least common multiple of r_2, r_1.

But since x,y are arbitrary and for each pair

$$xR_S \cap yR_S = r_1 R_S \cap r_2 R_S = [r_1, r_2] R_S$$ a principal

ideal, R_S is an HCF domain.

Proposition 10. A quasi local domain with Krull dimension 1 is a valuation domain iff it is an HCF domain.

Proof. If R is a domain as in the hypothesis and is HCF also, the result follows from Example 2 and from Lemma 8. The converse is obvious.

Corollary 6. For every minimal prime ideal P in an HCF domain R, R_P is a rank one valuation domain.

Proof. By Lemma 9 R_P is an HCF domain and since P is minimal, R_P is a one dimensional quasi local domain and so by Proposition 10, the result follows.

A simple but worthy of mention fact is recorded as

Proposition 11. If R is an integral domain in which every non zero non unit is expressible as a product of a finite number of quanta then the sufficient condition for R to be a GUFD is that it is an HCF domain.

Proof. By Lemma 8 above, every quantum of R in the hypothesis is a prime quantum. Thus every element x in R (other than zero or a unit) is expressible as the product of a finite number of prime quanta.

Let $x = p_1 p_2 \ldots p_n$, where p_i are prime quanta. Then if (say) p_1, p_2 are not distinct then by (3) of Lemma 1, $p_1 p_2$ is a prime quantum similar to p_1 and p_2, and after a finite number of steps we are able to express x as a prime quantum or as the product of a finite number of distinct prime quanta.

Cor 7. An atomic HCF domain is a UFD.

Now we have enough material to be able to prove