Greedy Algorithms: Grouping Children

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Algorithmic Toolbox
Data Structures and Algorithms

Outline

Many children came to a celebration. Organize them into the minimum possible number of groups such that the age of any two children in the same group differ by at most one year.

Outline

MinGroups(C)

```
$m \leftarrow$ len($C$)
for each partition into groups
$C = G_1 \cup G_2 \cup \cdots \cup G_k$:
    good $\leftarrow$ true
    for $i$ from 1 to $k$:
        if $\max(G_i) - \min(G_i) > 1$:
            good $\leftarrow$ false
    if good:
        $m \leftarrow \min(m, k)$
return $m$
```

Running time

Lemma

The number of operations in `MinGroups`(C) is at least 2^n, where n is the number of children in C.

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- $C = G_1 \cup G_2$
- For each $G_1 \subset C$, $G_2 = C \setminus G_1$
- Size of C is $\boldsymbol{n}$
- Each item can be included or excluded from G_1
- There are 2^n different G_1
- Thus, at least 2^n operations $\square$

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  operations!
- We will improve this significantly

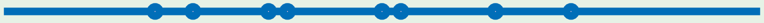
Outline

Covering points by segments

Input: A set of n points $x_1, \ldots, x_n \in \mathbb{R}$.

Output: The minimum number of segments of unit length needed to cover all the points.

Example



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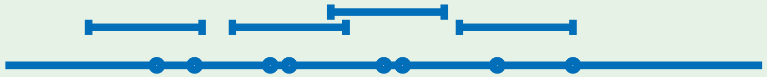

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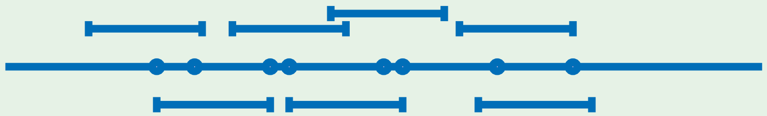
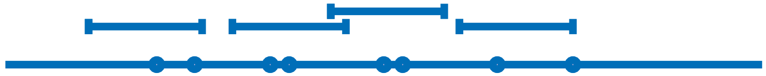


Safe move: cover the leftmost point with a unit segment which starts in this point.



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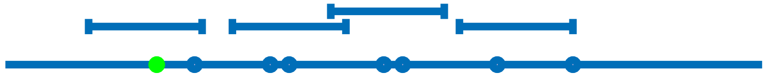

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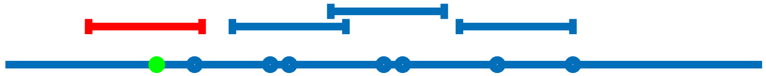

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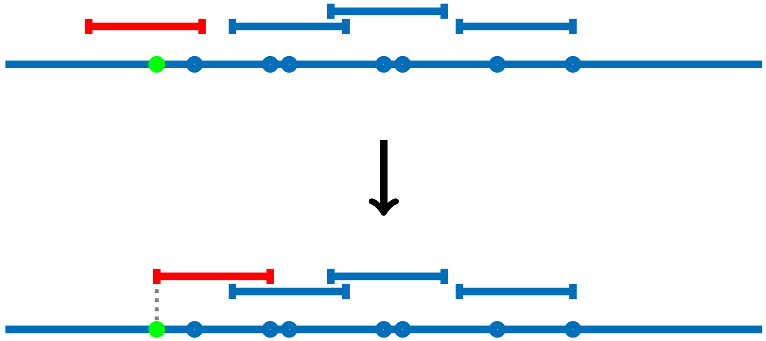

Assume $x_1 \le x_2 \le \ldots \le x_n$

```
PointsCoverSorted($x_1, \ldots, x_n$)

$R \leftarrow \{\}$, $i \leftarrow 1$
while $i \le n$:
    $[\ell, r] \leftarrow [x_i, x_i + 1]$
    $R \leftarrow R \bigcup \{[\ell, r]\}$
    $i \leftarrow i + 1$
    while $i \le n$ and $x_i \le r$:
        $i \leftarrow i + 1$
return $R$
```

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- For each i, at most 1 new segment
- Overall, running time is $O(n)$ $\square$

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- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`
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- Sort + `PointsCoverSorted` is $O(n \log n)$

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- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$
- Fast for $n = 10\,000\,000$
- Huge improvement!

Conclusion

- Straightforward solution is exponential
- Important to reformulate the problem in mathematical terms
- Safe move is to cover leftmost point
- Sort in $O(n \log n)$ + greedy in $O(n)$