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COMMUNICATIONS IN ALGEBRA, 13(8), 1699-1712 (1985)

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THE V-OPERATION AND INTERSECTIONS OF QUOTIENT RINGS OF INTEGRAL DOMAINS

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Throughout this note the letters D and K denote a commutative integral domain with 1 and its quotient field respectively.

Let F(D) denote the set of fractional ideals of D. The opeation v: F(D) $\to$ F(D) defined by $A \to (A^{-1})^{-1} = A_v$ is a <u>*-operation</u> (of [2]) called the <u>v-operation</u>. A fractional ideal $A \in F(D)$ is called a <u>v-ideal</u> if $A = A_v$. Further, $A \in F(D)$ is called a <u>t-ideal</u> if for each finitely generated ideal $B \subset A$, $B_v \subset A$. A maximal t-ideal is an integral ideal maximal w.r.t. being a t-ideal. A maximal t-ideal is prime. Let $\{D_i\}_{i \in I}$ be a family of overrings (rings between D and K) of D with $D = \cap D_i$. Then, according to [2] Theorem 32.5, the operation $A \to \bigcap_i AD_i$ on F(D) is a *-operation. This operation is said to be <u>induced</u> by the family $\{D_i\}$. In this note we record the consequences of the following theorem.

Theorem 1. Let $\{M_i\}_{i \in I}$ be a family of prime ideals of D such that $D = \cap D_{M_i}$. If A is an integral ideal of D with $A^{-1} \neq D$ then A_v, and hence A, is contained in at least one M_i.

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Theorem 1 improves a result of Tang [7]. This result says that, for every finitely generated ideal A of D with $A^{-1} \neq D$, A must be contained in a prime ideal P which is minimal over $(a):(b) = \{r \in D \mid rb \in (a)\}$ for some $a, b \in D$ with $b \notin aD$ ((ii) Theorem E of [7]). Theorem 1 shows that the restriction of being finitely generated is superfluous.

Using Theorem 1 we prove that in a noetherian domain D each maximal prime of a principal ideal (cf [5]) is a maximal t-ideal. We also show that for every fractional ideal A of a noetherian domain D, $A_v = \cap (AD_P)_v$ where P ranges over maximal primes of principal ideals of D. Moreover if A is an integral v-ideal of a noetherian domain D we show that $A = \cap (AD_{P_i} \cap D)$ where P_i ranges over maximal primes of principal ideals of D which contain A. Clearly if each of the P_i is also minimal over A then $A = \cap (AD_{P_i} \cap D)$ is a unique primary decomposition of A. From this it follows that if D is a noetherian domain whose maximal t-ideals are of rank one then every integral v-ideal of D has a unique primary decomposition. We also discuss analogues of these results for generalizations of noetherian domains.

Two *-operations $*_1$ and $*_2$ are said to be equivalent over D if for each finitely generated fractional ideal A, $A^{*_1} = A^{*_2}$. Let $D = \cap D_i$ where $\{D_i\}_{i \in I}$ is a family of quotient rings of D. We study, in the spirit of Theorem 1, the situation in which the *-operation induced by $\{D_i\}_{i \in I}$ is equivalent to the v-operation.

This note is planned as follows. In section 1 we provide a brief introduction to the theory of *-operations. In section 2 we prove Theorem 1 and use it to study v-ideals in noetherian domains. Finally in section 3 we study the various cases where a *-operation is equivalent to the v-operation. Any unexplained notions will be standard as in [2] or [5].

1. The *-operations - an introduction

To facilitate the reading of this note we give, in this section, a brief out-line of the theory of *-operations. These operations have been treated in detail by Gilmer [2] (sections 32 and 34) and by Jaffard [4].

Let $F(D)$ be the set of fractional ideals of D. A function $*: F(D) \to F(D)$ described as $A \to A^*$, is called a *-operation on D if for all $A, B \in F(D)$

(1). $A = A^*$ and $(AB)^* = A B^*$ if A is principal

(2). $A \subset A^*$ and if $A \subset B$ then $A^* \subset B^*$

(3). $(A^*)^* = A^*$.

If $A \in F(D)$ with $A^* = A$ then A is called a *-ideal. In particular A^* is a *-ideal. A *-ideal A is called a *-ideal of finite type if $A = B^*$ where B is a finitely generated fractional ideal. The *-operation on the product of two fractional ideals A, B satisfies the following equations:

$$(AB)^* = (A^* B)^* = (A B^*)^* = (A^* B^*)^*.$$

Every *-operation gives rise to another operation $*_f$, say, defined by $A^{*f} = \cup B^*$ where B ranges over all finitely generated D-submodules of A. Clearly a $*_f$-operation satisfies (1) - (3) and hence is a *-operation again. It is also clear that if A is a *-ideal then it is a $*_f$-ideal. A *-operation is said to be of finite character if it coincides with its $*_f$ operation. It is easy to establish that each $*_f$-operation is of finite character.

