Journal of Algebra and its Applications

ON ALMOST VALUATION RING PAIRS

--Manuscript Draft--

Manuscript Number:	JAA-D-19-00779R1
Full Title:	ON ALMOST VALUATION RING PAIRS
Article Type:	Research Paper
Keywords:	Commutative ring, integral domain, total quotient ring, quotient field, almost valuation domain, almost valuation ring, AV-ring pair, valuation domain,, ring extension, P-extension, normal pair, Prufer domain, root extension, overring, integrality, alge
Abstract:	If A \subseteq B are (commutative) rings, [A, B] denotes the set of intermediate rings and (A, B) is called an almost valuation (AV)-ring pair if each element of [A, B] is an AV-ring. Many results on AV-domains and their pairs are generalized to the ring-theoretic setting. Let R \subseteq S be rings, with \overline{R}_S denoting the integral closure of R in S. Then (R, S) is an AV-ring pair if and only if both (R, \overline{R}_S) and (\overline{R}_S, S) are AV-ring pairs. Characterizations are given for the AV-ring pairs arising from integrally closed (resp., integral; resp., minimal) ring extensions R \subseteq S. If (R, S) is an AV-ring pair, then R\subseteq S is a P-extension. The AV-ring pairs (R, S) arising from root extensions are studied extensively. Transfer results for the \AV-ring" property are obtained for pullbacks of (B, I, D) type, with applications to pseudo-valuation domains, integral minimal ring extensions, and integrally closed maximal non-AV subrings. Several sucient conditions are given for (R, S) being an AV-ring pair to entail that S is an overring of R, but there exist domain-theoretic counter-examples to such a conclusion in general. If (R, S) is an AV-ring pair and R \subset S satisfies FCP, then each intermediate ring either contains or is contained in \overline{R}_S. While all AV-rings are quasi-local going-down rings, examples in positive characteristic show that an AV-domain need not be a divided domain or a universally going-down domain.
Response to Reviewers:	Please find the revised version and the cover letter.

ON ALMOST VALUATION RING PAIRS

DAVID E. DOBBS AND NOÔMEN JARBOUI

ABSTRACT. If $A \subseteq B$ are (commutative) rings, $[A, B]$ denotes the set of intermediate rings and (A, B) is called an almost valuation (AV)-ring pair if each element of $[A, B]$ is an AV-ring. Many results on AV-domains and their pairs are generalized to the ring-theoretic setting. Let $R \subseteq S$ be rings, with $\overline{R}_S$ denoting the integral closure of R in S. Then (R, S) is an AV-ring pair if and only if both $(R, \overline{R}_S)$ and $(\overline{R}_S, S)$ are AV-ring pairs. Characterizations are given for the AV-ring pairs arising from integrally closed (resp., integral; resp., minimal) ring extensions $R \subseteq S$. If (R, S) is an AV-ring pair, then $R \subseteq S$ is a P-extension. The AV-ring pairs (R, S) arising from root extensions are studied extensively. Transfer results for the "AV-ring" property are obtained for pullbacks of (B, I, D) type, with applications to pseudo-valuation domains, integral minimal ring extensions, and integrally closed maximal non-AV subrings. Several sufficient conditions are given for (R, S) being an AV-ring pair to entail that S is an overring of R, but there exist domain-theoretic counter-examples to such a conclusion in general. If (R, S) is an AV-ring pair and $R \subseteq S$ satisfies FCP, then each intermediate ring either contains or is contained in $\overline{R}_S$. While all AV-rings are quasi-local going-down rings, examples in positive characteristic show that an AV-domain need not be a divided domain or a universally going-down domain.

1. INTRODUCTION

All rings considered below are commutative and unital, often (integral) domains; all inclusions of rings, ring extensions, and ring/algebra homomorphisms are unital. If $R \subseteq S$ are rings, $[R, S]$ denotes the set of intermediate rings of this extension (that is, $[R, S] := \{T \mid T$ is a

---

2010 *Mathematics Subject Classification.* Primary 13B99, 13B21; Secondary 13G05, 12F05, 13F05, 13A15.

Key words and phrases. Commutative ring, integral domain, total quotient ring, quotient field, almost valuation domain, almost valuation ring, AV-ring pair, valuation domain,, ring extension, P-extension, normal pair, Prüfer domain, root extension, overring, integrality, algebraic extension, prime ideal, factor ring, pullback, minimal ring extension, characteristic, pseudo-valuation domain, analytic indeterminate, maximal non-AV subring, reduced ring, regular element, going-down.

ring and $R \subseteq T \subseteq S\}$) and $\overline{R}_S$ denotes the integral closure of R in S. If R is a ring, Spec(R) (resp., Max(R)) denotes the set of prime (resp., maximal) ideals of R and dim(R) denotes the (Krull) dimension of R. If R is a nonzero ring, Reg(R) denotes the set of non-zero-divisors of R (also known as the regular elements of R) and tq(R) denotes the total quotient ring of R. If R is a domain, we typically use qf(R), rather than tq(R), to denote the quotient field of R. By an *overring* of a nonzero ring R, we mean any ring E such that $R \subseteq E \subseteq \mathrm{tq}(R)$; thus, $[R, \mathrm{tq}(R)]$ is the set of all overrings of R. Studies of overrings have been central in much of multiplicative ideal theory (cf. [30]), including studies of Prüfer domains and their generalizations. One of those generalizations, the notion of an almost Prüfer domain (in short, an AP-domain) and its quasi-local case of an almost valuation domain (in short, an AV-domain), was introduced by D. D. Anderson and M. Zafrullah in [3], with sequels in [2] and [4]. The definition of an AP-domain will be recalled in Section 7, as our main interest here is in AV-domains and their generalization, AV-rings. Recall that a domain R is called an AV-*domain* if, for every nonzero $u \in \mathrm{qf}(R)$, there exists an integer $n \geq 1$ such that either $u^n \in R$ or $u^{-n} \in R$. Recently, in [7], N. Ouled Azaiez and M. A. Moutui introduced the concept of an *almost valuation domain pair* (in short, an AV-*domain pair*), as follows: if $R \subseteq S$ are domains, then (R, S) is called an AV-*domain pair* if each element of $[R, S]$ is an AV-domain. The most natural example of an AV-domain pair is given by (V, W) where V is a valuation domain and W is an overring of V. Section 2 of [7] is devoted to a study of AV-domain pairs. The main goal of the present paper is to introduce and study the notion of AV-ring pairs in a way that generalizes essentially all the domain-theoretic results in [7, Section 2] to the context of arbitrary (commutative) rings and, going beyond that, to deepen our understanding of AV-domains and AV-rings, as well as the pairs thereof.

Recall that a ring R is called an *almost valuation ring* (in short, an AV-*ring*) if, for all elements a and b of R, there exists an integer $n \geq 1$ such that either $a^nR \subseteq b^nR$ or $b^nR \subseteq a^nR$. A domain is an AV-ring if and only if it is an AV-domain. The most natural example of an AV-ring is given by a valuation ring (in the sense of [36], that is, a ring R such that, whenever a and b are non-zero-divisors of R, either $aR \subseteq bR$ or $bR \subseteq aR$). Any almost valuation ring is quasi-local [34, Proposition 2.2]. If $R \subseteq S$ are rings, we will say that (R, S) is an *almost valuation ring pair* (in short, an AV-*ring pair*) if each element of $[R, S]$ is an AV-ring. It is clear that if $R \subseteq S$ are domains, then (R, S) is an AV-ring pair if and only if (R, S) is an AV-domain pair. For rings $R \subseteq S$, Theorem 2.4 establishes that (R, S) is an AV-ring pair if (and only if) both $(R, \overline{R}_S)$ and $(\overline{R}_S, S)$ are AV-ring pairs. Thus, the study of AV-ring pairs separates naturally into two cases, which are determined by whether $R \subseteq S$ is an integrally closed extension (in the sense that R is integrally closed in S) or $R \subseteq S$ is an integral extension. For the first case, the AV-ring pairs are characterized in Corollary 2.8. That work is materially aided by the result (Proposition 2.2) that any AV-ring pair (R, S) has the property that $R \subseteq S$ must be a P-extension, in the sense of [31]; and the known fact that the integrally closed P-extensions $R \subseteq S$ are essentially the same as the normal pairs (R, S). (Recall from [12] that (R, S) being a *normal pair* means that $R \subseteq S$ are rings such that each element of $[R, S]$ is integrally closed in S.) Whereas the domain-theoretic approach in [7] makes use of the fact that any AV-domain pair (R, S) entails that $R \subseteq S$ is an algebraic extension of domains [7, Theorem 3.2 (2)], the access to P-extensions that is provided by Proposition 2.2 is one of the theoretical agents that allows us to extend many results from domains to rings (and to discover some new phenomena as well).

As for the second case (the AV-ring pairs arising from integral ring extensions), a characterization is given in Proposition 3.9 (b). Our most extensive results for the integral case are for root extensions. (Recall that a ring extension $R \subseteq S$ is called a *root extension* if, for each $\xi \in S$, there exists an integer $n \geq 1$ (possibly depending on ξ) such that $\xi^n \in R$. Of course, any root extension is an integral extension and the converse is false. Root extensions have played a fundamental role in studies of AV-domains. For instance, it was shown, i.a., in [3, Theorem 5.6] that if R is a domain, with R' denoting the integral closure of R (in its quotient field), then R is an AV-domain if and only if $R \subseteq R'$ is a root extension and R' is a valuation domain.) Our basic result along these lines is Proposition 3.2 (c), which states the following. If $R \subseteq S$ is a root extension of rings, then: (R, S) is an AV-ring pair $\Leftrightarrow$ R is an AV-ring $\Leftrightarrow$ S is an AV-ring; and if, in addition, S is a reduced ring, then these equivalent conditions are also equivalent to $(R, \mathrm{tq}(S))$ being an AV-ring pair. Pooling results from the various cases, Proposition 3.4 gives numerous characterizations of (R, S) being an AV-ring pair subject to the assumption that $R \subseteq \overline{R}_S$ is a root extension. As an interesting consequence of this approach, one finds new characterizations of AV-domain pairs such that $R \subseteq \overline{R}_S$ is a root extension: see, especially, condition (6) in Proposition 3.5 (c). We wish to point out two addtional aspects to the significance of Proposition 3.2

(c): not only does it extend some earlier domain-theoretic observations from [3] but, far more importantly, it leasds naturally to the question of whether (R, S) being an AV-ring pair entails S being (R-algebra isomorphic to) an overring of R. Example 3.3 answers this question in the negative, by constructing, for each prime number p, two families of examples of root extensions of (almost) valuation domains (but not fields) of characteristic p, $R \subset S$, such that S is not an overring of R. Despite Example 3.3, we have found another noteworthy situation that is in keeping with the literature's focus on the AV-domain pairs that arise from overring extensions. Specifically, Theorem 2.10 establishes that if $R \subseteq S$ is an integrally closed ring extension and $\mathrm{tq}(R)$ is a von Neumann regular ring, then (R, S) is an AV-ring pair if and only if R is an AV-ring and S is an overring of R.

Some of the methods of constructing AV-domain pairs in [7, Section 2] involved pullbacks (perhaps implicitly). For instance, [7, Corollary 3.4] established that if $R \subseteq S$ are domains sharing a common nonzero ideal I, then (R, S) is an AV-domain pair if (and only if) R is an AV-domain (the point being that S is an overring of R). Accordingly, it is natural to ask to characterize when a prominent type of pullback (the so-called "(B,I,D)" type $R = D \times_{S/I} S$) is an AV-domain. Such a characterization was obtained by Mimouni [41, Theorem 2.9]. For convenience, this result is restated below as Theorem 4.1. The main goal of Section 4 is to obtain a ring-theoretic generalization of Mimouni's result. This is achieved in Theorem 4.5. This characterization (of the AV-ring pairs issuing from the partners in a pullback of (B, I, D) type) is achieved at the minor cost of assuming that the common ideal I contains a regular element of S. This assumption allows for the technical arguments leading up to the proof of Theorem 4.5 (and coincidentally ensures that S is an overring of R). One application of Theorem 4.5 (though it could also have been obtained directly) concerns pseudo-valuation domains (PVDs), in the sense of [32]. Specifically, Proposition 4.9 shows that if (R, M) is a PVD with canonically associated valuation overring V, then: R is an AV-domain $\Leftrightarrow$ $R' = V$ and the field extension $R/M \subseteq V/M$ is a root extension $\Leftrightarrow$ every overring of R is a PVD and $(R, \mathrm{qf}(R))$ is an AV-domain pair $\Leftrightarrow$ either $R = V$ or V/M is purely inseparable over R/M or V/M is algebraic over some finite field. Besides using Theorem 4.5, the proof of Proposition 4.9 also appeals to the pullback-theoretic characterization of PVDs in [5, Proposition 2.6] and to Nagata's characterization of the field extensions that are root extensions [42]. For convenience, that result of Nagata is stated below as Proposition 4.6. As discussed below, additional applications of Theorem 4.5 arise in Sections 5 and

6 in studying integral minimal ring extensions and integrally closed maximal non-almost valuation subrings, respectively.

Recall from [28] that distinct rings $R \subset S$ are said to form a *minimal ring extension* if $[R, S] = \{R, S\}$. A minimal ring extension is either integrally closed or integral. Subject to the assumption that $\mathrm{tq}(R)$ is a von Neumann regular ring, Theorem 2.10 characterizes the integrally closed minimal ring extensions $R \subset S$ such that (R, S) is an AV-ring pair by the condition that "R is an AV-ring and S is an overring of R." The proof of Theorem 2.10 makes use of our above-mentioned work on normal pairs and P-extensions. Accordingly, most of Section 5 involves studying integral minimal ring extensions. Principal tools there include the Ferrand-Olivier classification of the minimal ring extensions of a field [28, Lemme 1.2], the crucial maximal ideal of a minimal ring extension (in the sense of [28, Théorème 2.2]), and a study of whether certain ring extensions that arise naturally are root extensions. Theorem 5.4 includes the following characterizations. If $R \subset S$ is an integral minimal ring extension such that the conductor $(R : S)$ contains an element of $\mathrm{Reg}(S)$, then: (R, S) is an AV-ring pair $\Leftrightarrow$ R is an AV-ring $\Leftrightarrow$ S is an AV-ring and $R/(R : S) \subset S/(R : S)$ is a root extension. The proof of Theorem 5.4 makes use of Theorem 4.5. As noted in Corollary 5.5, another equivalence that was given in Theorem 5.4 takes the following particularly succinct form in the domain-theoretic context: if $R \subset S$ is an integral minimal ring extension and S is a domain, then: (R, S) is an AV-ring pair $\Leftrightarrow$ S is an AV-domain, $\mathrm{char}(R/(R : S)) > 0$, and either $R/(R : S) \subset S/(R : S)$ is a purely inseparable field extension or $S/(R : S)$ is an algebraic field extension of some finite field. Much of our work to that point culminates in Theorem 5.6: if (R, S) is an AV-ring pair and $R \subseteq \overline{R}_S$ satisfies the finite chain property (also known as FCP or FC), then each element of $[R, S]$ either contains or is contained in $\overline{R}_S$.

The focus in Sections 6 and 7 is almost completely domain-theoretic. Section 6 seeks to characterize the (distinct) domains $R \subset S$ such that R is a maximal non-almost valuation subring of S. (The relevant definition is recalled from [7] at the beginning of Section 6.) Such a characterization was given in [7, Theorem 3.9] (resp., [7, Theorem 3.10]) in the special case where R is (resp., is not) integrally closed and $S = \mathrm{qf}(R)$. We give the desired characterization for the general context of (distinct) domains $R \subset S$ in Theorem 6.1 (resp., Theorem 6.4) if the ring extension $R \subset S$ is (resp., is not) integrally closed. The proofs in Section 6 use, *i.a.*, some results from [35] and some of our above-mentioned work on P-extensions, normal pairs and pullbacks.

