QUESTION 2202: I read in a review of an article of yours that you are very sentimental about rings of polynomials of the type $D+XD_S[X]$ and $D+XL[X]$. Any explanation?

ANSWER: I split the answer to your question into two parts. The first part is to do with (A) Why I like those constructions and the second is (B) to answer the "sentimental" part.

(A). Let D be an integral domain, $K=qf(D)$, L an extension field of K and let X be an indeterminate. The rings $D+XK[X]$, $D+XL[X]$ and $D+XD_S[X]$ can often be useful as convenient example builders. Here $D+XD_S[X]=\{f\in D_S[X]|\ f(0)\in D\}=D^{(S)}$, $D+XK[X]=\{f\in K[X]|\ f(0)\in D\}=T$. These rings were studied in [6] that essentially started in an effort to provide a counterexample to a conjecture of Sheldon's. The $D+XL[X]$ construction was studied in [7], under the guise of the general $D+M$ constructions.

(1) For (D,M) quasi-local, the ring $R=D+XL[X]$ is h-local.

By [3, Corollary 17], the ring R is of finite character. With reference to the discussion before [3, Corollary 17], maximal ideals of R are of the form $M+XL[X]$ and of the form $(1+Xg(X))R$ where the primes of the form $(1+Xg(X))R$ are of height one. Hence no two primes of R contain a nonzero prime ideal. This makes $D+XL[X]$ h-local.

(2) For (D,M) t-local the ring $R=D+XL[X]$ is t-h-local.

By [3, Corollary 17], the ring R is of finite character. With reference to the discussion before [3, Corollary 17], maximal t-ideals of R are of the form $M+XL[X]$ and of the form $(1+Xg(X))R$ where the primes of the form $(1+Xg(X))R$ are of height one. Hence no two t-primes of R contain a nonzero prime ideal. This makes $D+XL[X]$ t-h-local.

(3) By Corollary 3.13 of [4], $D+XK[X]$ is an AGCD domain if and only if D is.

(4) Let D be an AV-domain with quotient field K then $D+XK[X]$ is t-h-local by (2) and $D+XK[X]$ is an AGCD domain, and hence an APVMD, by (3).

(5) Let D be an AV-domain with quotient field K then $D+XK[X]$ is t-h-local by (2) and $D+XK[X]$ is an AB-domain by Theorem 4.9 of [**?**],hence a AGCD domain and hence a APVMD.

Call a domain D a tightly irreducible divisor finite (tidf) domain if for each nonzero non unit x of D, x is divisible by at least one and at most a finite number of irreducible elements. These domains have been treated in [8]. It is easy to see that if D is a domain of finite t-character such that every maximal t-ideal is generated by a prime, then D is a tidf domain. (Take a non-zero non unit, x in D. Then x must be in at least one maximal t-ideal and so divisible by at least one prime and is in at most a finite number of maximal t-ideals and so is divisible by at least one prime and is divisible by at most a finite number of distinct primes.) Here's simple example using the $D+XD_S[X]$ construction.

(6) If (V,M) is a a discrete rank 2 valuation domain and if S is the multiplicative set generated by the generator p of M, then the ring $D=V+XV_S[X]$ is a tidf domain.

