QUESTION (HD 2101) (1) Let D be a domain. How to show that $D[X^2, X^3] \cong D[Y,Z]/(Y^2 - Z^3)$?

(2) Let $R = Z_{(p)} + (X;Y)Q[[X,Y]]$ and $M = (X,Y)Q[[X,Y]]$. Why is $R[1/p] = Q[[X,Y]] = R_M$?

ANSWER: There are two answers to each of **(1)** and **(2)**. One based on a suggestion by Shiqi Xing of Sichuan Normal University, Chengdu, China and the other as a comment on these questions by Tiberiu Dumitrescu of Universitatea Bucuresti, Romania.

Xings Answer: Let $\phi : D[Y,Z] \to D[X^2, X^3]$ defined by $\phi(Y) = X^3$ and $\phi(Z) = X^2$. Then ϕ is onto. Indeed $\ker\phi \supseteq (Y^2 - Z^3)$.We need to show that every element $g = g(Y,Z)$ in $\ker\phi$ is divisible by $(Y^2 - Z^3)$. For this we first note that $g(0,0) = (0)$. That is there is no nonzero constant term. Express g as a polynomial in Y as: $g = f_n(Z)Y^n + f_{n-1}(Z)Y^{n-1} + ... + f_0(Z)$. Suppose that g is not divisible by $(Y^2 - Z^3)$. Then, $g = q(Y,Z)(Y^2 - Z^3) + f(Y,Z)$ and degree of f is less than 2. Hence $\partial f = 1$ or 0 in Y. So $f = f_1(Z)Y + f_0(Z)$. Because $g(Y,Z) = (0)$ on setting $Y = X^3$ and $Z = X^2$, we have $f_1(X^2)X^3 + f_0(X^2) = (0)$. But this forces $f_1(X^2)X^3 = 0 = f_0(X^2)$, because one is of odd degree and the other of even. Of course as $X^3 \neq 0$ we have $f_1(X^2) = (0)$. Whence $f_1(Z) = (0)$ and $f_0(Z) = 0$. Forcing $f(Y,Z) = 0$. But then $g = q(Y,Z)(Y^2 - Z^3) \in (Y^2 - Z^3)$.

(2). Note that $\cap p^n R = \cap(p^n Z_{(p)} + (X;Y)Q[[X;Y]]) \supseteq M = (X;Y)Q[[X;Y]]$ and there is no prime between $\cap p^n R$ and $M = (X;Y)Q[[X;Y]]$ because $\cap p^n Z_{(p)} = (0)$ and indeed there is no prime between pR and M Thus $M = \cap p^n R$. But as M misses powers of p we have $R_M = R[1/p] = Q + M = Q + (X,Y)[[X,Y]] = Q[[X,Y]]$.

Tiberiu's comments around hd2101

Question 1. Let D be a domain. How to show that

$$D[X^2, X^3] = D[Y,Z]/(Y^2 - Z^3)?$$

We can approach this question as follows. Let A be a domain and $d \in A$ such that $\sqrt{d} \notin A$. We get the well-known isomorphism

$$(*) \qquad A[Y]/(Y^2 - d) \simeq A[\sqrt{d}].$$

Indeed, if u is the ring epimorphism $A[Y] \to A[\sqrt{d}]$ sending Y into $\sqrt{d}$ and if $f \in ker(u)$, we divide f by $Y^2 - d$, to get $f = (Y^2 - d)g + aY + b$ with $g \in A[Y]$, $a, b \in A$, to get $0 = f(\sqrt{d}) = a + b\sqrt{d}$, so $a = b = 0$, thus $f = (Y^2 - d)g$. This is essentially the proof in HD2101. Taking $A = D[Z]$ and $d = Z^3$ in $(*)$, we get

$$D[Z][Y]/(Y^2 - Z^3) \simeq D[Z][\sqrt{Z^3}] \simeq D[X^2, X^3]$$

thus answering Question 1. The case $D = \mathbb{Z}$, that is,

$$(**) \qquad \mathbb{Z}[X^2, X^3] = \mathbb{Z}[Y,Z]/(Y^2 - Z^3)$$

is generic in the sense that, given a commutative ring B, we just tensor $(**)$ by B over $\mathbb{Z}$ to get $B[X^2, X^3] = B[Y,Z]/(Y^2 - Z^3)$. Note that $(**)$ can be obtained from the epimorphism of rings

$$\mathbb{Z}[Y,Z]/(Y^2 - Z^3) \to \mathbb{Z}[X^2, X^3]$$

observing that both rings are two-dimensional domains.

Question 2. Let $R = \mathbb{Z}_{(p)} + (X,Y)\mathbb{Q}[[X;Y]]$ and $M = (X,Y)\mathbb{Q}[[X,Y]]$. Why is

$$R[1/p] = \mathbb{Q}[[X,Y]] = R_M?$$

The first equality is clear because $\mathbb{Z}_{(p)}[1/p] = \mathbb{Q}$. The second equality comes from the following basic fact (for $D = R$, $S = \{p^n \mid n \geq 1\}$ and $P = M$).

If D is a domain, $S \subset D$ a multiplicative set and P a prime ideal of D disjoint if S, then QE is a prime ideal of the fraction ring $E = D_S$ and $D_P = E_Q$.

I am thankful to these gentlemen for coming to my rescue at a time when it is hard to concetrate due to my current health problems.