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# COMPLETELY INTEGRALLY CLOSED PRÜFER $v$ -MULTIPLICATION DOMAINS

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**ABSTRACT.** We study the effects on  $D$  of assuming that the power series ring  $D[[X]]$  is a  $v$ -domain or a PVMD. We show that a PVMD  $D$  is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $t$ -invertible  $t$ -ideal  $I$  of  $D$ . Using this, we show that if  $D$  is an AGCD domain, then  $D[[X]]$  is integrally closed if and only if  $D$  is a completely integrally closed PVMD with torsion  $t$ -class group. We also determine several classes of PVMDs for which being Archimedean is equivalent to being completely integrally closed and give some new characterizations of integral domains related to Krull domains.

## INTRODUCTION

The aim of this paper is to show that  $D$  is a  $v$ -domain when  $D[[X]]$  is a  $v$ -domain and to prove the following two results and record their consequences. Throughout,  $D$  is an integral domain with quotient field  $K$ . Other necessary definitions will be provided later.

**Theorem 0.1.** *Let  $D$  be an integral domain that is an intersection of localizations at divisorial prime ideals. If  $D[[X]]$  is a Prüfer  $v$ -multiplication domain (PVMD), then  $D$  is a  $v$ -domain that is an intersection of essential discrete rank-one valuation domains, and thus is completely integrally closed.*

**Theorem 0.2.** *A PVMD  $D$  is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $t$ -invertible  $t$ -ideal  $I$  of  $D$ .*

Several classes of integral domains of interest, such as Noetherian, Krull, Mori, and the so-called H-domains that have maximal  $t$ -ideals divisorial, fall under the umbrella of integral domains that are intersections of localizations at divisorial prime ideals. We show that an H-domain  $D$  is a Krull domain if and only if  $D[[X]]$  is a PVMD. As a consequence of Theorem 0.2, we show that if  $D$  is an almost GCD (AGCD) domain, then  $D[[X]]$  is integrally closed if and only if  $D$  is a completely integrally closed PVMD with torsion  $t$ -class group. We also show that if  $D$  is an AGCD domain such that  $D[[X]]$  is integrally closed and every nonzero nonunit of  $D$  has only finitely many minimal prime ideals, then  $D$  is a locally finite intersection of rank-one valuation domains. We isolate a property of integral domains of finite  $t$ -character and use it in combination with complete integral closure to give some new characterizations of integral domains related to Krull domains and their generalizations. We also answer a recently asked question about the ring of power series over a Krull-like PVMD.

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As our work involves star operations, it seems pertinent to give the reader an idea of some of the notions involved. Let  $D$  be an integral domain with quotient field  $K$ , and let  $F(D)$  (resp.,  $f(D)$ ) be the set of nonzero fractional ideals (resp., nonzero finitely generated fractional ideals) of  $D$ .

A *star operation*  $*$  on  $D$  is a function  $*$ :  $F(D) \rightarrow F(D)$  that satisfies the following properties for every  $I, J \in F(D)$  and  $0 \neq x \in K$ :

- (i)  $(x)^* = (x)$  and  $(xI)^* = xI^*$ ,
- (ii)  $I \subseteq I^*$ , and  $I^* \subseteq J^*$  whenever  $I \subseteq J$ , and
- (iii)  $(I^*)^* = I^*$ .

An  $I \in F(D)$  is called a *\*-ideal* if  $I^* = I$  and a *\*-ideal of finite type* if  $I = J^*$  for some  $J \in f(D)$ . A star operation  $*$  is said to be of *finite character* if  $I^* = \bigcup \{J^* \mid J \subseteq I \text{ and } J \in f(D)\}$ . For  $I \in F(D)$ , let  $I_d = I$ ,  $I^{-1} = (D :_K I) = \{x \in K \mid xI \subseteq D\}$ ,  $I_v = (I^{-1})^{-1}$ ,  $I_t = \bigcup \{J_v \mid J \subseteq I \text{ and } J \in f(D)\}$ , and  $I_w = \{x \in K \mid xJ \subseteq I \text{ for some } J \in f(D) \text{ with } J_v = D\}$ . A  $v$ -ideal is sometimes also called a *divisorial ideal*. The functions defined by  $I \mapsto I_d$ ,  $I \mapsto I_v$ ,  $I \mapsto I_t$ , and  $I \mapsto I_w$  are all examples of star operations. Given two star operations  $*_1, *_2$  on  $D$ , we say that  $*_1 \leq *_2$  if  $I^{*_1} \subseteq I^{*_2}$  for every  $I \in F(D)$ . Note that  $*_1 \leq *_2$  if and only if  $(I^{*_1})^{*_2} = (I^{*_2})^{*_1} = I^{*_2}$  for every  $I \in F(D)$ . The  $d$ -operation,  $t$ -operation, and  $w$ -operation all have finite character,  $d \leq \rho \leq v$  for every star operation  $\rho$ , and  $\rho \leq t$  for every star operation  $\rho$  of finite character. We will often use the two facts that  $(IJ)^* = (IJ^*)^* = (I^*J^*)^*$  for every star operation  $*$  and  $I, J \in F(D)$  and  $I_v = I_t$  for every  $I \in f(D)$ . An  $I \in F(D)$  is said to be *\*-invertible* if  $(II^{-1})^* = D$ . If  $I$  is  $*$ -invertible for  $*$  of finite character, then both  $I^*$  and  $I^{-1}$  are  $v$ -ideals of finite type. The reader in need of more introduction may consult [42] or [24, Sections 32 and 34].

For a star operation  $*$ , a *maximal \*-ideal* is an integral  $*$ -ideal that is maximal among proper integral  $*$ -ideals. Let  $\text{*-Max}(D)$  be the set of maximal  $*$ -ideals of  $D$ . For a star operation  $*$  of finite character, it is well known that a maximal  $*$ -ideal is a prime ideal; every proper integral  $*$ -ideal is contained in a maximal  $*$ -ideal; and  $\text{*-Max}(D) \neq \emptyset$  if  $D$  is not a field. Moreover,  $t\text{-Max}(D) = w\text{-Max}(D)$ ;  $I_w = \bigcap_{M \in t\text{-Max}(D)} ID_M$  for every  $I \in F(D)$ ; and  $I_w D_M = ID_M$  for every  $I \in F(D)$  and  $M \in t\text{-Max}(D)$ .

Recall that an integral domain  $D$  with quotient field  $K$  is *completely integrally closed* if whenever  $rx^n \in D$  for  $x \in K$ ,  $0 \neq r \in D$ , and every integer  $n \geq 1$ , then  $x \in D$ . Equivalently,  $D$  is completely integrally closed if and only if  $(II^{-1})_v = D$  for every  $I \in F(D)$  [24, Theorem 34.3]. We will use the well-known facts that a completely integrally closed domain is integrally closed [24, Theorem 13.1(2)], an intersection of completely integrally closed domains is completely integrally closed,  $D[[X]]$  is completely integrally closed if and only if  $D$  is completely integrally closed [24, Theorem 13.9], a Krull domain is completely integrally closed, and a valuation domain  $D$  is completely integrally closed if and only if  $D$  has rank at most one [24, Theorem 17.5(3)].

We say that an integral domain  $D$  is a *Prüfer  $v$ -multiplication domain* (PVMD) if every nonzero finitely generated ideal  $I$  of  $D$  is  $t$ -invertible, i.e.,  $(II^{-1})_t = D$  for every  $I \in f(D)$ . If  $D_P$  is a valuation domain for a nonzero prime ideal  $P$  of  $D$ , then  $P$  is necessarily a  $t$ -ideal of  $D$ . For PVMDs, the converse is true. Indeed, Griffin [26, Theorem 5] showed that  $D$  is a PVMD if and only if  $D_M$  is a valuation domain for every maximal  $t$ -ideal  $M$  of  $D$ . As indicated in [42], Kang [30] showed that an

integrally closed domain  $D$  is a PVMD if and only if  $t = w$  over  $D$ . An integral domain  $D$  is a *v-domain* if every nonzero finitely generated ideal  $I$  of  $D$  is *v*-invertible, i.e.,  $(II^{-1})_v = D$  for every  $I \in f(D)$ . Equivalently,  $D$  is a PVMD (resp., *v-domain*) if and only if for every  $I \in f(D)$ , there is a  $J \in f(D)$  (resp.,  $J \in F(D)$ ) such that  $(IJ)_v = D$ . Thus, a PVMD is a *v-domain*. Note that a *v-domain* (and hence a PVMD) is integrally closed and a completely integrally closed domain is a *v-domain*. It can be shown that  $D$  is a PVMD (resp., *v-domain*) if and only if every nonzero two-generated ideal of  $D$  is *t*-(resp., *v*-)invertible [2, Theorem 2.2]. From this, it is easy to conclude that a *v-domain*  $D$  is a PVMD if and only if  $aD \cap bD$  is a *v*-ideal of finite type for every  $0 \neq a, b \in D$ .

A valuation overring  $V$  of an integral domain  $D$  is called an *essential valuation domain* if  $V = D_P$  for some prime ideal  $P$  of  $D$  ( $P$  is called an *essential* or *valued-prime ideal*). We call  $D$  an *essential domain* if  $D = \bigcap_{P \in F} D_P$  for some family  $F$  of essential prime ideals of  $D$ . An essential domain is integrally closed, and a PVMD is an essential domain since  $D = \bigcap_{M \in t\text{-Max}(D)} D_M$ .

An integral domain  $D$  is called a *GCD domain* if  $(a) \cap (b)$  is principal for every  $0 \neq a, b \in D$  and an *almost GCD (AGCD) domain* if for every  $0 \neq a, b \in D$ , there is an integer  $n \geq 1$  such that  $(a^n) \cap (b^n)$  is principal. Thus, a GCD domain is a PVMD in which every *v*-ideal of finite type is principal, and a GCD domain is an AGCD domain. AGCD domains were introduced in [40] and further studied in [10]. It is well known that  $D$  is an AGCD domain if and only if for every  $0 \neq a_1, \dots, a_s \in D$ , there is an integer  $k \geq 1$  such that  $(a_1^k, \dots, a_s^k)_v$  is principal [10, Remark after Lemma 3.3].

