Assignment No. 8: **Inverse Systems**

Reg. No. : 2016-EE-189

Task 1

Statement:

Consider a system $H(z)$ which has the following poles and zeros:

- 4 poles at $z = 0$
- 4 zeros at $z = 0.9e^{j\,0.6\pi},\ 0.9e^{-j\,0.6\pi},\ 1.25e^{j\,0.8\pi},\ 1.25e^{-j\,0.8\pi}$

Consider a signal $x[n]$,

$$x[n] \ = \ \text{sinc}\left(\frac{\pi}{16}(n-50)\right)\cos(\omega_c\, n)$$

Question (1):

Draw the magnitude, phase and group delay response of the system using MATLAB. Do not use MATLAB's builtin functions.

Answer (1):

The system $H(z)$ described above will have the following expression,

$$H(z) \ = \ \frac{(z-0.9e^{j\,0.6\pi})(z-0.9e^{-j\,0.6\pi})(z-1.25e^{j\,0.8\pi})(z-1.25e^{-j\,0.8\pi})}{z^4}$$

$$H(z) \ = \ (1-0.9e^{j\,0.6\pi}\,z^{-1})(1-0.9e^{-j\,0.6\pi}\,z^{-1})(1-1.25e^{j\,0.8\pi}\,z^{-1})(1-1.25e^{-j\,0.8\pi}\,z^{-1})$$

Note that it is of the form,

$$H(z) \ = \ \left(1-a_1\,z^{-1}\right)\left(1-a_1{}^*\,z^{-1}\right)\left(1-a_2\,z^{-1}\right)\left(1-a_2{}^*\,z^{-1}\right)$$

where $a_1 = 0.9e^{j\,0.6\pi}$ and $a_2 = 1.25e^{j\,0.8\pi}$. All of its 4 poles are at $z = 0$; 2 zeros (conjugate pair) are inside the unit-circle, and the other 2 (also conjugate pair) are outside the unit-circle.

The code, along with results (inline), to compute & plot various responses from $H(z)$ expression above, starts on the next page.

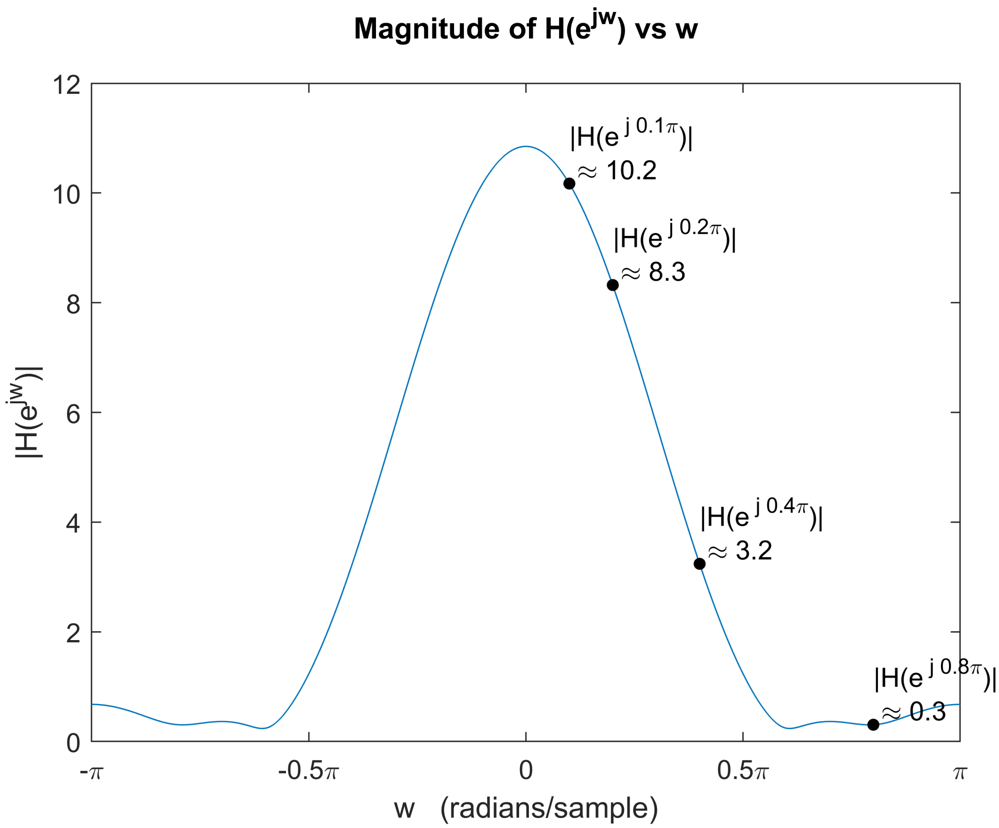
Note: The various functions used in the code (markOnPlot(), plot2SeqDTFT(), plot3SeqDTFT()) have been defined at the end of report.

```
% <task> plot magnitude and phase response of given H(z)

% define z-domain expression (symbolically)
syms z;
a1 = 0.9*exp(1i*0.6*pi);
a2 = 1.25*exp(1i*0.8*pi);
Hz_N = (z-a1)*(z-conj(a1))*(z-a2)*(z-conj(a2));     % numerator of H(z)
Hz_D = z^4;                                         % denominator of H(z)
Hz = Hz_N/Hz_D;

% display H(z) expression
disp('H(z) = '); disp(vpa(Hz, 4))
```

