse Z-transform in eq. (IV) derived on paper.

```
% save h[n] expression as function-handle for later use
func_h = matlabFunction(h)
```

```
func_h = function_handle with value:
    @(n)(-5.0e-1).^n.*8.367812335691269+(n-1.0==0.0).*2.0-(n==0.0).*7.367812335691269
```

```
% [method 2] using residuez() (partial fraction decomposition)

% b's and a's in poly. frac H(z)=B(z)/A(z) are,
b = [1 -2*cos(w0) 1];
a = [1 0.5];

% compute partial fraction form of H(z) using residuez()
% <syntax> [r, p, k] = residuez(b, a)
[R, p, C] = residuez(b, a)
```

```
R = 8.3678
p = -0.5000
C = 1×2
   -7.3678    2.0000
```

```
% display partial fraction sum symbolically
Hz = sum(R./(1-p*z^-1));
if ~isempty(C)
    Hz = Hz + sum(C.*z.^-(0:length(C)-1));
end
Hz = vpa(Hz, 5)
```

Hz =

$$\frac{8.3678}{\frac{0.5}{z}+1.0}+\frac{2.0}{z}-7.3678$$

Which matches the partial fraction decomposition in eq. (II) derived on paper.

This result can be used to easily compute inverse Z-transform of $H(z)$, using Z-transform properties and pairs, as follows:

$$H(z) \;=\; \frac{8.3678}{1+0.5z^{-1}}+2z^{-1}-7.3678$$

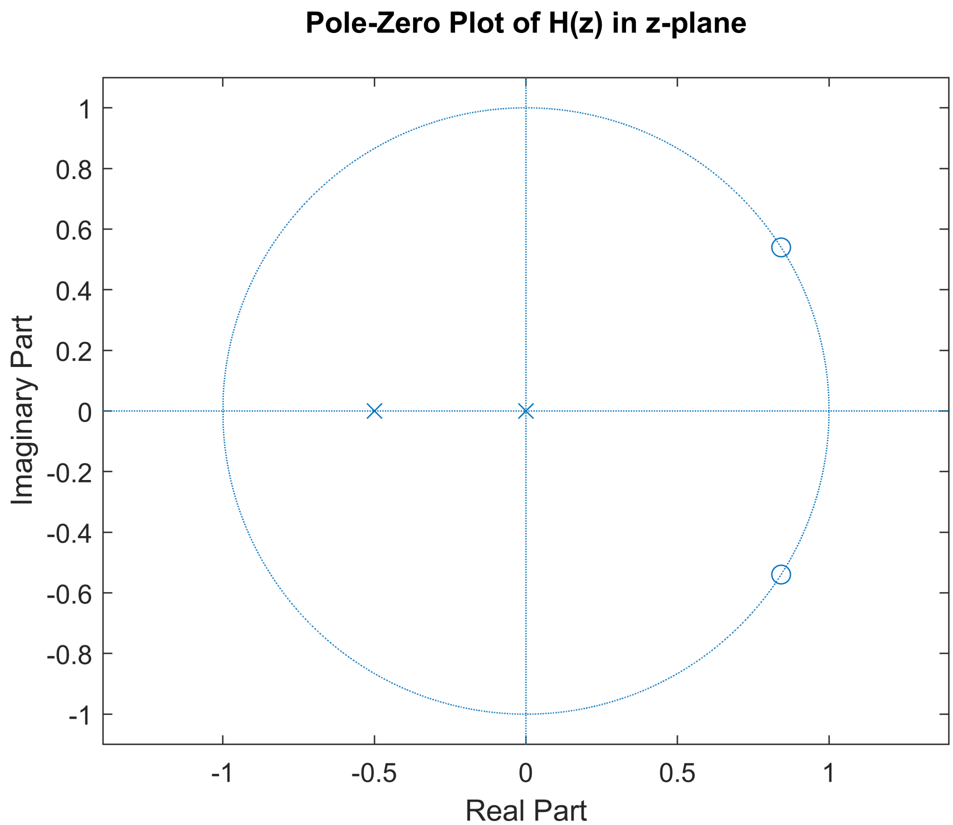
Assuming ROC_h is $|z|>0.5$, the inverse Z-transform is,

$$\begin{aligned} h[n] \;&=\; Z^{-1}\{\, H(z)\,\} \\ &=\; 8.3678\,(-0.5)^n\,u[n]+2\,\delta[n-1]-7.3678\,\delta[n] \\ &=\; 2\,\delta[n-1]-7.3678\,\delta[n]+8.3678\,(-0.5)^n\,u[n] \end{aligned}$$

Which matches the inverse Z-transform in eq. (IV) derived on paper.

```
% >>> task <<< plot all poles and zeros of H(z) in z-plane

% coefficients (b's and a's) in poly. frac H(z)=B(z)/A(z) are,
b = [1 -2*cos(w0) 1];
a = [1 0.5];

% plot pole-zero map of H(z) using zplane() with b's and a's from H(z)=B(z)/A(z)
% <syntax> zplane(b, a)
```

```
zplane(b, a);
title({'Pole-Zero Plot of H(z) in z-plane';''});
```



Question (3):

Substitute $z = e^{j\omega}$ in $H(z)$ and calculate its Fourier Transform $H(e^{j\omega})$ on paper. Plot the response using MATLAB.

What kind of a system do you think it is? (Question 4)

Answer (3):

The work done on paper, i.e. deriving $H(e^{j\omega})$, has been attached starting at the next page.

