Chapter 1
On $\star$-Semi-Homogeneous Integral Domains

D. D. Anderson and Muhammad Zafrullah

Abstract Let $\star$ be a finite character star-operation defined on an integral domain D. A nonzero finitely generated ideal of D is $\star$-*homogeneous* if it is contained in a unique maximal $\star$-ideal. And D is called a $\star$-*semi-homogeneous* ($\star$-*SH*) *domain* if every proper nonzero principal ideal of D is a $\star$-product of $\star$-homogeneous ideals. Then D is a $\star$-semi-homogeneous domain if and only if the intersection $D = \bigcap_{P\in\star\text{-}\mathrm{Max}(D)} D_P$ is independent and locally finite where $\star$-$\mathrm{Max}(D)$ is the set of maximal $\star$-ideals of D. The $\star$-SH domains include h-local domains, weakly Krull domains, Krull domains, generalized Krull domains, and independent rings of Krull type. We show that by modifying the definition of a $\star$-homogeneous ideal we get a theory of each of these special cases of $\star$-SH domains.

1.1 Introduction

Many important types of integral domains have a representation of the form $D = \bigcap_{P\in\mathscr{F}} D_P$ where $\mathscr{F}$ is a set of prime ideals of D that is (1) independent, that is, two distinct elements of $\mathscr{F}$ do not contain a common nonzero prime ideal and (2) has finite character (or is locally finite), that is, each nonzero element of D is contained in at most finitely many elements of $\mathscr{F}$. These domains called $\mathscr{F}$-IFC domains were the subject of [10]. Suppose that D is an $\mathscr{F}$-IFC domain. If $\mathscr{F} = \mathrm{Max}(D)$, the set of maximal ideals of D, we get the h-local domains of Matlis [20] while if $\mathscr{F} = X^{(1)}(D)$, the set of height-one prime ideals of D, we get weakly Krull domains

D.D. Anderson
The University of Iowa, Department of Mathematics, Iowa City, Iowa, 52242-1419 e-mail: dan-anderson@uiowa.edu

Muhammad Zafrullah
Idaho State University, Department of Mathematics, Pocatello, ID 83209 e-mail: mzafrullah@usa.net

[5]. We can further put conditions on D_P for $P \in \mathscr{F}$. If each D_P is a valuation domain we get the independent rings of Krull type (IRKT) of Griffin [15], generalized Krull domains if further $\mathscr{F} = X^{(1)}(D)$, and finally Krull domains when each D_P is a DVR.

Now in [10] we began with a representation $D = \bigcap_{P \in \mathscr{F}} D_P$ and its induced star-operation $\star_{\mathscr{F}}$ given by $A^{\star_{\mathscr{F}}} = \bigcap_{P \in \mathscr{F}} AD_P$ for a nonzero fractional ideal A of D. (Needed results about star-operations are reviewed in Section 1.2.) We showed that D is an $\mathscr{F}$-IFC domain if and only if each nonzero proper principal ideal of D (or equivalently, each nonzero proper ideal A of D with $A = A^{\star_{\mathscr{F}}}$) has a representation of the form $A = (I_1 \cdots I_n)^{\star_{\mathscr{F}}}$ where each I_i is contained in a unique element of $\mathscr{F}$. In this paper we change the point of view. We begin with an integral domain D and $\star$ a finite character star-operation on D so $D = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} D_P$ where $\star$-$\mathrm{Max}(D)$ is the set of maximal $\star$-ideals of D. We define a nonzero finitely generated ideal I of D to be $\star$-homogeneous if I is contained in a unique element of $\star$-$\mathrm{Max}(D)$ and D to be a $\star$-semi-homogeneous ($\star$-SH) domain if each proper nonzero principal ideal Dx of D has a representation $Dx = (I_1 \cdots I_n)^\star$ where I_i is $\star$-homogeneous. We show (Theorem 4) that D is a $\star$-SH domain if and only if D is a $\star$-$\mathrm{Max}(D)$-IFC domain, that is, the representation $D = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} D_P$ is independent and of finite character. In this case each nonzero finitely generated ideal I with $I^\star \neq D$ has a representation $I^\star = (I_1 \cdots I_n)^\star$ where each I_i is a $\star$-homogeneous ideal (Theorem 6). We also show that for any domain D if a proper $\star$-ideal I has a representation as a $\star$-product of $\star$-homogeneous ideals, then I has a representation $I = (J_1 \cdots J_n)^\star$ where $J_1, \ldots, J_n$ are pairwise $\star$-comaximal $\star$-homogeneous ideals and that this representation is unique in the sense that if $I = (K_1 \cdots K_m)^\star$ where $K_1, \ldots, K_m$ are pairwise $\star$-comaximal $\star$-homogeneous ideals of D, then $n = m$ and after re-ordering $J_i^\star = K_i^\star$ for $i = 1, \ldots, n$.

Our approach in this paper is to add additional conditions to the definition of a $\star$-homogeneous ideal I (such as for each $\star$-homogeneous ideal $J \supseteq I$ (or perhaps just for I itself) $J^\star$ is $\star$-invertible or principal, or some $(J^n)^\star$ is principal) to get a "$\star$-β-homogeneous ideal". We then say that a $\star$-β-homogeneous ideal I has type 1 (resp., type 2) if $\sqrt{I} = M(I)$ where $M(I)$ is the unique $\star$-maximal ideal containing I (resp., $I^\star = (M(I)^n)^\star$ for some $n \geq 1$). We define D to be a "$\star$-β-SH domain" (resp., $\star$-β-SH domain of type i, $i = 1,2$) if each proper nonzero principal ideal of D is a $\star$-product of $\star$-β-homogeneous ideals (resp., $\star$-β-homogeneous ideals of type i, $i = 1,2$). For example, we call the $\star$-homogeneous ideal I $\star$-super-homogeneous if for each $\star$-homogeneous ideal $J \supseteq I$, J is $\star$-invertible. We show (Theorem 10) that D is a $\star$-super-SH domain if and only if D is an $\star$-IRKT, that is, $D = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} D_P$ is independent and of finite character and each D_P is a valuation domain. As a second example, we show (Theorem 7) that D is a $\star$-SH domain of type 1 if and only if D is a $\star$-weakly Krull domain, that is, D is weakly Krull and $\star$-$\mathrm{Max}(D) = X^{(1)}(D)$.

So here we define a class of integral domains by requiring that each proper nonzero principal ideal is a $\star$-product of a certain kind of $\star$-homogeneous ideal. As a bonus we get that if I is a finitely generated nonzero ideal with $I^\star \neq D$, then $I^\star$ is actually a $\star$-product of this kind of $\star$-homogeneous ideal. Moreover, if a proper

$\star$-ideal I is a $\star$-product of this kind of $\star$-homogeneous ideal, we can write I as a $\star$-product of pairwise $\star$-comaximal $\star$-homogeneous ideals of that kind and this representation is unique in the sense previously mentioned. Also within this class of $\star$-β-SH domains, by slightly changing the definition of a $\star$-β-homogeneous ideal, we get $\star$-β-SH domains with trivial or torsion $\star$-class group $C\ell_\star(D)$.

Of course we can also vary the star-operation. Two important star-operations are are the d-operation $A \to A_d = A$ and the t-operation $A \to A_t = \bigcup\{J_v | J \subseteq I$ is a nonzero finitely generated ideal$\}$ where $J_v = (J^{-1})^{-1}$. A d-SH domain is just an h-local domain while t-SH domains (not called that) were the subject of [7]. By varying the kind of $\star$-homogeneous ideal (and possibly adding a type) and varying the star-operation we get a whole host of various important integral domains including h-local domains, weakly Krull domains, Krull domains, Dedekind domains, generalized Krull domains, independent rings of Krull and these classes of domains that have trivial or torsion $\star$-class group.

1.2 Star-operations and $\mathscr{F}$-IFC-domains

Let D be an integral domain with quotient field K. Let $F(D)$ (resp., $f(D)$) be the set of nonzero (resp., nonzero finitely generated) fractional ideals of D. A *star-operation* $\star$ on D is a closure operation on $F(D)$ that satisfies $D^\star = D$ and $(xA)^\star = xA^\star$ for $A \in F(D)$ and $x \in K^* := K \setminus \{0\}$. With $\star$ we can associate a new star-operation $\star_s$ given by $A \to A^{\star_s} := \bigcup\{B^\star | B \subseteq A, B \in f(D)\}$ for $A \in F(D)$. We say that $\star$ has *finite character* if $\star = \star_s$. Three important star-operations are the *d-operation* $A \to A_d := A$, the *v-operation* $A \to A_v := (A^{-1})^{-1} = \bigcap\{Dx | Dx \supseteq A, x \in K^*\}$ where $A^{-1} = \{x \in K | xA \subseteq D\}$, and the *t-operation* $t := v_s$. Here d and t have finite character. A fractional ideal $A \in F(D)$ is a *$\star$-ideal* (resp., *finite type $\star$-ideal*) if $A = A^\star$ (resp., $A = A_1^\star$ for some $A_1 \in f(D)$). If $\star$ has finite character and $A^\star$ has finite type, then $A^\star = A_1^\star$ for some $A_1 \in f(D)$ with $A_1 \subseteq A$. A fractional ideal $A \in F(D)$ is *$\star$-invertible* if there exists a $B \in F(D)$ with $(AB)^\star = D$; in this case we can take $B = A^{-1}$. For any $\star$-invertible $A \in F(D)$, $A^\star = A_v$. If $\star$ has finite character and A is $\star$-invertible, then $A^\star$ is a finite type $\star$-ideal and $A^\star = A_t$. Given two fractional ideals $A, B \in F(D)$, $(AB)^\star$ is their *$\star$-product*. Note that $(AB)^\star = (A^\star B)^\star = (A^\star B^\star)^\star$. Given two star-operations $\star_1$ and $\star_2$ on D, we write $\star_1 \leq \star_2$ if $A^{\star_1} \subseteq A^{\star_2}$ for all $A \in F(D)$. So $\star_1 \leq \star_2 \Leftrightarrow A^{\star_1\star_2} = A^{\star_2} \Leftrightarrow A^{\star_2\star_1} = A^{\star_2}$ for all $A \in F(D)$. For any finite character star-operation $\star$ on D we have $d \leq \star \leq t$. For an introduction to star-operations, the reader is referred to [14, Section 32]. For a more detailed treatment see [16] and [18].

Suppose that $\star$ is a finite character star-operation on D. Then a proper $\star$-ideal is contained in a maximal $\star$-ideal and a maximal $\star$-ideal is prime. We denote the set of maximal $\star$-ideals of D by $\star\text{-}\mathrm{Max}(D)$, the set of maximal ideals of D by $\mathrm{Max}(D)$, and the set of height-one prime ideals of D by $X^{(1)}(D)$. We have $D = \bigcap_{P \in \star\text{-}\mathrm{Max}(D)} D_P$.

Let $\mathscr{F}$ be a nonempty collection of nonzero prime ideals of D. We say that $\mathscr{F}$ is a *defining family of primes for D* if $D = \bigcap_{P\in\mathscr{F}} D_P$. So for a finite character star-operation $\star$ on D, $\star$-$\mathrm{Max}(D)$ is a defining family of primes for D. We say that the intersection $D = \bigcap_{P\in\mathscr{F}} D_P$, or the set $\mathscr{F}$ of prime ideals itself, is of *finite character*, or is *locally finite*, if each nonzero element of D is in at most finitely many $P \in \mathscr{F}$. This is equivalent to each nonzero element of D (or of K) being a unit in almost all D_P, $P \in \mathscr{F}$. We will say that the finite character star-operation $\star$ is *locally finite* if $D = \bigcap_{P\in\star\text{-}\mathrm{Max}(D)} D_P$ is locally finite. The defining family of primes $\mathscr{F}$ is *independent* if for distinct $P,Q \in \mathscr{F}$, there does not exist a nonzero prime ideal m with $m \subseteq P\cap Q$. This is equivalent to $D_PD_Q = K$ [10, Lemma 4.1]. If $\mathscr{F}$ is independent, then $\mathscr{F}$ is an anti-chain. We say that a finite character star-operation $\star$ is *independent* if $\star$-$\mathrm{Max}(D)$ is independent. Note that if two prime $\star$-ideals contain a nonzero prime ideal, they actually contain a (nonzero) prime $\star$-ideal. Indeed, if P is a nonzero prime ideal and $0 \neq x \in P$, we can shrink P to a prime ideal P' minimal over Dx, and P' is a prime $\star$-ideal. For a finite character star-operation $\star$ on D, we call D a *$\star$-h-local domain* if $\star$ is independent and locally finite, that is, each proper principal ideal is contained in only finitely many maximal $\star$-ideals and each prime $\star$-ideal is contained in a unique maximal $\star$-ideal. For the case of $\star = d$, we just get the h-local domains of Matlis [20]. We say that D is a *$\mathscr{F}$-IFC domain* if $\mathscr{F}$ is an independent, finite character defining family of prime ideals for D. Thus for a finite character star-operation $\star$ on D, D being a $\star$-h-local domain is the same thing as D being a $\mathscr{F}$-IFC domain for $\mathscr{F} = \star$-$\mathrm{Max}(D)$.

