Notes on Partially Ordered Monoids

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Abstract: In this paper, we study Conrad's F-condition for lattice-ordered monoids, Riesz monoids, and pre-Riesz monoids.

Key words: Lattice-ordered monoid, Riesz monoid, Pre-Riesz monoid, F-condition, Homogeneous element, Basis

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1 Introduction

Let M=(M,+,$\leq$,0) be a partially ordered monoid(p.o.monoid) with + not necessarily commutative. For a general p.o.monoid M, we call an element $a \in M^+$ primal if for all a_1, $a_2 \in M^+$, $a \leq a_1 + a_2$ implies that $a = b_1 + b_2$, for some b_1, $b_2 \in M^+$ such that $b_i \leq a_i$, $i = 1, 2$. A directed p.o. monoid (directed monoid for short) M is called a Riesz monoid if every element of M^+ is primal. It is easy to see that a lattice-ordered monoid M which is also a divisibility monoid satisfying the order-cancellation law and $a + (b \wedge c) = (a + b) \wedge (a + c)$ and $(a \wedge b) + c = (a + c) \wedge (b + c)$ is a Riesz monoid. By "a partially ordered monoid M satisfying weak (m,n)-interpolation property or the weak Riesz interpolation property (WRIP)", we mean that if for given $x_1,\ x_2,\ ...,\ x_m;\ y_1,\ y_2,\ ...,\ y_n \in M$ with $x_i \leq y_j$ and the inverses of x_i exist in M for all integers $i \in [1,\ m]$, $j \in [1,\ n]$, there is a $z \in M$ such that $x_i \leq z \leq y_j$ for all integer pairs$(i,\ j) \in [1,\ m] \times [1,\ n]$. By "a partially ordered monoid M satisfying (m,n)-interpolation property or the Riesz interpolation property (RIP)", we mean that if for given $x_1,\ x_2,\ ...,\ x_m;\ y_1,\ y_2,\ ...,\ y_n \in M$ with $x_i \leq y_j$ for all integers $i \in [1,\ m]$, $j \in [1,\ n]$, there is a $z \in M$ such that $x_i \leq z \leq y_j$ for all integer pairs$(i,\ j) \in [1,\ m] \times [1,\ n]$. Note that if a p.o.monoid M satisfies (RIP), then it satisfies (WRIP). So a Riesz monoid M which satisfies the order-cancellation law has the (WRIP). A directed monoid M is a refinement directed monoid (refinement monoid for short), if $a_0 + a_1 = b_0 + b_1$ in M^+ implies the existence of $c_{i,j} \in M^+$, for i, $j = 0, 1$, such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$, for all $i = 0, 1$. Then we have every refinement divisibility monoid (cf. Definition 7) M is a Riesz divisibility monoid. Two positive elements x, $y \in M$ are disjoint if $x \wedge y = 0$. Call an element $x \in M$ homogeneous if x is strictly positive, i.e., $x > 0$, and for all h, $k \in (0,\ x]$, h and k are non-disjoint. In section 4, we extend Conrad's work[1] on basic elements, bases and Conrad's F-condition in the framework of lattice-ordered groups to lattice-ordered monoids showing finally that a lattice-ordered monoid satisfying Conrad's F-condition has a basis. In section 5, we generalize Mott, Rashid and Zafrullah's work[2] on homogeneous elements, bases and Conrad's F-condition in the framework of Riesz groups to Riesz monoids. In section 6, we investigate Yang and Zafrullah's work[3] on homogeneous elements, bases and Conrad's F-condition in the framework of pre-Riesz groups and generalize these notions to pre-Riesz monoids showing finally that a Riesz monoid satisfying the (pR) and Conrad's F-condition has a basis.

2 Preliminaries

Definition 1 [4] *A poset is a set in which a binary relation $\leq$ is defined, which satisfies for all x, y, z the following conditions:*

P1. For all x, $x \leq x$. *(Reflexive)*

P2. If $x \leq y$ and $y \leq x$, then $x = y$. *(Antisymmetry)*

P3. If $x \leq y$ and $y \leq z$, then $x \leq z$. *(Transitivity)*

Definition 2 [4] *A quasi-ordering of a set S as a relation $\leq$ satisfies P1 and P3, but not necessarily P2. The couple $(S, \leq)$ is then called a quasi-ordered set.*

Definition 3 [4] *The least upper bound of a poset X is denoted l.u.b. X or sup X. By P2, sup X is unique if it exists. The notion of a lower bound of X and the greatest lower bound (g.l.b. X or inf X) of X are defined dually. Again by P2, inf X is unique if it exists.*

Definition 4 [4] *A lattice is a poset P any two of whose elements have a g.l.b. or "meet" denoted $x \wedge y$, and a l.u.b. or "join" denoted by $x \vee y$. A lattice L is complete when each of its subsets X has a l.u.b. and a g.l.b. in L.*

Definition 5 [4] *A quasi-ordered groupoid is a quasi-ordered set M with a binary operation for which we prefer $+$ as the notation, and the quasi-order is compatible with the binary operation:*

$$a \leq b \Rightarrow x + a \leq x + b,\ a + x \leq b + x, \tag{2.1}$$

for all a, b, $x \in M$. When binary operation is commutative or associative, M is called a commutative quasi-ordered groupoid or quasi-ordered semigroup, respectively. A quasi-ordered semigroup M with identity(or "neutral element") 0 such that

$$x + 0 = 0 + x = x,\ \forall x \in M, \tag{2.2}$$

is called a quasi-ordered monoid.