The set  $t\text{-inv}(D)$  of *t*-invertible fractional *t*-ideals of  $D$  is an abelian group under the *t*-multiplication  $I * J = (IJ)_t$ . Its subset  $P(D)$  of nonzero principal fractional ideals is a subgroup of  $t\text{-inv}(D)$ . The quotient group  $t\text{-inv}(D)/P(D)$  is called the *class group* (or *t-class group*) of  $D$  and is usually denoted by  $Cl_t(D)$ . The group  $Cl_t(D)$  was introduced in [15], where it was pointed out that  $Cl_t(D)$  is the divisor class group when  $D$  is a Krull domain and  $Cl_t(D)$  is the ideal class group when  $D$  is a Prüfer domain. Also, it was shown in [40, Corollary 3.8 and Theorem 3.9] that an integrally closed AGCD domain is a PVMD with torsion *t*-class group and that a PVMD with torsion *t*-class group is an AGCD domain. Thus, an integral domain  $D$  is a PVMD with torsion *t*-class group if and only if  $D$  is an integrally closed AGCD domain. For more on the *t*-class group, see [12].

In Section 1, we show that if  $D[[X]]$  is a *v-domain*, then  $D$  is a *v-domain*, but not necessarily conversely. We also show that if  $D$  is an H-domain (every maximal *t*-ideal of  $D$  is divisorial), then  $D[[X]]$  is a PVMD if and only if  $D$  is a Krull domain. This answers a question recently raised in [21]. In Section 2, we show that if an integral domain  $D$  is completely integrally closed, then  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every ideal  $I$  of  $D$  with  $I_v \subsetneq D$  and that the complete integral closure  $D''$  of a PVMD  $D$  is a generalized ring of fractions  $D_S$ , where  $S$  is the multiplicatively closed set of *t*-invertible *t*-ideals  $I$  of  $D$  such that  $\bigcap_{n=1}^{\infty} (I^n)_v \neq (0)$ , thus establishing Theorem 0.2. In this section, we also determine several special classes of PVMDs  $D$ , including PVMDs with torsion *t*-class group, whose being completely integrally closed requires only that  $D$  be Archimedean, i.e.,  $\bigcap_{n=1}^{\infty} (x^n) = (0)$  for every nonunit  $x \in D$ . In Sections 3 and 4, we continue the work begun in Section 2 and use the notion of a potent maximal *t*-ideal from [5] to provide new characterizations of integral domains of interest, such as UFDs, PIDs, Krull domains, and generalized

Krull domains. In Section 5, we investigate several “Archimedean-like” conditions for an integral domain.

### 1. WHEN $D[[X]]$ IS A $v$ -DOMAIN

We will need the following results from [17] on ideals in power series rings, in connection with star operations.

**Lemma 1.1.** ([17, Proposition 2.1]) *Let  $I$  be a nonzero fractional ideal of an integral domain  $D$ .*

- (1)  $(ID[[X]])^{-1} = I^{-1}[[X]] = (I[[X]])^{-1}$ .
- (2)  $(ID[[X]])_v = I_v[[X]] = (I[[X]])_v$ .

Lemma 1.1 was attributed to D. F. Anderson and B.G. Kang in [17]. Using Lemma 1.1, we first prove the following result.

**Proposition 1.2.** *If  $D$  is an integral domain such that  $D[[X]]$  is a  $v$ -domain, then  $D$  is a  $v$ -domain.*

*Proof.* Let  $D[[X]]$  be a  $v$ -domain; so  $(JJ^{-1})_v = D[[X]]$  for every nonzero finitely generated ideal  $J$  of  $D[[X]]$ . In particular, let  $J = ID[[X]]$  for  $I \in f(D)$ . Then  $D[[X]] = ((ID[[X]])(ID[[X]])^{-1})_v \subseteq (I[[X]]I^{-1}[[X]])_v \subseteq ((II^{-1})[[X]])_v = (II^{-1})_v[[X]] \subseteq D[[X]]$  by Lemma 1.1; so  $(II^{-1})_v[[X]] = D[[X]]$ . Thus,  $(II^{-1})_v = D$  for every  $I \in f(D)$ ; so  $D$  is a  $v$ -domain.  $\square$

To see that the converse of Proposition 1.2 is not true, note that any  $v$ -domain  $D$  with a nonunit  $x$  such that  $\bigcap_{n=1}^{\infty} (x^n) \neq (0)$  can serve as a counterexample. Suppose that  $D[[X]]$  is a  $v$ -domain. Then, in particular,  $D[[X]]$  is integrally closed. But, by [32, Theorem 0.1] (or [24, Theorem 13.10 and Proposition 13.11]),  $D[[X]]$  is integrally closed implies that  $D$  is integrally closed and  $\bigcap_{n=1}^{\infty} (x^n) = (0)$  for every nonunit  $x \in D$ . The presence of a nonunit  $x$  with  $\bigcap_{n=1}^{\infty} (x^n) \neq (0)$  will contradict this. Now, take  $D$  to be a rank-two valuation domain and  $x \in D$  a nonunit that is not in the height-one prime ideal of  $D$ ; so  $\bigcap_{n=1}^{\infty} (x^n) \neq (0)$ . Then  $D$  is a PVMD, and hence a  $v$ -domain, but  $D[[X]]$  is not integrally closed, and thus not a  $v$ -domain.

**Remark 1.3.** In general, for ideals  $I$  and  $J$  of  $D$ ,  $I[[X]]J[[X]] \subseteq IJ[[X]]$ , but  $I[[X]]J[[X]] \neq IJ[[X]]$ . For an example showing that generally  $I[[X]]J[[X]] \neq IJ[[X]]$ , see [8, page 352].

**Corollary 1.4.** *If  $D$  is an integral domain such that  $D[[X]]$  is completely integrally closed, then  $D$  is completely integrally closed.*

The proof entails noting that  $D[[X]]$  is completely integrally closed if and only if  $(JJ^{-1})_v = D[[X]]$  for every non-zero ideal  $J$  of  $D[[X]]$ , and taking, in particular,  $J = ID[[X]]$  for any nonzero ideal  $I$  of  $D$  as in the above proof. However, it is well known that  $D[[X]]$  is completely integrally closed if and only if  $D$  is completely integrally closed.

An integral domain is called a *generalized Krull domain* (cf. [24, page 524]) if it is a locally finite intersection of essential rank-one valuation domains. This terminology goes back at least to Griffin [27], and such rings were considered by Ribenboim [36]. Popescu [35] introduced the notion of a generalized Dedekind domain via localizing systems. Nowadays, the following equivalent definition is usually given: an integral domain is a *generalized Dedekind domain* if it is a strongly discrete Prüfer domain (i.e.,  $P \neq P^2$  for every prime ideal  $P$ ) and every (prime) ideal

$I$  has  $\sqrt{I} = \sqrt{(a_1, \dots, a_n)}$  for some  $a_1, \dots, a_n \in I$  (or equivalently, every principal ideal has only finitely many minimal prime ideals). (To add to the confusion, Zafrullah [41] defined an integral domain to be a generalized Dedekind domain if every divisorial ideal is invertible. In [7], these rings were called pseudo-Dedekind domains in analogy with pseudo-principal ideal domains, i.e., integral domains in which every divisorial ideal is principal.) Based on the strongly discrete definition of a generalized Dedekind domain, El Baghdadi [19] defined an integral domain  $D$  to be a generalized Krull domain if it is a strongly discrete PVMD (i.e.,  $D_M$  is a strongly discrete valuation domain for every maximal  $t$ -ideal  $M$  of  $D$ ) and every principal ideal has only finitely many minimal prime ideals, or equivalently,  $D$  is a PVMD with  $P \neq (P^2)_t$  and  $P = \sqrt{J}_t$  for some finitely generated ideal  $J$  of  $D$  for every prime  $t$ -ideal  $P$  of  $D$ . To avoid confusion, we (and hopefully others), will use the terminology “generalized Krull domain” as defined by Griffin and will call the generalized Krull domains as defined by El Baghdadi *Krull-like PVMDs*. In particular, Krull-like PVMDs are integrally closed, and generalized Krull domains are completely integrally closed. While both generalized Krull domains and Krull-like PVMDs are, of course, PVMDs, neither definition implies the other. For example, while any valuation domain  $D$  is a PVMD,  $D$  is a generalized Krull domain (resp., Krull-like PVMD) if and only if  $D$  has rank at most one (resp., is strongly discrete).

In [21, Question 2.4(2)], El Baghdadi and Kim asked the following question: If  $D$  is a Krull-like PVMD domain, is  $D[[X]]$  a Krull-like PVMD? The next example gives a negative answer to their question. The question of [21] can also be answered in another way via Corollary 1.9.

**Example 1.5.** Let  $D$  be a Krull domain that is not a field. Then, for every multiplicative subset  $S$  of  $D$  with at least one nonunit of  $D$ , the ring  $R = D + YD_S[Y]$  is a Krull-like PVMD such that  $R[[X]]$  is not a Krull-like PVMD. (For a specific example, let  $R = \mathbb{Z} + Y\mathbb{Q}[Y]$ .) To see this, note that if  $D$  is a Krull domain, then  $D + YD_S[Y]$  is a PVMD for every multiplicative subset  $S$  of  $D$  [4, Corollary 2.7]. Also, as  $D$  is a Krull domain and  $D + YD_S[Y]$  is a PVMD,  $R = D + YD_S[Y]$  is a Krull-like PVMD [20, Proposition 3.4]. Now, let  $d \in D$  be one of the promised nonzero nonunits in  $S$ . Then  $(0) \neq YD_S[Y] \subseteq \bigcap_{n=1}^{\infty} d^n R$ , and as in the discussion concerning the failure of the “converse” of Proposition 1.2,  $R[[X]]$  is not integrally closed. Thus,  $R[[X]]$  is not a Krull-like PVMD.

We now give the proof of Theorem 0.1 from the Introduction.

*Proof.* (of Theorem 0.1) Let  $D = \bigcap_{P \in F} D_P$ , where  $F$  is a set of divisorial prime ideals of  $D$ , and suppose that  $D[[X]]$  is a PVMD, and hence a  $v$ -domain. Every  $P \in F$  is divisorial, and so  $P[[X]]$  is a divisorial ideal of  $D[[X]]$  by Lemma 1.1. It is well known that if  $P$  is a prime ideal of  $D$ , then  $P[[X]]$  is a prime ideal of  $D[[X]]$ . Also, every divisorial ideal is a  $t$ -ideal. So for  $P \in F$ , the prime ideal  $P[[X]]$  is a  $t$ -ideal of the PVMD  $D[[X]]$ . Thus,  $D[[X]]_{P[[X]]}$  is a valuation domain, and hence  $D_P$  is an essential discrete rank-one valuation domain [13, Theorem 1]. Thus,  $D_P$  is completely integrally closed for every  $P \in F$ , and hence  $D = \bigcap_{P \in F} D_P$  is completely integrally closed. That  $D$  is a  $v$ -domain follows from Proposition 1.2 or the fact that a completely integrally closed domain is a  $v$ -domain.  $\square$

We do not know if the integral domain  $D$  in Theorem 0.1 is actually a PVMD, but for a  $v$ -domain  $D$  to be a PVMD, all we need to check is that  $D$  is a  $v$ -finite

conductor domain, i.e.,  $aD \cap bD$  is a  $v$ -ideal of finite type for every  $0 \neq a, b \in D$  [23, Corollary 4]. Thus, we have the following result.

