On Chevalley's Extension Theorem

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Dedicated ti the memory of Paul Cohn

Abstract. Professor Daniel Anderson informed me, recently, that there is an error in the proof of Theorem 56 of Kaplansky's book on Commutative Rings. His (Dan's) reason was "He (Kaplansky) orders by reverse inclusion but in the last line uses inclusion, so we don't contradict maximality (which is minimality)". The aim of this short note is to indicate that while Dan Anderson appears to be correct in pointing out an error in the proof of Theorem 56 of [**7**], the statement of the theorem is a correct consequence of a Theorem of Chevalley's.

Professor Daniel Anderson informed me, recently, via [**1**], that there is an error in the proof of Theorem 56 of Kaplansky's book on Commutative Rings. His (Dan's) reason was "He (Kaplansky) orders by reverse inclusion but in the last line uses inclusion, so we don't contradict maximality (which is minimality)". Looking at the theorem and its proof, I realized that I had seen a similar result elsewhere. After some search I found Chevalley's Extension Theorem as Theorem 3.1.1 of [**6**]. The aim of this short note is to indicate that while Dan Anderson appears to be correct in pointing out an error in the proof of Theorem 56 of [**7**], the Theorem is correct as stated as it follows from a theorem of Chevalley's. We include, below, Chevalley's theorem and its proof to indicate how it is related to another theorem in [**7**].

It is hard to believe that there would be such an error in [**7**], but Dan Anderson is a very serious and respected Mathematician, and a student of Kaplansky's. Consider this another reason why this note got written.

I give the statement and a redo of the proof, below, of Chevalley's Extension Theorem.

Theorem 1. *Given that K is a field, let $R \subseteq K$ be a subring of K and let $P \subseteq R$ be a prime ideal of R. Then there exists a valuation ring O of K such that $R \subseteq O$ and $M \cap R = P$, where M is the maximal ideal of O.*

Proof. We use the standard notation R_P for localization of R at P. Let $\sum = \{(A, I)|R_P \subseteq A \subseteq K,\ pR_P \subseteq I \subseteq A\}$ where A is a ring and I a proper ideal of A. Then $\sum \neq \phi$, because $(R_P, pR_P) \in \sum$. Moreover $\sum$ may be partially ordered as

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follows: for all $(A_j, I_j) \in \sum$, $(j = 1, 2)$ we declare $(A_1, I_1) \leq (A_2,\, I_2) \Leftrightarrow A_1 \subseteq A_2$ and $I_1 \subseteq I_2$. For each chain $\{(A_j, I_j) | j \in J$ where J is an index set$\}$ we have an upper bound $(\cup A_j, \cup I_j) \in (\sum, \leq)$. By Zorn's Lemma, $\sum$ has a maximal element (O, M). Observe that $R \subseteq R_P \subseteq O$, and since PR_P is the maximal ideal of R_P we have $M \cap R_P = PR_P$ and consequently $M \cap R = P$. So, to complete the proof, it remains to show that (O, M) is a valuation domain. From the maximality of (O, M) we first conclude that O is a local ring. Assume now that O is not a valuation ring. Then there is $x \in K \backslash \{0\}$ such that $x, x^{-1} \notin O$. But then $O \subsetneq O[x]$, $O[x^{-1}]$. The maximality of (O, M) implies therefore that $M[X] = O[x]$ and $M[x^{-1}] = O[x^{-1}]$. But then there exist $a_0, ..., a_n; b_0, ..., b_m \in M$ such that

$1 = \sum_{i=0}^{n} a_i x^i$ and $1 = \sum_{i=0}^{m} b_i x^{-i}$ ....(i) with n, m minimal.

Suppose for a start, that $m \leq n$. As $b_0 \in M$, we have $\sum_{i=1}^{m} b_i x^{-i} = 1 - b_0 \in O \backslash M$ (a nonzero non unit). Or, dividing both sides of the previous equation by $1 - b_0$ we get, $\sum_{i=1}^{m} \frac{b_i}{1-b_0} x^{-i} = 1$. Thus we have $\sum_{i=1}^{m} c_i x^{-i} = 1$ ..... (ii) where $c_i = \frac{b_i}{1-b_0}$.

Multiplying both sides of (ii) by x^n we get $\sum_{i=1}^{m} c_i x^{n-i} = x^n$....(iii)

Now from (i) we have $1 = \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n-1} a_i x^i + a_n x^n$ ....(iv)

Substituting in (iv) the value of x^n from (iii) we get

$1 = \sum_{i=0}^{n-1} a_i x^i + a_n \sum_{i=1}^{m} c_i x^{n-i}$ ....(v)

Because $m \leq n$, powers p of x in each summand on the right of (v) are $0 \leq p \leq n - 1$. But this contradicts the minimality of n in expressing 1 as a polynomial in x. If, on the other hand, we take $n \leq m$, then arguing in a similar fashion, we get a contradiction to the minimality of m. □

Let $R \subseteq S$ be an extension of rings and let I be a proper ideal of R. Let us say that I survives in S if I generates a proper ideal of S, i.e., if $IS \neq S$.

Corollary 1. *(Kaplansky Theorem 56). Let K be a field, R a subring of K, and I an ideal in $R, I \neq R$. Then there exists a valuation domain V, $R \subseteq V \subseteq K$, such that K is the quotientfield of V and I survives in V.*

Proof. Because $I \neq R$ there is a prime ideal P of R such that $I \subseteq P$. Now by Theorem 1 there is a valuation domain (V, M) such that $P = M \cap R$, i.e, P survives in V and consequently I survives in V. □

The proof of Theorem 1 can be slightly modified to produce another interesting corollary.