The following results, which can be traced back to [2] and [4], will be of use to us.

(i). Let $\{D_i\}_{i \in I}$ be a family of overrings of D with $D = \cap D_i$. Then $A \to \bigcap_i AD_i$ is a *-operation and for all $A \in F(D)$, $AD_i = A^*D_i$.

(ii). If a *-operation is of finite character and if an integral ideal A is a *-ideal it belongs to at least one <u>maximal *-ideal</u>. Maximal *-ideals are defined in the same way as maximal t-ideals.

(iii). If * is of finite character then a maximal *-ideal is prime.

We shall be mainly concerned with the v-operation, the special *-operation defined in the introduction. Note here that the t-operation is the v_f-operation. As we have already observed, a v-ideal is a t-ideal. In fact it can be easily shown that

(iv). For every *-operation and for each A in F(D) $A^* \subset A_v$ and $(A_v)^* = A_v$. In other words a v-ideal is a *-ideal for every *-operation on F(D).

(v). If A is a v-ideal then so are

$$A :_K B = \{x \in K \mid xB \subset A\}$$

$$A : B = \{x \in D \mid xB \subset A\}$$

(vi). Finally for each $A \in F(D)$, A^{-1} is a *-ideal for every *-operation on D. Thus $(A^{-1})^* = A^{-1} = (A^{-1})_v$. That is A^{-1} is a v-ideal. Moreover, it is easy to see that $(A_v)^{-1} = A^{-1}$.

2. <u>Theorem 1 and primary decomposition in noetherian domains</u>

We use (i) and (iv) of section 1 to prove Theorem 1.

<u>Proof of Theorem 1</u>. The family $\{D_{M_i}\}_{i \in I}$ of quotient rings of

D defines a *-operation $A \to \bigcap_i AD_{M_i} = A^*$. If A_v is not contained in any of these primes then $(A_v)^* \doteq \bigcap_i A_v D_{M_i} = \bigcap D_{M_i} = D$. Consequently $A_v = D$. But $A^{-1} = (A_v)^{-1} \neq D$ a contradiction.

Let $a, b \in D$ such that $b \notin aD$, then a minimal prime of $aD:bD$ is called an <u>associated prime of aD</u> (cf. [1]) or, as we shall call it, an <u>associated prime (of D)</u>. Clearly if P is minimal over $aD:bD$ then PD_P is minimal over $(aD:bD)D_P = aD_P : bD_P$. Now since, by (v) of section 1, $aD_P:bD_P$ is a v-ideal it must be contained in a maximal t-ideal. Due to minimality PD_P becomes the maximal t-ideal containing $aD_P:bD_P$. From this it was concluded in [8] that an associated prime ideal P of D is such that PD_P is a t-ideal. Using lemma 4 of [8] we can show that an associated prime P is itsey a t-ideal. Tang [7] proves that $D = \bigcap D_P$ where P ranges over associated primes of D. Tang also proves in [7] that if A is finitely generated ideal of D with $A^{-1} \neq D$ then A is contained in at least one associated prime of D ([7] Th. E). Using Theorem 1 we improve this result to the following corollary.

Corollary 1. Let A be any integral ideal of D and suppose that $A^{-1} \neq D$. Then A is contained in at least one associated pri-me of D.

Proof. According to Theorem 1 the proof follows from the fact that $D = \bigcap D_P$ where P ranges over the associated primes of D.

Corollary 2. Let P be a finitely generated maximal t-ideal of D and let $\{M_i\}_{i \in I}$ be a family of prime ideals of D with $D = \bigcap_i D_{M_i}$. Then P is contained in at least one M_i.

Proof. The proof follows from the fact that $P = P_t = P_v$; because P is finitely generated.

A prime ideal P of D is called a maximal prime of an ideal A if P is maximal within the zero divisors of D/A. Moreover a prime ideal Q is called a prime divisor of A if there exists a multiplicative set S in D such that QD_S is a maximal prime of AD_S. According to Nagata [6] (p. 19) the maximal primes of A are precisely the maximal elements of the set of prime divisors of A. According to Lemma 2 of [1] and 7.5 of [6] prime divisors and the associated primes of (an ideal and hence of) a principal ideal coincide in a noetherian domain. Now since maximal primes of an ideal are maximal among prime divisors we conclude that in a noetherian domain every maximal prime of a principal ideal is an associated prime of a principal ideal. We recall from [5], Theorem 123, that $D = \cap D_P$ where P ranges over maximal primes of principal ideals and that for noetherian domains this intersection is locally finite. Note that $\cap D_P$ is locally finite if each non-zero non-unit of D is contained in only a finite number of prime ideals P appearing in $\cap D_P$. We use this observation to prove the following result.