Section 7 begins with an easy fact (Corollary 7.2) that seems to have escaped notice in the literature, namely, that each AP-domain is a going-down domain (in the sense of [13], [23]). In particular, each AV-domain is a quasi-local going-down domain. In fact, Proposition 2.1 shows that each AV-ring is a going-down ring (in the sense of [17]). Since each of these assertions has a false converse, we spend the rest of Section 7 seeking to determine how or whether the class of AV-domains (R, M) fits within some much-studied classes of going-down domains. The characteristic of R/M plays a role in that work. The need for such a role may be expected from Corollary 4.9: if (R, M) is a PVD such that $\mathrm{char}(R/M) = 0$, then R is an AV-domain (if and) only if R is a valuation domain. (Of course, no such result is possible in prime characteristic $p > 0$: consider $R := \mathbb{F}_p + X\mathbb{F}_{p^2}[[X]]$.) Recall from [15, Proposition 2.1 and Example 2.9] that each divided domain is a quasi-local going-down domain but the converse is false. Seeking a possible relation between AV-domains and divided domains, Example 7.4 gives an example of an AV-domain that is not a divided domain, but Proposition 7.6 shows that if (R, M) is an AV-domain containing (a copy of) $\mathbb{Q}$ as a subring (so that $\mathrm{char}(R/M) = 0$), then R is a divided domain. Next, recall from [21, Proposition 2.2 (a) and Remark 2.5 (b)] that each universally going-down domain is a going-down domain but the converse is false (even in the quasi-local case). Seeking a possible relation between AV-domains and universallly going-down domains, Example 7.7 gives an example of a one-dimensional AV-domain of characteristic 2 that is not a universally going-down domain, but Proposition 7.8 shows that if R is an AV-domain containing (a copy of) $\mathbb{Q}$ as a subring, then R is a universally going-down domain. We leave open the question of a more precise positioning of the class of AV-domains within the larger class of quasi-local going-down domains.

As usual, for any prime-power q, $\mathbb{F}_q$ denotes the field of cardinality q; for any nonzero ring R, $\mathrm{char}(R)$ denotes the characteristic of R; and $\subset$ denotes proper inclusion. Any undefined terminology is standard, as in [30] and [37].

2. Basic results

We begin with a straightforward result that was stated without proof in [34, page 810]. For the sake of completeness, we provide a proof of it.

Lemma 2.1. *Let R be an* AV*-ring and I an ideal of R. Then R/I is an* AV*-ring.*

Proof. Put $\widetilde{R} := R/I$. Let $\alpha, \beta \in R/I$. Pick $a, b \in R$ such that $\alpha = a + I$ and $\beta = b + I$. Since R is an AV-ring, there exists an integer $n \geq 1$ such that either $a^n R \subseteq b^n R$ or $b^n R \subseteq a^n R$. If $a^n R \subseteq b^n R$, then

$$\alpha^n \widetilde{R} = (a^n + I)(R/I) = (a^n R + I)/I \subseteq (b^n R + I)/I = \beta^n \widetilde{R}.$$

Similarly, if $b^n R \subseteq a^n R$, then $\beta^n \widetilde{R} \subseteq \alpha^n \widetilde{R}$. The proof is complete. $\square$

The next result is the first step in generalizing [7, Theorem 3.2 (2)] from domains to arbitrary (commutative) rings. Recall from [31] that a ring extension $R \subseteq S$ is called a P-*extension* if each element u of S satisfies a polynomial in $R[X]$ at least one of whose coefficients is a unit of R (equivalently, each $u \in S$ satisfies a polynomial in $R[X]$ whose coefficients generate the unit ideal of R).

Proposition 2.2. *Let $R \subseteq S$ be rings such that each subring of S that properly contains R is an* AV*-ring* (*for instance, such that* (R, S) *is an* AV*-ring pair*). *Then $R \subseteq S$ is a* P*-extension. If, in addition, R is integrally closed in S, then (R, S) is a normal pair.*

Proof. For the first assertion, it will suffice to show that if T is any ring such that $R \subset T \subseteq S$, the extension $R \subset T$ is residually algebraic. (The validity of this reduction in the context of arbitrary rings has a complicated history. That can best be understood by combining [16, Theorem] and the comments preceding [11, Theorem 2.5], while also noting that the proof of [6, Theorem 2.3] is valid for arbitrary rings. The rest of our proof of the first assertion is motivated by the proof that (i) $\Leftrightarrow$ (iv) in [6, Theorem 2.3].) To that end, let $Q \in \mathrm{Spec}(T)$ and set $P := Q \cap R$. Our task is to show that $R/P \subseteq T/Q$ is an algebraic extension (of domains). Suppose not. Choose $z \in T$ such that $z + Q \in T/Q$ is transcendental over R/P. Then the domain $A := (R/P)[z + Q]$ is not quasi-local and so A is not an AV-ring. On the other hand, $R/P \subset A \subseteq T/Q$, and so a standard homomorphism theorem provides a (uniquely determined) ring H such that $R + Q \subset H \subseteq T$ and $A = H/Q$. By assumption, H is an AV-ring, and hence so is its factor ring A, by Lemma 2.1. This contradiction completes the proof of the first assertion.

The proof of the "in addition " assertion follows at once from the fact that if $R \subseteq S$ are rings, then (R, S) is a normal pair if and only if $R \subseteq S$ is a P-extension such that R is integrally closed in S. (This fact has been discovered several times: cf. [38, Theorem 5.2, page 47], [8, Theorem 1].) $\square$

As a consequence of Proposition 2.2, part (b) of the next corollary characterizes the AV-domain pairs in which the bottom ring is a field and the top ring is a domain. More generally, part (a) of Corollary 2.3 allows the top ring to be any ring of Krull dimension 0.

Corollary 2.3. *Let R be a field and let S be a nonzero R-algebra. View $R \subseteq S$ as usual. Then:*

(*a*) (R, S) *is an* AV*-ring pair if and only if $R \subseteq S$ is an integral extension and S is a quasi-local* (*necessarily zero-dimensional*) *ring.*

(*b*) *Assume, in addition, that S is a domain. Then (R, S) is an* AV*-domain pair if and only if S is a field which is algebraic over R.*

Proof. (a) Suppose first that (R, S) is an AV-ring pair. Then S, being an AV-ring, must be quasi-local. Also, by Proposition 2.2, $R \subseteq S$ is a P-extension. Thus, since R is a field, S is algebraic (hence integral) over R. So (cf. [37, Theorem 48]), $\dim(S) = \dim(R)$ $(= 0)$.

For the converse, suppose that $R \subseteq S$ is an integral extension and S is quasi-local (so $\dim(S) = 0$). We must show that if $T \in [R, S]$, then T is an AV-domain. As $T \subseteq S$ is an integral extension, it follows from the hypotheses on S that T is quasi-local and $\dim(T) = 0$. Hence, each element of T is either a unit of T or nilpotent. As noted in [34, Example 2.3], any ring T with these properties must be an AV-ring.

(b) Any field is quasi-local; and a domain is a field if and only if it is zero-dimensional. Consequently, by (a), (R, S) is an AV-domain pair $\Leftrightarrow$ $R \subseteq S$ is an algebraic extension of domains and S is a quasi-local zero-dimensional domain $\Leftrightarrow$ S is a field which is algebraic over R. $\square$

The next theorem is the main result of this section. It establishes a kind of transitivity result which allows the study of AV-ring pairs to be separated into two cases, namely, where R is integrally closed in S and where S is integral over R. The former (resp., latter) case will be characterized later in this section (resp., in the following section.)

Theorem 2.4. *Let $R \subseteq S$ be rings. Then the following conditions are equivalent:*

(*1*) (R, S) *is an* AV*-ring pair;*

(*2*) $(R, \overline{R}_S)$ *and* $(\overline{R}_S, S)$ *are* AV*-ring pairs.*

Proof. The implication (1)$\Rightarrow$(2) is trivial. For the converse, assume (2). Without loss of generality, $R \subset \overline{R}_S$ and $\overline{R}_S \subset S$. It suffices to show that if $T \in [R, S]$ and T and $\overline{R}_S$ are incomparable under inclusion, then T is an AV-ring. Consider the ring $A := \overline{R}_S \cap T$. As $A \in [R, \overline{R}_S]$ and $(R, \overline{R}_S)$ is an AV-ring pair, A is an AV-ring.

We claim that $A \subseteq T$ is a P-extension. As $(R, \overline{R}_S)$ and $(\overline{R}_S, S)$ are AV-ring pairs, Proposition 2.2 ensures that $R \subseteq \overline{R}_S$ and $\overline{R}_S \subseteq S$ are each P-extensions. It follows that $R \subset S$ is also a P-extension, as it was shown independently in [8] and [44] that the class of P-extensions is stable under juxtaposition. In particular, $A \subseteq T$ is a P-extension, thus proving the above claim.

It is easy to check that A is integrally closed in T. Since $A \subseteq T$ is a P-extension, it follows from earlier comments that (A, T) is a normal pair. Hence, by the pullback-theoretic characterization of normal pairs having a quasi-local base ring [25, Theorem 6.8], $T = A_Q$ for some $Q \in \mathrm{Spec}(A)$. Consequently, by [34, Corollary 2.8], T inherits from A the property of being an AV-ring. The proof is complete. □

The next preparatory result generalizes a domain-theoretic result of Mimouni [41].

Lemma 2.5. *Let $R \subseteq S$ be rings, let $I \in \mathrm{Max}(S)$ such that $I \subset R$, put $D := R/I$, and assume also that both S and D are* AV*-rings and that $S/I = \mathrm{qf}(D)$. Then R is an* AV*-ring.*

Proof. Let $a, b \in R$. As S is an AV-ring, there exists an integer $n \geq 1$ such that either $a^n S \subseteq b^n S$ or $b^n S \subseteq a^n S$. Thus, either there exists $x \in S$ such that $a^n = b^n x$ or there exists $y \in S$ such that $b^n = a^n y$. We will suppose that $a^n = b^n x$, leaving the similar argument for the case where $b^n = a^n y$ to the reader.

Since D is an AV-domain with quotient field S/I, there exists an integer $m \geq 1$ such that either $(x+I)^m \in D$ or $x \notin I$ with $(x+I)^{-m} \in D$. If $(x + I)^m \in D$, then $x^m \in R$, whence $a^{nm}R = (b^n x)^m R = b^{nm}x^m R \subseteq b^{nm}R$, as desired. Therefore, without loss of generality, we may assume that $x \notin I$ and $(x + I)^{-m} \in D$. As S is quasi-local (since it is an AV-ring), x is a unit of S; that is, $x^{-1} \in S$. Working in $S/I = \mathrm{qf}(D)$, we have

$$x^{-m} + I = (x + I)^{-m} \in D = R/I,$$

whence $x^{-m} \in R+I = R$. As $a^n = b^n x$ leads to $a^{nm} = b^{nm}x^m$, we have, working in S, that $b^{nm} = a^{nm}x^{-m} \in a^{nm}R$, and so $b^{nm}R \subseteq a^{nm}R$. This completes the proof. □

Corollary 2.8 will characterize the AV-ring pairs that arise from integrally closed ring extensions. The next two results accomplish even more.

Theorem 2.6. *Let (R, S) be a normal pair. Then the following conditions are equivalent:*

(*1*) *(R, S) is an* AV*-ring pair;*
(*2*) *R is an* AV*-ring;*
(*3*) *R is quasi-local and S is an* AV*-ring.*

Proof. (1) $\Rightarrow$ (2): Trivial.

(2) $\Rightarrow$ (3): Assume (2). Then R is quasi-local, since the prime ideals of any AV-ring are linearly ordered by inclusion [34, Proposition 2.2]. Hence, by [25, Theorem 6.8] (and the "normal pair" hypothesis), there exists a divided prime ideal Q of R such that $S = R_Q$. Since the class of AV-rings is stable under localization at prime ideals [34, Corollary 2.8], it follows that S is an AV-ring.

(3) $\Rightarrow$ (1): Assume (3). Once again using [25, Theorem 6.8], we infer the existence of a divided prime ideal Q of R such that $S = R_Q$ and R/Q is a valuation domain with quotient field S/Q. As the "divided prime ideal" condition gives $Q = QR_Q = QS \in \mathrm{Max}(S)$ and R/Q is an AV-ring, Lemma 2.5 yields that R is an AV-ring. It remains only to prove that if T is any ring such that $R \subset T \subseteq S$, then T is an AV-ring. This, in turn, follows by applying the above-proved implication (2) $\Rightarrow$ (3) to the pair (R, T) (after noting that (R, T) inherits the "normal pair" property from (R, S)). The proof is complete. $\square$

Corollary 2.7. *Let $R \subseteq S$ be rings. Then the following conditions are equivalent:*

(*1*) (R, S) *is an* AV*-ring pair;*

(*2*) $(R, \overline{R}_S)$ *is an* AV*-ring pair and* $R \subseteq S$ *is a* P*-extension;*

(*3*) $(R, \overline{R}_S)$ *is an* AV*-ring pair and* $\overline{R}_S \subseteq S$ *is a* P*-extension*

Proof. (1) $\Rightarrow$ (2): Assume (1). Then the first (resp., second) assertion in (2) is trivial (resp., is a consequence of Proposition 2.2).

(2) $\Rightarrow$ (3): Trivial.

(3) $\Rightarrow$ (1): Assume (3). Then, by Theorem 2.4, it suffices to show that $(\overline{R}_S, S)$ is an AV-ring pair. As $\overline{R}_S$ is integrally closed in S and $\overline{R}_S \subseteq S$ is a P-extension, it follows from (the second assertion of) Proposition 2.2 that $(\overline{R}_S, S)$ is a normal pair. Therefore, since $\overline{R}_S$ is an AV-ring, it follows from the implication that (2) $\Rightarrow$ (1) in Theorem 2.6 (as applied to the base ring $\overline{R}_S$) that $(\overline{R}_S, S)$ is an AV-ring pair. The proof is complete. $\square$

Corollary 2.8. *Let $R \subseteq S$ be rings, with R integrally closed in S. Then the following conditions are equivalent:*

(*1*) (R, S) *is an* AV*-ring pair;*

(*2*) R *is an* AV*-ring and* $R \subseteq S$ *is a* P*-extension.*

Proof. By hypothesis, $\overline{R}_S = R$, and so an application of Corollary 2.7 completes the proof. $\square$

It seems worth noting that neither of the assertions in the statement of condition (2) of Corollary 2.8 can be deleted. Indeed, $\mathbb{Z} \subset \mathbb{Q}$ is a

P-extension although $\mathbb{Z}$ is not an AV-ring (it *is* an AP-domain); and if F is any field, the polynomial ring extension $F \subset F[X]$ is not a P-extension although F is an AV-ring. We continue this glance at the domain-theoretic setting for one more result, which generalizes [7, Theorem 3.2 (2)] from the "overring" to the "algebraic" setting.

Corollary 2.9. *Let $R \subseteq S$ be domains. Then the following conditions are equivalent:*

(*1*) (R, S) *is an* AV*-domain pair;*

(*2*) $(R, \overline{R}_S)$ *is an* AV*-domain pair and* $R \subseteq S$ *is an algebraic extension.*

Proof. The implication (1) ⇒ (2) follows immediately from Proposition 2.2 (or Corollary 2.7). For the converse, assume (2). As $R \subseteq S$ is algebraic, an application of [37, Exercise 35, page 44] shows that S is (R-algebra isomorphic to) an overring of $\overline{R}_S$. Thus, each element of $[\overline{R}_S, S]$ is also an overring of $\overline{R}_S$. As each overring of an AV-domain is an AV-domain [34, Proposition 3.1] (or [7, Theorem 3.2 (1)]) and $\overline{R}_S$ is an AV-domain, it follows that $(\overline{R}_S, S)$ is an AV-domain pair. Then an application of Theorem 2.4 shows that (R, S) is an AV-domain pair. The proof is complete. □

In regard to the comment that preceded Corollary 2.9, we note that the next section will give some examples of AV-domain pairs (R, S) such that S is not an overring of R. To close this section, we give two results that identify some contexts in which any AV-ring pair *must* arise from an overring extension. The first of these results, Theorem 2.10, gives a far-reaching generalization of all of [7, Theorem 3.2] from the domain-theoretic setting to the class of base rings R such that $\mathrm{tq}(R)$ is a von Neumann regular ring. This assumption on R will be used to ensure (thanks to [27, Propositions 3.4 and 2.4]) that any normal pair (R, S) has the property that S is an overring of R. Continuing the "necessarily overring" theme (but without the assumption that $\mathrm{tq}(R)$ is von Neumann regular), Corollary 2.11 characterizes the AV-ring pairs (R, S) such that $R \subset S$ is an integrally closed minimal ring extension, leaving the study of the AV-ring pairs arising from integral minimal ring extensions as the topic of Section 5 below.