The work done on paper starts from the next page.

$$H(z) = \frac{1 - 2\cos(\omega_0)\, z^{-1} + z^{-2}}{1 + 0.5z^{-1}}$$

to obtain FT $H(e^{j\omega})$, substitute $z = e^{j\omega}$,

$$H(e^{j\omega}) = \frac{1 - 2\cos(\omega_0)\, e^{-j\omega} + e^{-j2\omega}}{1 + 0.5e^{-j\omega}}$$

again, using long division as before,

$$H(e^{j\omega}) = 2e^{-j\omega} - 4\cos(\omega_0) - 4 + \frac{4\cos(\omega_0) + 5}{1 + 0.5e^{-j\omega}}$$

— eq. (VI)

for $\omega_0 = \frac{2000\pi}{F_s} = \frac{2000\pi}{11025}$ r/s, FT $H(e^{j\omega})$ becomes,

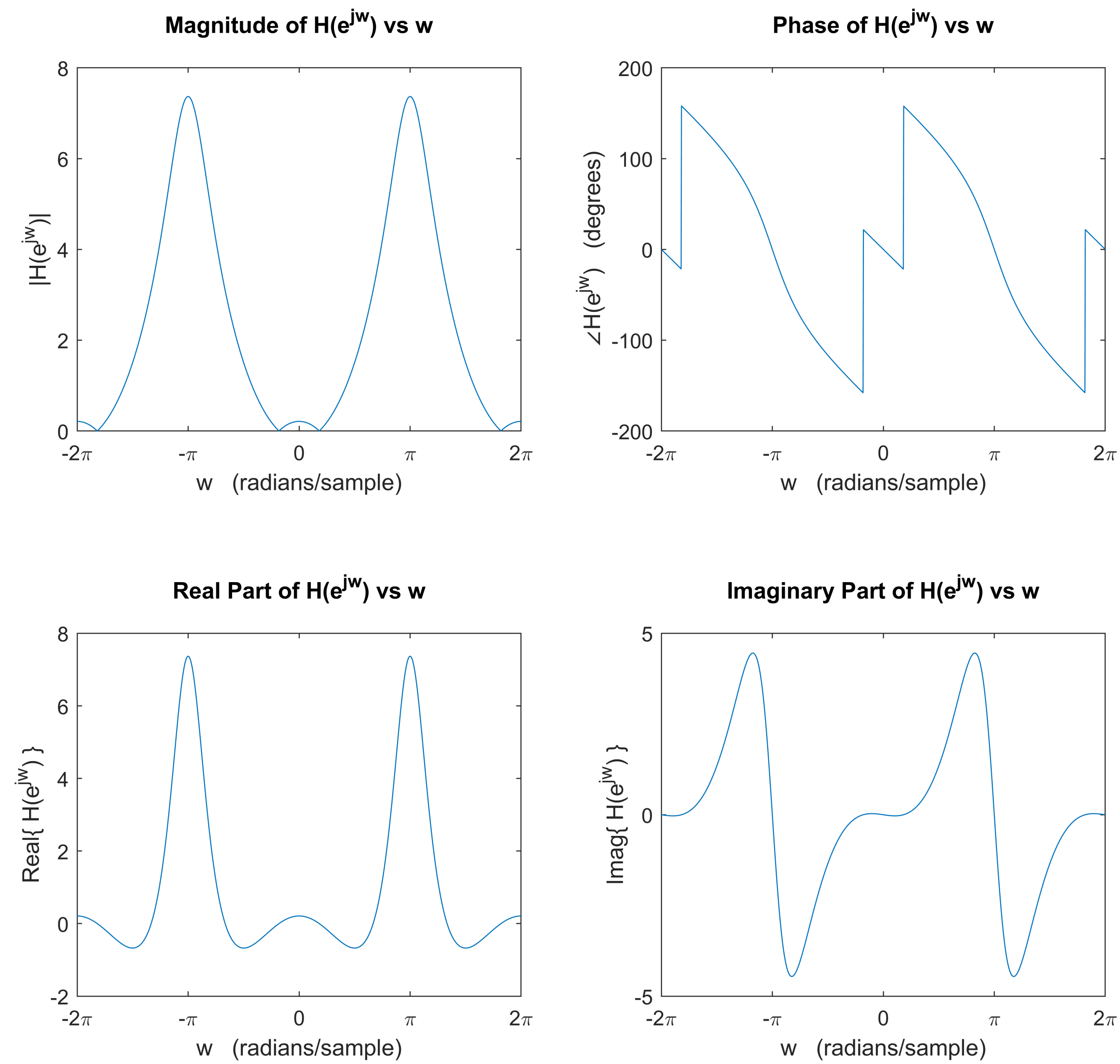
$$H(e^{j\omega}) = 2e^{-j\omega} - 7.3678 + \frac{8.3678}{1 + 0.5e^{-j\omega}}$$

— eq. (VII)

which is the desired FT $H(e^{j\omega})$.

The expression for $H(e^{j\omega})$, given by eq. (VI) derived on paper, has been used in the code below to plot the frequency response $H(e^{j\omega})$ of the system.

```
% initialize frequency vector for computing freq. resp. H(w)
w = -2*pi:0.01:2*pi;

% compute freq. resp. H(w) using derived equation eq. (VI) on paper
Hw = 2*exp(-1i*w) - (4*cos(w0)+4) + (4*cos(w0)+5)./(1+0.5*exp(-1i*w));

% obtain magnitude and phase of freq. resp. H(w)
Hw_mag = abs(Hw);                           % get the magnitude of H(w)
Hw_ang = angle(Hw)*180/pi;                  % get the phase (angle) of H(w) in degrees

% obtain real and imaginary parts of freq. resp. H(w)
Hw_real = real(Hw);                         % get the real part of H(w)
Hw_imag = imag(Hw);                         % get the imaginary part of H(w)

% plot magnitude and phase of freq. resp. H(w),
% and also its real and imaginary parts
fig = figure; set(fig,'Units','normalized','Position',[0 0 1 1.6]);

% plot magnitude of H(w) vs w
subplot(2, 2, 1);
plot(w, Hw_mag);
xlabel('w   (radians/sample)');
ylabel('|H(e^{jw})|');
title({'Magnitude of H(e^{jw}) vs w';''});
setDTFTradialAxis()

% plot phase of H(w) vs w
subplot(2, 2, 2);
plot(w, Hw_ang);
xlabel('w   (radians/sample)');
ylabel('∠H(e^{jw})   (degrees)');
title({'Phase of H(e^{jw}) vs w';''});
setDTFTradialAxis()

% plot real part of H(w) vs w
subplot(2, 2, 3);
plot(w, Hw_real);
xlabel('w   (radians/sample)');
ylabel('Real\{ H(e^{jw}) \}');
title({'';'Real Part of H(e^{jw}) vs w';''});
setDTFTradialAxis()

% plot imaginary part of H(w) vs w
subplot(2, 2, 4);
plot(w, Hw_imag);
xlabel('w   (radians/sample)');
ylabel('Imag\{ H(e^{jw}) \}');
title({'';'Imaginary Part of H(e^{jw}) vs w';''});
setDTFTradialAxis()
```



Question (4):

Based on the shape of it's response, what kind of a system do you think it is?