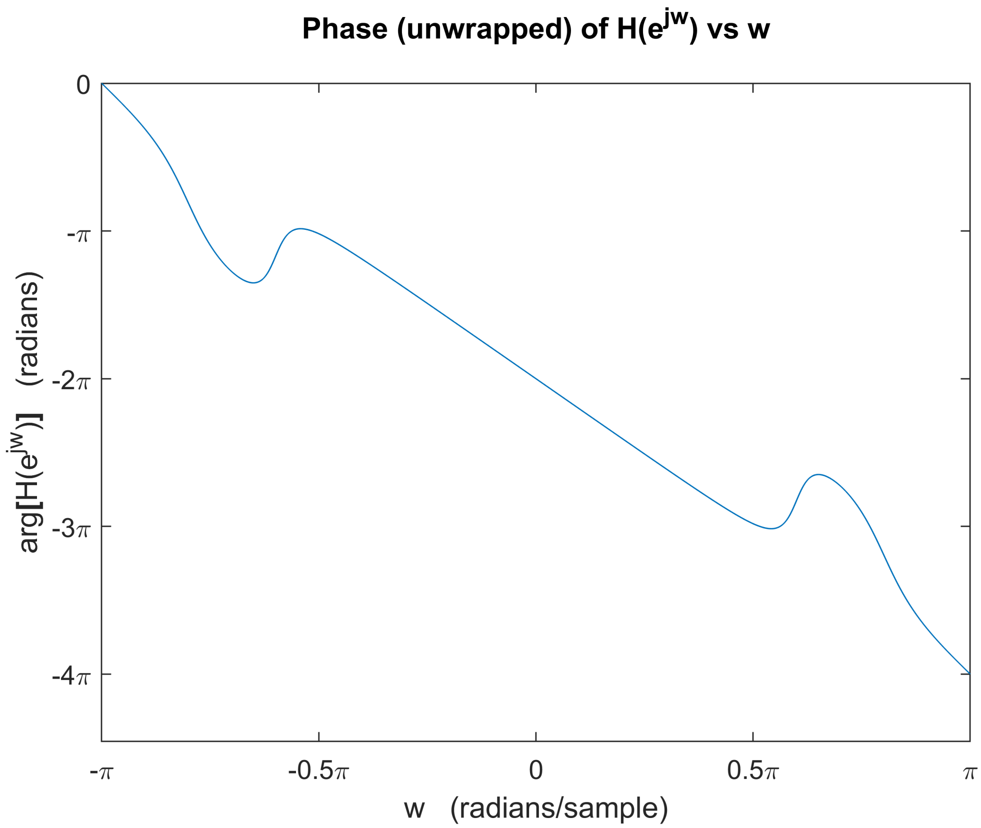
H(z) =

$$\frac{(z+1.011-0.7347\,i)\,(z+1.011+0.7347\,i)\,(z+0.2781-0.856\,i)\,(z+0.2781+0.856\,i)}{z^4}$$

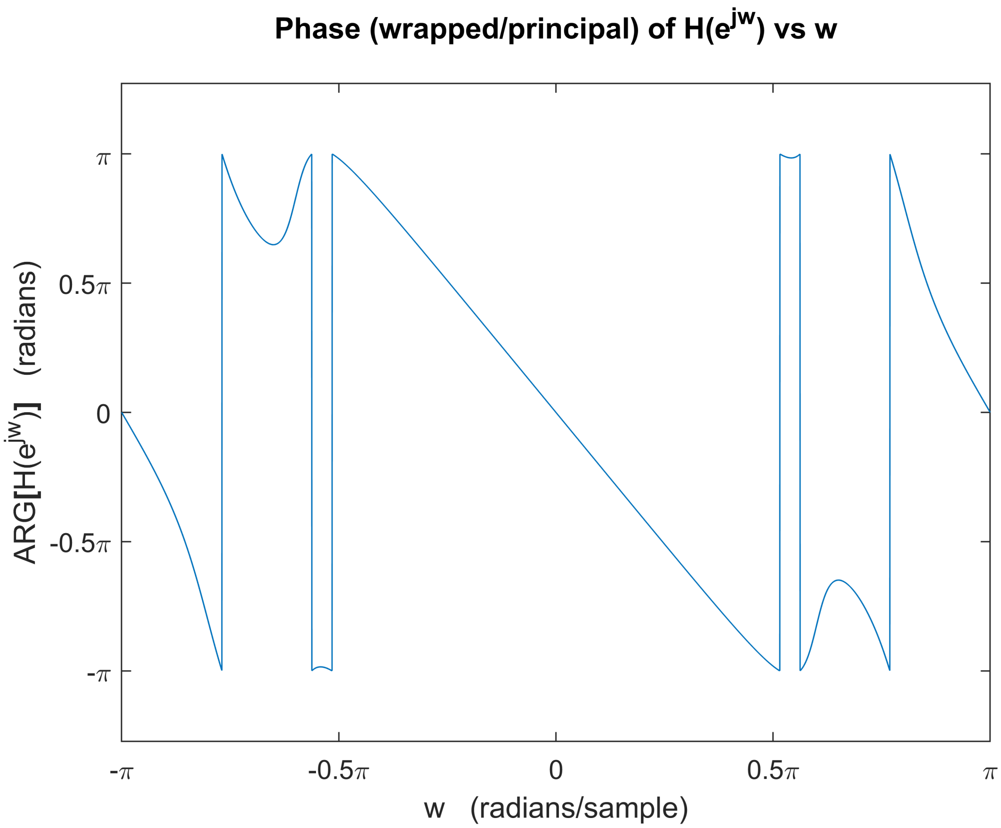
```
% obtain frequency response H(w) expression from H(z)
% by substituting z = e^(jw)
syms w_;
Hw = subs(Hz, {z}, {exp(1i*w_)});

% compute frequency response H(w) for w = [-pi, pi]
dw = 0.001;
w = -pi:dw:pi;
Hw = subs(Hw, {w_}, {w});
Hw = double(Hw);

% obtain magnitude and phase of H(w)
Hw_mag = abs(Hw);
Hw_arg = phase(Hw);                     % unwrapped phase (in radians)
Hw_ARG = angle(Hw);                     % wrapped phase (in radians)

% frequencies to be noted/marked on the response plots
wvals = [0.1, 0.2, 0.4, 0.8]*pi;

% plot magnitude of H(w) vs w
fig = figure;
plot(w, Hw_mag);
xlabel('w   (radians/sample)');
ylabel('|H(e^{jw})|');
title({'Magnitude of H(e^{jw}) vs w';''});
setDTFTradialAxis(1);
markOnPlot(wvals, w, Hw_mag, dw, [], 0.65, {'|H(e^{ j ', '\pi})|', ''}, pi);
```




```
% plot phase (unwrapped) of H(w) vs w
fig = figure;
plot(w, Hw_arg);
xlabel('w   (radians/sample)');
ylabel('arg{\bf[}H(e^{jw}){\bf]}   (radians)');
title({'Phase (unwrapped) of H(e^{jw}) vs w';''});
setDTFTradialAxis(1, 1, 1);
```




```
% plot phase (wrapped/principal) of H(w) vs w
fig = figure;
plot(w, Hw_ARG);
xlabel('w   (radians/sample)');
ylabel('ARG{\bf[}H(e^{jw}){\bf]}   (radians)');
title({'Phase (wrapped/principal) of H(e^{jw}) vs w';''});
setDTFTradialAxis(1, 0.5, 1);
```