Remark 1 *Usually, + is preferred as the notation for binary operation, even when the binary operation is non-commutative.*

Theorem 1 [5] *Every commutative monoid M is endowed with its algebraic quasi-ordering (only satisfying reflexive and transitivity conditions), defined by $a \leq b$ iff there exists $c \in M$ such that $b = c + a$, i.e., $b \in M + a$.*

Proof For all a, b, $c \in M$, M satisfies reflexive condition for $a = 0 + a$ and $0 \in M$. If $a \leq b$ and $b \leq c$, then there exist $x, y \in M$ such that $b = x + a$ and $c = y + b$. So we have $c = (y + x) + a$, i.e., $a \leq c$. Thus M satisfies transitivity condition. If $a \leq b$, then there exist $x \in M$ such that $b = x + a$. So we have $c + b = (c + x) + a = x + (c + a)$ and $b + c = (x + a) + c = x + (a + c)$, i.e., $c + a \leq c + b$ and $a + c \leq b + c$. Therefore, we have the conclusion. □

Definition 6 [4] *A partially ordered groupoid (or m-poset) is a poset M with a binary operation (for which we prefer + as a notation) and the partial order is compatible with the binary operation:*

$$a \leq b \Rightarrow x + a \leq x + b,\ a + x \leq b + x, \tag{2.3}$$

for all a, b, $x \in M$. When binary operation is commutative or associative, M is called a commutative partially ordered groupoid or partially ordered semigroup, respectively. A partially ordered semigroup M with identity (or "neutral element") 0 such that

$$x + 0 = 0 + x = x,\ \forall x \in M, \tag{2.4}$$

is called a partially ordered monoid, or a p.o. monoid.

Note that any p.o.group G is a p.o.monoid; moreover the positive cone $P = G^+$ and negative cone $-P = G^-$ of any p.o.group G are also p.o.monoids.

Definition 7 [4] *A divisibility monoid is a p.o.monoid M in which*

$$b \in M + a \Leftrightarrow a \leq b \Leftrightarrow b \in a + M.$$

A divisibility monoid is a p.o. monoid in which the induced quasi order is the partial order. This forces every element of a divisibility monoid positive, i.e., ≥ 0.

Lemma 1 [4] *In any divisibility monoid, $a + b = 0$ implies $a = b = 0$.*

Proof Trivially, $0 \leq a, b$. While $a + b = 0$ implies $a, b \leq 0$. Hence (by *P2*), $a + b = 0$ implies $a = b = 0$. Hence, the monoid identity is a universal lower bound.

Definition 8 [4] *A multiplicative semilattice, or m-semilattice is a semilattice M under $\vee$ with a binary operation such that*

$$a + (b \vee c) \;=\; (a+b) \vee (a+c) \text{ and } (a \vee b) + c \;=\; (a+c) \vee (b+c) \tag{2.5}$$

for all $a, b, c \in M$. If M is a lattice with a binary operation and (2.5) holds, then M is called an "m-lattice" or a l.o. groupoid. A l.o.groupoid which is a semigroup (monoid) under binary operation is called a l.o.semigroup (resp., l.o.monoid).

Note that (2.3) follows trivially from (2.5): if $b \leq c$, then $a + c = a + (b \vee c) = (a + b) \vee (a + c)$, whence $a + b \leq a + c$. In other words, any m-semilattice is trivially a p.o.groupoid (i.e., m-poset).

Note also that any lattice ordered group G satisfies (2.5) and its dual

$$a + (b \wedge c) = (a + b) \wedge (a + c) \text{ and } (a \wedge b) + c = (a + c) \wedge (b + c) \tag{2.6}$$

for all $a, b, c \in G$. However, in general, the dual of a l.o. semigroup (resp., l.o. monoid) is not a l.o. semigroup (resp., l.o. monoid). In fact, any l.o.group G is a l.o. monoid that satisfies both (2.5) and (2.6); moreover the positive cone $P = G^+$ and negative cone $-P = G^-$ of any l.o.group G are also l.o.monoids.

Definition 9 [4] *A p.o.monoid M is said to be upper directed if for all $a, b \in M$, there is at least one element $c \in M$ such that $a, b \leq c$. We can define "lower directed" in a dual fashion. A p.o.monoid M is said to be directed if it is both upper directed and lower directed, i.e., bidirected. Moreover, a divisibility monoid M is said to be directed if it is both upper directed and lower directed, i.e., bidirected.*

Note that an upper directed monoid M may not be lower directed, so a upper directed monoid M may be not directed.

