# Cohen Type Theorems for a Commutative Ring

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Abstract. Let $R$ be a commutative ring with $1 \neq 0$. We show that if every prime ideal containing a proper ideal is principal (resp., invertible, finitely generated locally principal), then $I$ is a finite product of principal (resp., invertible, finitely generated locally principal) prime ideals. Let $R$ be an integral domain and $*$ a finite character star operation on $R$. We show that if every prime $*$-ideal containing a proper $*$-ideal $I$ is $*$-invertible, then $I$ is a finite $*$-product of $*$-invertible prime $*$-ideals and hence is $*$-invertible.

Let $R$ be a commutative ring with $1 \neq 0$. It is well known that an ideal maximal with respect to not being principal (resp., invertible, finitely generated) is prime [**8**, Exercise 10, sec.1-1] (resp., [**8**, Exercise 36, sec.1-4], [**8**, Theorem 7]). A similar proof shows that an ideal maximal with respect to not being finitely generated locally principal is prime. In the case of an integral domain $R$ and a finite character star operation $*$, a similar proof shows that a nonzero $*$-ideal $I$ maximal with respect to not being $*$-invertible is a prime $*$-ideal [**9**, Proposition 2.1]. Necessary definitions related to star operations will be provided below.

Now the set of non-principal (resp., non-invertible, non-(finitely generated locally principal), non-(finitely generated)) ideals containing a given non-principal (resp., non-invertible, non-(finitely generated locally principal, non-(finitely generated)) ideal is inductive. So by Zorn's Lemma a non-principal (resp., non-invertible, non-(finitely generated locally principal, non-(finitely generated)) ideal is contained in an ideal maximal with respect to this property which is necessarily prime. Stated in another way this says that if every prime ideal containing an ideal is principal (resp., invertible, finitely generated locally principal, finitely generated), then so is $I$. In the case of a finite character star operation $*$ on an integral domain $R$ we have that the set of non-$*$-invertible ideals containing a nonzero non-$*$-invertible ideal is inductive [**9**, Proposition 2.1]. So by Zorn's Lemma, $I$ is contained in a $*$-ideal maximal with respect to this property which is necessarily prime [**9**, Proposition 2.1]. Put another way, if every prime $*$-ideal containing a nonzero $*$-ideal $I$ is $*$-invertible, then $I$ is $*$-invertible.

The purpose of this paper is to give a stronger version of these results. For example, instead of showing that if every prime ideal containing an ideal $I$ is principal, then $I$ is principal we show that $I$ is actually a product of principal prime ideals. Similar results are given for cases in which some kind of locally principal

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property is involved. But first a brief introduction to star operations. Our terminology is standard as in Gilmer's [**6**, Sections 32 and 34] for star operations and as in [**8**] for any other topics, and occasionally we would refer to [**11**]. We provide a quick introduction to the star operations below.

Let $R$ denote an integral domain with quotient field $K$ and let $F(R)$ be the set of nonzero fractional ideals of $R$. A star operation $*$ on $R$ is a function $*\colon F(R) \to F(R)$ such that for all $A, B \in F(R)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c) $(A^*)^* = A^*$.

We note that for $A, B \in F(R)$ $(AB)^* = (A^*B)^* = (A^*B^*)^*$, and call it the $*$-product. A fractional ideal $A \in F(R)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where $B$ is a finitely generated fractional ideal. Any nonzero intersection of $*$-ideals for any star operation $*$ is again a star ideal. A star operation $*$ is said to be of *finite character* or *of finite type* if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. A star operation $*$ is of finite type if and only if for each ideal $A \in F(R)$, $x \in A^*$ implies that there is a finitely generated $F \subseteq A$ with $x \in F^*$. For $A \in F(R)$ define $A^{-1} = \{x \in K \mid xA \subseteq R\}$ and call $A \in F(R)$ $*$-*invertible* if $(AA^{-1})^* = R$. Clearly every invertible ideal is a $*$-invertible $*$-ideal for every star operation $*$. If $*$ is of finite character and $A$ is $*$-invertible, then $A^*$ is of finite type. The most well known examples of star operations are the $v$-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the $t$-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$ and the $d$-operation defined by $A \mapsto A$ for all $A \in F(R)$. By definition $t$ is of finite character and so is $d$. Also, for every star operation $*$ of finite character every $*$-invertible $*$-ideal is a $t$-invertible $t$-ideal, see [**11**, Theorem 1.1]. Also let $I, J$ be integral ideals with $J \subseteq I$, if $I$ is $*$-invertible, then there is an integral ideal $B$ such that $J^* = (IB)^*$. A domain $R$ is a Krull domain if and only if every nonzero ideal of $R$ is $t$-invertible if and only if every $t$-ideal of $R$ is $t$-invertible [**9**, Theorem 2.5].

THEOREM 1. *(1) Let $R$ be a commutative ring and let $I$ be a proper ideal of $R$. If every prime ideal of $R$ containing $I$ is principal (resp., invertible, finitely generated locally principal), then $I$ is a finite product of prime ideals that are principal (resp., invertible, finitely generated locally principal), (2) Let $R$ be an integral domain, $*$ a finite character star operation on $R$ and $I$ a proper $*$-ideal of $R$. If every prime $*$-ideal containing $I$ is $*$-invertible, then $I$ is a finite $*$-product of $*$-invertible prime $*$-ideals. Hence if every prime $*$-ideal containing $I$ is principal (resp., invertible), then $I$ is a finite product of principal (resp., invertible) prime ideals.*

PROOF. (1) It is enough to do the finitely generated locally principal case. So suppose that $I$ is a proper ideal such that every prime ideal containing $I$ is finitely generated locally principal. Suppose that $I$ is not a finite product of finitely generated locally principal ideals. Let $S$ be the set of proper ideals containing $I$ and not a product of finitely generated locally principal ideals, ordered by inclusion. Since a finite product of finitely generated ideals locally principal prime ideals is finitely generated, the set $S$ is inductive. Hence by Zorn's Lemma $S$ has a maximal element $P$. By hypothesis, $P$ cannot be prime. So, let $xy \in P$ with $x, y \notin P$. Then $(P, x) \supsetneq P$ and $(P : x) = (P : (P, x)) \supsetneq P$. By the maximality, in $S$, of $P$, both $(P, x)$ and $(P : x)$ are each either $R$ or a finite prosuct of finitely generated locsally principal ideals. But $P = (P, x)(P : x)$ as the equality holds locally.

(Let $M$ be a maximal ideal. Then $(P,x)_M$ is principal and $P_M \subseteq (P,x)_M$, so, $P_M = (P,x)_M(P_M : (P,x)_M) = (P,x)_M(P_M : \frac{x}{1}) = (P,x)_M(P : x)_M = ((P,x)(P : x))_M$.) So $P$ is a finite product of finitely generated locally principal prime ideals, a contradiction.

(2) The proof is similar to that of (1). Suppose that $I$ is not a $*$-product of $*$-invertible prime $*$-ideals. Then as above the set $S = \{J \mid J$ is a proper $*$-ideal that is not a $*$-product of $*$-invertible prime $*$-ideals$\}$ is inductive and hence by Zorn's Lemma has a maximal element $P$ which is a proper $*$-ideal. By hypothesis $P$ cannot be prime. So let $xy \in P$ with $x, y \notin P$. Then $(P,x)^* \supsetneq P$. So either $(P,x)^* = R$ or $(P,x)^*$ is a $*$-product of $*$-invertible prime $*$-ideals of $R$ and hence $*$-invertible. So there is an ideal $C$ of $R$ with $P = (C(P,x))^*$ where $(P,y) \subseteq C$. (We can take $C = (P(P,x)^{-1}, y)$.) Then $C^* \supsetneq P$ and so is $R$ or a finite product of $*$-invertible prime $*$-ideals. But then $P$ is a finite $*$-product of $*$-invertible prime $*$-ideals, a contradiction. □

Recall from the introduction that $R$ is a Krull domain if and only if every nonzero ideal and hence every $t$-ideal of $R$ is $t$-invertible. Now Theorem 1 shows that every $t$-ideal of a domain $R$ is $t$-invertible if every prime $t$-ideal is $t$-invertible, and conversely, a known result [**9**, Theorem 2.5], but ours is an easy corollary as the remarks after [**9**, Theorem 2.5] show. Next, Theorem 1 indicates that in a domain $R$, every $t$-ideal is invertible if all prime $t$-ideals are invertible, and obviously conversely. Recall that an integral domain $R$ is a locally factorial Krull domain if and only if every $t$-ideal of $R$ is invertible (see e.g. [**5**, Theorem 2.9]). Thus Theorem 1 characterizes locally factorial Krull domains as domains whose prime $t$-ideals are invertible. Next Theorem 1 characterizes domains whose $t$-ideals are principal as domains whose prime $t$-ideals are all principal. But as each minimal prime of a principal ideal is a $t$-ideal, free of charge, we conclude that every nonzero prime ideal contains a nonzero principal prime, which forces $R$ to be a UFD. Indeed it is customary to call a Krull domain a $t$-Dedekind domain and a UFD a $t$-PID and Theorem 1 leaves no doubt about the terminology being correct.

