QUESTION (HD 2008) Someone has sent me a copy of a page from a Korean website. This page has so graciously awarded a zero rating to my paper "On finite conductor domains" [Z, Manuscripta Math. 24(1978) 191-203]. How should I respond?

ANSWER: I have actually seen the whole site, thanks for sending the one page though. For, when I saw it first I did not think it meant much. It really doesn't mean much. For there may still be some websites that mention me as a student. (Some folks think I'm a student of Dan Anderson's and some think I am a student of Evan Houston's. Things happen, you can't stop people from being mean and/or ignorant. Of course, this kind of things happen more to you if your name includes "Muhammad" as a part. Heck, I once surprised a Head of Department encouraging students in my class to give me the works in the upcoming "teaching evaluations".

Now let's come to the "zero-rated paper" and its contents to see what is so wrong or right in the paper. In this paper I prove, as Lemma 4, the following result. (Let me warn you. If you want to make some sense of the following material, read up sections 32 and 34 of Gilmer's book [5].)

Lemma 1 *Let A be a nonzero ideal of D and let S be a multiplicative set in D. If A is finitely generated, then*
(1). $(AD_S)^{-1} = A^{-1}D_S$.
(2). $(AD_S)_v = (A_vD_S)_v$.

(Part (2) was improved to (3) *If A_v is of finite type, then* $(AD_S)_v = (A_vD_S)_v$, in Corollary 1.6 of a later paper by Malik, Mott and myself [9]

This is what I write in my paper on "Putting t-invertibility to use" [15]:

"The notion of ideal systems, as introduced by Prüfer and Krull, had not taken any real shape when it was hijacked by Lorenzen [8] into partially ordered groups, where it really took its pre-Griffin shape. Griffin's work [6], [7], brought rings, essentially integral domains, into the picture. With rings came their usual questions, and one of them was that of localization. Aubert [1] produced a very general and very brief description of how an ideal system would fare under localization." Obviously that was not enough, there was a need and my result filled the gap. It is worth noting that there was no formula, before the one offered in Lemma 4 of [12]. So I was the first to connect the v-operation on D to that on D_S, where S is a multiplicative set and $D_S = \{\frac{d}{s}|$ where $d \in D$ and $s \in S\}$. There is no denying that after the introduction of my formula, multiplicative ideal theory has never looked back. Now, instead of being appreciated, I am having to fend off attacks directly from unknown Koreans and indirectly from their handlers who hate the fact that someone named Muhammad Zafrullah helped shape what is known as Multipluicative Ideal Theory (MIT). Frankly, I feel like the old man in the "Old Man and the Sea" of Hemingway, trying to fight off the sharks coming for his big catch. My Oars are all broken, harpoons gone and my hands are both scarred. I am left with no choice but to leave my catch to its fate and lie down on the damp, uneven, floor of my boat to sleep, dreaming of the times when fairness will reign and looking at the size of the

carcass fair minded folks will imagine and appreciate the size of my catch. That is when they'll will come to realize the profound effect this formula of mine has had on MIT.

Now let's get back to business at hand, the zero-rated paper and what else was in it that draws the ire of the folks behind that silly website.

Lemma 5 of [12] shows that I knew the importance of PD_S being a t-ideal and in Lemma 6 of [12] I show that an associated prime P of a principal ideal, i.e. a prime ideal that is minimal over a proper nonzero ideal of the type $(a) :_D (b)$, is such that PD_P is a maximal t-ideal of D_P. Then I use this fact to produce an essential domain, i.e. a domain D with a family $\mathcal{F}$ of primes such that D_P as a valuation domain for each $P \in \mathcal{F}$ and $D = \cap_{P \in \mathcal{F}} D_P$. Then in Lemma 8 of [12] I show when an essential domain becomes a PVMD. Recall that a domain D is a PVMD if every nonzero finitely generated ideal of D is such that $(AA^{-1})_v = D$ and $A^{-1} = B_v$ for some finitely generated fractional ideal B of D. (Or every finitely generated nonzero ideal A of D is t-invertible, that is $(AA^{-1})_t = D$.) The other important thing that I did in that paper was including a simple example of a Schreier domain that is not a GCD domain.

Recall that an element of an integral domain D is called primal if x is such that for all $y, z \in D$, $x|yz$ implies $x = rs$ such that $r|y$ and $s|z$, where r, s belong to D also. An integrally closed integral domain, whose nonzero elements are all primal was called a Schreier domain by Paul Cohn in [2]. It was shown in [2] that a GCD domain is a Schreier domain and that if D is a Schreier domain then so is $D[X]$. It was noted in [3] that because the properties of being integrally closed and Schreier are both first order properties, they are both preserved in polynomial ring formation and direct limits and consequently if D is Schreier, S a multiplicative set of D and X an indeterminate over D_S, then $D^{(S)} = D + XD_S[X]$ is Schreier. The example is the following.

