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CONTENTS OF POLYNOMIALS AND INVERTIBILITY

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INTRODUCTION

Let D be a commutative integral domain with identity, let $D[X]$ denote the ring of polynomials in one variable with coefficients in D, and let K denote the quotient field of D. For a polynomial $f \in K[X]$, the *D-content* of f, denoted by A_f, is defined to be the fractional ideal of D generated by the coefficients of f. Thus, $A_f = (a_0, a_1, \cdots, a_n)D$ if $f = a_0 + a_1X + \cdots + a_nX^n$.

Let f and g be two polynomials in $K[X]$ and let fg denote their product. Several mathematicians, including Gauss and Prüfer [11], have studied the connection between the content ideals A_f, A_g, and A_{fg}. For any integral domain D, it is always the case that $A_{fg} \subseteq A_fA_g$. In her dissertation at the University of Chicago [13], Tsang studied polynomials $f \in R[X]$, where R is a commutative ring with identity, such that $A_fA_g = A_{fg}$ for all $g \in R[X]$. She called such polynomials *Gaussian*. Dan Anderson [1] observed that over a commutative ring with identity, a polynomial f is Gaussian if A_f is locally principal. In particular, f is Gaussian if A_f is invertible. Tsang proved that an integral domain D is a Prüfer domain if and only if $A_{fg} = A_f A_g$ for all polynomials $f, g \in D[X]$. Although she never published her result, R. Gilmer obtained the result independently and published it in [6] and [7, p. 347].

The purpose of this paper is to extend the investigations listed above. For a ring R, we use R^* to denote the set of nonzero elements of R.

1. POLYNOMIALS WITH INVERTIBLE CONTENT

We begin by making the following simple observation about the sets $S = \{f \in K[X] | A_f \text{ is invertible }\}$ and $S_D = S \cap D[X]$.

PROPOSITION 1.1. The sets S and S_D are closed under multiplication.

Proof. Immediate since $A_{fg} = A_f A_g$ and since a product of invertible ideals is invertible.

For $f, g \in S$ define $f \sim g$ if and only if $A_f = A_g$. Then the relation $\sim$ is an equivalence relation on S. Let $S/\sim$ denote the set of equivalence classes of elements of S under the relation $\sim$ and let $[f]$ denote the class of all polynomials $g \in S$ such that $f \sim g$. Next, define a binary operation $\circ$ on $S/\sim$ by the equation $[f] \circ [g] = [fg]$. It is immediate that $\circ$ is well-defined. In fact, we observe the following.

PROPOSITION 1.2. The set of equivalence classes $S/\sim$ forms an abelian group under the binary operation $\circ$. Moreover, $(S/\sim, \circ)$ is isomorphic to the group $I(D)$ of all invertible fractional ideals of D.

Proof. Obviously $\circ$ is associative and the identity in $(S/\sim, \circ)$ is $[1] = [h]$, where h is any polynomial in $K[X]$ such that $A_h = D$. We need only verify the existence of inverses in $(S/\sim, \circ)$. If $f \in S, A_f = (a_0, a_1, \cdots, a_n)D$ is invertible and $1 = \sum_{i=0}^{n} a_i a_i'$ where $a_i' \in [D : A_f]_K = \{t \in K | t a_i \in D \text{ for all } i\} = A_f^{-1}$. Let $g \in K[X]$ where $g = a_0' + a_1' X + \cdots + a_n' X^n$. Clearly $A_g = A_f^{-1}$.

Define the map $\theta : (S/\sim, \circ) \to I(D)$ by $\theta([f]) = A_f$. Since $[f] = [g]$ if and only if $A_f = A_g$, we see that θ is both well-defined and injective. Moreover, θ is surjective since any ideal $I \in I(D)$ is finitely generated, and hence $I = A_f$ for suitable $f \in S$. Finally, θ is a homomorphism by Proposition 1.1.

Clearly θ maps $\{[k] | k \in K^*\}$ onto the subgroup $P(D)$ of all principal fractional ideals of D. Therefore, we have the next corollary.

