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ON ALMOST VALUATION DOMAIN PAIRS

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Abstract. In this note, we point out some false results by N. Ouled Azaiez and M. A. Moutui [Almost valuation property in bi-amalgamations and pairs of rings, J. Algebra Appl. 11 (6)(2019) 1950104, 14 pp] regarding AV-domain pairs. Contrary to the authors's claim, we show by means of explicit counterexamples that if (R, S) is an AV-domain pair and R is not a field, then R and S may have different quotient fields. Our positive results include characterizations of the domain extensions $R \subset S$, with L the quotient field of S and each element of S that is integral over R having a power in R, such that (R, L) is an AV-domain pair.

1. Introduction

All rings considered in this note are commutative and unital, usually (integral) domains; all inclusions of rings and ring extensions are unital. If D is a domain with quotient field L, we write $\mathrm{qf}(D) = L$ and, by an *overring* of D, we mean any ring E such that $D \subseteq E \subseteq \mathrm{qf}(D)$. Studies of overrings have been central in much of multiplicative ideal theory (cf. [6]), including studies of Prüfer domains and their generalizations. One of those generalizations, the notion of an almost Bézout domain and its quasi-local case of an almost valuation domain (in short, an AV-domain), was introduced by D. D. Anderson and M. Zafrullah in [2], with sequels in [1] and [3]. Recall that a domain D is called an *AV-domain* if, for every nonzero $u \in \mathrm{qf}(D)$, there exists a positive integer n such that either $u^n \in D$ or $u^{-n} \in D$. Recently, in [5], N. Ouled Azaiez and M. A. Moutui introduced the concept of an *almost valuation domain pair* (in short, an AV-*domain pair*), as follows. If $R \subseteq S$ are domains, then (R, S) is called an AV-domain pair if each ring T such that $R \subseteq T \subseteq S$ is an AV-domain, (The most natural example of

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an AV-domain pair is given by (V, W) where V is a valuation domain and W is an overring of V.) In [5, Proposition 3.2 (2)], Ouled Azaiez and M. A. Moutui purported to show that if (R, S) is an AV-domain pair and R is not a field, then S is an overring of R. However, that assertion is false. Indeed, in Example 2.2, we construct two classes of counterexamples to that incorrect assertion from [5]. As explained in Remark 2.3 (a), three specific "results" from [5] are shown to be false by the counterexamples in Example 2.2. Remark 2.3 (b) identifies the incorrect step in the published "proof" of Case 2 of [5, Proposition 3.2 (2)] which accounts for all three of the erroneous "results" that we have found in [5].

Although our main purpose in this note is to point out the above-mentioned errors in [5], we also give some positive results. The most important of these, in Proposition 3.3, involves the following much-studied concept. A ring extension $A \subseteq B$ is called a *root extension* if, for each $\xi \in B$, there exists a positive integer n (possibly depending on ξ) such that $\xi^n \in A$. Of course, any root extension is an integral extension and the converse is false. Root extensions play a fundamental role in studying AV-domains. For instance, it was shown, i.a., in [2, Theorem 5.6] that if R is a domain, with R' denoting the integral closure of R (in its quotient field), then R is an AV-domain if and only if $R \subseteq R'$ is a root extension and R' is a valuation domain. An easier fact, which was observed without proof in [2], is that if $R \subseteq S$ is a root extension of domains, then: R is an AV-domain $\Leftrightarrow$ S is an AV-domain $\Leftrightarrow$ (R, S) is an AV-domain-pair. For the sake of completeness, we include a proof of these equivalences in Proposition 2.1, but that result also includes a new equivalence, namely, the condition that $(R, \mathrm{qf}(S))$ is an AV-domain-pair. Proposition 3.3 includes a generalization of that result, with the earlier assumption that $R \subseteq S$ is a root extension being replaced by the assumption that $R \subseteq \overline{R}_S$ is a root extension, where $\overline{R}_S$ denotes the integral closure of R in S. The significance of this result is two-fold: not only does it extend our earlier extension of some observations from [2] (in Proposition 2.1) but, far more importantly, it also applies to some non-overring (and non-field) extensions $R \subset S$ such as the counterexamples to some work from [5] that were presented in Example 2.2.

For any domain R, we let $\mathrm{qf}(R)$ denote the quotient field of R. As usual, for any prime number p, $\mathbb{F}_p$ denotes the field of cardinality p; for any domain D, $\mathrm{char}(D)$ denotes the characteristic of D; and $\subset$ denotes proper inclusion. Any undefined terminology is standard, as in [6] and [9].

2. Initial results and counterexamples to [5]

We begin by collecting some useful facts. In Proposition 2.1, parts (a) and (b), as well as the equivalence (1) $\Leftrightarrow$ (2) in part (d), were stated without proof in [2, page 301]. For the sake of completeness, we include the easy proofs of those facts.

Proposition 2.1. (*a*) (*D. D. Anderson and Zafrullah*) *Let R be an* AV*-domain and let S be an overring of R. Then S is an* AV*-domain.*

(*b*) (*D. D. Anderson and Zafrullah*) *Let $R \subseteq S$ be a root extension of domains such that R is an* AV*-domain. Then S is an* AV*-domain.*

(*c*) (*D. D. Anderson and Zafrullah*) *Let $R \subseteq S$ be a root extension of domains such that S is an* AV*-domain. Then R is an* AV*-domain.*

(*d*) *Let $R \subseteq S$ be a root extension of domains. Then the following conditions are equivalent:*

(*1*) *R is an* AV*-domain;*
(*2*) *S is an* AV*-domain;*
(*3*) *(R, S) is an* AV*-domain pair;*
(*4*) *$(R, \mathrm{qf}(S))$ is an* AV*-domain pair.*

Proof. (a) Consider any nonzero element $x \in \mathrm{qf}(S)$. Then $x \in \mathrm{qf}(R)$ since S is an overring of R. Hence, since R is an AV-domain, there exists a positive integer n such that either $x^n \in R$ or $x^{-n} \in R$. Consequently either $x^n \in S$ or $x^{-n} \in S$, and so S is an AV-domain.

(b) Consider any nonzero element $x \in \mathrm{qf}(S)$. As $R \subseteq S$ is a root extension, so is $\mathrm{qf}(R) \subseteq \mathrm{qf}(S)$. (This was observed without proof in [1, page 549]. For completeness, we provide the details. If $\xi \in \mathrm{qf}(S)$, then $\xi = uv^{-1}$ with $u, v \in S$ and $v \neq 0$. Pick positive integers p, q with $u^p, v^q \in R$. As $u^{pq}, v^{pq} \in R$, we have $\xi^{pq} = u^{pq}(v^{pq})^{-1} \in \mathrm{qf}(R)$, as desired.) Thus, there exists a positive integer n such that $x^n \in \mathrm{qf}(R)$. Hence, since R is an AV-domain, there exists a positive integer m such that either $x^{nm} \in R \subseteq S$ or $x^{-nm} \in R \subseteq S$. Thus, S is an AV-domain.

(c) Consider any nonzero element $x \in \mathrm{qf}(R)$. Then $x \in \mathrm{qf}(S)$. Hence, since S is an AV-domain, there exists a positive integer k such that either $x^k \in S$ or $x^{-k} \in S$. Thus, since $R \subseteq S$ is a root extension, there exist positive integers n' and m' such that either $x^{kn'} \in R$ or $x^{-km'} \in R$. Hence, either $x^{kn'm'}$ $(= (x^{kn'})^{m'}) \in R$ or $x^{-kn'm'}$ $(= (x^{-km'})^{n'}) \in R$. Thus, R is an AV-domain.

(d) For any rings $R \subseteq A \subseteq B \subseteq S$, it is clear that the ring extension $A \subseteq B$ inherits the "root extension" property from $R \subseteq S$. Hence, the equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow from (b) and (c). As (4) $\Rightarrow$ (3) trivially, it will suffice to show that (2) $\Rightarrow$ (4).

Assume that S is an AV-domain. Our task is to show that if T is a ring such that $R \subseteq T \subseteq \mathrm{qf}(S)$, then T is an AV-domain. To that end, consider any nonzero element $x \in \mathrm{qf}(T)$. Then $x \in \mathrm{qf}(S)$. As S is an AV-domain, there exists a positive integer n such that either $x^n \in S$ or $x^{-n} \in S$. Thus, since $R \subseteq S$ is a root extension, there exist positive integers m and k such that either $x^{nm} \in R$ or $x^{-nk} \in R$. Therefore, either x^{nmk} $(= (x^{nm})^k) \in R \subseteq T$ or x^{-nmk} $(= (x^{-nk})^m) \in R \subseteq T$. Hence, T is an AV-domain, as desired. The proof is complete. □

If R is a domain but not an AV-domain, it is nonetheless the case that $S := \mathrm{qf}(R)$ is an AV-domain. Thus, the equivalence (1) $\Leftrightarrow$ (2) in Proposition 2.1 (d) cannot be expected to hold for an arbitrary pair of domains $R \subseteq S$. What led to that equivalence holding in Proposition 2.1 (d) was the assumption that $R \subseteq S$ is a root extension. We show next, however, that the "root extension" assumption is not enough to prevent the existence of counterexamples to some of the assertions in [5]. In particular, each of the ring extensions $R \subset S$ constructed in Example 2.2 is a counterexample which shows that each of Proposition 3.2 (2), Theorem 3.3 (2) and Corollary 3.4 (ii) in [5] is wrong.

Example 2.2. *Let p be a prime number. Then:*

(*a*) *There exists a root extension $R \subseteq S$ of domains such that neither R nor S is a field, $(R, \mathrm{qf}(S))$ is an AV-domain pair, $\mathrm{char}(R) = p$, and S is not contained in the quotient field of R (that is, S is not an overring of R). It can be further arranged that R and S are each valuation domains and that $u^p \in R$ for each $u \in S$.*

One way to produce domains $R \subset S$ with the above behavior is the following. Let X be an analytic indeterminate over a field F of characteristic p. Put $S := F[[X]]$, the ring of formal power series in X over F; and put $R := F[[X^p]]$.

(*b*) *Another way to produce domains $R \subset S$ with the behavior that was stipulated above in (a) is the following. Let $k \subset K$ be a purely inseparable field extension of characteristic p and exponent 1 (that is, for every $\alpha \in K$, $\alpha^p \in k$, where $p = \mathrm{char}(k)$). Let X be an analytic indeterminate over K. Put $R := k[[X]]$ and $S := K[[X]]$.*

Proof. (a) It is well known that $S = F[[X]]$ is a (discrete rank 1) valuation domain but not a field. The same conclusion holds for $R = F[[X^p]]$, since X^p is an analytic indeterminate over F. Moreover, since X cannot be expressed as a quotient of elements from $F[[X^p]]$, it follows that S is not contained in the quotient field of R. Thus, as R is an AV-domain, Proposition 2.1 (d) reduces our task to proving that $u^p \in R$ for each $u \in S$. This, in turn, holds, for we can write $u = \sum_{i=0}^{\infty} a_i X^i$

with each $a_i \in F$ and then

$$u^p = \sum_{i=0}^{\infty} (a_i)^p (X^i)^p = \sum_{i=0}^{\infty} (a_i)^p (X^p)^i \in F[[X^p]] = R,$$

as required.

(b) As in the proof of (a), both $R = k[[X]]$ and $S = K[[X]]$ are valuation domains that are not fields. Note that if $\xi \in K \setminus k$, then ξ cannot be expressed as a quotient of elements from $k[[X]]$, and so S is not contained in the quotient field of R. With minor changes, the rest of the proof of (a) (including the appeal to Proposition 2.1 (d) and the displayed calculation of u^p) easily carries over to the present context. □

