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CONSIDERATIONS OF QUASI-STATIONARITY IN ELECTROPHYSIOLOGICAL SYSTEMS

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Conditions under which a time varying electromagnetic field problem (such as arises in electrophysiology, electrocardiography, etc.) can be reduced to the conventional quasi-static problem are summarized. These conditions are discussed for typical physiological parameters.

1. *Introduction.* Bioelectric sources originate in the electrochemical activity of individual cells. A consequence of this activity is an electric potential field which is established in the volume conductor exterior to the cell. This field is, in general, time varying. Studies which have been made in classical electrophysiology as well as in applications to electrocardiography, electromyography and electroencephalography, etc. are based on a quasi-statical model. That is, it is assumed that at each instant of time the potential field satisfies Laplace's equation, and that the boundary conditions are those which would exist if the source condition were stationary. The following is essentially a tutorial examination and summary of the bases for quasi-static assumptions in bioelectric studies.

We start with a general formulation for the electric field in an infinite volume conductor due to an impressed current density source $\mathbf{J}_s$ whose temporal behavior is harmonic at an angular frequency ω. The medium is assumed, initially, to be linear, homogeneous and isotropic and characterized by physical parameters μ, σ and ε. To the degree that the medium possesses linearity, the results apply to a linear combination of harmonic frequencies and hence, to a periodic or to an aperiodic source through the use of a Fourier series or integral, respectively.

The electric and magnetic fields are found by solution of the inhomogeneous Helmholtz equations. The scalar potential Φ and the vector potential $\mathbf{A}$ are given by

$$\mathbf{A}(x, y, z) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}_s'(x', y', z')e^{-jkR}}{R} \, dV'(x', y', z'), \tag{1}$$

$$\Phi(x, y, z) = \frac{1}{4\pi(\sigma + j\omega\varepsilon)} \int_{V'} \frac{\rho'(x', y', z')e^{-jkR}}{R} \, dV'(x', y', z'), \tag{2}$$

where

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2. \tag{3}$$

The unprimed variables refer to the field point and the primed variables to the source point. In this equation

$$k^2 = \omega^2 \mu \varepsilon_c = \omega^2 \mu \varepsilon (1 + \sigma/j\omega\varepsilon), \tag{4}$$

and ε_c is a complex dielectric constant that includes the effect of conductivity and losses, that is, $\varepsilon_c = \varepsilon(1 + \sigma/j\omega\varepsilon)$. An alternate form for (4) is

$$k^2 = -j\omega\mu\sigma_c = -j\omega\mu\sigma(1 + j\omega\varepsilon/\sigma), \tag{5}$$

where $\sigma_c = \sigma(1 + j\omega\varepsilon/\sigma)$. This form is more appropriate for media that are essentially resistive. We also have

$$\nabla \cdot \mathbf{J}_s' = -\rho'$$

where $\mathbf{J}_s'$ is the applied current density and ρ' is a current volume source density. Finally, the electric field is found from the vector and scalar potentials $\mathbf{A}$ and Φ by

$$\mathbf{E} = -j\omega\mathbf{A} - \nabla\Phi. \tag{6}$$

It should be noted that complex phasor notation is used. That is, the quantities $\mathbf{A}$, $\mathbf{E}$, Φ, ρ, $\mathbf{J}$, ε_c, are complex and, in general, functions of ω.

The usual implications of quasi-stationarity involve assumptions to simplify the above equations. These assumptions will be enumerated in the following under four main headings: Propagation Effects, Capacitative Effect, Inductive Effects and Boundary Considerations. To evaluate the error involved in making these simplifications, it will be necessary to utilize representative data for the parameters appearing in (1) through (6), and this will be considered first.

2. *Electrical properties of biological materials.* Typical values for the conductivity of biological materials as reported by Rush *et al.* (1963) are given in

Table I. In computations involving conductivity, we shall choose $\sigma = 0.2$ mho/m as representing a mean value.

TABLE I*

Conductivity of Biological Tissues

Tissue	Mean Conductivity (mho/m)
Blood	0.67
Lung	.05
Liver	.14
Fat	.04
Human Trunk	0.21

* From Rush *et al.* (1963).