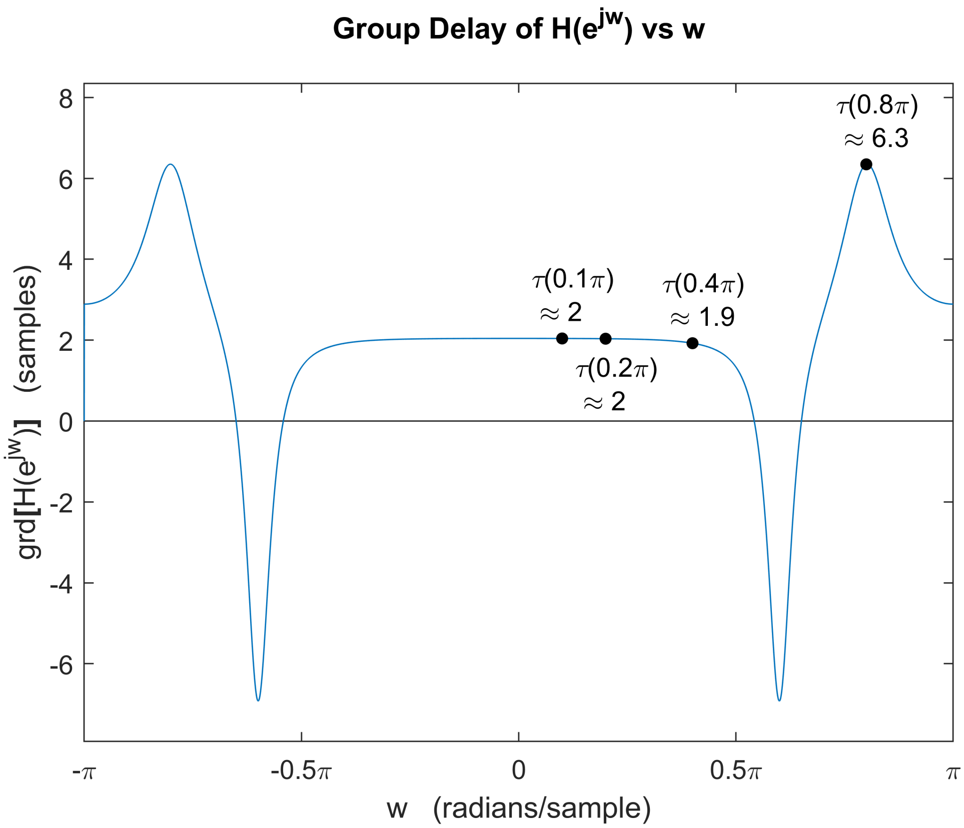
Now, the group delay for $H(e^{j\omega})$ is given by,

$$\tau(\omega) \;=\; \mathrm{grd}[H(e^{j\omega})] \;=\; -\frac{d}{d\omega}\left\{\arg[H(e^{j\omega})]\right\}$$

where the derivative of (unwrapped) phase may be computed using,

$$\frac{d}{d\omega}\left\{\arg[H(e^{j\omega})]\right\} \;=\; \lim_{\Delta\omega \to 0} \frac{\arg[H(e^{j\,(\omega+\Delta\omega)})] - \arg[H(e^{j\omega})]}{\Delta\omega}$$

```
% <task> plot group-delay of given H(z)

dw = (w(end)-w(1))/(length(w)-1);          % compute the w-step-size 'dw' used in w-array

% obtain group-delay of H(w)
Hw_grd = -([Hw_arg 0]-[0 Hw_arg])/dw;
Hw_grd = Hw_grd(1:(length(Hw_grd)-1));

% plot group-delay of H(w) vs w
fig = figure;
plot(w, Hw_grd);
xlabel('w   (radians/sample)'); ylabel('grd{\bf[}H(e^{jw}){\bf]}   (samples)');
title({'Group Delay of H(e^{jw}) vs w';''});
setDTFTradialAxis(1); ylim([min(Hw_grd)-1 max(Hw_grd)+2]);
markOnPlot(wvals, w, Hw_grd, dw, -0.07*pi, [1 -1 1 1]*1.1, {'\tau(', '\pi)', ''}, pi);
```



Question (2):

Is the system stable? Is the system causal?