Definition 10 [5] *We say that a monoid M has the cancellation law, if*

$$a + b = a + c \Rightarrow b = c, \tag{2.7}$$

$$b + a = c + a \Rightarrow b = c \tag{2.8}$$

for all $a, b, c \in M$.

Definition 11 [5] *We say that a p.o. monoid M has the order-cancellation law, if*

$$a + b \leq a + c \Rightarrow b \leq c, \tag{2.9}$$

$$b + a \leq c + a \Rightarrow b \leq c \tag{2.10}$$

for all $a, b, c \in M$.

Remark 2 *It is easy to verify that the cancellation law is equivalent to the order cancellation law in a p.o. group. However, the two cancellation laws are different in a p.o. monoid. For instance, let* $\leq$ *be the the usual order on the monoid* $(\mathbb{N}, +, 0)$ *with cancellation law, define* $0 \preceq x \in \mathbb{N}$ *if and only if* $x \neq 1$*, and* $\forall x, y \in \mathbb{N} \setminus \{0\}$, $x \preceq y$ *if and only if* $x \leq y$*. Then* $(\mathbb{N}, \preceq)$ *is a p.o. monoid,* $0 + 1 \preceq 1 + 1$*, but* $0 \not\preceq 1$.

3 Riesz Monoids

We call an element a of a p.o.monoid M is positive if $a \geq 0$, and denote the positive cone of M by $M^+ = \{a \in M \mid \mathrm{a} \geq 0\}$.

Definition 12 *We call two positive elements* a, b *in a p.o.monoid* M *disjoint if for every* $c \leq a$, b *we have* $c \leq 0$*, the identity of* M*, i.e.,* $a \wedge b = 0$.

Let D be a a one dimensional local Noetherian domain with quotient field K that is not a UFD. Set $M = \{dD : d \in D \setminus \{0\}\}$. Let p, q be two non-associated irreducible elements of D. Then within $M, cD \leq pD, qD$ implies that $cD = D$, the identity of M. However in $G(D) = \{kD : k \in K \setminus \{0\}\}$, pD, qD are not disjoint for $p|q^n$ in D for some n because D is one dimensional local. This is because if two elements of a p.o. group are disjoint then so are their powers. So, care must be taken when dealing with disjoint elements in monoids which are the positive cones of p.o. groups. One way of avoiding confusion here is calling coprime the elements that are monoid disjoint in a monoid M that is the positive cone of a p.o. group.

Definition 13 *For a general p.o.monoid* M*, we call an element* $a \in M^+$ *primal if for all* a_1, $a_2 \in M^+$, $a \leq a_1 + a_2$ *implies that* $a = b_1 + b_2$*, for some* b_1, $b_2 \in M^+$ *such that* $b_i \leq a_i$, $i = 1, 2$.

Definition 14 *A directed p.o.monoid (directed monoid for short)* M *is called a Riesz monoid if every element of* M^+ *is primal.*

Lemma 2 *Let* M *be a p.o.monoid satisfying (2.6), for* a, b, $c \in M$*, if* $a \leq b + c$ *and* $a \wedge b = 0$ *with* $c \geq 0$*, then* $a \leq c$.

Proof Adding c to both sides of $a \wedge b = 0$, we have $(a + c) \wedge (b + c) = c$ for (2.6). Now $a \leq a + c$, $a \leq b + c$ implies $a \leq (a + c) \wedge (b + c) = c$.

Theorem 2 *A l.o.monoid* M *which is also a divisibility monoid satisfying the order-cancellation law and (2.6) is a Riesz monoid.*

Proof M is a directed monoid, being a l.o.monoid. So if we want to prove that M is a Riesz monoid, we only need to show that every element of M^+ is primal. For x, $a_1,\ a_2 \in M^+$, if $x \leq a_1\ +\ a_2$, we want to prove $x = b_1\ +\ b_2$, for some $b_1,\ b_2 \in M^+$ and $b_i \leq a_i$, i = 1, 2. For this, set $b_1\ = x \wedge a_1$. Then $b_1 \leq x,\ a_1$, and $b_1 \in M^+$. This means there exist b_2, $c \in M^+$ such that $x = b_1\ +\ b_2$ and $a_1 = b_1\ +\ c$ for M is a divisibility monoid. Now $b_1 + b_2 \leq a_1 + a_2$, $a_1\ = b_1\ +\ c$ and M satisfying order-cancellation law, gives $b_2 \leq c + a_2$. Moreover, we have $b_2 \wedge c$=0 since M satisfying order-cancellation law and $b_1 + b_2 \wedge c = (b_1 + b_2 \wedge b_1 + c =$ $(x \wedge a_1$= b_1. Therefore, $b_2 \leq c + a_2$ and $b_2 \wedge c$=0 implies $b_2 \leq a_2$ by Lemma 2. This shows that M is a Riesz monoid.