**Corollary 1.6.** *If  $D$  is a  $v$ -finite conductor domain and  $D[[X]]$  is a  $v$ -domain, then  $D$  is a PVMD.*

In some instances, the  $v$ -finite conductor property gets provided by indirect means.

**Corollary 1.7.** *Let  $D$  be an integral domain that is a locally finite intersection of localizations at divisorial prime ideals. Then  $D[[X]]$  is a PVMD if and only if  $D$  is a Krull domain.*

*Proof.* Suppose that  $D[[X]]$  is a PVMD. By the proof of Theorem 0.1,  $D$  is a locally finite intersection of discrete rank-one valuation domains. Thus,  $D$  is a Krull domain. Conversely, if  $D$  is a Krull domain, then  $D[[X]]$  is a Krull domain [24, Corollary 44.11], and hence a PVMD.  $\square$

Of course, there is yet another way that the  $v$ -finite conductor condition becomes available free of charge. Recall that an integral domain  $D$  is called an  $H$ -domain if for every nonzero ideal  $I$  of  $D$ ,  $I^{-1} = D$  implies that there is a finitely generated ideal  $F \subseteq I$  such that  $F^{-1} = D$ . It was shown by Houston and Zafrullah [29, Theorem 2.4] that  $D$  is an  $H$ -domain if and only if every maximal  $t$ -ideal of  $D$  is divisorial.  $H$ -domains were introduced by Glaz and Vasconcelos in [25], where it was shown that a completely integrally closed  $H$ -domain is a Krull domain [25, 3.2d]. Indeed, a Krull domain is an  $H$ -domain. In fact, a Krull-like PVMD is an  $H$ -domain since every maximal  $t$ -ideal is  $t$ -invertible [19, Corollary 3.6], and hence divisorial. Thus, a Krull-like PVMD is a Krull domain if and only if it is completely integrally closed. With this introduction, we state the following result.

**Corollary 1.8.** *The following statements are equivalent for an  $H$ -domain  $D$ .*

- (1)  $D[[X]]$  is a PVMD.
- (2)  $D$  is completely integrally closed.
- (3)  $D$  is a Krull domain.
- (4)  $D[[X]]$  is completely integrally closed.

*Proof.* (1)  $\Rightarrow$  (2) Being an  $H$ -domain,  $D$  is an intersection of localizations at divisorial prime ideals. By Theorem 0.1,  $D[[X]]$  is a PVMD implies that  $D$  is completely integrally closed.

(2)  $\Rightarrow$  (3) A completely integrally closed  $H$ -domain is a Krull domain [25, 3.2d].

(3)  $\Rightarrow$  (4)  $D$  is a Krull domain implies that  $D[[X]]$  is a Krull domain [24, Corollary 44.11], and thus  $D[[X]]$  is completely integrally closed.

(4)  $\Rightarrow$  (1)  $D[[X]]$  is completely integrally closed implies that  $D$  is completely integrally closed, and a completely integrally closed  $H$ -domain is a Krull domain [25, 3.2d]. Thus,  $D[[X]]$  is a Krull domain, and hence a PVMD.  $\square$

**Corollary 1.9.** *Let  $D$  be a Krull-like PVMD. Then  $D[[X]]$  is a Krull-like PVMD if and only if  $D$  is a Krull domain.*

*Proof.* We have already observed that a Krull-like PVMD is an  $H$ -domain. The corollary now follows directly from Corollary 1.8.  $\square$

Corollary 1.9 shows that the answer to the El Baghdadi-Kim question is, generally no. Specifically, let  $D$  be a Krull-like PVMD that is not a Krull domain (e.g.,

$\mathbb{Z} + Y\mathbb{Q}[Y]$  as in Example 1.5, or a strongly discrete rank-two valuation domain). Then  $D[[X]]$  is not a Krull-like PVMD. (We are thankful to Said El-Baghdadi for support in the form of advice and references for this section.)

## 2. WHEN $D[[X]]$ IS INTEGRALLY CLOSED FOR $D$ A PVMD

An integral domain  $D$  is *Archimedean* if  $\bigcap_{n=1}^{\infty} (x^n) = (0)$  for every nonunit  $x \in D$ . According to [32, Theorem 0.1] (or [24, Theorem 13.10 and Proposition 13.11]), if  $D[[X]]$  is integrally closed, then  $D$  is integrally closed and Archimedean. Although a completely integrally closed domain is Archimedean [24, Corollary 13.4] (cf. Corollary 2.4), an Archimedean domain need not be completely integrally closed since any one-dimensional domain, Noetherian domain, or more generally, an integral domain satisfying ACCP is Archimedean. However, we do not know of an example of a PVMD, let alone a Prüfer domain,  $D$  such that  $D[[X]]$  is integrally closed, but  $D$  is not completely integrally closed, or such that  $D$  is Archimedean, but not completely integrally closed. So there seems to be no harm in putting forward a “lame” conjecture that if  $D$  is a PVMD such that  $D[[X]]$  is integrally closed, then  $D$  is completely integrally closed, which is implied by our second “lame” conjecture that a PVMD  $D$  is completely integrally closed if and only if  $D$  is Archimedean. These conjectures are “lame” in that there is very little hope of them being true. Yet, there is every hope of generating interest in producing counterexamples to them.

**Conjecture 2.1.** Let  $D$  be a PVMD. Then  $D[[X]]$  is integrally closed if and only if  $D[[X]]$  is completely integrally closed (if and only if  $D$  is completely integrally closed).

**Conjecture 2.2.** Let  $D$  be a PVMD. Then  $D$  is completely integrally closed if and only if  $D$  is Archimedean.

These conjectures certainly hold for valuation domains and are somewhat supported by the fact, that will soon become apparent, that if  $D$  is a GCD domain (in fact, an AGCD domain) and  $D[[X]]$  is integrally closed, then  $D$  must be completely integrally closed. This follows from the fact that a GCD domain  $D$  is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} (x^n) = (0)$  for every nonunit  $x \in D$  (see Corollary 2.7(1)).

We next define an “Archimedean-like” condition that is equivalent to being completely integrally closed. We say that an integral domain  $D$  is *strongly Archimedean* if  $\bigcap_{n=1}^{\infty} (a/b)^n = (0)$  for every  $a, b \in D$  with  $(b) \not\subseteq (a)$ . A strongly Archimedean domain is certainly Archimedean. However, a Noetherian domain is always Archimedean, but is strongly Archimedean if and only if it is (completely) integrally closed. Related “Archimedean-like” conditions will be studied in Section 5.

**Proposition 2.3.** An integral domain  $D$  is strongly Archimedean if and only if  $D$  is completely integrally closed.

*Proof.* Let  $0 \neq a, b \in D$ . Then  $\bigcap_{n=1}^{\infty} (a/b)^n \neq (0)$  if and only if  $b/a \in D''$ , the complete integral closure of  $D$ , and  $(b) \subseteq (a)$  if and only if  $b/a \in D$ . Thus,  $D$  is strongly Archimedean if and only if  $D$  is completely integrally closed.  $\square$

**Corollary 2.4.** Let  $D$  be a completely integrally closed domain. Then  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every ideal  $I$  of  $D$  with  $I_v \subsetneq D$ . In particular, a completely integrally closed domain is Archimedean.

*Proof.* Since  $I_v$  is a proper divisorial ideal of  $D$ ,  $I \subseteq (a/b)$  for some  $a, b \in D$  with  $(b) \not\subseteq (a)$ . Thus,  $(I^n)_v \subseteq (a/b)^n$  for every integer  $n \geq 1$ . Hence,  $\bigcap_{n=1}^{\infty} (I^n)_v \subseteq \bigcap_{n=1}^{\infty} (a/b)^n = (0)$  since a completely integrally closed domain is strongly Archimedean. The “in particular” statement is clear.  $\square$

The following result sets the stage for a possible resolution of Conjecture 2.2, at least in some special cases. For better reading, however, we include some explanation of the notions mentioned in the proposition that follows.

A nonempty family  $S$  of nonzero ideals of an integral domain  $D$  is said to be a *multiplicative system of ideals* if  $IJ \in S$  for every  $I, J \in S$ . If  $S$  is a multiplicative system of ideals, then the set of ideals of  $D$  each containing some ideal of  $S$  is still a multiplicative system, which is called the *saturation of  $S$* , and is denoted by  $Sat(S)$ . A multiplicative system  $S$  is said to be *saturated* if  $S = Sat(S)$ . If  $S$  is a multiplicative system of ideals, then the overring  $D_S = \bigcup \{ (D :_K J) \mid J \in S \}$  of  $D$  is called the *generalized ring of fractions* (or *generalized transform*) of  $D$  with respect to  $S$ . Indeed,  $D_S = D_{Sat(S)}$ , and note that  $\{J_v \mid J \in S\} \subseteq Sat(S)$ .

**Proposition 2.5.** *Let  $D$  be a PVMD. Then the complete integral closure  $D''$  of  $D$  is the generalized ring of fractions  $D_S$ , where  $S = \{I \mid I \subseteq D \text{ is } t\text{-invertible and } \bigcap_{n=1}^{\infty} (I^n)_v \neq (0)\}$  is a multiplicative system of ideals.*

*Proof.* We first show that  $S$  is a multiplicative system of ideals. Let  $I, J \in S$ , and let  $0 \neq x \in \bigcap_{n=1}^{\infty} (I^n)_v$  and  $0 \neq y \in \bigcap_{n=1}^{\infty} (J^n)_v$ . Then  $IJ$  is  $t$ -invertible and  $0 \neq xy \in (I^n)_v (J^n)_v \subseteq ((I^n)_v (J^n)_v)_v = (I^n J^n)_v = ((IJ)^n)_v$  for every integer  $n \geq 1$ . Thus,  $\bigcap_{n=1}^{\infty} ((IJ)^n)_v \neq (0)$ ; so  $IJ \in S$ .

We now show that  $D'' = D_S$ . Let  $x \in D_S$ . Then  $xI \subseteq D$  for some  $I \in S$ ; so  $x^n(I^n)_v = (x^n I^n)_v = ((xI)^n)_v \subseteq D$  for every integer  $n \geq 1$ . Let  $0 \neq d \in \bigcap_{n=1}^{\infty} (I^n)_v$ . Then  $dx^n \in D$  for every integer  $n \geq 1$ , and so  $x \in D''$ . Thus,  $D_S \subseteq D''$ . For the reverse inclusion, suppose that  $a/b \in D''$  for  $0 \neq a, b \in D$ . Then, there is a  $0 \neq d \in D$  such that  $d(a/b)^n \in D$  for every integer  $n \geq 1$ . Hence,  $d \in (b^n) : (a^n)$  for every integer  $n \geq 1$ . Since  $D$  is a PVMD and  $(b^n) : (a^n)$  is divisorial, we have  $(b^n) : (a^n) = ((b^n) : (a^n))_w = \bigcap_{M \in t\text{-Max}(D)} ((b^n) : (a^n)) D_M = \bigcap_{M \in t\text{-Max}(D)} ((b) : (a))^n D_M = (((b) : (a))^n)_w$  for every integer  $n \geq 1$  because  $D_M$  is a valuation domain for every  $M \in t\text{-Max}(D)$ . Thus,  $(b^n) : (a^n) = (((b) : (a))^n)_v$  for every integer  $n \geq 1$  because  $(b^n) : (a^n)$  is divisorial and  $w \leq v$ . Hence,  $0 \neq d \in \bigcap_{n=1}^{\infty} ((b^n) : (a^n)) = \bigcap_{n=1}^{\infty} (((b) : (a))^n)_v$ . Thus,  $I = (b) : (a) \in S$  because  $(b) : (a)$  is  $t$ -invertible and  $0 \neq d \in \bigcap_{n=1}^{\infty} (I^n)_v$ . Hence,  $a/b \in D_S$  because  $(a/b)I = (a/b)((b) : (a)) \subseteq D$ ; so  $D'' \subseteq D_S$ . Thus,  $D'' = D_S$ .  $\square$

We can now give the proof of Theorem 0.2 from the Introduction.

*Proof.* (of Theorem 0.2) Let  $D$  be a completely integrally closed PVMD. Then, by Corollary 2.4,  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper divisorial ideal  $I$  of  $D$ , and thus  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $t$ -invertible  $t$ -ideal  $I$  of  $D$ . Alternatively, one can use Proposition 2.5. Conversely, suppose that  $D$  is a PVMD with  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $t$ -invertible  $t$ -ideal  $I$  of  $D$ . Then  $S = \{D\}$ ; so  $D = D_S = D''$  by Proposition 2.5. Thus,  $D$  is completely integrally closed.  $\square$

Recall that an integral domain  $D$  is a *generalized GCD (GGCD) domain* if every finite type  $v$ -ideal of  $D$  is invertible. GGCD domains were studied in [1], where it was shown that the complete integral closure of a GGCD domain is an



invertible generalized transform. Proposition 2.5 is an extension of that result. Because a GGCD domain is a PVMD in which every  $t$ -invertible  $t$ -ideal is actually invertible, [1, Theorem 5] and its corollary [1, Corollary 3] become special cases of Proposition 2.5 and Theorem 0.2, respectively. Of course, the next corollary is true for any completely integrally closed integral domain.

**Corollary 2.6.** *If a PVMD  $D$  is completely integrally closed, then  $D$  is Archimedean.*

**Corollary 2.7.** (1) ([14, Theorem 3.1]) *A GCD domain  $D$  is completely integrally closed if and only if  $D$  is Archimedean.*

(2) ([1, Corollary 3]) *A GGCD domain  $D$  is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} I^n = (0)$  for every proper invertible ideal  $I$  of  $D$ .*

(3) ([24, Corollary 26.9]) *A Prüfer domain  $D$  is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} I^n = (0)$  for every proper invertible ideal  $I$  of  $D$ .*

*Proof.* Note that a GCD (resp., GGCD or Prüfer) domain  $D$  is a PVMD in which every  $t$ -invertible  $t$ -ideal is principal (resp., invertible).  $\square$

We would, of course, like to resolve the two conjectures one way or another. One way of doing that would be to establish the connection, if one exists, between a PVMD  $D$  being completely integrally closed and its Kronecker function ring  $T$  (or the ring  $D\{X\} = D[X]_{N_v}$  [30]) being completely integrally closed. For the Kronecker function ring  $T$  (or  $D\{X\}$ ) is a Bezout domain, which being a GCD domain, is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} (x^n) = (0)$  for every nonunit  $x \in T$  (or  $D\{X\}$ ). In the absence of any insight in that direction, we are reduced to making the best of the situation.

If we can link every proper  $t$ -invertible  $t$ -ideal  $I$  of a PVMD  $D$  with a nonunit  $x \in D$  such that  $\bigcap_{n=1}^{\infty} (I^n)_v \subseteq (x^m)$  for every integer  $m \geq 1$ , then  $D$  being Archimedean would be equivalent to  $D$  being completely integrally closed. This can be done in two distinct ways, one computational and the other theoretical; we pursue both courses.

**Lemma 2.8.** *Let  $I$  and  $J$  be ideals of an integral domain  $D$ .*

(1) *If  $I \subseteq J$ , then  $\bigcap_{n=1}^{\infty} I^n \subseteq \bigcap_{n=1}^{\infty} J^n$  and  $\bigcap_{n=1}^{\infty} (I^n)_v \subseteq \bigcap_{n=1}^{\infty} (J^n)_v$ .*

(2) *If  $I^k \subseteq (x)$  for some  $x \in D$  and integer  $k \geq 1$ , then*

$$\bigcap_{n=1}^{\infty} I^n \subseteq \bigcap_{n=1}^{\infty} (I^n)_v \subseteq \bigcap_{n=1}^{\infty} (x^n).$$

*Proof.* (1) is obvious. For (2), first note that  $\bigcap_{n=1}^{\infty} (I^{nk})_v = \bigcap_{n=1}^{\infty} (I^n)_v$  and  $\bigcap_{n=1}^{\infty} I^n \subseteq \bigcap_{n=1}^{\infty} (I^n)_v$ . If  $I^k \subseteq (x)$ , then  $(I^{nk})_v \subseteq (x^n)$  for every integer  $n \geq 1$ , and thus  $\bigcap_{n=1}^{\infty} I^n \subseteq \bigcap_{n=1}^{\infty} (I^n)_v = \bigcap_{n=1}^{\infty} (I^{nk})_v \subseteq \bigcap_{n=1}^{\infty} (x^n)$ .  $\square$

**Proposition 2.9.** *Let  $D$  be a PVMD such that for every  $0 \neq a_1, \dots, a_s \in D$  with  $(a_1, \dots, a_s)_v \neq D$ , there is an integer  $k \geq 1$  and a nonunit  $d \in D$  such that  $(a_1^k, \dots, a_s^k) \subseteq (d)$ . Then  $D$  is completely integrally closed if and only if  $D$  is Archimedean.*

*Proof.* A completely integrally closed domain is always Archimedean. For the converse, suppose that  $D$  is Archimedean, that is,  $\bigcap_{n=1}^{\infty} (x^n) = (0)$  for every nonunit  $x \in D$ . Now, take a proper  $t$ -invertible  $t$ -ideal  $I$  of  $D$ . Then  $I = (a_1, \dots, a_s)_v$  for some  $0 \neq a_1, \dots, a_s \in D$ . By hypothesis, there is an integer  $k \geq 1$  and a nonunit  $d \in D$  such that  $(a_1^k, \dots, a_s^k) \subseteq (d)$ , and thus  $(a_1^k, \dots, a_s^k)_v \subseteq (d)$ . Now, as  $D$  is a PVMD, we have  $(a_1^k, \dots, a_s^k)_v = ((a_1, \dots, a_s)^k)_v$  [10, Lemma 3.3]; so  $I$  is  $t$ -invertible and  $(I^k)_v \subseteq (d)$ . Hence,  $\bigcap_{n=1}^{\infty} (I^n)_v \subseteq \bigcap_{n=1}^{\infty} (d^n)$  by Lemma 2.8(2). But

$D$  is Archimedean, and thus  $\bigcap_{n=1}^{\infty} (I^n)_v \subseteq \bigcap_{n=1}^{\infty} (d^n) = (0)$  for every  $t$ -invertible  $t$ -ideal  $I$  of  $D$ . Hence,  $D$  is completely integrally closed by Theorem 0.2.  $\square$

As a repeat corollary, we conclude that a GCD domain  $D$  is completely integrally closed if and only if  $D$  is Archimedean because in a GCD domain,  $I_v$  is principal for every nonzero finitely generated ideal  $I$  of  $D$ . This is, of course, a known result ([14, Theorem 3.1]). Next, an integral domain  $D$  with the *QR property* (every overring of  $D$  is a quotient ring) is known to be a Prüfer domain such that for every nonzero finitely generated ideal  $I$  of  $D$ , there is an  $i \in I$  and an integer  $n \geq 1$  such that  $I^n \subseteq (i)$  [34, Theorem 5]. Thus, we have the following corollary to Proposition 2.9.

**Corollary 2.10.** *A QR domain  $D$  is completely integrally closed if and only if  $D$  is Archimedean.*

We can do somewhat better than Corollary 2.10. Recall that an integral domain  $D$  with quotient field  $K$  is called a  *$t$ -QR domain* if every  $t$ -linked overring of  $D$  is a quotient ring of  $D$ . Here, in an extension  $D \subseteq R \subseteq K$ ,  $R$  is said to be  *$t$ -linked over  $D$*  if  $A^{-1} = D$  implies  $(AR)^{-1} = R$  for every nonzero finitely generated ideal  $A$  of  $D$ . Obviously, every flat overring is  $t$ -linked; so every overring of a Prüfer domain is  $t$ -linked. In [18, Theorem 1.3], it was shown that a PVMD  $D$  has the  $t$ -QR property if and only if for every nonzero finitely generated ideal  $I$  of  $D$ , there is a  $b \in I_v$  and an integer  $n \geq 1$  such that  $I^n \subseteq (b)$ . Since in a Prüfer domain every nonzero finitely generated ideal is a  $v$ -ideal, this characterization reduces to that given by Pendleton in [34, Theorem 5] for QR domains. Thus, we have the following result as well. (We are thankful to Tiberiu Dumitrescu for reminding us of [18].)

**Corollary 2.11.** *A  $t$ -QR PVMD  $D$  is completely integrally closed if and only if  $D$  is Archimedean.*

Proposition 2.9 clearly points to the following result once we note that an integrally closed AGCD domain is a PVMD. Also, note that Noetherian domains are Archimedean and there are Noetherian AGCD domains that are not integrally closed.

**Corollary 2.12.** *An integrally closed AGCD domain  $D$  is completely integrally closed if and only if  $D$  is Archimedean.*

It was shown in [40] that an integrally closed AGCD domain is a PVMD with torsion  $t$ -class group and that a PVMD with torsion  $t$ -class group is an AGCD domain. Also, since a Prüfer domain is a PVMD, a Prüfer domain with torsion class group is an AGCD domain. Hence, a Prüfer domain  $D$  with torsion class group is completely integrally closed if and only if  $D$  is Archimedean. These facts are mentioned here because the QR property was mentioned in [32]. Of course, Ohm did not know about AGCD domains, nor about  $t$ -QR domains, at the time of writing [32]. The results in this section greatly expand the scope of his work from mere QR domains to  $t$ -QR and AGCD domains. Thus, we state the following result.

**Proposition 2.13.** *The following statements are equivalent for an AGCD domain  $D$ .*

- (1)  $D[[X]]$  is integrally closed.
- (2)  $D$  is completely integrally closed.

(3)  $D[[X]]$  is completely integrally closed.

*Proof.* (1)  $\Rightarrow$  (2)  $D[[X]]$  is integrally closed implies that  $D$  is integrally closed and Archimedean by [32, Theorem 0.1] (or [24, Theorem 13.10 and Proposition 13.11]). Thus,  $D$  is completely integrally closed by Corollary 2.12.

(2)  $\Rightarrow$  (3)  $D$  is completely integrally closed implies that  $D[[X]]$  is completely integrally closed.

(3)  $\Rightarrow$  (1) This is obvious since a completely integrally closed domain is integrally closed.  $\square$

**Corollary 2.14.** *Let  $D$  be an AGCD domain. Then  $D[[X]]$  is (completely) integrally closed if and only if  $D$  is a completely integrally closed PVMD with torsion  $t$ -class group.*

Corollaries 2.10 - 2.12 use Proposition 2.9 to give several classes of PVMDs in which being completely integrally closed is equivalent to being Archimedean. The next result shows that this equivalence actually holds as long as  $D''$ , the complete integral closure of  $D$ , is a quotient ring of  $D$ . Since  $D''$  is always a  $t$ -linked overring of  $D$  [6, Proposition 2.5], Corollary 2.11 also follows from Proposition 2.15.

**Proposition 2.15.** *Let  $D$  be an integral domain with complete integral closure  $D'' = D_S$  for a multiplicative subset  $S$  of  $D$ . Then  $D$  is completely integrally closed if and only if  $D$  is Archimedean.*

*Proof.* A completely integrally closed domain is always Archimedean. Conversely, suppose that  $D$  is Archimedean. Let  $s \in S$ . Then  $1/s \in D_S = D''$ ; so  $\bigcap_{n=1}^{\infty} (s^n) \neq (0)$ . Thus,  $s$  is a unit of  $D$  since  $D$  is Archimedean; so  $D = D_S = D''$  is completely integrally closed.  $\square$

We end this section with a slight generalization of Proposition 2.5 and Theorem 0.2.

**Proposition 2.16.** *Let  $D$  be an essential domain. Then the complete integral closure  $D''$  of  $D$  is the generalized ring of fractions  $D_S$ , where  $S = \{I \mid I \subseteq D \text{ is } v\text{-invertible and } \bigcap_{n=1}^{\infty} (I^n)_v \neq (0)\}$  is a multiplicative system of ideals.*

*Proof.* Let  $D$  be an essential domain with  $F$  the set of essential prime ideals defining  $D$ , and let  $*$  be the star operation induced on  $D$  by  $F$ , i.e.,  $I^* = \bigcap_{P \in F} ID_P$  for every  $I \in F(D)$ .

The proof is similar to the proof of Proposition 2.5, but we replace the  $w$ -operation by  $*$  as defined in the above paragraph. Note that  $(b) : (a) = a^{-1}((a) \cap (b))$  is  $v$ -invertible for every  $0 \neq a, b \in D$  since  $(a) \cap (b)$  is  $v$ -invertible. The last statement follows because  $((a) \cap (b))(a, b)D_P = (ab)D_P$  for every  $P \in F$  since  $D_P$  is a valuation domain, and thus  $((a) \cap (b))(a, b)^* = (ab)$  implies  $((a) \cap (b))(a, b)_v = (ab)$  since  $*$   $\leq$   $v$ .  $\square$

**Corollary 2.17.** *An essential domain  $D$  is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $v$ -invertible  $v$ -ideal  $I$  of  $D$ .*

*Proof.* Let  $D$  be completely integrally closed. Then  $D_S = D'' = D$  by Proposition 2.16, and thus  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $v$ -invertible  $v$ -ideal  $I$  of  $D$ . Alternatively, use Corollary 2.4

Conversely, suppose that  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $v$ -invertible  $v$ -ideal  $I$  of  $D$ . Then, in particular, we have that  $(a) : (b)$ , where  $0 \neq a, b \in D$  with  $(b) \not\subseteq (a)$ , is  $v$ -invertible (see the proof of Proposition 2.16), and so  $\bigcap_{n=1}^{\infty} (((a) : (b))^n)_v = (0)$ . Since  $D$  is an essential domain,  $((a) : (b))^n_v = (a^n) : (b^n) = ((a^n) : (b^n))_v$  for every integer  $n \geq 1$ ; so  $\bigcap_{n=1}^{\infty} ((a^n) : (b^n))_v = (0)$ . But,  $(a^n) : (b^n) = (a^n/b^n) \cap D$ , and so  $(0) = \bigcap_{n=1}^{\infty} ((a^n/b^n) \cap D) = (\bigcap_{n=1}^{\infty} (a^n/b^n)) \cap D$ , which forces  $\bigcap_{n=1}^{\infty} (a/b)^n = (0)$  for every  $a, b \in D$  with  $(b) \not\subseteq (a)$ . Thus,  $D$  is strongly Archimedean, and hence completely integrally closed by Proposition 2.3.  $\square$

**Corollary 2.18.** *An essential domain  $D$  is completely integrally closed if and only if  $\bigcap_{n=1}^{\infty} (((a) : (b))^n)_v = (0)$  for every  $0 \neq a, b \in D$  with  $(b) \not\subseteq (a)$ .*

*Proof.* This follows from the proof of Corollary 2.17 since in an essential domain,  $((a) : (b))^n_v = (a^n) : (b^n)$  for every integer  $n \geq 1$ .  $\square$

### 3. ALTERNATIVE APPROACH

The alternative approach is offered here not just to replicate the results in the previous section, but actually to expand the scope of the study. We use this approach, for instance, to provide some new characterizations of Krull domains and their specializations such as UFDs, PIDs, locally factorial Krull domains, and generalized Krull domains. We do this by concentrating on integral domains whose maximal  $t$ -ideals are potent and mixing them with complete integral closure.

Call a maximal  $t$ -ideal  $P$  of an integral domain  $D$  *potent* if it contains a nonzero finitely generated ideal that is not contained in any other maximal  $t$ -ideal. Next, call a  $v$ -ideal  $I$  of finite type a *rigid ideal* if  $I$  is contained in one and only one maximal  $t$ -ideal. Thus,  $P$  is potent if and only if it contains a rigid ideal. Let us call a rigid ideal contained in a maximal  $t$ -ideal  $P$  a  *$P$ -ideal*. It was shown in [5, Theorem 1.1] that if  $D$  is of *finite  $t$ -character*, i.e., every nonzero nonunit belongs to only finitely many maximal  $t$ -ideals, then every maximal  $t$ -ideal of  $D$  is potent. Note that every rigid ideal in a PVMD is a  $t$ -invertible  $t$ -ideal. Also, recall that an integral domain  $D$  is a *ring of Krull type* (cf. [24, page 537]) if  $D$  is a locally finite intersection of essential valuation domains. Hence, a generalized Krull domain is a ring of Krull type. A ring of Krull type is a PVMD, and thus integrally closed, but need not be completely integrally closed.

**Lemma 3.1.** *Let  $P$  be a potent maximal  $t$ -ideal of a PVMD  $D$ .*

- (1) *The set of all  $P$ -ideals of  $D$  is totally ordered by inclusion.*
- (2) *If a  $t$ -invertible  $t$ -ideal  $I$  of  $D$  is contained in  $P$ , then  $I$  is contained in a  $P$ -ideal of  $D$ .*
- (3) *For all  $P$ -ideals  $I$  and  $J$  of  $D$ , the ideal  $(IJ)_v$  is a  $P$ -ideal of  $D$ .*
- (4) *For every  $P$ -ideal  $J$  of  $D$ ,  $\bigcap_{n=1}^{\infty} (J^n)_v$  is a prime ideal of  $D$  contained in  $P$ .*
- (5) *If  $J$  is a  $P$ -ideal of  $D$  and  $Q$  is a prime ideal of  $D$  contained in  $P$  with  $J \not\subseteq Q$ , then  $Q \subseteq \bigcap_{n=1}^{\infty} (J^n)_v$ .*

*Proof.* (1) Let  $I$  and  $J$  be  $P$ -ideals of  $D$ . Then  $I_w = \bigcap_{Q \in t\text{-Max}(D)} ID_Q = D \cap ID_P$ , and likewise,  $J_w = D \cap JD_P$ . Also, as  $I$  is a  $t$ -ideal,  $I_w = I$  because  $w \leq t$ . Hence,  $I = D \cap ID_P$ , and similarly,  $J = D \cap JD_P$ . Then, since  $D$  is a PVMD and  $P$  is a maximal  $t$ -ideal,  $D_P$  is a valuation domain; so  $ID_P \subseteq JD_P$  or  $JD_P \subseteq ID_P$ , say  $ID_P \subseteq JD_P$ . Thus,  $I = D \cap ID_P \subseteq D \cap JD_P = J$ .

(2) Because  $P$  is potent, there is a  $P$ -ideal  $J$  of  $D$  contained in  $P$ . Then  $(I + J)_v$  is a  $P$ -ideal of  $D$  containing  $I$ .

(3) Obvious.

