Splitting sets and Factorization

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Preamble

By factorization of a nonzero nonunit of an integral domain D, we generally mean, expressing the element as a product of other, simpler, elements.

The “simpler” elements are often, irreducible elements or atoms, i.e. ones that cannot be further expressed as products of two or more non units.

We study factrization because factorization properties of nonzero nonunits of an integral domain often lead to nice conclusions about the properties of integral domain.

The factorization I would be talking about is more to do with expressing nonzero elements x of an integral domain D as products $x = uv$ where u comes from one kind of elements and v comes from another kind of elements.

Under certain circumstances, this kind of factorization also leads to integral domains with some interesting properties.

Terminology and notation.

I would often use “domain D” or just “D” to mean an “integral domain D” . The letter K is reserved for the field of fractions of D.

I would use $F(D)$ to denote the set of nonzero fractional ideals of D. For $A \in F(D)$, I will use the notation $A^{-1} = \{x \in K : xA \subseteq D\}$, $A_v = (A^{-1})^{-1}$and $A_t = \cup\{F_v : F$ is a nonzero subideal of $A\}$.

The function, $*$ on $F(D)$, denoted by $A \to A_*$ is called a star operation if it has the following properties: For all $a \in K\backslash\{0\}, A, B \in F(D)$

(1) $(a)^* = (a)$, $(aA)^* = aA^*$, (2) $A \subseteq A^*$ and $A \subseteq B \Rightarrow A^* \subseteq B^*$ (3) $(A^*)^* = A^*$. For a star operation the following can also be shown to hold:

(4) $(AB)^* = (A^*B)^* = (A^*B^*)^*$ and (5) For $A_i \in F(D)$, if $\sum A_i \in F(D)$, then

$(\sum A_i)^* = (\sum (A_i)^*)^*$.

The operations $A \to A_v$ and $A \to A_t$ are the well known star operations. For further details you may consult Gilmer's [Multiplicative Ideal Theory, Marcel Dekker, New York, 1972].

Two elements $x, y \in D$ are called v-coprime if $(x,y)_v = D$ or equivalently if $xD \cap yD = xyD$. It can be shown that if r and s are v-coprime to x then so is their product and conversely. $\{(x,rs)_v = (x,rx,rs)_v = (x,(rx,rs)_v)_v. = (x,r)_v$ if x,s are v-coprime. Also, $(x,r)_v = (x,rs,r)_v = ((x,rs)_v,r)_v.)$. Indeed, using these equations we can show that if $x \mid rs$ and $(s,x)_v = D$ then $x \mid r$ and that if $(x,rs)_v = D$ then $(x,r)_v = (x,s)_v = D.\}$

Splitting (Multiplicative) Sets

A saturated multiplicative set S of D is said to be a splitting multiplicative set if each

$x \in D \backslash \{0\}$ can be written as $x = ds$ where $s \in S$ and d is v-coprime to every member of S. For ease of expression we shall call d and s, "S –disjoint factors of x".

A splitting set S is said to be an lcm-splitting set if each element of S has an lcm with each element of D.

Examples: (i) A saturated multiplicative set S generated by principal primes of a domain D that satisfies ACC on principal ideals.

(ii) Not all sets of primes generate splitting sets. Let (V, pV) be a discrete rank one valuation domain with maximal ideal pV and let K be the quotient field of V. Then, in the domain $D = V + XK[[X]]$, we cannot write X as a product of S –disjoint factors, where S is the saturation of $\{p^n\}_{n=0}^{\infty}$ in D.

(iii) Not all splitting sets have to be generated by primes. Let (V, M) be a rank one non-discrete valuation domain. In $V[X]$ the set $V \backslash \{0\}$ is a splitting set. Then there is the theorem: Let D be a GCD domain and let S be a multiplicative set of $D,$ the construction $D + XD_S[X]$ is a GCD domain if and only if S is a splitting set.[Zafrullah, J. Pure Appl. Algebra 50 (1988), no. 1, 93–107].

(iv) All the splitting sets we have met so far happen to be lcm-splitting sets. Recall that D is a weakly factorial domain (WFD) if every nonzero nonunit of D is expressible as a product of primary elements [Anderson and Mahaney, J. Pure Appl. Algebra 54 (1988), no. 2-3,141–154]. A WFD does not have to be a GCD domain, now consider the following: An integral domain D is a WFD if and only if every saturated multiplicative set in D is a splitting multiplicative set, [AZ, Proc. Amer.Math. Soc. 109 (1990), no. 4, 907–913]. So we have splitting sets that are not lcm-splitting.