Theorem 2.10. *Let $R \subseteq S$ be rings such that R is integrally closed in S and $\mathrm{tq}(R)$ is a von Neumann regular ring. Then the following conditions are equivalent:*

(*1*) (R, S) *is an* AV*-ring pair;*

(*2*) R *is an* AV*-ring and* S *is an overring of* R.

Proof. As every overring of an AV-ring is an AV-ring [34, Proposition 3.1], we have that (2) $\Rightarrow$(1). For the converse, assume (1). Then, of course, R is an AV-ring. It remains only to prove that S is an overring of R. By the above comments, it follows from results in [27] (and the hypothesis on $\mathrm{tq}(R)$) that it suffices to prove that (R, S) is a normal pair. This, in turn, follows from the final assertion in Proposition 2.2, since R is integrally closed in S and (R, S) is an AV-ring pair. The proof is complete. □

Corollary 2.11. *Let $R \subset S$ be an integrally closed minimal ring extension. Then the following conditions are equivalent:*

(*1*) (R, S) *is an* AV*-ring pair;*
(*2*) R *is an* AV*-ring;*
(*3*) R *is quasi-local and* S *is an* AV*-ring;*
(*4*) R *is an* AV*-ring and* S *is an overring of* R.

Proof. As any integrally closed minimal ring extension $R \subset S$ produces a normal pair (R, S), Theorem 2.6 gives (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). As (4) $\Rightarrow$ (2) trivially, it remains only to prove that (3) $\Rightarrow$ (4). Assume (3). Then, since $R \subset S$ is an integrally closed minimal ring extension and R (being an AV-ring) is quasi-local, [26, Theorem 3.1] ensures that S is an overring of R. The proof is complete. □

3. Root extensions and AV-ring pairs

Let $R \subseteq S$ be nonzero rings. It is clear that $\mathrm{Reg}(S) \cap R \subseteq \mathrm{Reg}(R)$. However, the reverse inequality need not hold. In other words, it need not be the case that $\mathrm{Reg}(S) \cap R = \mathrm{Reg}(R)$. (In other other words, it need not be the case that S is a torsion-free R-module, in the classical sense of that term.) This equality is often desirable, as it is equivalent to the statement that the universal mapping property of rings of fractions allows the inclusion map $R \hookrightarrow \mathrm{tq}(S)$ to extend to a (unique, necessarily injective) ring homomorphism $\mathrm{tq}(R) \to \mathrm{tq}(S)$, in which case we use that injection to view $\mathrm{tq}(R) \subseteq \mathrm{tq}(S)$. It is natural to ask if there are relevant kinds of base rings R and ring extensions $R \subseteq S$ admitting an embedding of $\mathrm{tq}(R)$ in $\mathrm{tq}(S)$ in this way. We begin this section with just such a result. First, two asides: recall that a ring is said to be a *reduced ring* if it has no nonzero nilpotent elements; and, to avoid trivialites, we will assume that $\mathrm{tq}(0) = 0$, which is obviously an AV-ring.

Lemma 3.1. *Let $R \subseteq S$ be a root extension of rings with S (and hence R) being a reduced ring. Then* $\mathrm{Reg}(S) \cap R = \mathrm{Reg}(R)$ *and* $\mathrm{tq}(R) \subseteq \mathrm{tq}(S)$ *is a root extension.*

Proof. For the first assertion, we need only show that if $x \in \mathrm{Reg}(R)$, then $x \in \mathrm{Reg}(S)$. Suppose that this fails, and pick a nonzero element $y \in S$ such that $xy = 0$. Since $R \subseteq S$ is a root extension, there exists an integer $n \geq 1$ such that $y^n \in R$. Then $x^n y^n = (xy)^n = 0$. As $\mathrm{Reg}(R)$ is a multiplicatively closed set, $x^n \in \mathrm{Reg}(R)$, and so $y^n = 0$. Since S is reduced, $y = 0$, the desired contradiction. This completes the proof of the first assertion.

In light of the first assertion, it follows from the above comments that we may view $\mathrm{tq}(R) \subseteq \mathrm{tq}(S)$. It remains only to show that this is a root extension. Given $u \in \mathrm{tq}(S)$, we must find an integer $e \geq 1$ such that $u^e \in \mathrm{tq}(R)$. Write $u = a/z$, with $a \in S$ and $z \in \mathrm{Reg}(S)$. Since $R \subseteq S$ is a root extension, there exist integers $n \geq 1$ and $m \geq 1$ such that $a^n \in R$ and $z^m \in R$. Note that $(a^n)^m \in R^m = R$. Moreover, since $\mathrm{Reg}(S)$ is a multiplicatively closed set, z^m and $(z^m)^n$ are elements of $\mathrm{Reg}(S)$ (and of R). Consequently, by the first assertion, $(z^m)^n \in \mathrm{Reg}(R)$. Since

$$u^{nm} = \frac{(a^n)^m}{(z^m)^n} \in \mathrm{tq}(S),$$

it follows that $u^{nm} \in \mathrm{tq}(R)$. Therefore, taking $e := nm$ completes the proof. □

By adding a "root extension" hypothesis, we next deepen the study of a theme from Section 2, while continuing the project of generalizing some domain-theoretic observations and results from [3] and [7] to the ring-theoretic context.

Proposition 3.2. (*a*) *Let $R \subseteq S$ be a root extension of rings such that R is an* AV*-ring. Then S is an* AV*-ring.*

(*b*) *Let $R \subseteq S$ be a root extension of rings such that S is an* AV*-ring. Then R is an* AV*-ring.*

(*c*) *Let $R \subseteq S$ be a root extension of rings. Then the following conditions are equivalent:*

(*1*) *R is an* AV*-ring;*

(*2*) *S is an* AV*-ring;*

(*3*) *(R, S) is an* AV*-ring pair;*

If, in addition, S is a reduced ring, then the above conditions (*1*)–(*3*) *are equivalent to:*

(*4*) *$(R, \mathrm{tq}(S))$ is an* AV*-ring pair.*

Proof. (a) Consider two nonzero elements $x, y \in S$. As $R \subseteq S$ is a root extension, there exists an integer $n \geq 1$ such that $x^n, y^n \in R$. Since R is an AV-ring, there exists an integer $m \geq 1$ such that either x^{nm} divides y^{nm} in R or y^{nm} divides x^{nm} in R. Therefore, either x^{nm} divides y^{nm} in S or y^{nm} divides x^{nm} in S. Thus, S is an AV-ring.

(b) Consider two nonzero elements $x, y \in R$. Then $x, y \in S$. Hence, since S is an AV-ring, there exists a positive integer k such that either $x^k = ay^k$ or $y^k = bx^k$ for some elements $a, b \in S$. Since $R \subseteq S$ is a root extension, there exists a positive integer n such that $a^n, b^n \in R$. Hence, either $x^{kn} = a^n y^{kn}$ or $y^{kn} = b^n x^{kn}$. Since $a^n, b^n \in R$, this completes the proof that R is an AV-ring.

(c) For any rings $R \subseteq A \subseteq B \subseteq S$, it is clear that the ring extension $A \subseteq B$ inherits the "root extension" property from $R \subseteq S$. Hence, the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from (a) and (b).

Next, assume that S is reduced. Then by Lemma 3.1 (and the discussion preceding it), we have $(R \subseteq)$ $\mathrm{tq}(R) \subseteq \mathrm{tq}(S)$. Since (4) $\Rightarrow$ (3) trivially, it will suffice to show that (2) $\Rightarrow$ (4).

Assume that S is an AV-ring (and that S is reduced). Our task is to show that if T is a ring such that $R \subseteq T \subseteq \mathrm{tq}(S)$, then T is an AV-ring. To that end, consider two nonzero elements $x, y \in T$. Then $x, y \in \mathrm{tq}(S)$. As $\mathrm{tq}(S)$ inherits the "AV-ring" property from S by [34, Proposition 3.1], there exists an integer $n \geq 1$ and $\alpha, \beta \in \mathrm{tq}(S)$ such that either $x^n = \alpha y^n$ or $y^n = \beta x^n$. Without loss of generality, we assume that $x^n = \alpha y^n$. Since $\mathrm{tq}(R) \subseteq \mathrm{tq}(S)$ is a root extension by Lemma 3.1, there exists an integer $m \geq 1$ such that $\alpha^m \in \mathrm{tq}(R)$. Thus, $\alpha^m = c/d \in \mathrm{tq}(R)$, for some elements $c \in R$ and $d \in \mathrm{Reg}(R)$. As S is an AV-ring, there exist $u, v \in S$ and an integer $k \geq 1$ such that either $c^k = ud^k$ or $d^k = vc^k$. Again as $R \subseteq S$ is a root extension, there exists a positive integer p such that $u^p, v^p \in R \subseteq T$. Therefore, either $x^{nmkp} = u^p y^{nmkp}$ or $y^{nmkp} = v^p x^{nmkp}$. (The handling of the latter possibility involved a somewhat subtle additional use of Lemma 3.1. In detail, if $d^k = vc^k$, one obtains $x^{nmkp} = y^{nmkp}/v^p$ *enroute* to showing that $y^{nmkp} = v^p x^{nmkp}$, and this use of "fractional" notation is legitimate since $v^p \in \mathrm{Reg}(S) \cap R = \mathrm{Reg}(R)$, the underlying point being that $vc^k = d^k \in \mathrm{Reg}(R) \subseteq \mathrm{Reg}(S)$ ensures that v is an element of the multiplicatively closed set $\mathrm{Reg}(S)$.) Thus, either $x^{nmkp}T \subseteq y^{nmkp}T$ or $y^{nmkp}T \subseteq x^{nmkp}T$. This proves that T is an AV-ring. The proof is complete. □

If R is a domain but not an AV-domain, it is nonetheless the case that $S := \mathrm{qf}(R)$ is an AV-domain. Thus, the equivalence (1) $\Leftrightarrow$ (2) in Proposition 3.2 (c) cannot be expected to hold for an arbitrary pair of domains (let alone, an arbitrary pair of rings) $R \subseteq S$. What led to that equivalence holding in Proposition 3.2 (c) was the assumption that $R \subseteq S$ is a root extension.

It seems natural to ask whether a ring extension $R \subseteq S$ satisfying the conditions in Proposition 3.2 must be an overring extension. While

Theorem 2.10 and Corollary 2.11 may lead one to expect an affirmative answer, Corollary 2.3 (b) shows that an affirmative answer is not possible if one allows the ambient rings to form an algebraic field extension. In fact, we show next that the answer is in the negative even if one restricts attention to root extensions of domains that are far from being fields.

Example 3.3. *Let p be a prime number. Then:*

(a) *There exists a root extension $R \subseteq S$ of domains such that neither R nor S is a field, $(R, \mathrm{qf}(S))$ is an* AV*-domain pair,* $\mathrm{char}(R) = p$, *and S is not contained in the quotient field of R (that is, S is not an overring of R). It can be further arranged that R and S are each valuation domains and that $u^p \in R$ for each $u \in S$.*

One way to produce domains $R \subset S$ with the above behavior is the following. Let X be an analytic indeterminate over a field F of characteristic p. Put $S := F[[X]]$, the ring of formal power series in X over F; and put $R := F[[X^p]]$.

(b) *Another way to produce domains $R \subset S$ with the behavior that was stipulated above in (a) is the following. Let $k \subset K$ be a purely inseparable field extension of characteristic p and exponent* 1 *(that is, for every $\alpha \in K$, $\alpha^p \in k$, where $p = \mathrm{char}(k)$). Let X be an analytic indeterminate over K. Put $R := k[[X]]$ and $S := K[[X]]$.*

Proof. (a) It is well known that $S = F[[X]]$ is a (discrete rank 1) valuation domain but not a field. The same conclusion holds for $R = F[[X^p]]$, since X^p is an analytic indeterminate over F. Moreover, since X cannot be expressed as a quotient of elements from $F[[X^p]]$, it follows that S is not contained in the quotient field of R. Thus, as R is an AV-domain, Proposition 3.2 (c) reduces our task to proving that $u^p \in R$ for each $u \in S$. This, in turn, holds, for we can write $u = \sum_{i=0}^{\infty} a_i X^i$ with each $a_i \in F$ and then

$$u^p = \sum_{i=0}^{\infty} (a_i)^p (X^i)^p = \sum_{i=0}^{\infty} (a_i)^p (X^p)^i \in F[[X^p]] = R,$$

as required.

(b) As in the proof of (a), both $R = k[[X]]$ and $S = K[[X]]$ are valuation domains that are not fields. Note that if $\xi \in K \setminus k$, then ξ cannot be expressed as a quotient of elements from $k[[X]]$, and so S is not contained in the quotient field of R. With minor changes, the rest of the proof of (a) (including the appeal to Proposition 3.2 (c) and the displayed calculation of u^p) easily carries over to the present context. $\square$

We next show how to use the "root extension" property to give a sufficient condition for (R, S) to an AV-ring pair without entailing that S is necessarily an overring of R. In fact, the condition given in Propositions 3.4 and 3.5 are satisfied by the (example) extensions $R \subset S$ that were constructed in parts (a) and (b) of Example 3.3. Despite the above example, it seems natural to ask, in the spirit of Proposition 3.2 (c), when given an AV-domain pair (R, S), for (i) a sufficient condition that S be an overring of R and (ii) a sufficient condition that $(R, \mathrm{qf}(S))$ be an AV-domain pair. Such sufficient conditions will be given in Proposition 3.5. First, Proposition 3.4 summarizes what can be said when one collects what has already been done above.

Proposition 3.4. *Let $R \subseteq S$ be rings such that $R \subseteq \overline{R}_S$ is a root extension. Then the following ten conditions are equivalent:*

(*1*) (R, S) *is an* AV*-ring pair;*
(*2*) $(R, \overline{R}_S)$ *is an* AV*-ring pair and* $(\overline{R}_S, S)$ *is an* AV*-ring pair;*
(*3*) $(R, \overline{R}_S)$ *is an* AV*-ring pair and* $R \subseteq S$ *is a* P*-extension;*
(*4*) $(R, \overline{R}_S)$ *is an* AV*-ring pair and* $\overline{R}_S \subseteq S$ *is a* P*-extension;*
(*5*) R *is an* AV*-ring and* $(\overline{R}_S, S)$ *is an* AV*-ring pair;*
(*6*) $\overline{R}_S$ *is an* AV*-ring and* $(\overline{R}_S, S)$ *is an* AV*-ring pair;*
(*7*) R *is an* AV*-ring and* $R \subseteq S$ *is a* P*-extension;*
(*8*) $\overline{R}_S$ *is an* AV*-ring and* $R \subseteq S$ *is a* P*-extension;*
(*9*) R *is an* AV*-ring and* $\overline{R}_S \subseteq S$ *is a* P*-extension;*
(*10*) $\overline{R}_S$ *is an* AV*-ring and* $\overline{R}_S \subseteq S$ *is a* P*-extension.*

If, in addition, $\overline{R}_S$ is a reduced ring, then the above conditions (1)–(10) are equivalent to each of (11)–(13):

(*11*) $(R, \mathrm{tq}(\overline{R}_S))$ *is an* AV*-ring pair and* $(\overline{R}_S, S)$ *is an* AV*-ring pair;*
(*12*) $(R, \mathrm{tq}(\overline{R}_S))$ *is an* AV*-ring pair and* $R \subseteq S$ *is a* P*-extension;*
(*13*) $(R, \mathrm{tq}(\overline{R}_S))$ *is an* AV*-ring pair and* $\overline{R}_S \subseteq S$ *is a* P*-extension.*

Proof. Combine Theorem 2.4, Corollary 2.7 and Proposition 3.2. □

Proposition 3.5 will show how adding a "root extension" hypothesis can lead to additional equivalent conditions in the domain-theoretic context. For domains $R \subseteq S$ with $R \subseteq \overline{R}_S$ being a root extension, the thirteen conditions in Proposition 3.4 remain equivalent (of course), but we will focus in Proposition 3.5 on equivalences in the domain-theoretic setting where "P-extension" conditions are replaced by algebraicity conditions.

Proposition 3.5. *Let $R \subseteq S$ be domains such that $R \subseteq \overline{R}_S$ is a root extension. Let $(\overline{R}_S)'$ denote the integral closure of $\overline{R}_S$ (in its quotient field). Then the following conditions are equivalent:*

(1) (R, S) *is an* AV*-domain pair;*

(2) R *is an* AV*-domain and* $R \subseteq S$ *is an algebraic extension;*

(3) $\overline{R}_S$ *is an* AV*-domain and* $R \subseteq S$ *is an algebraic extension;*

(4) $(R, \overline{R}_S)$ *is an* AV*-domain pair and* $R \subseteq S$ *is an algebraic extension;*

(5) $(R, \mathrm{qf}(S))$ *is an* AV*-domain pair;*

(6) S *is both an* AV*-domain and an overring of* $\overline{R}_S$*, and* $(\overline{R}_S)'$ *is a valuation domain.*

Proof. Also suppose, for the moment, that R is a field. In view of Proposition 2.2 and basic facts about integral extensions of domains, we see that each of the conditions (1)-(6) is equivalent to $R \subseteq S$ being an algebraic field extension. Thus, we can assume henceforth that R is not a field.

Since $R \subseteq \overline{R}_S$ is a root extension, it follows from Proposition 3.2 (c) that R is an AV-domain $\Leftrightarrow$ $\overline{R}_S$ is a AV-domain $\Leftrightarrow$ $(R, \overline{R}_S)$ is an AV-domain pair $\Leftrightarrow$ $(R, \mathrm{qf}(\overline{R}_S))$ is an AV-domain pair. Also, by using Proposition 2.2 and the well known clearing-of-denominators argument, we see that each of the conditions (1)-(5) implies that $\mathrm{qf}(\overline{R}_S) = \mathrm{qf}(S)$ and that $R \subset S$ is an algebraic extension. It is now straightforward to check that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5); and that (1) $\Leftrightarrow$ (4) by virtue of Corollary 2.9.

(1) $\Rightarrow$ (6): Assume (1). Then $R \subseteq S$ is an algebraic extension by Proposition 2.2. Also, by the reasoning in the second paragraph of this proof, $\mathrm{qf}(S) = \mathrm{qf}(\overline{R}_S)$. Of course, (1) implies that S is an AV-domain. Finally, since $\overline{R}_S$ is a AV-domain, it follows from [3, Theorem 5.6] that $(\overline{R}_S)'$ is a valuation domain. This completes the proof that (1) $\Rightarrow$ (6).

It will suffice to prove that (6) $\Rightarrow$ (3). Observe that $\overline{R}_S = (\overline{R}_S)' \cap S$. Assume (6). Then $\overline{R}_S$ is an intersection of two overrings (namely, $(\overline{R}_S)'$ and S) that happen to be AV-domains with comparable integral closures. Therefore, by [4, Lemma 2], $\overline{R}_S$ is an AV-domain. It remains only to prove that $R \subseteq S$ is an algebraic extension. Of course, S is algebraic over $\overline{R}_S$, since S is an overring of $\overline{R}_S$. Moreover, $\overline{R}_S$ is integral, hence algebraic, over R. So, by the transitivity of algebraicity, $R \subseteq S$ is algebraic, which completes the proof. $\square$

It is natural to ask how essential it is to have the hypothesis "$R \subset \overline{R}_S$ is a root extension" in a result having the flavor of Proposition 3.4. Along those lines, we raise the following question.

Question. If (R, S) is an AV-domain pair such that S is integral over R and R is not a field, must $R \subseteq S$ be a root extension?

The (root) integral extensions in parts (a) and (b) of Example 3.3 are rather "large." One may ask if the fact that those examples lead to (R, S) being an AV-ring (domain) pair could be verified by replacing S with somewhat smaller rings (domains). Parts (a) and (b) of Proposition 3.9 will give some elementary answers to that question. First, we present a few results on a certain kind of direct limit which are of some independent interest and serve to motivate the focus on finite-type algebras in Proposition 3.9.

When a set of rings $\{A_i \mid i \in I\}$ is indexed by a poset I, it will be convenient to say that $\{A_i \mid i \in I\}$ is (*inclusion-*) *directed* if the following conditions hold: whenever $i \leq j$ in I, then $A_i \subseteq A_j$ and there exists $k \in I$ such that $i \leq k$ and $j \leq k$; and there exists some "universal" ring that contains each A_i as a subring. It is easy to check that if $\{A_i \mid i \in I\}$ is an inclusion-directed set of rings inside some "universal" ring $\mathcal{U}$, then $\cup_{i\in I} A_i$ is also a subring of $\mathcal{U}$.

Proposition 3.6. *Let $\{A_i \mid i \in I\}$ and $\{B_i \mid i \in I\}$ each be an inclusion-directed set of rings such that $A_i \subseteq B_i$ for each $i \in I$. Put $A := \cup_{i\in I} A_i$ and $B := \cup_{i\in I} B_i$. If (A_i, B_i) is an* AV*-ring pair for each $i \in I$, then (A, B) is an* AV*-ring pair.*

Proof. We will show that if $T \in [A, B]$, then T is an AV-ring. For each $i \in I$, note that $T_i := T \cap B_i$ is a ring. In fact, $\{T_i \mid i \in I\}$ is an inclusion-directed set of rings such that $\cup_{i\in I} T_i = T$. Moreover, for each $i \in I$,

$$A_i = A \cap A_i \subseteq T \cap A_i \subseteq T \cap T_i = T_i \subseteq B_i,$$

that is, $T_i \in [A_i, B_i]$. Hence, by hypothesis, T_i is an AV-ring for each $i \in I$. Therefore, it suffices to prove that A is an AV-ring. To that end, let $c, d \in A$. Pick $i, j \in I$ such that $c \in A_i$ and $d \in A_j$. Since $\{A_i \mid i \in I\}$ is inclusion-directed, there exists $k \in I$ such that $c, d \in A_k$. As A_k is an AV-ring, there exists an integer $n \geq 1$ such that either $c^n \in A_k d^n$ or $d^n \in A_k c^n$. Hence, either $c^n \in Ad^n$ or $d^n \in Ac^n$, as desired. □

Corollary 3.7. *Let $\{R_i \mid i \in I\}$ be an inclusion-directed set of rings such that R_i is an* AV*-ring for each $i \in I$. Then $\cup_{i\in I} R_i$ is also an* AV*-ring.*

Proof. It suffices to apply Proposition 3.6 with $A_i := R_i$ and $B_i := R_i$ for each $i \in I$. □

We next isolate an application of Proposition 3.6 which includes the fact that if one forms an inclusion-directed union of domains each of which has the property that each of its overrings is an AV-domain, then the union inherits that property.

Corollary 3.8. *Let $\{R_i \mid i \in I\}$ be an inclusion-directed set of domains such that some "universal" domain contains each R_i as a subring, and put $R := \cup_{i \in I} R_i$. For each $j \in I$, let L_j be a field containing R_j as a subring, suppose that $\{L_i \mid i \in I\}$ forms an inclusion-directed set of rings such that some "universal" field contains each L_i as a subring, and put $L := \cup_{i \in I} L_i$. (For instance, the last assumption holds if $L_j = \mathrm{qf}(R_j)$ for each $j \in I$, in which case $L = \mathrm{qf}(R)$.) If (R_i, L_i) is an* AV*-domain pair for each $i \in I$, then (R, L) is an* AV*-domain pair.*

Proof. By Proposition 3.6, it suffices to address the parenthetical assertion. It is clear that if $R_i \subseteq R_j$, then $\mathrm{qf}(R_i) \subseteq \mathrm{qf}(R_j)$. Thus, $\{\mathrm{qf}(R_i) \mid i \in I\}$ forms an inclusion-directed set of fields. It remains only to show that the union of this set of fields is precisely $\mathrm{qf}(R)$. This general fact has been observed before (cf. [22]). The proof is complete. □

Recall that Corollary 2.8 characterized the integrally closed ring extensions $R \subseteq S$ such that (R, S) is an AV-ring pair. While this section has accumulated much information for the analagous question for root extensions, such extensions are quite rare within the class of integral extensions. Thus, it is seems reasonable to ask for a characterization of the integral ring extensions $R \subseteq S$ such that (R, S) is an AV-ring pair. Part (b) of Prioposition 3.9 provides such a characterization.

Proposition 3.9. *Let $R \subseteq S$ be rings. Then:*

(*a*) *The following conditions are equivalent:*

(*1*) *(R, S) is an* AV*-ring pair;*

(*2*) *T is an* AV*-ring for each finite-type R-subalgebra T of S;*

(*3*) *$R[a, b]$ is an* AV*-ring for all subsets $\{a, b\}$ of S having cardinality at most* 2.

(*b*) *If S is an integral ring extension of R, then the above conditions (1), (2) and (3) are also equivalent to the following condition:*

(*4*) *T is an* AV*-ring for each module-finite R-subalgebra T of S.*

Proof. The following two facts explain why (b) is a consequence of (a): (2) ⇒ (4) in general, since each module-finite R-subalgebra of S is a finite-type R-algebra; and when S is integral over R, (4) implies (2) since each finite-type R-subalgebra of S is then a module-finite R-algebra. Next, for a proof of (a), note first that the implications (1) ⇒ (2) ⇒ (3) are trivial. Therefore, it remains only to prove that (3) ⇒ (1). Assume (3). We will prove that each $T \in [R, S]$ is an AV-ring.

Let $a, b \in T$. Since (3) ensures that $R[a, b]$ is an AV-ring, there exists an integer $n \geq 1$ and an element $\lambda \in R[a, b]$ such that either $a^n = \lambda b^n$ or $b^n = \lambda a^n$. As $R[a, b] \subseteq T$, we have $\lambda \in T$, so that either $Ta^n \subseteq Tb^n$ or $Tb^n \subseteq Ta^n$, as desired. The proof is complete. □

The next result applies some of the above ideas to subrings rather than to extension rings.

Corollary 3.10. *Let R be a ring and let A be the prime subring of R (that is, A is the smallest subring of R). Then the following conditions are equivalent:*

(*1*) (A, R) *is an* AV*-ring pair;*
(*2*) *Every subring of R is an* AV*-ring;*
(*3*) $A[r_1, r_2]$ *is an* AV*-ring for all $r_1, r_2 \in R$.*

Proof. It is clear that (1) ⇔ (2) ⇒ (3). It remains only to prove that (3) ⇒ (1). An application of Proposition 3.9 (a) completes the proof. □

For an example of a ring R that satisfies the equivalent conditions in Corollary 3.10 (but need not be a domain), one could take R to be any quasi-local ring of Krull dimension 0 that is integral over its prime subring, for instance any finite local ring. Indeed, any subring T of R inherits the assumptions on R. Hence, each element of T is either nilpotent or a unit of T, and so it is easy to check that T is an AV-ring.

Remark 3.11. (a) The referee has kindly requested that we look at a recent reprint [1] by D. D. Anderson, S. Xing and M. Zafrullah, presumably in regard to possible overlap between our work and [1]. Information that we accessed online indicates that [1] became available on arXiv.org on December 4, 2019. Our records indicate that we submitted the first version of the present paper to this journal during the last week of November, 2019 and that the journal acknowledged receipt of it on December 3, 2019. While it seems clear to us that our work has quite different purposes from those of [1], both papers have AV-domains and treatments involving rings of formal power series as common points of interest. For instance, it follows from [1, Theorem 13 (8)] that if $k \subset K$ is a purely inseparable field extension of characteristic p, then $k + XK[[X]]$ is an AV-domain. We do not believe that the innovative reasoning in [1] which supports that conclusion has any significant overlap with our Example 3.3. We wish to stress that the above statement and proof of Example 3.3 are unchanged from our original submission in November, 2019. We believe that any overlapping discoveries that may be discerned involving Example 3.3 and [1] were innocent, independent and almost simultaneous. We wish also to add that a more general version of the transfer result on root extensions of domains in [1, Theorem 2 (1)] also appeared in our original submission (where it had been motivated by material in [3] and a preliminary version of [7]) and that a ring-theoretic generalization of that transfer result was given above in Proposition 3.2 (c).

(b) The following comments are also unchanged from our submission in November, 2019. The construction and conclusion in Example 3.3 (a) should be contrasted with the following result of D. D. Anderson and Zafrullah [3, Example 4.15]. Let S be a primitive numerical monoid (nowadays usually called a "numerical semigroup"), that is, an additive submonoid of the set of nonnegative integers under addition such that $\mathrm{GCD}\{S\} = 1$. Then for any field F of characteristic $p > 0$, the ring $F[[X^s \mid s \in S]]$ is an AV-domain whose integral closure is $F[[X]]$. (Actually, to get the "AV-domain" conclusion from the cited result in [3], one also needs to note that quasi-local API-domain $\Rightarrow$ quasi-local AB-domain $\Rightarrow$ AV-domain.) Note, however, that despite the similarities in notation, the AV-domain $R = F[[X^p]]$ that was constructed in Example 3.3 (a) cannot be described as $F[[X^s \mid s \in S]]$ for a numerical semigroup S, the point being that for any prime number p, the set of elements in the additive abelian group $p\mathbb{Z}$, when viewed in $\mathbb{Z}$, has greatest common divisor $p \neq 1$.

4. AV-ring pairs arising from pullbacks

In [41, Theorem 2.9], Mimouni proved a result which we restate below as Theorem 4.1. The first goal of this section is to generalize that domain-theoretic result to arbitrary rings: see Theorem 4.5. As other pullback-theoretic studies have already arisen (sometimes implicitly) in work on AV-domains (cf. [7, Corollary 3.4, Theorem 3.6, Corollary 3.7]), it seems appropriate to determine when pseudo-valuation domains lead to AV-domains or AV-domain pairs. (Recall from [5, Proposition 2.6] that PVDs can be characterized as the pullbacks arising from a valuation domain V and a subfield of the residue field of V.) Such results on PVDs are given below in Proposition 4.7 and Corollary 4.9.

Theorem 4.1. (*Mimouni*) *Let M be a maximal ideal of a domain S and let D be a proper subring of S/M. Then the pullback $D \times_{S/M} S$ is an* AV*-domain if and only if both S and D are* AV*-domains and $\mathrm{qf}(D) \subseteq S/M$ is a root extension.*

Our path to a generalization of Theorem 4.1 will begin with the following lemma whose easy proof is left to the reader.

