manuscripta math. 17, 55 - 66 (1975)

SEMIRIGID GCD DOMAINS

Muhammad Zafrullah[1]

Let R be a commutative integral domain. An element x of R is called <u>rigid</u> if for all r,s dividing x ; r divides s or s divides r . In our terminology, R is <u>semirigid</u> if each non zero non unit of R is a finite product of rigid elements. We show that semirigid GCD domains have a type of unique factorization, and are a known generalization of Krull domains.

Let R be a commutative integral domain. An element x in R is called <u>rigid</u> if for every $r , s|x$; $r|s$ or $s|r$ (see [2] p.129 for a general definition). The aim of this note is to study the properties of rigid non units and of their finite products in GCD domains. It turns out that in a GCD domain, a finite product of rigid elements can be expressed as a product of mutually co-prime rigid elements uniquely up to associates. We use this result to extend the concept of Unique Factorization.

---

(1) Basic ideas are taken from the author's Doctoral Thesis submitted to the University of London in 1974.

To keep the discussion simple and compact we include in the following some conventions and definitions.

I. Throughout this note, a <u>ring</u> will mean a commutative ring with unity and R will denote a commutative integral domain. We shall use (a,b) to denote the GCD of a and b and $a|b$ to denote "a divides b". The properties of GCD domains are assumed to be well known and are freely used. In case of doubt, the reader is referred to [6], pp. 32, 33 and 41, 42.

II. By a rigid element we shall mean a <u>non zero non unit</u> rigid element from now on.

III. <u>DEFINITION 1</u>: <u>An element</u> $x \in R$ <u>will be called semirigid if</u> x <u>can be expressed as a product of a finite number of rigid elements</u>.

IV. We shall regard a unit as an empty product of rigid elements and thus a semirigid element.

According to Cohn [2], R is rigid if every element of R is rigid. With a slight difference we give the following definition.

V. <u>DEFINITION 2</u>: <u>An intergral domain</u> R <u>will be called a Semirigid Domain if every non zero element of</u> R <u>is semirigid</u>.

Semirigid Domains are of interest because of the fact that an <u>irreducible element</u> which we prefer to call an atom is rigid. Consequently, an atomic(2) integral domain is semirigid. Now, UFD's being atomic GCD domains (cf. [3]) are a special case of Semirigid GCD domains, and this encourages the study of general Semirigid GCD domains. In the course of our study we find that not only have the semirigid GCD domains a type of unique factorization but also they are a known generalization of Krull domains (cf. Remarks 2 (i)).

We restrict our study of rigid and semirigid elements to GCD domains because the rigid elements being a generalization of atoms do not promise any better behaviour otherwise.

LEMMA 1. *In a* GCD *domain* R *the following are valid.*

(1) *If* r,s *are two non coprime rigid elements of* R

(a) Any two non unit factors r_1, s_1 respectively of r and s are non coprime.

(b) $r|s$ or $s|r$,

(c) rs is again a rigid element.

(2) *To each rigid element* $r \in R$, *there is associated a prime ideal* $P(r) = \{x \in R | x$ is non coprime to $r\}$.

(3) *If* r,s *are rigid elements in* R *then* $P(r) = P(s)$ *iff* r,s *are non coprime.*

(4) *If* r *is a rigid element in* R *then the localization* $R_{P(r)}$ *is a valuation domain.*

---

(2) An integral domain in which every non zero non unit is expressible as a product of atoms, is called atomic.

Remark 1. It may be noted that a non unit factor of a rigid element is again rigid.

PROOF OF LEMMA 1

1 (a) Let d be a non unit common factor of r and s. Then since r and s are rigid and $r_1, d|r$ and $s_1, d|s$ we have $r_1|d$ or $d|r_1$ and $s_1|d$ or $d|s_1$. We consider the four possibilities arising from the above observation, and show that the result follows from each case.

(i) If $r_1|d$ and $d|s_1$ then $r_1|s_1$, i.e. $(r_1,s_1) \neq 1$,

(ii) if $r_1|d$ and $s_1|d$ then d being rigid $(r_1,s_1) \neq 1$,

(iii) if $d|r_1$ and $s_1|d$ then again $(r_1,s_1) \neq 1$,

(iv) if $d|r_1$ and $d|s_1$ then since d is a non unit $(r_1,s_1) \neq 1$.