Answer (2):

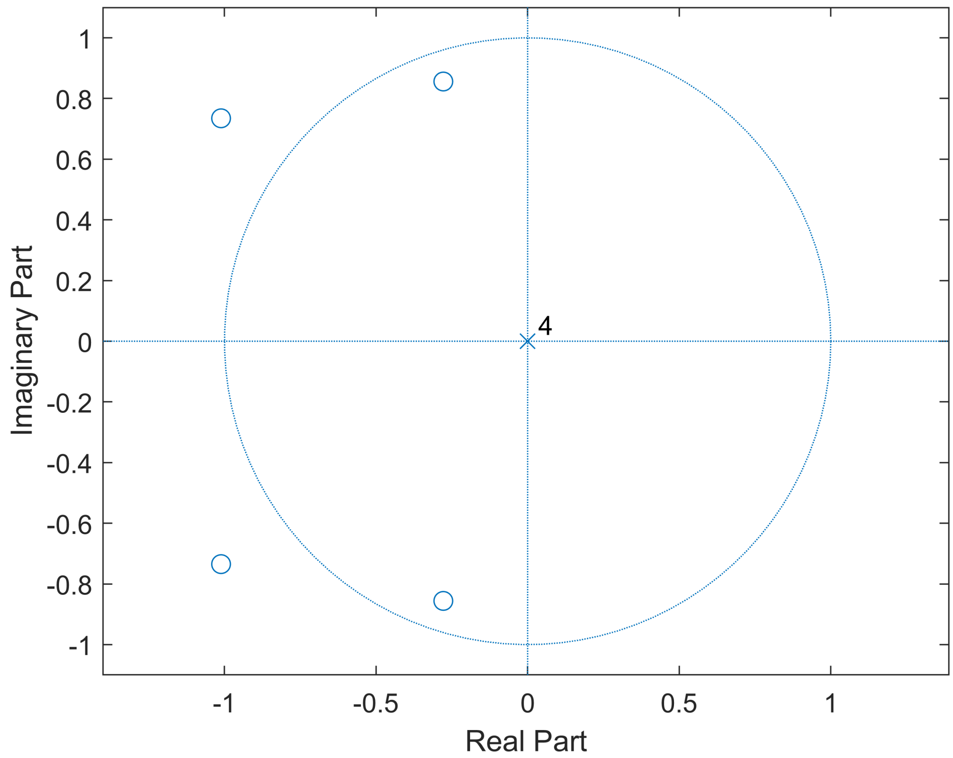
Consider the pole-zero plot of this system $H(z)$, as drawn below.

```
% extract CCDE coefficients (b's and a's) from H(z) expression
[N, D] = numden(Hz);
b = coeffs(N, 'All');   a = coeffs(D, 'All');
b = double(b./a(1));    a = double(a./a(1));
disp('b ='); disp(b);   disp('a ='); disp(a);
```

```
b =
    1.0000    2.5788    3.4975    2.5074    1.2656

a =
     1     0     0     0     0
```

```

% plot the pole-zero map of H(z) using extracted b's and a's
figure;
zplane(b, a);
title({'Pole-Zero Plot of H(z) in z-plane';''});
```

Pole-Zero Plot of H(z) in z-plane



As can be seen above, all poles of $H(z)$ lie within the unit-cricle. Furthermore, all 4 poles lie at origin. This means the only possible ROC is the one extending outward from $|z| > 0$. So, ROC is all of z-plane except $z = 0$.

Since the unit-circle lies in ROC, the system is stable. As here, ROC extends outward from outermost pole $(z = 0)$, so $h[n]$ is a right-sided function (possibly with $h[n] = 0 \;\forall\; n < 0$). Hence the system is also causal.

So the system represented by $H(z)$ is causal and stable.

Question (3):

Pass $x[n]$ through $H(z)$ for $\omega_c = 0.1\pi,\ 0.2\pi,\ 0.4\pi,\ 0.8\pi$ and obtain the output signal $y[n]$.

How does the magnitude of the output signal change with change in ω_c? Can you explain the difference in magnitude corresponding to different ω_c using the magnitude response of $H(z)$.

Answer (3):

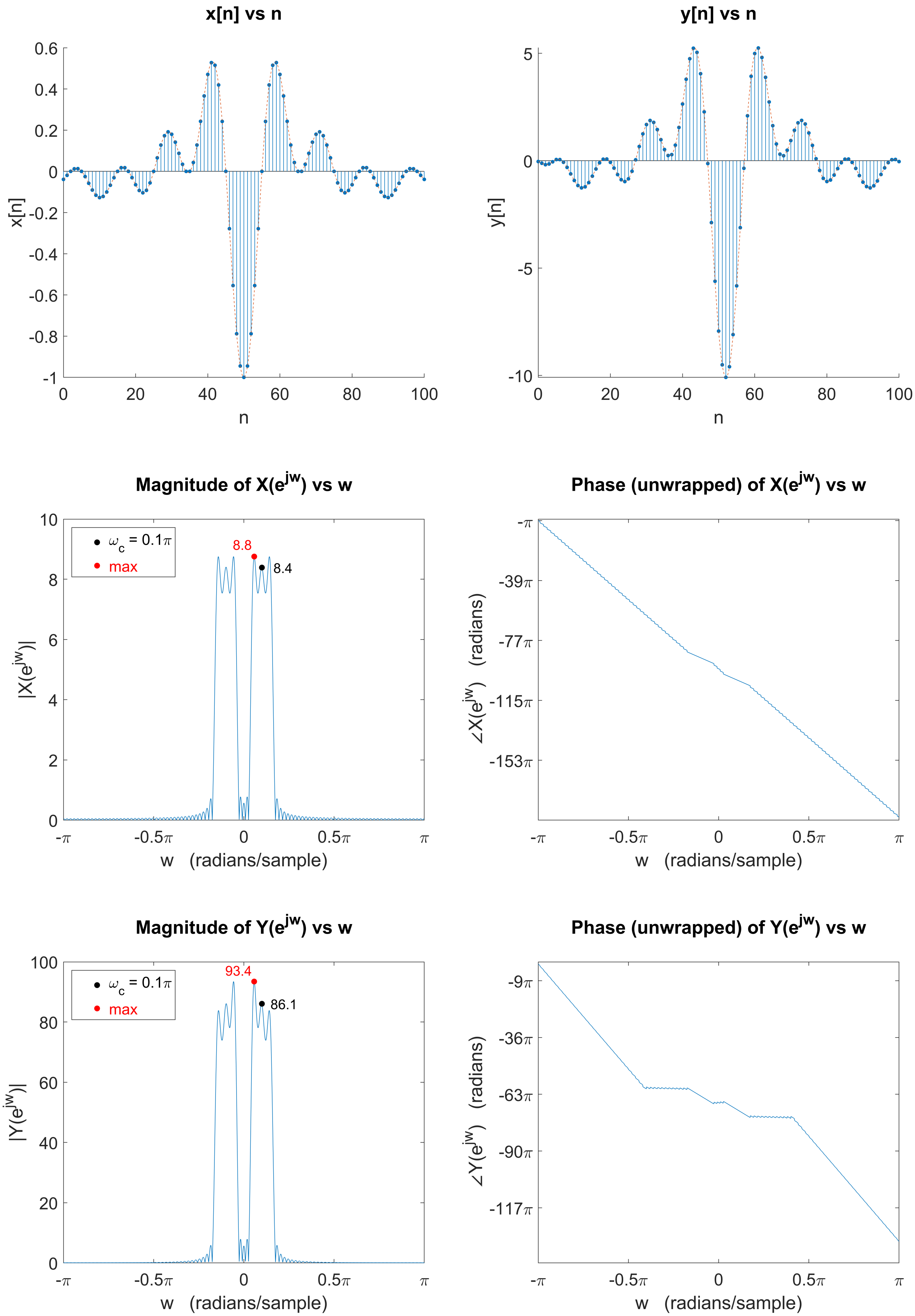
Since MATLAB defines $\operatorname{sinc}()$ function as $\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$, we'll define $x[n]$ as, $x[n] = \operatorname{sinc}\left(\dfrac{n-50}{16}\right)\cos(\omega_c n)$,