Example 2 *Let V be a rank two valuation domain with quotient field K, let $S = D \backslash P$ where P is the height one prime of V and let X be an indeterminate over V_S. Then by the above comment, $V^{(S)} = V + XV_S[X]$ is Schreier.*

Now, Schreier domains, other than GCD domains, are not very well known and so any and every example of those domains is welcome. The above example, Example 2, especially when V is taken to be a discrete rank two valuation ring, is more welcome for its simplicity and so has often been used.

I indicate below some of the impact that the zero rated paper [12] has had on the subsequent literature.

Recall that a domain D is a finite conductor domain if for all $a, b \in D \backslash \{0\}$ we have $aD \cap bD$ finitely generated. Recall also that a prime ideal P is essential if D_P is a valuation domain. Also that D is essential if there is a family $\mathcal{F}$ of essential primes of D such that $D = \cap_{P \in \mathcal{F}} D_P$. In Lemma 7 of [12] I show that an integrally closed FC domain is essential. This led me to study integral domains with property P: every associated prime of a principal ideal is essential. Coupled with the fact that if $Ass(D)$ represents the set of associated primes of principal ideals of D, then $D = \cap_{P \in Ass(D)} D$ the property P meant that I had

an essential domain at hand. Showing that any ring of fractions of a domain with property P had property P was a breeze and I thought I had another characterization of a PVMD. Being unable to do that I consulted Professor Joe Mott, my usual talking board those days. He alerted me to the existence of an example of a non-PVMD domain that turned out to have the same properties as my domain with property P. This is how the paper [10] was born. As you can see [12] and Lemma 1 had a significant role to play in the development of [10].

P-domains were characterized by Ira Papick in [11] as indicated in the following. Let $D \subseteq T$ denote an extension of integral domains, where D is a subring of T. Call $u \in T$ super-primitive if u satisfies a polynomial f such that $A_f^{-1} = D$. Here, A_f denotes the "content of f", the ideal generated by coefficients of f. Recall that an element $u \in T$ is called primitive if u satisfies a polynomial f with $A_f = D$ and that u is primitive if and only if $D \subseteq D[u]$ satisfies INC. This result was proved in the more general setting of commutative rings with 1 by Dobbs in [4]. Here "$D \subseteq T$ satisfies INC" stands for "incomparability", that is taken to mean: For all $Q_1, Q_2 \in Spec(T), (Q_1 \cap D = Q_2 \cap D$ and $Q_1 \neq Q_2) \Rightarrow$ Q_1 and Q_2 are incomparable, under set inclusion. So $D \subseteq D[u]$ satisfies INC if distinct comparable primes of $D[u]$ do not contract to the same prime of D.

Now let $D \subseteq T$ be an extension of domains and let $P \in Spec(D)$. Let's say $D \subseteq T$ satisfies INC at P if distinct comparable primes of T do not contract to P. (INC has been isolated from the result, dubbed as folklore, attributed to Graham Evans, in [4]: Let $R \subseteq T$ be an extension of rings. Then the following are equivalent.

(1). T is integral over R, (2) For any extension of rings $R \subseteq A \subseteq B \subseteq T$, the extension $A \subseteq B$ satisfies both LO and INC. Here LO: For every prime ideal P of A there is a prime ideal Q of B such that $Q \cap A = P$ and (3) For any inclusion of rings $R \subseteq A \subseteq T$ and for any $u \in T$, the extension $A \subseteq A[u]$ satisfies LO and INC.)

In Corollary 2.2 of [11] the author establishes the following.

Proposition P1 Let $D \subseteq T$ be an extension of domains and let $u \in T$. If u is super-primitive over D, then $D \subseteq D[u]$ satisfies INC on $\mathcal{P}(D)$.

Then in Corollary 2.3 of [11] Papick proves the following result.

Proposition P2. The following statements are equivalent for a domain D with quotient field K. (1) D is a P-domain, (2) D is integrally closed and $D \subseteq D[u]$ satisfies INC on $\mathcal{P}(D)$ for each $u \in \widetilde{K}$ where $\widetilde{K}$ denotes the algebraic closure of K and (3) D is integrally closed and $D \subseteq D[u]$ satisfies INC on $\mathcal{P}(D)$ for each $u \in K$.

If you read the paper, you will find it totally devoted to P-domains and on how to get from P-domains to PVMDs. For instance he proves the following result in Proposition 2.5 of [11].

Proposition P3. Let $\overline{K}$ denote the algebraic closure of the quotient field of D. Then D is a PVMD if and only if D is integrally closed and u is super primitive for all $u \in \overline{K}$.

