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# Cellular automata models for general traffic conditions on a line

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**Abstract**

By the use of the methods proposed in a recent article we were able to shed some light on the experimentally obtained flux-density relation in traffic flow. We suggest an order parameter showing the existence of two regimes in freeway traffic. We also introduce the case of traffic at signal condition, analysing the effects of convenient traffic parameters on the flux-density diagram.

## 1. Introduction

Traditionally, the treatment of the problem of traffic flow was based on the analysis of experimental data. However, the poverty of data in critical regions led to a series of different interpretations about the behaviour of relations among the fundamental traffic parameters, density and flux. These include proposals so divergent as the "reverse lambda", discontinuous, and continuous forms for flow-occupancy relationships ( [1] and references therein). In part this is caused by the absence of criteria in the experimental methods used, showing the lack of an adequate theoretical approach to the problem to guide this research. Besides, some earlier analyses seem not to have taken precisely into consideration these same methods, as it was argued in [1]. This question remains open until now.

The first theoretical models were based on fluid and nonlinear equations [2–6]. They were well suited to treat traffic at very high or low car densities, but could not be trivially extended to the whole interval of observable densities. So they could not be of

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much help in clearing up the precise flow-occupancy relationship. For this reason, and for the fact that we intend to examine the particular problem of signalized ways, we base our work in the recently introduced cellular automata (CA) traffic models [7].

In the next section we give a review of this model. In Section 3 we define the convenient parameters and show the diagrams for freeway transit. Nextly we introduce the case of traffic under signal constraints. In the conclusion we hope to give a possible explanation to the experimental problem about the critical region of the flux-density diagram.

## 2. The model

The use of a CA approach to model traffic flowing along a line was made by K. Nagel and M. Schreckenberg [7]. They obtained a nice agreement with experimental data, which signed the correctness of the proposal.

The CA were constituted of an array of $L$ sites, each site being occupied by one car (with integer velocity $v$) or being empty. The parallel update rules were the following:

(i) $v_j \rightarrow v_j + 1$ if the distance of the $j$th car to the next car is greater than $v_j + 1$, and taking $v_j = v_{max}$ as a limit.
(ii) $v_j \rightarrow d - 1$ if the distance to the next car is $d \leq v_j$.
(iii) $v_j \rightarrow v_j - 1$ with a random probability $p$ if $v_j \geq 1$.
(iv) The $j$th car is advanced $v_j$ sites.

Other works then followed using this technique [8,9]. In this paper, we will join some new rules to the basic automatum presented above, in order to have a wider comprehension of the cases we want to study. They will be introduced sequentially along the text.

## 3. Free traffic

Here we consider a line with periodic boudary condition (closed circuit), with a random initial distribution of car positions and null initial velocities. We want to study the properties of the fundamental (flux × density) diagram. So, we define the parameters

$$\bar{\rho}_i = \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} n_i(t) \quad , \tag{1}$$

and

$$\bar{q}_i = \frac{1}{T} \sum_{t=t_0+1}^{t_0+T} m_i(t) \quad . \tag{2}$$

The first expression represents the density of cars on the site $i$ ($S_i$) over a time period $T$; $t_0$ is the relaxation time, usually taken as $t_0 = 10 \times L$, following the prescription

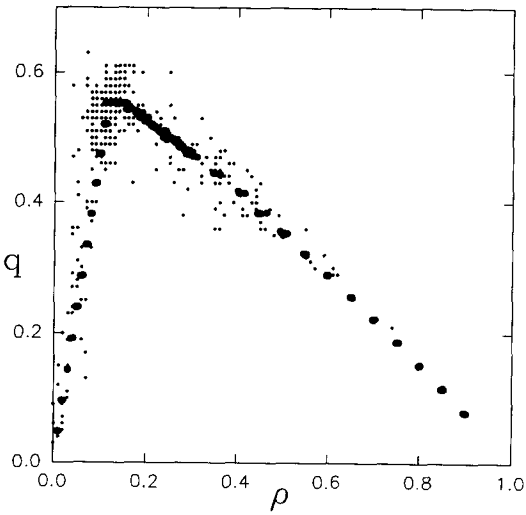


Fig. 1. Flux × density diagram for free traffic with $v_{max} = 5$, $p = 0.2$ and $L = 1000$. Small circles represent simulations for $T = 100$ and big circles represent simulations for $T = 10000$.

of [7]; $n_i(t)$ is zero if $S_i$ is empty and one if it is occupied at time $t$. The second expression represents the flux of cars on $S_i$; $m_i(t)$ is one if at time $t-1$ there was a car behind or at $S_i$ and at $t$ it is found after $S_i$ (i.e., a car is detected passing by $S_i$) and zero otherwise.