COROLLARY 1.3. The factor group $(S/\sim)/\{[k] | k \in K^*\}$ is isomorphic to the Picard group $I(D)/P(D) = \text{Pic}(D)$.

REMARKS. (1) Though the groups $S/\sim$ and $I(D)$ are isomorphic, the sets $S/\sim$ and $I(D)$ do not have identical properties; for example, for any fractional ideals A, B of D such that $AB \in I(D)$, we have that both A and B are in $I(D)$. But we show in Theorem 1.5 that $S/\sim$ has this property if and only if D is integrally closed.

(2) For a finitely generated R-module M, where R is a commutative ring with identity, let $\mu(M)$ denote the minimum number of generators of M. Then the isomorphism between the groups $S/\sim$ and $I(D)$ can be used to make the beautiful observation that if I is a finitely generated ideal and J is an invertible ideal, then $\mu(IJ) \leq \mu(I) + \mu(J) - 1$. Indeed, Dan Anderson [2] used this observation to obtain several nice conclusions about the minimum number of generators of a product of invertible ideals.

Let us determine conditions that guarantee that S is a saturated multiplicatively closed set in $K[X]$. We begin with the following observation.

PROPOSITION 1.4. Suppose $b \in K$ is integral over D. Then $b \in D$ if and only if the fractional ideal $(1, b)D$ is invertible.

Proof. Clearly if $b \in D$, then $(1, b)D = D$. Conversely, suppose $(1, b)$ is invertible. Then there are elements $c, d \in (1, b)^{-1}$ such that $c + bd = 1$. We note that c, d, bc and bd are in D. Since b is integral over D, b satisfies a monic polynomial in $D[X]$. Hence, we have the equation:

$$(1) \qquad b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0, \quad \text{where each } a_i \in D.$$

By multiplying equation (1) by d^{n-1}, we conclude that $b^n d^{n-1} \in D$. Moreover, $c + bd = 1$ implies that $c\lambda + (bd)^{n-1} = 1$ where $\lambda \in D$. Therefore, $b = bc\lambda + b^n d^{n-1} \in D$ as we wished to prove.

Recall that a multiplicatively closed set T in a commutative ring R is said to be *saturated* if for $a, b \in R$, $ab \in T$ implies $a \in T$ and $b \in T$.

THEOREM 1.5. Let D be an integral domain with quotient field K. Then the following are equivalent:

(1) D is integrally closed in K.

(2) S is a saturated multiplicative system in $K[X]$.

(3) S_D is a saturated multiplicative system in $D[X]$.

(4) If $f \in D[X]$ is linear and $g \in D[X]$ is such that $fg \in S$, then both f and g are in S.

(5) If $\alpha \in K$ is integral over D, then the fractional ideal $(1, \alpha)D$ is invertible.

(6) If $f, g \in K[X]$ are such that $A_{fg} \subseteq D$, then $A_f A_g \subseteq D$.

Proof. (1) implies (2). Let $f, g \in K[X]$ be such that $fg \in S$. Since D is integrally closed, $D = \bigcap_\alpha V_\alpha$ is an intersection of valuation o... ings V_α of

D. If A is a nonzero fractional ideal of D, define $A^{\omega} = \bigcap_{\alpha} AV_{\alpha}$. In particular, if A is an invertible fractional ideal of D, then we have $A^{\omega} = \bigcap_{\alpha} AV_{\alpha} = A(\bigcap_{\alpha} V_{\alpha}) = AD = A$. Thus, since A_{fg} is invertible, we have that $A_{fg} = (A_{fg})^{\omega} = \bigcap_{\alpha} A_{fg}V_{\alpha} = \bigcap_{\alpha}(A_fV_{\alpha})(A_gV_{\alpha})$ because $A_{fg}V_{\alpha} = (A_fV_{\alpha})(A_gV_{\alpha})$ for each valuation domain V_{α} (recall that a valuation domain is a Prüfer domain). But then

$$A_{fg} = \bigcap_{\alpha}(A_fV_{\alpha})(A_gV_{\alpha}) \supseteq \left(\bigcap_{\alpha} A_fV_{\alpha}\right)\left(\bigcap_{\alpha} A_gV_{\alpha}\right) = A_f^{\omega}A_g^{\omega} \supseteq A_fA_g.$$

As A_{fg} is always contained in A_fA_g, we conclude $A_fA_g = A_{fg}$. The invertibility of A_{fg} implies that of A_f and A_g. Therefore, $f \in S$ and $g \in S$.

Clearly (2) implies (3) and (3) implies (4). So we prove (4) implies (5). Assume α is integral over D. Then α satisfies a monic polynomial $f \in D[X]$. Hence $f = (X - \alpha)g$ where $g \in K[X]$. If $\alpha = a/b$ where $a, b \in D$, then by clearing the denominators of the coefficients we obtain the equation:

$$bdf = (bX - a)(dg), \quad \text{where } d \in D^*.$$

Now $A_{bdf} = bdA_f = bdD$ since f is monic, and since $bd \neq 0$, we have $bdf \in S$. Therefore, by (4), $(bX - a) \in S$ and then $A_{bX-a} = (a, b)D$ invertible implies that $(1, \alpha)D$ is invertible.

Proposition 1.4 shows (5) implies (1). To prove (1) implies (6), assume $f, g \in D[X]$ are such that $A_{fg} \subseteq D$. Since D is integrally closed, $D = \bigcap_{\alpha} V_{\alpha}$ where V_{α} is a valuation overring of D for each α. Hence

$$\begin{aligned} A_fA_g &\subseteq (A_fA_g)^{\omega} = \bigcap_{\alpha}(A_fA_g)V_{\alpha} \\ &= \bigcap_{\alpha}(A_{fg}V_{\alpha}) \subseteq \bigcap_{\alpha}(DV_{\alpha}) = D. \end{aligned}$$

Finally, (6) implies (1). Let $\alpha \in K$ be integral over D. Then α satisfies some monic polynomial $f \in D[X]$. Hence $f = (X - \alpha)g$ where $g \in K[X]$.

Then $A_f = D$ since f is monic. Therefore by (6), $A_{X-\alpha}A_g \subseteq D$. But since $A_{X-\alpha} \supseteq D$ and $A_g \supseteq D$, we have $A_{X-\alpha}A_g = D$ and $A_{X-\alpha} = (1,\alpha)D$ is invertible.

The following lemma will be useful in the next theorem.

LEMMA 1.6. Let D be an integral domain and let $\bar{D}$ denote the integral closure in the quotient field K of D. Then $S_{\bar{D}} = S_D^\circ$, the saturation of S_D in $\bar{D}[X]$.

Proof. Since $S_{\bar{D}}$ is saturated and $S_D \subseteq S_{\bar{D}}$, we have that $S_D^\circ \subseteq S_{\bar{D}}$. Conversely, suppose $f \in S_{\bar{D}}$. Since $A_f\bar{D}$ is invertible, we can choose $g \in K[X]$ such that $A_{fg}\bar{D} = \bar{D}$. Let $N_D = \{h \in D[X] | A_hD = D\}$ and let $N_{\bar{D}} = \{h \in \bar{D}[X] | A_h\bar{D} = \bar{D}\}$. Then $N_{\bar{D}}$ is the saturation of N_D in $\bar{D}[X]$, see, for instance, the proof of Theorem 3 in [8]. Thus $fgh \in N_D$ for some $h \in \bar{D}[X]$. By clearing denominators of g and h, we obtain that $fk \in S_D$ for some $k \in D[X]$. Thus $f \in S_D^\circ$ and hence $S_{\bar{D}} = S_D^\circ$.

In [8] Gilmer and Hoffman remark that at that time there were two characterizations of Prüfer domains in terms of polynomials. We list other characterizations in the next theorem.

THEOREM 1.7. Let D be an integral domain and let $\bar{D}$ denote the integral closure in the quotient field K of D. Then the following are equivalent:

(1) $\bar{D}$ is a Prüfer domain.

(2) $S_D^\circ = D[X]\backslash\{0\}$.