Theorem 2. Let D be a noetherian domain and let S be the set of maximal primes of principal ideals of D. Then S is precisely the set of maximal t-ideals of D. Moreover for any integral ideal A of D, $A_v = \cap A_vD_P$ where P ranges over S.

Proof. Since in a noetherian domain every prime ideal is finitely generated, by Corollary 2, every maximal t-ideal M is contained in some member P of S. Now $M \subset P$ and P being a t-ideal imply that $M = P$. Further since $P \in S$ are mutually incomparable, because of their maximality, each $P \in S$ is a maximal t-ideal. That $A_v = \cap A_vD_P$, is obvious.

Corollary 3. Let D be a noetherian domain. Then the following hold.

(i). For each ideal A of D which is not contained in any maximal prime of a principal ideal, $A_v = D$.

(ii). For each ideal A which properly contains a maximal prime of a principal ideal, $A_v = D$.

(iii). If A is an integral v-ideal then $A = \cap (AD_{P_i} \cap D)$ where P_i ranges over the finite set of maximal primes of principal ideals which contain A.

Proof. (i) and (ii) are obvious applications of Theorem 2. For (iii) we observe that A is contained in only a finite number of maximal primes of principal ideals. Let $P_1, P_2, \ldots, P_n$ be the set of these prime ideals. Then

$A_v = A = \cap AD_P = (AD_{P_1} \cap \ldots \cap AD_{P_n}) \cap \{D_P \mid P \not\supseteq A\}$. Now
$A = A \cap D = (AD_{P_1} \cap D) \cap \ldots \cap (AD_{P_n} \cap D)$.

Note that $A_v = \cap A_v D_P$ of Theorem 2 does not impart any real information, because this will happen for any integral domain and for any family $\{P_i\}_{i \in I}$ of primes with $D = \cap D_{P_i}$. The situation can be slightly improved by using the noetherian property, or a generalization of it. An integral domain D is called <u>coherent</u> if for any two finitely generated ideals A and B the ideal $A \cap B$ is finitely generated. Clearly a noetherian domain is coherent.

Lemma 1. Let D be a coherent integral domain. Then for any finitely generated fractional ideal A of D, $(AD_S)_v = A_v D_S$.

Proof. In [8], ((i) of Lemma 4) it is shown that if A is a finitely generated fractional ideal then $A^{-1}D_S = (AD_S)^{-1}$. Now if $A = (x_1, \ldots, x_n)$ then $A^{-1} = \cap (1/x_i)$ and because of the coherence

$\cap (1/x_i)$ is finitely generated. Consequently Lemma 4 of [8] can be applied to A^{-1} also and so

$(AD_S)_v = ((AD_S)^{-1})^{-1} = (A^{-1}D_S)^{-1} = (A^{-1})^{-1}D_S = A_vD_S$.

Theorem 3. Let D be a coherent integral domain and let $\{D_i\}_{i \in I}$ be a family of quotient rings of D such that $D = \cap D_i$. Then for each finitely generated fractional ideal A of D, $A_v = \bigcap_i (AD_i)_v$.

Proof. Since $A \to \bigcap_i AD_i$ is a $*$-operation $A_v = (A_v)^* = \bigcap_i A_vD_i$. By Lemma 1, $A_vD_i = (AD_i)_v$.

Corollary 4. If A is a fractional ideal of a noetherian domain then $A_v = \cap(AD_P)_v$ where P ranges over maximal primes of principal ideals of D.

Note that if in (iii) of Corollary 3, each of P_i is minimal over A then $A = \bigcap_i (AD_{P_i} \cap D)$ is a unique primary decomposition. Consequently if D is a noetherian domain with all maximal t-ideals of rank one then every integral v-ideal of D has unique primary decomposition. A more general result is possible.

Theorem 4. Let D be an integral domain such that every maximal t-ideal of D is of rank one. Suppose also that every non-zero non-unit of D belongs to only a finite number of maximal t-ideals. Then every integral v-ideal of D has a unique primary decomposition.