Proof. As noted in [13] $D = V + XV_S[X]$ is a Schreier domain. So every atom in D is a prime. Let $f \in D + XD_S[X]$. Then $f = a + Xf(X)$, where $f \in V_S[X]$. Obviously f is divisible by p whenever a is a non unit of V, and hence of D, and if a is a unit we have $f = 1 + xf[X]$. Being of finite degree f can be written as a product of at most a finite number of factors and at least one of those factors is irreducible. Thus every nonzero non unit is divisible by at least one prime. Next let $\{f_\alpha\}$ be the set of all the atoms dividing $f = a + Xf(X)$. Then f_α being irreducible, $\{f_\alpha\} = \{p\} \cup \{1 + Xf_\alpha(X)\}$. Now by the degree consideration you cannot have infinitely many distinct primes of the form $1 + Xf_\alpha(X)$ dividing f. Whence D is a tidf domain. An alternative method would be to show, as in Lemma 2.2 of [14], that every nonzero non unit f of D can be uniquely written as $f = gh$ where $h(0) = 1$ and g has no factor k such that $k(0) = 1$. Thus every factor of g is of the form $a + Xf(X)$ where a is a non unit and hence divisible by p and by no other irreducible element, as to be irreducible other than p, the element would have to be of the form $b + Xf(X)$ where b is not divisible by p and the only elements in V not divisible by p are the units. So the only irreducible factors of f are either p or irreducible irreducible factors of h and h is a product of at most a finite number of irreducible elements.

(7) If we allow V to be of rank greater than one but with maximal ideal idempotent, then the $V + XV_S[X]$ construction above gives us an idf domain that is not a tidf domain. The reasoning is the same except that we note that in $f = gh$, the element g is not divisible by any irreducible elements of D.

(8) Theorem A. A GCD domain D is a tidf domain if and only if D is of finite t-character and every maximal t-ideal of D is generated by a prime element.

Recall that a nonzero non unit element r in D is rigid if for all $x, y|r$ we have $x|y$ or $y|x$. Indeed an atom and hence a prime is rigid and a factor of a rigid element is rigid. Also note that if a rigid element r belongs to a maximal t-ideal M, then $x \in M$ if and only if x is divisible by some non-unit factor of r. The reason is that if M is a maximal t-ideal, then $x \in M$ if and only if $(x, m)_v \subseteq M$ for some $m \in M$. Now if D is a GCD domain $(x, r)_v$ is the GCD of and if $r \in M$, then $x \in M$ if and only if $(x, r)_v =$GCD$(x, r) \in M$. Indeed it can be shown that a nonzero non unit x of a GCD domain is rigid if and only if x belongs to a unique maximal t-ideal, see [11]. But then the GCD of (x, t) must be a non-unit factor of r. Based on this it was shown in Theorem A of [12] that a GCD domain is a ring of Krull type if and only if every nonzero non unit of D is divisible by at least one and at most a finite number of rigid elements.

Proof of Theorem A. Let D be a tidf GCD domain. Let M be a maximal t-ideal of D such that M is not generated by a prime and let $x \in M \backslash \{0\}$. Since D is tidf x is divisible by at least one and at most a finite number of non associated primes. Since the prime elements in D generate maximal t-ideals x belongs to all the primes generated by the irreducible factors of x. Let $p_1, p_2, ..., p_n$ be all the non associate prime factors of x. Correspondingly, $x \in M \cup p_i D$ where p_i are all the non associate primes dividing x. By prime avoidance there is $y \in M \backslash \cup p_i D$. Because M is a maximal t-ideal $(x, y)_v =$GCD$(x, y) = d \in M$. Now d is a nonzero non-unit in M that is not divisible by any of the primes that divide x, hence by no primes and hence by no irreducible elements, forcing d to be a unit, a contradiction. As this contradiction arose from the assumption that M was not generated by a prime we conclude that every maximal t-ideal of D is generated by a prime. Combining this with the tidf property we conclude that D is of finite t-character and a finite t-character domain is a ring of Krull type. For the converse note that if D is a GCD ring of Krull type such that each maximal t-ideal of D is generated by a principal prime then by an earlier remark, D is tidf.

(9) Recall that D is a generalized Krull domain (GKD) if D is a locally finite intersection of localizations at height one primes with D_P a valuation domain for each height one prime. Using the $D + XD_S[X]$ construction I showed in [12] that if D is a GCD GKD, then $D + XD_S[X]$ is a ring of Krull type if and only if $S = \{d^i\}_{i \in N}$ for a nonzero non-unit of D. (It would work even if d were a unit, but then the $D + XD_S[X]$ would reduce to $D[X]$ which would be a GKD a restricted ring of Krull type.) Now let D be a UFD and d a nonzero non-unit, $S = \{d^i\}_{i \in N}$, you have a ready-made GCD ring of Krull type whose maximal t-ideals are generated by primes, for an example of a tidf domain. Of course if you have a GCD GKD with at least one maximal t-ideal idempotent, then you can get an example of an idf domain that is not tidf.