Definition 15 *A p.o.monoid M satisfies the weak (m,n)-interpolation property or the weak Riesz interpolation property (WRIP), if for given $x_1,\ x_2;\ y_1,\ y_2,\ ...,\ y_n \in M$ with $x_i\ \leq\ y_j$ and the inverses of x_i exist in M for all integers $i \in$ [1, 2], $j \in$ [1, n], there is a $z \in M$ such that*

$$x_i\ \leq\ z\ \leq\ y_j, \tag{3.11}$$

for all integer pairs$(i,\ j) \in$ [1, 2] $\times$ [1, n].

Definition 16 *A p.o.monoid M satisfies the (m,n)-interpolation property or the Riesz interpolation property (RIP), if for given $x_1,\ x_2,\ ...,\ x_m;\ y_1,\ y_2,\ ...,\ y_n \in M$ with $x_i\ \leq\ y_j$ for all integers $i \in$ [1, m], $j \in$ [1, n], there is a $z \in M$ such that*

$$x_i\ \leq\ z\ \leq\ y_j, \tag{3.12}$$

for all integer pairs$(i,\ j) \in$ [1, m] $\times$ [1, n].

Note that if a p.o.monoid M satisfies the (RIP), then the (WRIP) holds for M.

Theorem 3 *A Riesz monoid M which satisfying the order-cancellation law has the (WRIP).*

Proof When (m, n)=(1, 1), (1, 2) or (2, 1), M has the (WRIP). In fact, if we can prove M has the (WRIP) for (m, n)=(2, 2), we can prove the (WRIP) for (2, n) by induction on n. Now, we only need to prove that the (WRIP) for (2, 2). As y_1 + (-x_1) = [y_1 + (-x_2)] + [x_2 + (-x_1)] $\leq$ [y_1 + (-x_2)] + [y_2 + (-x_1)] and y_1 + (-x_1), y_1 + (-x_2), y_2 + (-x_1) $\in M^+$, there exist z_1, $z_2 \in M^+$ satisfying $z_1 \leq y_1$ + (-x_2), $z_2 \leq y_2$ + (-x_1) and y_1 + (-x_1) = $z_1 + z_2$. From this, we have $x_1 \leq z_2 + x_1 \leq y_2$, $z_1 + z_2 + x_1 = y_1 \geq z_1 + x_2 \Rightarrow z_2 + x_1 \geq x_2$, and $y_1 = z_1 + z_2 + x_1 \geq z_2 + x_1$. Then set $z = z_2 + x_1$, we have the conclusion.

Definition 17 *A directed monoid M is a refinement directed monoid (refinement monoid for short), if $a_0 + a_1 = b_0 + b_1$ in M^+ implies the existence of $c_{i,j} \in M^+$, for $i, j = 0, 1$, such that $a_i = c_{i,0} + c_{i,1}$ and $b_i = c_{0,i} + c_{1,i}$, for all $i = 0, 1$.*

Theorem 4 *Every refinement divisibility monoid M is a Riesz divisibility monoid.*

Proof When M is a refinement divisibility monoid, if we want to prove M is a Riesz divisibility monoid, we only need to prove every element of M^+ is primal. For $a, a_0, a_1 \in M^+$, if $a \leq a_0 + a_1$, then there exists $b \in M$ such that $a + b = a_0 + a_1$ for M is divisibility monoid. Note that every element of a divisibility monoid is a positive element, so $b \in M^+ = M$. Then there exist $c_{i,j} \in M^+$, s.t., $a = c_{0,0} + c_{0,1}$, $b = c_{1,0} + c_{1,1}$, $a_0 = c_{0,0} + c_{1,0} \geq c_{0,0}$ and $a_1 = c_{0,1} + c_{1,1} \geq c_{0,1}$. So when we set $b_i = c_{0,i}$, $i = 0, 1$, we get the conclusion.

4 The Bases and Conrad F-Condition in Lattice-ordered Monoids

Definition 18 *For M a l.o.monoid, $x \in M$ is homogeneous if $x > 0$ and all h, k with $0 < h, k \leq x$ are non-disjoint.*

Definition 19 *Let M be a l.o.monoid. Call a $S \subseteq M$ disjoint if every element x of S satisfies $x > 0$ and for every pair of distinct strictly positive elements $x, y \in S$, $x \wedge y = 0$. A set that is maximal w.r.t. being disjoint and which contains homogeneous elements only is called a basis of M.*

Lemma 3 *A nonempty subset S of a l.o.monoid M is a basis if and only if S is disjoint and $(S \backslash \{s\}) \cup \{x, y\}$ is non-disjoint for any $s \in S$ and for any $\{x, y\} \subseteq (M \backslash S) \cup \{s\}$, with $x \neq y$.*