(3) $D[X]_{S_D}$ is a field.

(4) Each nonzero element $\alpha \in K$ satisfies a polynomial $f \in D[X]$ such that A_fD is invertible.

Proof. (1) implies (2) follows from Lemma 1.6 and the fact that eve finitely generated ideal of $\bar{D}$ is invertible.

The fact $D[X]_{S_D} = D[X]_{S_{\bar{D}}}$ yields that (2) implies (3). Next we sho that (3) implies (4). Every $\alpha \in K^*$ satisfies some nonzero polynomia $f \in D[X]$. Therefore, since $D[X]_{S_D}$ is a field, $fg = c$ where $g \in D[X]$ an $c \in S_D$. Then α satisfies c.

Finally, (4) implies (1). To prove that $\bar{D}$ is a Prüfer domain we need on show that each nonzero ideal of $\bar{D}$ with two generators is invertible. L $(a, b)\bar{D}$ be such an ideal where, without loss of generality, we may assum $a \neq 0$ and $b \neq 0$. Let $\alpha = a/b$. Then by hypothesis, α satisfies a polynomi $f \in D[X]$ where $A_f D$ is invertible. Then $f = (X - \alpha)g$ where $g \in K[X$ Then for a suitable $d \in D^*$, we get $df = (bX - a)(dg)$ where $dg \in \bar{D}[X]$. No that $df \in S_D$. Since $\bar{D}$ is integrally closed, Theorem 1.5 implies that $S_{\bar{D}}$ saturated. Hence $bX - a \in S_{\bar{D}}$ and $(a, b)\bar{D}$ is invertible.

REMARK. If D is a 1-dimensional Northerian domain then $\bar{D}$ is Dedekind and hence Prüfer domain. Therefore, $D[X]_{S_D}$ is always a field.

For the next two results, let us set the hypothesis, notation, and term nology. Assume D is a domain such that $D = \bigcap_\alpha V_\alpha$ where $\{V_\alpha\}$ is a fam of valuation overrings of D. For a fractional ideal I of D, define $I^\omega = \bigcap_\alpha I$ We say that an ideal I is ω-invertible if $(II^{-1})^\omega = D$.

PROPOSITION 1.8. Suppose $f \in D[X]$ is a nonzero polynomial su that $(A_{fg})^\omega = A_{fg}$ for all $g \in D[X]$. Then f is Gaussian.

Proof. We observe that for any $g \in D[X]$

$$\begin{aligned} A_{fg} = (A_{fg})^\omega = \bigcap_\alpha A_{fg} V_\alpha = \bigcap_\alpha A_f V_\alpha A_g V_\alpha \\ \supseteq \Big(\bigcap_\alpha A_f V_\alpha\Big)\Big(\bigcap_\alpha A_g V_\alpha\Big) = A_f^\omega A_g^\omega \supseteq A_f A_g \supseteq A_{fg}. \end{aligned}$$

Hence $A_{fg} = A_f A_g$ and f is Gaussian.

THEOREM 1.9. Let $f \in D[X]$ be a nonzero polynomial such that

(1) A_f is ω-invertible.

(2) $A_f^{-1} = (A_g)^{\omega}$ for some $g \in K[X]$.

(3) For all $h \in D[X]$, $(A_{fh})^{\omega} = A_{fh}$.

Then D is a Prüfer domain.

Proof. First we assert that A_f is invertible. We have that

$$A_f A_g = A_{fg} = (A_{fg})^{\omega} = \bigcap_{\alpha} A_{fg} V_{\alpha} = \bigcap_{\alpha} A_f A_g V_{\alpha}$$
$$= (A_f A_g)^{\omega} = (A_f (A_g)^{\omega})^{\omega} = (A_f A_f^{-1})^{\omega} = D.$$

Thus A_f is invertible.