(b) Suppose on the contrary that $r \nmid s$ and $s \nmid r$ and let $d = (r,s)$ be the GCD. Then $r = r_1d$ and $s = s_1d$ where r_1 and s_1 are coprime non units, but this contradicts (a) above.

(c) Suppose that rs is not rigid, then there exist non units $x, y|rs$ such that $x \nmid y$ and $y \nmid x$. Since R is a GCD domain we can assume without loss of generality that x and y are coprime. Recalling also from [3] that if $a|bc$ in a GCD domain then $a = a_1a_2$ where $a_1|b$ and $a_2|c$, we write

$$\left.\begin{array}{l} x = r_1\ s_1 \\ y = r_2\ s_2 \end{array}\right\} r_i | r \quad \text{and} \quad s_i | s \ .$$

We obtain a contradiction by showing that either x or y is a unit. Suppose that r_1 is a non unit, then since $(x,y) = 1$ $(r_1,y) = 1$ and so $(r_1,r_2) = 1 = (r_1,s_2)$. But r is rigid, so either $r_1|r_2$ or $r_2|r_1$ and hence r_2 is a unit. Moreover by (a) above, s_2 is also a unit and thus y is a unit. This contradicts the assumption that rs is non rigid (i.e. can have mutually coprime non unit factors).

2. If $x, y \in P(r)$ then $(x,r) = r_1 \neq 1 \neq (y,r) = r_2$ and since r is rigid either $r_1|r_2$ or $r_2|r_1$, so one of the r_1, r_2 divides $(x + y, r)$ and $x + y \in P(r)$. It is now easy to verify that $P(r)$ is an ideal. Moreover $P(r)$ is prime since if $(xy,r) \neq 1$ then either $(x,r) \neq 1$ or $(y,r) \neq 1$ (because of the GCD property).

3. The proof is immediate from 1(a).

4. For every two elements $x, y \in P(r)$ their GCD being non coprime to r is again in $P(r)$ which according to Sheldon [7], is a sufficient condition for $R_{P(r)}$ to be a valuation domain.

For semirigid elements we state the following.

THEOREM 2. *Let x be a semirigid element in a GCD domain R. Then x can be written as a product of mutually coprime rigid elements uniquely up to associates* *and up to the order in which the rigid elements appear in the factorization.*

PROOF

Let $x = \rho_1 \rho_2 \ldots \rho_s$, where ρ_i are rigid. Grouping ρ_i into sets of non coprime rigid elements and applying 1(c) of Lemma 1 to each set we can write $x = r_1 r_2 \ldots r_n$ where r_i are mutually coprime rigid elements.

Let $x = s_1 s_2 \ldots s_m$ be another expression of x as a product of mutually coprime rigid elements, then $r_1 r_2 \ldots r_n = s_1 s_2 \ldots s_m$. Now $r_1 | s_1 s_2 \ldots s_m = x$ and since s_i are mutually coprime, r_1 is non coprime to at most one of s_i (cf. Lemma 1). Thus $r_1 | s_j$ for some j. Similarly, considering $s_j | r_1 r_2 \ldots r_n$ we conclude that $s_j | r_1$. In other words, each r_i is an associate of some s_j, and so on, $n \leq m$. Similarly we can work out the reverse conclusion and so the factorization $x = r_1 \ldots r_n$ is unique up to associates.

DEFINITION 3. *If R is a GCD domain, $x \in R$ and $x = r_1 r_2 \ldots r_n$ where r_i are mutually coprime rigid elements then we call x, canonically represented and $r_1 r_2 \ldots r_n$ the canonical representation or factorization of x.*

Now if we take a unit to be its own canonical representation, we have the

COROLLARY 3. *Every non zero element of a semirigid GCD domain has a canonical factorization which is* *unique in the sense of Theorem 2.*

From Corollary 3 follows a classical result.