(10) Enough with examples of GCD tidf domain. Note that a Krull domain is a tidf domain, being an FFD. Recall that Corollary 2.6 of [2] says.

Proposition B. If D is a Krull domain, then $D^{(S)} = D + XD_S[X]$ is a ring of Krull type if and only if $|\{P \in X^1(D) | P \cap S \neq \phi\}| < \infty$. (Here $X^1(D)$ denotes the set of height one primes of D.)

For a start let's take D a Krull domain with at least one principal prime, p, taking S the saturation of $\{p^i\}_{i \in N}$ let's construct the ring of Krull type $D + XD_S[X]$. Note that S is a splitting set in D and in $D[X]$ and $D[X]_S = (D^{(S)})_S$.

Lemma C. An element $f \in D^{(S)}$ is an atom if $f = p$ or if f is not divisible by p in $D^{(S)}$ and $p^t f$ is an atom in $D[X]$. Suppose also that t is the least number such that $p^t f \in D[X]$.

Proof. Supposed for some power of p say p^t we have $fp^t \in D[X]$. Since p does not divide f we have $f = a + Xg(X)$, $a \in D$, where p does not divide a. Now suppose that $fp^t = p^t a + p^t Xg(X) \in D[X]$. (making $p^t g(X) \in D[X]$). If in $D[X]$, $p^t a + p^t Xg(X) = h(X)k(X)$, where h and k are non units in $D[X]$. Then $h = h_0 + Xh_1(X)$, $k = k_0 + Xk_1(X)$ and comparing coefficients we get $h_0 k_0 = p^t a$ in D, forcing $t = r + s$, for $r, s \in N$ such that $p^r | h_0$ and $p^s | k_0$. But then $p^{-r}h(X), p^{-s}k(X) \in D^{(S)}$ and $p^{-r}h(X), p^{-s}k(X)$ are obviously non-units in $D^{(S)}$ and as $fp^t = h(X)k(X)$ we have $f = p^{-r}h(X)p^{-s}k(X)$. Next if, p does not divide f in $D^{(S)}$ and for a least such t, fp^t is an atom in $D[X]$, then f is an atom in $D^{(S)}$. For if $f = hk$ in $D^{(S)}$ and for a least t we have $fp^t \in D[X]$, then one can show that for some $r, s \in N$, minimal such that $p^r h(X), p^s k(X) \in D[X]$ we must have $r + s = t$. But this contradicts the assumption that fp^t is an atom in $D[X]$. Now these observations are made to help establish that if $f \in D^{(S)}$ is such that $p \nmid f$ and f has a factorization of length n and if $t \in N$ is the least such that $p^t f \in D[X]$ then $p^t f$ has a factorization of length n in $D[X]$. For our purposes the following lemma would suffice.

Lemma D. Let p be a prime element in a Krull domain D, S the saturation of $\{p^i\}_{n\in N}$, X an indeterminate over D_S and that $f \in D^{(S)} = D + XD_S[X]$ with $f(0)$ such that $p \nmid f$ in $D^{(S)}$. Then f is an atom or a product of atoms in $D^{(S)}$.

The idea of the proof is that if f is not an atom in $D^{(S)}$, then $f = f_1 f_2$ in $D^{(S)}$ where degrees of $\deg(f_i)$ add up to the degree of f. Since $p \nmid f$ we conclude that $p \nmid f_i$. By an induction argument each of f_i is a product of atoms and so is f. (You can also note that if $f = f_1....f_n$ and all the degrees are positive and add up as $\sum \deg f_i = \deg f$ there would come a stage, for $n \leq \deg f$, that none of the factors can be factorized any further.)