To look for a possible transition between a free and a jammed phase, we also define a parameter

$$\bar{M}_i = 1 - \frac{1}{2T\bar{\rho}_i} \sum_{t=t_0+1}^{t_0+T} l_i(t) \quad , \tag{3}$$

where $l_i(t)$ is one if at time $t-1$ $S_i$ is occupied (empty) and at time $t$ it is empty (occupied); $l_i(t)$ is zero if at both times $S_i$ is occupied or empty. This choice of parameter is made on the consideration that a jammed regime means that all cars are grouped in long clusters.

Obviously, for free traffic, as this system is a homogeneous one due to the considered boundary conditions, none of these parameters will be position dependent. So, along this section the subindex $i$ will be omitted.

In Fig. 1 we show the cases with $T = 100$ and $T = 10000$ ($L = 1000$). The first presents the natural dispersion of points that is expected for the low accuracy caused by the small interval of time for the measurements. This kind of dispersion is exactly what is seen in the experimental data (see for exemple the comparison of Figs. 4 and 5 of Ref. [7]). It is not easy to establish any conclusion from such experimental diagrams, what has led to the discussion of what would be the correct form of the flux-density relation [1]. With higher values of $T$, we see that we can improve this until the obtention of a well defined curve is achieved. This is the most favourable point of this approach.

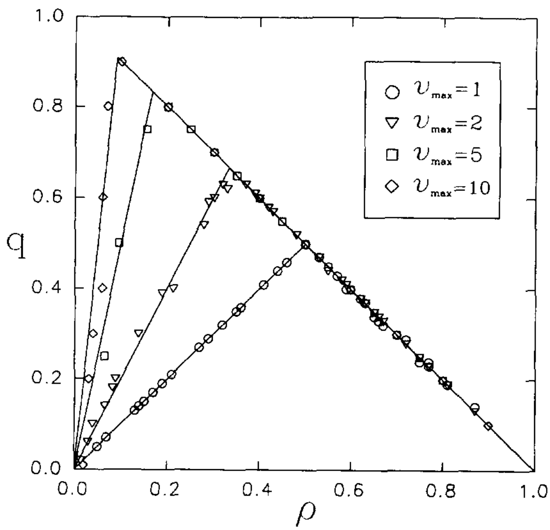


Fig. 2. Flux × density diagram for free traffic with $p = 0$. Lines are solutions of Eq. (4), while points are taken from simulations ($L = 1000, T = 1000$) for some different $v_{max}$.

In the low density limit, the exact solution for any $v_{max}$ and $p$ can be easily obtained by analising the evolution of the automatum for one car, and appears as

$$q = (v_{max} - p)\rho \ , \tag{4}$$

and we see that in this limit $p$ just scales the maximum velocity to a lower value. At high density, the asymptotic behaviour can be obtained by the reasoning that this system presents a kind of "particle-hole" symmetry, with a hole appearing with a $v_{max} = 1$ independently of the $v_{max}$ of the particle. So, in this limit,

$$q = (1-p)(1-\rho) \ . \tag{5}$$

For $p = 0$ we observe by numerical results that only the two regimes above exist. We always find $q = v_{max}\rho$ if $\rho \leq 1/(1+v)$ and $q = 1-\rho$ otherwise. So, the exact result for this CA when $p = 0$, but with an arbitrary $v_{max} = v$, is given by the implicit equation

$$\begin{aligned} q = {} & \frac{v}{1+v} + \left(\rho - \frac{1}{1+v}\right)\frac{(\sqrt{2}v - \sqrt{1+v^2})}{(\sqrt{2}+\sqrt{1+v^2})} \\ & - \frac{(\sqrt{2}v + \sqrt{1+v^2})}{(\sqrt{2}+\sqrt{1+v^2})}\left|\left(\rho - \frac{1}{1+v}\right) + \left(q - \frac{v}{1+v}\right)\frac{(\sqrt{2}v - \sqrt{1+v^2})}{(\sqrt{2}+\sqrt{1+v^2})}\right| \ . \end{aligned} \tag{6}$$