Suppose that $\mathscr{F}$ is a defining family of primes for D. Then the operation $A \longrightarrow A^{\star_{\mathscr{F}}} := \bigcap_{P\in\mathscr{F}} AD_P$ is a star-operation on D which has finite character if $\mathscr{F}$ is locally finite [2, Theorem 1]. (However, $\star_{\mathscr{F}}$ may have finite character without $\mathscr{F}$ being locally finite.) Moreover, $A^{\star_{\mathscr{F}}}D_P = AD_P$ for $A \in F(D)$ and $P \in \mathscr{F}$. Thus if D is a $\mathscr{F}$-IFC domain, $\star_{\mathscr{F}}$ has finite character and $\star_{\mathscr{F}}$-$\mathrm{Max}(D) = \mathscr{F}$. In the case where $\star$ is a finite character star-operation on D and $\mathscr{F} = \star$-$\mathrm{Max}(D)$, $\star_{\mathscr{F}} = \star_w$ where $\star_w$ is the star-operation defined by $A \to A^{\star_w} := \{x \in K | xJ \subseteq A$ for some $J \in f(D)$ with $J^\star = D\} = \bigcap_{P\in\star\text{-}\mathrm{Max}(D)} AD_P$ for $A \in F(D)$. Here $\star_w$ has finite character, $\star_w \leq \star$, and $(A\cap B)^{\star_w} = A^{\star_w} \cap B^{\star_w}$ for $A,B \in F(D)$. Also, $\star$-$\mathrm{Max}(D) = \star_w$-$\mathrm{Max}(D)$ and hence $A \in F(D)$ is $\star$-invertible if and only if it is $\star_w$-invertible. Moreover, for a $\star$-invertible (or $\star_w$-invertible) ideal $A \in F(D)$, $A^\star = A^{\star_w} = A_t = A_v$. For results on the $\star_w$-operation see [4].

We have the following result relating $\star$ and $\star_w$.

Theorem 1. *Let $\star_1$ and $\star_2$ be two finite character star-operations on an integral domain D. Then the following conditions are equivalent.*

1. $\star_{1w} = \star_{2w}$.
2. $\star_1$-$\mathrm{Max}(D) = \star_2$-$\mathrm{Max}(D)$.
3. $A^{\star_1} = D \Leftrightarrow A^{\star_2} = D$ *for* $A \in F(D)$.
4. $A^{\star_1} = D \Leftrightarrow A^{\star_2} = D$ *for* $A \in f(D)$.

5. $P^{\star_{1w}} = P^{\star_{2w}}$ *for each nonzero prime ideal* P *of* D.

Proof. $(1) \Rightarrow (2)$ $\star_1$-$\mathrm{Max}(D) = \star_{1w}$-$\mathrm{Max}(D) = \star_{2w}$-$\mathrm{Max}(D) = \star_2$-$\mathrm{Max}(D)$. $(2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (1) \Rightarrow (5)$ Clear. $(5) \Rightarrow (2)$ We have $\star_{1w}$-$\mathrm{Max}(D) = \star_{2w}$-$\mathrm{Max}(D)$ and hence as in $(1) \Rightarrow (2)$ we have $\star_1$-$\mathrm{Max}(D) = \star_2$-$\mathrm{Max}(D)$.

We next briefly review some of the material from [10] concerning $\mathscr{F}$-IFC domains. So let D be an integral domain and $\mathscr{F}$ a defining family of primes for D. For an ideal A of D let $m(A) = \{P \in \mathscr{F} | A \subseteq P\}$ and call A *unidirectional* if $|m(A)| = 1$. Suppose that A is unidirectional. If P is the unique element of $\mathscr{F}$ containing A, we say that A is *unidirectional pointing to* P. The following theorem sums up some of the results from [10].

Theorem 2. *Let* $\mathscr{F}$ *be a defining family of prime ideals for the integral domain* D *and let* $\star_{\mathscr{F}}$ *be the star-operation given by* $A^{\star_{\mathscr{F}}} = \bigcap_{P \in \mathscr{F}} AD_P$ *for* $A \in F(D)$.

1. *If* A *is unidirectional pointing to* $P \in \mathscr{F}$, *then* $A^{\star_{\mathscr{F}}} = AD_P \cap D$. *Conversely, suppose that* $\mathscr{F}$ *is independent. Let* $P \in \mathscr{F}$. *Then for a nonzero ideal* $A \subseteq P$, $AD_P \cap D$ *is unidirectional pointing to* P.
2. *Two nonzero ideals* A *and* B *of* D *are* $\star_{\mathscr{F}}$*-comaximal* (*i.e.,* $(A+B)^{\star_{\mathscr{F}}} = D$) *if and only if* $m(A) \cap m(B) = \emptyset$.
3. *If a* $\star_{\mathscr{F}}$*-ideal* A *of* D *is expressible as a finite* $\star_{\mathscr{F}}$*-product of unidirectional ideals, then* A *is uniquely expressible* (*up to order*) *as a* $\star_{\mathscr{F}}$*-product of pairwise* $\star_{\mathscr{F}}$*-comaximal unidirectional* $\star_{\mathscr{F}}$*-ideals.*
4. *The following conditions are equivalent.*

   a. $\mathscr{F}$ *is an independent defining family of finite character, i.e.,* D *is a* $\mathscr{F}$*-IFC domain.*
   b. *Every proper integral* $\star_{\mathscr{F}}$*-ideal of* D *is* (*uniquely*) *expressible as a finite* $\star_{\mathscr{F}}$*-product of* (*pairwise* $\star_{\mathscr{F}}$*-comaximal*) *unidirectional* ($\star_{\mathscr{F}}$*-*) *ideals.*
   c. *Every proper integral principal ideal of* D *is* (*uniquely*) *expressible as a finite* $\star_{\mathscr{F}}$*-product of* (*pairwise* $\star_{\mathscr{F}}$*-comaximal*) *unidirectional* ($\star_{\mathscr{F}}$*-*) *ideals.*
   d. *Every nonzero prime ideal of* D *contains a nonzero element* x *such that* Dx *is* (*uniquely*) *expressible as a finite* $\star_{\mathscr{F}}$*-product of* (*pairwise* $\star_{\mathscr{F}}$*-comaximal*) *unidirectional* ($\star_{\mathscr{F}}$*-*) *ideals.*

Proof. (1) [10, Lemma 2.3], (2) Clear, (3) [10, Lemma 2.6], (4) Combine [10, Proposition 2.7] and [10, Theorem 2.1].

1.3 $\star$-homogeneous Ideals

For $\mathscr{F}$-IFC domains we considered $\star_{\mathscr{F}}$-product representations of $\star_{\mathscr{F}}$-ideals. In this paper we change our point of view. We begin with a finite character star-operation $\star$ on the integral domain D and consider $\star$-product representations of $\star$-ideals. We make the following fundamental definition.

Definition 1. Let $\star$ be finite character star-operation on the integral domain D. An ideal I of D is *$\star$-homogeneous* if I is a nonzero finitely generated ideal and I is contained in a unique maximal $\star$-ideal.

Suppose that I is a $\star$-homogeneous ideal of D. If P is the unique maximal $\star$-ideal containing I we say that I is *P-$\star$-homogeneous*. We will often denote the unique maximal $\star$-ideal containing I by $M(I)$. We say that two $\star$-homogeneous ideals I and J are *similar*, denoted $I \sim J$, if $M(I) = M(J)$.

Suppose that $\star$ is a finite character star-operation on the integral domain D. So $D = \bigcap_{P \in \star\text{-}\operatorname{Max}(D)} D_P$, that is, $\star$-$\operatorname{Max}(D)$ is a defining family of primes for D and hence for $\mathscr{F} = \star$-$\operatorname{Max}(D)$, the star-operation $\star_{\mathscr{F}}$ given by $A \longrightarrow A^{\star_{\mathscr{F}}} = \bigcap_{P \in \mathscr{F}} AD_P$ is just the $\star_w$-operation. So $\star_{\mathscr{F}} = \star_w$ is a finite character star-operation on D and $\star_w \leq \star$, that is, $A^{\star_w} \subseteq A^\star$ for all $A \in F(D)$. Note that I is P-$\star$-homogeneous if and only if I is P-$\star_w$-homogeneous if and only if I is a finitely generated unidirectional ideal pointing to P.

The next two propositions give some results concerning $\star$-homogeneous ideals.

Proposition 1. *Let D be an integral domain, I a nonzero finitely generated ideal of D, and $\star$ a finite character star-operation on D.*

1. *Suppose that $I^\star \neq D$. Then I is $\star$-homogeneous if and only if for (finitely generated) ideals J and K of D with $J, K \supseteq I$ and $J^\star, K^\star \neq D$, we have $(J+K)^\star \neq D$.*
2. *For I $\star$-homogeneous, $M(I) = \{x \in D | (I,x)^\star \neq D\}$.*
3. *If I is $\star$-homogeneous, $I^\star D_{M(I)} \cap D = I^\star$.*
4. *If I is $\star$-homogeneous and $A_1, \ldots, A_n$ are pairwise $\star$-comaximal ideals of D with $A_1 \cdots A_n \subseteq I^\star$, then some $A_i \subseteq I^\star$.*

Proof. 1. First note that since $\star$ has finite character, if there are ideals $J, K \supseteq I$ with $J^\star, K^\star \neq D$, but $(J+K)^\star = D$, then there are finitely generated ideals J and K with this property. ($\Rightarrow$) Suppose that I is $\star$-homogeneous. If $J, K \supseteq I$ with $J^\star, K^\star \neq D$, then necessarily $J, K \subseteq M(I)$, so $(J+K)^\star \neq D$. ($\Leftarrow$) Let M_1 and M_2 be maximal $\star$-ideals containing I. Then $(M_1 + M_2)^\star \neq D$, so $M_1 = M_2$. Hence I is $\star$-homogeneous.

2. Here $M(I)$ is the unique maximal $\star$-ideal containing I. If $x \in M(I)$, then $(I,x) \subseteq M(I)$ and hence $(I,x)^\star \neq D$. Conversely, if $(I,x)^\star \neq D$, then (I,x) is contained in a maximal $\star$-ideal P that also contains I, so $P = M(I)$. Hence $x \in (I,x) \subseteq M(I)$.
3. Clearly $I^\star D_{M(I)} \cap D \supseteq I^\star$. Let $x \in I^\star D_{M(I)} \cap D$, so $x = i/s$ where $i \in I^\star$ and $s \notin M(I)$. So $xs \in I^\star$. Now $s \notin M(I)$ implies $(I,s)^\star = D$, so $Dx = (Ix, sx)^\star \subseteq I^\star$.
4. By induction it suffices to do the case $n = 2$. So suppose that A and B are $\star$-comaximal ideals of D with $AB \subseteq I^\star$. We cannot have both $A, B \subseteq M(I)$, so say $B \nsubseteq M(I)$. Then $A \subseteq AD_{M(I)} \cap D = ABD_{M(I)} \cap D \subseteq I^\star D_{M(I)} \cap D = I^\star$.

Proposition 2. *Let $\star$ be a finite character star-operation on the integral domain D. For $\star$-homogeneous ideals I and J of D, the following are equivalent.*

1. *$I \sim J$.*

2. $(I+J)^\star \neq D$.
3. IJ *is* $\star$*-homogeneous.*

If (1)*,* (2)*, or* (3) *holds, then* $IJ \sim I \sim J$*. Thus if* $I_1,\dots,I_n$ *are* $\star$*-homogeneous ideals of* D *with* $I_1,\dots,I_n$ *all similar, then* $I_1\cdots I_n$ *is* $\star$*-homogeneous and* $I_1\cdots I_n \sim I_1 \sim \cdots \sim I_n$*.*

Proof. (1)$\Rightarrow$(2) $I,J \subseteq M(I) = M(J) \Rightarrow I+J \subseteq M(J)$ and hence $(I+J)^\star \neq D$. (2)$\Rightarrow$(1) Now $(I+J)^\star \neq D$ implies $I+J$ is contained in a maximal $\star$-ideal P. But since $I,J \subseteq P$ we must have $M(I) = P$ and $M(J) = P$, so $M(I) = M(J)$. (1)$\Rightarrow$(3) IJ is finitely generated and $(IJ)^\star \neq D$. Let P be a maximal $\star$-ideal containing IJ. Since P is prime, we have, say $I \subseteq P$. So $P = M(I)$. So IJ is $\star$-homogeneous with $M(IJ) = M(I)$. (3)$\Rightarrow$(1) Suppose that $I \not\sim J$, so $M(I)$ and $M(J)$ are two distinct maximal $\star$-ideals containing IJ, a contradiction.