Lemma 4.2. *Let $R \subseteq S$ be rings having a common ideal I. Then $R \subseteq S$ is a root extension if and only if $R/I \subseteq S/I$ is a root extension.*

Lemma 4.3. *Let $R \subseteq S$ be rings, let $I \in \mathrm{Max}(S)$ such that $I \subset R$, put $D := R/I$, and assume also that $S/I = \mathrm{qf}(D)$. Then the following conditions are equivalent:*

(*1*) *R is an* AV*-ring;*

(*2*) *Both S and D are* AV*-rings.*

Proof. Lemma 2.5 established that (2) $\Rightarrow$ (1). We turn to the proof that (1) $\Rightarrow$ (2). Assume (1). It follows from Lemma 2.1 that D is an AV-ring and, hence, an AV-domain. It remains to prove that S is an AV-ring. To that end, consider the pullback $T := S \times_{S/I} D'$. Since D is an AV-domain, [3, Theorem 5.6] ensures that D' is a valuation domain and $D \subseteq D'$ is a root extension. As $T/I = D'$, it follows from Lemma 4.2 that $R \subseteq T$ inherits the "root extension" property from $D \subseteq D'$. Hence, by Proposition 3.2 (a), T inherits the "AV-ring" property from R. Moreover, as $(T/I, S/I) = (D', \mathrm{qf}(D))$ is a normal pair, so is (T, S) (by a result of Rhodes: cf. [38, Proposition 5.8, page 52]). Consequently, Theorem 2.6 ensures that S inherits the "AV-ring" property from T. The proof is complete. $\square$

Lemma 4.4. *Let $R \subseteq S$ be rings and let M be an ideal of S such that $M \subseteq R$ and M contains a regular element of S. Assume also that $D := R/M$ is a field. Then the following conditions are equivalent:*

(*1*) *R is an* AV*-ring;*

(*2*) *S is an* AV*-ring and $R \subseteq S$ is a root extension.*

Proof. The implication (2) $\Rightarrow$ (1) follows from Proposition 3.2 (b). For the converse, assume (1). We will show first that $R \subseteq S$ is a root extension, specifically that if $x \in S$, then there exists an integer $n \geq 1$ such that $x^n \in R$. If $x \in M$, then $x^1 = x \in M \subset R$, as desired. Hence, without loss of generality, we may assume that $x \notin M$. By hypothesis, we can pick an element $a \in M \cap \mathrm{Reg}(S)$. Then, since R is an AV-ring, there exists an integer $n \geq 1$ such that either $(ax)^n$ divides a^n in R or a^n divides $(ax)^n$ in R. Since $a^n \in \mathrm{Reg}(S)$, it is clear that if a^n divides $(ax)^n$ in R, then $x^n \in R$. Thus, we may asume that $(ax)^n$ divides a^n in R. Again using $a^n \in \mathrm{Reg}(S)$, we get that x^n has a multiplicative inverse, say v, in R. If $v \in M$, then $1 = vx^n \in MS \subseteq M$, which is a contradiction (since $M \in \mathrm{Max}(R)$). Hence, $v \in R \setminus M$. Consequently, in the field D, $\overline{v} := v + M \in R/M = D$ has a multiplicative inverse, say $r + M$ for some $r \in R$. As

$$1 + M = \overline{v}(r + M) = (v + M)(r + M) = (vr + M),$$

$1 - vr \in M$. Multiplying through by x^n, we get $x^n - r = x^n - x^n vr \in x^n M \subseteq SM = M \subset R$, so that $x^n \in r + M = R$, as desired. This completes the proof that $R \subseteq S$ is a root extension. Then, since R is an AV-ring, it follows from Proposition 3.2 (a) that S is an AV-ring. This completes the proof. □

Given the hypotheses in Lemma 4.4, there is another way to prove that R being an AV-ring implies that S is an AV-ring. (This same comment will also apply at the appropriate point in the upcoming proof of Theorem 4.5.) To wit: having chosen an element $w \in M \cap \mathrm{Reg}(S)$, one shows easily that $\mathrm{Reg}(S) \cap R = \mathrm{Reg}(R)$, whence $\mathrm{tq}(R) \subseteq \mathrm{tq}(S)$ canonically, and so $s = (sw)/w \in \mathrm{tq}(R)$ for each $s \in S$. As S is then an overring of R, S inherits the "AV-ring" property from R.

Theorem 4.5. *Let $R \subseteq S$ be rings and let I be an ideal of S such that $I \in \mathrm{Spec}(R)$ and I contains a regular element of S. Put $D := R/I$, $k := \mathrm{qf}(D)$ and $E := S/I$. Suppose also that $k \subseteq E$. Then the following conditions are equivalent:*

(*1*) *R is an* AV*-ring;*

(*2*) *S and D are* AV*-rings and $k \subseteq E$ is a root extension;*

Proof. (1) ⇒ (2): Assume (1). Then, by Lemma 2.1, D is an AV-ring. Consider the pullback $T := k \times_E S \in [R, S]$. Note that $T/I = k$ and $R = D \times_k T$. As R is an AV-ring, it follows from Lemma 4.3 that T is an AV-ring. Hence, in view of the definition of T, it follows from Lemma 4.4 that S is an AV-ring and $T \subseteq S$ is a root extension. Therefore, by Lemma 4.2, $T/I \subseteq S/I$ is a root extension; that is, $k \subseteq E$ is a root extension.

(2) ⇒ (1): Assume (2). Once again, consider $T := k \times_E S$. Recall that $T/I = k$ and $S/I = E$. Hence, by Lemma 4.2, $T \subseteq S$ is a root extension. As S is an AV-ring, it therefore follows from Lemma 4.4 that T is an AV-ring. It now follows from Lemma 4.3 that R is an AV-ring, as desired. □

Proposition 4.7 (d) will give a different kind of result, where it is shown that each overring of a certain kind of AV-domain is that same kind of AV-domain. The section will close with a corollary which indicates that our earlier emphasis (as in, for instance, Example 3.3) on AV-domains with residue class fields of positive characteristic was opportune and that the theory for AV-domain pairs with residue class fields of characteristic 0 may be somewhat meager in comparison.

The relevant condition on the base domain R in Proposition 4.7 is that R is a PVD and an AV-domain. Recall from [32] that a quasi-local domain (R, M) is said to be a *pseudo-valuation domain* (in short, a PVD) if M is the maximal ideal of some valuation overring V of R. Recall also from [5, Proposition 2.5] that if (R, M) is a PVD and V is a valuation overring of R such that M is the maximal ideal of V, then V is uniquely determined (as being the conductor $(M :_{\mathrm{qf}(R)} M)$) and V is called the *canonically associated valuation overring* of (the PVD) R. Also, it was shown in [5, Proposition 2.6] that the class of PVDs consists (up to isomorphism) of the pullbacks D of the form $D = k \times_{W/N} W$ where (W, N) is a valuation domain and k is a subfield of W/N (and that W is then the canonically associated valuation overring of D). First, it is convenient to isolate the followng useful result.

Proposition 4.6. (*Nagata* [42]) *Let $K \subset L$ be fields. Then $K \subset L$ is a root extension if and only if either L is purely inseparable over K or L is algebraic over some finite field.*

One useful consequence of Proposition 4.6 is that if $K \subset L$ is a field extension and a root extension (with $K \neq L$), then these fields have positive characteristic.

The next result will characterize the PVDs that are AV-domains and it will also show that when these conditions hold for a base ring, they also hold for each of its overrings.

Proposition 4.7. *Let (R, M) be a* PVD *and let V denote the canonically associated valuation overring of R. Put $k := R/M$ and $F := V/M$. Then the following conditions are equivalent:*

(*1*) *R is an* AV*-domain;*

(*2*) *$R' = V$ and the field extension $k \subseteq F$ is a root extension;*

(*3*) *Every overring of R is a* PVD *and $(R, \mathrm{qf}(R))$ is an* AV*-domain pair;*

(*4*) *At least one of the following three conditions holds: $R = V$; F is purely inseparable over k; F is algebraic over some finite field.*

Proof. It has been noted, for a PVD, (D, N), with canonically associated valuation overring W, that D' is a valuation domain (if and) only if $D' = W$. For the sake of completeness we prove this fact next. Since D' is the intersection of all the valuation overrings of D and D' is assumed to be a valuation domain, W is then an overring of (the valuation domain) D'. Hence, there exists $P \in \mathrm{Spec}(D')$ such that $W = (D')_P$ (cf. [37, Theorem 65]). We have $\mathrm{Spec}(D) = \mathrm{Spec}(W) = \mathrm{Spec}(D')$ (with the first equality following from the assumptions on D and W and the second equality following from the lying-over and incomparable properties of integral ring extensions). By the Lying-over Theorem, $N \subseteq P$ (for otherwise, no prime ideal of W would meet D in N, a contradiction), and so $P = N$. Then $W = (D')_N = D'$, as desired.

Consider the following condition on the ambient data:
(2)$'$: R' is a valuation domain and the field extension $k \subseteq F$ is a root extension.
By combining [3, Theorem 5.6] and Lemma 4.2, we get (1) $\Leftrightarrow$ (2)$'$; and by the preceding paragraph, (2)$'$ $\Leftrightarrow$ (2). Hence, (1) $\Leftrightarrow$ (2).

Next, we claim that if $R' = V$, then each overring of R is a PVD. To prove this claim, it suffices to combine the following three general facts: each overring of a PVD, D, must be comparable under inclusion with the canonically associated valuation overring W of D; each integral overring of a PVD is a PVD (cf. [32, Theorem 1.7]); and each overring of a valuation domain is an (almost) valuation domain.

Recall that each overring of an AV-domain is an AV-domain. Therefore, by combining the claim that was established in the preceding paragraph with the already-proven equivalence (1) $\Leftrightarrow$ (2), we see that (2) $\Rightarrow$ (3). Moreover, (3) $\Rightarrow$ (1) trivially. We have now shown that (1), (2), and (3) are equivalent. Next, note that $k = F$ if and only if $R = V$. Thus, by Proposition 4.6, $k \subseteq F$ is a root extension if and only if at least one of the following three conditions holds: $R = V$; F is purely inseparable over k; F is algebraic over some finite field. Hence, (2) $\Rightarrow$ (4). Therefore, to complete the proof, it will suffice to show that (4) implies (2). Hence, by combining Proposition 4.6 with the first paragraph of this proof, it will suffice to show that if (4) holds, then V is integral over R. This implication will be proved in the next paragraph.

By an easy calculation that is part of the folklore of pullbacks (cf. [29, Corollary 1.5 (5)]), V is integral over R if and only if V/M is integral over R/M; that is, if and only if the field extension $k \subseteq F$ is algebraic. It is clear that this property holds under either of the first two possibilities mentioned in the statement of condition (4). Therefore, our task is reduced to proving that if F is algebraic over a finite field L, then F is algebraic over k. Let Λ be the prime subfield of F; that is, Λ is the unique smallest subfield of F. (If $p := \mathrm{char}(F)$, then $\Lambda \cong \mathbb{F}_p$.) It is clear that Λ is a subfield of both L and k. As $\Lambda \subseteq L \subseteq F$ and the classical theory of finite fields ensures that L is algebraic over Λ, the transitivity of algebraicity gives that F is algebraic over Λ. Therefore, since $\Lambda \subseteq k \subseteq F$, we get that F is indeed algebraic over k, thus completing the proof. $\square$

The reader may have observed that there are several ways to use familar tools from commutative ring theory or multiplicative ideal theory to organize a proof of Proposition 4.7. Remark 4.8 will collect two relevant alternate arguments for part of that proof. The result given in 4.8 (b) is perhaps of some more general interest as it does not assume that the ambient rings are domains.

Remark 4.8. (a) Another approach to proving Proposition 4.7 could have used the fact that if R is a domain, then R' is a valuation domain if and only if only if each overring of R is quasi-local. This fact can be gleaned from some characterizations of quasi-local i-domains that are due to Papick [43, Corollary 2.15 and Proposition 2.34].

(b) As noted in the proof of Proposition 4.7, a result from the folklore of pullbacks states that V is integral over R if and only if the field extension $k \subseteq F$ is algebraic. Thus, one way to prove the implication (3) $\Rightarrow$ (2) in Proposition 4.7 is to combine [3, Theorem 5.6] with the proposition that is stated in the next paragraph.

Proposition: Let $R \subseteq V$ be nonzero rings such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ (as sets). Fix a common maximal ideal M of R and V. Put $k := R/M$ and $F := V/M$. Supose also that each element of $[R, V]$ is quasi-local. Then the field extension $k \subseteq F$ is algebraic.

We will prove the contrapositive of the above assertion. Without loss of generality, $R \neq V$ (for otherwise, $k = F$, in which case, the assertion is clear). Then, by [5, Lemma 3.2], R and V are each quasi-local with (unique) maximal ideal M. Pick an element $x \in V$ such that $\overline{x} := x + M \in F$ is transcendental over k. Then the domain $\Gamma := k[\overline{x}]$ is not quasi-local. Let $\varphi : V \to F$ denote the canonical surjection. It will suffice to prove that the domain $B := \varphi^{-1}(\Gamma) \in [R, V]$ is not quasi-local. This can be established by viewing B as the pullback $\Gamma \times_F V$, applying [29, Theorem 1.4] to that pullback description of R to describe $\mathrm{Spec}(B)$ up to homeomorphism in the Zariski topology, and then using the implied description of $\mathrm{Spec}(B)$ as a poset under inclusion. The latter description shows that as a poset, $\mathrm{Spec}(B)$ can be obtained as an identification space by "gluing" $\mathrm{Spec}(\Gamma)$ on top of $\mathrm{Spec}(V)$, with $0 \in \mathrm{Spec}(\Gamma)$ being identified with $M \in \mathrm{Spec}(V)$. The upshot is that the canonical map $\mathrm{Spec}(\Gamma) \to \mathrm{Spec}(B)$ induces an order-isomorphism $\mathrm{Max}(\Gamma) \to \mathrm{Max}(B)$. Therefore, since Γ has infinitely many maximal ideals, so does B. This completes the proof and ends the remark.

We close the section with an application of Proposition 4.7 which illustrates that the class of PVDs that are AV-domains is less diverse in case their residue fields have characteristic 0. We will pursue this theme for some other classes of AV-domains in Section 7.

Corollary 4.9. *Let (R, M) be a* PVD *such that* $\mathrm{char}(R/M) = 0$. *Then R is an* AV*-domain if and only if R is a valuation domain.*

Proof. Since every valuation domain is an AV-domain, it suffices to prove the "only if" assertion. Let R be an AV-domain. Let V denote the canonically associated valuation overring of R. Since $\mathrm{char}(R/M) = 0$, it follows from condition (4) in Proposition 4.7 that $R = V$. In particular, R is a valuation domain. □

Another equivalent condition that could have been mentioned in Corollary 4.9 is "R coincides with its canonically associated valuation overring." This fact has nothing to do with characterstic or with AV-domains. Indeed, it is clear (and well known) that a PVD is a valuation domain if and only if it coincides with its canonically associated valuation overring.

5. AV-ring pairs and integral minimal ring extensions

Recall that Corollary 2.11 characterized when an integrally closed ring extension $R \subset S$ has the property that (R, S) is an AV-ring pair. It seems natural to ask if an analgous characterization is available in the case of an integral minimal ring extension $R \subset S$. Theorem 5.4 gives such a result in case the crucial maximal ideal of $R \subset S$ contains a regular element of S. Our path to Theorem 5.4 involves the next three results.

Lemma 5.1. *Let k be a field and X an indeterminate over k. Then $k \subset k[X]/(X^2)$ is a root extension if and only if* $\mathrm{char}(k) \neq 0$.

Proof. Let $\alpha := X + (X^2)$. Then $\alpha \neq 0$ and $\alpha^2 = 0$.

Suppose first that $k \subset k[X]/(X^2)$ is a root extension. Then there exists an integer $n \geq 1$ such that $(1 + \alpha)^n \in k$. Since $\alpha^2 = 0$, we have $(1 + \alpha)^n = 1 + n\alpha$. Since $1 + n\alpha \in k$ and $\{1, \alpha\}$ is a basis for $k[X]/(X^2)$ as a k-vector space, we get $n = 0 \in k$, whence $\mathrm{char}(k) \neq 0$. This completes the proof of the "only if" assertion.