(4) Note that  $J$  being a  $t$ -invertible  $t$ -ideal,  $JD_P$  is principal, and so  $J^n D_P$  is principal. Also, being a  $t$ -invertible  $t$ -ideal,  $(J^n)_v D_P = (J^n D_P)_v = J^n D_P$  because  $J^n D_P$  is principal. Now, as  $D_P$  is a valuation domain,  $\bigcap_{n=1}^{\infty} J^n D_P = \bigcap_{n=1}^{\infty} (J^n)_v D_P$  is a prime ideal of  $D_P$ , say  $QD_P$  for  $Q \subseteq P$  a prime ideal of  $D$ . Thus,  $Q = QD_P \cap D = (\bigcap_{n=1}^{\infty} (J^n)_v D_P) \cap D = \bigcap_{n=1}^{\infty} ((J^n)_v D_P \cap D) = \bigcap_{n=1}^{\infty} (J^n)_v$ . For the last equality, we used the fact that  $(J^n)_v D_P \cap D = (J^n)_v$  since  $(J^n)_v$  is a  $P$ -ideal of  $D$  and  $ID_P \cap D = I$  for every  $P$ -ideal of  $D$  as shown in the proof of (1).

(5) Indeed, if  $J \not\subseteq Q$ , then  $JD_P \not\subseteq QD_P$ , and so  $Q \subseteq QD_P \subseteq JD_P$  because  $D_P$  is a valuation domain. Thus,  $Q \subseteq QD_P \subseteq (J^n)_v D_P$ , and hence  $Q \subseteq \bigcap_{n=1}^{\infty} (J^n)_v$  as in the proof of (4).  $\square$

These considerations immediately give the following result since a ring of Krull type is a PVMD.

**Proposition 3.2.** *The following statements are equivalent for a ring  $D$  of Krull type.*

- (1)  $D$  is completely integrally closed.
- (2)  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $t$ -invertible  $t$ -ideal  $I$  of  $D$ .
- (3)  $\bigcap_{n=1}^{\infty} (J^n)_v = (0)$  for every rigid ideal  $J$  of  $D$ .
- (4) Every maximal  $t$ -ideal of  $D$  has height one.
- (5)  $D$  is a locally finite intersection of rank-one valuation domains.

*Proof.* (1)  $\Rightarrow$  (2) This follows from Theorem 0.2 since a ring of Krull type is a PVMD, or use Corollary 2.4.

(2)  $\Rightarrow$  (3) Obvious.

(3)  $\Rightarrow$  (4) Let  $P$  be a maximal  $t$ -ideal of  $D$  and suppose that there is a nonzero prime ideal  $Q$  of  $D$  contained in  $P$  with  $x \in P \setminus Q$ . By [5, Theorem 1.1], there exists a  $P$ -ideal  $J$  of  $D$ . Then  $L = (J, x)_v$  is a (rigid)  $P$ -ideal of  $D$  not contained in  $Q$ . Thus, by Lemma 3.1(5),  $Q \subseteq \bigcap_{n=1}^{\infty} (L^n)_v = (0)$ ; so  $Q = (0)$ , a contradiction. Hence, every maximal  $t$ -ideal of  $D$  has height one.

(4)  $\Rightarrow$  (5) A ring of Krull type is a locally finite intersection of localizations at maximal  $t$ -ideals, and the localizations are valuation domains. By (4), each of these localizations has rank one because every maximal  $t$ -ideal has height one.

(5)  $\Rightarrow$  (1) Since a rank-one valuation domain is completely integrally closed,  $D$  is an intersection of completely integrally closed domains, and thus is completely integrally closed.  $\square$

The above result provides various characterizations of generalized Krull domains since a generalized Krull domain is a ring of Krull type, most of them are well known. We now prove some results that are new. Yet, to facilitate the realization of those results, we need to bring in some other notions and results.

Recall from [11] that a family  $F$  of (nonzero) prime ideals of an integral domain  $D$  is called a *defining family of primes for  $D$*  if  $D = \bigcap_{P \in F} D_P$ . If, further, every nonzero nonunit of  $D$  belongs to at most finitely many members of  $F$ , then  $F$  is of *finite character*, and if no two members of  $F$  contain a common nonzero prime ideal, then  $F$  is *independent*. An integral domain  $D$  is *independent of finite character*  $F$  (or an  *$F$ -IFC domain*) if it has a defining family  $F$  of prime ideals that is independent and of finite character. In [11], we denoted by  $*_F$  the star operation induced on  $D$  by the family  $\{D_P\}_{P \in F}$ , i.e.,  $I^{*_F} = \bigcap_{P \in F} ID_P$  for every  $I \in F(D)$ . We also called an integral ideal  $I$  of  $D$   *$*_F$ -unidirectional* if  $I$  belongs to a unique

member of  $F$ . If  $F$  consists of all the maximal  $t$ -ideals of  $D$ , then  $*_F = w$ ; and if  $D$  is a PVMD, then the rigid ideals are precisely the  $w$ -invertible unidirectional  $w$ -ideals because a  $t$ -invertible  $t$ -ideal is a  $w$ -invertible  $w$ -ideal. If  $D$  is a GCD domain, then the rigid ideals are precisely the ones generated by rigid elements of  $D$  [38, Lemma 1]. (For an integral domain  $D$ ,  $r \in D$  is said to be *rigid* if  $r|t$  and  $s|t$  for  $s, t \in D$  implies that either  $r|s$  or  $s|r$ ). While the unidirectional ideals served a somewhat limited purpose in [11], results proved in [11] can have some interesting uses. One of the results that we intend to use is the following.

**Theorem 3.3.** ([11, Theorem 3.3]) *Let  $F$  be a defining family of mutually incomparable prime ideals of an integral domain  $D$  such that  $*_F$  is of finite character. Then the following statements are equivalent.*

- (1)  *$F$  is independent of finite character.*
- (2) *Every nonzero prime ideal of  $D$  contains an element  $x$  such that  $(x)$  is a  $*_F$ -product of unidirectional  $*_F$ -ideals.*
- (3) *Every nonzero prime ideal of  $D$  contains a unidirectional  $*_F$ -invertible  $*_F$ -ideal.*
- (4) *For  $P \in F$  and  $0 \neq x \in P$ ,  $x D_P \cap D$  is  $*_F$ -invertible and unidirectional.*
- (5)  *$F$  is independent and for every nonzero ideal  $I$  of  $D$ ,  $I^{*F}$  is of finite type whenever  $I D_P$  is finitely generated for every  $P \in F$ .*

**Theorem 3.4.** *A PVMD  $D$  is a generalized Krull domain if and only if  $D$  is completely integrally closed and every maximal  $t$ -ideal of  $D$  is potent.*

*Proof.* Suppose that the PVMD  $D$  is completely integrally closed and every maximal  $t$ -ideal of  $D$  is potent. We first show that every maximal  $t$ -ideal  $P$  of  $D$  has height one. Since  $P$  is potent, it contains at least one rigid ideal  $J$ . Since  $D$  is completely integrally closed,  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every  $t$ -invertible  $t$ -ideal  $I$  of  $D$  by Theorem 0.2; in particular,  $\bigcap_{n=1}^{\infty} (J^n)_v = (0)$ . As in the proof of (3)  $\Rightarrow$  (4) of Proposition 3.2, one can show that  $P$  has height one. Thus, every nonzero prime ideal of  $D$  contains a maximal  $t$ -ideal. Note that  $*_F = w$ , where  $F = t\text{-Max}(D)$ , and hence  $*_F$  has finite character. Since every maximal  $t$ -ideal is potent, Theorem 3.3(3) holds. Thus,  $F$  has finite character by Theorem 3.3(1); so  $D$  is a generalized Krull domain. The converse is obvious.  $\square$

Since a Prüfer domain is a special case of a PVMD in which  $t$ -invertible is simply invertible, we have the following corollary.

**Corollary 3.5.** *A Prüfer domain  $D$  is a generalized Krull domain if and only if  $D$  is completely integrally closed and every maximal ideal of  $D$  is potent.*

A happy fallout from Theorem 3.4 that may not be termed as corollaries is the following set of results. Note that the “completely integrally closed” hypothesis is needed in Corollary 3.7 (let  $D$  be a rank-two valuation domain with principal maximal ideal).

**Theorem 3.6.** *Let  $D$  be an integral domain that is not a field.*

- (1)  *$D$  is a PID if and only if every maximal ideal  $M$  of  $D$  is principal and  $\bigcap_{n=1}^{\infty} M^n = (0)$ .*
- (2)  *$D$  is a Dedekind domain if and only if every maximal ideal  $M$  of  $D$  is invertible and  $\bigcap_{n=1}^{\infty} M^n = (0)$ .*
- (3)  *$D$  is a UFD if and only if every maximal  $t$ -ideal  $M$  of  $D$  is principal and  $\bigcap_{n=1}^{\infty} M^n = (0)$ .*

(4)  $D$  is a locally factorial Krull domain if and only if every maximal  $t$ -ideal  $M$  of  $D$  is invertible and  $\bigcap_{n=1}^{\infty} M^n = (0)$ .

(5)  $D$  is a Krull domain if and only if every maximal  $t$ -ideal  $M$  of  $D$  is  $t$ -invertible and  $\bigcap_{n=1}^{\infty} (M^n)_v = (0)$ .

*Proof.* For each of (1) - (5), the “ $\Rightarrow$ ” implication is well known and easy to prove.

(2) (resp., (1)) ( $\Leftarrow$ ) Here,  $D$  is an integral domain in which every nonzero prime ideal is invertible (resp., principal). It is then well known (via a Zorn’s Lemma argument, see [31, Exercise 36, page 44] (resp., [31, Exercise 10, page 8])) that every nonzero ideal of  $D$  is invertible (resp., principal).

(5) ( $\Leftarrow$ ) Let  $M$  be a maximal  $t$ -ideal of  $D$ . Then, by hypothesis,  $M$  is  $t$ -invertible and  $\bigcap_{n=1}^{\infty} (M^n)_v = (0)$ . Thus,  $MD_M$  is principal; so  $Q = \bigcap_{n=1}^{\infty} (MD_M)^n$  is a prime ideal of  $D_M$ . As in the proof of Lemma 3.1(4),  $Q \cap D = \bigcap_{n=1}^{\infty} (M^n)_v$ . Hence,  $Q \cap D = (0)$ ; so  $Q = (0)$ , and thus  $\text{ht}(M) = \text{ht}(MD_M) = 1$ . Hence, every maximal  $t$ -ideal of  $D$  has height one and is  $t$ -invertible. Thus, every prime  $t$ -ideal of  $D$  is  $t$ -invertible; so  $D$  is a Krull domain [29, Theorem 2.3].

(4) (resp., (3)) ( $\Leftarrow$ ) Note that invertible (resp., principal) implies  $t$ -invertible, and so by (5) ( $\Leftarrow$ ),  $D$  is a Krull domain in which every prime  $t$ -ideal is invertible (resp., principal), and hence  $D$  is locally factorial (resp., a UFD).  $\square$

**Corollary 3.7.** *Let  $D$  be a completely integrally closed PVMD that is not a field.*

(1)  $D$  is a PID (resp., Dedekind domain) if and only if every maximal ideal of  $D$  is principal (resp., invertible).

(2)  $D$  is a UFD (resp., locally factorial Krull domain, Krull domain) if and only if every maximal  $t$ -ideal of  $D$  is principal (resp., invertible,  $t$ -invertible).

Now we return to the main theme of this work and prove a result with reference to power series.

**Proposition 3.8.** *Let  $D$  be a PVMD such that every maximal  $t$ -ideal of  $D$  is potent and some power of every integral  $t$ -invertible  $t$ -ideal is contained in a proper principal integral ideal. Then the following statements are equivalent.*

(1)  $D[[X]]$  is integrally closed.

(2)  $D$  is Archimedean.

(3)  $D$  is a generalized Krull domain in which some power of every proper integral  $t$ -invertible  $t$ -ideal is contained in a proper principal integral ideal.

(4)  $D$  is completely integrally closed.

*Proof.* (1)  $\Rightarrow$  (2) This follows from [32, Theorem 0.2] (or [24, Proposition 3.11]).

(2)  $\Rightarrow$  (4) This follows from Lemma 2.8 and Theorem 0.2.

(4)  $\Rightarrow$  (3) This follows from Theorem 3.4.

(3)  $\Rightarrow$  (2) A generalized Krull domain is completely integrally closed, and hence Archimedean.

(4)  $\Rightarrow$  (1)  $D$  completely integrally closed implies that  $D[[X]]$  is completely integrally closed, and thus integrally closed.  $\square$

**Corollary 3.9.** *Let  $D$  be a GCD domain such that every maximal  $t$ -ideal of  $D$  is potent. If  $D[[X]]$  is integrally closed, then  $D$  is a locally finite intersection of (essential) valuation domains.*

**Corollary 3.10.** (cf. [32, Corollary 1.9]) *Let  $D$  be an integral domain that is a finite intersection of valuation domains. If  $D[[X]]$  is integrally closed, then  $D$  is a*

*finite intersection of rank-one valuation domains, and hence is a one-dimensional Bezout domain.*

Generalized Krull domains do not behave as well as Krull domains in at least one respect. While  $D$  is a Krull domain implies that  $D[[X]]$  is a Krull domain,  $D$  is a generalized Krull domain implies that  $D[[X]]$  is a generalized Krull domain only when  $D$  is a Krull domain. This result of [33, Theorem 2.5] shows that  $D$  is a Krull domain when  $D[[X]]$  is a generalized Krull domain. So it does not matter whether we assume that  $D[[X]]$  is a generalized Krull domain as defined by Griffin or defined by El Baghdadi (which we call a Krull-like PVMD),  $D$  has to be a Krull domain.

#### 4. UNIQUE REPRESENTATION DOMAINS

In the absence of a clear answer to the two conjectures for PVMDs in general, we look for special cases, as we have done above. One special case is when a PVMD is a unique representation domain. A *packet* of an integral domain  $D$  is a  $t$ -invertible  $t$ -ideal of  $D$  having prime radical. Then  $D$  is called a *unique representation domain* (URD) [20] if every proper  $t$ -invertible  $t$ -ideal of  $D$  can be uniquely expressed as a  $t$ -product of pairwise  $t$ -comaximal packets. (In [39], the term URD was used in the more restrictive sense as a GCD domain that is also a URD.) We note that for  $D$  a PVMD and  $I$  a proper  $t$ -invertible  $t$ -ideal of  $D$ , if  $I$  is a  $t$ -product of a finite number of packets, then  $I$  can be uniquely expressed as a  $t$ -product of a finite number of pairwise  $t$ -comaximal packets [20, Theorem 1.1] and if  $I = (I_1 \cdots I_n)_t$  is such an expression, then  $I = I_1 \cap \cdots \cap I_n$ , and that  $I$  has such a representation precisely when  $I$  has only finitely many minimal prime ideals [20, Theorem 1.2]. Moreover, a PVMD is a URD if and only if every nonzero principal ideal has only finitely many minimal prime ideals [20, Theorem, 1.9]. It may be hoped that this approach will come in handy if it may look hard to decide on the potency of maximal  $t$ -ideals, but there is this finiteness condition. It is well known that a minimal prime ideal  $P$  of a principal ideal  $(x)$  is a prime  $t$ -ideal, and so  $D_P$  is a valuation domain when  $D$  is a PVMD.

**Lemma 4.1.** *Let  $D$  be a PVMD and  $I$  a packet of  $D$  with unique minimal prime ideal  $P$ . Then  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  if and only if  $P$  has height one.*

*Proof.* ( $\Leftarrow$ ) Suppose that  $\text{ht}(P) = 1$ . Let  $M \supseteq P$  be a maximal  $t$ -ideal of  $D$ . Thus,  $ID_M$  is a principal ideal of the valuation domain  $D_M$  and  $ID_M \subseteq PD_M$ , where  $\text{ht}(PD_M) = 1$ . Then  $(0) = \bigcap_{n=1}^{\infty} (ID_M)^n = \bigcap_{n=1}^{\infty} (I^n D_M) = \bigcap_{n=1}^{\infty} (I^n)_v D_M \supseteq \bigcap_{n=1}^{\infty} (I^n)_v$ ; so  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$ .

( $\Rightarrow$ ) Suppose that  $\text{ht}(P) > 1$ . Let  $M \supseteq P$  be a maximal  $t$ -ideal of  $D$ . Now,  $(I^n)_v D_M = I^n D_M = (ID_M)^n$  for every integer  $n \geq 1$  and  $\sqrt{ID_M} = PD_M$ . Since  $D_M$  is a valuation domain,  $Q' = \bigcap_{n=1}^{\infty} (I^n)_v D_M = \bigcap_{n=1}^{\infty} (ID_M)^n$  is a prime ideal of  $D_M$ ; in fact,  $Q'$  is the unique prime ideal directly below  $PD_M$ . Thus,  $Q' = QD_M$ , where  $Q$  is the unique prime ideal of  $D$  directly below  $P$ . So  $\text{ht}(P) > 1$  implies  $Q \neq (0)$ . Suppose that  $N$  is a maximal  $t$ -ideal of  $D$  with  $N \not\supseteq P$ ; so  $(I^n)_v D_N = D_N$ . Hence,  $QD_N \subseteq \bigcap_{n=1}^{\infty} (I^n)_v D_N$ . Thus,  $Q = \bigcap_{M \in t\text{-Max}(D)} QD_M = \bigcap_{M \in t\text{-Max}(D)} (\bigcap_{n=1}^{\infty} (I^n)_v D_M) = \bigcap_{n=1}^{\infty} (\bigcap_{M \in t\text{-Max}(D)} (I^n)_v D_M) = \bigcap_{n=1}^{\infty} (I^n)_v = (0)$ , a contradiction.  $\square$



**Corollary 4.2.** *Let  $D$  be a PVMD, and let  $I$  be a proper  $t$ -invertible  $t$ -ideal of  $D$  such that  $I$  has only a finite number of minimal prime ideals  $P_1, \dots, P_m$ . Then  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  if and only if some  $P_i$  has height one.*

*Proof.* By [20, Theorem 1.2],  $I = (I_1 \cdots I_m)_t$ , where  $I_1, \dots, I_m$  are  $t$ -comaximal packets each with  $I_i$  contained in a unique minimal prime ideal  $P_i$ . By  $t$ -comaximality, we have  $(I^n)_v = (I_1^n \cdots I_m^n)_v = (I_1^n)_v \cap \cdots \cap (I_m^n)_v$ . Hence,  $\bigcap_{n=1}^{\infty} (I^n)_v = (\bigcap_{n=1}^{\infty} (I_1^n)_v) \cap \cdots \cap (\bigcap_{n=1}^{\infty} (I_m^n)_v)$ . Thus,  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  if and only if some  $\bigcap_{n=1}^{\infty} (I_i^n)_v = (0)$ , if and only if  $\text{ht}(P_i) = 1$ .  $\square$

**Proposition 4.3.** *Let  $D$  be a PVMD URD. Then  $D$  is a generalized Krull domain if and only if  $D$  is completely integrally closed.*

*Proof.* A generalized Krull domain is completely integrally closed. Conversely, let  $D$  be a PVMD URD that is completely integrally closed. Let  $x$  be a nonzero nonunit of  $D$ . Then  $(x) = (I_1 \cdots I_m)_t$ , where  $I_1, \dots, I_m$  are pairwise  $t$ -comaximal  $t$ -ideals with  $P_i$  the unique minimal prime ideal containing  $I_i$ . Then  $P_1, \dots, P_m$  are the minimal prime ideals of  $(x)$ . Now  $D$  is completely integrally closed; so  $\bigcap_{n=1}^{\infty} (I_i^n)_v = (0)$  for every  $1 \leq i \leq m$  by Corollary 2.4. Hence, by Lemma 4.1, every  $P_i$  has height one. So every prime ideal minimal over  $(x)$  has height one and there are only finitely many of them. To show that  $D$  is a generalized Krull domain, it is enough to show that  $D$  has no maximal  $t$ -ideal of height greater than one. By way of contradiction, assume that  $M$  is a maximal  $t$ -ideal of  $D$  with a nonzero prime ideal  $Q \subsetneq M$ . Let  $x \in M \setminus Q$ . We can shrink  $M$  to a prime ideal  $P$  minimal over  $(x)$ ; so  $P$  has height one. Now the prime ideals contained in  $M$  are totally ordered. Since  $x \notin Q$ , we must have  $Q \subsetneq P$ , a contradiction.  $\square$

Because a ring of Krull type is a PVMD URD [20, Corollary 1.10], we have the following corollary.

**Corollary 4.4.** (cf. Proposition 3.2) *A ring  $D$  of Krull type is a generalized Krull domain if and only if  $D$  is completely integrally closed.*

An integral domain  $D$  is a *generalized UFD* (GUFD) if  $D$  is a GCD domain that satisfies (1) every nonzero nonunit of  $D$  is expressible as a finite product of rigid elements of  $D$  and (2) every rigid element  $r$  of  $D$  is such that for every factor  $s$  of  $r$ ,  $r|s^n$  for some integer  $n \geq 1$ . Equivalently,  $D$  is a generalized Krull domain with  $Cl_t(D) = 0$ . The interested reader may consult [3] for various other characterizations of these integral domains.