Let's go back and look at the condemned paper again. While serving its purpose, of linking the v-operation of D with the v-operation of D_S, Lemma 4 of that paper (Lemma 1 here) had a problem. It would not show that if M is a prime t-ideal and S a multiplicative set such that $M \cap S = \phi$, then MD_S is a t-ideal. (The existence of P domains is due to that fact.) Noting that some folks were actually trying to prove that if M is a maximal t-ideal of then so is MD_M of D_M, I wrote [13] giving examples of locally GCD domains that a are non-PVMD and reasoning as follows. Let D be such that for each maximal ideal M of D we have D_M a GCD domain. If it were the case that for a maximal t-ideal P we have PD_P a maximal t-ideal of D_P then taking the maximal t-ideal P contained in a maximal ideal M we conclude that D_P is a quotient ring of D_M and hence a GCD domain. Now PD_P being a t-ideal makes D_P a t-local domain and it is easy to prove that a GCD t-local domain is a valuation domain. But that makes D a PVMD, as D is a PVMD if and only if D_P is a valuation domain for each maximal t-ideal P. Thus if there is a locally GCD domain D that is not a PVMD, then D must have a maximal t-ideal P such that PD_P is not a t-ideal. (See [13, Proposition 4.3] and note that the paragraph after [13, Proposition 4.3], links the formula squarely with the existence of non-well behaved prime t-ideals.)

The examples were based on the, so called, $D + XD_S[X]$ construction. The construction, though already mentioned, goes as: Let D be an integral domain, let S be a multiplicative set in D and let X be an indeterminate over D_S. Then the set: $\{f(X) \in D_S[X] | f(0) \in D\}$ is a subring of $D_S[X]$ denoted by $D + XD_S[X]$ or $D^{(S)}$. In [13], I first proved the following result.

Proposition P3A. Let D be a GCD domain and let S be a saturated multiplicative set in D. Then $D^{(S)}$ is a GCD domain if and only if for each PF prime P of D with $P \cap S = \phi$ there exists $d \in P$ such that d is not divisible by any non unit member of S.

Here, by a PF prime, or prime filter prime, of a GCD domain, we mean a prime ideal P such that for each pair x, y in P we have $GCD(x, y) \in P$.

The purpose of the above proposition was to offer an alternative to Theorem 1.1 of [3]. (That theorem states: $D^{(S)}$ is a GCD domain if and only if D is a GCD domain and $GCD(d, X)$ exists in $D^{(S)}$ for all $d \in D \backslash \{0\}$). What it really gave us was the notion of a splitting set as a saturated multiplicative set S of a domain D such that for each $d \in D \backslash \{0\}$ we have $d = rs$ where $s \in S$ and $(r) \cap (t) = (rt)$ for all $t \in S$. In fact it gave us the following corollary to [13, Theorem 1].born

Proposition P3B. [13, Corollary 1.5] Let S be a saturated multiplicative set of a GCD domain D and let X be an indeterminate over D_S. Then the following are equivalent. (1) $D^{(S)}$ is a GCD domain, (2) S is a splitting set of D and (3) For each prime P with $P \cap S = \phi$ there is at least one $d \in P$ such that d is not divisible by any non unit from S.

Using the freedom afforded by Theorem 1 of [13] I proved the following result.

Proposition P3C (cf., [13, Theorem 2.1]) Let D be a P-domain and let S be a multiplicative subset of D such that every associated prime ideal of D that

intersects S, intersects S in detail. Then $D^{(S)}$ is a P-domain.

Of course this wasn't enough in that every GCD domain is a P-domain. But then, every locally GCD domain is a P-domain. On the other hand, if D is a GCD domain then it is patent that $D^{(S)}$ is a Schreier domain as established in [3]. Now as a PVMD that is also Schreier is a GCD domain, we must rule out $D^{(S)}$ being GCD. So if there is a GCD domain D such that $D^{(S)}$ does not meet the requirement of [13, Theorem 1], for some saturated multiplicative set S, we have an example. The requirement of [13, Theorem 1] for $D^{(S)}$ to be GCD was that D was GCD and S had this property: for each PF-prime P of D with $P \cap S = \phi$ there is at least one $d \in P$ such that d is not divisible by any non unit in S. These observations led to the following statement.

Proposition P3C. [13, Theorem 2.4] Let D be a GCD domain. Suppose that S is a multiplicative set in D such that each PF prime that intersects S intersects it in detail. If there exists a PF prime P disjoint from S such that every element of P is divisible by at least one non unit from S, then $D^{(S)}$ is a P-domain that is not a PVMD.