Answer (4):

Consider the DTFT plot in the range $\omega = [-\pi,\ \pi]$ (i.e. one period of the response).

From the plot of real part of $H(e^{j\omega})$ in above figure, the system appears to be a sinc filter, with the sinc-center shifted to $-\pi$ and π.

Based on the plot of magnitude part of $H(e^{j\omega})$, the system can be considered to be allowing most frequencies through but reducing the strength of very lower frequencies (near 0), meanwhile giving relatively more magnitude to higher frequencies (near $\pm\pi$).

Note that at $\omega = \{0.1814\pi, \ -0.1814\pi\}$, there is a sharp dip (a cutoff for a single point), where magnitude drops to nearly zero. Hence that frequency will effectively be removed from an input given to this system.

Given the sampling frequency $F_s = 11025 \text{ samples/second}$, this radial frequency 0.1814π corresponds to,

$$F = \frac{\Omega}{2\pi} = \frac{\omega \times F_s}{2\pi} = \frac{0.1814\pi \times 11025}{2\pi} \text{ Hz} \approx 1 \text{ kHz}$$

Hence, besides affecting the magnitudes of various frequencies, this filter will essentially remove the frequency component at $\omega = \pm 0.1814\pi$, corresponding to $F \approx 1 \text{ kHz}$ for an input sampled at $F_s = 11025 \text{ samples/second}$.

Task 2

Statement:

Write a function like you have written in previous lab to compute output of the system $H(z)$, assuming the system is initially at rest. Plot the impulse response of the system.

Methodology:

A function "$\text{get\_system\_out}()$" that implements a CCDE (computes its output $y[n]$ for $0 \le n < \text{len}(x)$, where $x[n]$ is the input) given the system's CCDE coefficients (b's and a's) has already been implemented in assignment 3.

To compute system impulse response using $H(z)$, the CCDE coeffiecients must first be extracted, and then they can be used to invoke the regular "$\text{get\_system\_out}()$" function with input $x[n] = \delta[n]$. This process has been implemented in the respective 2 steps, using the below functions "$\text{get\_zsys\_coeffs}()$" and "$\text{get\_sys\_imp\_resp}()$".

Code (*get_zsys_coeffs.m*):

Firstly, the following function "$\text{get\_zsys\_coeffs}()$" (on next page) accepts a $H(z)$ expression in the form,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^{M} b_m z^{-m}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{B(z)}{A(z)}$$

and extracts the system CCDE coefficients (b's and a's).

```matlab
% <function>
% computes system CCDE coefficients (b's and a's) from the system
% Z-domain frequency response expression H(z).
%
% <syntax>
% [b, a] = get_zsys_coeffs(Hz)
%
% <I/O>
% b = M+1 coefficients { b0, b1, ..., bM } for { x[0], x[-1], ..., x[-M] }
% a = N+1 coefficients { a0, a1, ..., aN } for { y[0], y[-1], ..., y[-N] }
% Hz = system H(z) expression (type 'sym') of form B(z)/A(z)
%    = (b0 + b1.z^-1 + ..... + bM.z^-M)/(a0 + a1.z^-1 + ..... + aN.z^-N)

function [b, a] = get_zsys_coeffs(Hz)

    % Hz must not contain any other symvars besides 'z'
    syms z;
    if ~isequal(symvar(Hz), z)
        error("Only H(z) expressions with single default symvar 'z' are acceptable");
    end

    % NOTE:
    % Hz must be of form B(z)/A(z)
    %    = (b0 + b1.z^-1 + ..... + bM.z^-M)/(a0 + a1.z^-1 + ..... + aN.z^-N)
    % for the following method to work

    syms repl;                          % replacement symvar

    % replace z^-1 with replacement var 'repl'
    % to keep its degree consistent in the next step
    H2 = subs(Hz, z^-1, repl);

    % separate numerator and denominator polynomials
    [N, D] = numden(H2);

    % extract coefficients b's and a's
    b = coeffs(N);
    a = coeffs(D);

    % normalize the coefficients
    a = a./a(1);
    b = b./b(1);

end
```

Next, for the impulse response, input $x[n] = \delta[n]$ is given to the system, where $\delta[n]$ is 1 for $n = 0$ and 0 for $n \neq 0$. This is done by the next function "$\mathrm{get\_sys\_imp\_resp}()$". Note that "$\mathrm{get\_sys\_imp\_resp}()$" will make use of above "$\mathrm{get\_zsys\_coeffs}()$" function to convert provided $H(z)$ to system coefficients (b's and a's).

Code (*get_sys_imp_resp.m*):

So now, the following function "$\text{get\_sys\_imp\_resp}()$" accepts a $H(z)$ expression, uses the previous function "$\text{get\_zsys\_coeffs}()$" to get its coefficients (b's and a's), and then calls the "$\text{get\_system\_out}()$" function with input $x[n] = \delta[n]$, and the extracted coefficients.

Alternatively, the function may be provided the coefficients directly as a cell-array $\{b, a\}$ instead of a $H(z)$ symbolic expression.