Remark 2.3. (a) Let us recall three assertions from [5] that pertain to a given pair of domains $R \subset S$. First, it follows from the statement of [5, Proposition 3.2 (2)] that if (R, S) is an AV-domain pair, then S is an overring of R (that is, $\mathrm{qf}(R) = \mathrm{qf}(S)$). Second, according to [5, Theorem 3.3 (2)], if R is not a field, then (R, S) is an AV-domain pair (if and) only if R is an AV-domain such that S is an overring of R. Third, according to [5, Corollary 3.4 (ii)], if R is not a field and K is a field containing R (as a subring), then (R, K) is an AV-domain pair (if and) only if R is an AV-domain such that $K = \mathrm{qf}(R)$. Each of the three preceding consequences of "results" in [5] is wrong. In fact, it is now clear from the first paragraph of the statement of Example 2.2 that each of the extensions $R \subset S$ constructed in parts (a) and (b) of Example 2.2 is a counterexample to each of those three false "results" in [5].

(b) It is natural to try to identify the errors in reasoning that led to the mistaken assertions in [5] that were exposed in (a). We believe that all those mistaken assertions stem from a mistake in the handling of Case 2 in the published "proof" of [5, Proposition 3.2 (2)]. Let us analyze that "proof" in detail, using the notation from [5]. If one expresses α^j as its unique K-linear combination of the elements of the K-basis B, using coefficients $k_i \in K$, then equating coefficients leads to $1 = a_j k_q \beta$, so that $\beta^{-1} = a_j k_q \in K$. It seems to us that the unexplained notation r_j in [5] must therefore be $a_j k_q$. While the argument in [5] requires that (the unidentified) r_j satisfy $r_j \in R$, there is no reason to believe that $a_j k_q \in R$. When we noticed this difficulty with the published "proof" of Case 2, we attempted to find a counterexample in the simplest possible subcase, namely, where $q = 2 = n$. (Recall that $q \geq n \geq 2$ in Case 2.) This led us to the construction of the counterexample to [5, Proposition 3.2 (2)] that was given in Example

2.2 (a) in case the characteristic is $p = 2$ and the field F is $\mathbb{F}_2$. It was then easy to extend our reasoning to permit the characteristic to be any prime number p and the field F to be any field of characteristic p, just as presented above in Example 2.2 (a).

(c) The construction and conclusion in part (a) of Example 2.2 should be contrasted with the following result of D. D. Anderson and Zafrullah [2, Example 4.15]. Let S be a primitive numerical monoid (nowadays usually called a "numerical semigroup"), that is, an additive submonoid of the set of nonnegative integers under addition such that $\mathrm{GCD}\{S\} = 1$. Then for any field F of characteristic $p > 0$, the ring $F[[X^s \mid s \in S]]$ is an AV-domain whose integral closure is $F[[X]]$. (Actually, to get the "AV-domain" conclusion from the cited result in [2], one also needs to note that quasi-local API-domain $\Rightarrow$ quasi-local AB-domain $\Rightarrow$ AV-domain.) Note, however, that despite the similarities in notation, the AV-domain $R = F[[X^p]]$ that was constructed in part (a) of Example 2.2 cannot be described as $F[[X^s \mid s \in S]]$ for a numerical semigroup S, the point being that for any prime number p, the set of elements in the additive abelian group $p\mathbb{Z}$, when viewed in $\mathbb{Z}$, has greatest common divisor $p \neq 1$.

3. Further results

Despite the counterexamples to [5, Proposition 3.2 (2)] that were given in Example 2.2, it seems natural to ask, in the spirit of Proposition 2.1 (d), when given an AV-domain pair (R, S), for (i) a sufficient condition that S be an overring of R and (ii) a sufficient condition that $(R, \mathrm{qf}(S))$ be an AV-domain pair. Such sufficient conditions will be given in Propositions 3.2 and 3.3, respectively. First, it is convenient to collect a few useful facts in the following result. Proposition 3.1 (a) is a special case of [5, Proposition 3.2 (1)]. Proposition 3.1 (c) is a special case of [10, Theorem 2.9].

Proposition 3.1. (*a*) (*Ouled Azaiez and Moutui*) *Let (R, S) be an* AV-*domain pair. Then S is algebraic over R. If, in addition, R is a field, then S is a field* (*which is algebraic over R*).

