On minimal primes of a star ideal

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Abstract. We show that a proper ideal I of a ring R is principal (resp., invertible, finitely generated locally principal) if every prime ideal containing I has those properties and as a byproduct we conclude that if there is an ideal I, in a ring R, that is not principal ((resp., not invertible, not finitely generated locally principal) then I must be contained in a prime ideal of corresponding description. Dedekind domains and PIDs can be characterized using the about results and via star operations we extend the sway of the above result to include Krull domains, locally factorial Krull domains and UFDs.

The aim of this note is to indicate a few applications and variations of the following theorem.

Theorem 1. *Let R be a commutative ring and I a proper ideal of R. Suppose that every prime ideal containing I is principal (resp., invertible, finitely generated locally principal) Then I is a product of principal (resp., invertible, finitely generated locally principal) prime ideals and hence is principal (resp., invertible,finitely generated locally principal).*

Proof. Note that the case of finitely generated locally principal covers the other cases, so providing a proof for this case would be sufficient. Let I be contained in only finitely generated locally principal prime ideals. Pass to $\overline{R} = R/I$. So every prime ideal of $\overline{R}$ is finitely generated locally principal. By Cohen's Theorem $\overline{R}$ is Noetherian. So $(\overline{0})$ has a reduced primary decomposition $(\overline{0}) = \overline{Q_1} \cap \overline{Q_2} \cap ... \cap \overline{Q_n}$ where each Q_i is a P_i-primary ideal of R with $I \subseteq Q$. Since P_i is finitely generated locally principal, Q_i is a power of P_i, [**1**, Lemma 1]. Moreover for each maximal ideal M of $\overline{R}$, $\overline{R}_M$ is a local Noetherian ring with each prime ideal principal. $\overline{R}_M$ is DVR or SPIR. Thus the ideals $\overline{Q_1}, \overline{Q_2}, ..., \overline{Q_n}$ are pairwise co-maximal and hence, so are $Q_1, Q_2, ..., Q_n$. So, $I = Q_1 \cap Q_2 \cap ... \cap Q_n = Q_1 Q_2, ... Q_n$ is a product of finitely generated locally principal ideals. □

This result characterizes on the one hand rings all of whose proper ideals are products of principal (resp., invertible, finitely generated locally principal) primes as those rings whose nonzero prime ideals are principal (resp., invertible, finitely generated locally principal) and on the other provides an indirect proof of the

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following statement: Let R be a commutative ring and I a proper ideal of R. If I is not principal (resp., not-invertible, not finitely generated locally principal) then I is contained in a prime ideal that is not principal (resp., not-invertible, not finitely generated locally principal), thus bypassing tricky proofs of some well-known "Cohen-like" statements.

Indeed, among integral domains the obvious beneficiaries of this result are the PIDs and the Dedekind domains which are characterized by "every nonzero ideal is principal (resp., invertible)". Now PIDs and Dedekind domains are linked with UFDs and Krull domains, via the so called star operations. In what follows, to prove our results, we use a star operations version of the following theorem of Anderson: Let I be an ideal of a ring R with $I \neq R$. If every minimal prime ideal of I is finitely generated then I has only finitely many minimal primes [**2**, Theorem]. Here, by a minimal prime of an ideal I we mean a prime ideal minimal among the set of prime ideals containing I.While restricting ourselves to star operations of finite type (introduction to follow) we prove the following version of [**2**, Theorem].

Theorem 2. *Let R be an integral domain, $*$ a star operation of finite type defined on R and let I be an ideal of R with $I^* \neq R$. If every minimal prime ideal over $I^*$is a $*$-ideal of finite type then I has only finitely many minimal primes.*

A word about some terminology, by "ideal" we mean a fractionary ideal and an ideal that is contained in the ring is called an integral ideal or just an ideal if we talk about a ring, with $1 \neq 0$, allowing zero divisors. For reasons to be explained later we select the so called t-operation for the star operations version of Theorem 1.

Theorem 3. *Let R be an integral domain. If J is a t-ideal of R such that J is not contained in any nonprincipal (resp., non-invertible, non-t-invertible) prime t-ideal then J is principal (resp., invertible, t-invertible). Consequently, every non-principal (resp., non-invertible, non-t-invertible) nonzero ideal is contained in a nonprincipal (resp., non-invertible, non-t-invertible) prime ideal.*

Apart from Theorem 2 we shall need the following result as a tool.