Proof. Here $D = \cap D_P$ where P ranges over maximal t-ideals of D. So $A = A_v = \cap AD_P = (AD_{P_1} \cap D) \cap \ldots \cap (AD_{P_n} \cap D)$ where P_i are the rank one maximal t-ideals which contain A. Now since each P_i is a minimal prime and hence a minimal prime of A, $Q_i = AD_{P_i} \cap D$ is a P_i-primary component of A which appears in each primary decomposition of A, if any exists. From this if tollows that $A = Q_1 \cap Q_2 \cap \ldots \cap Q_n$ is a unique primary decomposition of A.

Corollary 5. In a Krull domain D, for every integral ideal A, A_v has a unique primary decomposition.

3. The equivalence of a *-operation with the v-operation

Recall that two *-operations $*_1$ and $*_2$ are equivalent on D if for all finitely generated $A \in F(D)$ $A^{*1} = A^{*2}$. It is natural to ask that if $\{P_i\}_{i \in I}$ is a family of prime ideals of D with $D = \cap D_{P_i}$, under what conditions is the *-operation $A \to \cap AD_{P_i}$ equivalent to the v-operation. The following result provides an answer to this question.

Theorem 5. Let $\{D_i\}_{i \in I}$ be a family of quotient rings of D such that $D = \bigcap_i D_i$. The operation $A \to \cap AD_i$ is equivalent to the v-operation if and only if for all finitely generated $A \in F(D)$, $AD_i = A_v D_i$.

Proof. Let $A^* = A_v$ for all finitely generated $A \in F(D)$. Then $A_v D_i = A^* D_i = AD_i$ (cf. 32.5 of [2]). Conversely if $AD_i = A_v D_i$ for each finitely generated $A \in F(D)$ then $\cap AD_i = \cap A_v D_i = (A_v)^* = A_v$.

The case of an integral domain with a family $\{D_i\}_{i \in I}$ of overrings with $D = \cap D_i$ and $AD_i = A_v D_i$ for each finitely generated A is too general and hence is not of immediate interest. So we restrict our attention to the case where, in each D_i for every finitely generated $A \in F(D)$, $AD_i = (A D_i)_v = A_v D_i$.

Corollary 6. Let D_i be a family of quotient rings of D with $D = \cap D_i$. Suppose that in each D_i every finitely generated ideal is a v-ideal. Then the *-operation $A \to \cap AD_i$ is equivalent to the v-operation.

Proof. Let A be a finitely generated fractional ideal of D. Then $AD_i = (AD_i)_v$ and by Lemma 4 of [8] $(AD_i)_v = (A_v D_i)_v$. Further as $(A_vD_i)_v = AD_i$ we have $A_vD_i \subset AD_i$ and as, in D, $A \subset A_v$ we have $AD_i \subset A_vD_i$ and hence $AD_i = A_vD_i$ for each $i \in I$. Now by Theorem 5 the result follows.

The class of integral domains whose finitely generated fractional ideals are v-ideals is again a very large class of rings. Before we try to characterise this class it seems best to indicate some applications of Corollary 6 in cases of permanent interest. Recall that an integral domain D is a Prüfer domain if every finitely generated ideal of D is invertible. Recall also that D is a Bezout domain if every finitely generated ideal of D is principal. Because an invertible ideal is a v-ideal we conclude that in both Prüfer and Bezout domains finitely generated ideals are v-ideals. Thus we have the following corollary.

Corollary 7. Let $\{D_i\}_{i \in I}$ be a family of quotient rings of D such that $D = \cap D_i$. If each D_i is a Prüfer (Bezout) domain then the $*$-operation induced by $\{D_i\}$ is equivalent to the v-operation.

A more familiar special case of Prüfer (Bezout) domains is the class of valuation rings i.e. the integral domains whose ideals are linearly ordered under inclusion. A valuation overring V of D is called essential if V is a quotient ring of D. A direct consequence of Corollary 7 is the following well known result.

Corollary 8. (cf. 44.13 of [2]). Let $\{V_i\}_{i \in I}$ be a family of essential valuation overrings of D such that $D = \cap V_i$. Then the $*$-operation $A \to \cap AV_i$ is equivalent to the v-operation.

To this point we have studied the equivalence of a $*$-operation induced by a special family of quotient rings of D with the v-operation. It is natural to look for integral domains for which every family $\{D_i\}_{i \in I}$ of quotient rings with $D = \cap D_i$ induces a $*$-operation equivalent to the v-operation. For this we have the following result.

Theorem 6. The following are equivalent for a commutative integral domain D.

(1) Every finitely generated fractional ideal of D is a v-ideal.

(2) For every finitely generated fractional ideal A and for every multiplicative set S, $AD_S = A_v D_S$.