Let $h \in D[X]$. By (3), we have that $A_{fh} = (A_{fh})^{\omega} = (A_f A_h)^{\omega} = (A_f (A_h)^{\omega})^{\omega} \supseteq A_f (A_h)^{\omega} \supseteq A_f A_h \supseteq A_{fh}$. Hence $A_f A_h = A_f (A_h)^{\omega}$. As A_f is invertible, we have $A_h = (A_h)^{\omega}$. Thus, $(A_{hk})^{\omega} = A_{hk}$ for all $h, k \in D[X]$. By Propositon 1.8, every $h \in D[X]$ is Gaussian. Tsang's theorem then tells us that D is a Prüfer domain.

If A is a fractional ideal of a domain D, define $A_v = (A^{-1})^{-1}$ and say that A is a v-ideal if $A = A_v$, and A is v-invertible if $(AA^{-1})_v = D$. See [7] for many well known properties of the v-operation.

Suppose D is a Krull domain. Then $D = \bigcap_{\alpha} V_{\alpha}$ where $\{V_{\alpha}\}$ is the family of DVR's obtained by localizations of D at height one prime ideals. We take the ω-operation with respect to this family.

COROLLARY 1.10. Let D be a Krull domain. Suppose that there is a nonzero polynomial $f \in D[X]$ such that $(A_{fg})^{\omega} = A_{fg}$ for all $g \in D[X]$. Then D is a Dedekind domain.

Proof. Since a Krull domain is completely integrally closed, each finitely generated ideal is v-invertible [7, p. 421] with a v-inverse of finite type. Moreover, the ω-operation in this case is same as the v-operation [7, p. 542]. Hence conditions (1) and (2) of Theorem 1.9 are satisfied; since condition (3) is assumed by hypothesis, D is a Prüfer domain. A domain that is both Prüfer and Krull is a Dedekind domain [7, p. 536].

If A is a fractional ideal of a domain D, define $A_t = \cup B_v$, where B runs through all finitely generated D-submodules of A. Then A is said to be t-invertible if $(AA^{-1})_t = D$. An integral domain D is called a Prüfer v-multiplication domain (PVMD) if the set $H(D)$ of v-ideals of finite type is a group under the v-multiplication: $(AB)_v = (A_vB_v)_v = (A_vB)_v$, or equivalently, if each finitely generated fractional ideal of D is t-invertible. If D is a PVMD, then there is a family $\{V_\alpha\}$ of essential valuation overrings of D such that $D = \bigcap_\alpha V_\alpha$ [9].

COROLLARY 1.11. Let D be a PVMD. Suppose there exists a nonzero polynomial $f \in D[X]$ such that $(A_{fg})^\omega = A_{fg}$ for all $g \in D[X]$. Then D is a Prüfer domain.

Proof. Since the ω-operation with respect to $\{V_\alpha\}$ is equivalent to v-operation [7, p. 553] and since D is a PVMD, we have conditions (1), (2) and (3) of Theorem 1.9.

2. POLYNOMIALS WITH v-INVERTIBLE CONTENT

In this section we prove several results for polynomials f where A_f is v-invertible; these results are parallel to those obtained in the preceeding section. Our first observation, an immediate corollary of Theorem 1.5, extends Gauss' Lemma and Theorem 34.8 of [7]. The result is due originally to Querré [12]; we offer a new proof.

PROPOSITION 2.1. An integral domain D is integrally closed if and only if for any two polynomials $f, g \in K[X]$, $(A_{fg})_v = (A_f A_g)_v$.

Proof. If D is integrally closed, $D = \bigcap_\alpha V_\alpha$ where each V_α is a valuation overring of D. Then

$$(A_{fg})^\omega = \bigcap_\alpha (A_{fg} V_\alpha) = \bigcap_\alpha (A_f A_g) V_\alpha = (A_f A_g)^\omega.$$

But then

$$(A_{fg})_v = \left((A_{fg})^\omega\right)_v = \left((A_f A_g)^\omega\right)_v = (A_f A_g)_v.$$

Conversely, suppose $(A_{fg})_v = (A_f A_g)_v$. Then $A_{fg} \subseteq D$ if and only if $A_f A_g \subseteq D$. Therefore, by (6) of Theorem 1.5, D is integrally closed.