The last statement is now immediate.

We next give a uniqueness result for $\star$-products of $\star$-homogeneous ideals. Compare with Theorem 2(3) ([10, Lemma 2.6]).

Theorem 3. *Let* D *be an integral domain and* $\star$ *a finite character star-operation on* D*. Let* I *be an ideal of* D*. If* I *is a* $\star$*-product of* $\star$*-homogeneous ideals of* D*, then* I *is uniquely expressible* (*up to order*) *as a* $\star$*-product of pairwise* $\star$*-comaximal* $\star$*-ideals* $(J_1^\star\cdots J_s^\star)^\star$ *where each* J_i *is* $\star$*-homogeneous.*

Proof. Suppose $I = (I_1\cdots I_n)^\star$ where I_i is $\star$-homogeneous. Let $M(I_{i_1}),\dots,M(I_{i_s})$ be the distinct maximal $\star$-ideals among $M(I_1),\dots,M(I_n)$. For $1 \leq \ell \leq s$, put $J_\ell{:=}\prod\{I_j | I_j \sim I_{i_\ell}\}$. So $J_1,\dots,J_s$ are $\star$-homogeneous ideals of D that are pairwise $\star$-comaximal and $I = (J_1\cdots J_s)^\star = (J_1^\star\cdots J_s^\star)^\star$. Suppose that we have another representation $I = (K_1\cdots K_t)^\star = (K_1^\star\cdots K_t^\star)^\star$ where $K_1,\dots,K_t$ are pairwise $\star$-comaximal $\star$-homogeneous ideals of D. Now $K_1\cdots K_t \subseteq (J_1\cdots J_s)^\star \subseteq J_1^\star$, so by Proposition 1, some $K_i \subseteq J_1^\star$. Reordering, we can take $i = 1$, so $K_1 \subseteq J_1^\star$. Reversing the roles of the J_i's and K_i's, we have some $J_i \subseteq K_1^\star \subseteq J_1^\star$. By $\star$-comaximality, $i = 1$, so $J_1 \subseteq K_1^\star$ and hence $J_1^\star = K_1^\star$. Continuing we see that each J_i matches up to a K_j with $J_i^\star = K_j^\star$. Likewise each K_i matches up to a J_j with $K_i^\star = J_j^\star$. Thus $s = t$ and after re-ordering $J_i^\star = K_i^\star$ for $i = 1,\dots,s$.

We next define $\star$-SH domains. We will see that a $\star$-SH domain is the same thing as a $\star$-*h*-local domain.

Definition 2. Let D be an integral domain and $\star$ a finite character star-operation on D. Then D is a $\star$*-semi-homogeneous* ($\star$*-SH*) *domain* if every proper nonzero principal ideal of D is a finite $\star$-product of $\star$-homogeneous ideals of D.

So by Theorem 3, D is a $\star$-SH domain if and only if each proper nonzero principal ideal Dx of D has a unique representation (up to order) as a finite $\star$-product of pairwise $\star$-comaximal $\star$-ideals $Dx = (J_1^\star\cdots J_s^\star)^\star$ $(= (J_1\cdots J_s)^\star)$ where J_i is $\star$-homogeneous. We next use our results from [10] to get some characterizations of $\star$-SH domains.

Theorem 4. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent.*

1. *D is a $\star$-SH domain.*
2. *D is a $\star$-$\mathrm{Max}(D)$-IFC domain, that is, D is a $\star$-h-local domain.*
3. *D is a $\star_w$-SH domain.*

Proof. (1)$\Leftrightarrow$(3) Since $\star$-$\mathrm{Max}(D) = \star_w$-$\mathrm{Max}(D)$, an ideal is $\star$-homogeneous if and only if it is $\star_w$-homogeneous. Let x be a nonzero nonunit of D. Now in a representation $Dx = (I_1 \cdots I_n)^\star$ (resp., $Dx = (J_1 \cdots J_m)^{\star_w}$) where each I_i (resp., J_i) is $\star$-homogeneous (resp., $\star_w$-homogeneous), $I_1 \cdots I_n$ (resp., $J_1 \cdots J_m$) is $\star$-invertible (resp., $\star_w$-invertible). But an ideal I is $\star$-invertible if and only if it is $\star_w$-invertible and in this case $I^\star = I_t = I^{\star_w}$. Thus $Dx = (I_1 \cdots I_n)^{\star_w}$ (resp., $(J_1 \cdots J_m)^\star$). So Dx is a $\star$-product of $\star$-homogeneous ideals if and only if it is a $\star_w$-product of $\star_w$-homogeneous ideals. (2)$\Leftrightarrow$(3) Let $\mathscr{F} = \star$-$\mathrm{Max}(D)$, so $\star_{\mathscr{F}} = \star_w$. By [10, Proposition 2.7], D is a $\mathscr{F}$-IFC domain if and only if for each nonzero nonunit $x \in D$, Dx is a $\star_{\mathscr{F}} = \star_w$-product of unidirectional ideals. Now a $\star_w$-homogeneous ideal is unidirectional. And if $Dx = (I_1 \cdots I_n)^{\star_w}$ where each I_i is unidirectional, then I_i is $\star_w$-invertible and hence $I_i^{\star_w} = (I_i')^{\star_w}$ for some finitely generated ideal $I_i' \subseteq I_i$. So I_i' is $\star_w$-homogeneous and $Dx = (I_1' \cdots I_n')^{\star_w}$.

Theorem 5. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent.*

1. *D is a $\star$-SH domain.*
2. *$\star$ is locally finite and independent.*
3. *Every nonzero prime ideal of D contains a nonzero element x such that Dx is a $\star$-product of $\star$-homogeneous ideals.*
4. *Every nonzero prime ideal of D contains a $\star$-invertible $\star$-homogeneous ideal of D.*
5. *For $P \in \star$-$\mathrm{Max}(D)$ and $0 \neq x \in P$, $xD_P \cap D = I^\star$ for some $\star$-invertible P-$\star$-homogeneous ideal I.*
6. *$\star$ is independent and if A is a nonzero ideal of D with AD_P finitely generated for each $P \in \star$-$\mathrm{Max}(D)$, then $A^\star$ is a finite type $\star$-ideal.*

Proof. (1)$\Leftrightarrow$(2) Theorem 4.

Note that for each i, $2 \leq i \leq 5$, (i) is equivalent to (i') where (i') is (i) with $\star$ replaced by $\star_w$. By [10, Theorem 3.3], $(2')$-$(5')$ are equivalent and hence (2)-(5) are equivalent.

(2)$\Rightarrow$(6) Now by hypothesis, $\star$ is independent and by [10, Theorem 3.3] $A^{\star_w}$ is a finite type $\star_w$-ideal. Hence $A^\star$ is a finite type $\star$-ideal. (6)$\Rightarrow$(5) Let $P \in \star$-$\mathrm{Max}(D)$ and $0 \neq x \in P$. Put $A{:=}xD_P \cap D$. Let $Q \in \star$-$\mathrm{Max}(D) \backslash \{P\}$. Since $\star$ is independent, $D_P D_Q = K$, the quotient field of D. Thus $AD_Q = (xD_P \cap D)D_Q = xD_P D_Q \cap D_Q = xK \cap D_Q = D_Q$. So P is the only maximal $\star$-ideal containing A. Since AD_M is finitely generated for each $M \in \star$-$\mathrm{Max}(D)$, $A^\star = A_1^\star$ for some finitely generated ideal A_1 of D. Moreover, since $\star$ has finite character we can take $A_1 \subseteq A$. Since P is the only maximal $\star$-ideal containing A, the same is true for A_1 and $A_2{:=}(A_1,x)$. So A_2 is P-$\star$-homogeneous. Also, $AD_Q = D_Q = A_2D_Q$ for $Q \in \star\text{-}\operatorname{Max}(D)\setminus\{P\}$ and $AD_P = xD_P \subseteq A_2D_P$, so $AD_P = A_2D_P$. Hence $A = AD_P \cap D = \bigcap_{Q\in\star\text{-}\operatorname{Max}(D)} AD_Q = \bigcap_{Q\in\star\text{-}\operatorname{Max}(D)} A_2D_Q = A_2^{\star_w}$. As in the proof of (5)⇒(4) of [10, Theorem 3.3], A_2 is $\star_w$-invertible. Thus A_2 is $\star$-invertible and so $A = A_2^{\star_w} = A_2^{\star}$.

We next note that in a $\star$-SH domain every proper finite type $\star$-ideal is a $\star$-product of $\star$-homogeneous ideals.

Theorem 6. *Let D be a $\star$-SH domain and I a nonzero finitely generated ideal of D with $I^\star \neq D$. Then $I^\star$ is uniquely expressible (up to order) as a $\star$-product $(J_1^\star \cdots J_n^\star)^\star$ of pairwise $\star$-comaximal $\star$-ideals $J_1^\star,\dots,J_n^\star$ where each J_i is $\star$-homogeneous.*

Proof. Since D is a $\star$-SH domain, $\star$ is locally finite by Theorem 5. Let $M_1,\dots,M_n$ be the maximal $\star$-ideals contained I and put $I_i{:=}ID_{M_i}\cap D$. So $I^{\star_w} = I_1\cap\cdots\cap I_n$ and hence $I^\star = (I_1\cap\cdots\cap I_n)^\star$. Since $\star$ is independent (Theorem 5) Theorem 2 gives that M_i is the unique maximal $\star$-ideal containing I_i. So $I_1,\dots,I_n$ are pairwise $\star$-comaximal and thus $(I_1\cap\cdots\cap I_n)^\star = (I_1\cdots I_n)^\star$. By Theorem 5, $I_i^\star$ has $\star$-finite type, so $I_i^\star = J_i^\star$ where J_i is $\star$-homogeneous. Now $J_1,\dots,J_n$ are pairwise $\star$-comaximal $\star$-homogeneous ideals with $I^\star = (J_1^\star\cdots J_n^\star)^\star$. Uniqueness follows from Theorem 3.

In [5] an integral domain D was defined to be *weakly Krull* if $D = \bigcap_{P\in X^{(1)}(D)} D_P$ and the intersection is locally finite. Thus D is weakly Krull if D is a $\mathscr{F}$-IFC domain for $\mathscr{F} = X^{(1)}(D)$. We generalize this definition as follows.

Definition 3. Let D be an integral domain and $\star$ a finite character star-operation on D. Then D is a $\star$-*weakly Krull domain* ($\star$-*WKD*) if D is a $\star$-h-local domain for which $X^{(1)}(D) = \star\text{-}\operatorname{Max}(D)$.

Thus D is a $\star$-WKD if and only if D is weakly Krull and $X^{(1)}(D) = \star\text{-}\operatorname{Max}(D)$. Note that for D weakly Krull, $t\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. Thus a weakly Krull domain is the same thing as a t-*WKD*. At the other extreme, D is a d-WKD if and only if $\dim D = 1$ and each nonzero element of D is in at most finitely many maximal ideals. If $\star_1$ and $\star_2$ be two finite character star-operations on D with $\star_1 \leq \star_2$, then D a $\star_1$-WKD implies that D is a $\star_2$-WKD. Evidently D is a $\star$-WKD if and only if it is a $\star_w$-WKD.

To give our characterization of $\star$-weakly Krull domains we need the following definition.

Definition 4. Let D be an integral domain and $\star$ a finite character star-operation on D. Then a $\star$-homogeneous ideal I of D has *type 1* if $M(I) = \sqrt{I^\star}$. And D is a *type 1 $\star$-SH domain* if each nonzero proper principal ideal of D is a $\star$-product of type 1 $\star$-homogeneous ideals.

It is easy to see that a $\star$-homogeneous ideal I has type 1 if and only if for each $\star$-homogenous ideal $A \supseteq I$, there exists an $n \geq 1$ with $A^n \subseteq I^\star$.

