QUESTION (**HD1002**): Everyone tells me that an integral domain that satisfies ACCP is atomic but no one shows me how. Could you please?

ANSWER. Frankly the explanation is so easy that I am surprised that you asked. But since you asked, I put down, below, an answer along with some relevant information.

Let's recall, for the sake of other readers, that an element x in an integral domain D is ireducible, or an atom, if x is nonzero nonunit and x cannot be expressed as a product of two nonunits. That is a nonzero nonunit x is irreducible if and only if $x = ab$ implies that a is a unit or b is a unit. An integral domain D is said to be atomic if every nonzero nonunit of D is expressible as a finite product of atoms of D. Finally an integral domain D is said to satisfy the ascending chain condition on principal ideals (ACCP) if every ascending chain of (proper) integral principal ideals terminates (finitely).

Now suppose that D satisfies ACCP and let x be a nonzero nonunit of D. Consider $xD = x_0D \subsetneq x_1D \subsetneq x_2D \subsetneq \ldots$ Note that $x_iD \subsetneq x_{i+1}D$ if and only if $x_{i+1} \mid x_i$ and $x_i \nmid x_{i+1}$. Now because of ACCP there is an n such that $xD = x_0D \subsetneq x_1D \subsetneq x_2D \subsetneq \ldots \subsetneq x_nD$ and there is no entry possible, of a proper principal ideal, in the chain, beyond x_nD. This obviously means that for any $y \in D$ with $x_nD \subsetneq yD$ we must have $yD = D$. This means that x_n is irreducible, because if $x_n = ab$ where both a, b are nonunits then $x_nD \subsetneq aD$. Thus we have verified the following observation: If an integral domain D satisfies ACCP then every nonzero nonunit of D is divisible by at least one atom.

Coming back to the nonzero nonunit x, in a domain D with ACCP, that we started with we know that x is divisible by an atom a_1. Consider $\frac{x}{a_1}$. If $\frac{x}{a_1}$ is a unit then x is an associate of an irreducible element and hence an ireducible element. If $\frac{x}{a_1}$ is not a unit then there is an atom a_2 dividing $\frac{x}{a_1}$ and we have $xD \subsetneq \frac{x}{a_1}D$. Take $\frac{x}{a_1a_2}$ and note that if $\frac{x}{a_1a_2}$ is a unit then x is a product of two atoms, if not continue and note that $xD \subsetneq \frac{x}{a_1}D \subsetneq \frac{x}{a_1a_2}D$. Continuing in this manner, assuming at each step that the quotient is a nonunit we can set up an ascending chain of proper principal ideals such that $xD \subsetneq \frac{x}{a_1}D \subsetneq \frac{x}{a_1a_2}D \subsetneq \ldots \subsetneq \frac{x}{a_1a_2\ldots a_{m-1}}D$ terminates at $\frac{x}{a_1a_2\ldots a_{m-1}}D$. That, as before means that $\frac{x}{a_1a_2\ldots a_{m-1}}$ is an atom a_m esuring that $x = a_1a_2\ldots a_{m-1}a_m$.

Thus we conclude that if D satisfies ACCP then every nonzero nonunit of D is expressible as a finite product of atoms.

Now that we have started talking about these things let's make a few notes.

(1). There are non atomic domains D such that every nonzero nonunit of D is divisible by an atom. One good example is a valuation domain V of Krull dimension ≥ 2 with a principal maximal ideal $M = pV$. Any nonzero element x of a non-maximal prime ideal P is divisible by every power of p and so cannot be expressed as a finite product of atoms.

(2) There are atomic domains that do not satisfy ACCP, so ACCP is a sufficient condition for atomicity of a domain but not a necessary one. The Example, which is due to Anne Grams is somewhat elaborate and cannot be reproduced here. Yet, in case you are interested, I include a reference to the article in which it appeared: [Grams, Atomic domains and ascending chain conditions on principal ideals, Math. Proc. Cambridge Philos. Soc. 75(1974) 321-329. Anne Grams was a student of Robert Gilmer.