It computes the impulse response $h[n]$ of the system for the n-values $0 \leq n \leq P$.

```
% <function>
% computes system impulse response h[n] for 0 <= n <= P using get_system_out(),
% given symbolic H(z) or numeric {b, a} CCDE coefficients, x (for n>=0),
% and initial conditions (N aux y values, M initial x values).
%
% <syntax>
% [h, nh] = get_sys_imp_resp(Hz|{b,a}, P)
% [h, nh] = get_sys_imp_resp(Hz|{b,a}, P, N_aux, M_init)
%
% <I/O>
% h = filter impulse resp. { h[-N], h[-N+1], ..., h[-1], h[0], h[1], ..., h[P] }
% Hz = system H(z) expression (type 'sym') of form B(z)/A(z)
%    = (b0 + b1.z^-1 + ..... + bM.z^-M)/(a0 + a1.z^-1 + ..... + aN.z^-N)
% {b, a} = system CCDE coefficients (type 'cell' of size [1,2])
% N_aux = N auxiliary y values { y[-1], y[-2], ...., y[-N] }
% M_init = M initial x values { x[-1], x[-2], ...., x[-M] }

function [h, nh] = get_sys_imp_resp(Hz, P, N_aux, M_init)

    % acceptable number of inputs
    if ~any([2 4] == nargin)
        error('Wrong Number of Input Arguments. Allowed: 2, 4.');
    end

    % check whether H(z) or the {b, a} coefficients are provided
    % in either case, extract the coefficients (b's and a's)
    if and(isequal(class(Hz), 'cell'), isequal(size(Hz), [1,2]))
        b = Hz{1};
        a = Hz{2};
    elseif isequal(class(Hz), 'sym')
        [b, a] = get_zsys_coeffs(Hz)
    else
        error("H(z) must either be a sym exp. or a cell of size 1x2");
    end

    % if initial conditions not provided, assume zero i.c. (rest)
    if (nargin == 2)
        N_aux = zeros(1, length(a)-1);
        M_init = zeros(1, length(b)-1);
    end
```

```
    % generate input sequence x[n] = impluse of length P+1
    % <syntax> [x, n] = impseq(n0, n1, n2) where n1 <= n0 <= n2
    [delta, n] = impseq(0, 0, P);

    % compute the impulse response h[n] using get_system_out()
    [h, nh] = get_system_out(b, a, delta, N_aux, M_init);

    % plot the impulse response h[n] of the system
    figure;
    stem(nh, h, 'filled', 'MarkerSize', 4);
    xlabel('n'); ylabel('h[n]');
    xlim([0 length(delta)]);
    title({'Impulse Response h[n] of given system';''});

end
```

Plotting the impulse response of system represented by given $H(z)$:

The following code uses the above function "$\mathrm{get\_sys\_imp\_resp}()$" on the system symbolic expression $H(z)$, to get and plot impulse response $h[n]$, and also compares the results to $h[n]$ expression (from inverse Z-transform of $H(z)$) computed in Task 1.

```
% >>> task <<< plot impulse response h[n] of sys. respresented by given H(z)

% define sampling frequency Fs and corresponding w0
Fs = 11025;
w0 = 2000*pi/Fs;

% define system H(z) symbolically
syms z;
syms w_0;
Hz = vpa((1-2*cos(w_0)*(z^-1)+(z^-2))/(1+0.5*(z^-1)));

% substitute w0 value to get H(z) only in terms of z
Hz = vpa(subs(Hz, {w_0}, {w0}), 5)
```

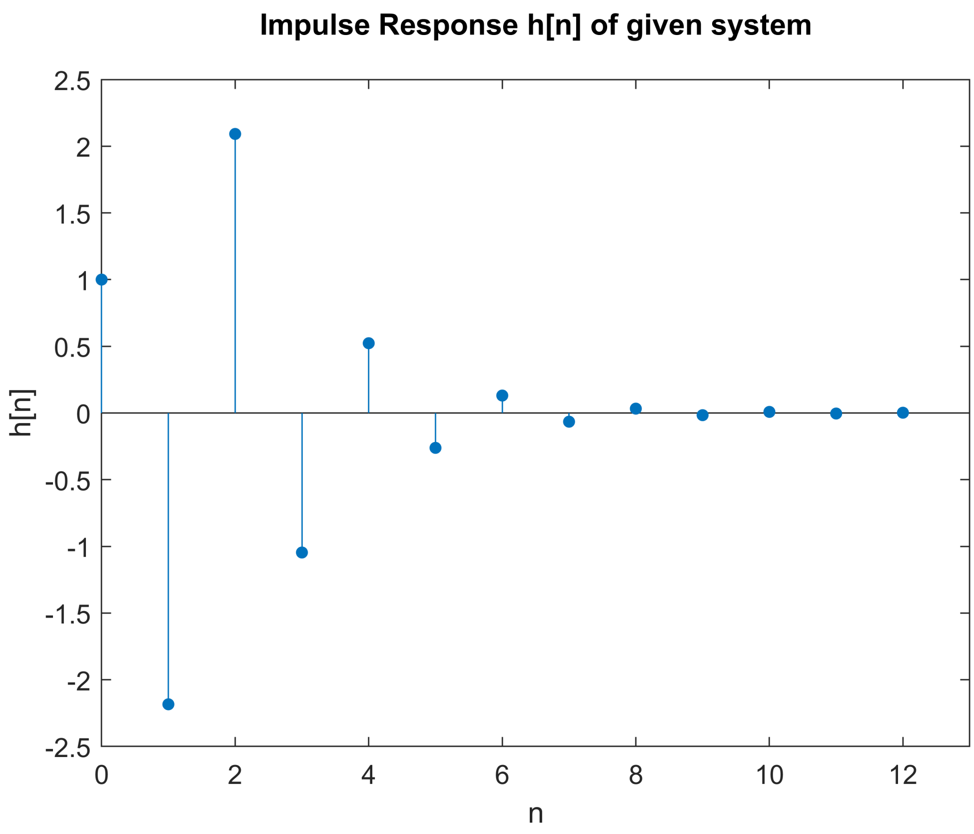
Hz =

$$\frac{\frac{1}{z^2} - \frac{1.6839}{z} + 1.0}{\frac{0.5}{z} + 1.0}$$

```
% call get_sys_imp_resp() with above expression Hz to get/plot its impulse resp. h[n]
% <syntax> [h, nh] = get_sys_imp_resp(Hz, P)
[h, nh] = get_sys_imp_resp(Hz, 12);
```