Theorem 4. *If P is a t-invertible prime t-ideal containing a t-ideal I then either (a) $I = (P^{(n)}J)_t$ where $J \nsubseteq P$ or (b) there is a non-t- invertible prime t-ideal between P and I. (Here $P^{(n)} = P^nR_P \cap R$. Obviously if P is invertible then $P^{(n)} = P^n$).*

Now to get on with the proofs we need some working introduction to the star operations. Our terminology is standard as in Gilmer's [**5**, Sections 32 and 34] for star operations and as in [**9**] for any other topics, and occasionally we would refer to [**13**]. We provide a quick introduction to the star operations below.

Let R denote an integral domain with quotient field K and let $F(R)$ be the set of nonzero fractional ideals of R. A star operation $*$ on R is a function $*\colon F(R) \to F(R)$ such that for all $A, B \in F(R)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,
(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,
(c) $(A^*)^* = A^*$.

We note that for $A, B \in F(R)$ $(AB)^* = (A^*B)^* = (A^*B^*)^*$. Let's call $(AB)^*$ the $*$-product. A fractional ideal $A \in F(R)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal.

Any intersection of $*$-ideals for any star operation $*$ is again a star ideal. A star operation $*$ is said to be of *finite character or of finite type* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. A star operation $*$ is of finite type if and only if for each ideal $A \in F(R)$, $x \in A^*$ implies that there is a finitely generated $F \subseteq A$ with $x \in F^*$. For $A \in F(R)$ define $A^{-1} = \{x \in K \mid xA \subseteq R\}$ and call $A \in F(R)$ $*$-*invertible* if $(AA^{-1})^* = R$. Clearly every invertible ideal is a $*$-invertible $*$-ideal for every star operation $*$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$. (Note that if A is nonzero finitely generated then $A_t = A_v$.) Given two star operations $*_1, *_2$ we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(R)$. Note that $*_1 \leq *_2$ if and only if $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. By definition t is of finite character, $t \leq v$ while $\rho \leq t$ for every star operation ρ of finite character. Thus if $I \in F(R)$ is $*$-invertible for any star operation of finite character then I is t-invertible. . Also, for every star operation $*$ of finite character every $*$-invertible $*$-ideal is a t-invertible t-ideal.[**13**, Theorem 1.1]. Also let I, J be integral ideals with $J \subseteq I$, if I is $*$-invertible then there is an integral ideal B such that $J^* = (IB)^*$. Finally if S is a ring of fractions of R and A is t-ideal of S then the contraction $A \cap R$ is a t-ideal of R [**13**, page 436]. This can be derived as a consequence of the following result. If A is an ideal of R and S a multiplicative set of R then $(AR_S)_t = (A_tR_S)_t$ [**8**, Lemma 3.4].

If $*$ is a star operation of finite character then using Zorn's Lemma we can show that an integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $*$-*max*(R). It can also be easily established that for a star operation $*$ of finite character on R we have $R = \bigcap\limits_{M \in *\text{-}\max(R)} R_M$. Using t-max(R) we can define a star operation w as $A \mapsto A^w = \bigcap_{P \in t-\max(D)} AR_P$, and it is a finite character star operation. This operation was studied in [**12**]. A v-ideal A of finite type is t-invertible if and only if A is t-locally principal i.e. for every $M \in t$-max(R) we have AR_M principal. A domain R is a Krull domain if and only if every nonzero ideal of R is t-invertible if and only if every t-ideal of R is t-invertible [**10**, Theorem 2.5], if and only if every prime t-ideal is t-invertible.

Next, every t-invertible prime t-ideal is a maximal t-ideal [**7**, Proposition 1.3] and as we have already indicated for a finite character operation $*$, a $*$-invertible $*$-ideal is a t-invertible t-ideal. Thus, if $*$ is of finite type then every $*$-invertible prime $*$-ideal Q is a maximal t-ideal. Thus every (nonzero) principal prime and every invertible prime is a maximal t-ideal. It was shown by Kang [**8**] that R is a Krull domain if and only if every minimal prime of a principal ideal of R is t-invertible. This result can be improved to: R is a Krull domain if and only if every prime t-ideal is t-invertible.