Proof Let S be a basis and suppose that for some $s \in S$, $(S \backslash \{s\}) \cup \{x, y\}$ is disjoint for some $\{x, y\} \subseteq (M \backslash S) \cup \{s\}$, with $x \neq y$. Then $x \wedge s > 0$ and $y \wedge s > 0$ because S is a maximal set of disjoint elements. Thus $x \wedge s, y \wedge s \in (0, s]$. Therefore, $0 < (x \wedge s) \wedge (y \wedge s) = x \wedge y \wedge s = 0$ for s is homogeneous, a contradiction. Conversely, suppose that S is disjoint and satisfies the condition in the lemma. If $S \cup \{x\}$ is disjoint for some $x \in M \backslash S$, then for any $s \in S$, $(S \backslash \{s\}) \cup \{s, x\}$ is disjoint and $s \neq x$, a contradiction. Therefore, S is a maximal disjoint set. If $s \in S$ and s is not homogeneous, then there exist at least one pair of elements $0 < x, y < s$ such that $x \wedge y = 0$. But then $x, y \notin S$, $x \neq y$ and $(S \backslash \{s\}) \cup \{x, y\}$ is disjoint, a contradiction. Thus, S is a maximal disjoint set consisting of homogeneous elements, i.e., S is a basis.

Definition 20 *Let M be a l.o.monoid. We call a set $S \subseteq M$ independent if S consists of mutually disjoint homogeneous elements.*

Note that if we have an ascending chain $\{F_\alpha\}_{\alpha \in I}$ of independent subsets of M under inclusion then the union of such a chain is again an independent set. For if $x \in \cup F_\alpha$, then x must be homogeneous because x belongs to one of F_α. Also if there are two elements x, $y \in \cup F_\alpha$ with $x \wedge y \neq 0$, then for some $\alpha \in I$ we have x, $y \in F_\alpha$. This observation leads to the following statement:

Lemma 4 *Let S be a nonempty independent set in a l.o.monoid M. Then there is a maximal independent set containing S.*

Theorem 5 *A nontrivial l.o.monoid M has a basis if and only if (P): each $0 < x \in M$ exceeds at least one homogeneous element. Every basis of M is a maximal independent subset and every maximal independent subset of M is a basis provided M has a basis.*

Proof Let $S = \{0 < a_\gamma \colon \gamma \in \Gamma\}$ be a basis for M, and consider $0 < x \in M$. There exists a $\gamma \in \Gamma$ such that $x \wedge a_\gamma > 0$, for otherwise S is not a maximal set of disjoint elements. But this means that $0 < x \wedge a_\gamma \leq x$ and $x \wedge a_\gamma$ is homogeneous for a_γ is homogeneous and $x \wedge a_\gamma \in (0,\, a_\gamma]$. Thus, M satisfies the (P), and clearly S is a maximal independent subset of M.
Conversely, suppose that M satisfies the (P). By Lemma 4, there exists a maximal independent subset $T = \{0 < a_\gamma \colon \gamma \in \Gamma\}$ of M and by the property $T \neq \varnothing$. All we need show is that T is a maximal set of disjoint elements. Suppose on the contrary that there is an element $0 < x \in M \backslash T$ such that $x \wedge a_\gamma = 0$ for all $\gamma \in \Gamma$. But then by the property (P), x exceeds a homogeneous element h, and h is disjoint with a_γ for all $\gamma \in \Gamma$. Therefore, $T \cup \{h\} \supsetneqq T$ and $T \cup \{h\}$ is an independent subset of M, but this is contrary to our choice of T.
Conrad's F-condition on a l.o.monoid reads thus: Each strictly positive element x in a l.o.monoid M is greater than at most a finite number of (mutually) disjoint positive elements.

Theorem 6 *If a nontrivial l.o.monoid M satisfies Conrad's F-condition, then M has a basis.*

Proof Suppose that the condition holds but M has no basis. Then by Thm. 5, there is at least one $0 < y \in M$ such that no homogeneous element is contained in $\{x \in M \colon 0 < x \leq y\}$. Then there exist two disjoint elements x_1, y_1 with $0 < x_1$, $y_1 < y$. None of x_1, y_1 exceeds a homogeneous element for otherwise y would. So, say, $0 < x_2$, $y_2 < x_1$ with $x_2 \wedge y_2 = 0$. Since $x_1 \wedge y_1 = 0$ and $y_2 < x_1$ we have $y_1 \wedge y_2 = 0$. Next $0 < x_3$, $y_3 < x_2$ with $x_3 \wedge y_3 =$

0. We can conclude that y_1, y_2, y_3 are mutually disjoint. Similarly producing x_i $'s$, y_i $'s$ and using induction we can produce an infinite sequence $\{y_i\}$ of mutually disjoint element less than y. Contradicting the assumption that M satisfies Conrad's F-condition.

Corollary 1 *For a l.o.monoid M, the following are equivalent:*
(i) M satisfies Conrad's F-condition.
(ii) Every strictly positive element exceeds at least one and at most a finite number of homogeneous elements that are mutually disjoint.

Proof (i) $\Rightarrow$ (ii) can be obtained from the definition of the Conrad F-condition, Theorems 5 and 6. For (ii) $\Rightarrow$ (i), suppose that (ii) holds yet M does not satisfy (i). Then there is $0 < x \in M$ that exceeds an infinite sequence $\{x_i\}$ of mutually disjoint strictly positive elements of M. Now each of x_i exceeds at least one homogeneous element h_i. For $\{x_i\}$ are mutually disjoint, $\{h_i\}$ are mutually disjoint which cause a contradiction. Thus, we have the conclusion.

