QUESTION HD0901: In multiplicative ideal theory , we often deal with Picard groups.

I want to know completely about the Picard groups. I also see in some materials that it has connections with Algebraic geometry. Please guide me to know about the Picard groups deeply. Which readings you suggest to be most useful?

ANSWER: Wanting to know "completely" about the Picard group is somewhat ambitious, because the Picard group is almost everywhere, in different setups and different guises. As you have informed me that they are mentioned in algebraic geometry a lot. They can be found in category theory and of course as you have already indicated they are in multiplicative ideal theory. To be simple and straightforward, in ring theory which includes multiplicative ideal theory, the Picard group is a generalization of the ideal class group of a Dedekind domain. I would just stay within commutative rings to keep things clear.

The ideal class group (or the Picard group) of an integral domain R is the quotient group $Inv(R)/P(R)$ where $Inv(R)$ is the group of invertible ideals of R under ordinary multiplication of ideals and $P(R)$ is the group of (nonzero) principal fractional ideals of R. It really depends upon who is talking about the Picard group of a domain, for example a French Mathematician will most probably call a Picard group a Cartier group. (Cartier was a French Mathematician who studied what are known as Cartier divisors and it turns out that the Cartier group is the same as the Picard group.) In any case in Dedekind domains the Picard group measures as to how far the Dedekind domain is from being a PID, or a Prufer domain is from being a Bezout domain, because the Picard group of a domain D is trivial precisely when every invertible ideal of D is principal. The Picard group of a domain with only finitely many maximal ideals is zero see Theorem 60 of Kaplansky [K, Commutative Rings, Allyn and Bacon 1970], so a semi-local Dedekind domain is a PID and a semilocal Prufer domain is Bezout. Note that the Picard group is usually considered a group under addition, tradition and expediency may be the reasons. So $Pic(D) = 0$ is written to denote that the Picard group is trivial.

Let R be a commutative ring with $1 \neq 0$ and let M be an R module. The R-module $Hom_R(M, R) = M^*$ is called the dual of M. It can be shown that M is finitely generated projective of constant rank one if and only if $M \otimes M^* \cong M^* \otimes M \cong R$. (For ease of expression we shall call *invertible* a finitely generated module that is projective of constant rank one.) Note that M is invertible if and only if M^* is, and that if M and N are invertible then so are $M \otimes N$ and $N \otimes M$, (in particular, R is invertible) we conclude that if $[M]$ denotes the class of all modules isomorphic to M then the set $G = \{[M] : M$ is a projective rank one R-module$\}$ is a group under the operation $[M] * [N] = [M \otimes N]$. This group of classes is called the Picard group of R, denoted by $Pic(R)$. (You may read Chapter 3, of J.R. Silvester's book “Introduction to Algebraic K-theory", Chapman and Hall 1981.)

To see how the ideal class group of a domain can be extracted from the above definition, note that if R is a domain a finitely generated projective of constant rank one R-module M is isomorphic to an invertible ideal and for a nonzero fractional ideal I of a domain R, $Hom_R(I,R) = I^{-1}$. Also because a fractional ideal of R is an R-module it can be shown that $I \cong xI$ for all $x \in K\backslash\{0\}$, where K is the quotient field of R. Also that, for I, J any two R-submodules of K the quotient field of R, $I \cong J$ if and only if $J = xI$ for some $x \in K\backslash\{0\}$.Thus $[I] = \{xI : x \in K\backslash\{0\}\} = IP(R)$ and so $Pic(R) = \{[I] : I$ ranges over invertible ideals of $R\} = Inv(R)/P(R)$.

Now why are the commutative ring theorists interested in the Picard group? One reason of interest is that the Picard group is used in algebraic geometry. This is a good enough reason for me too. I now try to describe the kind of work that has been done. Mostly people have been interested in exploring the relationship between $Pic(R)$ and $Pic(R[X])$ where X is a nonempty set of indeterminates over R and R is a commutative ring with $1 \neq 0$. The main question was: when is $Pic(R) \cong Pic(R[X])$? The domains for which this was the case were called seminormal. In this connection you may read the paper by James Coykendall and references there. This paper can be found at: http://www.math.ndsu.nodak.edu/faculty/coykenda/picard.pdf

Note however that Coykendall restricts himself to integral domains, but some of the references that he mentions have no such restraint. The other area of interest is the study of the Picard group of a monoid domain $D[S]$, where S is a monoid with certain properties, with reference to the Picard group of D. David Anderson has written a number of papers on this topic. One such paper is referenced in the above mentioned paper of Coykendall's.