Some solutions for different $v$ are plotted in Fig. 2. The point that we observe is that, by the definitions we used for $q$ and $\rho$, all the curves in the graphic of $q$ against $\rho$, independently of $v$ and $p$ , cannot transpose the line $q = 1-\rho$. This condition is fulfilled by the solution of Eq. (4), where the flux rises until it meets this line in the point

$$\rho = 1/(1+v) \ , \qquad q = v/(1+v) \tag{7}$$

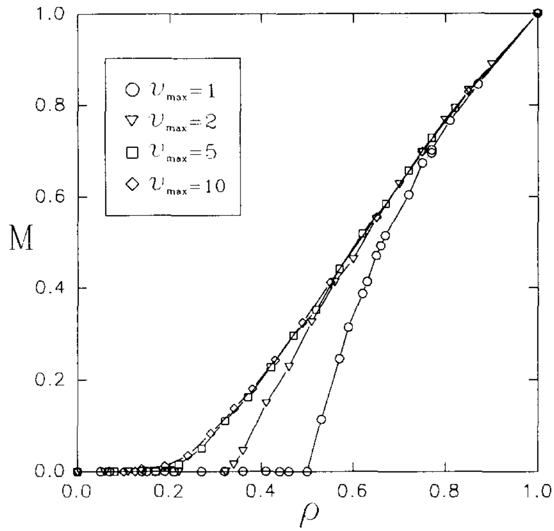


Fig. 3. Order parameter as a function of the density for different $v_{max}$ and $p = 0$. The full lines are guides to the eye.

and then falls along it. This agrees with the particular solution obtained in Ref. [8] in the limit when we have $v_{max} = 1$.

Although our solution is restricted to the case $p = 0$, the independence of this no-crossing statement on the values of $p$ happens because a non-zero value of $p$ just makes the probability of occurence of higher velocities low. So, for a given density, the value of the flux has to be lower (or equal) than in the case of a null $p$.

Here, we make a change in relation to the basic CA of last section. We do not take a fixed value of $v_{max}$ for all cars, but rather a $v_{max}$ distribution between a low, $v_L$, and a high, $v_H$, velocity.

In this numerical treatment, the change of a fixed $v_{max}$ to a random distribution of maximum velocities showed that the shape of the curves remained unchanged, but now the $v_{max}$ that appears in the formulas above should be understood as the lowest value of the distribution ($v_L$). An interesting point is that the existence of a sole car with $v_L$ is sufficient to make this change, no matter how high is $v_H$.

The order parameter is shown in Fig. 3, where it is plotted against $\rho$ for several values of $v_{max}$ ($p = 0$). There we see two distinct regions. The first, where the parameter is zero, is associated with a free regime. The cars are able to develop their maximum velocity, which means that there is no correlation between them. After a transition point the order parameter is not zero anymore, representing the region associated with a jammed phase[2]. The critical density $\rho_c$ is a function of the maximum velocity and for $p = 0$ it is given by Eq. (5). We show this phase diagram in Fig. 4. As an observation, we comment that the value of $L = 1000$ is already sufficient to assure the thermodynamic

[2] A more detailed and complete analysis of the criticality of this model is under preparation.

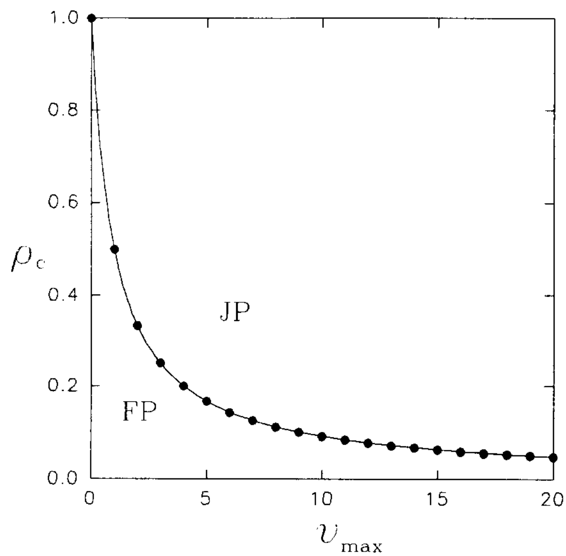


Fig. 4. Diagram of critical density (at the transition point) against maximum velocity showing a free (FP) and a jammed (JP) phase. The values for this curve were taken from Eq. (5).

limit, i.e., increasing this value does not change these results.