5 The Bases and Conrad F-Condition in Riesz Monoids

Proposition 1 *Let M be a p.o.monoid. If M satisfies the (RIP), then the following property holds. (pR): if 0, $g \leq x_1$, x_2, ..., $x_n \in M$ with $g \nleq 0$, then there exists $r \in M$ such that 0 $< r \leq x_1$, x_2, ..., x_n.*

Proof From Def. 16, there is $r \in M$ such that 0, $g \leq r \leq x_1$, x_2, ..., x_n. Now $r \geq 0$ and $r \neq 0$ because of g.

Definition 21 *For M a Riesz monoid, $x \in M$ is homogeneous if $x > 0$ and all h, k with $0 < h$, $k \leq x$ are non-disjoint.*

Definition 22 *Let M be a Riesz monoid. Call a set $S \subseteq M$ disjoint if every element x of S satisfies $x > 0$ and for every pair of distinct strictly positive elements x, $y \in S$, $x \wedge y = 0$. A set that is maximal w.r.t. being disjoint and which contains homogeneous elements only is called a basis of M.*

Lemma 5 *A nonempty subset S of a Riesz monoid M which satisfies the (RIP) is a basis if and only if S is disjoint and $(S\backslash\{s\}) \cup \{x, y\}$ is non-disjoint for any $s \in S$ and for any $\{x, y\} \subseteq (M\backslash S) \cup \{s\}$, with $x \neq y$.*

Proof Let S be a basis and suppose that for some $s \in S$, $(S\backslash\{s\}) \cup \{x,\ y\}$ is disjoint for some $\{x,\ y\} \subseteq (M\backslash S) \cup \{s\}$, with $x \neq y$. Then $x \wedge s \neq 0$ and $y \wedge s \neq 0$ because S is a maximal set of disjoint elements. By Prop. 1, this leads to the existence of $0 < t \leq x,\ s$ and $0 < u \leq y,\ s$. For s is homogeneous, there is $w \in M$ with $0 < w \leq t,\ u,\ x,\ y$, a contradiction. Conversely, suppose that S is disjoint and satisfies the condition in the lemma. If $S \cup \{x\}$ is disjoint for some $x \in M\backslash S$, then for any $s \in S$, $(S\backslash\{s\}) \cup \{s,\ x\}$ is disjoint and $s \neq x$, a contradiction. Therefore, S is a maximal disjoint set. If $s \in S$ and s is not homogeneous, then there exist at least one pair of elements $0 < x,\ y < s$ such that $x \wedge y = 0$. But then x, $y \notin S$, $x \neq y$ and $(S\backslash\{s\} \cup \{x,\ y\}$ is disjoint, a contradiction. Thus, S is a maximal disjoint set consisting of homogeneous elements, i.e., S is a basis.

Definition 23 *Let M be a Riesz monoid. We call a set $S \subseteq M$ independent if S consists of mutually disjoint homogeneous elements.*

Note that if we have an ascending chain $\{F_\alpha\}_{\alpha\ \in\ I}$ of independent subsets of M under inclusion then the union of such a chain is again an independent set. For if $x \in \cup F_\alpha$, then x must be homogeneous because x belongs to one of F_α. Also if there are two elements $x,\ y \in \cup F_\alpha$ with $x \wedge y \neq 0$, then for some $\alpha \in I$ we have $x,\ y \in F_\alpha$. This observation leads to the following statement:

Lemma 6 *Let S be a nonempty independent set in a Riesz monoid M. Then there is a maximal independent set containing S.*

Theorem 7 *A nontrivial Riesz monoid M satisfying the (RIP) has a basis if and only if (P): each $0 < x \in M$ exceeds at least one homogeneous element. Every basis of M is a maximal independent subset and every maximal independent subset of M is a basis provided M has a basis.*

Proof Let $S = \{0 < a_\gamma\colon\ \gamma \in \Gamma\}$ be a basis for M, and consider $0 < x \in M$. There exists a $\gamma \in \Gamma$ such that $x \wedge a_\gamma \neq 0$, for otherwise S is not a maximal set of disjoint elements. By Prop. 1, this means that there is $0 < h \leq x,\ a_\gamma$, and h is homogeneous for a_γ is homogeneous and $h \in (0,\ a_\gamma]$. Thus, M satisfies the (P), and clearly S is a maximal independent subset of M.

Conversely, suppose that M satisfies (P). By Lemma 6, there exists a maximal independent subset $T = \{0 < a_\gamma\colon\ \gamma \in \Gamma\}$ of M and by the property $T \neq \varnothing$. All we need show is that T is a maximal set of disjoint elements. Suppose on the contrary that there is an element $0 < x \in M\backslash T$ such that $x \wedge a_\gamma = 0$ for all $\gamma \in \Gamma$. But then by the property (P), x exceeds

a homogeneous element h, and h is disjoint with a_γ for all $\gamma \in \Gamma$. Therefore, $T \cup \{h\} \supsetneqq T$ and $T \cup \{h\}$ is an independent subset of M, but this is contrary to our choice of T.
Conrad's F-condition on a Riesz monoid reads thus: Each strictly positive element x in a Riesz monoid M is greater than at most a finite number of (mutually) disjoint positive elements.