b = $\begin{pmatrix} 1.0 & -1.6839061678458620008314028382301 & 1.0 \end{pmatrix}$

a = $\begin{pmatrix} 1.0 & 0.5 \end{pmatrix}$




```
% >>> task <<< compare this impulse response h[n] with inverse Z-tranform
% expression (h[n]) of H(z) found using iztrans(), for n>=0

% truncate above calculated h[n] to start at n=0
h = h(find(nh==0):end);
nh = nh(find(nh==0):end);

% call h[n] expression function-handle stored in func_h() to directly compute
% some h[n] values from inverse Z-transform of H(z)
h2 = func_h(nh);

% will return 1 if both h[n] are equal
isequal(round(h, 10), round(h2, 10))
```

```
ans = logical
   1
```

The next task starts from the next page.

Task 3

Statement:

Download the provided file almostcaught.wav.

Play the file in MATLAB using sound command and listen to the legend speaking with a tone.

Pass the sampled sound data from the wave file through your system. Plot the DTFT of the sound data before and after passing through the system.

Methodology:

The given system is represented by,

$$H(z) = \frac{1 - 2\cos(\omega_0) z^{-1} + z^{-2}}{1 + 0.5 z^{-1}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{a_0 + a_1 z^{-1}}$$

Upon comparison, it can be seen that coefficients of the system CCDE are,

$$\begin{aligned} b &= \{ b_0 , b_1 , b_2 \} = \{ 1, \ -2\cos(\omega_0), \ 1 \} \\ a &= \{ a_0 , a_1 \} = \{ 1, 0.5 \} \end{aligned}$$

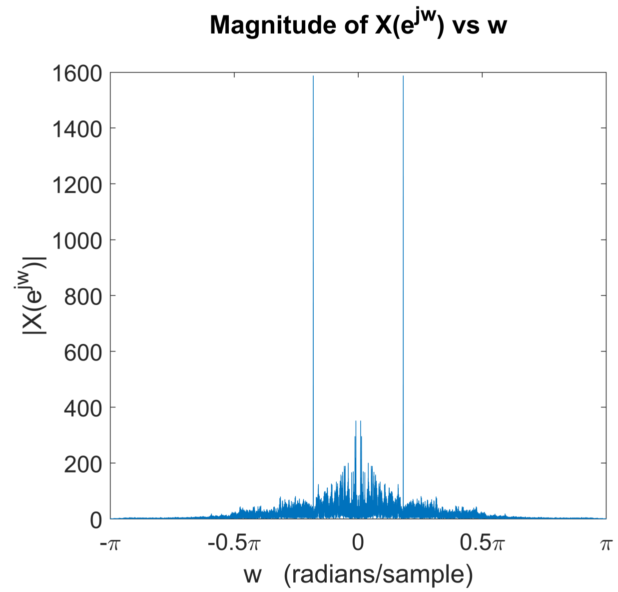
By providing these coefficient arrays to function "$\text{get\_system\_out}()$" (made in assignment 3) that implements any CCDE, the system output $y[n]$ can be computed for any given input $x[n]$.

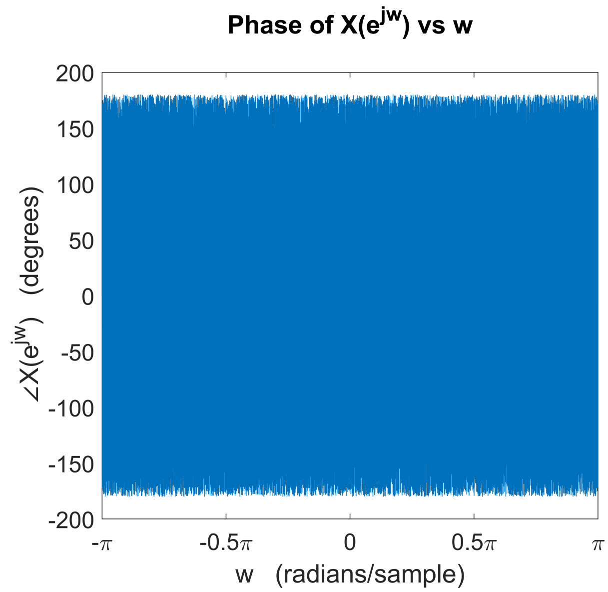
In the following case, input $x[n]$ will be the samples of provided .wav file. Here, $\omega_0 = 2000\pi / F_s$, and sampling rate $F_s = 11025 \text{ samples/second}$.

Code (with results):