There are trivial splitting sets, for each D, such as $D\backslash\{0\}$ and $U(D)$ the set of all units.

(v) We can make new splitting sets from given ones. Let S be a splitting set of D and consider $T = \{x \in D : x$ is v-coprime to every member of $S\}$. Then T is a splitting multiplicative set.[AAZ, J. Pure Appl. Algebra 74 (1991), no. 1, 17–37].

The set T is said to be the m-complement of the splitting set S. It can be shown that $D = D_S \cap D_T$ [Prop. 1.1 AZ,PAMS, 129,8 (2000),2209-2217].

Properties of Splitting Sets

By way of preparation let me mention that $P(D)$ denotes the group of nonzero principal fractional ideals of D partially ordered in the usual way.

Theorem([Theorem 2,AAZ, JPAA]). Let S be a saturated multiplicative set of D. Then TFAE:

(1). S is a splitting set.

(2). $<S>$ is a cardinal summand of $P(D)$. That is there is a subgroup H of $P(D)$ such that $P(D) =< S > \oplus_c H$. Indeed $H =< T >$

where T is the m-complement of S.

(3). If A is an integral principal ideal of D_S then $A \cap D$ is principal.

(4). The set $T = \{t \in D : (t,s)_v = D$ for all $s \in S\}$ is a splitting set.

From this point on S will denote a splitting set of D and T the m-complement of S.

From this theorem, among other things, we conclude that if we discover a nontrivial splitting set S we have another, its m-complement, nontrivial splitting set T, such that for all $d \in D \backslash \{0\}$, $d = st$ and this factorization is unique up to associates.

Now this information can be put to use from the factorization point of view in many ways. For example:

1) If the members of S and T are expressible as products of atoms then each nonzero nonunit of D is expressible as a product of atoms.

2) If S and T are factorial monoids then so

is $ST = D\backslash\{0\}$. This is essentially Nagata's Theorem for UFD's.

3) If nonunit members of S and T are finite products of primary elements of D then nonunit elements of $ST = D\backslash\{0\}$ are finite products of primary elements of D.

Now the effectiveness of the splitting sets does not stop at the factorization properties. It extends its sway over the monoid $f(D)$ of nonzero finitely generated ideals of D albeit in a special way.

To see it more clearly we need introduction to some more terminology. For a star operation $*$, a fractionary ideal $A \in F(D)$ is said to be a $*$-*ideal* if $A = A^*$, *a star ideal of finite type* if $A = B^*$ for a finitely generated $B \in F(D)$ and $*$-*invertible* if there is $B \in F(D)$ such that $(AB)^* = D$. An integral domain in which every finitely generated ideal is t-invertible is called a Prufer v-multiplication domain (PVMD).

It is well known that an integral ideal

maximal w.r.t. being a t-ideal is a prime ideal and that D is a PVMD if and only if for eaxh maximal t-ideal P of D, D_P is a valuation domain. Here are some auxiliary results that will be of use soon.

(1). It was shown in [AAZ,JPAA] that if S is a splitting set and A is an integral ideal of D then $(AD_S)_t = A_t D_S$ so every maximal t-ideal P of D with $P \cap S = \phi$ extends to a maximal ideal of D_S.

(2). Proposition ([Lemma 3.1, AAZ, JPAA]). Let S be a splitting multiplicative set of D and let T be the m-complement of S. For $s_1, s_2, ..., s_n \in S$ and $t_1, t_2, .., t_n \in T$, $(\sum s_i t_i D)_v = ((\sum s_i D)(\sum t_i D))_v$.

Let us add a minor lemma to this proposition.

(3). Lemma . Let S be a splitting multiplicative set of D and let T be the m-complement of S in D. For each nonzero finitely generated ideal A of D and for each $t \in T$ we have $(A, t)_v = (t_1, t_2, ...t_r, t)_v$ where

$t_i \in T$.

Proof. Let us first note that if $s \in S$ and $t, u \in T$ then $(t, su)_v = (t, tu, su)_v = (t, (tu, su)_v)_v = (t, (t, s)_v u$ $= (t, u)_v$ because $(t, s)_v = D$. Now let $A = (a_1, a_2, ...a_r) = (s_1t_1, s_2t_2, ..., s_rt_r),$ where $s_i \in S$ and $t_i \in T$. Then $(t, A)_v =$ $(t, s_1t_1, s_2t_2, ..., s_rt_r)_v =$ $((t, s_1t_1)_v, s_2t_2, ..., s_rt_r)_v = (t, t_1, s_2t_2, ...s_rt_r)_v.$ Now invoking induction we get the desired result.