Theorem 7. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent.*

1. *D is a $\star$-weakly Krull domain.*
2. *D is a $\star$-h-local domain and each $\star$-homogeneous ideal has type 1.*
3. *Every proper principal ideal of D is a $\star$-product of type 1 $\star$-homogeneous ideals, that is, D is a type 1 $\star$-SH domain.*
4. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of type 1 $\star$-homogeneous ideals.*

Proof. (1)⇒(2) By definition a $\star$-weakly Krull domain is $\star$-h-local. Let I be a $\star$-homogeneous ideal of D. Since $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$, $M(I)$ is a minimal prime over $I^\star$ and as any prime ideal minimal over $I^\star$ is a $\star$-ideal, $M(I)$ is the unique prime ideal minimal over $I^\star$. Hence $M(I) = \sqrt{I^\star}$, so I has type 1.

(2)⇒(3) Clear since in a $\star$-h-local domain every proper principal ideal is a $\star$-product of $\star$-homogeneous ideals (Theorem 4).

(3)⇒(1) Certainly (3) gives that D is a $\star$-SH domain and hence $\star$-h-local (Theorem 4). We show $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. Let M be a maximal $\star$-ideal. Suppose that there exists a nonzero prime ideal $Q \subsetneq M$. Let $0 \neq x \in Q$. Shrinking Q to a prime ideal minimal over Dx we can assume that Q is a $\star$-ideal. Now $Dx = (I_1 \cdots I_n)^\star$ where each I_i is a type 1 $\star$-homogeneous ideal. Now $I_1 \cdots I_n \subseteq Q$, so some $I_i \subseteq Q$ and hence $I_i^\star \subseteq Q$. But $M(I_i) = \sqrt{I_i^\star} \subseteq Q \subsetneq M$, a contradiction. Thus $\star\text{-}\operatorname{Max}(D) \subseteq X^{(1)}(D)$ and hence we have equality since each height-one prime ideal is a $\star$-ideal.

(4)⇒(3) Clear. (2)⇒(4) This follows from Theorem 6 since a $\star$-h-local domain is a $\star$-SH domain.

Invoking Theorem 3 we see that in a $\star$-weakly Krull domain a nonzero finitely generated ideal I with $I^\star \neq D$ has a unique representation (up to order) $I^\star = (J_1^\star \cdots J_n^\star)^\star$ where $J_1, \ldots, J_n$ are pairwise $\star$-comaximal type 1 $\star$-homogeneous ideals.

Now a Krull domain is a weakly Krull domain (or equivalently, a t-WKD) in which D_P is a DVR for each $P \in X^{(1)}(D)$. With this in mind we make the following definition.

Definition 5. Let D be an integral domain and $\star$ a finite character star-operation on D. Then D is a $\star$-*Krull domain* if D is a $\star$-weakly Krull domain and D_P is a DVR for each $P \in \star\text{-}\operatorname{Max}(D)$.

Evidently D is a $\star$-Krull domain if and only if D is a Krull domain and $\star\text{-}\operatorname{Max}(D) = X^{(1)}(D)$. Thus a Krull domain is the same thing as a t-Krull domain. At the other extreme, a d-Krull domain is a Dedekind domain. If $\star_1$ and $\star_2$ are finite character star-operations on D with $\star_1 \leq \star_2$, then D $\star_1$-Krull implies that D is $\star_2$-Krull.

Our characterization of $\star$-Krull domains requires the following definition.

Definition 6. Let D be an integral domain and $\star$ a finite character star-operation on D. A $\star$-homogeneous ideal I of D has *type 2* if $I^\star = (M(I)^n)^\star$ for some $n \geq 1$. And D is a *type 2 $\star$-SH domain* if each nonzero proper principal ideal of D is a $\star$-product of type 2 $\star$-homogeneous ideals.

Theorem 8. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-Krull domain.*
2. *Every proper $\star$-ideal of D is a $\star$-product of prime $\star$-ideals of D.*
3. *Every proper principal ideal of D is a $\star$-product of prime $\star$-ideals of D.*
4. *Every proper $\star$-ideal of D is a $\star$-product of type 2 $\star$-homogeneous ideals of D.*
5. *Every proper principal ideal of D is a $\star$-product of type 2 $\star$-homogeneous ideals of D, that is, D is a type 2 $\star$-SH domain.*

Proof. (1)⇒(4) D is $\star$-Krull, so D is a Krull domain and $\star\text{-Max}(D) = X^{(1)}(D)$. For $A \in F(D)$, $A^{\star_w} = \bigcap_{P \in X^{(1)}(D)} AD_P = A_t$, so $A^{\star_w} = A^\star = A_t$. Let $P \in X^{(1)}(D)$. Choose $x \in P \backslash P^2$. Let $Q_1, \dots, Q_n$ be the other height-one primes containing x and choose $y \in P \backslash (Q_1 \cup \cdots \cup Q_n)$. So $(x,y)^\star = (x,y)^{\star_w} = \bigcap_{Q \in X^{(1)}(D)} (x,y)D_Q = P$. Put $H(P){:=}(x,y)$, so $H(P)$ is a type 2 $\star$-homogeneous ideal. Let A be a proper $\star$-ideal of D. Then $A = \bigcap_{P \in X^{(1)}(D)} AD_P = P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)}$ where $P_1, \dots, P_s$ are the height-one primes containing A and $P_i^{(n_i)} = P_i^{n_i} D_{P_i} \cap D$. But $P_1^{(n_1)} \cap \cdots \cap P_s^{(n_s)} = (P_1^{n_1} \cdots P_s^{n_s})_t = (P_1^{n_1} \cdots P_s^{n_s})^\star = ((H(P_1)^\star)^{n_1} \cdots (H(P_s)^\star)^{n_s})^\star = (H(P_1)^{n_1} \cdots H(P_s)^{n_s})^\star$.

(4)⇒(2)⇒(3), (4)⇒(5)⇒(3) Clear.

(3)⇒(1) Let x be a nonzero nonunit of D. So $Dx = (P_1 \cdots P_n)^\star$ where P_i is a prime $\star$-ideal of D. Then P_i is $\star$-invertible so $P_i = H(P_i)^\star$ where $H(P_i)$ is a finitely generated ideal contained in P_i. Thus $H(P_i)$ is a type 2 $\star$-homogeneous ideal and hence a type 1 $\star$-homogeneous ideal. So each proper principal ideal of D is a $\star$-product of type 1 $\star$-homogeneous ideals. By Theorem 7, D is a $\star$-WKD. Let $P \in X^{(1)}(D)$; we need to show that D_P is a DVR. Let $0 \neq x \in P$, so $Dx = (P_1 \cdots P_n)^\star$ where P_i is a prime $\star$-ideal which is $\star$-invertible. Now some $P_i \subseteq P$ and hence $P_i = P$, so P is $\star$-invertible. Thus $(PP^{-1}) \not\subseteq P$, so $PP^{-1}D_P = D_P$ and hence PD_P is invertible and therefore principal. Since $\operatorname{ht} P = 1$, D_P is a DVR.

Once again we can invoke Theorem 3 to get the appropriate uniqueness result for pairwise $\star$-comaximal type 2 $\star$-homogeneous ideals in Theorem 8. We leave it to the reader to show that in a $\star$-Krull domain if $(P_1 \cdots P_n)^\star = (Q_1 \cdots Q_m)^\star$ where the P_i's and Q_i's are maximal $\star$-ideals, then $n = m$ and after reordering $P_i = Q_i$ for each i.

The notion of a Krull domain can be generalized in a number of ways. We have already defined $\star$-Krull domains and $\star$-weakly Krull domains. An integral domain D is an *independent ring of Krull type* (*IRKT*) [15] if D is a $\mathscr{F}$-IFC domain for some defining family $\mathscr{F}$ of primes where D_P is a valuation domain for each $P \in \mathscr{F}$. For a finite character star-operation $\star$ on P, we call D a *$\star$-independent ring of Krull type* (*$\star$-IRKT*) if D is a $\mathscr{F}$-IFC domain for $\mathscr{F} = \star\text{-Max}(D)$, that is, D is $\star$-*h*-local, and for each $P \in \star\text{-Max}(D)$, D_P is a valuation domain. Thus D is a $\star$-IRKT if and only if D is an IRKT where $\mathscr{F} = \star\text{-Max}(D)$. A d-IRKT is just a finite character, independent Prüfer domain. At the other extreme, a t-IRKT is just an IRKT. If $\star_1$ and $\star_2$ are finite character star-operations on D with $\star_1 \leq \star_2$ and D is a $\star_1$-IRKT, then D is a $\star_2$-IRKT, see Proposition 3 below. Recall that D is a *$P\star MD$* if each nonzero finitely generated ideal of D is $\star$-invertible, or equivalently, D_M is a valuation domain for each $M \in \star\text{-}\mathrm{Max}(D)$. Thus a $\star$-IRKT is a $P\star MD$. In fact, D is a $\star$-IRKT if and only if D is a $\star$-h-local P$\star$MD. A *PvMD* is usually defined to be a v-domain (each nonzero finitely generated ideal of D is v-invertible) in which A^{-1} is a finite type v-ideal for each nonzero finitely generated ideal A of D. Thus a *PvMD* is just a *PtMD* and a $P\star MD$ is a *PvMD*. Of course a *PdMD* is just a Prüfer domain.

The integral domain D is a *generalized Krull domain* (*GKD*) if $D = \bigcap_{P \in X^{(1)}(D)} D_P$ is locally finite and for each $P \in X^{(1)}(D)$, D_P is a valuation domain, that is, D is weakly Krull and for each $P \in X^{(1)}(D)$, D_P is a valuation domain. Let $\star$ be a finite character star-operation on D. We call D a *$\star$-generalized Krull domain* (*$\star$-GKD*) if $D = \bigcap_{P \in X^{(1)}(D)} D_P$ locally finite, $\star\text{-}\mathrm{Max}(D) = X^{(1)}(D)$, and D_P is a valuation domain for each $P \in X^{(1)}(D)$, or equivalently, D is $\star$-weakly Krull and for each $P \in X^{(1)}(D)$, D_P is a valuation domain, that is, D is a $\star$-GKD if and only if D is a GKD and $\star\text{-}\mathrm{Max}(D) = X^{(1)}(D)$. So D is a d-GKD if and only if D is a one-dimensional finite character Prüfer domain. At the other extreme, a t-GKD is just a GKD. If $\star_1$ and $\star_2$ are two finite character star-operations on D with $\star_1 \leq \star_2$, then D a $\star_1$-GKD implies that D is a $\star_2$-GKD.

Proposition 3. *Let D be an integral domain and $\star_1$ and $\star_2$ be finite character star-operations on D with $\star_1 \leq \star_2$. If D is a $\star_1$-IRKT, then D is a $\star_2$-IRKT.*

Proof. Let $P \in \star_2\text{-}\mathrm{Max}(D)$. Then $P^{\star_1} \subseteq P^{\star_2} = P$, so $P^{\star_1} \neq D$ and hence P is contained in a maximal $\star_1$-ideal Q. Moreover, Q is unique since $\star_1$ is independent. Also, D_Q is a valuation domain and hence so is $D_P = (D_Q)_{P_Q}$. Note that $\star_2$ is independent. Suppose that m is a nonzero prime ideal with $m \subseteq M_1, M_2$, two maximal $\star_2$-ideals. Then M_i is contained in a maximal $\star_1$-ideal M_i'. Since $m \subseteq M_1' \cap M_2'$, $M_1' = M_2'$ as $\star_1$ is independent. But then $M_1, M_2 \subseteq M_1'$ and $D_{M_1'}$ is a valuation domain. So M_1 and M_2 are comparable. Here $M_1 = M_2$. So $\star_2$ is independent. We next show that $\star_2$ is locally finite. Suppose some $0 \neq x \in D$ is contained in an infinite number of maximal $\star_2$-ideals $\{Q_n\}_{n=1}^{\infty}$. Now each Q_n is contained in a maximal $\star_1$-ideal P_n. Now if $P_n = P_m$, then Q_n and Q_m are comparable since D_{P_n} is a valuation domain, so $Q_n = Q_m$. Thus x is contained in infinitely many maximal $\star_1$-ideals, a contradiction.

The following diagram gives the various implications between the different generalizations of Krull domains.

$$
\begin{array}{ccccc}
\text{Dedekind} & \Longrightarrow & \star\text{-Krull} & \Longrightarrow & \text{Krull} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{one-dimensional} & \Longrightarrow & \star\text{-GKD} & \Longrightarrow & \text{GKD} \\
\text{finite character Prüfer} & & & \searrow\!\!\searrow \qquad \swarrow\!\!\swarrow & \\
\Downarrow & & \Downarrow & \star\text{-WKD} \Rightarrow \text{weakly Krull} & \Downarrow \\
\text{finite character} & & & & \\
\text{independent Prüfer} & \Longrightarrow & \star\text{-IRKT} & \Longrightarrow & \text{IRKT} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{Prüfer} & \Longrightarrow & P\star MD & \Longrightarrow & PvMD
\end{array}
$$

To characterize $\star$-IRKTs using $\star$-homogeneous ideals we need the following definition.

Definition 7. Let D be an integral domain and $\star$ a finite character star-operation on D. A $\star$-homogeneous ideal I of D is $\star$-*super-homogeneous* if each $\star$-homogeneous ideal containing I is $\star$-invertible. The $\star$-super-homogeneous ideal I has *type 1* (resp., *type 2*) if I has type 1 as a $\star$-homogeneous ideal, that is, $\sqrt{I^\star} = M(I)$ (resp., $I^\star = (M(I)^n)^\star$ for some $n \geq 1$). The domain D is a $\star$-*super-SH domain* (resp., *type 1* $\star$-*super-SH domain*, *type 2* $\star$-*super-SH domain*) if every nonzero proper principal ideal of D is a $\star$-product of $\star$-super-homogeneous ideals (resp., of type 1, of type 2).

Note that if I is $\star$-super-homogeneous, then each finitely generated ideal containing I is $\star$-invertible. Now by [17, Theorem 1.11] a product of similar $\star$-super-homogeneous ideals is again $\star$-super-homogeneous. Thus the proof of Theorem 3 gives the corresponding uniqueness result for $\star$-products of $\star$-super-homogeneous ideals.

Theorem 9. *Let $\star$ be a finite character star-operation on the integral domain D and let $J_1, \ldots, J_n$ be a set of $\star$-super-homogeneous ideals of D. Then the $\star$-product $(J_1 \cdots J_n)^\star$ can be expressed uniquely, up to order, as a $\star$-product of pairwise $\star$-comaximal $\star$-super-homogeneous ideals.*

We next give several characterizations of $\star$-IRKTs.

Theorem 10. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-IRKT.*
2. *D is $\star$-h-local and every $\star$-homogeneous ideal is $\star$-invertible.*
3. *D is $\star$-h-local and every $\star$-homogeneous ideal is $\star$-super-homogeneous.*
4. *Every proper nonzero principal ideal is a $\star$-product of $\star$-super-homogeneous ideals, that is, D is a $\star$-super-SH domain.*
5. *If I is a nonzero finitely generated ideal with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-super-homogeneous ideals.*

Proof. (1)⇒(2),(3) Let I be a $\star$-homogeneous ideal of D and let $J \supseteq I$ be a finitely generated ideal of D. Then JD_P is principal for each $P \in \star\text{-Max}(D)$ since D_P is a valuation domain. Thus J is $\star$-invertible. (2)⇒(1) Let $P \in \star\text{-}\operatorname{Max}(D)$. We need to show that D_P is a valuation domain. It suffices to show that for $x, y \in P \setminus \{0\}$, $(x,y)D_P$ is principal. Let $A = (x,y)D_P \cap D$. By Theorem 5, $A^\star$ is a finite type $\star$-ideal. So $A^\star = A_1^\star$ where $A_1 \subseteq A$ is finitely generated. Now P is the unique maximal $\star$-ideal containing A and hence the unique maximal $\star$-ideal containing A_1. So by hypothesis A_1, and hence A, is $\star$-invertible. So $(x,y)D_P = AD_P$ is principal. (3)⇒(4) This is immediate since for a $\star$-h-local domain each proper nonzero principal ideal is a $\star$-product of $\star$-homogeneous ideal by Theorem 4. (4)⇒(1) Every proper nonzero principal ideal of D is a $\star$-product of $\star$-homogeneous ideals, so by Theorem 4, D is $\star$-h-local. Let $P \in \star\text{-}\operatorname{Max}(D)$. We need that D_P is a valuation domain. Let $0 \neq x \in P$, so $Dx = (I_1 \cdots I_n)^\star$ where I_i is $\star$-super-homogeneous. Let $I = \prod\{I_i | I_i \text{ is } P\ \star\text{-homogeneous}\}$. Then $xD_P \cap D = I^\star$. By [17, Theorem 1.11], I is $\star$-super-homogeneous. Let $0 \neq y \in P$. Then again $yD_P \cap D = J^\star$ for some $\star$-super-homogeneous ideal J of D. But by [17, Theorem 1.11] for two P-$\star$-super-homogeneous ideals I and J of D, $I^\star$ and $J^\star$ are comparable. Thus $xD_P \cap D$ and $yD_P \cap D$ are comparable, so D_P is a valuation domain. (5)⇒(4) Clear. (1)⇒(5) Let I be a nonzero finitely generated ideal of D with $I^\star \neq D$. By (1)⇒(3) it is enough to show $I^\star$ is a $\star$-product of $\star$-homogeneous ideals. But this follows from Theorem 6.

Using Theorems 9 and 10 we get the following result.

Proposition 4. *Let D be an integral domain and $\star$ a finite character star-operation on D. Suppose that D is a $\star$-IRKT. Let $a, b \in D^*$ with $(a,b)^\star \neq D$. Then $(a,b)^\star = (I_1 \cdots I_n)^\star$ where $I_1, \ldots, I_n$ are pairwise $\star$-comaximal $\star$-super-homogeneous ideals of D containing (a,b) such that $(a,b)D_{M(I_i)} = I_i D_{M(I_i)} = aD_{M(I_i)}$ or $bD_{M(I_i)}$.*

Proof. Now by Theorems 9 and 10 $(a,b)^\star = (I_1 \cdots I_n)^\star$ where $I_1, \ldots, I_n$ are pairwise $\star$-comaximal $\star$-super-homogeneous ideals of D. Put $I_i' = I_i + (a,b)$. Then $M(I_i') = M(I_i)$, each I_i' is a $\star$-super-homogeneous ideal, and $I_i' \supseteq (a,b)$. Now $I_1 \cdots I_n \subseteq I_1' \cdots I_n' = (I_1 + (a,b)) \cdots (I_n + (a,b)) \subseteq I_1 \cdots I_n + (a,b)$, so $(I_1' \cdots I_n')^\star = (I_1 \cdots I_n)^\star$. Thus we can replace I_i by I_i' and hence assume that $(a,b) \subseteq I_i$. Since (a,b) and $I_1 \cdots I_n$ are $\star$-invertible we have $(a,b)^{\star_w} = (a,b)^\star = (I_1 \cdots I_n)^\star = (I_1 \cdots I_n)^{\star_w}$. So $(a,b)D_{M(I_i)} = (a,b)^{\star_w} D_{M(I_i)} = (I_1 \cdots I_n)^{\star_w} D_{M(I_i)} = I_1 \cdots I_n D_{M(I_i)} = I_i D_{M(I_i)}$. Now $D_{M(I_i)}$ is a valuation domain, so either $(a,b)D_{M(I_i)} = aD_{M(I_i)}$ or $(a,b)D_{M(I_i)} = bD_{M(I_i)}$.

Using Theorem 10 we get several characterizations of $\star$-GKDs.

Theorem 11. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent.*

1. *D is a $\star$-GKD.*
2. *D is a $\star$-IRKT and a $\star$-WKD.*
3. *D is a $\star$-IRKT and every $\star$-super-homogeneous ideal has type 1.*
4. *D is a $\star$-WKD and every $\star$-homogeneous ideal is $\star$-invertible.*
5. *D is $\star$-h-local and every $\star$-homogeneous ideal is $\star$-super-homogeneous and has type 1.*

6. *Every proper nonzero principal ideal of D is a $\star$-product of $\star$-super-homogeneous ideals of type 1, that is, D is a type 1 $\star$-super-SH domain.*
7. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of type 1 $\star$-super-homogeneous ideals.*

Proof. (1)⇔(2) Clear. (2)⇔(3) First note that by Theorem 10, for a $\star$-IRKT the notions of $\star$-homogeneous and $\star$-super-homogeneous coincide. Then use Theorem 7. (2)⇔(4) Theorem 10. (4)⇔(5)⇔(6) Combine Theorems 7 and 10. (7)⇒(6) Clear. (5)⇒(7) Theorem 6.

Once again we can invoke Theorem 3 to get the appropriate uniqueness result for pairwise $\star$-comaximal type 1 $\star$-super-homogeneous ideals in Theorem 10.

By Theorem 8 D is a $\star$-Krull domain if and only if D is a type 2 $\star$-SH domain. Now in a $\star$-Krull domain a nonzero finitely generated ideal I is $\star$-homogeneous if and only if $I^\star = P^{(n)}$ for some $P \in X^1(D)$ and $n \geq 1$. Hence I is $\star$-homogeneous if and only if it is a type 2 $\star$-super homogeneous ideal. Thus a type 2 $\star$-super-SH domain is the same thing as a $\star$-Krull domain and if I is a nonzero finitely generated ideal of D with $I^\star \neq D$, $I^\star$ is a $\star$-product of type 2 $\star$-super-homogeneous ideals.

Let $\star$ be a finite character star-operation on the integral domain D. We define D to be $\star$-*Bezout* if for $a,b \in D^*$, $(a,b)^\star$ is principal. It easily follows that D is $\star$-Bezout if and only if $A^\star$ is principal for each nonzero finitely generated (fractional) ideal A of D. If $\star_1$ and $\star_2$ are finite character star-operations on D, then $D\,\star_1$-Bezout implies that D is $\star_2$-Bezout. A d-Bezout domain is just a Bezout domain while a t-Bezout domain is a GCD domain. We also define D to be a $\star$-*Prüfer domain* if for $a,b \in D^*$, $(a,b)^\star$ is invertible. Using [19, Exercise 22, page 43], it is easy to see that D is $\star$-Prüfer if and only if $A^\star$ is invertible for each nonzero finitely generated (fractional) ideal A of D. Again if $\star_1 \leq \star_2$ are finite character star-operations on D, then $D\ \star_1$-Prüfer implies that D is $\star_2$-Prüfer. A d-Prüfer domain is just a Prüfer domain while a t-Prüfer domain is a generalized GCD domain (GGCD domain). GGCD domains were introduced in [1] and studied in more detail in [3]. We have $\star$-Bezout ⇒ $\star$-Prüfer ⇒ P$\star$MD.

Storch [21] defined a Krull domain D to be *almost factorial* if for $a,b \in D^*$ there exists an $n = n(a,b) \geq 1$ with $a^nD \cap b^nD$ principal. The second author initiated a general theory of almost factoriality in [22]. There he defined an integral domain D to be an *almost GCD domain* (*AGCD domain*) if for $a,b \in D^*$, there exists an $n = n(a,b) \geq 1$ with $a^nD \cap b^nD$ principal, or equivalently, $(a^n,b^n)_v$ $(= (a^n,b^n)_t)$ principal. This investigation was continued in [9]. In that paper an integral domain D was defined to be an *almost Bezout domain* (*AB domain*) (resp., *almost Prüfer domain* (*AP domain*)) if for $a,b \in D^*$, there exists an $n = n(a,b) \geq 1$ with (a^n,b^n) principal (resp., invertible). It was shown that D is almost Bezout (resp., almost Prüfer) if and only if for $a_1,\ldots,a_s \in D^*$; there exists an $n = n(a_1,\ldots,a_s) \geq 1$ with $(a_1^n,\ldots,a_s^n)$ principal (resp., invertible). Briefly mentioned in [9] was the notion of an *almost generalized GCD domain* (*AGGCD domain*). Here D is a AGGCD domain if for $a,b \in D^*$ there exists an $n = n(a,b) \geq 1$ with $a^nD \cap b^nD$ invertible, or equivalently, $(a^n,b^n)_v$ $(= (a^n,b^n)_t)$ is invertible.

With the definitions in the previous two paragraphs in mind, we make the following definitions. Let D be an integral domain and $\star$ a finite character star-operation on D. We say the D is a *$\star$-almost Bezout domain* (resp., *$\star$-almost Prüfer domain, almost P$\star$MD*) if for $a,b \in D^*$, there exists an $n = n(a,b) \geq 1$ with $(a^n, b^n)^\star$ principal (resp., invertible, $\star$-invertible). (More generally, we could call D a *$\star_2$-almost P$\star_1$MD* if $(a^n,b^n)^{\star_2}$ is $\star_1$-invertible.) If $\star_1 \leq \star_2$ are finite character star-operations on D, then D $\star_1$-almost Bezout (resp., $\star_1$-almost Prüfer, almost P$\star_1$MD) implies D is $\star_2$-almost Bezout (resp., $\star_2$-almost Prüfer, almost P$\star_2$MD). A d-almost Bezout domain (resp., d-almost Prüfer domain) is just an almost Bezout domain (resp., almost Prüfer domain), while a t-almost Bezout domain (resp., t-almost Prüfer domain) is just an AGCD domain (resp., AGGCD domain).