PROPOSITION 2.2. Let D be an integral domain and suppose $f \in D[X] \backslash \{0\}$. If A_f is v-invertible, then for each polynomial $g \in D[X] \backslash \{0\}$, $(A_{fg})_v = (A_f A_g)_v$.

Proof: Assume that f and g are as stated in the hypothesis. By Dedekind-Mertens Lemma [6] or [7, p. 343], there is a positive integer k such that $A_f^{k+1} A_g = A_f^k A_{fg}$. Multiplying by $\left((A_f)^{-1}\right)^k$ and applying the v-operation, we conclude $(A_f A_g)_v = (A_{fg})_v$.

COROLLARY 2.3. The set $V_D = \{f \in D[X] | A_f \text{ is } v\text{-invertible}\}$ is closed under multiplication.

Proof: Suppose $f, g \in V_D$. Then $(A_{fg})_v = (A_f A_g)_v$ by Proposition 2.2. But then $\left(A_f^{-1} A_g^{-1} (A_{fg})_v\right)_v = \left((A_f^{-1} A_f)(A_g^{-1} A_g)\right)_v = D$.

Now we obtain a result analogous to Theorem 1.5.

THEOREM 2.4. *The set V_D is saturated if and only if D is integrally closed.*

Proof: If D is integrally closed and $fg \in V_D$, then by Proposition 2.1 $(A_{fg})_v = (A_f A_g)_v$. But since A_{fg} is v-invertible, for $C = (A_{fg})^{-1}$, we conclude $(A_{fg}C)_v = D = (A_f A_g C)_v$, so both A_f and A_g are v-invertible.

Conversely, suppose V_D is saturated and suppose $\alpha = a/b \in K^*$ is integral over D where $a, b \in D^*$. Then α satisfies a monic polynomial $f \in D[X]$ and $f = (X - \alpha)g$ where $g \in K[X]$. Clearing denominators we have $dbf = (bX - a)(dg)$, where $dg \in D[X]$. But then $A_{dbf} = dbA_f = dbD$ since f is monic. Hence, $(bX - a)dg \in V_D$ and since V_D is saturated, $b(X - \alpha) = bX - a \in V_D$ so that $A_{X-\alpha}$ is v-invertible. But $D = (A_f)_v = (A_{(X-\alpha)g})_v = (A_{X-\alpha}A_g)_v$ by Proposition 2.2. Thus, $A_{(X-\alpha)}A_g \subseteq D$. Therefore, the product $(1, \alpha)(1, g_{n-1}, \cdots, g_0)D \subseteq D$, where $g = X^n + g_{n-1}X^{n-1} + \cdots + g_0$. But since α is in the product, we conclude $\alpha \in D$.

Next we prove a theorem that is closely related to Theorem 1.7. Recall that a v-domain is an integral domain for which $(AA^{-1})_v = D$ for all finitely generated v-ideals A of D [3].

THEOREM 2.5. *Suppose D is an integrally closed domain. Then the following are equivalent:*

(1) *D is a v-domain.*

(2) *$V_D = D[X]\setminus\{0\}$.*

(3) *$D[X]_{V_D}$ is a field.*

(4) *Each nonzero element $\alpha \in K$ satisfies a polynomial $f \in D[X]$ such that A_f is v-invertible.*

Proof: First we show (1) implies (2). If $f \in D[X]^*$, then $(A_f)_v$ is a v-ideal of finite type. Since D is a v-domain, A_f is v-invertible so $f \in V_D$.

Clearly (2) implies (3) and the proof that (3) implies (4) is similar to that in Theorem 1.7.

REMARK. We observe that (3) implies (4) for an arbitrary integral domain.

To prove (4) implies (1) we need the following lemma and then the proof follows the same pattern as in Theorem 1.7 where invertible is replaced by v-invertible.

LEMMA 2.6. An integral domain D is a v-domain if and only if every nonzero fractional ideal with two generators is v-invertible.

Proof. Obviously if D is a v-domain then every two generated nonzero ideal is v-invertible.

