Lemma 5. Let x be a non zero non unit with a single minimal subvalued prime q in an HCF domain and let $\{P_\beta\} \subset \{P_\alpha\}$ be the family of valued primes containing x, then for every element y which is contained only in the intersection of a subfamily of $\{P_\beta\}$ such that $y \notin q$ then $y^n | x$ for all n.

Proof. Let x and y be as in the hypothesis, then for each n, $xy^n \in q$ and xy^n has q as its minimal subvalued prime (any minimal subvalued prime of y is some subvalued prime containing q).

Now suppose that $y \nmid x$ and let $d = (x,y)$ where $x = x_1 d$, $y = y_1 d$ and $(x_1, y_1) = 1$, then since $y \notin q$, $d \notin q$ and so $xy/d^2 \in q$ and q is the single minimal subvalued prime of xy/d^2. But $xy/d^2 = x_1 y_1$ where $(x_1, y_1) = 1$. In other words xy/d^2 has a single minimal prime and is expressible as a product of two co-prime non units, a contradiction of (p_1) unless y_1 is a unit i.e. $y|x$. Similarly we can proceed with y^n and can show that $y^n | x$ for each n.

To show that (p_2) also holds for x of Lemma 4, we first note that q being a prime ideal, $x_1 \in q$ or $x_2 \in q$, and we have two cases to consider:

(a) $x_1 \in q$, and $x_2 \notin q$ (or $x_2 \in q$ and $x_1 \notin q$)

(b) $x_1, x_2 \in q$.

If (a) holds, x_2 belongs to a subfamily of the valued primes containing x and by Lemma 5, $x_2^n | x$ for each n, i.e. $x_2 | x_1$. And in case (b) holds, $x_1, x_2 \in q$ implies that x_1, x_2 both have q as their minimal subvalued prime and that $(x_1, x_2) = d \in q$ (R is an HCF domain and R_q is a valuation domain). Now if $(x_1, x_2) = d$ then $x_1 = x_1' d$, $x_2 = x_2' d$ where $(x_1', x_2') = 1$ i.e. at least one of x_1', x_2' is not in q. This in turn gives rise to the following two cases: