$(\gamma,\lambda) \neq 1$ i.e. if $xy \in P(r)$ then $x \in P(r)$ or $y \in P(r)$ and this establishes (3).

(4) If $P(r) = P(s)$ then since $r \in P(r)$, $(r,s) \neq 1$.

Conversely let $(r,s) \neq 1$ then by (1), $r|s$ or $s|r$. If $r|s$ then $(x,r) \neq 1$ implies that $(x,s) \neq 1$, that is $P(r) \subseteq P(s)$. If on the other hand $(x,s) \neq 1$ then by the HCF property $x = x_1 s_1$ and $s = s_1 s_2$, where $(x_1, s_2) = 1$. Since $s_1 | s$, s_1 is a rigid element which is non co-prime to r(since we have assumed that $r|s$) that is $(x,s) \neq 1$ implies that $(x,r) \neq 1$ i.e. $P(s) \subseteq P(r)$ and combining the two inclusion relations the result follows.

(5) Since R is an HCF domain, $R_{P(r)}$ is an HCF domain (cf Lemma 9, Ch. 1). To prove that a quasi-local HCF domain ($R_{P(r)}$ in this case) is a valuation domain, all we have to show is that no two non units of this domain ($R_{P(r)}$) are co-prime. Suppose on the contrary that there exist x,y in $P(r)R_{P(r)}$, such that $(x,y) = 1$ and let

$$x = u_1/v_1 \ ; \ y = u_2/v_2 \ (\text{ we can assume that } (u_i, v_i) = 1).$$

Now since v_1, v_2 are units in $R_{P(r)}$ we get $(u_1, u_2) = 1$ in $R_{P(r)}$, that is $(u_1, u_2) \notin P(r)R_{P(r)}$. But since we assumed that x,y are non units in $R_{P(r)}$, $u_1, u_2 \in P(r)$ and so $(u_i, r) = r_i$ $(i = 1,2.)$ are such that $r_i \neq 1$ that is $d = (u_1, u_2)$ is a multiple of r_1 or of r_2 in R (since r_i are factors of a rigid element r) and thus $(u_1, u_2) = d \in P(r)$ i.e. u_1, u_2 are non co-prime in $R_{P(r)}$ a contradiction establishing that no two non units in $R_{P(r)}$ are co-prime which implies the result.

2. Semirigid Domains.

Using Lemma 1, we first prove the