Needless to say that using the above proposition a number of useful examples were constructed in [13]. However, what really needs said here is to stress the fact that [13, Theorem 1] was stated without expressly saying that the multiplicative set S was saturated but, as Evan Houston pointed out, it does not seem right. On the other hand I find it easier to go along with Evan's suggestion, while being unable to find a counter example. This situation demands an open problem.

Problem. Find an example to show that, in the statement of [13, Theorem 1] it is necessary to assume that S is a saturated multiplicative set.

After [13] it became necessary to characterize well behaved prime t-ideals (prime t-ideals P such that PD_P is a t-ideal of D_P) and well behaved domains D, that is domains D such that each prime t-ideal P of D is well behaved. To this end I wrote [14], proving results such as the following.

Proposition P4 (cf. [14, Proposition 1.1]) A nonzero prime ideal P of D is a well behaved prime t-ideal if, and only if, for every finitely generated subideal F of P there exist elements $a \in P$ and $b \in D$ with $a \nmid bs$ for all $s \in D \backslash P$, such that $F \subseteq \frac{a}{b} D$.

Proposition P5 (cf. [14, Proposition 1.2]) An integral domain D is well behaved if, and only if, D_S is a well behaved domain for every multiplicative set S of D.

Proposition P5 (cf. [14, Proposition 1.4]) Let D be an integral domain such that for every finitely generated nonzero ideal A and for every multiplicative set S of D with $A \cap S = \phi$, $A_v D_S$ is divisorial. Then D is well behaved.

This paper also includes first ever example of a non-t-maximal prime t-ideal that is not well behaved, in [14, Proposition 2.5]. The example, based on the $D + XD_S[X]$ construction, is a little involved. Nowadays a much simpler example can be constructed as follows.

Example E2. Let X and Y be two indeterminates over Q the field of rational numbers. Let Z be the ring of integers, let p be a prime number in Z and consider the ring $D = Z_{(p)} + (X, Y)Q[[X, Y]]$.

Illustration: Obviously the ring D is a quasi-local ring with maximal ideal $M = pD$. The ideal $N = (X, Y)Q[[X, Y]]$, being a contraction of the maximal ideal $(X, Y)Q[[X, Y]]$ of the regular local ring $Q[[X, Y]]$, is a prime ideal. It is easy to see that the ideal $N = \cap p^n D$. Being an intersection of principal ideals, the ideal N is a divisorial ideal and hence a t-ideal. Yet if $S = \{p^n | n \in N\}$ we have $Q[[X, Y]] = D_S$ and in this ring the ideal $(X, Y)Q[[X, Y]]$ is a height two prime ideal which is not a t-ideal, as in $Q[[X, Y]]$ we have $(X, Y)_v = D$. Indeed $ND_S = ND_N = (X, Y)Q[[X, Y]]$.

In the same paper it was shown that for D a PVMD, $D^{(S)} = D + XD_S[X]$ is a PVMD if and only if $D^{(S)}$ is well behaved, [14, Proposition 3.3].

Recently, Evan Houston wrote to me about the possibility of studying "well behaved t-ideals". Between the two of us we have quite a few results, with the possibility of further expansion. The point is the zero rated paper still has something to offer.

Now this diatribe of mine would be totally one-sided and hence ineffective if I do not try to explore the reasons why someone should zero rate my paper. I did hear a couple of times a reason as: The result proved in the paper was an exercise. Yes! I redid an exercise and showed that an integrally closed finite conductor domain is a PVMD, using some new techniques. But then you do not block new ideas because some of their consequences lead to new solutions of some old exercises. I did not hide the fact that I was redoing an exercise. In fact this is what I wrote in Remark 3 of [12]: We note that this theorem ([12, Theorem 2]) appears as an exercise in [5] (cf Ex 21 p. 432.) But since our proof is designed to draw some extra benefits the repetition seems to be in order.

Having seen some of the extra benefits, folks should have shut up. But it did not happen. So, let's look for some other reasons. Earlier on Dan Anderson had marked me as a source of new ideas and had noted that I had no support in terms of a teacher going all out after you if you stole from me. So, he trained his student B.G. Kang to lift my results by changing definitions etc.. Kang, a very intelligent fellow actually came up with some good results, some of which I mentioned in my survey [15]. One of them was: *For any nonzero ideal A of D, $(AD_S)_t = (A_t D_S)_t$.* Now it's not my fault that in spite of efforts on the part of some Koreans, and some of my support, it did not fly. Now I do not want to get into this muck too deep. Dan and his minions the South Koreans think they have given me a lot of grief, but I am going in full knowledge of the fact that after a while we'll all be forgotten as new ways of doing things will develop. If someone tries to push the meanness a bit further, someone else might dig the whole thing up and find out the truth. Until then I have this.

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