It is also required that the complex conductivity $\sigma(1 + j\omega\varepsilon/\sigma)$ be considered. This, in turn, involves utilization of typical ratios of displacement to conduction current, that is, $j\omega\varepsilon/\sigma$. Table II lists representative values as reported by Schwan and Kay (1957). Further discussion is given in the following sections.

TABLE II†

Averages of Ratio of Capacitive to Resistive Current for Various Frequencies and Body Tissues

	10 KC	100 KC	1000 KC	10,000 KC
Lung	0.15	0.025	0.05	0.14
Fatty Tissue		.01	.03	.15
Liver	.20	.035	.06	.20
Heart Muscle	0.10	0.04	0.15	0.32

† From Schwan and Kay (1957).

The highest component frequency of significance in bioelectric systems is of the order of 1 Kc. This probably relates to a maximum rise time for action potentials of around 1 ms. We shall choose 1 Kc. in the following numerical computations requiring a maximum value of frequency.

Finally, we note the absence of magnetic materials in biological systems so that the permeability is the free space value of $4\pi \times 10^{-7}$ henries/meter. The maximum value of R corresponds to an overall dimension of the human body: for simplicity $R_{\max} = 1$ m.

3. *Propagation effects.* The time required for changes in the source to propagate to a field point is represented by the phase delay e^{-jkR} in equations (1) and (2). Since

$$e^{-jkR} = 1 - jkR - \frac{(kR)^2}{2!} - j\frac{(kR)^3}{3!} \cdots, \tag{7}$$

propagation effects can be ignored if $kR \ll 1$ because e^{-jkR} is then approximately constant (i.e., unity). In this event, the integral in (2) is precisely that for static fields. Utilizing the data in Section 2 and setting the magnitude of $(1 + j\omega\varepsilon/\sigma)$ equal to the conservative value of $\sqrt{2}$ yields

$$kR_{\max} = (1 - j)\sqrt{2000\pi \times 4\pi \times 10^{-7} \times 0.2} = 0.0397\,(1 - j).$$

Thus, the magnitude of e^{-jkR} is unity to within a four percent error, while the phase angle error of 0.0397 rad. (2.3°) is clearly negligible. The numerical result is roughly the same as that reported by Geselowitz (1963) in a similar analysis.

4. *Capacitive effects.* The nature of the conductive medium is described by its conductivity and dielectric permittivity. The mathematical expression is given by equation (2) where the coefficient $(\sigma + j\omega\varepsilon)$ can be written as $\sigma(1 + j\omega\varepsilon/\sigma)$ and the conductivity viewed as a complex phasor quantity. In the quasi-static approximation, a purely resistive medium is required. This is necessary in order to justify the assumption that with complex time variations, the field quantities at all points are in synchrony. (Such an assumption is not strictly necessary under *harmonic conditions* as is clear from equation (2) since the formulation includes the appropriate phase relationships.) The quantity $\sigma(1 + j\omega\varepsilon/\sigma)$ will be real so long as $j\omega\varepsilon/\sigma \ll 1$. A consultation of Table II reveals that the inequality is satisfied fairly well. The values of this ratio at 10 cps which do not appear too small are reported to be conservative and Schwan and Kay (1957) conclude that the medium can be considered to be resistive.

5. *Inductive effects.* The component of electric field that arises from magnetic induction is given in (6) by the term $j\omega\mathbf{A}$. We wish to compare the importance of this term relative to that expressed by $\nabla\Phi$. We shall do this by considering the specific case of a differential current source element and assume that if $|\omega\mathbf{A}| \ll |\nabla\Phi|$ under these conditions then distributed sources, such as arise in electrophysiology, which are the superposition of such elements, would also satisfy the inequality.‡

Thus, for a source element $\mathbf{J}_s\,dV$ we have, from (1),

$$\mathbf{A} = \frac{\mu}{4\pi}\frac{\mathbf{J}_s'\,dV'}{R}. \tag{8}$$

‡ Special electrical devices, such as inductors, are capable of setting up quasi-static electric fields related to magnetic effects alone, i.e., capactive and propagation effects are negligible. In the analysis of the magnetocardiograph, for example, quasi-static inductive effects are the sole source of electric field. Bioelectric sources may be characterized by current double layers located at cellular membranes; special geometry which would enhance magnetic effects, such as those present in a solenoid, does not arise. Consequently, the ratio $|\omega\mathbf{A}/\nabla\Phi|$ for a current element should be a satisfactory measure for typical electrophysiological distributed sources.