Theorem 8 *If a nontrivial Riesz monoid M satisfies the (RIP) and Conrad's F-condition, then M has a basis.*

Proof Suppose that the condition holds but M has no basis. Then by Thm. 7, there is at least one $0 < y \in M$ such that no homogeneous element is contained in $\{x \in M\colon 0 < x \leq y\}$. Then there exist two disjoint elements x_1, y_1 with $0 < x_1, y_1 < y$. None of x_1, y_1 exceeds a homogeneous element for otherwise y would. So, say, $0 < x_2, y_2 < x_1$ with $x_2 \wedge y_2 = 0$. Since $x_1 \wedge y_1 = 0$ and $y_2 < x_1$ we have $y_1 \wedge y_2 = 0$. Next $0 < x_3, y_3 < x_2$ with $x_3 \wedge y_3 = 0$. We can conclude that y_1, y_2, y_3 are mutually disjoint. Similarly producing $x_i\,'s$, $y_i\,'s$ and using induction we can produce an infinite sequence $\{y_i\}$ of mutually disjoint element less than y. Contradicting the assumption that M satisfies Conrad's F-condition.

Corollary 2 *For a Riesz monoid M that satisfies the (RIP), the following are equivalent:*
(i) M satisfies Conrad's F-condition.
(ii) Every strictly positive element exceeds at least one and at most a finite number of homogeneous elements that are mutually disjoint.

Proof (i) $\Rightarrow$ (ii) can be obtained from the definition of the Conrad F-condition, Theorems 7 and 8. For (ii) $\Rightarrow$ (i), suppose that (ii) holds yet M does not satisfy (i). Then there is $0 < x \in M$ that exceeds an infinite sequence $\{x_i\}$ of mutually disjoint strictly positive elements of M. Now each of x_i exceeds at least one homogeneous element h_i. For $\{x_i\}$ are mutually disjoint, $\{h_i\}$ are mutually disjoint, which causes a contradiction. Thus, we have the conclusion.

6 The Bases and Conrad F-Condition in pre-Riesz Monoids

Proposition 2 *In a Riesz monoid M which satisfies the order-cancellation law, the following property holds. (WpR): if 0, $g \leq x_1, x_2, \ldots, x_n \in M$ with $g \nleq 0$ and the inverse of g exists in M, then there exists $r \in M$ such that $0 < r \leq x_1, x_2, \ldots, x_n$.*

Proof M satisfying the (WRIP) from Thm. 3. By the (WRIP), there is $r \in M$ such that $0, g \leq r \leq x_1, x_2, \ldots, x_n$. Now $r \geq 0$ and $r \neq 0$ because of g.

Note that if a partially ordered monoid M satisfies the (pR), then the (WpR) holds.

Definition 24 *Call a directed monoid M a pre-Riesz monoid if M satisfies the property the (WpR).*

Corollary 3 *A Riesz monoid satisfying the order-cancellation law is a pre-Riesz monoid.*

Proof By Def. 14, Prop. 2, and Def. 24, we get the desired conclusion.

Definition 25 *For M a pre-Riesz Monoid, $x \in M$ is homogeneous if $x > 0$ and all h, k with $0 < h$, $k \leq x$ are non-disjoint.*

Definition 26 *Let M be a pre-Riesz monoid. Call a set $S \subseteq M$ disjoint if every element x of S satisfies $x > 0$ and for every pair of distinct strictly positive elements x, $y \in S$, $x \wedge y = 0$. A set that is maximal w.r.t. being disjoint and which contains homogeneous elements only is called a basis of M.*

Lemma 7 *A nonempty subset S of a pre-Riesz monoid M which satisfies the (pR) is a basis if and only if S is disjoint and $(S\backslash\{s\}) \cup \{x, y\}$ is non-disjoint for any $s \in S$ and for any $\{x, y\} \subseteq (M\backslash S) \cup \{s\}$, with $x \neq y$.*

Proof Let S be a basis and suppose that for some $s \in S$, $(S\backslash\{s\}) \cup \{x, y\}$ is disjoint for some $\{x, y\} \subseteq (M\backslash S) \cup \{s\}$, with $x \neq y$. Then $x \wedge s \neq 0$ and $y \wedge s \neq 0$ because S is a maximal set of disjoint elements. By Prop. 1, this leads to the existence of $0 < t \leq x, s$ and $0 < u \leq y, s$. For s is homogeneous, there is $w \in M$ with $0 < w \leq t, u, x, y$, a contradiction. Conversely, suppose that S is disjoint and satisfies the condition in the lemma. If $S \cup \{x\}$ is disjoint for some $x \in M\backslash S$, then for any $s \in S$, $(S\backslash\{s\}) \cup \{s, x\}$ is disjoint and $s \neq x$, a contradiction. Therefore, S is maximal disjoint set. If $s \in S$ and s is not homogeneous, then there exist at least one pair of elements $0 < x, y < s$ such that $x \wedge y = 0$. But then $x, y \notin S$, $x \neq y$ and $(S\backslash\{s\} \cup \{x, y\}$ is disjoint, a contradiction. Thus, S is a maximal disjoint set consisting of homogeneous elements, i.e., S is a basis.