As already mentioned the Picard group is a group of isomorphism (or equivalence) classes. Any group that can be interpreted as a group of isomorphism (or equivalence) classes can be called a class group. I would write about these class groups once I have given you an idea of where to look for geometry related Picard groups.

For the geometric connection of the Picard group you should consult a book on algebraic geometry. (I recommend Basic Algebraic Geometry, by Shafarevich, and published by Springer, and I recommend that you attend a course on algebraic geometry before attempting anything.) If you have however attended such a course then you may look up http://www.warwick.ac.uk/~maseap/arith/notes/picard.pdf

Other class groups. For other class groups you would need a dose of star operations. Let $F(D)$ denote the set of nonzero fractional ideals of D. A star operation $*$ on D is a function $*\colon F(D) \to F(D)$ such that for all $A, B \in F(D)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c) $(A^*)^* = A^*$.

For $A, B \in F(D)$ we define $*$-*multiplication* by $(AB)^* = (A^*B)^* = (A^*B^*)^*$. A fractional ideal $A \in F(D)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal. A star operation $*$ is said to be of *finite character* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(D)$ define $A^{-1} = \{x \in K \mid xA \subseteq D\}$ and call $A \in F(D)$ $*$-*invertible* if $(AA^{-1})^* = D$. Clearly every invertible ideal is $*$-

invertible for every star operation $*$, because $D^* = D$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The most well known examples of star operations are: the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$, and the d operation defined by $A \mapsto A$, for all $A \in F(D)$. Given two star operations $*_1, *_2$ we say that $*_1 \leq *_2$ if $A^{*_1} \subseteq A^{*_2}$ for all $A \in F(D)$. Note that $*_1 \leq *_2$ if and only if $(A^{*_1})^{*_2} = (A^{*_2})^{*_1} = A^{*_2}$. By definition t is of finite character, $t \leq v$ while $\rho \leq t$ for every star operation ρ of finite character; thus a v-ideal is a $*$-ideal for every star operation $*$ and so is a nonzero principal fractional ideal. The v-operation is also important in that, for any star operation $*$, and for A a $*$-invertible fractional ideal we have $A^* = A_v$ (see page 433 of Zafrullah's [Z, Putting t-invertibility to use, in Non-Noetherian commutative ring theory, 429–457, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000, by Chapman and Glaz].

If $*$ is a star operation of finite character then using Zorn's Lemma we can show that a proper integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every proper integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $* - max(D)$. It can also be easily established that for a star operation $*$ of finite character on D we have $D = \bigcap_{M \in * - \max(D)} D_M$. A v-ideal A of finite type is t-invertible if and only if A is t-locally principal i.e. for every $M \in t - \max(D)$ we have AD_M principal. An integral domain D is called a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. I recommend looking up sections 32 and 34 of Gilmer's book [G, "Multiplicative Ideal Theory" Dekker, 1972], and [Z] for star operations.

For a given star operation $*$ define $Inv_*(D) = \{A \in F(D) : A$ is a $*$-invertible $*$-ideal$\}$. Since the $*$-multiplication of two $*$-invertivle $*$-ideals is again a $*$-invertible $*$-ideal, the $*$-multiplication is obviously associative, and since D is the identity and the inverse is assured by the definition of $*$-multiplication, we conclude that $Inv_*(D)$ is an abelian group under $*$-multiplication. Next let $P(D) = \{xD : x \in K\backslash\{0\}\}$ be the group of principal fractional ideals of D. Since each principal fractional ideal is a $*$-ideal for any star operation $*$, $P(D)$ is a subgroup of $Inv_*(D)$ for every star operation $*$. The quotient group $Inv_*(D)/P(D)$ is called the $*$-class group, $Cl_*(D)$. Looking at the similarity of definitions one could call a $*$-class group a $*$-Picard group. Now you can have a $*$-class group for any star operation that you fancy. If you set $* = d$ the identity operation that takes each $A \in F(D)$ to A, you have the d-class group which is clearly the Picard group. If, on the other hand, you put $* = v$ or $* = t$ you get the v-class group or the t-class group, respectively. As it usually happens, the notion of a t-class group appeared in a paper by A. Bouvier [Le groupe des classes d'un anneau integre, in « 107 eme Congres des Societes Savantes, Brest, (1982) Vol. 4, pp. 85-92], much before the more general $*$-class group that was introduced by David Anderson in [A1, Comm. Algebra 16(4)(1988), 805-847].