We mention two useful results from [9]. First, let $\star$ be a finite character star-operation on D. Let $\{a_\alpha\} \subseteq D^*$ and $n \geq 1$. If $(\{a_\alpha\})$ is $\star$-invertible, then $(\{a_\alpha^n\})^\star = ((\{a_\alpha\})^n)^\star$. In particular, $(\{a_\alpha^n\})$ is also $\star$-invertible. This is stated for the case $\star = t$ in [9, Lemma 3.3]. The proof carries over mutatis mutandis for a general finite character star-operation $\star$. Next, for an integral domain D, the following conditions are equivalent [9, Theorem 6.8]: (1) D is n-root closed (i.e., for $x \in K$ with $x^n \in D, x \in D$), (2) for $\{a_\alpha\} \subseteq D^*$, $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$, (3) for $\{a_\alpha\} \subseteq D^*$, $(\{a_\alpha^n\})_v = ((\{a_\alpha\})^n)_v$, and (4) for $a,b \in D^*$, $(a^n,b^n)_t = ((a,b)^n)_t$. Thus if D is integrally closed, $(\{a_\alpha^n\})_t = ((\{a_\alpha\})^n)_t$ for all $\{a_\alpha\} \subseteq D^*$ and $n \geq 1$.

Using the first mentioned result of the previous paragraph, the proof of [9, Lemma 4.3] can easily be modified to show that for an integral domain D and finite character star-operation $\star$ on D, if D is $\star$-almost Bezout (resp., $\star$-almost Prüfer, almost P$\star$MD) and $a_1,\dots,a_s \in D^*$, then there exits an $n = n(a_1,\dots,a_s) \geq 1$ with $(a_1^n,\dots,a_s^n)^\star$ principal (resp., invertible, $\star$-invertible). Thus for D integrally closed, D is $\star$-almost Bezout (resp., $\star$-almost Prüfer, almost P$\star$MD) if and only if for A a nonzero finitely generated (fractional) ideal of D, there exists an $n = n(A) \geq 1$ with $(A^n)^\star$ principal (resp., invertible, $\star$-invertible). The implication ($\Leftarrow$) does not require that D be integrally closed. Indeed, if $(A^n)^\star$ is $\star$-invertible, A is $\star$-invertible and hence for $A = (a,b)$, $(a^n,b^n)^\star = ((a,b)^n)^\star$. Conversely, suppose that D is integrally closed and let $A = (a_1,\dots,a_s)$. Then for some $n \geq 1$, $(a_1^n,\dots,a_s^n)$ is $\star$-invertible and hence $(a_1^n,\dots,a_s^n)^\star = (a_1^n,\dots,a_s^n)_t$. Thus $(A^n)_t \supseteq (a_1^n,\dots,a_s^n)^\star = (a_1^n,\dots,a_s^n)_t = (A^n)_t$.

Let $\star$ be a finite character star-operation on D. The set $\star$-Inv(D) of $\star$-invertible fractional $\star$-ideals forms a group under the $\star$-product $I \star J{:=}(IJ)^\star$ with subgroup Princ(D), the set of nonzero principal fractional ideals of D. The quotient group $C\ell_\star(D){:=}\star\text{-Inv}(D)/\text{Princ}(D)$ is called the *$\star$-class group of D*, see [11]. For $\star = d$, we have the usual class group $C(D)$, while for $\star = t$, we have the t-class group introduced by Bouvier [12] and further studied in [13]. For a Krull domain, $C\ell_t(D)$ is just the usual divisor class group. Suppose that $\star_1 \leq \star_2$ are finite character star-operations on D. Then we have natural inclusions $C(D) \subseteq C\ell_{\star_1}(D) \subseteq C\ell_{\star_2}(D) \subseteq C\ell_t(D)$. Let Inv$(D)$ be the subgroup of $\star$-Inv(D) consisting of invertible ideals of D. The group $LC\ell_\star(D){:=}\star\text{-Inv}(D)/\text{Inv}(D)$ is called the local *$\star$-class group of D*.

Proposition 5. *Suppose that D is a $\star$-IRKT. Then the following conditions are equivalent.*

1. *D is $\star$-almost Bezout (resp., $\star$-almost Prüfer).*
2. *$C\ell_\star(D)$ is torsion (resp., $LC\ell_\star(D)$ is torsion).*
3. *For each $\star$-super-homogeneous ideal A of D, there exists a natural number $n = n(A)$ with $(A^n)^\star$ principal (resp., invertible).*
4. *D is an AGCD (resp., AGGCD domain).*
5. *$C\ell_t(D)$ is torsion (resp., $LC\ell_\star(D)$ is torsion).*

Proof. We do the $\star$-almost Bezout case, the $\star$-almost Prüfer case is similar. Now D being a $\star$-IRKT is integrally closed. Hence D is $\star$-almost Bezout if and only if for each nonzero finitely generated ideal A of D, $(A^n)^\star$ is principal for some $n \geq 1$. Also, each nonzero finitely generated ideal of D is $\star$-invertible. So (1)⇒(2)⇒(3). (3)⇒(1) Let A be a nonzero finitely generated ideal of D. If $A^\star = D$, we can take $n = n(A) = 1$. So suppose that $A^\star \neq D$. Then by Theorem 10, $A^\star = (I_1 \cdots I_m)^\star$ where each I_i is $\star$-super-homogeneous. By hypothesis, there exists an n_i with $(I_i^{n_i})^\star$ is principal. Then for $n = n_1 \cdots n_m$, $(A^n)^\star = ((I_1^{n_1})^{n/n_1} \cdots (I_m^{n_m})^{n/n_m})^\star$ is principal. (1)⇒(4) Here D is $\star$-almost Bezout. Since $\star \leq t$, D is t-almost Bezout, that is, an AGCD domain. (4)⇔(5) This follows since D is integrally closed. (5)⇒(2) Here $C\ell_\star(D) \subseteq C\ell_t(D)$ so $C\ell_t(D)$ torsion gives that $C\ell_\star(D)$ is torsion.

Definition 8. Let D be an integral domain and $\star$ a finite character star-operation on D. A $\star$-homogeneous ideal I of D is a *$\star$-almost factorial-homogeneous ideal* (*$\star$-af-homogeneous ideal*) (resp., *$\star$-locally almost factorial-homogeneous ideal* (*$\star$-laf-homogeneous ideal*)) if for each $\star$-homogeneous ideal $J \supseteq I$, there exists an $n = n(J) \geq 1$ with $(J^n)^\star$ principal (resp., invertible). The integral domain D is a *$\star$-af-SH domain* (resp., *$\star$-laf-SH domain*) if for each nonzero nonunit $x \in D$, Dx is expressible as a $\star$-product of finitely many $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals).

Thus a $\star$-homogeneous ideal I is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous) if and only if for each finitely generated (or equivalently, each finite type $\star$-ideal) $J \supseteq I$, some $(J^n)^\star$ is principal (resp., invertible). Note that a $\star$-af-homogeneous ideal (resp., $\star$-laf-homogeneous ideal) is actually $\star$-super-homogeneous. In the spirit of Theorems 3 and 9 we have the following uniqueness result for $\star$-products of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals).

Theorem 12. *Let D by an integral domain and $\star$ a finite character star-operation on D. Let I be an ideal of D. If I is a $\star$-product of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals) of D, then I is uniquely expressible (up to order) as a $\star$-product of pairwise $\star$-comaximal $\star$-ideals $(J_1^\star \cdots J_s^\star)^\star$ where each J_i is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous).*

Proof. We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. The uniqueness of the product $(J_1^\star \cdots J_s^\star)^\star$ follows from Theorem 3. To show the existence of the product, the proof of Theorem 3 shows that it suffices to prove that the product IJ of two similar $\star$-af-homogeneous ideals I and J is again $\star$-af-homogeneous. Of course IJ is $\star$-homogeneous. Let $C \supseteq IJ$ be $\star$-homogeneous ideal of D. Then $E{:=}C+I$ is $\star$-homogeneous. So there exists a $n \geq 1$ with $(E^n)^\star$ principal.

Thus E is $\star$-invertible. So $(CE^{-1}+IE^{-1})^\star = D$ where $C \subseteq CE^{-1} \subseteq D$ and $I \subseteq IE^{-1} \subseteq D$. Thus $(CE^{-1})^\star = D$ or $(IE^{-1})^\star = D$. In the first case, $C^\star = E^\star$ and hence $(C^n)^\star = (E^n)^\star$ is principal. So we can assume that $(IE^{-1})^\star = D$. Then $I^\star = E^\star \supseteq C \supseteq IJ$ so $D \supseteq (CI^{-1})^\star \supseteq J^\star$. Choose a finitely generated ideal $L \supseteq J$ with $(CI^{-1})^\star = L^\star$. So there exists an $m \geq 1$ with $(L^m)^\star$ principal. So $((CI^{-1})^m)^\star$ is principal. Choose n with $(I^n)^\star$ principal. Then $(C^{mn})^\star = (((CI^{-1})^m)^n(I^n)^m)^\star$ is principal.

We next give a characterization of AGCD $\star$-IRKTs (resp., AGGCD $\star$-IRKTs) using $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals). Of course we could enlarge the list of equivalences via Proposition 5.

Theorem 13. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-af-SH domain (resp., $\star$-laf-SH-domain).*
2. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals).*
3. *D is an AGCD $\star$-IRKT (resp., AGGCD $\star$-IRKT).*
4. *D is an $\star$-SH domain and every $\star$-homogeneous ideal is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous).*
5. *D is a $\star$-IRKT with $C\ell_\star(D)$ torsion (resp., $LC\ell_\star(D)$ torsion) (equivalently, $C\ell_t(D)$ torsion (resp., $LC\ell_t(D)$ torsion)).*
6. *D is $\star$-h-local and for each $\star$-homogeneous ideal I of D there exists an $n \geq 1$ with $(I^n)^\star$ principal (resp., invertible).*

Proof. We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. (3)⇒(2) By Theorem 10 $I^\star$ is a $\star$-product of $\star$-super-homogeneous ideals. By Proposition 5 $C\ell_\star(D)$ is torsion. Hence each $\star$-super-homogeneous ideal is a $\star$-af-homogeneous ideal. So $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals. (2)⇒(1) Clear. (1)⇒(3) Since a $\star$-af-homogeneous ideal is $\star$-super-homogeneous, D is an $\star$-IRKT by Theorem 10. It remains to show that D is an AGCD domain. Let a be a nonzero nonunit of D. So $Da = (I_1 \cdots I_n)^\star$ where I_i is $\star$-af-homogeneous (and hence $\star$-super-homogeneous). By Theorem 12 we can take $I_1,\dots,I_n$ to be pairwise $\star$-comaximal. Now for each i, $i = 1,\dots,n$, there exists an $n_i \geq 1$ with $(I_i^{n_i})^\star$ principal. Hence for a suitable $m \geq 1$ $Da^m = Da_1 \cdots Da_n$ where Da_i is $\star$-super-homogeneous and $Da_1,\dots,Da_n$ are pairwise $\star$-comaximal. Thus $Da_1 \cdots Da_n = Da_1 \cap \cdots \cap Da_n$. Let a,b be nonzero nonunits of D. By the previous remarks, there is an $m \geq 1$ with $Da^m = Da_1 \cdots Da_n = Da_1 \cap \cdots \cap Da_n$ and $Db^m = Db_1 \cdots Db_n = Db_1 \cap \cdots \cap Db_n$ where either Da_i and Db_i are similar $\star$-super-homogeneous ideals of D or exactly one of Da_i, Db_i is a $\star$-super-homogeneous ideal and the other is D, and $Da_1,\dots,Da_n$ (resp., $Db_1,\dots,Db_n$) are pairwise $\star$-comaximal. Now if Da_i and Db_i are both $\star$-super-homogeneous ideals, being similar, they are comparable [17, Theorem 1.11]. Thus in either case $Da_i \cap Db_i$ is a principal $\star$-super-homogeneous ideal. Thus $Da^m \cap Db^m = (Da_1 \cap Db_1) \cap \cdots \cap (Da_n \cap Db_n) = (Da_1 \cap Db_1) \cdots (Da_n \cap Db_n)$ is principal. So D is an AGCD. (4)⇒(1) Clear. (2)⇒(4) Let I be a $\star$-homogeneous ideal of D. Then $I^\star = (I_1 \cdots I_n)^\star$ where I_n is $\star$-af-homogeneous. Of course $I_1,\dots,I_n$ must be similar. By the proof of Theorem 12 a product of similar $\star$-af-homogeneous ideals is again $\star$-af-homogeneous. Thus $I_1 \cdots I_n$ and hence I is $\star$-af-homogeneous. (3)⇔(5) Proposition 5. (6)⇔(3) Combine Theorem 10 and Proposition 5.