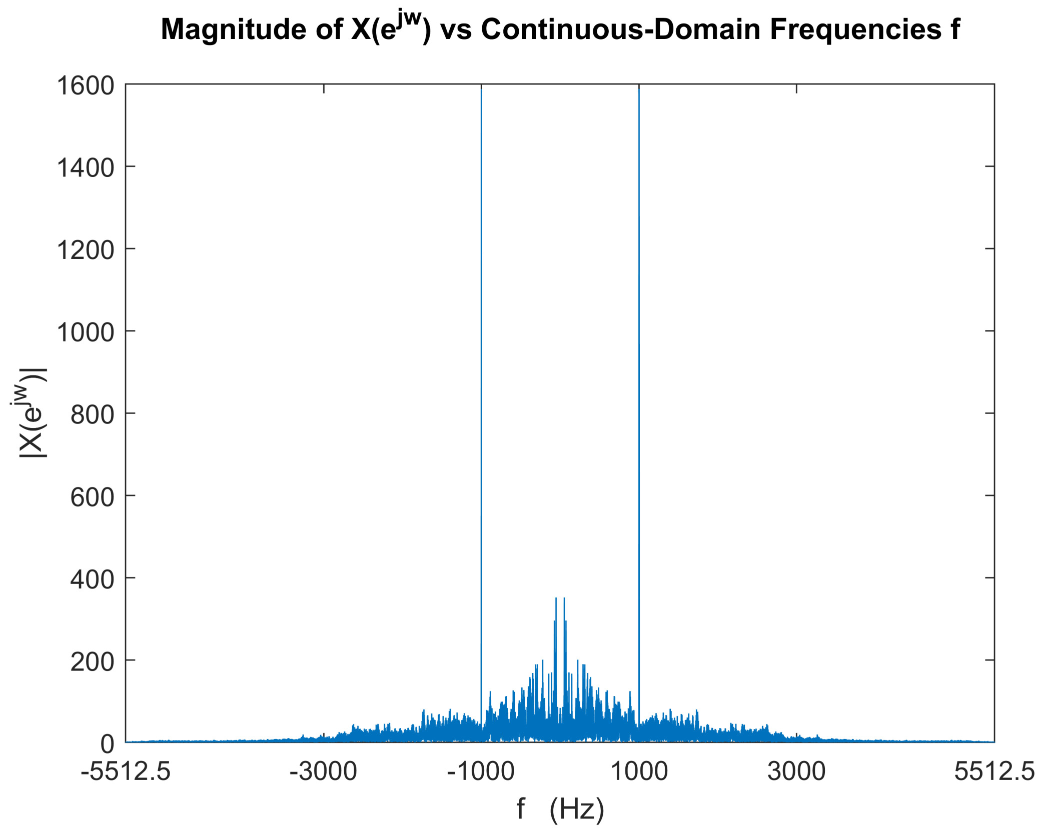
```
% >>> task <<< read and play the provided almostcaught.wav file
% in MATLAB using the sound command, with Fs = 11025 samples/second

% define sampling frequency Fs and corresponding w0
Fs = 11025;
w0 = 2000*pi/Fs;

% read audio data from almostcaught.wav file
[x, Fs] = audioread('almostcaught.wav');
duration = (length(x)-1)/Fs;

% get and play the audio x(t) using sound command, using above Fs
sound(x, Fs);
% pause execution while the sound is playing
pause(duration);
```

```
% >>> task <<< plot DTFT of sound data before passing through system (DTFT of x[n])

% initialize frequency vector for computing DTFT X(w)
w = -pi:2*pi/(length(x)-1):pi;              % for higher/proper DTFT resolution
f = Fs * w/(2*pi);                          % corresponding cont. domain frequencies
x = x.';                                    % convert x[n] to row-vector

% NOTE:
% ------------------
% Since the sound data x[n] is quite large, much higher frequency (w) resolution
% is needed for DTFT, to properly identify any tones. This requires some
% consideration in the method by which DTFT is computed, discussed below.
% ------------------
% (1) Matrix-multiplication method,
%               Xw = x * exp(-1i * n' * w);
%     cannot be used since the matrix n'*w will be too large to store in memory.
% ------------------
% (2) For-Loop method,
%               Xw = zeros(1, length(w));
%               for i = 1:length(w)
%                   Xw(i) = x *  exp(-1i * w(i) * n');
%               end
%     cannot be used since it will be quite slow due to many values of w.
% ------------------
% (3) Instead, use the MATLAB in-built function fftshift(x) to first compute
%     the DFT of x[n] (using FFT algorithm), and then use fftshift(dft) to
%     transform this DFT to DTFT of x[n], as follows,
%               Xw = fftshift(fft(x));
%     This is the method used below for DTFT computation.
% ------------------

% compute DTFT X(w) using fft() and fftshift()
Xw = fftshift(fft(x));

% obtain magnitude and phase of DTFT X(w)
Xw_mag = abs(Xw);                           % get the magnitude of X(w)
Xw_ang = angle(Xw)*180/pi;                  % get the phase (angle) of X(w) in degrees

% plot magnitude and phase of DTFT X(w)
fig = figure; set(fig,'Units','normalized','Position',[0 0 1.4 1]);

% plot magnitude of X(w) vs w
subplot(1, 2, 1);
plot(w, Xw_mag);
xlabel('w   (radians/sample)'); ylabel('|X(e^{jw})|');
title({'Magnitude of X(e^{jw}) vs w';''}, 'FontSize', 16);
setDTFTradialAxis(1); ax = gca; ax.FontSize = 16;

% plot phase of X(w) vs w
subplot(1, 2, 2);
plot(w, Xw_ang);
xlabel('w   (radians/sample)'); ylabel('∠X(e^{jw})   (degrees)');
title({'Phase of X(e^{jw}) vs w';''}, 'FontSize', 16);
setDTFTradialAxis(1); ax = gca; ax.FontSize = 16;
```