Definition 27 *Let M be a pre-Riesz monoid. We call s set $S \subseteq M$ independent if S consists of mutually disjoint homogeneous elements.*

Note that if we have an ascending chain $\{F_\alpha\}_{\alpha \in I}$ of independent subsets of M under inclusion then the union of such a chain is again an independent set. For if $x \in \cup F_\alpha$, then x

must be homogeneous because x belongs to one of F_α. Also if there are two elements x, $y \in \cup F_\alpha$ with $x \wedge y \neq 0$, then for some $\alpha \in I$ we have x, $y \in F_\alpha$. This observation leads to the following statement:

Lemma 8 *Let S be a nonempty independent set in a pre-Riesz monoid M. Then there is a maximal independent set containing S.*

Theorem 9 *A nontrivial pre-Riesz monoid M satisfying the (pR) has a basis if and only if (P): each $0 < x \in M$ exceeds at least one homogeneous element. Every basis of M is a maximal independent subset and every maximal independent subset of M is a basis provided M has a basis.*

Proof Let $S = \{0 < a_\gamma \colon \gamma \in \Gamma\}$ be a basis for M, and consider $0 < x \in M$. There exists a $\gamma \in \Gamma$ such that $x \wedge a_\gamma \neq 0$, for otherwise S is not a maximal set of disjoint elements. By Prop. 1, this means that there is $0 < h \leq x$, a_γ, and h is homogeneous for a_γ is homogeneous and $h \in (0, a_\gamma]$. Thus, M satisfies (P), and clearly S is a maximal independent subset of M. Conversely, suppose that M satisfies the (P). By Lemma 8, there exists a maximal independent subset $T = \{0 < a_\gamma \colon \gamma \in \Gamma\}$ of M and by the property $T \neq \varnothing$. All we need show is that T is a maximal set of disjoint elements. Suppose on the contrary that there is an element $0 < x \in M \backslash T$ such that $x \wedge a_\gamma = 0$ for all $\gamma \in \Gamma$. But then by the property (P), x exceeds a homogeneous element h, and h is disjoint with a_γ for all $\gamma \in \Gamma$. Therefore, $T \cup \{h\} \supsetneqq T$ and $T \cup \{h\}$ is an independent subset of M, but this is contrary to our choice of T.
Conrad's F-condition on a pre-Riesz monoid reads thus: Each strictly positive element x in a pre-Riesz monoid M is greater than at most a finite number of (mutually) disjoint positive elements.

Theorem 10 *If nontrivial pre-Riesz monoid M satisfies the (pR) and Conrad's F-condition, then M has a basis.*

Proof Suppose that the condition holds but M has no basis. Then by Thm. 9, there is at least one $0 < y \in M$ such that no homogeneous element is contained in $\{x \in M \colon 0 < x \leq y\}$. Then there exist two disjoint elements x_1, y_1 with $0 < x_1, y_1 < y$. None of x_1, y_1 exceeds a homogeneous element for otherwise y would. So, say, $0 < x_2, y_2 < x_1$ with $x_2 \wedge y_2 = 0$. Since $x_1 \wedge y_1 = 0$ and $y_2 < x_1$ we have $y_1 \wedge y_2 = 0$. Next $0 < x_3, y_3 < x_2$ with $x_3 \wedge y_3 = 0$. We can conclude that y_1, y_2, y_3 are mutually disjoint. Similarly producing $x_i\,'s$, $y_i\,'s$ and using induction we can produce an infinite sequence $\{y_i\}$ of mutually disjoint elements less than y. Contradicting the assumption that M satisfies Conrad's F-condition.

Corollary 4 *For a pre-Riesz monoid M which satisfies the (pR), the following are equivalent:*
(i) M satisfies Conrad's F-condition.
(ii) Every strictly positive element exceeds at least one and at most a finite number of homogeneous elements that are mutually disjoint.

Proof (i) $\Rightarrow$ (ii) can be obtained by the definition of the Conrad F-condition, Theorems 9 and 10. For (ii) $\Rightarrow$ (i), suppose that (ii) holds yet G does not satisfy (i). Then there is $0 < x \in M$ that exceeds an infinite sequence $\{x_i\}$ of mutually disjoint strictly positive elements of M. Now each of x_i exceeds at least one homogeneous element h_i. For $\{x_i\}$ are mutually disjoint, $\{h_i\}$ are mutually disjoint, which causes a contradiction. Thus, we have the conclusion.

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