We would like to point out now the similarity between our results about the phase description above and what was obtained by Nagatani [10]. This happens because our parameter M is directly related to the parameter which he called "mean velocity" ($\langle v \rangle = 1 - M$ when $v_{max} = 1$), since the ergodicity of the CA admits identifying spatial means of measurements done at a fixed time with temporal means of measurements done at a fixed position. We observe that the characteristics of this transition are already completely described just considering an one-lane traffic model. It seems that the inclusion of an extra lane, introduced by Nagatani, does not bring any further insight on them.

## 4. Traffic light

Our motivation for the analysis of traffic under a stop-and-go condition lies in the problem noticed by Lighthill and Whitham [2] of the existence of a non-functional relation between flux and density obtained from avaiable experimental data (see Fig. 19 of Ref. [2]). This was against the theoretical results, and none satisfactory explanation was given by them.

Our changes in the basic CA of Section 2 begin by the inclusion of N traffic lights equally spaced along the array. We define the time of duration of green, yellow and red lights as $T_g$, $T_y$ and $T_r$, respectively. The duration of the full cycle is $T_c$. During $T_g$ all cars move freely following the rules of the basic CA. During $T_r$ the signal behaves as a car with zero velocity, stopping the cars behind it. $T_y$ avoids the possibility of having a car with velocity greater than the distance $d_s$ to the signal just at the moment of change

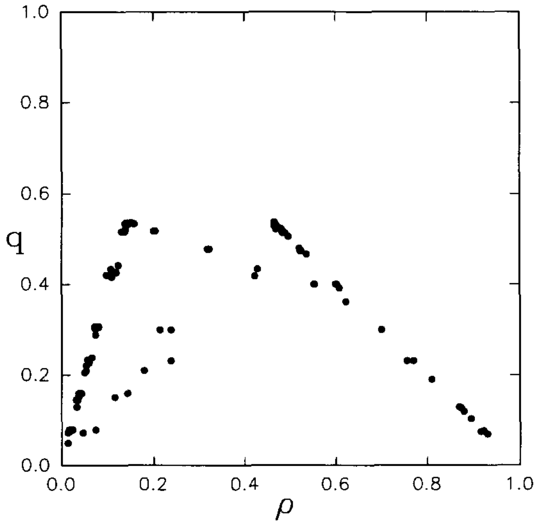


Fig. 5. Flux × density diagram for signalized traffic with $v_{max} = 4$, $p = 0$, $L = 1000$, $T_g/T_c = 24/40$, $N = 4$. The simulation was made with several radars in different positions, $d_r = 10, 30, 70, 125, 210$.

from green to red light. During $T_y$ we consider the car immediately before the signal (with velocity $v$ and distance $d_s$ to the signal) which will obey the rules:

(i) If $v(T_y - 1) > d_s$ then the car will follow the rules of the basic CA.

(ii) If $v(T_y - 1) \leq d_s$ then the car will follow the rules for the red signal.

A typical flux-density graphic is shown in Fig. 5. It was obtained with several "radars" (where the measurements are made) at different positions in relation to each signal. This has to be taken into consideration because the signals create an inhomogeneity in the system. The existence of two well defined curves can be now easily explained. The one at higher densities is related to measurements made at distances from the traffic light smaller than the mean size of the queue generated by the signal. The other is related to measurements made at distances bigger than this mean size. So for a given number of cars, there will be two classes of measurements giving two different densities for the same flux. We notice that both curves have the same maximum.

In Fig. 6 we show the curve obtained by only one radar at a distance $d_r$ from a signal. We see that when the number of cars increases so that the mean queue size gets higher than $d_r$, the behaviour of the measured flux changes from the low to the high density curve of Fig. 5.