Corollary 2. *Let K be a field and R a subring of K. Let $u \in K \backslash \{0\}$, and let I be an ideal in R, $I \neq R$. Then I survives either in $R[u]$ or in $R[u^{-1}]$.*

Proof. Suppose that I survives in neither. Then $IR[u] = R[u]$ and $IR[u^{-1}] = R[u^{-1}]$. Then there exist $a_0, ..., a_n; b_0, ..., b_m \in I$ such that

$1 = \sum_{i=0}^{n} a_i x^i$ and $1 = \sum_{i=0}^{m} b_i x^{-i}$ ....(I) with n, m minimal.

From the second expression in (I) we have $\sum_{i=1}^{m} b_i x^{-i} = 1 - b_0$ ... (II)

Assuning that $m \leq n$ and multiplying (II) by x^n, throughout, we get $\sum_{i=1}^{m} b_i x^{n-i} = (1 - b_0)x^n$ ...(III)

Next, multiplying the first equation in (I) by $(1 - b_0)$ we have

$(1 - b_0) = \sum_{i=0}^{n-1} a_i (1 - b_0) x^i + a_n (1 - b_0) x^n$ ... (IV)

Now, substituting the value of $(1 - b_0)x^n$ in (IV) and re-writing we get

$1 = b_0 + \sum_{i=0}^{n-1} a_i (1 - b_0) x^i + a_n \sum_{i=1}^{m} b_i x^{n-i}$ ...(V)

Since every power of x that appears in (V) is less than n we conclude that 1 can be expressed as a polynomial of degree less than n and that contradicts the minimality of n. Finally assuming that $n \leq m$ and reversing the roles of m and n in the above calculations we get a similar contradiction. $\square$

Now let us change the wording of Corollary 2 to see that with the same wording as in the proof of Corollary 2 we can prove.

COROLLARY 3. *Let $R \subseteq T$ be rings, let u be a unit in T, and let I be an ideal in R, $I \neq R$. Then I survives either in $R[u]$ or in $R[u^{-1}]$.*

Observe that Corollary 3 is precisely Theorem 55 of [**7**].

Finally, thanks to Kaplansky's students and disciples Chevalley's Extension Theorem gets cited a lot, in the form of Theorem 56 of [**7**], in Multiplicative Ideal Theory, and the paper [**5**] is no exception. Now if there is a comment about the veracity of Theorem 56 of [**7**], from a big gun like Dan Anderson, it would seriously undermine the confidence in all the papers using that theorem, with [**5**] included and that is another reason for jotting down the above few lines. I hope I have been able to establish the veracity of the statement of Theorem 56 of [**7**]. Of course if the ordering is reversed in the proof of Theorem 56 of [**7**], to fit Dan's requirement, then the proof will become all right, but then it would clearly appear to have been taken from Chevalley's Extension Theorem!

In addition to [**6**], one can find Chevalley's Theorem, without a reference to Chevalley, as Lemma 4.3 in [**3**] with statement:

THEOREM 2. *Let K be a field, R a subring of K and $\mathfrak{a}$ a nonzero proper ideal in R. Then there is a proper subring V with an ideal $\mathfrak{p}$ such that $(V, \mathfrak{p})$ is maximal among pairs dominating $(R, \mathfrak{a})$. Further, any such maximal pair $(V, \mathfrak{p})$ consists of a valuation ring $V \neq K$ and its maximal ideal $\mathfrak{p}$.*

Here a pair $(R, \mathfrak{a})$, where R is a subring of K and $\mathfrak{a}$ an ideal of R is said to dominate another pair $(R', \mathfrak{a}')$, if $R \supseteq R'$ and $\mathfrak{a} \supseteq \mathfrak{a}'$. In [**4**] Theorem 2 appears as Lemma 9.4.3, with exactly the statement as above and with a proof similar to that of Theorem 1. Yet in this book Paul does mention Chevalley, including [**2**] as a reference at the end of the book. Theorem 1 in [**2**] comes close to what is called Chevalley's Extension Theorem in [**6**]. Though couched in a totally different language, the proof has a similar flavor. If there is any doubt, Remark 1 after the proof removes it by saying that Neither the definition of a V-ring in R, nor the proof of Theorem 1, makes any use of the fact that R is a field of algebraic functions of one variable. It follows that our proof of Theorem 1 yield a result which is valid for any pair of fields (K, R) such that K is a subfield of R. (For our purposes, a V-ring is a valuation domain $\mathcal{V}$ and a place is its maximal ideal.)

Finally, there has been a spate of some clever Mathematicians taking the easy way out by presenting "new results" by a mere change of terminology and adopting the results they fancy, from someone who they think may not have a voice. Here are two of the examples that I am painfully aware of: https://lohar.com/mithelpdesk/hd2004.pdf and https://lohar.com/mithelpdesk/hd2006.pdf

In my opinion this cannot happen without the help of a supportive referee and as this plague seems to be rampant in the so called multiplicative ideal theory and among the students of Dan Anderson's, I am beginning to see Kaplansky's Theorems 55 and 56 of [**7**], in a similar light, a heist from an unsuspecting fellow's work.

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