There is another class group that was known way before the t-class group which has some very important applications, in number theory and geometry. This group, for a domain D, is called the divisor class group and is denoted by $Cl(D)$. To define it we need just the v-operation. (Gilmer has a more general definition in section 34 of [G] but we shall leave it as further reading.)

Define on $F(D)$ the relation ˜ as $A\tilde{}B$, if and only if $A_v = B_v$ (or equivalently $A^{-1} = B^{-1}$) for $A, B \in F(D)$. Obviously ˜ is an equivalence relation, usually called Artin's equivalence. In fact Van der Waerden calls ˜ "quasi-equality" at page 185 [VW, Algebra Volume 2, Ungar, New York, 1970]. If you read [VW] remember that Van der Waerden was dealing only with integrally closed Noetherian domains. In any case after taking a few hints of historical nature let's get back to the task at hand.

For each $A \in F(D)$ denote by $div(A)$ the, divisorial, eqivalence class of A, i.e. the set of all $X \in F(D)$ with $X_v = A_v$. Denote by $\mathcal{D}(D) = \{div(A) : A \in F(D)\}$. Define on $\mathcal{D}(D)$ the operation $+$ by $div(A) + div(B) = div(AB)$. It is easy to see that $\mathcal{D}(D)$ is a commutative semigroup with identity $div(D)$. Now by this definition, $A \in F(D)$ has an inverse if there is a $B \in F(D)$ such that $div(A) + div(B) = div(D)$. This of course requires that $div(AB) = div(D)$ which means $AB\tilde{}D$ which happens if and only if $(AB)_v = D$ i.e. A is v-invertible. It can be shown that in this case $B_v = A^{-1}$ see e.g. page 433 of Zafrullah [Z]. So, $\mathcal{D}(D)$ being a group requires that every $A \in F(D)$ is v-invertible. But it is well known that every $A \in F(D)$ is v-invertible if and only if D is completely integrally closed, as indicated in Theorem 34.3 of [G]. Thus $\mathcal{D}(D)$ is a group under $+$, defined above, if and only if D is completely integrally closed. Now for D completely integrally closed define $\mathcal{P}(D) = \{div(A), A \in F(D) : div(A)$ contains a nonzero principal fractional ideal$\}$. It is easy to see that $\mathcal{P}(D)$ is a subgroup of $\mathcal{D}(D)$ under $+$. Now define the divisor class group as the quotient group $Cl(D) = \mathcal{D}(D)/\mathcal{P}(D)$. Now it is easy to see that if, for D completely integrally closed, we define $\varphi : \mathcal{D}(D) \to Inv_v(D)$ by $\varphi(div(A)) = A_v$ then $\varphi(div(A) + div(B)) = \varphi(div(AB)) = (AB)_v$ and φ is a group isomorphism, it being easy to see that φ is well-defined and is onto and one-one. Now $\mathcal{D}(D) \cong Inv_v(D)$ and in a similar fashion $\varphi(\mathcal{P}(D)) = P(D)$ the set of nonzero principal fractional ideals of D under multiplication. Thus, for a completely integrally closed integral domain D we can represent the divisor class group by $Inv_v(D)/P(D)$ the v-class group $Cl_v(D)$. It may however be noted that, as we have seen above, the v-class group can be defined for any integral domain D and that $Cl_v(D)$ is a divisor class group only when D is completely integrally closed.

The divisor class group becomes very useful when D is a Krull domain; which is known to be completely integrally closed. Recall that D is a Krull domain if D is a locally finite intersection of localizations at height one prime ideals and if localization at each height one prime is a discrete valuation domain. A good source for divisor class groups is Robert Fossum's [F, The Divisor Class Group of a Krull domain, Springer, New York, 1973]. One among the many uses of the divisor class group of a Krull domain D is the fact that a Krull domain is a UFD if and only if $Cl(D) = Cl_v(D)$ is trivial. Thus a Krull domain D is a UFD if and only if every divisorial ideal of D is principal. If D is Krull and $Cl(D)$ is torsion we get a generalization of UFDs called Almost Factorial Domain (fast factoriell ringe) [F, page 33]. In [F] it was shown that a Krull domain D is almost factorial if and only if for each pair $f, g \in D\backslash\{0\}$ there is a natural number n such that $f^n D \cap g^n D$ is principal. The almost factorial domains were introduced by Storch [S, Fastfactorielle Ringe, Schriftenreihe Math. Inst. Univ. Munster, Heft 36(1967)].

Now it is well known that D is a Krull domain if and only if every nonzero ideal of D is t-invertible (see e.g. Theorem 2.3 of Houston and Zafrullah's [HZ, Michigan Math. J. 35(1988), 291-300]). Also note that every t-invertible t-ideal of any domain is actually a v-invertible v-ideal and hence divisorial. Further because every divisorial ideal is actually a t-ideal and hence in a Krull domain every divisorial ideal is actually a t-invertible t-ideal. Putting these observation together we conclude that for D Krull, $Inv_t(D) = Inv_v(D)$ and so $Cl_t(D) = Cl(D)$.

A PVMD is considered a good generalization of both of Prufer domains and Krull domains and the first impression was that the t-class group was a sort of generalization of the divisor class group. So the early impulse was to study it as such and find results corresponding to results on divisor class groups. Sure enough, for PVMDs the t-class group went smoothly and it was obvious that a PVMD D is a GCD domain if and only if $Cl_t(D)$ is trivial [B, Proposition 2]. But there was evidence that the t-class group could work for more general domains. Zafrullah [Z2, Manuscripta Math. 51(1985), 29-62] studied almost GCD (AGCD) domains as domains D such that for every pair $x, y \in D\backslash\{0\}$ there is a natural number $n = n(x, y)$ $x^n D \cap y^n D$ is principal. In [Z2] it was shown that a PVMD with torsion t-class group is AGCD and conversely. There was also a script [BZ1, On the class group] written by Bouvier based on my answers to to his questions while he was working on [B]. An improved version of that script appeared as Bouvier and Zafrullah's [BZ, On some class groups of an integral domain, Bull. Soc. Math. Grece. 29(1988), 45-59]. Driss Nour-El-Abidine and A. Rykaert wrote theses under the supervision of Bouvier at Univ. Claude Bernard, Lyon, France studying t-class groups. Later Anderson and Zafrullah in [J. Algebra, 142(2)(1991) 285-309] decided that every AGCD domain has torsion t-class group.

On the other hand a domain that satisfies ACC on divisorial ideals, called a Mori domain, is a generalization of both of Noetherian and Krull domains. Barucci and Gabelli studied the t-class groups of Mori domains. Their work is mentioned in David Anderson's second survey, "The class group and local class group of an integral domain" [A2, in Non-Noetherian commutative ring theory, 33-55, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000, by Chapman and Glaz]. The local class group is the group $Inv_t(D)/Inv(D)$ and it was Bouvier's work on the local class groups of Krull domains i.e. $\mathcal{D}(D)/Inv(D)$ [B2, Canad. Math. Bull. 26(1983) 13-19] that prompted me to suggest to him to work on the t-class groups.

There has been a lot of interest in the t-class groups and local class groups. Some of the contributors such as Marco Fontana and M.H. Park are mentioned in a recent paper by Anderson, Fontana and Zafrullah [J. Algebra 319(1)(2008), 272-295]. There you can also see the definition of the $*$-class group for the semistar operations and a decent introduction to semistar operations. In the Anderson-Fontana-Zafrullah paper you would also see, for the first time, a discussion of the v-class group of some integral domains. The notion of the t-class groups was also translated into the language of monoids by Halter-Koch [H-K, Ideal Systems, An introduction to ideal theory, Marcel Dekker, New York,1998]. There are many more contributors to the study of the t-class groups, such as Said El Baghdadi, M. Khalis, S. Kabbaj, G. Chang. If you read the literature you will find many more names. But I stop here because I think I have given enough introduction to the notion of $*$-class groups or $*$-Picard groups and to the literature.