Here is a "more constructive" alternative approach to (1) of Theorem 1.

THEOREM 2. *Let $R$ be a commutative ring and $I$ a proper ideal of $R$. Suppose that every prime ideal containing $I$ is principal (resp., invertible, finitely generated locally principal) Then $I$ is a product of principal (resp., invertible, finitely generated locally principal) prime ideals and hence is principal (resp., invertible,finitely generated locally principal). Consequently if $I$ is non-principal (resp., non-invertible, non-(finitely generated locally principal)), then $I$ is contained in a prime ideal that is non-principal (resp., non-invertible, non-(finitely generated locally principal)).*

PROOF. Note that the case of finitely generated locally principal covers the other cases, so providing a proof for this case would be sufficient. Let $I$ be contained in only finitely generated locally principal prime ideals. Pass to $\overline{R} = R/I$. So every prime ideal of $\overline{R}$ is finitely generated locally principal. By Cohen's Theorem $\overline{R}$ is Noetherian. So $(\overline{0})$ has a reduced primary decomposition $(\overline{0}) = \overline{Q_1} \cap \overline{Q_2} \cap \cdots \cap \overline{Q_n}$ where each $Q_i$ is a $P_i$-primary ideal of $R$ with $I \subseteq Q$. Since $P_i$ is finitely generated locally principal, $Q_i$ is a power of $P_i$ [**1**, Lemma 1]. Moreover, for each maximal ideal $M$ of $\overline{R}$, $\overline{R}_M$ is a local Noetherian ring with each prime ideal principal. So $\overline{R}_M$ is DVR or SPIR. Thus the ideals $\overline{Q_1}, \overline{Q_2}, ..., \overline{Q_n}$ are pairwise co-maximal and hence, so are $Q_1, Q_2, ..., Q_n$. So, $I = Q_1 \cap Q_2 \cap \cdots \cap Q_n = Q_1 Q_2 \cdots\ Q_n$ is a product of finitely generated locally principal ideals. $\square$

The following well-known result is an immediate corollary to Theorem 1.

THEOREM 3. *Let $R$ be a commutative ring with $1 \neq 0$. (1) Suppose that every prime ideal of $R$ is principal. Then $R$ is a principal ideal ring (PIR) and hence a direct product of PIDs and special principal ideal rings (SPIRs). (2) $R$ is a Dedekind domain if and only if every nonzero prime ideal of $R$ is invertible. (3) Suppose that every prime ideal of $R$ is finitely generated locally principal, then every ideal of $R$ is finitely generated locally principal and is a product of prime ideals; so $R$ is a general ZPI ring and hence is a finite direct product of Dedekind domains and SPIRs (The converse is obvious). (4) Let $R$ be an integral domain and $*$ a finite character star operation on $R$. Then every proper ideal of $R$ is $*$-invertible if and only if every nonzero prime $*$-ideal is $*$-invertible, if and only if every proper $*$-ideal is a $*$-product of $*$-invertible prime $*$-ideals. In case $* = t$ these conditions are equivalent to $R$ being a Krull domain.*

PROOF. (1) By Theorem 1, if every prime ideal is principal, then every ideal is principal. We only need remark that a PIR is a finite direct product of PIDs and SPIRs, for example, see [**8**, Exercise 8, sec.3-3].

(2) By Theorem 1, if every nonzero prime ideal is invertible every nonzero ideal is invertible, so $R$ is an integral domain. But a domain is Dedekind if and only if every nonzero ideal is invertible.

(3) By definition a general ZPI ring is a ring in which every ideal is a product of prime ideals. However a ring is a general ZPI ring if and only if it is a finite direct product of Dedekind domains and SPIRs [**6**, Theorem 39.2].

(4) The first part follows from Theorem 1. For the "In case" part note that for a finite character star operation $*$, a $*$-invertible $*$-ideal is a $t$-invertible $t$-ideal [**11**, Theorem 1.1]. So a domain satisfying the above conditions is at least a Krull domain [**9**, Theorem 2.5]. For the "if and only if every proper $t$-ideal is a $t$-product of prime $t$-ideals" part see [**3**, Corollary 3.2]. $\square$

REMARK 1. *Note that if $* = d$, then $*$-invertible becomes invertible and in this case (4) of Theorem 3 characterizes Dedekind domains. If we were to call a domain characterized by (4) of Theorem 3 a $*$-Krull domain, then all $*$-Krull domains lie between Krull domains and Dedekind domains. To identify a $*$-Krull domain strictly between Krull domains and Dedekind domains is made difficult by the fact that in a $*$-Krull domain $* = t$, as can be gleaned from* [**11**].