Now Φ and $\mathbf{A}$ must satisfy the Lorentz condition, namely,

$$\nabla \cdot \mathbf{A} = -j\omega\varepsilon_c\mu\Phi = -j\omega\varepsilon(1 + \sigma/j\omega\varepsilon)\mu\Phi. \tag{9}$$

Consequently,

$$\begin{aligned}\Phi &= -\frac{\mu}{j4\pi\omega\varepsilon_c\mu}\nabla\cdot\left(\frac{\mathbf{J}_s' \, dV'}{R}\right),\\ &= -\frac{dV'}{j4\pi\omega\varepsilon_c}\mathbf{J}_s'\cdot\nabla\left(\frac{1}{R}\right) = \frac{dV'}{j4\pi\omega\varepsilon_c}\frac{\mathbf{J}_s\cdot\mathbf{a}_R}{R^2}.\end{aligned} \tag{10}$$

If we let the z-axis coincide with the direction of $\mathbf{J}_s$, then

$$\Phi = \frac{\mathbf{J}_s \cos\theta}{j4\pi\omega\varepsilon_c R^2} \tag{11}$$

and

$$|\nabla\Phi| = \frac{J_s}{4\pi\omega\varepsilon_c R^3}. \tag{12}$$

The ratio of interest is then

$$\left|\frac{\omega\mathbf{A}}{\nabla\Phi}\right| = |\omega^2\mu\varepsilon_c R^2|^2 = |kR|^2. \tag{13}$$

Thus, for the inductive effect to be negligible $|kR|^2 \ll 1$ must be satisfied. So long as propagation effects can be ignored ($|kR| \ll 1$), this condition will automatically be met. For the numerical values used in Section 3, we obtain

$$|kR_{\max}|^2 = 0.0022. \tag{14}$$

6. *Boundary considerations.* Since the total current (conduction plus displacement) is solenoidal, the normal component at the interface between two media must be continuous. At the boundary between different tissues, since the displacement current can be ignored (see Table II), the rigorous condition that

$$\sigma_1(1 + j\omega\varepsilon_1/\sigma_1)E_{1n} = \sigma_2(1 + j\omega\varepsilon_2/\sigma_2)E_{2n} \tag{15}$$

reduces to

$$\sigma_1 E_{1n} = \sigma_2 E_{2n}, \tag{16}$$

where σ_1 and σ_2 are the conductivities of regions 1 and 2 respectively and E_{1n} and E_{2n}, the respective normal electric fields. The boundary condition expressed by (16) is the same as that for stationary (d.c.) conditions.

In many problems, however, one of the regions has zero conductivity, as for example, the space which surrounds the human body in the consideration of the electrocardiographic system. Thus, $\sigma_2 = 0$ results in $E_{1n} = 0$ according to (16); however, the rigorous formulation of (15) results in

$$\sigma_1(1 + j\omega\varepsilon_1/\sigma_1)E_{1n} = j\omega\varepsilon_2 E_{2n}, \quad (\sigma_2 = 0). \tag{17}$$

It seems reasonable to suppose that $E_{1n} \doteq 0$, provided that $(\omega\varepsilon_2/\sigma_1) \ll 1$. Since ε_2 is actually a free-space dielectric constant, we utilize $\varepsilon_0 = 9 \times 10^{-12}$ farads/m. and, with $\omega = 2000\pi$ and $\sigma_1 = 0.2$ mhos/m., $\omega\varepsilon_0/\sigma_1 = 3 \times 10^{-8}$, which is clearly negligible.