Proof. (of Theorem 2)Without loss of generality we can assume that I is a $*$-ideal. Then every prime ideal P minimal over I is a prime $*$-ideal [**6**, Proposition 1.1] and by the hypothesis, P is of finite type. Now let $S = \{(P_1P_2 \cdots P_n)^* \mid P_i$ is a prime ideal minimal over $I\}$. If for some $C = (P_1P_2 \cdots P_n)^* \in S$ we have $C \subseteq I$, then every prime ideal minimal over I contains some P_i and so $\{P_1, P_2, ..., P_n\}$ is the set of minimal primes of I, where $|\{P_1, P_2, ..., P_n\}| \leq n$. To establish that one of the $C \in S$ is indeed such that $C \subseteq I$ we arrange for a contradiction via Zorn's Lemma.

Let's assume that $C \nsubseteq I$ for any of the C in S and define $T = \{J \mid J$ is a $*$-ideal with $J \supseteq I$ and $C \nsubseteq J$ for any C in $S\}$. Then obviously T is nonempty as $I \in T$, and as members of S are $*$-ideals of finite type we conclude that the union of an ascending chain in T is in T. To see this let $\{U_\alpha\}$ be an ascending chain in T and let $(P_1P_2 \cdot \cdot \cdot P_n)^* = (x_1, x_2, ..., x_r)^*$ and suppose that $(x_1, x_2, ..., x_r)^* \subseteq \cup U_\alpha$. Then say $x_1 \in U_{\alpha_1}$, $x_2 \in U_{\alpha_2}, ..., x_r \in U_r$ where $U_{\alpha_1} \subseteq U_{\alpha_2} \subseteq ...U_{\alpha_r}$. This gives $\{x_1, x_2, ..., x_r\} \subseteq U_{\alpha_r} = \cup_{i=1}^r U_{\alpha_i}$. But as each U_α is a $*$-ideal $(x_1, x_2, ..., x_r)^* \subseteq U_{\alpha_r}$ a contradiction establishing that T is indeed inductive. So by Zorn's Lemma, T must have a maximal element Q. It is easy to show, as we demonstrate below, that Q is a prime ideal and so must contain a minimal prime P containing I. By the condition P must be a $*$-ideal of finite type and hence must be in S and this contradicts Q being in T. For the proof that Q is a prime, suppose that $xy \in Q$ and that $x, y \notin Q$. Now as Q is maximal in T, $(Q, x)^*, (Q, y)^*$ must violate the conditions defining T. Now as $(Q, x)^*, (Q, y)^*$ each contain I the only violation would be that each of $(Q, x)^*, (Q, y)^*$ contains a member from S. Let $C_1, C_2 \in S$ such that $C_1 \subseteq (Q, x)^*$ and $C_2 \subseteq (Q, y)^*$. But then $(C_1C_2)^* \subseteq ((Q, x)^*(Q, y)^*)^* = (Q, x)(Q, y))^* \subseteq Q$ because xy belongs to Q. But then Q belongs to T and $(C_1C_2)^* \in S$ a contradiction, establishing that Q is indeed a prime ideal. □

Remark 1. *Theorem follows from Sahandi's work in* [**11**]. *We have kept out proof because it is direct, short and lets the reader continue without having to struggle with new terminology. As Sahandi points out El Baghdadi and Gabelli* [**3**] *have proved Theorem for* $* = t$ *in the context of PVMDs.*