i.e. without the extra π in MATLAB.

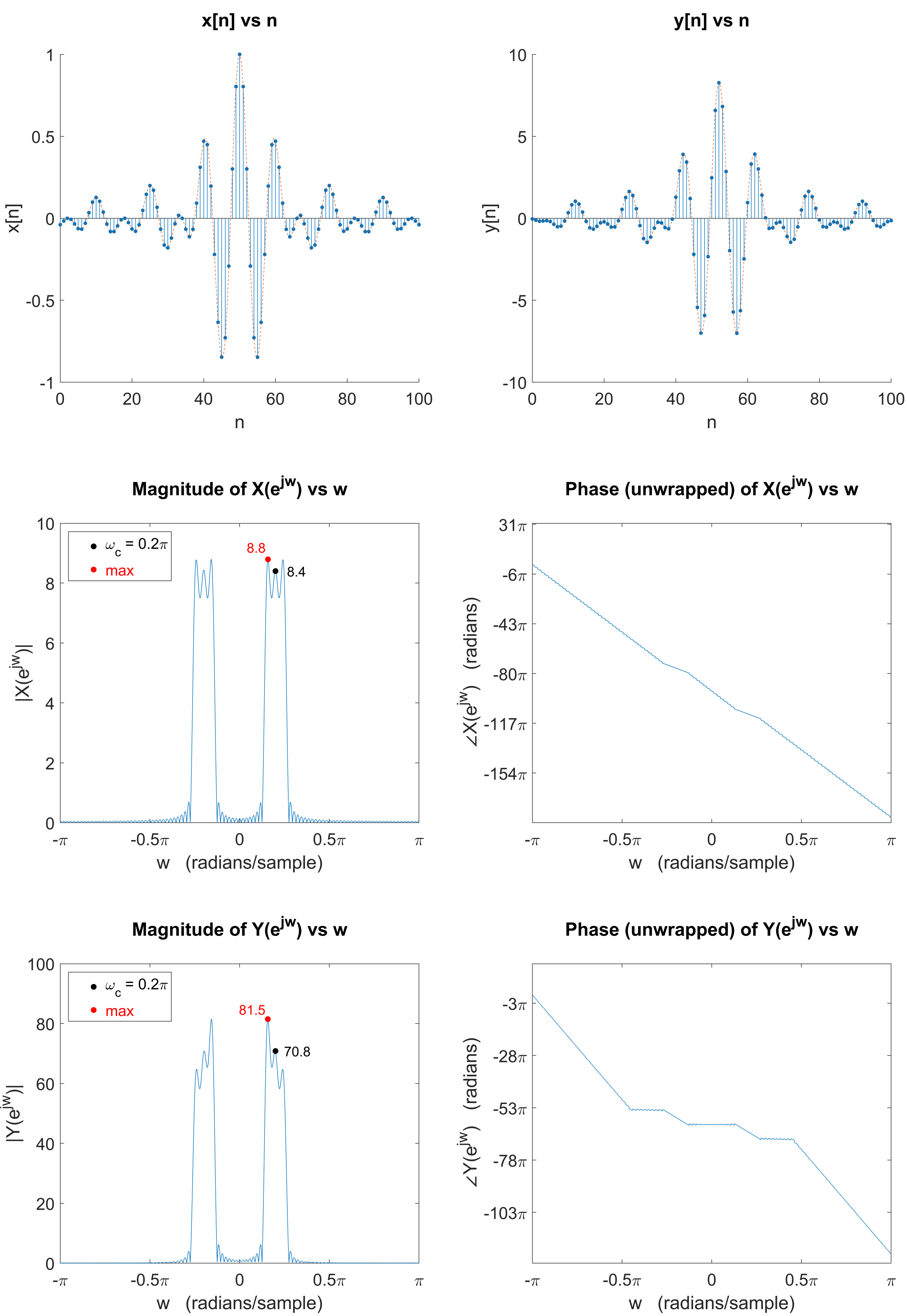
Note that in frequency-domain, this $\text{sinc}()$ function corresponds to a gate-pulse of certain width (W), which is shift-copied to $\omega = \pm\,\omega_c\,\text{rad/sample}$ due to the $\cos()$ term. Changing ω_c in $x[n]$ thus changes where the gate-pulse (+ve & -ve) is centered in $x[n]$'s spectrum.