For the "if" assertion, assume that $p := \mathrm{char}(k) > 0$. We will show that $\zeta^p \in k$ for each $\zeta \in k[X]/(X^2)$. Write $\zeta = a + b\alpha$ for some $a, b \in k$. Then $\zeta^p = (a + b\alpha)^p = a^p + b^p\alpha^p = a^p \in k$. Thus, $k \subset k[X]/(X^2)$ is a root extension, as desired. □

Proposition 5.2. *Let $R \subset S$ be a minimal ring extension. Then the following two conditions are equivalent:*

(*1*) *$R \subset S$ is a root extension;*

(*2*) *$k := R/(R : S)$ is a field with positive characteristic and either* (*i*) *$S/(R : S) \cong k[X]/(X^2)$ as k-algebras or* (*ii*) *$S/(R : S)$ is a purely* *inseparable field extension of k or* (*iii*) *$S/(R:S)$ is an algebraic field extension of some finite field.*

Proof. (1) $\Rightarrow$(2): Assume (1). As $R \subset S$ is an integral minimal ring extension, $(R:S)$ must be its crucial maximal ideal, by [28, Théorème 2.2]. Thus, $(R:S) \in \mathrm{Max}(R)$ and $k := R/(R:S)$ is a field. Since $k \subset S/(R:S)$ inherits the propery of being an integral minimal ring extension from $R \subset S$, the Ferrand-Olivier classification of the minimal ring extensions of a field [28, Lemme 1.2] ensures that $S/(R:S)$ is k-algebra isomorphic to one of $k \times k$, $k[X]/(X^2)$, or a minimal field extension of k. Also, Lemma 4.2 ensures that $k \subset S/(R:S)$ is a root extension. Note that $A \subset A \times A$ is not a root extension for any nonzero ring A. Then (2) follows by combining Proposition 4.6 and Lemma 5.1.

(2) $\Rightarrow$ (1): Assume (2). Then, again combining Proposition 4.6 and Lemma 5.1, we get that $R/(R:S) \subset S/(R:S)$ is a root extension. Hence, by Lemma 4.2, $R \subset S$ is a root extension, thus completing the proof. $\square$

Recall that if $R \subset S$ is a minimal ring extension, then either R is integrally closed in S or S is integral over R. Hence, by combining Proposition 5.2 with some of the above comments, one easily gets the following result.

Corollary 5.3. *Let $R \subset S$ be a minimal ring extension. Put $k := R/(R:S)$. Then the following two conditions are equivalent:*

(*1*) *$R \subset S$ is not a root extension;*

(*2*) *(Exactly) one of the following four conditions holds:*

(*a*) *R is integrally closed in S;*

(*b*) *k is a field and $S/(R:S) \cong k \times k$ as k-algebras ;*

(*c*) *k is a field of characteristic 0 and either $S/(R:S) \cong k[X]/(X^2)$ as k-algebras or $S/(R:S)$ is a field* (*in fact, a minimal field extension of k*)*;*

(*d*) *k is a field of positive characteristic and $S/(R:S)$ is a field that is neither a purely inseparable field extension of k nor an algebraic field extension of a finite field.*

Theorem 5.4. *Let $R \subset S$ be an integral minimal ring extension such that $(R:S)$ contains an element of $\mathrm{Reg}(S)$. Put $k := R/(R:S)$ and $E := S/(R:S)$. Then the following conditions are equivalent:*

(*1*) *(R,S) is an* AV*-ring pair;*

(*2*) *R is an* AV*-ring;*

(*3*) *S is an* AV*-ring and $k \subset E$ is a root extension;*

(*4*) *S is an* AV*-ring,* $\mathrm{char}(k) > 0$*, and either* (*i*) *$k \subset E$ is a purely* *inseparable field extension or* (*ii*) *E is an algebraic field extension of some finite field or* (*iii*) *$E \cong k[X]/(X^2)$ as k-algebras.*

Proof. The equivalence (2) $\Leftrightarrow$ (3) follows from Theorem 4.5. As minimality of the ring extension $R \subset S$ shows that (1) is equivalent to R and S being AV-rings, it is now clear that (3) $\Rightarrow$ (1) $\Rightarrow$ (2). Finally, Propositions 5.2 and 4.6 can be combined to prove that (3) $\Leftrightarrow$ (4). $\square$

We next isolate the application to the domain-theoretic context.

Corollary 5.5. *Let $R \subset S$ be an integral minimal ring extension such that S is a domain. Put $k := R/(R : S)$ and $E := S/(R : S)$. Then the following conditions are equivalent:*

(*1*) *(R, S) is an* AV*-domain pair;*

(*2*) *R is an* AV*-domain;*

(*3*) *S is an* AV*-domain and $k \subset E$ is a root extension;*

(*4*) *S is an* AV*-domain,* $\mathrm{char}(k) > 0$*, and either $k \subset E$ is a purely inseparable field extension or E is an algebraic field extension of some finite field.*

We close this section with an application of many of the above results. First, recall that a ring extension $R \subseteq S$ is said to satisfy FCP (the *finite chain property*) if, when $[R, S]$ is viewed as a poset under inclusion, each chain in $[R, S]$ is finite.

Theorem 5.6. *Let (R, S) be an* AV*-ring pair such that $R \subseteq \overline{R}_S$ satisfies* FCP*. Then $[R, S] = [R, \overline{R}_S] \cup [\overline{R}_S, S]$*

Proof. Of course, $[R, \overline{R}_S] \cup [\overline{R}_S, S] \subseteq [R, S]$. We turn to a proof of the reverse inclusion. Without loss of generality, $R \neq \overline{R}_S$, that is, $R \subset \overline{R}_S$. We will show that if $T \in [R, S]$, then either $T \in [\overline{R}_S, S]$ or $T \in [R, \overline{R}_S]$. Consider $A := \overline{R}_S \cap T$. As $A \in [R, S]$, the hypotheses ensure that A is an AV-ring and, hence, is quasi-local. Let M denote the (unique) maximal ideal of A. There are some cases to consider. If $A = \overline{R}_S$, then $\overline{R}_S \subseteq T$, and so $T \in [\overline{R}_S, S]$. Similarly, if $A = T$, it is clear that $T \in [R, \overline{R}_S]$. Thus, we may assume henceforth that $A \neq \overline{R}_S$ and $A \neq T$.

As (R, S) is an AV-ring pair, $R \subset S$ is a P-extension by Proposition 2.2. Then, *a fortiori*, $A \subset T$ is a P-extension. As A is integrally closed in T, it follows that (A, T) is a normal pair. Therefore, by [25, Theorem 6.8], $T = A_Q$ for some (necessarily non-maximal) prime ideal Q of A. As $A \subset \overline{R}_S$ inherits the FCP from $R \subset S$, there exists a finite maximal chain $A_0 = A \subset A_1 \subset \ldots \subset A_{n-1} \subset A_n = \overline{R}_S$ of rings in $[A, \overline{R}_S]$, for some integer $n \geq 1$. Since $A \subset \overline{R}_S$ is an integral extension, [28, Théorème 2.2 (ii)] ensures that $(A_{i-1} : A_i)$ is the crucial maximal ideal of the integral minimal ring extension $A_{i-1} \subset A_i$, for each $i \in \{1, 2, ..., n\}$. As maximal ideals must lie over maximal ideals in any integral extension, we get $(A_{i-1} : A_i) \cap A = M$ for each i, and so $MA_i \subseteq (A_{i-1} : A_i)A_i \subseteq A_{i-1}$. By iteration, we obtain $M^n\overline{R}_S \subseteq M^{n-1}A_{n-1} \subseteq \ldots \subseteq M^{n-n}A_0 = A$. Thus, $M^n \subseteq (A : \overline{R}_S)$. Hence, since Q does not contain M (and M is a prime ideal of A), it follows that $(A : \overline{R}_S) \nsubseteq Q$. Therefore, by an easy calculation, $A_Q = (\overline{R}_S)_Q$. Note that the canonical map $\overline{R}_S \to (\overline{R}_S)_Q$ must be an injection (for when it is composed with the inclusion map $(\overline{R}_S)_Q = A_Q = T \hookrightarrow S$, the composite map is the inclusion map $\overline{R}_S \hookrightarrow S$). Viewing this injection as an inclusion, we have $\overline{R}_S \subseteq (\overline{R}_S)_Q = T$, and so $T \in [\overline{R}_S, S]$, thus completing the proof. □

6. Maximal non-almost valuation subrings

Let $R \subset S$ be (distinct) domains. Recall from [7, Definition 3.8] that R is said to be *a maximal non-almost valuation subring of* S (in short, a *maximal non-*AV *subring of* S) if R is not an AV-domain while each subring of S that properly contains R is an AV-domain. In [7, Theorem 3.9], Ouled Azaiez and Moutui characterized when R is a maximal non-AV-subring of its quotient field. Here, we deal with the general case, that is, when the given domain S that contains R need not be an overring of R. We begin, in Theorem 6.1, by considering the subcase where R is integrally closed in S. First, recall that if $x \leq y$ in a poset $(\mathcal{P}, \leq)$, then $[x, y[$ denotes $\{z \in \mathcal{P} \mid x \leq y < z\}$.

Theorem 6.1. *Let $R \subset S$ be domains such that R is integrally closed in S. Then the following three conditions are equivalent:*

(*1*) *R is a maximal non-*AV *subring of S;*

(*2*) *R is a maximal non-quasi-local subring of S and S is an* AV-*domain;*

(*3*) *(R, S) is a normal pair, S is an* AV-*domain, R is a semi-quasi-local domain with exactly two distinct maximal ideals M and N, and* (*exactly*) *one of the following three conditions holds:*
(*i*) *$S = R_M$ and $[0, N[\subseteq [0, M[$,*
(*ii*) *$S = R_N$ and $[0, M[\subseteq [0, N[$,*
(*iii*) *There exists $Q \in \mathrm{Spec}(R)$ such that $Q \subset M \cap N$, $S = R_Q$ and $[0, M[= [0, N[$.*

Proof. (1) ⇒ (2): Assume (1). Then, by Proposition 2.2, (R, S) is a normal pair. Since all AV-domains are quasi-local, it will suffice to prove that R is not quasi-local. Suppose that this assertion fails. As (R, S) is a normal pair and (1) ensures that R is quasi-local and that

S is an AV-domain, Theorem 2.6 gives that R is an AV-domain, the desired contradiction (to (1)).

(2) $\Rightarrow$ (3): This implication is an immediate consequence of [35, Theorem 1].

(3) $\Rightarrow$ (1): Assume (3). Then R is not quasi-local and, hence, R is not an AV-domain. Thus, our task is to show that if T is a ring such that $R \subset T \subseteq S$, then T is an AV-domain. Under the hypotheses, R is a maximal non-quasi-local subring of S (cf. [35, Theorem 1]). In particular, T is a quasi-local domain. As it is also the case that (T, S) is a normal pair and S is an AV-domain, another appeal to Theorem 2.6 ensures that T is an AV-domain. The proof is complete. □

The following corollary recovers [7, Theorem 3.9].

Corollary 6.2. *Let R be an integrally closed domain, with $K := \mathrm{qf}(R)$. Then the following conditions are equivalent:*

(*1*) *R is a maximal non-*AV *subring of K;*

(*2*) *R is a maximal non-quasi-local subring of K;*

(*3*) *R is a semi-quasi-local Prüfer domain with exactly two distinct maximal ideals M and N, and $[0, M[= [0, N[$;*

(*4*) *R is integrally closed and R is a maximal non-valuation subring of K.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 6.1, while the equivalences (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) follow from [35, Corollary 1]. □

Theorem 6.3 shows how the above information can be sharpened if one assumes that S is not a field.

Theorem 6.3. *Let $R \subset S$ be domains such that R is integrally closed in S and S is not a field. Then the following conditions are equivalent:*

(*1*) *R is a maximal non-*AV *subring of S;*

(*2*) *(Exactly) one of the following two conditions holds:*

(*i*) *S is an* AV*-domain with maximal ideal M and there exists a semi-quasi-local Prüfer domain D such that $\mathrm{qf}(D) = S/M$, D has exactly two distinct maximal ideals m and m', $[0, m[= [0, m'[$, and $R := D \times_{S/M} S$;*

(*ii*) *S is an* AV*-domain, R is not quasi-local, and $R \subset S$ is an* (*integrally closed*) *minimal ring extension.*

Proof. (1) $\Rightarrow$ (2): This implication follows by combining Theorem 6.1 and [35, Theorem 2].

(2) $\Rightarrow$ (1): Assume (2). We will show that if T is any ring such that $R \subset T \subseteq S$, then T is an AV-domain. This *desideratum* clearly

holds when (ii) holds (for the minimality of $R \subset S$ would ensure that $T = S$). Thus, we may assume, without loss of generality, that (i) holds. Then, by applying [29, Theorem 1.4] to the pullback defining R and reasoning as in the proof of the proposition in Remark 4.8 (b), we see that the canonical map $\mathrm{Spec}(D) \to \mathrm{Spec}(R)$ induces an order-isomorphism $\mathrm{Max}(D) \to \mathrm{Max}(R)$. Consequently, R is not quasi-local, and so R is not an AV-domain. Put $H := T/M$. Then H is a domain such that $D \subset H \subseteq S/M = \mathrm{qf}(D)$ and $T = H \times_{S/M} S$. Also, since [35, Corollary 1] ensures that D is a maximal non-(almost) valuation subring of its quotient field, H must be an (almost) valuation domain. Hence, by Theorem 4.1 (or Theorem 4.5), T is an AV-domain. The proof is complete. □

The next result characterizes the subcase of the general case where the given ring extension $R \subset S$ is not integrally closed.

Theorem 6.4. *Let $R \subset S$ be domains such that R is not integrally closed in S. Then the following conditions are equivalent:*

*(1) R is a maximal non-*AV *subring of S;*

*(2) R is a maximal non-*AV *subring of $\overline{R}_S$, $R \subset S$ is an algebraic extension, and R_Q is an* AV*-domain for each non-maximal prime ideal Q of R such that $R_Q \subseteq S$.*

Proof. (1) ⇒ (2): Assume (1). Then by Proposition 2.2, $R \subset S$ is a P-extension and, hence, an algebraic extension. Our task is now to show that if Q is any non-maximal prime ideal of R such that $R_Q \subseteq S$, then R_Q is an AV-domain. By (1), it will suffice to prove that $R \subset R_Q \subseteq S$. This, in turn, follows since the first of these desired inclusions is indeed proper (because $Q \notin \mathrm{Max}(R)$) and the second of these inclusions has been assumed.

(2) ⇒ (1): Assume (2). The assertion becomes trivial if $\overline{R}_S = S$, and so we may assume, without loss of generality, that $\overline{R}_S \subset S$. The hypotheses ensure that S is an overring of $\overline{R}_S$ (cf. [37, Exercise 35, page 44]). Therefore, since $\overline{R}_S$ is an AV-domain, it follows that $(\overline{R}_S, S)$ is an AV-domain pair. Moreover, in view of integrality and Proposition 2.2, both $R \subset \overline{R}_S$ and $\overline{R}_S \subset S$ are P-extensions. Consequently, $R \subset S$ is also a P-extension. It remains to show that if T is any ring such that $R \subset T \subseteq S$, then T is an AV-domain.

Without loss of generality, $T \nsubseteq \overline{R}_S$ and $\overline{R}_S \nsubseteq T$. Consider the ring $A := \overline{R}_S \cap T$. If $A \neq R$, then A is an AV-domain and so T is also an AV-domain (the point being that T is an overring of A, as a consequence of the facts that $A \subset T$ is an algebraic extension and A is integrally closed in T). Hence, without loss of generality, $A = R$. Then, since R

is integrally closed in T and $R \subseteq T$ is a P-extension, (R,T) is a normal pair. Thus, by [25, Theorem 6.8], $T = R_Q$ for some $Q \in \mathrm{Spec}(R)$ such that $QR_Q = Q$. Note that $Q \notin \mathrm{Max}(R)$ (for otherwise, since Q is a divided prime ideal of R, Q would be the unique maximal ideal of R, so that $T = R_Q = R$, a contradiction). Hence, by (2), T is an AV-domain, thus completing the proof. □

The following immediate consequence of Theorem 6.4 gives a companion for [7, Theorem 3.10].

Corollary 6.5. *Let R be a domain that is not integrally closed, and put $K := \mathrm{qf}(R)$. Then the following conditions are equivalent:*

(*1*) *R is a maximal non-*AV *subring of K;*

(*2*) *R is a maximal non-*AV *subring of R' and R_Q is an* AV*-domain for each non-maximal prime ideal Q of R.*

7. Connections with locally divided domains

Since AV-domains are defined by a condition that involves divisibility, it is natural to ask if an AV-domain must be a divided domain. While this question will be answered in the negative in Example 7.4 below, we will obtain a positive answer for the case of AV-domains containing $\mathbb{Q}$ as a subring (see Proposition 7.6). What *can* be shown without any assumption involving characteristic 0 (see Proposition 7.1) is that an AV-domain must be a going-down domain. To motivate the relevance of Proposition 7.1, recall that any divided domain is a quasi-local going-down domain [15, Proposition 2.1] and that the converse is true in the seminormal case [15, Corollary 2.6].