Using this statement we can prove statements like:

Observation: Let S and T be as agreed. Suppose that for all $A \in f(D),\ A \cap S \neq \Phi$ implies that A is t-invertible. Then D is a (i) PVMD if and only if (ii) D_S is a PVMD if and only if (iii) for each $A \in f(D),\ A \cap T \neq \Phi$ implies that A is t-invertible.

The equivalence of (i) and (iii) follows from the fact that for each $A \in f(D), A_v = ((s_1, ..., s_n)(t_1, ..., t_n))_v$ which is t-inverytible iff each of $(s_1,...,s_n),(t_1,...,t_n)$ is t-invertible. Next (i) implies (ii) always. So this leaves (ii) implies (i) (or (iii)). To prove this we shall need the following results.

(1). Part of Theorem 2.10 [AZ,PAMS,2000]. If $S \neq D^*, U(D)$ is a splitting multiplicative set and T its m-complement in D then S partitions t-max(D) into $F = \{P \in t-Max(D) : P \cap S = \phi\}$ and $G = \{Q \in t - \max(D) :$ Q intersects S in detail$\}$ such that for $P \in F$ and $Q \in G$ if a nonzero element x belongs to $P \cap Q$ then $x = x_1x_2$ where $x_1 \in P \backslash X$ for every $X \in G$ and $x_2 \in Q \backslash Y$ for every $Y \in F$.

(2). If S and T are as above such that for each $A \in f(D)$ with $A \cap S \neq \phi,\ A$ is t-invertible then D_T is a PVMD. (It is enough to note that if A is a finite type t-ideal of D_T then there is a finitely generated ideal *a* of D such that $A = (aD_T)_t = a_tD_T$ and we can assume that $a_t \cap S \neq \phi$, so a_t and hence a_tD_T is t-invertible.)

Now, by (2), D_T is a PVMD and we are given that D_S is a PVMD. So we have $D = D_S \cap D_T$ and by (1) above every maximal t-ideal of D survives in exactly one of D_S, D_T where it is a maximal t-ideal and hence every maximal t-ideal P of D is such that D_P is a valuation domain. This of course means that D is a PVMD.

Now let S be lcm-splitting. Then, for each $A \in f(D)$ with $A \cap S \neq \phi,\ A_v$ is principal, so the above observation becomes slightly more than a well known Nagata type Theorem, on PVMD's. See, for example [Theorem 4.3,AAZ,JPAA,91].

Now let us take a closer look at the splitting set S defined in (2) above, as the splitting set such that for each $A \in f(D)$ with $A \cap S \neq \phi$ A is t-invertible. Let us , for the purposes of this talk, call such a set a $*$ set. Now for each s in the $*$ set $S,$ and for each $d \in D\backslash\{0\}$ we have (s,d) t-invertible which means that $sD \cap dD$ is t-invertible. So, a $*$ splitting set is like an lcm splitting set. We can say that x, y have t-lcm if $xD \cap yD$ is t-invertible. Thus a $*$ splitting set is a t-lcm splitting set. With some straightforward reasoning it can be shown that a t-lcm splitting set is indeed a $*$ splitting set. Now the question is does a t-lcm splitting set have the same nice properties as an lcm splitting has? One nice property I have shown, someone in the audience can explore the others. Then we can also consider d-lcm splitting sets, which can be defined analogously, if we agree to say that x and y have d-lcm if $xD \cap yD$ is invertible.

Finally, the question is why stop here why not also study splitting sets S such that for each $A \in F(D)$, with $A \cap S \neq \phi$, A is t-invertible. If S is such a splitting set, call it a $**$ set, it is easy to check that D is a Krull domain if and only if D_S is a Krull domain. Then there is the obvious question that if D

is such that every saturated multiplicative set is $**$, must D be Krull?

The next question about the splitting sets that could arise in certain situations is. How do splitting sets fare in ring extensions? For general extensions the situation is hopeless, yet in some particular cases we do have a positive response.

THEOREM. Let $D \subseteq E$ be an R_2 –stable extension of domains, where E is an overring of D. Let S be a saturated multiplicative set in D and let S' be the saturation of S in E.

(1). If S is a splitting set then so is S'.

(2). If S is an lcm splitting set then so is S'.

This theorem is a part of my paper with, Dumitrescu [lcm-splitting sets in some ring extensions, to appear in Proc. Amer. Math. Soc.].