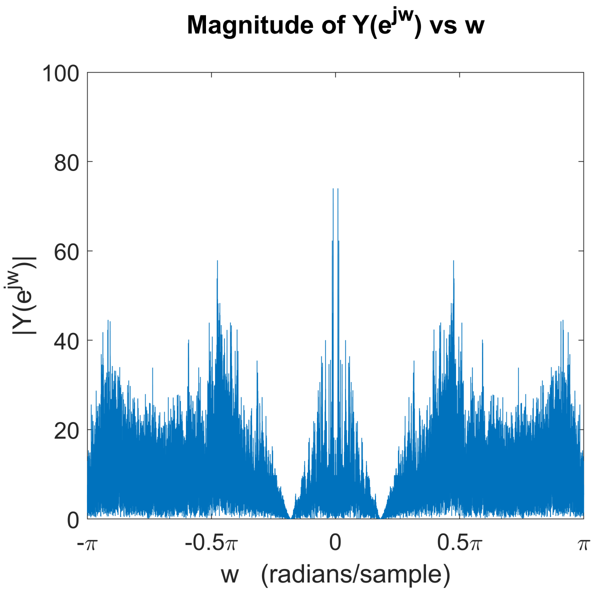


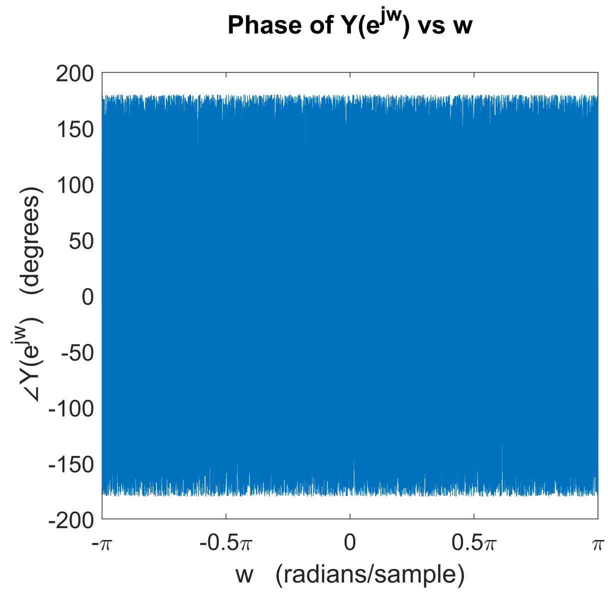

```
% plot magnitude of X(w) vs continuous-domain frequencies f
fig = figure;
plot(f, Xw_mag);
xlabel('f   (Hz)'); ylabel('|X(e^{jw})|');
xlim([min(f) max(f)]); xticks([min(f) -3000:2000:3000 max(f)]);
title({'Magnitude of X(e^{jw}) vs Continuous-Domain Frequencies f';''});
```



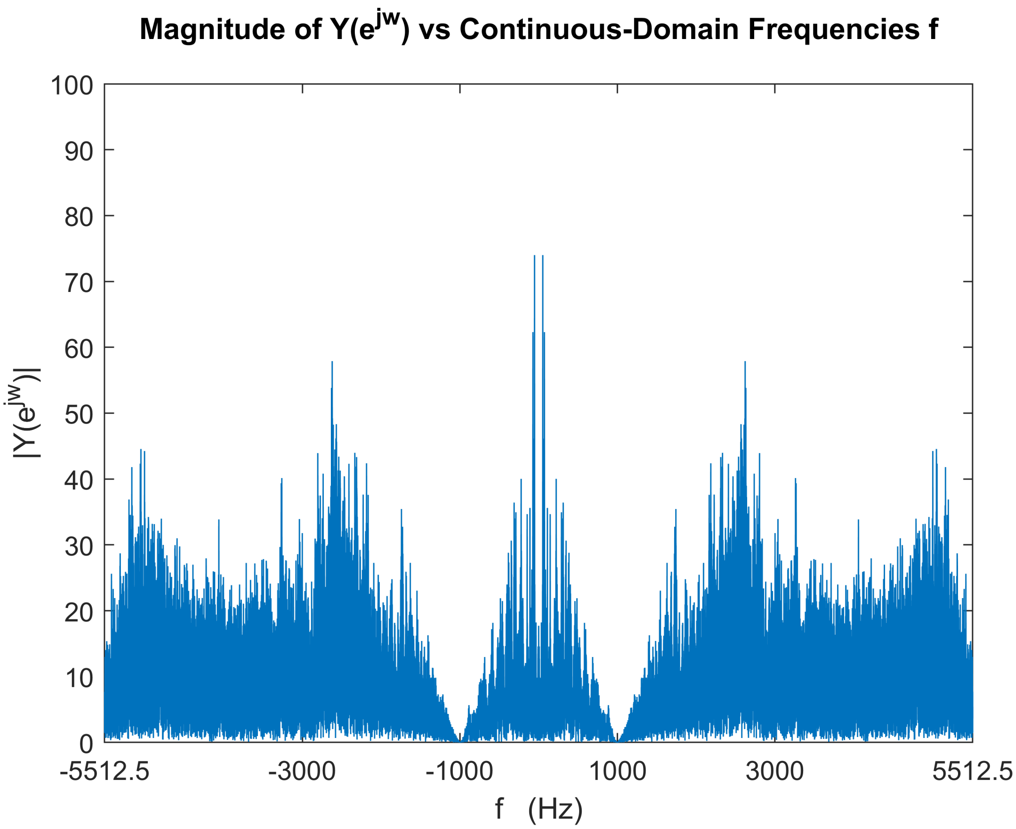
Note: Sharp tone of high magnitude can be seen in this spectrum $X(e^{j\omega})$, at about $F = \pm 1\,\mathrm{kHz}$.

```
% >>> task <<< pass the sound data through given system, using get_system_out(),
% with the system CCDE coefficients stated in methodology section

% b's and a's in poly. frac H(z)=B(z)/A(z) are (as shown in methodology section),
b = [1 -2*cos(w0) 1];
a = [1 0.5];

% assuming zero initial conditions (rest)
N_aux = zeros(1, length(a)-1);
M_init = zeros(1, length(b)-1);

% compute system output y[n] for 0 <= n < length(x), with input = sound data x[n],
% using get_system_out()
[y, ny] = get_system_out(b, a, x, N_aux, M_init);
duration = (length(y)-1)/Fs;

% get and play the audio y(t) using sound command, using given Fs
sound(y, Fs);
% pause execution while the sound is playing
pause(duration);

% >>> task <<< plot DTFT of sound data after passing through system (DTFT of y[n])

% initialize frequency vector for computing DTFT Y(w)
w = -pi:2*pi/(length(y)-1):pi;              % for higher/proper DTFT resolution
f = Fs * w/(2*pi);                          % corresponding cont. domain frequencies

% compute DTFT Y(w) using fft() and fftshift()
Yw = fftshift(fft(y));

% obtain magnitude and phase of DTFT Y(w)
Yw_mag = abs(Yw);                           % get the magnitude of Y(w)
Yw_ang = angle(Yw)*180/pi;                  % get the phase (angle) of Y(w) in degrees

% plot magnitude and phase of DTFT Y(w)
fig = figure; set(fig,'Units','normalized','Position',[0 0 1.4 1]);

% plot magnitude of Y(w) vs w
subplot(1, 2, 1);
plot(w, Yw_mag);
xlabel('w   (radians/sample)'); ylabel('|Y(e^{jw})|');
ylim([0 100]);
title({'Magnitude of Y(e^{jw}) vs w';''}, 'FontSize', 16);
setDTFTradialAxis(1);
ax = gca; ax.FontSize = 16;

% plot phase of Y(w) vs w
subplot(1, 2, 2);
plot(w, Yw_ang);
xlabel('w   (radians/sample)'); ylabel('∠Y(e^{jw})   (degrees)');
title({'Phase of Y(e^{jw}) vs w';''}, 'FontSize', 16);
setDTFTradialAxis(1);
ax = gca; ax.FontSize = 16;
```






```
% plot magnitude of Y(w) vs continuous-domain frequencies f
fig = figure;
plot(f, Yw_mag);
xlabel('f   (Hz)'); ylabel('|Y(e^{jw})|');
xlim([min(f) max(f)]); xticks([min(f) -3000:2000:3000 max(f)]); ylim([0 100]);
title({'Magnitude of Y(e^{jw}) vs Continuous-Domain Frequencies f';''});
```



Note: Compared to the magnitude plot of $X(e^{j\omega})$, this magnitude plot of $Y(e^{j\omega})$ is zoomed in on the y-axis.

Question (1):

Can you identify the tone in the sound spectrum before passing through the system?

Answer (1):

Yes, the tone can be visibly identified in sound data $x[n]$'s spectrum magnitude $|X(e^{j\omega})|$ (at Page 20, top-left figure). It occurs at about $\omega = \pm 0.5699 = \pm 0.1814\pi$ radians/second.

For the sampling rate $F_s = 11025$ samples/second, the DT radial frequencies $\omega = [-\pi, \pi]$ correspond to CT frequencies:

$$F = \left[-\frac{\Omega_s/2}{2\pi}, \frac{\Omega_s/2}{2\pi}\right] = \left[-\frac{F_s}{2}, \frac{F_s}{2}\right] = [-5512.5, 5512.5]\ \mathrm{Hz}$$

The bottom figure on Page 20 shows the spectrum magnitude $|X(e^{j\omega})|$ against respective CT frequencies (in Hertz). So, the tone identified at $\omega = \pm 0.1814\pi$ radians/second corresponds to a tone of CT frequency:

$$\frac{0.1814\pi \times 11025}{2\pi}\ \mathrm{Hz} \approx 1\ \mathrm{kHz}$$

in the audio signal. This tone is also audible in the audio playback signal $x(t)$ reconstructed from $x[n]$.

Question (2):

Can you identify the tone in the sound spectrum after passing through the system?

Answer (2):

As discussed in Task 1 Q4 (Page 12-13), the given filter $H(z)$ has a sharp dip (a cutoff for a single point) around $\omega = \pm 0.1814\pi$ radians/second (see Page 12, top-left figure), which for $F_s = 11025$ samples/second maps to CT frequency $F \approx 1\ \mathrm{kHz}$.

As answered in Q1 above, $1\ \mathrm{kHz}$ is also the frequency where the tone lies in the given audio sequence $x[n]$ (see Page 20, bottom figure).

So, after passing given audio sequence through the system $H(z)$, the output spectrum is given by,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

which means that resultant magnitude in $Y(e^{j\omega})$ at $F = 1\ \mathrm{kHz}$ is nearly zero (see Page 22, bottom figure). Hence, the tone at $F = 1\ \mathrm{kHz}$ has been removed in the output sequence $y[n] = h[n] * x[n]$, and is no longer present in respective (filtered) audio signal $y(t)$.

Question (3):

Do you hear the tone in the output of your system? Can you relate the output of your system to its (system's) Fourier Transform.

Answer (3):

As described in the above answer, the tone identified at $F = 1\,\mathrm{kHz}$ in sound data $x[n]$ has been removed in output $y[n]$ obtained after passing through given system. Hence, this tone is no longer audible in the output audio signal $y(t)$ reconstructed from $y[n]$.

The output Fourier Transform is $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$, magnitude shown in bottom figure at Page 22. Here, $H(e^{j\omega})$ is a sinc-filter which has high magnitude around the higher-frequencies (near $\pm\pi$) and lower magnitude at frequencies near 0, and has a sharp dip (magnitude ≈ 0) at $F = 1\,\mathrm{kHz}$. The magnitude of spectrum $X(e^{j\omega})$ of input audio sequence can be seen in bottom figure on Page 20; it has a sharp high-magnitude tone at $F = 1\,\mathrm{kHz}$, and has magnitude becoming smaller towards the higher-frequencies.

Upon multiplication with $H(e^{j\omega})$, the magnitude of frequencies below $1\,\mathrm{kHz}$ in $X(e^{j\omega})$ is relatively reduced in $Y(e^{j\omega})$, the magnitude of frequencies above $1\,\mathrm{kHz}$ in $X(e^{j\omega})$ is relatively increased in $Y(e^{j\omega})$ (with respect to the sinc pattern), and the magnitude at and around frequency $1\,\mathrm{kHz}$ is largely reduced. Particularly, at $1\,\mathrm{kHz}$, the magnitude is now nearly 0, thereby eliminating the $1\,\mathrm{kHz}$ tone in the given audio signal. This explains the spectrum-magnitude transformation from $X(e^{j\omega})$ (input) to $Y(e^{j\omega})$ (output), due to the system FT $H(e^{j\omega})$.