Proposition 7.1. *Let R be an* AV*-ring* (*resp., an* AV*-domain*)*. Then R is a going-down ring* (*resp., a going-down domain*)*.*

Proof. A domain is an AV-ring (resp., a going-down ring) if and only if it is an AV-domain (resp., a going-down domain). So, it suffices to prove that every AV-ring is a going-down ring. More, in fact, is true, as it was shown in [18, Proposition 2.27] that any pseudo-almost valuation ring (in the sense of [33]) is a going-down ring. □

Recall from [3] that a domain R is called an *almost Prüfer domain* (in short, an AP-*domain*) if, whenever $\{a,b\} \subseteq R \setminus \{0\}$, there exists an integer $n \geq 1$ such that (a^n, b^n) is an invertible ideal of R. It was shown in [3, Theorem 5.8] that a domain R is an AP-domain if and only if R_M is an AV-domain for each maximal ideal M of R. In particular, the AV-domains are the same as the quasi-local AP-domains. As it has often been noted in the literature that the property of being a going-down domain is a local property of domains, Proposition 7.1 has the following immediate consequence.

Corollary 7.2. *Every* AP-*domain is a going-down domain.*

Remark 7.3. (a) The converse of Corollary 7.2 is false, even in the quasi-local case. In other words, a quasi-local going-down domain need not be an AV-domain. (Thus, each of the two assertions in the statement of Proposition 7.1 has a false converse.) To see this, recall that a domain R is an AV-domain if and only if R' is a valuation domain and $R \subseteq R'$ is a root extension [3, Theorem 5.6]. Thus, since each domain of Krull dimension 1 is a going-down domain, it suffices to find an integrally closed quasi-local domain R of Krull dimension 1 which is not a valuation domain. As one of the first published applications of the classical $D+M$ construction, Krull produced such a domain R approximately 75 years ago: cf. the ring $D_1 := F + YF(X)[[Y]]$ in [30, Exercise 14, page 203]. In case the field F has characteristic 0, one can obtain another proof that D_1 is not an AV-domain by noting that D_1 is a PVD (regardless of whether $\mathrm{char}(F) = 0$) and then applying Proposition 5.7 (a).

(b) If one wants only a proof that every AV-domain is a going-down domain (and hence to obtain Corollary 7.2 as well), there is no need to consider rings with zero-divisors as in the above proof of Proposition 7.1. Indeed, there are several domain-theoretic proofs that every AV-domain is a going-down domain. In the next paragraph, we will sketch three such proofs. They all use the result that Prüfer domains constitute the best-known class of going-down domains. Recall that the definition of SGD ("simple going-down") rings in [13] was modeled after Richman's characterization of Prüfer domains. As facts about Prüfer domains will be used to motivate some of the rest of this paper, we believe that it is appropriate to pause to quote two sentences from the introduction to [14]. "A word about the propriety of motivating questions about going-down [domains] by results about Prüfer domains seems to be in order. The characterization of going-down [domains] in [23, Theorem 1] was in fact motivated by characterizations of Prüfer domains involving flatness (cf. [46, Theorem 4] and [13, Proposition 3.1])."

We next give three domain-theoretic proofs that if R is an AP-domain, then R is a going-down domain. By the comments preceding Proposition 7.2, we may assume (in each of the three proofs) that R is an AV-domain. Then, since R' is a valuation domain, one can conclude the first of the domain-theoretic proofs by citing [14, Corollary 2.5].

(Note that the cited result followed easily from a descent result [14, Theorem 2.4] because valuation domains, being the quasi-local Prüfer domains, are going-down domains.) For the second of the proofs, use the lying-over and incomparable properties of integral extensions (cf. [37, Theorem 44]) and the fact that $R \subseteq R'$ is a root extension to see that the canonical map $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is a bijection (that is, that the extension $R \subseteq R'$ is "unibranched," in the sense of [15]) and then conclude the second proof by citing [15, Lemma 2.3] (which applies because R', being a valuation domain, must also be a going-down domain). The third proof is really a reformulation of the second proof, as it proceeds by applying Papick's generalization of [14, Theorem 2.4] that was given in [43, Proposition 2.14], the point being that $R \subseteq R'$ is an i-extension (in the sense that $\mathrm{Spec}(R') \to \mathrm{Spec}(R)$ is an injection) and R' (being a valuation domain) is a Prüfer domain. This concludes the third proof.

(c) Since every going-down domain is a treed domain [13, Theorem 2.2] and the property of being a treed domain is a local property of domains, it follows from Corollary 7.2 that every AP-domain is a treed domain and, hence, that the set of prime ideals of any AV-domain is linearly ordered by inclusion. However, more is already known, as it was shown in [34, Proposition 2.2] that the set of prime ideals of any AV-ring are linearly ordered by inclusion.

(d) The converse of Proposition 7.1 is false. To see this, first recall that [17, Example 1] gives an example of a going-down ring R with a maximal ideal that contains two incomparable prime ideals. So, by localizing R at that maximal ideal, one obtains a quasi-local going-down ring (by [17, Proposition 2.1 (b)]) which cannot be an AV-ring (because its set of prime ideals is not linearly ordered by inclusion). This completes this remark.

Since all seminormal going-down domains (for instance, all Prüfer domains) are locally divided domains, Corollary 7.2 may lead one to ask whether all AP-domains are locally divided domains. Example 7.4 will show that the answer is in the negative, even in the quasi-local case.

Example 7.4. *There exists an* AP*-domain R which is not a locally divided domain. No such domain R can be seminormal. It can, however, be arranged that R is quasi-local, so that R is then an example of an* AV*-domain which is not a divided domain.*

Proof. Consider the ring B that was constructed in [24, Example 4.5]. Recall that $B :=\mathrm{Int}(A)$, the ring of integer valued polynomials on A, where A is any Noetherian local domain of Krull dimension 1 which has a finite residue field and is unibranched but is not analytically irreducible. It was shown in the first paragraph of the proof of [24, Example 4.5] that B is not a locally divided domain. Hence, we can pick a maximal ideal N of B such that B_N is not a divided domain. It remains only to prove that B is an AP-domain. (Indeed, it will then follow that B_N is an AV-domain [3, Theorem 5.8]; moreover, the assertion about seminormality would, after reducing the issue to the local case, be a consequence of combining Corollary 7.2 with [15, Corollary 2.6].) To that end, [3, Corollary 4.8 (2)] reduces our task is showing that $B \subseteq B'$ is a root extension and B' is a Prüfer domain. These two *desiderata* were established in the second paragraph of the proof of [24, Example 4.5], and so the proof is complete. $\square$

Despite Example 7.4, we will show in Proposition 7.6 that, in the spirit of Corollary 4.9, a positive result is available when one adds an assumption involving characteristic 0. The next result will serve to shape that assumption. Lemma 7.5 may already be known, but its proof is included for lack of a convenient reference.

Lemma 7.5. *Let R be a domain. Then the following conditions are equivalent:*
(*1*) *R contains* (*an isomorphic copy of*) *$\mathbb{Q}$ as a subring;*
(*2*) $\mathrm{char}(R/M) = 0$ *for all maximal ideals M of R;*
(*3*) $\mathrm{char}(R/P) = 0$ *for all prime ideals P of R.*

Proof. Suppose first that $\mathbb{Q} \subseteq R$ but $\mathrm{char}(R/P) = p \neq 0$ for some prime ideal P of R. Then p is in the kernel of $\mathbb{Z} \hookrightarrow R \to R/P$ (where we have composed the inclusion map with the canonical projection). So, p is in the kernel of $\mathbb{Q} \hookrightarrow R \to R/P \hookrightarrow \mathrm{qf}(R/P)$. But this ring homomorphism of fields is necessarily injective and $p \neq 0$, the desired contradiction.

Next, suppose that $\mathrm{char}(R/M) = 0$ for all maximal ideals M of R but R does not contain an isomorphic copy of $\mathbb{Q}$. As $\mathrm{char}(R) = 0$, we can view $\mathbb{Z}$ as a subring of R. Then some prime number p is not a unit of R. So, p (that is $p \cdot 1_R$) is an element of some maximal ideal M of R. Thus, $p + M = 0 \in R/M$. Hence, char(R/M)=p, the desired contradiction. This completes the proof. $\square$

Proposition 7.6. *Let R be an* AP*-domain* (*resp., an* AV*-domain*) *containing* (*an isomorphic copy of*) *$\mathbb{Q}$ as a subring. Then R is a locally divided domain* (*resp., a divided domain*).

Proof. By [15, Lemma 2.2 (a)], the quasi-local locally divided domains are the same as the divided domains. Also, we can view $\mathbb{Q} \subseteq R \subseteq R_M$

for all maximal ideals M of R. Thus, in view of [3, Theorem 5.8], we can assume that (R, M) is quasi-local, that is, an AV-domain, and our task is to show that R is a divided domain. By [3, Theorem 5.6], $V := R'$ is a valuation domain and $R \subseteq V$ is a root extension. Also, by Proposition 7.1 (or Corollary 7.2), R is a going-down domain.

Suppose that the assertion fails. Then, by [15, Corollary 2.6], there exist a prime ideal P of R and an element $w \in PR_P \setminus P$ such that w^2, w^3 are elements of R (and, hence, of P). Note that $w \in R' = V$ and $w \notin R$. Consider the element $\xi := 1 + w \in V$. As $R \subseteq V$ is a root extension, there exists an integer $n \geq 1$ such that $\xi^n \in R$. In fact, $n > 1$ (for otherwise, $1 + w \in R$, whence $w \in R$, a contradiction). Thus, by using the binomial theorem to express the fact that $(1+w)^n = \xi^n \in R$, we have

$$1 + nw + \sum_{i=2}^{n} r_i w^i = \xi^n \in R,$$

for some elements r_i (which are the canonical images of the binomial coefficients $\binom{n}{i}$ for $i = 2, \ldots, n$) of R. Since $w^2, w^3, \ldots, w^n \in R$, it follows that $nw \in R$. As $n^{-1} \in \mathbb{Q} \subseteq R$, we get that $w = n^{-1}(nw) \in RR = R$, the desired contradiction. The proof is complete. $\square$

Recall from [19] that a domain R is called a *locally pseudo-valuation domain* (in short, an LPVD) if R_M is a PVD for each maximal ideal M of R. Every Prüfer domain is an LPVD. Every LPVD is a seminormal locally divided domain, by [19, Corollary 2.3 and Remarks 2.4 (a)]. A sufficient condition for an LPVD to be a Prüfer domain was given in [19, Remark 2.11 (a)]. The following companion for that result can be obtained by combining Lemma 7.5 and Corollary 5.8: if a domain R contains an isomorphic copy of $\mathbb{Q}$ as a subring, then R is a Prüfer domain if and only if R is an AP-domain and an LPVD. The most important use of Lemma 7.5 for the rest of this paper will be to use its condition (1) as embodying the spirit of the "characteristic" hypothesis from Corollary 5.8 as we consider domains that may not be quasi-local.

Recall that the proof of Example 7.4 made heavy use of the proof of [24, Example 4.5]. A glance at the latter proof reveals that it made heavy use of the theory of universally going-down domains. Recall from [21] that a domain R is said to be a *universally going-down domain* if, for each overring T of R and each commutative R-algebra S, the canonical homomorphism $S \to S \otimes_R T$ satisfies GD. Any universally going-down domain is a going-down domain, but the converse is false. By using [20, Corollary 2.3], we see that a domain R is a universally going-down domain if and only if, for each overring T of R and each nonempty finite set $\{X_1, \ldots, X_n\}$ of commuting, algebraically independent indeterminates over R, the extension of polynomial rings $R[X_1, \dots, X_n] \subseteq T[X_1, \dots, X_n]$ satisfies GD. According to [21, Corollary 2.3], the integrally closed universally going-down domains are the Prüfer domains. However, as was shown in [21, Remark 2.5 (a)], not all universally going-down domains are integrally closed. For the proof of Example 7.7, the most important characterization of universally going-down domains is the following [21, Theorem 2.4]: a domain R is a universally going-down domain if and only if R' is a Prüfer domain and R' is the weak normalization R^* of R (inside R'). The definition of the weak normalization R^* of a domain R will be recalled from [20, page 421] as needed during the proof of Example 7.7.

As we move now from considerations of locally divided domains to considerations of universally going-down domains, the next two results may be viewed as analogues of Example 7.4 and Proposition 7.6.

Example 7.7. *There exists a domain R which is an* AP*-domain and an LPVD* (*hence, a locally divided domain*) *but is not a universally going-down domain. It can be arranged that R is quasi-local of Krull dimension* 1 *and* $\mathrm{char}(R) = 2$*, so that R is then an example of a one-dimensional domain of characteristic* 2 *which is a* AV*-domain and a PVD* (*hence a divided domain*) *but is not a universally going-down domain.*

Proof. One way to construct such a domain R is the following. Let $k := \mathbb{F}_2$. Inside a fixed algebraic closure F of k, let v be a primitive fifth root of unity. (Such v exists since 5 is relatively prime to 2: see [39, page 203]. It will become clear that we could replace 5 with any other odd prime number in the present construction.) Consider the field $L := k(v) \subseteq F$, and let X be an analytic indeterminate over L. Put $V := L[[X]]$. Then V is a DVR, in particular a valuation domain, with maximal ideal $M := XV$. We will show that $R := k + M$ has the asserted properties.

Of course, R is a PVD [30, Exercise 12 (4), page 203] (and hence an LPVD), of Krull dimension 1 [32, Example 2.1], and of characteristic 2. Moreover, since L is algebraic over the finite field k, it follows from the result of Nagata that was stated in Proposition 5.6 that $k \subset L$ is a root extension. Consequently, the field extension $R/M \subset V/M$ (which can be identified with $k \subset L$) is a root extension. An easy calculation then shows that $R \subset V$ is a root extension (and hence an integral extension). It follows that $R' = V$ since V, being a valuation domain, is integrally closed. (This can also be seen via [30, Exercise 11 (2), page 202], since L is algebraic over k.) Therefore, by [3, Theorem 5.6], R is an AV-domain (and hence an AP-domain).

It remains only to prove that R is not a universally going-down domain. As $R' = V$ is a valuation, hence Prüfer, domain, it follows from [21, Theorem 2.4] that, in order to prove that R is not a universally going-down domain, we need only show that V is not the weak normalization R^* of R (inside V). The construction of R^* in [20, page 421] shows, since M is the only nonzero prime ideal of R, that $R^* = \{u \in V \mid u^{2^n} \in R + M (= R)$ for some integer $n \geq 1\}$. (In applying that definition, we have used that the Jacobson radical of V_M is M.) Notice that v is a unit of V (since $v^5 = 1$). To obtain a contradiction from an assumption that $R^* = V$, argue as follows. Since $v \in R^*$ and v is a unit of V, it follows that there exists an integer $n \geq 1$ such that $v^{2^n} = 1 + e$ for some $e \in M$. As $v^{2^n}, 1 \in L$ and $V = L \oplus M$ is an additive internal direct sum, we get $v^{2^n} = 1$ (and $0 = e$). Since the order of v in the multiplicative subgroup consisting of the nonzero elements of F is 5, we get (by a familiar argument with the division algorithm) that 5 is an integral divisor of 2^n, which contradicts the Fundamental Theorem of Arithmetic. The proof is complete. $\square$

Proposition 7.8. *Let R be an* AP*-domain* (*for instance, an* AV*-domain*) *containing* (*an isomorphic copy of*) $\mathbb{Q}$ *as a subring. Then R is a universally going-down domain.*

Proof. Prior to the statement of Example 7.7, we noted a role played by polynomial rings in many variables in a characterization of universally going-down domains. It is straightforward to use that characterization to show that the property of being a universally going-down domain is a local property of domains. Therefore, by arguing as in the second and third sentences of the proof of Proposition 7.6, we can assume, without loss of generality, that R is quasi-local, that is, an AV-domain, with $V := R'$ being a valuation domain and $R \subseteq V$ being a root extension.