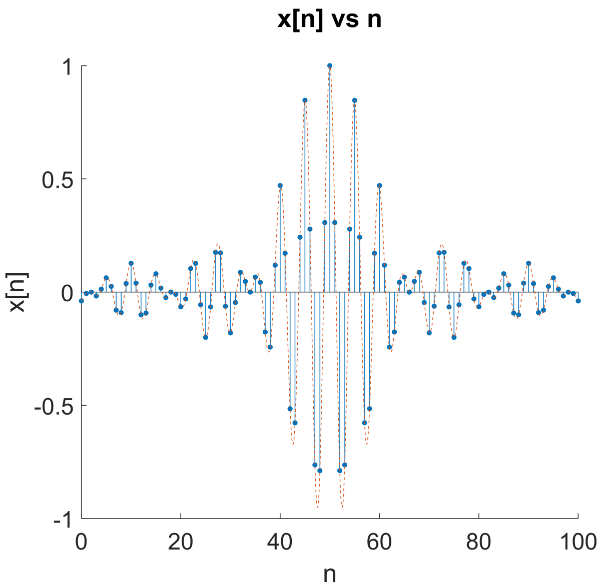
The code, along with the results, to pass $x[n]$ for each ω_c through $H(z)$, starts below.

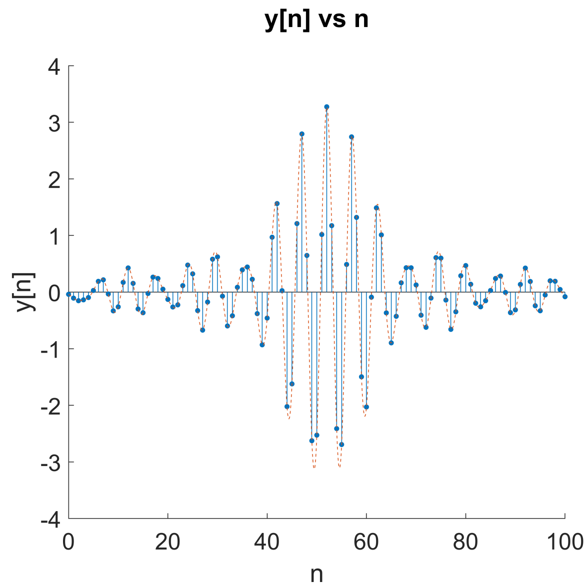
```
% <task> pass x[n] (for each wc value) through system H(z),
% and plot the output y[n] and its spectrum

wcvals = [0.1, 0.2, 0.4, 0.8]*pi;           % wc values (x[n] center frequencies)
n = 0:100;
x = cell(1, length(wcvals));                % to store x[n] arrays for different wc
y = cell(1, length(wcvals));                % to store y[n] arrays for different wc
ny = cell(1, length(wcvals));               % to store ny arrays for different wc

for i = 1:length(wcvals)

    % define x[n] = (sin(pi*(n-50)/16)./(pi*(n-50)/16)).*cos(wc*n)
    wc = wcvals(i);
    x{i} = sinc((n-50)/16).*cos(wc*n);

    % compute DTFT X(w)
    w = -pi:0.001:pi;                       % for 1 period [-pi, pi] of DTFT
    Xw = x{i} * exp(-1i * n' * w);          % using matrix-multiplication method

    % obtain the magnitude and phase (unwrapped, in radians) of DTFT X(w)
    Xw_mag = abs(Xw);
    Xw_arg = phase(Xw);

    % assuming zero initial conditions (rest)
    N_aux = zeros(1, length(a)-1);
    M_init = zeros(1, length(b)-1);

    % compute output y[n], with input x[n] to system represented by H(z),
    % using get_system_out() and extracted b's and a's
    [y{i}, ny{i}] = get_system_out(b, a, x{i}, N_aux, M_init);

    % compute DTFT Y(w)
    w = -pi:0.001:pi;                       % for 1 period [-pi, pi] of DTFT
    Yw = y{i} * exp(-1i * ny{i}' * w);      % using matrix-multiplication method

    % obtain the magnitude and phase (unwrapped, in radians) of DTFT Y(w)
    Yw_mag = abs(Yw);
    Yw_arg = phase(Yw);

    % plot sequences x[n] & y[n], and their DTFTs X(w) & Y(w) (magnitude and phase)
    plot2SeqDTFT(n, x{i}, wc, w, Xw_mag, Xw_arg, 'x', 'X', ...
        [0.2 0 -0.04 0.4, 0.15 0 -0.04 0.04*max(Yw_mag)], ...
        ny{i}, y{i}, Yw_mag, Yw_arg, 'y', 'Y', 'left', 'right');

end
```