These results lead to the following result.

**Theorem 4.5.** *The following statements are equivalent for a GCD domain  $D$  that is also a URD.*

- (1)  $D$  is an intersection of rank-one valuation domains.
- (2)  $D$  is completely integrally closed.
- (3)  $D[[X]]$  is integrally closed.
- (4)  $D$  is Archimedean.
- (5) Every packet of  $D$  is contained in a height-one prime ideal of  $D$ .
- (6)  $D$  is a GUFD.

*Proof.* (1)  $\Rightarrow$  (2)  $\Rightarrow$  (3)  $\Rightarrow$  (4) were established in [32, Theorem 0.2].

(4)  $\Rightarrow$  (5) follows from Lemma 4.1.

(5)  $\Rightarrow$  (6) In a GCD URD, every nonzero nonunit  $x$  is expressible as a finite product of mutually co-prime packets, i.e., elements with unique minimal prime ideals. Say  $x = x_1 \cdots x_n$ , where every  $(x_i)$  has a unique minimal prime  $P_i$ , which by (5) has height one. But then  $D_{P_i}$  is a rank-one valuation domain, making  $x_i$  a rigid element such that for every nonunit factor  $r$  of  $x_i$ ,  $x_i$  divides  $r^n$  for some integer  $n \geq 1$ . This makes every packet a prime quantum and  $D$  a GUFD, as described in [3, page 402] and in Zafrullah's doctoral dissertation [37].

(6)  $\Rightarrow$  (1) As shown in [3, Theorem 10], a GUFD is a generalized Krull domain, and thus is a locally finite intersection of rank-one (essential) valuation domains.  $\square$

Ohm proved the equivalence of (1) through (5) of Theorem 4.5 for finite intersections of valuation domains [32, Corollary 1.9]. As a GCD domain of finite  $t$ -character is a URD as well, we have the following repeat corollary.

**Corollary 4.6.** *A GCD domain  $D$  of finite  $t$ -character is a GUFD if and only if  $D[[X]]$  is integrally closed, equivalently, if and only if  $D$  is completely integrally closed.*

Of course, it would be interesting to see if  $D$  is a PVMD URD and  $D[[X]]$  is integrally closed implies that  $D$  is completely integrally closed. As it stands, we can only make decisions about GCD domains and AGCD domains, even for the URD case. We have kept the AGCD URD case as the last item because it is different from the GCD case in only a few minor details.

Call an integral domain  $D$  an *almost GUFD* if  $D$  is a generalized Krull domain with torsion  $t$ -class group. Of course, being a generalized Krull domain, every nonzero nonunit  $x \in D$  is expressible as a  $t$ -product  $(x) = (I_1 \cdots I_n)_t$ , where  $I_i = xD_{P_i} \cap D$  and  $P_i$  ranges over all the height-one prime ideals of  $D$  containing  $x$  [9, Corollary 2.3]. Now, as  $Cl_t(D)$  is torsion, there are integers  $n_i \geq 1$  such that  $(I_i^{n_i})_t$  is a principal  $P_i$ -primary ideal of  $D$ .

**Theorem 4.7.** *The following statements are equivalent for an AGCD domain  $D$  that is also a URD.*

- (1)  $D$  is a locally finite intersection of rank-one valuation domains.
- (2)  $D$  is an intersection of rank-one valuation domains.
- (3)  $D$  is completely integrally closed.
- (4)  $D[[X]]$  is integrally closed.
- (5)  $D$  is integrally closed and Archimedean.
- (6)  $D$  is integrally closed and every packet of  $D$  is contained in a height-one prime ideal of  $D$ .
- (7)  $D$  is an almost GUFD.

*Proof.* (1)  $\Rightarrow$  (2) is clear.

(2)  $\Rightarrow$  (3)  $\Rightarrow$  (4)  $\Rightarrow$  (5) were established in [32, Theorem 0.2].

(5)  $\Rightarrow$  (3) follows from Corollary 2.12.

(3)  $\Rightarrow$  (6) follows from Corollary 2.4 and Lemma 4.1.

(6)  $\Rightarrow$  (7) In an AGCD URD, every nonzero nonunit  $x$  is expressible as a finite  $t$ -product of mutually  $t$ -comaximal packets. Say  $(x) = (I_1 \cdots I_n)_t$ , where every  $I_i$  has a unique minimal prime ideal  $P_i$ , which by (6) has height one. Thus,  $D_{P_i}$  is a rank-one valuation domain, making  $I_i = xD_{P_i} \cap D$ . Hence, by [9, Corollary 2.3],  $D$  is a generalized Krull domain that is also an AGCD domain.

(7)  $\Rightarrow$  (1) This follows since a generalized Krull domain is a locally finite intersection of rank-one (essential) valuation domains.  $\square$

## 5. ARCHIMEDEAN-LIKE CONDITIONS

In this final section, we consider several “Archimedean-like” conditions on an integral domain  $D$ .

**Theorem 5.1.** *Consider the following statements for an integral domain  $D$ .*

- (1)  $D$  is completely integrally closed.
  - (2)  $\bigcap_{n=1}^{\infty} (a/b)^n = (0)$  for every  $a, b \in D$  with  $(b) \not\subseteq (a)$ , i.e.,  $D$  is strongly Archimedean.
  - (3)  $\bigcap_{n=1}^{\infty} ((a^n) : (b^n)) = (0)$  for every  $a, b \in D$  with  $(b) \not\subseteq (a)$ .
  - (4)  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $v$ -ideal  $I$  of  $D$ .
  - (5)  $\bigcap_{n=1}^{\infty} I^n = (0)$  for every proper  $v$ -ideal  $I$  of  $D$ .
  - (6)  $\bigcap_{n=1}^{\infty} ((a) : (b))^n_v = (0)$  for every  $0 \neq a, b \in D$  with  $(b) \not\subseteq (a)$ .
  - (7)  $\bigcap_{n=1}^{\infty} ((a) : (b))^n = (0)$  for every  $a, b \in D$  with  $(b) \not\subseteq (a)$ .
  - (8)  $\bigcap_{n=1}^{\infty} (x^n) = (0)$  for every nonunit  $x \in D$ , i.e.,  $D$  is Archimedean.
- Then we have the following implications.

$$\begin{array}{ccccccc}
 (1) & \Longleftrightarrow & (2) & \Longleftrightarrow & (3) & \implies & (4) \implies (5) \\
 & & & & & & \updownarrow \quad \updownarrow \\
 & & & & & & (6) \implies (7) \implies (8)
 \end{array}$$

*Proof.* (1)  $\Leftrightarrow$  (2) follows from Proposition 2.3.

(2)  $\Leftrightarrow$  (3) follows from the fact that  $(a^n) : (b^n) = (a/b)^n \cap D$  for every  $a, b \in D$  with  $(b) \not\subseteq (a)$  and integer  $n \geq 1$ .

(1)  $\Rightarrow$  (4) follows from Corollary 2.4.

(4)  $\Rightarrow$  (5) and (6)  $\Rightarrow$  (7) are both clear.

(4)  $\Leftrightarrow$  (6) and (5)  $\Leftrightarrow$  (7) Note that if  $I$  is a proper divisorial ideal of  $D$ , then  $I \subseteq (a/b) \cap D = (a) : (b)$  for some  $a, b \in D$  with  $(b) \not\subseteq (a)$ .

(7)  $\Rightarrow$  (8) is clear.  $\square$

The following examples show that none of the “ $\Rightarrow$ ” implications in the above theorem can be reversed.

**Example 5.2.** (a) ((4)  $\nRightarrow$  (3)) Let  $D = k[[X^2, X^3]]$  for a field  $k$  (or let  $D$  be any one-dimensional Noetherian Gorenstein domain that is not Dedekind). Then  $D$  is not completely integrally closed and every proper nonzero ideal of  $D$  is divisorial. Since  $D$  is Noetherian,  $\bigcap_{n=1}^{\infty} I^n = (0)$  for every proper nonzero ideal  $I$  of  $D$ , and as  $D$  is one-dimensional Gorenstein, every  $I^n$  is divisorial. Thus,  $\bigcap_{n=1}^{\infty} (I^n)_v = (0)$  for every proper  $v$ -ideal of  $D$ . Hence, (4)  $\nRightarrow$  (1); equivalently, (4)  $\nRightarrow$  (3).

(b) ((5)  $\nRightarrow$  (4)) [28, Example 1.5] gives a Noetherian domain  $D$  with a maximal  $t$ -ideal  $P$  such that  $(P^n)_v = P$  for every integer  $n \geq 1$ . Then  $\bigcap_{n=1}^{\infty} P^n = (0)$  since  $D$  is Noetherian, but  $\bigcap_{n=1}^{\infty} (P^n)_v = P \neq (0)$ . Thus, (5)  $\nRightarrow$  (4); so also (7)  $\nRightarrow$  (6). (We are thankful to Evan Houston for this example.)

(c) ((8)  $\nRightarrow$  (7)) Let  $V = K + M$  be a non-Noetherian (i.e., non-discrete) one-dimensional valuation domain with maximal ideal  $M$  and  $K$  a field that is a subring of  $V$ . Then  $M^2 = M$ ; so  $M^n = M$  for every integer  $n \geq 1$ . Suppose that  $K$  has a proper subfield  $k$ . Then  $D = k + M$  is also one-dimensional (but not a valuation domain), and thus satisfies (8). Let  $0 \neq m \in M$ ,  $\alpha \in K \setminus k$ ,  $b = m$ , and  $a = \alpha m$ . Then  $(b) \not\subseteq (a)$  and  $(a) : (b) = M$ ; so  $\bigcap_{n=1}^{\infty} ((a) : (b))^n = \bigcap_{n=1}^{\infty} M^n = M \neq (0)$ . Thus, (8)  $\nRightarrow$  (7).

For an essential domain, we have (3)  $\Leftrightarrow$  (6) (see the proof of Corollary 2.17), and thus statements (1) - (4), (6) are all equivalent. In a GCD domain, or more generally an integrally closed AGCD domain, statements (1) - (8) are all equivalent by Corollary 2.12. In (4) and (5), we may replace “ $I$  is a proper  $v$ -ideal of  $D$ ” with “ $I$  is an ideal of  $D$  with  $I_v \subsetneq D$ ”. Conjecture 2.2 is that statements (1) - (8) are all equivalent for a PVMD.

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