As $R' = V$ is a valuation, hence Prüfer, domain, it follows from [21, Theorem 2.4] that, in order to prove that R is a universally going-down domain, we need only show that V is the weak normalization R^* of R (inside V). As $\mathbb{Q} \subseteq R$, Lemma 7.5 gives that $\mathrm{char}(R/P) = 0$ for all $P \in \mathrm{Spec}(R)$. So, by a comment in [20, page 420] (and the definitions of the weak normalization and the seminormalization in [20, page 421]), that weak normalization is the same as the corresponding seminormalization. So, if $P \in \mathrm{Spec}(R)$ is lain over by (only) $\mathfrak{P} \in \mathrm{Spec}(V)$, we need only prove that $\mathrm{qf}(R/P) = \mathrm{qf}(V/\mathfrak{P})$.

Since $R \subseteq V$ is a root extension, so is $R+\mathfrak{P} \subseteq V$, hence so is $R/P \cong (R+\mathfrak{P})/\mathfrak{P} \subseteq V/\mathfrak{P}$; hence, by Lemma 3.1, so is $\mathrm{qf}(R/P) \subseteq \mathrm{qf}(V/\mathfrak{P})$. As these are fields of characteristic 0, the result of Nagata that was stated in Proposition 5.6 shows that these two fields are the same. The proof is complete. □

Prior to the statement of Example 7.7, we noted a role played by polynomial rings in many variables in a characterization of universally going-down domains. That brings to mind the following notion. Recall that a (not necessarily Noetherian) domain R is said to be a *universally catenarian domain* if, whenever $\{X_1, \ldots, X_n\}$ is a finite set of algebraically independent indeterminates over R and $P \subseteq Q$ are prime ideals of $R[X_1, \ldots, X_n]$, then all maximal chains of prime ideals of $R[X_1, \ldots, X_n]$ going from P to Q have the same finite length. It is clear that each universally catenarian domain is locally finite-dimensional (in short, LFD), in the sense that each of its prime ideals has finite height (cf. [9, page 212]). It is well known that the converse is false (that is, an LFD domain need not be universally caterarian), even in the Noetherian case. However, as a consequence of a celebrated result of Ratliff [45, (2.6)], any Noetherian domain of Krull dimension at most 1 must be universally catenarian. That result was generalized in [9, Theorem 6.2], which is a result on going-down domains with Prüferian integral closure that will play the key role in proving the final result of this section. A special case (and a precursor) of this tool is the fact that any LFD Prüfer domain must be a universally catenarian domain ([40, page 256], [10, Theorem 12]).

We show next that, unlike the situation for properties of "locally divided domain" and "universally going-down domain," the study of the property of "universally catenarian domain" within the universe of AP-domains does not require a separate treatment for the case of characteristic 0.

Corollary 7.9. *Let R be a going-down domain such that R' is a Prüfer domain (for instance, let R be an* AP*-domain; for instance, let R be an* AV*-domain). Then the following conditions are equivalent:*

(*1*) *R is a universally catenarian domain;*
(*2*) *R' is a universally catenarian domain;*
(*3*) *R' is* LFD*;*
(*4*) *R is* LFD.

Proof. The first "for instance" assertion follows from [3, Corollary 4.8 (2)] and Corollary 7.2; its subcase for a quasi-local R is the second "for instance" assertion. Given the hypothesis that R' is a Prüfer domain (and the hypothesis that R is a going-down domain R), the equivalence of (1), (2), and (3) follows as a special case of (some of the equivalences established in) [9, Theorem 6.2]. As noted above, the implication (1) $\Rightarrow$

(4) holds for any domain R. Finally, one easily obtains the implication (4) $\Rightarrow$ (3) for any domain R by using the incomparable property of the integral extension $R \subseteq R'$. The proof is complete. $\square$

References

[1] D. D. Anderson, S. Xing and M. Zafrullah, Almost discrete valuation domains, preprint.
[2] D. D. Anderson, K. R. Knopp and R. L. Lewin, Almost Bézout domains, II, J. Algebra **167** (1994) 547–556.
[3] D. D. Anderson and M. Zafrullah, Almost Bézout domains, J. Algebra **142** (2) (1991), 285–309.
[4] D. D. Anderson and M. Zafrullah, Almost Bézout domains, III, Bull. Math. Soc. Math. Roumanie **51** (99) (1) (2008), 3–9.
[5] D. F. Anderson and D. E. Dobbs, Pairs of rings with the same prime ideals, Canadian J. Math. **32** (2) (1980) 362–384.
[6] A. Ayache and A. Jaballah, Residually algebraic pairs of rings, Math. Z. **225** (1997), 49–65.
[7] N. Ouled Azaiez and M. A. Moutui, Almost valuation property in bi-amalgamations and pairs of rings, J. Algebra Appl. **11** (6) (2019), 1950104, 14 pp.
[8] M. Ben Nasr and N. Jarboui, New results about normal pairs of rings with zero-divisors, Ric. Mat. **63** (1) (2014), 149–155.
[9] A. Bouvier, D. E. Dobbs and M. Fontana, Universally catenarian integral domains, Adv. in Math. **72** (2) (1988), 211–238.
[10] A. Bouvier and M. Fontana, The catenarian property of the polynomial rings over a Prüfer domain, pp. 340–354, in "Séminaire d'A1gèbre Paul Dubreil et Marie-Paule Malliavin," Lecture Notes Math. **1146**, Springer, Berlin/New York, 1985.
[11] R. D. Chatham and D. E. Dobbs, On open ring pairs of commutative rings, Houston J. Math. Bull. **31** (1) (2005), 65–74.
[12] E. D. Davis, Overrings of commutative rings III: Normal pairs, Trans. Amer. Math. Soc. **182** (1973), 175–185.
[13] D. E. Dobbs, On going down for simple overrings, II, Comm. Algebra **1** (1974), 439–458.
[14] D. E. Dobbs, Ascent and descent of going-down rings for integral extensions, Austral. Bull. Math. **15** (2) (1976), 253–264.
[15] D. E. Dobbs, Divided rings and going-down, Pacific J. Math. **67** (1976), 353–363.
[16] D. E. Dobbs, On INC-extensions and polynomials with unit content, Can. Math. Bull. **23** (1980), 37–42.
[17] D. E. Dobbs, Going-down rings with zero-divisors, Houston J. Math. **23** (1) (1997), 1–12.
[18] D. E. Dobbs, A. El Khalfi and N. Mahdou, Trivial extensions satisfying certain valuation-like properties, Comm. Algebra **47** (5) (2019), 2060-2077.
[19] D. E. Dobbs and M. Fontana, Locally pseudo-valuation domains, Ann. Mat. Pura Appl. **134** (1983), 147-168.

[20] D. E. Dobbs and M. Fontana, Universally going-down homomorphisms of commutative rings, J. Algebra **90** (2) (1984), 410-429.
[21] D. E. Dobbs and M. Fontana, Universally going-down integral domains, Arch. Math. (Basel) **42** (5) (1984), 426-429.
[22] D. E. Dobbs, M. Fontana and I. J. Papick, Direct limits and going-down, Comment. Math. Univ. St. Paul. **31** (2) (1982), 129-135.
[23] D. E. Dobbs and I. J. Papick, On going down for simple overrings, III, Proc. Amer. Math. Soc. **54** (1976), 35–38.
[24] D. E. Dobbs and G. Picavet, Straight rings, Comm. Algebra **37** (3) (2009), 757-793.
[25] D. E. Dobbs, G. Picavet and M. Picavet-L'Hermitte, Characterizing the ring extensions that satisfy FIP or FCP, J. Algebra **371** (2012), 391–429.
[26] D. E. Dobbs and J. Shapiro, A classification of the minimal ring extensions of certain commutative rings, J. Algebra **308** (2) (2007), 800–821.
[27] D. E. Dobbs and J. Shapiro, Normal pairs with zero-divisors, J. Algebra Appl. **10** (2) (2011), 335–356.
[28] D. Ferrand and J.-P. Olivier, Homomorphismes minimaux d'anneaux, J. Algebra **16** (1970), 461–471.
[29] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura App. **123** (4) (1980), 331–355.
[30] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.
[31] R. Gilmer and J. F. Hoffmann, A characterization of Prüfer domains in terms of polynomials, Pacific J. Math. **60** (1) (1975), 81–85.
[32] J. R. Hedstrom and E. G. Houston, Pseudo-valuation domains, Pacific J. Math. **75** (1) (1978), 137–147.
[33] R. Jahani-Nezhad and F. Khosahayand, Pseudo-almost valuation rings. Bull. Iranian Math. Soc. **41** (4) (2015), 815–824.
[34] R. Jahani-Nezhad and F. Khosahayand, Almost valuation rings, Bull. Iranian Math. Soc. **43** (3) (2017), 807–816.
[35] N. Jarboui and S. Trabelsi, Some results about proper overrings of pseudo-valuation domains, J. Algebra Appl. **15** (5) (2016), Article ID: 1650099, 16 pp.
[36] M. Kabbour and N. Mahdou, On valuation rings, Comm. Algebra **39** (1) (2011), 176–183.
[37] I. Kaplansky, Commutative Rings, rev. ed., Univ. Chicago Press, Chicago, 1974.
[38] M. Knebusch and D. Zhang, Manis Valuations and Prüfer Extensions I, A New Chapter in Commutative Algebra, Lecture Notes Math. **1791**, Springer-Verlag, Berlin, 2002.
[39] S. Lang, Algebra, Addison-Wesley, Reading, MA, 1965.
[40] S. Malik and J. L. Mott, Strong S-domains, J. Pure Appl. Algebra **28** (1983), 249–264.
[41] A. Mimouni, Prüfer-like conditions and pullbacks, J. Algebra **279** (2) (2004), 685–693.
[42] M. Nagata, A type of integral extension, J. Math. Soc. Japan **20** (1968), 266–267.
[43] I. J. Papick, Topologically defined classes of going-down domains, Trans. Amer. Math. Soc. **219** (1976), 1–37.

[44] G. Picavet and M. Picavet-L'Hermitte, Quasi-Prüfer extensions of rings, pp. 307–336, in Rings, Polynomials, and Modules, Springer, Cham, 2017.
[45] L. J. Ratliff, Jr., On quasi-unmixed local domains, the altitude formula, and the chain condition for prime ideals, II, Amer. J. Math. **92** (1970), 99–144.
[46] F. Richman, Generalized quotient rings, Proc. Amer. Math. Soc. **16** (1965), 794–799.

Department of Mathematics, University of Tennessee, Knoxville, Tennessee 37996-1320, E-mail: dedobbs@comporium.net

Department of Mathematics and Statistics, College of Science, King Faisal University, P.O. Box 400, Al-Ahsa 31982, Saudi Arabia, E-mail: njarboui@kfu.edu.sa

March 20, 2020

To the editor and the referee,

We are submitting a revision of our paper. Please note that the title has been changed slightly, to reflect the fact that our emphasis is now on (commutative) rings, rather than on integral domains.

Most of this message will concern the nature of the revision, but first we would like to address "the elephant in the room." Perhaps, one of us is overly nervous, but he felt that the referee's report may have indicated a suspicion that we had fabricated some quotations from the paper by Ouled Azaiez and Motoui. So, as soon as this revision is submitted online by Noomen Jarboui, David Dobbs will send the communicating editor (Professor Bruce Olberding) a copy of the version of that paper, in the hope that he will forward it to the referee. You will see that all of the quotations and claims in the original version of our paper are supported by that earlier copy of the published paper. We feel that it is important for the referee to understand that we are serious, honest scholars.

But one must face reality. We acknowledge that, through no fault of our own, the original version of our paper was not critiquing a paper that had actually been published. The urgency that we had felt in needing to alert the community to the existence of errors in print was no longer a pressing matter, and so we have taken advantage of this opportunity for a revision to write a more ambitious work which is set more generally and addresses a number of new themes that are, nonetheless, tangibly related to the original submission. In the rest of this message, we will describe the nature of the revision and explain why we believe that you should regard it as an authentic revision of our original manuscript. First, we wish to sincerely thank the referee for advising us to make a substantial revision. The revision that we are submitting is much more substantial than the original manuscript, as it is now 43 pages (in pdf form), with 7 sections and 46 references. As we explain next, that greater length reflects a greater breadth and, we think, a greater depth.

The vision expressed in the attached revision is more comprehensive than in the earlier version. This can be seen by reading the abstract and the first few results in Section 2. Indeed, section 2 quickly gives the new result that the problem of understanding AV-ring pairs can be separated into the integrally closed case and the integral case, both cases are better addressed for rings (rather than for domains) by the new result that any AV-ring pair comes from a P-extension (also known in the literature as an INC-pair or a residually algebraic pair), and the integrally close case is then quickly characterized in conjunction with finding a role for normal pairs. As for Section 3, the result of ours on root extensions that the referee had liked the most is now Proposition 3.2 (c), where the result has been extended to rings and our earlier domain-theoretic result that the top ring in the AV-ring pair could go all the way up to the quotient field of S has been replaced by the analogue conclusion if S is a reduced ring. As we point out at several places in the revision, we have given ring-theoretic generalizations of every part of the result of Ouled Azaiez and Motoui that the referee had preferred to our earlier version of Proposition 3.2.

It seems that there may be another elephant in the room. The referee has asked us to look at a recent preprint by D. D. Anderson, Xing and Zafrullah. We have accessed a copy of this preprint which arrived at arViv.org on December 4, 2019. That date is subsequent to the journal's receipt of our first submission. There is no question that our Example 3.3 (which is unchanged from our earlier submission) was done independently of (and apparently slightly earlier than) anything that may be relevant in the paper by Anderson eet al. In Remark 3.11, we discuss what minor overlaps we could find, but if the referee is aware of other overlaps, we would be glad to address those also. Meanwhile, note that for the most part, the concerns of Anderson et al differ from our concerns, as Anderson et al seem to be more interested in low-dimensional Noetherian cases and, of course, most of our contexts do not allow use of the Cohen structure theory.

Very little of Sections 4-7 appeared in in any form in the original submission, and so we would like to briefly make the case that these sections belong in what could be viewed as a substantial revision. Section 4 gives (in Theorem 4.5) a ring-theoretic generalization of a domain-theoretic result of Mimouni on pullbacks. As the abstract and Introduction note, Theorem 4.5 finds a number of uses in the later sections. Note that most of Proposition 4.7 on PVDs appeared in our earlier submission (where it was treated via Mimouni's result), as did Corollary 4.9. However, the current form of Proposition 4.7 is stronger, as we now show that a PVD that is an AV-domain must lead to an AV-domain pair in which integrality of such a PVD inside its canonical valuation overring is a necessary condition (rather than an assumption, as had been the case in the original version). Corollary 4.9 had also been in the original version and serves as motivation for some of the new Section 7. Our final comment on Section 4 is that Ouled Azaiez and Motoui also treated pullbacks (sometimes implicitly) and our revision has made the role of pullbacks deeper and central to much of the AV-related studies.

Our comments on Sections 5 and 6 can be brief. As our new approaches in Section 2 have now enabled us to characterize the integrally closed minimal ring extensions there, it was natural to devote a section (Section 5) to characterizing the integral minimal ring extensions. Also, as Ouled Azaiez and Motoui had characterized when an integral domain R is a maximal non-AV subring of its quotient field K, it was natural for us to devote a section (Section 6) where we obtain generalizations in which K is replaced by an arbitrary integral domain containing R as a subring. Like our original paper, this puts into proper perspective the view that the quotient field is not always the appropriate universe of discourse in domain-theoretic studies of ring extensions. In short, the first few paragraphs of the introduction, together with the abstract, of our revision make clear that we have taken seriously the referee's advice for us to make a serous revision, and we hope that you will agree and find the revision to be acceptable for the JAA.

Sincerely yours,

David E. Dobbs and Noômen Jarboui