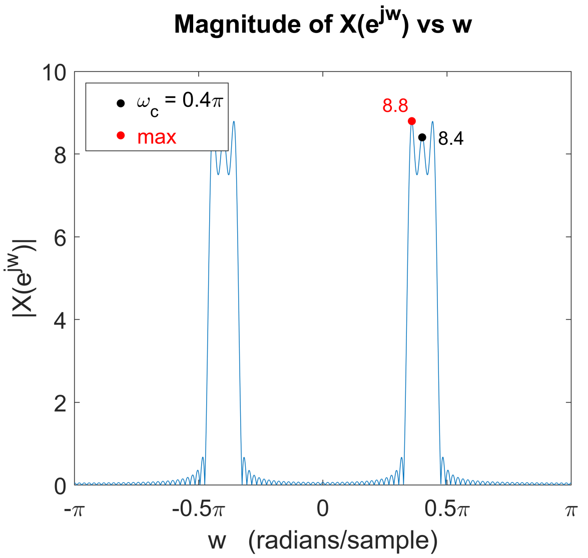


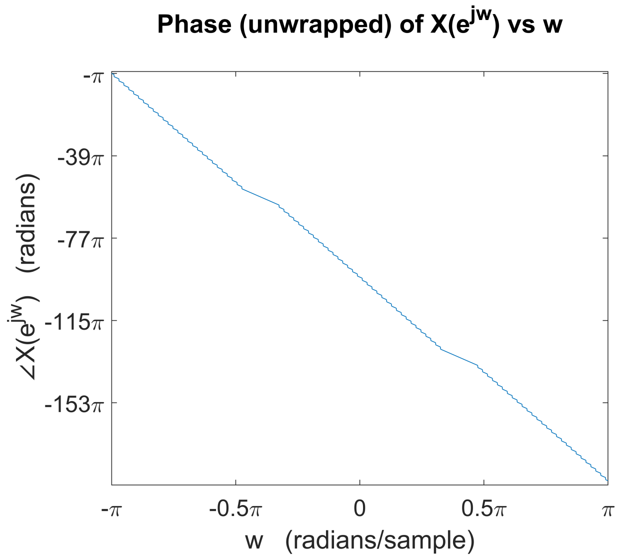
x[n] vs n
x[n]
n
y[n] vs n
y[n]
n
Magnitude of X(e^jw) vs w
|X(e^jw)|
ωc = 0.1π
max
8.8
8.4
w (radians/sample)
Phase (unwrapped) of X(e^jw) vs w
∠X(e^jw) (radians)
-π
-39π
-77π
-115π
-153π
w (radians/sample)
Magnitude of Y(e^jw) vs w
|Y(e^jw)|
ωc = 0.1π
max
93.4
86.1
w (radians/sample)
Phase (unwrapped) of Y(e^jw) vs w
∠Y(e^jw) (radians)
-9π
-36π
-63π
-90π
-117π
w (radians/sample)

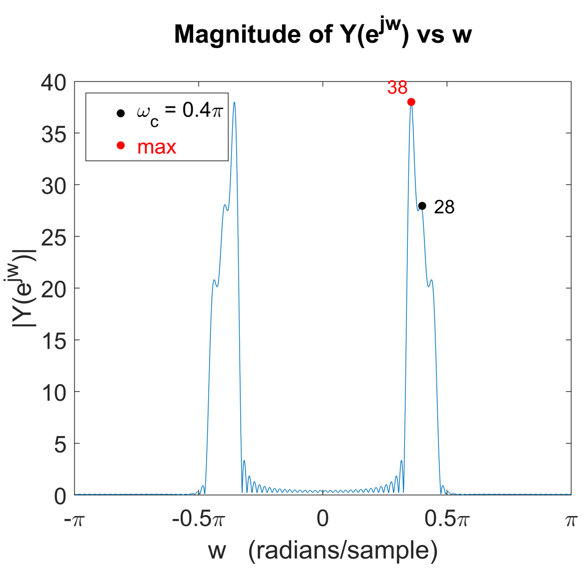

x[n] vs n
y[n] vs n
Magnitude of X(e^jw) vs w
ω_c = 0.2π
max
8.8
8.4
Phase (unwrapped) of X(e^jw) vs w
Magnitude of Y(e^jw) vs w
ω_c = 0.2π
max
81.5
70.8
Phase (unwrapped) of Y(e^jw) vs w
w (radians/sample)