An alternate approach to (1) of Theorem 1 is to use the following result [**2**, Theorem]: Suppose that every prime ideal minimal over an ideal $I$ is finitely generated, then there are only a finite number of prime ideals minmal over $I$. Indeed for (2) of Theorem 1 there is an alternate approach as well. This approach goes via the following finite character $*$-operation analog of [**2**, Theorem]. The alternate approach may appear later.

THEOREM 4. *Let $R$ be an integral domain, $*$ a star operation of finite type defined on $R$ and let $I$ be a nonzero ideal of $R$ with $I^* \neq R$. If every minimal prime ideal over $I^*$is a $*$-ideal of finite type, then $I$ has only finitely many minimal primes.*

Proof. Without loss of generality we can assume that $I$ is a $*$-ideal. Then every prime ideal $P$ minimal over $I$ is a prime $*$-ideal [**7**, Proposition 1.1] and by the hypothesis, $P$ is of finite type. Now let $S = \{(P_1P_2 \cdots P_n)^* \mid P_i$ is a prime ideal minimal over $I\}$. If for some $C = (P_1P_2 \cdots P_n)^* \in S$ we have $C \subseteq I$, then every prime ideal minimal over $I$ contains some $P_i$ and so $\{P_1, P_2, ..., P_n\}$ is the set of minimal primes of $I$, where $|\{P_1, P_2, ..., P_n\}| \ \leq n$. To establish that one of the $C \in S$ is indeed such that $C \subseteq I$ we arrange for a contradiction via Zorn's Lemma. Let's assume that $C \nsubseteq I$ for any of the $C$ in $S$ and define $T = \{J \mid J$ is a $*$-ideal with $J \supseteq I$ and $C \nsubseteq J$ for any $C$ in $S\}$. Then obviously $T$ is nonempty as $I \in T$, and as members of $S$ are $*$-ideals of finite type we conclude that the union of an ascending chain in $T$ is in $T$. To see this let $\{U_\alpha\}$ be an ascending chain in $T$ and let $(P_1P_2 \cdot \cdot \cdot P_n)^* = (x_1, x_2, ..., x_r)^*$ and suppose that $(x_1, x_2, ..., x_r)^* \subseteq \cup U_\alpha$. Then say $x_1 \in U_{\alpha_1}$, $x_2 \in U_{\alpha_2}, ..., x_r \in U_r$ where $U_{\alpha_1} \subseteq U_{\alpha_2} \subseteq ...U_{\alpha_r}$. This gives $\{x_1, x_2, ..., x_r\} \subseteq U_{\alpha_r} = \cup_{i=1}^r U_{\alpha_i}$. But as each $U_\alpha$ is a $*$-ideal $(x_1, x_2, ..., x_r)^* \subseteq U_{\alpha_r}$ a contradiction establishing that $T$ is indeed inductive. So by Zorn's Lemma, $T$ must have a maximal element $Q$. It is easy to show, as we demonstrate below, that $Q$ is a prime ideal and so must contain a minimal prime $P$ containing $I$. By the condition $P$ must be a $*$-ideal of finite type and hence must be in $S$ and this contradicts $Q$ being in $T$. For the proof that $Q$ is a prime, suppose that $xy \in Q$ and that $x, y \notin Q$. Now as $Q$ is maximal in $T$, $(Q, x)^*, (Q, y)^*$ must violate the conditions defining $T$. Now as $(Q, x)^*, (Q, y)^*$ each contain $I$ the only violation would be that each of $(Q, x)^*, (Q, y)^*$ contains a member from $S$. Let $C_1, C_2 \in S$ such that $C_1 \subseteq (Q, x)^*$ and $C_2 \subseteq (Q, y)^*$. But then $(C_1C_2)^* \subseteq ((Q, x)^*(Q, y)^*)^* = (Q, x)(Q, y))^* \subseteq Q$ because $xy$ belongs to $Q$. But then $Q$ belongs to $T$ and $(C_1C_2)^* \in S$ a contradiction, establishing that $Q$ is indeed a prime ideal. $\square$

Remark 2. *Theorem 4 follows from Sahandi's work in* [**10**]. *We have kept our proof because it is direct, short and lets the reader continue without having to struggle with new terminology. As Sahandi points out, El Baghdadi and Gabelli* [**4**] *have proved Theorem 4 for* $* = t$, *in the context of Prufer v-multiplication domains.*

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