(*b*) (*Nagata* [11]) *Let $K \subset L$ be fields. Then $K \subset L$ is a root extension if and only if either L is purely inseparable over K or L is algebraic over some finite field.*

(*c*) (*Mimouni*) *Let M be the maximal ideal of a quasi-local domain S and let D be a proper subring of S/M. Then the pullback $D \times_{S/M} S$ is an* AV-*domain if and only if both S and D are* AV-*domains and* $\mathrm{qf}(D) \subseteq S/M$ *is a root extension.*

One useful consequence of Proposition 3.1 (b) is that if $K \subset L$ is a field extension and a root extension (with $K \neq L$), then these fields have positive characteristic.

Proposition 3.2. *Let $R \subseteq S$ be domains such that R is integrally closed in S. Then the following conditions are equivalent:*
(1) (R, S) is an AV-domain pair;
(2) R is an AV-domain and S is an overring of R.

Proof. (1) $\Rightarrow$ (2): Assume (1). Then, of course, R is an AV-domain. Also, by Proposition 3.1 (a), S is algebraic over R. As R is integrally closed in S, it follows that S is contained in the quotient field of R, that is, that S is an overring of R [9, Exercise 35, page 44].

(2) $\Rightarrow$ (1): Assume (2). We must show that if T is a ring such that $R \subseteq T \subseteq S$, then T is an AV-domain. As T inherits from S the property of being an overring of R, an application of Proposition 2.1 (a) completes the proof. $\square$

We next show how to use the "root extension" property to give a sufficient condition for (R, S) to an AV-domain pair without entailing that S is necessarily an overring of R. In fact, the condition given in Proposition 3.3 is satisfied by the (counterexample) extensions $R \subset S$ that were constructed in parts (a) and (b) of Example 2.2.

Proposition 3.3. *Let $R \subset S$ be domains. Let $\overline{R}_S$ denote the integral closure of R in S and let $(\overline{R}_S)'$ denote the integral closure of $\overline{R}_S$ (in its quotient field). Suppose also that $R \subseteq \overline{R}_S$ is a root extension. Then the following conditions are equivalent:*
(1) (R, S) is an AV-domain pair;
(2) R is an AV-domain and $R \subset S$ is an algebraic extension;
(3) $\overline{R}_S$ is an AV-domain and $R \subset S$ is an algebraic extension;
(4) $(R, \overline{R}_S)$ is an AV-domain pair and $R \subset S$ is an algebraic extension;
(5) $(R, \mathrm{qf}(S))$ is an AV-domain pair;
(6) S is both an AV-domain and an overring of $\overline{R}_S$, and $(\overline{R}_S)'$ is a valuation domain.

Proof. Also suppose, for the moment, that R is a field. In view of Proposition 3.1 (a) and basic facts about integral extensions of domains, we see that each of the conditions (1)-(6) is equivalent to $R \subset S$ being an algebraic field extension. Thus, we can assume henceforth that R is not a field.

Since $R \subseteq \overline{R}_S$ is a root extension, it follows from Proposition 2.1 (d) that R is an AV-domain $\Leftrightarrow$ $\overline{R}_S$ is a AV-domain $\Leftrightarrow$ $(R, \overline{R}_S)$ is an

AV-domain pair $\Leftrightarrow$ $(R, \mathrm{qf}(\overline{R}_S))$ is an AV-domain pair. Also, by using Proposition 3.1 (a) and the well known clearing-of-denominators argument, we see that each of the conditions (1)-(5) implies that $\mathrm{qf}(\overline{R}_S) = \mathrm{qf}(S)$ and that $R \subset S$ is an algebraic extension. It is now straightforward to check that (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5); and that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Assume (2). Then, by the above remarks, $\mathrm{qf}(\overline{R}_S) = \mathrm{qf}(S)$. As $R \subseteq \overline{R}_S$ is a root extension, so is $\mathrm{qf}(R) \subseteq \mathrm{qf}(\overline{R}_S)$; that is, the field extension $\mathrm{qf}(R) \subseteq \mathrm{qf}(S)$ is a root extension. According to Proposition 3.1 (b), there are two cases (the second of which has three subcases).