Recall that we defined a $\star$-homogeneous ideal I to be of type 1 (resp., type 2) if $M(I) = \sqrt{I^\star}$ (resp., $I^\star = (M(I)^n)^\star$ for some $n \geq 1$). Thus by a *$\star$-af-homogeneous ideal of type 1* (resp., *type 2*), we mean a $\star$-af-homogeneous ideal that is type 1 (resp., type 2) as a $\star$-homogeneous ideal. And by a *$\star$-af-SH domain* of *type 1* (resp., *type 2*) we mean an integral domain in which each proper nonzero principal ideal is a $\star$-product of $\star$-af-homogeneous ideals of type 1 (resp., type 2). Of course we have the analogous definitions for $\star$-laf-homogeneous ideals. The next two theorems characterize these domains. Again we can invoke Theorem 3 to get the appropriate uniqueness results.

Theorem 14. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent.*

1. *D is a $\star$-af-SH domain of type 1 (resp., $\star$-laf-SH domain of type 1).*
2. *D is an AGCD $\star$-GKD (resp., AGGCD $\star$-GKD).*
3. *D is a $\star$-SH domain and each $\star$-homogeneous ideal is a $\star$-af-homogeneous ideal (resp., $\star$-laf-homogeneous ideal) of type 1.*
4. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals) of type 1.*
5. *D is a $\star$-GKD with $C\ell_\star(D)$ torsion (resp., $LC\ell_\star(D)$ torsion) or equivalently $C\ell_t(D)$ torsion (resp., $LC\ell_t(D)$ torsion).*

Proof. We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. (1)⇒(2) By Theorem 11 D is a $\star$-GKD since a $\star$-af-homogeneous ideal is $\star$-super-homogeneous. And by Theorem 13 D is an AGCD domain. (2)⇒(1) By Theorem 11 every nonzero proper principal ideal of D is a $\star$-product of $\star$-super-homogeneous ideals of type 1. Now a $\star$-GKD is a $\star$-IRKT and hence by Theorem 13 each $\star$-super-homogeneous ideal is $\star$-af-homogeneous. (3)⇒(1) Clear. (1)⇒(3) This follows from Theorem 13 once we observe that a product of similar type 1 $\star$-af-homogeneous ideals is again a $\star$-af-homogeneous ideal of type 1. (4)⇒(1) Clear. (3)⇒(4) Theorem 6 (2)⇔(5) Proposition 5.

Theorem 15. *Let D by an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-af-SH domain (resp., $\star$-laf-homogeneous-SH domain) of type 2.*
2. *D is an AGCD $\star$-Krull domain (resp., AGGCD $\star$-Krull domain).*
3. *D is a $\star$-SH domain and each $\star$-homogeneous ideal is a $\star$-af-homogeneous ideal (resp., $\star$-laf-homogeneous ideal) of type 2.*
4. *If I is a nonzero finitely generated ideal D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals (resp., $\star$-laf-homogeneous ideals) of type 2.*
5. *D is a $\star$-Krull domain with $C\ell_\star(D)$ torsion or equivalently $C\ell(D)$ torsion (resp., $LC\ell_\star(D)$ torsion or equivalently $LC\ell(D)$ torsion).*

Proof. We do the $\star$-af-homogeneous case, the $\star$-laf-homogeneous case is similar. (1)$\Rightarrow$(2) By Theorem 8 D is $\star$-Krull. And since a $\star$-af-SH domain of type 2 is certainly a $\star$-af-SH domain of type 1, Theorem 14 gives that D is an AGCD domain. (2)$\Rightarrow$(1) By Theorem 8 each proper nonzero principal ideal of D is a $\star$-product of $\star$-homogeneous ideals of type 2. Now a $\star$-Krull domain is certainly a $\star$-GKD, so by Theorem 14 each $\star$-homogeneous ideal is actually $\star$-af-homogeneous. So each proper nonzero principal ideal of D is a $\star$-product of $\star$-af-homogeneous ideals of type 2. (3)$\Rightarrow$(1) Clear. (1)$\Rightarrow$(3) This follows from Theorem 13 once we observe that a product of similar type 2 $\star$-af-homogeneous ideals is again a $\star$-af-homogeneous ideal of type 2. (4)$\Rightarrow$(1) Clear. (3)$\Rightarrow$(4) Theorem 6. (2)$\Leftrightarrow$(5) Proposition 5.

To give GCD domain and GGCD domain versions of Theorems 13–15 we need the following definitions.

Definition 9. Let D be an integral domain and $\star$ a finite character star-operation on D. An ideal I of D is *$\star$-factorial ($\star$-f)-homogeneous* (resp., *$\star$-locally factorial ($\star$-lf)-homogeneous*) if I if $\star$-homogeneous and for each $\star$-homogeneous ideal $J \supseteq I, J^\star$ is principal (resp., invertible). We say the D is a *$\star$-f-SH domain* (resp., *$\star$-lf-SH domain*) if each nonzero proper principal ideal of D is a $\star$-product of $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals).

Let D be an integral domain and $\star$ a finite character star-operation on D. Let I be a nonzero ideal of D. Then we have I $\star$-f-homogeneous (resp., $\star$-lf-homogeneous) $\Rightarrow$ I is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous) $\Rightarrow$ I is $\star$-super-homogeneous $\Rightarrow$ I is $\star$-homogeneous. Thus D a $\star$-f-SH domain $\Rightarrow$ D is a $\star$-af-SH domain $\Rightarrow$ D is a $\star$-super-SH domain $\Rightarrow$ D is a SH domain with similar implications for the "locally" case. Also, I $\star$-f-homogeneous (resp., $\star$-af-homogeneous) $\Rightarrow$ I is $\star$-lf-homogeneous (resp., $\star$-laf-homogeneous). So D a $\star$-f-SH domain (resp., $\star$-af-SH domain) $\Rightarrow$ D is a $\star$-lf-SH domain (resp., $\star$-laf-SH domain). We have also shown that a product of similar $\star$-af-homogeneous (resp., $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous) ideals is again $\star$-af-homogeneous (resp., $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous). Using this we showed that if an ideal I of D is a $\star$-product of $\star$-af-homogeneous (resp, $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous) ideals, then I is uniquely expressible (up to order) as a $\star$-product of pairwise $\star$-comaximal $\star$-ideals $(J_1^\star \cdots J_s^\star)^\star$ where each J_i is $\star$-af-homogeneous (resp., $\star$-laf-homogeneous, $\star$-super-homogeneous, $\star$-homogeneous). Not surprisingly we have an analogous result for $\star$-f-homogeneous ideals and $\star$-lf-homogeneous ideals.

Theorem 16. *Let D be an integral domain and $\star$ a finite character star-operation on D.*

1. *If I and J are similar $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals) of D, then IJ is $\star$-f-homogeneous (resp., $\star$-lf-homogeneous).*
2. *Let I be an ideal of D that is a $\star$-product of $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals). Then $I^\star$ is uniquely expressible (up to order) as a $\star$-product of pairwise $\star$-comaximal $\star$-ideals $(J_1^\star \cdots J_s^\star)^\star$ where each J_i is $\star$-f-homogeneous (resp., $\star$-lf-homogeneous).*

Proof. We do the $\star$-f-homogeneous case, the $\star$-lf-homogeneous case is similar. Once we prove (1), the proof of (2) is similar to the proofs of the $\star$-af-homogeneous, $\star$-super-homogeneous and $\star$-homogeneous cases (Theorem 12, 9, and 3, respectively). So let I and J be similar $\star$-f-homogeneous ideals. Let $C \supseteq IJ$ be a $\star$-homogeneous ideal. We need to show that $C^\star$ is principal. Since I and J are $\star$-super-homogeneous, so is their product IJ. Thus $I^\star, J^\star$, and $C^\star$ are comparable [17, Theorem 1.11]. If $C^\star \supseteq I^\star$, then $C+I \supseteq I$ is $\star$-homogeneous and hence $C^\star = (C+I)^\star$ is principal. Likewise $C^\star$ is principal when $C^\star \supseteq J^\star$. Thus without loss of generality we may assume that $I^\star \supseteq J^\star \supsetneq C^\star \supseteq C \supseteq IJ$. Now $D \supseteq I^\star I^{-1} \supseteq C^\star I^{-1} \supseteq J^\star$ where $I^{-1} = (I^\star)^{-1}$ is principal. So $CI^{-1} + J \supseteq J$ is $\star$-homogeneous and hence $(CI^{-1}+J)^\star$ is principal. But $(CI^{-1}+J)^\star = (CI^{-1})^\star = C^\star I^{-1}$ and hence $C^\star$ is principal since I^{-1} is.

We next give a characterization of GCD (resp., GGCD) $\star$-IRKTs using $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals).

Theorem 17. *Let D be an integral domain and $\star$ a finite character star-operation on D. The the following conditions are equivalent.*

1. *D is a $\star$-f-SH domain (resp., $\star$-lf-SH domain).*
2. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-f-homogeneous ideals (resp., $\star$-lf-homogeneous ideals).*
3. *D is a GCD (resp., GGCD) $\star$-IRKT.*
4. *D is a $\star$-Bezout (resp., $\star$-Prüfer) $\star$-IRKT.*
5. *D is a $\star$-SH domain and every $\star$-homogeneous ideal of D is $\star$-f-homogeneous (resp., $\star$-lf-homogeneous).*
6. *D is a $\star$-IRKT with $C\ell_\star(D) = 0$, or equivalently, $C\ell_t(D) = 0$ (resp., $LC\ell_\star(D) = 0$, or equivalently, $LC\ell_t(D) = 0$).*

Proof. We do the $\star$-f-homogeneous case, the $\star$-lf-homogeneous case is similar. (5)⇒(4) Since a $\star$-f-homogeneous ideal is $\star$-af-homogeneous, Theorem 13 gives that D is an AGCD $\star$-IRKT. Let I be a nonzero finitely generated ideal of D with $I^\star \neq D$. By Theorem 13 $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals each of which is $\star$-f-homogeneous by hypothesis and hence principal. Thus for each nonzero finitely generated ideal I of D, $I^\star$ is principal. So D is $\star$-Bezout. (4)⇒(3) A $\star$-Bezout domain is a GCD domain. (3)⇒(2) Let I be a nonzero finitely generated ideal of D with $I^\star \neq D$. Since D is an AGCD $\star$-IRKT, $I^\star$ is a $\star$-product of $\star$-af-homogeneous ideals. But since D is a GCD domain, $C\ell_t(D) = 0$; so $C\ell_\star(D) \subseteq C\ell_t(D)$ gives each $\star$-invertible ideal is principal. Thus a $\star$-af-homogeneous ideal is $\star$-f-homogeneous. (2)⇒(1) Clear. (1)⇒(3) In the proof of (1)⇒(3) of Theorem 13 we can take $m = 1$ and get that $Da \cap Db$ is principal. Thus D is a GCD domain. (3)⇒(4) D a GCD domain gives $C\ell_t(D) = 0$ and hence $C\ell_\star(D) = 0$. So D is $\star$-Bezout. (4)⇒(5) A $\star$-IRKT is a $\star$-SH domain. Let I be a $\star$-homogeneous ideal. If $J \supseteq I$ is $\star$-homogeneous, then $J^\star$ is principal since D is $\star$-Bezout. Thus I is $\star$-f-homogeneous. (3)⇒(6) This follows since $C\ell_t(D) = 0$ for D a GCD domain. (6)⇒(4) Suppose that $C\ell_t(D) = 0$. Let I be a nonzero finitely generated ideal of D. By Theorem 10 I is $\star$-invertible. Since $C\ell_\star(D) = 0$, $I^\star$ is principal. So D is $\star$-Bezout.

Combining Theorem 17 with previous results we have the following two theorems.

Theorem 18. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following are equivalent.*

1. *D is a $\star$-f-SH domain of type 1 (resp., type 2).*
2. *D is a GCD $\star$-GKD (resp., GCD $\star$-Krull domain, or equivalently a UFD $\star$-Krull domain, or UFD $\star$-GKD).*
3. *D is a $\star$-GKD (resp., $\star$-Krull domain) with $C\ell_\star(D) = 0$, or equivalently, $C\ell_t(D) = 0$.*

Proof. For the type 1 (resp., type 2) equivalences just combine Theorem 17 and Theorem 11 (resp., Theorem 8).