The meaning of the above criterion may be clarified by considering how boundary conditions are utilized. The field in the conductive medium can be thought of as arising from the primary sources $\mathbf{E}_0$ and secondary sources $\mathbf{E}_s$ while the total field in the external medium is $\mathbf{E}_e$. Application of (15) then gives

$$\sigma_1(1 + j\omega\varepsilon_1/\sigma_1)E_{on} + \sigma_1(1 + j\omega\varepsilon_1/\sigma_1)E_{sn} = j\omega\varepsilon_2 E_{en}. \tag{18}$$

Now E_{on}, E_{sn} and E_{en} are in the same order of magnitude so that if $|j\omega\varepsilon_2/\sigma_1| \ll 1$

$$\sigma_1(1 + j\omega\varepsilon_1/\sigma_1)E_{on} = -\sigma_1(1 + j\omega\varepsilon_1/\sigma_1)E_{sn}, \tag{19}$$

that is, the total normal current due to the applied field is equal and opposite that due to the secondary field. Depending on the method used equation (19) leads either to an integral equation formulation for the sources of the secondary field or a condition for the determination of series coefficients in a separable coordinate system. But equation (19) is equivalent to the requirement that

$$(E_o + E_s)_n = 0, \tag{20}$$

which corresponds to the stationary condition.

Summary. We summarize below the criteria for making the associated simplification:

Condition	Criteria	
Neglect Propagation Effects	$kR_{\max} \ll 1$	(21)
Neglect Capacitance Effects	$\omega\varepsilon/\sigma \ll 1$	(22)
Neglect Inductive Effects	$(kR)^2 \ll 1$	(23)
Set $E_{1n} = 0$	$\omega\varepsilon_0/\sigma_1 \ll 1$	(24)

A consequence of (21) through (23) is that

$$\Phi = \frac{1}{4\pi\sigma}\int_{V'} \frac{\rho'(x', y', z')}{R}\, dV', \tag{25}$$

$$\mathbf{E} = -\nabla\Phi. \tag{26}$$

The total current $\mathbf{J}$ is the sum of the source current $\mathbf{J}_s$ and the conduction current $\sigma\mathbf{E}$

$$\mathbf{J} = \mathbf{J}_s + \sigma\mathbf{E}. \tag{27}$$

Since $\mathbf{J}$ is solenoidal,

$$\nabla\cdot\mathbf{J} = 0 = \nabla\cdot\mathbf{J}_s + \nabla\cdot(\sigma\mathbf{E}); \tag{28}$$

and for homogeneous media we have, utilizing (26) and (28),

$$\nabla \cdot \mathbf{J}_s = \sigma \nabla^2 \Phi. \tag{29}$$

Taking the Laplacian of (25), we get

$$\nabla^2 \Phi = -\frac{\rho}{\sigma}. \tag{30}$$

Equations (29) and (30) are the conventional expressions used in the solution of electrophysiological problems subject to the boundary conditions of (16) or (24).

If all conditions (21) through (24) are satisfied, as is expected under normal conditions, then all field components will have the same temporal behavior, i.e., will be in synchrony. This can be shown formally by taking the Fourier transform of any field quantity (at any point) to obtain its temporal behavior. In all cases, it will be observed that the frequency dependent term is the same, namely $\rho'(x, y, z, \omega)$; consequently, the temporal dependence is always the same. This means that one can view the problem at any instant of time as if steady state conditions were in effect corresponding to a stationary source $\rho'(x, y, z, t_0)$ or $J'(x, y, z, t_0)$.

The weakest condition in the set of equations (21) through (24) is that $\omega\varepsilon/\sigma \ll 1$. If the remaining criteria are satisfied, one can still proceed by formally solving Laplace's equation, but σ is everywhere replaced by the complex phasor σ_c where

$$\sigma_c = \sigma(1 + j\omega\varepsilon/\sigma).$$

As noted, the boundary condition (18) continues to be satisfied as well as (20) at an outer boundary. In this case, the temporal behavior of the field does not necessarily coincide with that of the source. The actual Fourier transforms must be taken if this approach is followed since both ρ' and σ_c are frequency dependent. If (24) is not satisfied, then the outer region must be included in the specification of the problem, and condition (18) utilized at the interface. If, however, (21) is not satisfied, then the quasi-static approach breaks down and the general field equations (1), (2) and (6) must be utilized.