(3) For every finitely generated fractional ideal A of D and for every maximal ideal M, $AD_M = A_v D_M$.

(4) For any family $\{D_i\}$ of quotient rings of D with $D = \cap D_i$ the $*$-operation $A \to \cap AD_i$ is equivalent to the v-operation.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) are abvious. Further, (2)$\Rightarrow$ (4) follows from Theorem 5 and (4) $\Rightarrow$ (3) is a straightforward application of 32.5 of [2].

Restricting our attention to the more special case where D has the property that $(AD_S)_v = A_v D_S$ for all finitely generated A and for every multiplicative set S, we have the following result.

Theorem 7. Let D be an integral domain with the property that for every multiplicative set S and for every finitely generated fractional ideal A, $(AD_S)_v = A_v D_S$. Then the following are equivalent for D.

(1) Every finitely generated fractional ideal of D is a v-ideal.

(2) Every finitely generated ideal of D_S is a v-ideal.

(3) Every finitely generated fractional ideal of D_P is a v-ideal for each prime ideal P.

(4) Every finitely generated ideal of D_M is a v-ideal for each maximal ideal M of D.

Proof. (1) $\Rightarrow$ (2). Let A be a finitely generated fractional ideal of D. Then, by (1), $A = A_v$ and so $AD_S = A_vD_S$. But then by hypothesis $A_vD_S = (AD_S)_v$. Whence it follows that every finitely generated ideal A of D_S is a v-ideal for any multiplicative set S.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious. So we prove (4) $\Rightarrow$ (1) to complete the proof. (4) $\Rightarrow$ (1). Recall that for every integral (and hence for every fractional) ideal A, $A = \cap AD_M$ where M ranges over maximal ideals of D. Now by (4) and by hypothesis for every maximal ideal M, $AD_M = (AD_M)_v = A_vD_M$. Whence $A = \cap A_vD_M = A_v$.

We call D an <u>FGV-domain</u> if every finitely generated ideal of D is a v-ideal. As a result of this definition, every maximal ideal of an FGV-domain is in fact a t-ideal and hence a maximal t-ideal. We note that, according to Lemma 1, a coherent (and hence a noetherian) domain satisfies the hypothesis of Theorem 7. Thus we have the following corollary.

Corollary 9. A coherent (or noetherian) domain D is FGV if and only if for each maximal ideal M, D_M is an FGV-domain.

Theorem 7 is based on the notion of quasi-local FGV-domains. Clearly a valuation domain is an FGV-domain but the nature of our results demands that we should also look for non-valuation FGV-domains. We note that an integrally closed local FGV-domain is a valuation domain. This follows from the following theorem.

Theorem 8. An integrally closed FGV-domain is a Prufer domain.

Proof. Let K be the quotient field of D. Then as D is integrally closed, by 34.8 of [2], for all f, g ε K[X], $(A_{fg})_v = (A_fA_g)_v$, where A_f denotes the content of the polynomial f. But as D is an

FGV-domain $(A_{fg})_v = A_{fg}$ and $(A_f A_g)_v = A_f A_g$. Whence for all f, $g \in K[X]$, $A_{fg} = A_f A_g$ and this is a sufficient condition, by 28.6 of [2], for D to be a Prüfer domain.

So if we are looking for non-valuation quasilocal FGV-domains we are looking for quasi-local FGV-domains which are not integrally closed. In the following we give two examples of these integral domains.

(a). Pseudovaluation domains (PVD's). According to Hedstrom and Houston, and integral domain is a PVD if it is a quasi-local domain with maximal ideal M and if it satisfies one of the following equivalent properties, (cf. [3] Prop. 1.2).

(1). For each pair I, J of ideals of D either $I \subset J$ or $MJ \subset MI$.

(2). $x^{-1} M \subset M$ whener $x \in K - D$.

(3). D has a unique valuation overring V with maximal ideal M.

(4). There exists a valuation overring such that every prime ideal of D is also a prime ideal of V.

According to Corollary 1.8. of [3] the valuation domain of (3) and (4) above is two-generated as a D-module if and only if every finitely generated ideal of D is a v-ideal.

(b). Gorenstein rings. According to theorem 222 of [5], a one dimensional local domain R is a Gorenstein ring if and only if each ideal of R is a v-ideal. Thus a one-dimensional local Gorenstein domain is an example of an FGV domain. (I am thankful to the referee for suggesting this neat alternative for an earlier, rather involved, statement).

Acknowledgement

I thank Valentina Barucci and Stefania Gabelli for reading the script thoroughly and critically.

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Received: January 1984
Revised: April 1985