Proof. (of Theorem 4) The basic result that will prove this assertion is: If pR is a principal prime containing a nonzero ideal I such that $I \neq p^nJ$ for any n and any integral ideal J then there is a prime ideal $Q = \cap_{n=1}^\infty p^nR$ containing I such that $pR \supsetneq Q \supseteq I$. Of course Q is a t-ideal being an intersection of principal ideals and Q cannot be principal, because nonzero principal is invertible and so t-invertible and so a maximal t-ideal. But pR is already a maximal t-ideal properly containing Q. Now to draw the same conclusion for the prime P in the proposition note that P is t-invertible and hence a t-ideal of finite type and so PR_P is a principal prime containing IR_P. So, by the above, $IR_P = P^nR_P$ or there is a t-ideal $QR_P = \cap_{r=1}^\infty P^rR_P$ where Q cannot be principal. Next if $IR_P = P^nR_P$, then as P is t-invertible and hence of finite type $(P^n)_v$ is t-invertible. But as $P^{(n)}$ is a t-ideal, being contracted from a principal ideal, and as $P^{(n)} \sqsupseteq P^n \supseteq A^n$ where A is finitely generated such that $A_v = P$ we have $P^{(n)} \supseteq (P^n)_v$. On the other hand $(P^n)^w = \bigcap_{M \in t-\max(R)} P^nR_M = P^nR_P \cap R = P^{(n)}$. But $(P^n)^w \subseteq (P^n)_v$ whence $P^{(n)} = (P^n)_v$ and $P^{(n)}$ is t-invertible. Of course as $P^{(n)} \supseteq I$ we can find an integral ideal J such that $I = (P^{(n)}J)_t$. That $J \nsubseteq P$ follows from the fact that $IR_P = P^nR_P = (P^{(n)}J)_tR_P$. Because IR_P is principal and hence a t-ideal we can write $(P^{(n)}J)_tR_P = (P^{(n)}JR_P)_t$ by [**8**, Lemma 3.4]. But then $P^nR_P = (P^{(n)}J)_tR_P = P^{(n)}R_P(JR_P)_t$ which gives $P^nR_P = (P^{(n)}JR_P)_t = (P^{(n)}R_PJR_P)_t = P^{(n)}R_P(JR_P)_t$. Cancelling $P^nR_P = P^{(n)}R_P$ we get $R_P = (JR_P)_t$ and hence $JR_P = R_P$, because PR_P is a t-ideal. For the second case, note that $QR_P \cap R \supseteq I$ is a prime t-ideal being a contraction of a prime t-ideal and that $QR_P \cap R$ cannot be t-invertible because $QR_P \cap R$ is not a maximal t-ideal. □

Now we proceed to prove star operations analogue of Theorem 1, for domains as indicated above. For the proof of Theorem 3 note that (a) a prime ideal that is a t-invertible t-ideal must be a maximal t-ideal [**7**] and (b) by the condition on J the maximal t-ideals that contain J have to be t- invertible. Thus if a domain R does not have a t-invertible prime t-ideal then the t-ideal J of the theorem would have to be all of R.

PROOF. (of Theorem 3) As (nonzero) principal and invertible are both t-invertible it suffices to prove the result for t-invertible and then wave hands if necessary. Let J be a t-ideal as described in the theorem. If J is not contained in any t-invertible prime t-ideal then $J = R$. So let $J \neq R$ and let $\mathcal{F} = \{P : P$ is a t-invertible prime t-ideal containing $J\}$. By the condition on J and by Theorem 4, each of $P \in \mathcal{F}$ must be minimal over J and being t-invertible each $P \in \mathcal{F}$ must be a t-ideal of finite type. Hence by Theorem2, $\mathcal{F}$ must be finite. Let $\mathcal{F} = \{P_1, P_2, ..., P_n\}$. By minimality of each and by Theorem 4, $J = ((P_1^{(a_1)}P_2^{(a_2)}...P_n^{(a_n)})K)_t$ where K_t is not in any non-t-invertible prime t-ideals and hence is R. Now note that if each of the prime t-ideals P_i is invertible/principal then $J = ((P_1^{a_1}P_2^{a_2}...P_n^{a_n})K$, because an invertible ideal is a t-ideal and not being in any invertible/principal t-ideal forces K to be R. For the consequently part note that we cannot have a nonzero non-t-invertible ideal that is not in any non-t-invertible prime t-ideal. □

Recall from the introduction that R is a Krull domain if and only if every nonzero ideal and hence every t-ideal of R is t-invertible. Now Theorem 3 shows that every t-ideal of a domain R is t-invertible if there are no non-t-invertible prime t-ideals of R and conversely. Thus R is a Krull domain if and only if every prime t-ideal is t-invertible, a known result [**10**, Theorem 2.5], but ours is an easy corollary as the remarks after [**10**, Theorem 2.5] show. Next, Theorem 3 indicates that in a domain R, every t-ideal is invertible if there are no non-invertible prime t-ideals and obviously conversely. Recall that an integral domain R is a locally factorial Krull domain if and only if every t-ideal of R is invertible (see e.g. [**4**, Theorem 2.9]). Thus Theorem 3 characterizes locally factorial Krull domains as domains whose prime t-ideals are invertible. Next Theorem 3 characterizes domains whose t-ideals are principal as domains whose prime t-ideals are all principal. But as each minimal prime of a principal ideal is a t-ideal, free of charge, we conclude that every nonzero prime ideal contains a nonzero principal prime, which forces R to be a UFD. Indeed it is customary to call a Krull domain a t-Dedekind domain and a UFD a t-PID and Theorem 3 leaves no doubt about the terminology being correct.

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