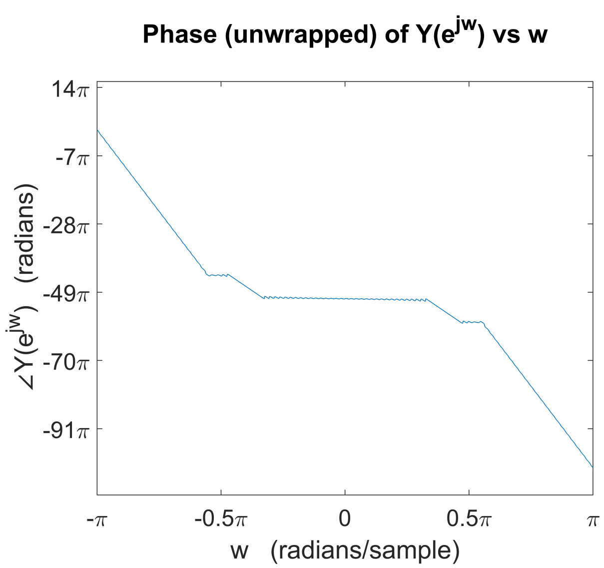

x[n] vs n
x[n]
n

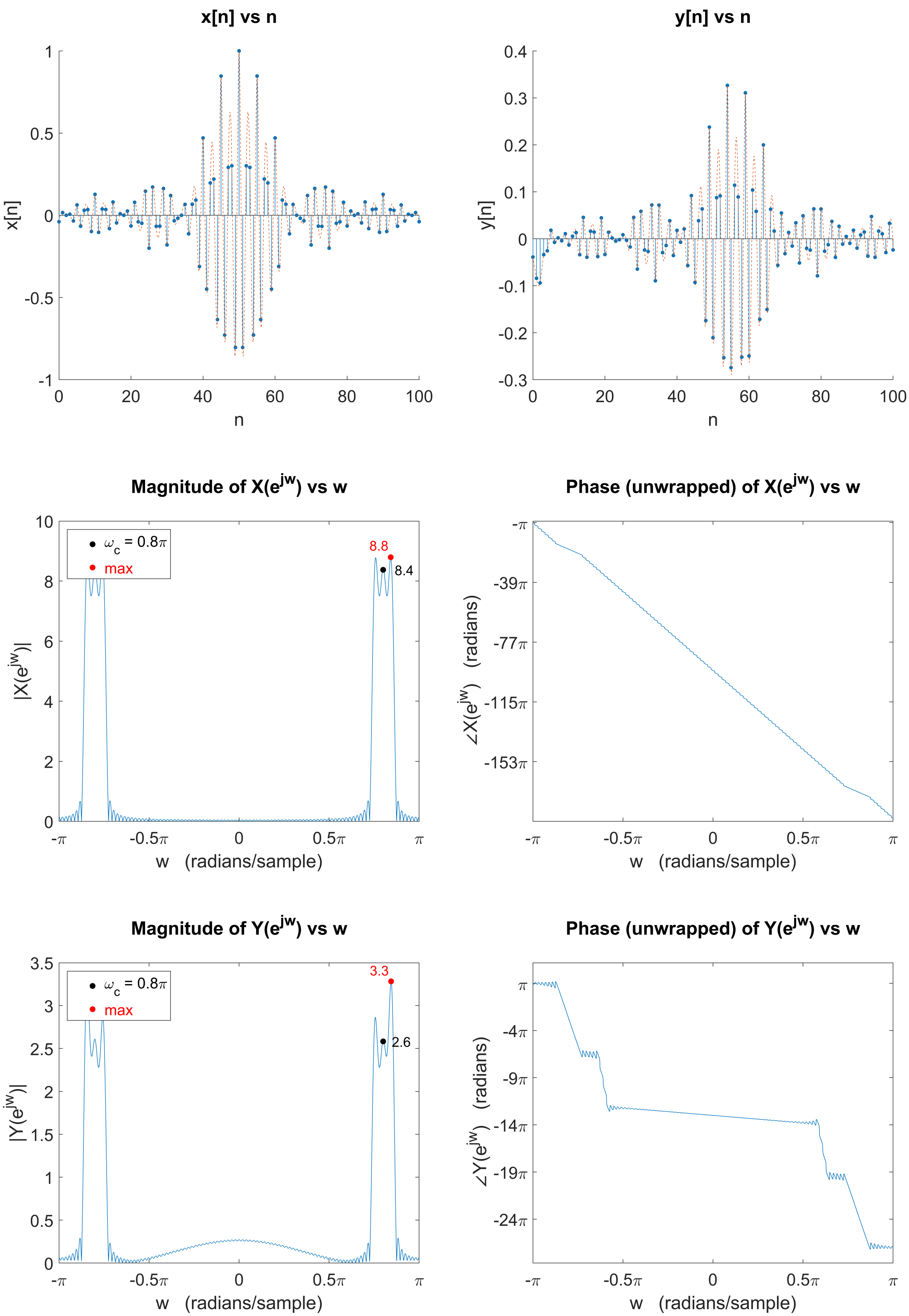

y[n] vs n
y[n]
n


Magnitude of X(e^jw) vs w
|X(e^jw)|
w (radians/sample)
ω_c = 0.4π
max
8.8
8.4


Phase (unwrapped) of X(e^jw) vs w
∠X(e^jw) (radians)
w (radians/sample)


Magnitude of Y(e^jw) vs w
|Y(e^jw)|
w (radians/sample)
ω_c = 0.4π
max
38
28


Phase (unwrapped) of Y(e^jw) vs w
∠Y(e^jw) (radians)
w (radians/sample)


x[n] vs n
y[n] vs n
Magnitude of X(e^jw) vs w
Phase (unwrapped) of X(e^jw) vs w
Magnitude of Y(e^jw) vs w
Phase (unwrapped) of Y(e^jw) vs w
ω_c = 0.8π
max
8.8
8.4
3.3
2.6
x[n]
y[n]
n
|X(e^jw)|
∠X(e^jw) (radians)
|Y(e^jw)|
∠Y(e^jw) (radians)
w (radians/sample)

In the magnitude plot of $H(e^{j\omega}) = H(\omega)$ (Page 3), the magnitude values i.e. gains for each of the ω_c frequencies have been marked. These gain values are as follows:

$$\begin{aligned}
|H(\;0.1\pi \;\text{ rad/sample }\;)| &\approx 10.2 \\
|H(\;0.2\pi \;\text{ rad/sample }\;)| &\approx 8.3 \\
|H(\;0.4\pi \;\text{ rad/sample }\;)| &\approx 3.2 \\
|H(\;0.8\pi \;\text{ rad/sample }\;)| &\approx 0.3
\end{aligned}$$

Since $H(z)$ is not an all-pass system, the various frequency components (centered around ω_c) in $x[n]$ are distorted (in magnitude) by different amounts. Following table summarizes the variation in magnitude from $X(e^{j\omega})$ to $Y(e^{j\omega})$ after being passed through $H(z)$:

	$\lvert X(e^{j\omega})\rvert$		$\lvert Y(e^{j\omega})\rvert$		$\frac{\lvert Y(e^{j\omega})\rvert}{\lvert X(e^{j\omega})\rvert} = \lvert H(e^{j\omega})\rvert$	
center frequency of x[n] (ω_c)	at ω_c	max	at ω_c	max	at ω_c	max
0.1π rad/sample	8.4	8.8	86.1	93.4	10.3	10.6
0.2π rad/sample	8.4	8.8	70.8	81.5	8.4	9.26
0.4π rad/sample	8.4	8.8	28	38	3.3	4.3
0.8π rad/sample	8.4	8.8	2.6	3.3	0.3	0.38

The ratio of output to input magnitude (shown in the last 2 columns of above table) corresponds to the gain provided by the system $H(z)$ to the input. Note that the gain values at center frequencies ω_c, calculated in above table, match (approximately) with the magnitude values noted further above from the $|H(e^{j\omega})|$ plot at respective frequencies. Thus the output magnitude is given by,

$$|Y(e^{j\omega})| = |X(e^{j\omega})|\,|H(e^{j\omega})|$$

And since $H(e^{j\omega})$ has different (widely-varying) magnitudes at different frequencies ω, this explains the changing magnitude of output $Y(e^{j\omega})$ based on the frequency-content in input $X(e^{j\omega})$ (centered around $\pm\omega_c$).

Next Question/Answer starts from the next page.