Case 1: $\mathrm{char}(R) = 0$. In this case, $\mathrm{qf}(R) = \mathrm{qf}(S)$. Thus, S is an overring of R. As R is an AV-domain, an application of Proposition 2.1 (a) gives that (R, S) is an AV-domain pair.

Case 2: $\mathrm{char}(R) = p > 0$. This case has three subcases.

• $\mathrm{qf}(R) = \mathrm{qf}(S)$: Then S is an overring of R, and so (R, S) is an AV-domain pair by Proposition 2.1 (a).

• $\mathrm{qf}(S)$ is algebraic over a finite field: Then, by transitivity of algebraicity, $\mathrm{qf}(S)$ is algebraic over $\mathbb{F}_p$. Hence, R is algebraic over $\mathbb{F}_p$, a contradiction since R is not a field. Thus, this subcase cannot actually arise.

• $\mathrm{qf}(S)$ is purely inseparable over $\mathrm{qf}(R)$: Then, for each $\alpha \in \mathrm{qf}(S)$, there exists a positive integer r such that $\alpha^{p^r} \in \mathrm{qf}(R)$. Let T be a ring contained between R and S. Our task is to prove that T is an AV-domain. Let $F := \mathrm{qf}(T)$. We must show that if u is a nonzero element of F, then there exists a positive integer ν such that either $u^\nu \in T$ or $u^{-\nu} \in T$. As $u \in \mathrm{qf}(S)$, the hypothesis of the present subcase provides a positive integer r such that $u^{p^r} \in \mathrm{qf}(R)$. Then, since R is an AV-domain, there exists a positive integer n such that either $u^{p^r n} = (u^{p^r})^n \in R \subseteq T$ or $u^{-p^r n} = (u^{p^r})^{-n} \in R \subseteq T$. Thus, $\nu := p^r n$ has the desired property, T is an AV-domain, and the proof that (2) $\Rightarrow$ (1) is complete.

(1) $\Rightarrow$ (6): Assume (1). Then $R \subset S$ is an algebraic extension by Proposition 3.1 (a). Also, by the reasoning in the second paragraph of this proof, $\mathrm{qf}(S) = \mathrm{qf}(\overline{R}_S)$. Of course, (1) implies that S is an AV-domain. Finally, since $\overline{R}_S$ is a AV-domain, it follows from [2, Theorem 5.6] that $(\overline{R}_S)'$ is a valuation domain. This completes the proof that (1) $\Rightarrow$ (6).

It will suffice to prove that (6) $\Rightarrow$ (3). Observe that $\overline{R}_S = (\overline{R}_S)' \cap S$. Assume (6). Then $\overline{R}_S$ is an intersection of two overrings (namely, $(\overline{R}_S)'$ and S) that happen to be AV-domains with comparable integral closures. Therefore, by [3, Lemma 2], $\overline{R}_S$ is an AV-domain. It remains only to prove that $R \subseteq S$ is an algebraic extension. Of course, S

is algebraic over $\overline{R}_S$, since S is an overring of $\overline{R}_S$. Moreover, $\overline{R}_S$ is integral, hence algebraic, over R. So, by the transitivity of algebraicity, $R \subseteq S$ is algebraic, which completes the proof. $\square$

It is natural to ask how essential it is to have the hypothesis "$R \subset \overline{R}_S$ is a root extension" in a result having the flavor of Proposition 3.3. Along those lines, we raise the following question.

Question. If (R, S) is an AV-domain pair such that S is integral over R and R is not a field, must $R \subseteq S$ be a root extension?

The results to this point in this section have concerned finding sufficient conditions on an AV-domain pair (R, S) for S to be an overring of R. Proposition 3.4 (b) will give a different kind of result, where one is given a certain kind of quasi-local domain (R, M), considers its overring $S := (M :_{\mathrm{qf}(R)} M)$, and shows that integrality of the extension $R \subseteq S$ is sufficient to entail that (R, S) is an AV-domain pair. We will close with a corollary which indicates that our earlier emphasis (as in, for instance, Example 2.2) on AV-domains with residue class fields of positive characteristic was opportune and that the theory for AV-domain pairs with residue class fields of characteristic 0 may be somewhat meager in comparison.