Proposition E. Let p be a prime element in a Krull domain D, S the saturation of $\{p^i\}_{n\in N}$, X an indeterminate over D_S and let $D^{(S)} = D + XD_S[X]$. Then $D^{(S)}$ is a a tidf domain.

Proof. Indeed $D^{(S)}$ is a ring of Krull type by Proposition B. Now a typical nonzero non unit of $D^{(S)}$ can be written as $f = a + Xg(X)$ where $g(X) \in D_S[X]$. If $a \neq 0$, we can write $a = p^r b$ where b is not divisible by p, because p is a prime in the Krull domain D and $a \in D$. But then $f = p^r(b + Xg'(X)) = p^r f'$ where f' is not divisible by p and thus f is a product of irreducible elements of $D^{(S)}$, by Lemma D. If $a = 0$, we can write $f = p^u X^r h(X)$ where $u \in Z$ the ring of integers and $h \in D^{(S)}$ where $p \nmid h$, this is because $XD_S[X]$ is a height one prime. But then, by Lemma D, h is a product of atoms of $D^{(S)}$ and $p^u X^r$ is divisible by only one atom, p. Thus, in this case too f is divisible by at least one atom and by at most a finite number of non-associate atoms.

(11) I recall writing, recently the following:

Proposition XD. Let D be a domain with quotient field K, let X be an indeterminate over L an extension field of K and let $R = D + XL[X]$ and $S = D + XL[[X]]$. Then the following hold.

(1) Given that D is not a field, then R is a tightly idf domain if and only if D has at least one and at most a finite number of atoms.

(2). Given that D is a field R is tightly idf if and only if $|L^*/D^*| < \infty$.

(3) Given that D is not a field, S has n atoms if and only if D has $n > 0$ atoms.

(4) Given that D is a field, S is tightly idf if and only if $|L^*/D^*| < \infty$.

Proof. A general element of $D + XL[X]$ is of the form $(hX^r)(1 + Xg(X))$, where $h \in L$ and $g(X) \in L[X]$. Of these $1 + Xg(X)$ is a product of powers of finitely many height one primes in $L[X]$ and hence in $D + XL[X]$. Let n_g be the number of prime divisors of $1 + Xg(X)$ and let n_D be the number of atoms in D. If $r > 0$, then the number of irreducible divisors of $(hX^r)(1 + Xg(X))$ is $n_D + n_g$. If on the other hand $r = 0$, then $h \in D$ and so the number of irreducible divisors of h is $n_h \leq n_D$ and the number of irreducible divisors of $h(1 + Xg(X))$ is m such that $1 \leq m \leq n_D + n_g$. Conversely D must have at most a finite number of irreducible elements because $X \in R$ is divisible by every element of D.

(2) Suppose that D is a field. Then a typical element of R is $f(X) = (hX^r)(1 + Xg(X))$. If $r = 0$, $h \in D$ the number of distinct irreducible divisors is n_g and hence finite. (Indeed if $r = 0$, and $n_g = 0$, then $f(X)$ is a unit.) On

the other hand if $r = 1$, hX is irreducible and if $r \geq 2$ then hX^r has finitely many irreducible divisors if and only if $|L^*/D^*| < \infty$ as in [1].(The number of distinct irreducible divisors depends upon the distinct cosets of L^*/D^*.

(3) A typical element of S is $f(X) = (hX^r)(1 + Xg(X))$ where $g(X)$ is a power series in $L[[X]]$ and so $(1 + Xg(X))$ is a unit in $L[[X]]$ and hence in $D = XL[[x]]$. And X being divisible by every nonzero element of D must have as many irreducible divisors as n_D. On the other hand if $(hX^r)(1+Xg(X))$ has n irreducible divisors hX^r has n irreducible divisors. Because D is not a field, X is not irreducible. So the only irreducible divisors of $(hX^r)(1 + Xg(X))$ are the irreducible elements of D. whence $n = n_D$.