A change on the number of signals does not modify any of the conclusions above. In the same way, the important features of the fundamental diagram are not sensible to the independent variation of $T_g$ and $T_c$, but only to the variation of the relation $T_g/T_c$. So this is the principal parameter on which the CA will depend. To study this dependence, we define the variables $q_M$, the maximum value of the flux; $\rho_M$, which is the mean between the values of $\rho$ at the two points of maximum flux; and $v_M = q_M/\rho_M$, which is the mean maximum velocity of the system. In Fig. 7 we see the relation between

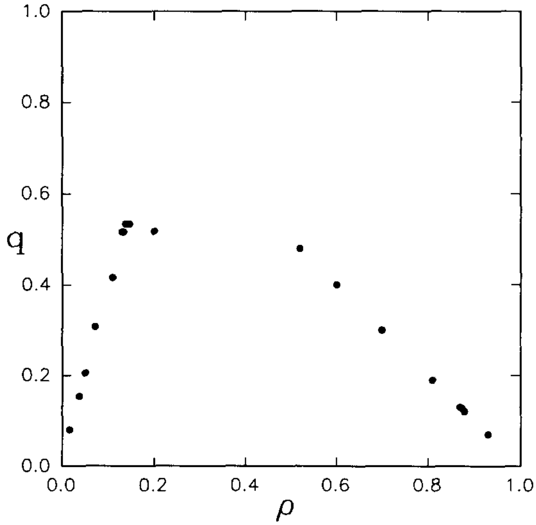


Fig. 6. Same system of Fig. 5 but using a simulation with only one radar($d_r = 75$).

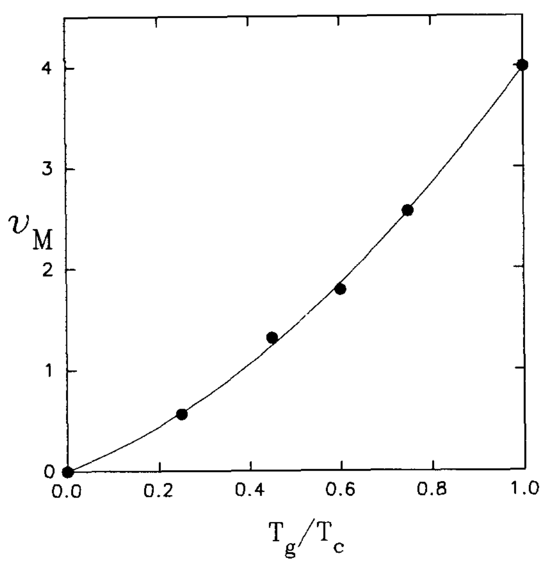


Fig. 7. Mean maximum velocity as a function of $T_g/T_c$. The line is a quadratic fitting for the data with a standard deviation of 4%.

$v_M$ and $T_g/T_c$. It shows how the velocity of the cars is so strongly dependent on $T_g/T_c$, falling rapidly from the maximum that occurs, obviously, when $T_g/T_c = 1$. For example, at a typical value of $T_g/T_c = 0.5$, $v_M$ would be only approximately 36% of its maximum for free traffic.

## 5. Conclusion

The question for the interpretation of the shape of the fundamental diagram can now be answered in the following way: the "reverse Λ" and discontinuous forms are not plausible in the case of freeway traffic; the continuous and continuous differentiable form will occur with a non-zero $p$, while the "inverse V" form will happen if $p = 0$. An interesting point to be stressed is that in the analysis of experimental data the value of $p$ can be discovered if we restrict ourselves to the limits of high and low densities (using the expressions (6) and (7), and admitting a calibration for the experimental diagrams such as that done in Ref. [7]) and so avoiding the troublesome intermediary region where the spreading of points does not allow a precise conclusion.

As $p$ is associated to an aleatory braking that is not caused by the presence of a car ahead, we can imagine that this factor would be related to the dirigibility conditions of the roads. So, for good freeways, $p$ would be null and we would have a more adequate description for the flux-density relation by an "inverse V" form.

Also the question for the interpretation of the non-functional relation between flux and density in the case of signalized traffic would be answered by the existence of two distinct curves for the same system. This seems not to have been obtained by any theoretical model until now.

So, our results seem to be qualitatively in accordance with the known experimental data, clarifying some intriguing points in traffic research.

## Acknowledgement

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