The relevant sufficient condition on the base domain R in Proposition 3.4 is that R is a PVD. Recall from [8] that a quasi-local domain (R, M) is said to be a *pseudo-valuation domain* (in short, a PVD) if M is the maximal ideal of some valuation overring V of R. Recall also from [4, Proposition 2.5] that if (R, M) is a PVD and V is a valuation overring of R such that M is the maximal ideal of V, then V is uniquely determined (as being the conductor $(M :_{\mathrm{qf}(R)} M)$) and V is called the *canonically associated valuation overring* of (the PVD) R. Also, it was shown in [4, Proposition 2.6] that the class of PVDs consists (up to isomorphism) of the pullbacks D of the form $D = k \times_{W/N} W$ where (W, N) is a valuation domain and k is a subfield of W/N (and that W is then the canonically associated valuation overring of D).

Proposition 3.4. *Let (R, M) be a* PVD, *and let V denote the canonically associated valuation overring of R. Then:*

(a) *R is an* AV-*domain if and only if at least one of the following three conditions holds: $R = V$; V/M is purely inseparable over R/M; R/M is algebraic over some finite field.*

(b) *Assume, in addition, that $R \subseteq V$ is an integral extension. Then (R, V) is an* AV-*domain pair if and only if at least one of the following* *three conditions holds: $R = V$; V/M is purely inseparable over R/M; R/M is algebraic over some finite field.*

Proof. (a) As any valuation domain (in particular, any field) must be a AV-domain, both V and R/M are AV-domains. Therefore, by applying Proposition 3.1 (c) to the pullback $R = R/M \times_{V/M} V$, we see that R is an AV-domain if and only if the field extension $R/M \subseteq V/M$ is a root extension. (Note that Proposition 3.1 (c) cannot be applied in case $R/M = V/M$, but the assertion also holds in that case, for $R/M = V/M$ implies that $R = V$ is (almost) valuation and $\mathrm{qf}(R/M) \subseteq V/M$, being the identity map on V/M, is then a root extension.) Finally, since $R/M = V/M$ (if and) only if $R = V$, an application of Proposition 3.1 (b) completes the proof of (a).

(b) It is known that any integral overring of a PVD must be a PVD (cf. [8, Theorem 1.7]). Thus, the hypothesis that $R \subseteq V$ is integral ensures that each ring E contained between R and V is a PVD (necessarily having associated valuation overring V). If φ denotes the canonical surjection $V \to V/M$, the collection of such E consists of the rings of the form $\varphi^{-1}(F) = F \times_{V/M} V$ as F runs through the set of rings (fields) contained between R/M and V/M. Hence, by (a), (R, V) is an AV-domain pair if and only if, for each field F contained between R/M and V/M, at least one of the following three conditions holds: $\varphi^{-1}(F) = V$; V/M is purely inseparable over $\varphi^{-1}(F)/M$ $(= F)$; $\varphi^{-1}(F)/M$ $(= F)$ is algebraic over some finite field. It is clear that any given F satisfies (at least) one of these three conditions if the field R/M satisfies the analogous condition. Therefore, an application of (a) completes the proof. ☐

An interesting consequence of Proposition 3.4 is that if R is a PVD with canonically associated valuation overring V and the ring extension $R \subseteq V$ is integral, then (R, V) is an AV-domain pair.

Corollary 3.5. *Let (R, M) be a* PVD *such that* $\mathrm{char}(R/M) = 0$. *Then R is an* AV*-domain if and only if R is a valuation domain.*

Proof. Since every valuation domain is an AV-domain, it suffices to prove the "only if" assertion. Let R be an AV-domain. Let V denote the canonically associated valuation overring of R. Since $\mathrm{char}(R/M) = 0$, it follows from Proposition 3.4 (a) that $R = V$. (Indeed, if $R \neq V$, then $R/M \neq V/M$ and it cannot be the case that either V/M is purely inseparable over R/M or R/M is algebraic over some finite field.) As it is clear (and well known) that a PVD is a valuation domain if and only if it coincides with its canonically associated valuation overring, the proof is complete. ☐

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