(4) The proof is straightforward.

So you can see that these constructions work to give simple examples of some new ideas and that, for me, is a good enough reason to like them.

(B). I am not sure which article you are talking about. The only possible review that I can think of was by Marco Fontana of either on [**?**] "Facets on rings between $D[X]$ and $K[X]$" or on " Various facets of rings between $D[X]$ and $K[X]$[**?**]. A footnote at the first page of [**?**] read: #This article was to have appeared in full in the recent Dekker publication, "Commutative Ring Theory and Applications," in Volume 231 of the Dekker Lecture Notes series; however, part of the article was inadvertently omitted from that volume. The editors of the Dekker volume and Communications in Algebra agreed to remedy the problem by republishing this article, which was originally received in October 2001.

Let me give you a bit of history on this. I was already unhappy about the editors changing the title without consulting me and when the paper appeared "amputated" I was literally mad. By then I had had my kidney transplant and one of the transplant medicines was a steroid that often gives me uncontrollable temper. There was a suggestion to "move on" and that set off a tirade of emails from me and the result was the publication in Comm. Algebra of the second paper, mentioned above. Why was I so unhappy about a paper being amputated? It wasn't that paper, it was a couple of other unlucky papers. Perhaps it was my bad luck, the paper [6] "The construction $D+XD_S[X]$ [J. Algebra 53(1978), 423-439] went to a referee who "sat" on the paper for three years while Brewer and Rutter cherry picked the results from it they could understand and without adequate reference to where they got the ideas from published their paper in 1976. Frankly, if it wasn't for the factorization and dimension theory related stuff, these thugs had run away with the juicier results. I have pointed out the results that they swiped in https://lohar.com/researchpdf/D+M%20constructions.pdf

Then there was [7]! We tried to publish it in several standard journals, but there seemed a stonewall to have gone up. That is where my mistake and bad luck came in. Instead of settling for some lower tier European journal, I suggested that we publish it in the Pakistani journal. For, to that point the late Professor L.M. Chawla was the editor the standard of the journal was high and he never thought that being an Ahmadi, I was a personna nongratta. In any case the paper came out mauled, and the new editor informed me that he'd decided to "proofread" it himself. With [?] happening something similar, I thought I must do something. Of course, I wasn't completely innocent as the abstract of [?] started with: "Since the circulation, in 1974, of the first draft of "The construction $D+XD_S[X]$, J.Algebra 53 (1978), 423-439"a number of variations of this construction have appeared. Some of these are: The generalized $D+M$ construction, the $A+(X)B[X]$ construction, with X a single variable or a set of variables, and the $D+I$ construction (with I not necessarily prime). These constructions have proved their worth not only in providing numerous examples and counter examples in commutative ring theory, but also in providing statements that often turn out to be forerunners of results on general pullbacks." In these opening remarks I had pointed out the priority of $D+XD_S[X]$ over $D+M$ and let loose a hidden barb about folks who would produce a pullback result as soon as I had produced a result in one of these constructions. Of course the powers that be were not amused. Now why am I so unhappy with Brewer and Rutter? Because they set off a trend of "Steal if one of the authors is Muhammad". Here's a scheme of examples. Under the influence of Dan and David Anderson, a lot of Koreans think research is " Take a result of Zafrullah, change terminology and publish mentioning references that would get a "sympathetic" referee. Now for a concrete example, look up the tail end part of https://lohar.com/mithelpdesk/hd2006.pdf

See how the slick fellow has changed the terminology to get his big theorems. The paper to look for is [9]. If you have a couple of results in a paper that are in doubt the whole paper must be in doubt. I have a feeling that somehow the fellow preempted my paper with Malik and Mott **??**. But, in Math feelings don't count, and there is a result in [9], that is a definite improvement on **??**. So, willy nilly, even I have to mention this paper of Kang's, when there is need.

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