QUESTION: (HD 1504) Did anyone ever look at domains with the property that if the gcd exists for a given pair, then the LCM exists for that given pair or if the gcd exists for a given pair it is a linear combination? This question was proposed by Professor Daniel Anderson.

ANSWER: I'll take parts of the question one by one.

The domains in which the following holds: "if the gcd exists for a given pair, then the LCM exists for that given pair".

It is patent that if for $a, b \in D\backslash\{0\}$ $LCM(a, b)$ exists then $GCD(a, b)$ exists. Thus we are looking at domains D in which $GCD(a, b) \Leftrightarrow LCM(a, b)$ exists.

It is easy to see that if $GCD(a, b)$ and $LCM(a, b)$ both exist then $LCM(a, b)D = \frac{ab}{GCD(a,b)}D = (a) \cap (b)$

Next if $GCD(a, b) = d$ then $a = a_1d$ and $b = b_1d$ where $GCD(a_1, b_1) = 1$. So, in these domains, $GCD(a, b) = d \Longleftrightarrow LCM(a, b) = a_1b_1d$, where a_1, b_1 are as described. Thus, in these domains, $GCD(x, y) = 1 \Longleftrightarrow LCM(x, y)D = xyD = (x) \cap (y) = xy(x, y)^{-1}$. Or, in these domains, $GCD(x, y) = 1 \Longleftrightarrow xyD = xy(x, y)^{-1}$. Cancelling xy we get $GCD(x, y) = 1 \Longleftrightarrow D = (x, y)^{-1}$ and as $(x, y)^{-1} = D \Longleftrightarrow ((x, y)^{-1})^{-1} = (x, y)_v = D$. When $(x, y)_v = D$ we say that x, y are v-coprime as we say that x, y are coprime when $GCD(x, y) = 1$. Thus in the domains in question any two coprime elements are v-coprime. Again if $GCD(a, b) = d$ then $a = a_1d$ and $b = b_1d$ where $GCD(a_1, b_1) = 1$ and so $(a, b)_v = d(a_1, b_1)_v = dD = GCD(a, b)D$ and $((a, b)_v)^{-1} = \frac{1}{ab}((a) \cap (b))$ and from $(a, b)_v = dD$ we get $((a, b)_v)^{-1} = (\frac{1}{d})$. Comparing, $\frac{1}{ab}((a) \cap (b)) = \frac{1}{d}D$ or $((a) \cap (b)) = \frac{ab}{d}D$. Thus a domain D in which $GCD(a, b)$ exists implies $LCM(a, b)$ exists is precisely the domain in which x, y coprime implies x, y v-coprime.

Now these domains do have a name! In [MZ, On Prufer v-multiplication domains, Manuscripta Math. 35(1981), 1-26], on page 18, a domain D is said to satisfy Property λ if any two coprime elements of D are v-coprime. The property appears to be quite toothless. But works wonders in the following situations.

(1) When D is atomic, i.e. every nonzero non unit of D is expressible as a finite product of irreducible elements.

Proposition 6.4 of [MZ] says: An atomic integral domain D is a UFD if and only if D satisfies the property λ.

In more general situations Corollary 6.5 of [MZ] says: If an integral domain D satisfies property λ then every atom of D is a prime.

(2) Of course every GCD domain satisfies property λ. But the property λ can be seen in a generalization of GCD domains, the so called pre-Schreier domains of [Z, Comm. Algebra 15(9) (1987), 1895-1920]. Using the proof of Lemma 2.1 of [Z1, J. Pure Appl. Algebra 65(1990) 199-207] we can establish that every pair of coprime elements of a pre-Schreier domain is v-coprime.

(3) Another generalization of GCD domains, the so-called Prufer v-multiplication domain PVMD does not generally satisfy the λ property. In fact, even a Prufer domain, a specialization of PVMDs, does not satisfy the λ property. This can be seen by taking a non-PID Dedekind domain D. Because D is not a PID, by

Proposition 6.4 of [MZ] D does not satisfy λ.