Recall that an integral domain D is *locally factorial* if D_M is a UFD for each maximal ideal M of D. And D is called a *π-domain* if each proper nonzero principal ideal of D is a product of (necessarily invertible) prime ideals. For an integral domain D the following are equivalent: (1) D is a π-domain, (2) D is a locally factorial Krull domain, and (3) D is a Krull domain with $LC\ell(D) = 0$ [1, Theorem 1].

Theorem 19. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-lf-SH domain of type 1 (resp., type 2).*
2. *D is a GGCD $\star$-GKD (resp., GGCD $\star$-Krull domain, or equivalently a locally factorial $\star$-Krull domain, or locally factorial $\star$-GKD).*
3. *D is a $\star$-GKD (resp., $\star$-Krull domain) with $LC\ell_\star(D) = 0$, or equivalently, $LC\ell_t(D) = 0$.*

Proof. For the type 1 (resp., type 2) equivalence just combine Theorem 17 and Theorem 11 (resp., Theorem 8).

We next wish to characterize $\star$-SH domains with $C\ell_\star(D) = 0$ or $C\ell_\star(D)$ torsion (resp., $LC\ell_\star(D) = 0$ or $LC\ell_\star(D)$ torsion). For this we need to define yet more types of $\star$-homogeneous ideals.

Definition 10. Let D be an integral domain and $\star$ a finite character star-operation on D. An ideal of I of D is *$\star$-weakly factorial-($\star$-wf-) homogeneous* (resp., *$\star$-almost weakly factorial-($\star$-awf-) homogeneous*, *$\star$-weakly locally factorial ($\star$-wlf-) homogeneous*, *$\star$-weakly almost locally factorial ($\star$-walf-) homogeneous*) if (1) I is $\star$-homogeneous and (2) if I is $\star$-invertible, then $I^\star$ is principal (resp., $(I^n)^\star$ is principal for some $n \geq 1$, $I^\star$ is invertible, $(I^n)^\star$ is invertible for some $n \geq 1$). And D is called a *$\star$-wf-SH domain* (resp., *$\star$-awf-SH domain*, *$\star$-wlf-SH domain*,*$\star$-walf-SH domain*) if each proper nonzero principal ideal of D is a $\star$-product of $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous, $\star$-wlf-homogeneous, $\star$-walf-homogeneous) ideals.

Theorem 20. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is an $\star$-wf-SH domain (resp., $\star$-awf-SH domain).*
2. *If I is a nonzero finitely generated ideal of D with $I^\star \neq D$, then $I^\star$ is a $\star$-product of $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals.*
3. *D is a $\star$-SH domain with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion).*

Proof. We do the case for $C\ell_\star(D) = 0$, the $C\ell_\star(D)$ torsion case is similar. (3)⇒(2) Since D is an $\star$-SH domain, by Theorem 6 $I^\star = (I_1 \cdots I_n)^\star$ where I_i is $\star$-homogeneous. Now if I_i is $\star$-invertible, then $I_i^\star$ is principal. Thus I_i is $\star$-wf-homogeneous. (2)⇒(1) Clear. (1)⇒(3) It suffices to show that if A is a finitely generated nonzero $\star$-invertible integral ideal with $A^\star \neq D$, then $A^\star$ is principal. As in the proof of Theorem 6, $A^\star = ((AD_{M_1} \cap D) \cdots (AD_{M_n} \cap D))^\star$ where $M_1, \ldots, M_n$ are the maximal $\star$-ideals containing A. Now $AD_{M_i} \cap D$ is $\star$-invertible, so $AD_{M_i} \cap D = (AD_{M_i} \cap D)^{\star_w} = (AD_{M_i} \cap D)^\star$. Hence $AD_{M_i} \cap D$ is a $\star$-invertible $\star$-ideal. So $(AD_{M_i} \cap D)_{M_i} = a_i D_{M_i}$ for some $a_i \in D$. Now by hypothesis $Da_i = (I_1 \cdots I_s)^\star$ where each I_j is $\star$-wf-homogeneous. Hence $I_j^\star = Dx_j$ for some $x_j \in D$. So $Da_i = Dx_1 \cdots Dx_s$ where Dx_j is $\star$-homogeneous. By combining similar factors we can assume that $Dx_1, \ldots, Dx_s$ are pairwise $\star$-comaximal. Now some $M(Dx_j) = M_i$. By Proposition 1 $x_j D_{M_i} \cap D = x_j D$. Now $a_i D_{M_i} = x_j D_{M_i}$ and hence $AD_{M_i} \cap D = a_i D_{M_i} \cap D = x_j D_{M_i} \cap D = x_j D$. So $A^\star$ is principal.

We have a companion theorem for the "locally" case. The proof is left to the reader.

Theorem 21. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-wlf-SH domain (resp., $\star$-walf-SH domain).*
2. *If I is a nonzero finitely generated ideal with $I^\star \neq D$ then $I^\star$ is a $\star$-product of $\star$-wlf-homogeneous (resp., $\star$-walf-homogeneous) ideals.*
3. *D is a $\star$-SH domain with $LC\ell_\star(D) = 0$, (resp., $LC\ell_\star(D)$ torsion).*

Let D be an integral domain and $\star$ a finite character star-operation on D. It is evident that a $\star$-product of similar $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals is again $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous). Thus if an ideal is a $\star$-product of $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals, it is a $\star$-product of pairwise $\star$-comaximal $\star$-wf-homogeneous (resp., $\star$-awf-homogeneous) ideals. Similar results hold for the "locally" case. Let us call an element $x \in D$ $\star$-homogeneous if Dx is $\star$-homogeneous. We have the following element-wise characterization of $\star$-SH domains with $C\ell_\star(D) = 0$ or torsion.

Theorem 22. *Let D be an integral domain and $\star$ a finite character star-operation on D. Then the following conditions are equivalent.*

1. *D is a $\star$-SH domain with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion).*
2. *For each nonzero nonunit $x \in D$, x (resp., x^n for some $n = n(x) \geq 1$) is a product of $\star$-homogeneous elements.*
3. *For each nonzero nonunit $x \in D$, x (resp., x^n for some $n = n(x) \geq 1$) can be written uniquely up to order as a product of pairwise $\star$-comaximal $\star$-homogeneous elements.*

Proof. For both cases it is clear that (2)⇔(3) and (1)⇒(2). And it is immediate from Theorem 20 that if each nonzero nonunit of D is a product of $\star$-homogeneous elements, then D is a $\star$-SH domain with $C\ell_\star(D) = 0$. So suppose that D is an integral domain with the property that for each nonzero nonunit x, some power of x is a product of $\star$-homogeneous elements. Let x be a nonzero nonunit of D. Then some x^n is a product of $\star$-homogeneous elements. Thus x^n, and hence x, is contained in only finitely many maximal $\star$-ideals. So $\star$ is locally finite. Suppose that M_1 and M_2 are distinct maximal $\star$-ideals and there is a nonzero prime ideal $P \subseteq M_1 \cap M_2$. Let $0 \neq x \in P$. So some x^n is a product of $\star$-homogeneous elements. Thus P contains a $\star$-homogeneous element which is absurd since $P \subseteq M_1 \cap M_2$. So $\star$ is independent. By Theorem 4, D is an $\star$-SH domain. Let A be a nonzero finitely generated integral $\star$-invertible ideal of D with $A^\star \neq D$. It suffices to show that for some $n \geq 1$, $(A^n)^\star$ is principal. But this follows from an easy modification of the proof of (1)⇒(3) of Theorem 20.

We note that the notions of type 2 $\star$-f-SH domain (resp., type 2 $\star$-af-SH domain) and type 2 $\star$-wf-SH domain (resp., type 2 $\star$-waf-SH domain) coincide, they are both equivalent to being $\star$-Krull with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion). Also, the notions of type 2 $\star$-lf-SH domain (resp., type 2 $\star$-laf-SH domain) and type 2 $\star$-wlf-SH domain (resp., type 2 $\star$-walf-SH domain) coincide, they are both equivalent to being $\star$-Krull with $LC\ell_\star(D) = 0$ (resp., $LC\ell_\star(D)$ torsion). However, this is not the case for type 1. Now a type 1 $\star$-f-SH domain (resp., type 1 $\star$-af-SH domain) is a $\star$-GKD with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion). And a type 1 $\star$-wf-SH domain (resp., type 1 $\star$-waf-SH domain) is a $\star$-weakly Krull domain with $C\ell_\star(D) = 0$ (resp., $C\ell_\star(D)$ torsion). Finally a type 1 $\star$-lf-SH domain (resp., type 1 $\star$-wlf-SH domain) is a $\star$-GKD with $LC\ell_\star(D) = 0$ (resp., $\star$-weakly Krull domain with $LC\ell_\star(D) = 0$) and a type 1 $\star$-laf-SH domain (resp., type 1 $\star$-walf-SH domain) is a $\star$-GKD domain (resp., $\star$-Krull domain) with $LC\ell_\star(D)$ torsion. An integral domain is *weakly factorial* [6] if each nonzero nonunit is a product of primary elements. An integral domain D is weakly factorial if and only if D is weakly Krull and $C\ell_t(D) = 0$ [8, Theorem]. Also, the following are equivalent: (1) D is a weakly factorial GCD domain, (2) D is a weakly factorial GKD, and (3) D is a GCD GKD [6, Theorem 20]. For a Noetherian domain D, D is integrally closed weakly factorial if and only if D is factorial. For any field K, $K[[X^2, X^3]]$ is weakly factorial but not factorial and hence is a type 1 $\star$-wf-SH domain, but not a type 1 $\star$-f-SH domain (for $K[[X^2, X^3]]$, $d = t$).

References

1. Anderson, D. D.: π-domains, overrings, and divisorial ideals. Glasgow Math. J. **19**, 199-203 (1978)
2. Anderson, D. D.: Star-operations induced by overrings. Comm. Algebra. **16**, 2535–2553 (1988)
3. Anderson, D. D., Anderson, D. F.: Generalized GCD domains. Comment. Math. Univ. St. Pauli. **27** 215–221 (1979)

4. Anderson, D. D., Cook, S. J.: Two star operations and their induced lattices. Comm. Algebra. **28**, 2461–2476 (2000)
5. Anderson, D. D., Houston E. G., Zafrullah, M.: t-linked extensions, the t-class group, and Nagata's theorem. J. Pure Appl. Algebra. **86**, 109–124 (1993)
6. Anderson, D. D., Mahaney, L. A.: On primary factorization. J. Pure Appl. Alg. **54**, 141–154 (1988)
7. Anderson, D. D., Mott, J. L., Zafrullah, M.: Finite character representations of integral domains. Boll. Un. Mat. Ital. B(7)6, 613–630 (1992)
8. Anderson, D. D., Zafrullah, M.: Weakly factorial domains and groups of divisibility. Proc. Amer. Math. Soc. **109** 907–913 (1990)
9. Anderson, D. D., Zafrullah, M.: Almost Bezout domains. J. Algebra. **142**, 285–309 (1991)
10. Anderson, D. D., Zafrullah, M.: Independent locally-finite intersections of localizations. Houston J. Math. **25**, 109–124 (1999)
11. Anderson, D. F.: A general theory of class groups. Comm. Algebra. **16**, 805–847 (1988)
12. Bouvier, A.: Le groupe de classes d'un anneau integre. 107 erne Congres National des Societe Savantes. Brest, France, Fasc. **IV**, 85–92 (1982)
13. Bouvier, A., Zafrullah, M.: On some class groups of an integral domain. Bull. Soc. Math. Greece. **29**, 45–49 (1988)
14. Gilmer, R.: Multiplicative Ideal Theory. Marcel Dekker, New York, (1972)
15. Griffin, M.: Rings of Krull type. J. Reine Angew. Math. **229**, 1–27 (1968)
16. Halter-Koch, F.: Ideal Systems - An Introduction to Multiplicative Ideal Theory. Marcel Dekker, New York (1998)
17. Houston, E. G., Zafrullah, M.: $\star$ Super potent domains. J. Comm. Algebra, to appear
18. Jaffard, P.: Les Systèmes d' Ideáux. Dunod, Paris (1962)
19. Kaplansky, I.: Commutative Rings (revised edition). Polygonal Publishing, Washington, NJ. (1994)
20. Matlis, E.: Torsion-free Modules. The University of Chicago Press, Chicago (1972)
21. Storch, U.: Fastfaktorielle Ringe. Schritenreiche Math. Inst. Univ. Munster. Vol. 36, Munster (1967)
22. Zafrullah, M.: A general theory of almost factoriality. Manuscripta Math. **51**, 29–62 (1985)