It is more reasonable to assume that electrophysiological media are inhomogeneous. We assume that the region under consideration is composed of a finite number of subregions each of which is homogeneous. The problem, then, is to find solutions to the vector or scalar Helmholtz equations

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu_o \mathbf{J}_s, \tag{31}$$

$$\nabla^2 \Phi + k^2 \Phi = \frac{\nabla \cdot \mathbf{J}_s}{j\omega\varepsilon_c} = \frac{-\rho}{\sigma + j\omega\varepsilon}, \tag{32}$$

where k is defined by (4) and σ and ε (and hence k) are constant within each subregion. One can integrate (31) and (32) by the standard Green's theorem technique applied to the inhomogeneous region (Smythe, 1950, p. 53; Stratton, 1941, p. 424). The result for (32) [a similar expression is obtained for components of (31)] is

$$\Phi(p) = \frac{1}{4\pi}\int_{V'} \frac{\rho'}{(\sigma + j\omega\varepsilon)} \frac{e^{-jkR}}{R}\, dV' + \frac{1}{4\pi}\sum_i \int_{S_i} \left\{\left[\frac{e^{-jk_1R}}{R}\frac{\partial\Phi_{1i}}{\partial n_{1i}} - \Phi_{1i}\frac{\partial}{\partial n_{1i}}\left(\frac{e^{-jk_1R}}{R}\right)\right] + \left[\frac{e^{-jk_2R}}{R}\frac{\partial\Phi_{2i}}{\partial n_{2i}} - \Phi_{2i}\frac{\partial}{\partial n_{2i}}\left(\frac{e^{-jk_2R}}{R}\right)\right]\right\} dS_i, \quad (33)$$

where S_i is a component surface separating subscripted region 1 from region 2; n_{1i} and n_{2i} are the surface normals drawn outward from regions 1 and 2 respectively. Assuming that (22) and (21) are satisfied in all regions and noting that $\Phi_{1i} = \Phi_{2i}$ at each interface, we get

$$\Phi(p) = \frac{1}{4\pi}\int_{V'} \frac{\rho'}{\sigma R}\, dV' + \frac{1}{4\pi}\sum_i \int_{S_i} \frac{1}{R}\frac{\partial[\Phi_{1i} - \Phi_{2i}]}{\partial n_{1i}}\, dS_i. \quad (34)$$

Comparing (34) with (25) shows that the difference is in the creation of equivalent sources at the phase boundaries. But the existence of these additional sources do not affect arguments leading to criteria (23) and (24); here quasi-static conditions prevail if equations (21) through (24) are satisfied for all regions of an inhomogeneous body. In this event $E = -\nabla\Phi$, and (34) becomes

$$\Phi(p) = \frac{1}{4\pi}\int_{V'} \frac{\rho'}{\sigma R}\, dV' - \frac{1}{4\pi}\sum_i \int_{S_i} \frac{E_{n1} - E_{n2}}{R}\, dS_i, \quad (35)$$

which corresponds to the result given by Geselowitz (1963).

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LITERATURE

Geselowitz, D. B. 1963. "The Concept of an Equivalent Cardiac Generator." *Biomed. Sci. Instrumentation*, **1**, 325–330

Rush, S., J. A. Abildskov and R. McFee. 1963. "Restivity of Body Tissue at Low Frequencies." *Circulation Res.*, **12**, 40–50.

Schwan, H. P. and C. F. Kay. 1957. "The Conductivity of Living Tissues." *Ann. N.Y. Acad. Sci.*, **65**, 1007–1013.

Smythe, W. R. 1950. *Static and Dynamic Electricity*. New York: McGraw-Hill.

Stratton, J. A. 1941. *Electromagnetic Theory*. New York: McGraw-Hill.

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