(4) Cohn [C, Bezout rings and their subrings, Proc. Cambridge Philos. Soc. 64 (1968), 251-264] called a domain D a pre-Bezout ring if for every pair $x, y \in D$, x, y coprime implies that x and y are comaximal. Now x, y being co-maximal means the GCD, 1, is a linear combination of x and y. And as $d = GCD(a, b) = dGCD(a_1, b_1)$ where a_1, b_1 are coprime, we conclude that pre-Bezout domains are precisely the domains in which the GCD of two elements a, b is a linear combination of a, b. (This much answers the part: if the gcd exists for a given pair it is a linear combination.) The pre-Bezout property was generalized to the GCD-Bezout property in [PT, Divisibility properties related to star operations on integral domains, Int. Electron. J. Algebra 12 (2012), 53-74] where Park and Tartarone study domains in which the GCD of a finite set of elements, if it exists, is a linear combination of those elements. Of interest to me is the fact that pre-Bezout and GCD-Bezout domains all satisfy the λ property.

That leaves: If LCM m of a, b exists when is m a linear combination of a, b? The answer, with a tongue in the cheek, is yes! Always. As we can always have $mD = a_1b_1d(1, x)$ for some x in D. But of course in the pre-Bezout domains case we can have $mD = a_1b_1d(a_1, b_1)$. In any case in the pre-Bezout domains this also is the case that if LCM of a, b exists, then GCD of a, b is a linear combination of a, b. Now note that, as we have already seen $(a) \cap (b)$ is principal if and only if $(a, b)_v$ is principal. Thus the domains in which LCM(a,b) exists implies GCD (a,b) is a linear combination of a,b are precisely the domains in which a, b v-coprime implies a, b co-maximal. These domains were discussed in [HZ, J. Algebra 423 (1)(2015) 93-113].

Comment added on 2-9-2020. About that Park-Tartarone paper on GCD-Bezout domains [Int. Electron. J. Algebra 12 (2012), 53-74]. I had a brief look into it again and realized that a so-called GCD-Bezout domain is nothing but the Special pre-Bezout domains of [DZ, J. Pure Appl. Algebra, 214 (2010), 2087-2091]. Let me elaborate on it. Indeed D may be assumed to be different from its field of quotients. Now reading the comments between Corollaries 11 and 12 of the DZ paper one gathers that D is a Special pre-Bezout (spre-Bezout) domain if and only if for every finite set of elements $x_1, ..., x_n$ in D the ideal $(x_1, ..., x_n)$ being primitive implies that $(x_1, ..., x_n) = D$. (Here $(x_1, ..., x_n)$ is primitive if $(x_1, ..., x_n) \subseteq xD$ implies that x is a unit.) On the other hand Park and Tartarone say that D is a GCD Bezout domain if whenever $GCD(x_1, ..., x_n) = d$ exists "we have a Bezout identity" which, in plain Math, means we have $d = (x_1, ..., x_n)$.

Now let's start. Spre-Bezout implies GCD-Bezout. Let d be a GCD of $(y_1, ..., y_n)$ and write $y_i = x_i d$. Then $(y_1, ..., y_n) = (x_1, ..., x_n)d$, where $(x_1, ..., x_n)$ is primitive because d is a GCD of the y_i. So, by the spre-Bezout property $(x_1, ..., x_n) = D$ forcing $(y_1, ..., y_n) = dD$ and this means that d is a linear combination of y_i or d satisfies the Bezout identity or whatever scholarly speak you want to speak. Conversely suppose that D is a GCD-Bezout domain and let $(x_1, ..., x_n)$ be a primitive ideal in D. Then 1 is a GCD of $(x_1, ..., x_n)$ and so by the GCD-Bezout property 1 is a linear combination of $x_1, ..., x_n$. That is $(x_1, ..., x_n) = D$. Oddly, after Corollary 12, the authors of DZ talk about the PSP property and lo and behold PSP property has good coverage in that Park-Tartarone paper.