Conversely, suppose that every nonzero ideal with two generators is v-invertible. Consider an ideal with three generators: $A = (x_1, x_2, x_3)D$. Now for ideals I, J, K in any commutative ring R, $(I+J+K)(IK+IJ+JK) = (J+K)(K+I)(I+J)$ so $(x_1, x_2, x_3)(x_1x_2, x_1x_3, x_2x_3) = (x_1, x_2)(x_2, x_3)(x_1, x_3)$. Because each factor of the right hand side is v-invertible by hypothesis, so is each factor of the left hand side. From this, we conclude (x_1, x_2, x_3) is v-invertible. Continue by induction.

REMARK. The above argument is a version of an argument that Prüfer [11, p. 7] used to prove that a domain is a Prüfer domain if and only if each ideal with two generators is invertible. Remarkable here is the fact that using the t-invertible version of the argument we can prove that an integral domain D is a PVMD if and only if every nonzero ideal with two generators is t-invertible. (Also see lemma 1.7 of [10]).

3. POLYNOMIALS WITH t-INVERTIBLE CONTENT

Now let us define $T = \{f \in K[X] | A_f$ is t-invertible$\}$ and $T_D = T \cap D[X]$. Using arguments similar to those in Proposition 1.1, Corollary 2.3, Theorem 1.5, and Theorem 2.4 we can show that T and T_D are closed under multiplication and T_D is saturated if and only if D is integrally closed.

THEOREM 3.1. For an integrally closed domain D the following are equivalent.

(1) D is a PVMD.

(2) $T_D = D[X]\backslash\{0\}$.

(3) $D[X]_{T_D}$ is a field.

(4) Each $\alpha \in K^*$ satisfies a polynomial $f \in D[X]$ such that A_f is t-invertible.

For this proof we need only apply lemma 1.7 of [10] or the observation in the preceding remark.

We conclude this paper with the observation that an equivalence relation may be defined on T by $f \sim g$ if and only if $(A_f)_t = (A_g)_t$. Then $T/\sim$ is a group under the operation $[f] \circ [g] = [fg]$. This group is associated to the t-class group of D. For the definition of the t-class group and related results, see [4] or [14].

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REFERENCES

1. D. D. Anderson, Some remarks on the ring $R(X)$, Comment. Math. Univ. St. Pauli XXVI -2 (1977), 137-140.

2. D. D. Anderson, On a result of M. Martin concerning the number of generators of products of invertible ideals, Comm. Algebra 15(1988), 1765-1768.

3. D. D. Anderson, D. F. Anderson, D. Costa, D. Dobbs, J. Mott, and M. Zafrullah, Some characterizations of v-domains and related topics, to appear in Colloq. Math.

4. A. Bouvier, Le groupe des classes d'un anneau integré, Congrés national des Societés savantés, Brest IV (1982), 85-92.

5. A.T. Dixon, A Polynomial Ring Localization, University of Missouri-Columbia, Dissertation, 1987.

6. R. Gilmer, Some applications of the Hilfsatz von Dedekind-Mertens, Math. Scand. 20 (1967), 240-244.

7. R. Gilmer, Multiplicative Ideal Theory, Marcel Dekker, New York, 1972.

8. R. Gilmer and J. Hoffman, A characterization of Prüfer domains in terms of polynomials, Pacific J. Math. 60 (1975), 81-85.

9. M. Griffin, Some results on v-multiplication rings, Canad. J. Math. 19(1967), 710-722.

10. S. Malik, J. Mott, and M. Zafrullah, On t-invertibility, Comm. Algebra 16 (1988), 149-170.

11. H. Prüfer, Untersuchen über Teilbarkeitseigenschaften in Körpern, J. Reine angew. Math. 168 (1932), 1-36.

12. J. Querré, Ideaux divisoriels d'un anneau de polynomes, J. Algebra 64 (1980), 270-284.

13. H. Tsang, Gauss' Lemma, University of Chicago, Dissertation, 1965.

14. M. Zafrullah, A general theory of almost factoriality, Manuscripta Math. 51 (1985), 29-62.

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