COROLLARY 4. *An element x of a UFD can be expressed uniquely as $x = u\, p_1^{a_1} p_2^{a_2} \dots p_n^{a_n}$, where u is a unit and p_i are distinct primes.*

PROOF. By Corollary 3, x has a canonical factorization $x = r_1\, r_2 \dots r_n$, where $r_i = p_{i1}\, p_{i2} \dots p_{ia_i}$ say. Now r_i being rigid implies that each p_{ij} is an associate of some prime p_i. Let $p_{ij} = \alpha_{ij}\, p_i$. Then $r_i = \alpha_i\, p_i^{a_i}$ where α_i is a unit. Putting $u = \alpha_1\, \alpha_2 \dots \alpha_n$ we have

$$x = u\, p_1^{a_1}\, p_2^{a_2} \dots p_n^{a_n}\,.$$

This seems to be an appropriate point to give some examples of Semirigid GCD domains which are not UFD's.

EXAMPLE 1. An arbitrary valuation domain V is the simplest example. Because V is a GCD domain and a non zero element of V is either rigid or a unit.

EXAMPLE 2. Let V be a valuation domain and x be an indeterminate over V then the ring of polynomials $V[x]$ is a Semirigid GCD domain.

ILLUSTRATION:- It is well known that if R is GCD then so is $R[x]$ where x is an indeterminate. So $V[x]$ is a GCD domain. To see that $V[x]$ is semirigid let

$$\sum_{i=0}^{n} v_i\, x^i$$

be a non zero non unit in $V[x]$. We can write $\sum v_i x^i = v(\sum v'_i x^i)$ where v is the GCD of v_i and $v'_i = v_i/v$. It is easy to verify that $\sum v'_i x^i$ is primitive and a product of atoms and so is semirigid. And obviously since v is rigid, $v(\sum v'_i x^i) = \sum v_i x^i$ is semirigid.

Finally we show that semirigid GCD domains are a known generalization of Krull domains.

THEOREM 5. Let R be a Semirigid GCD domain then R has a family of prime ideals

$$\Phi = \{P_\alpha\}_{\alpha \in I}$$

such that

IKT_1. R_{P_α} is a valuation domain for each $\alpha \in I$,

IKT_2. Each non zero non unit $x (\in R)$ is contained in at most a finite number of P_α.

IKT_3. $P_\alpha \cap P_\beta$ does not contain a non zero prime ideal whenever $P_\alpha \neq P_\beta$. $\alpha, \beta \in I$.

IKT_4. $R = \bigcap R_{P_\alpha}$, $\alpha \in I$.

PROOF. Let S be the set of all rigid elements of R. Since R is a GCD domain, being non coprime is an equivalence relation over S (cf. 1(a) of Lemma 1). As a result, S can be partitioned into a set Γ of equivalence classes. We represent each class $\gamma_\alpha \in \Gamma$ by one of its elements r_α($\alpha \in I$ - an index set). By (2) and (3) of Lemma 1, to each $\gamma_\alpha \in \Gamma$ is associated a prime ideal $P(r_\alpha)$. Thus we construct a family of prime ideals

$$\Phi = \left\{ P(r_\alpha) \mid r_\alpha \in \gamma_\alpha \in \Gamma \right\}$$

in relation to the factorization properties of R . We now proceed to show that IKT_{1-4} hold for Φ .

IKT_1: Obviously by (4) of Lemma 1, $R_{P(r_\alpha)}$ is a valuation domain.

IKT_2: Every non zero non unit element of R has canonical factorization and $x \in P(r_\alpha)$ only if x is non coprime to r_α .

IKT_3: Suppose that there is a non zero prime ideal P contained in $P(r_\alpha) \cap P(r_\beta)$, where $P(r_\alpha) \neq P(r_\beta)$ and for $x \in P$ let $x = r_1 r_2 \ldots r_n$ be the canonical factorization of x . Then $x \in P(r_\alpha) \cap P(r_\beta)$ and so x is non coprime to both r_α and r_β . Let r_1 and r_2 be equal to r_α and r_β respectively. Then $x = r_1 y$ where $y = r_2 \ldots r_n \notin P(r_\alpha)$ and $r_1 \notin P(r_\beta)$ whence r_1 , $y \notin P$ contradicting the assumption that P is a prime.

IKT_4: Suppose that $R' = \bigcap R_{P(r_\alpha)}$ and let $u/v \in R'-R$. Since R is a GCD domain, we can assume that $(u,v) = 1$. Let v be a non unit and let $v = r_1 r_2 \ldots r_n$ be its canonical representation, then v is a non unit in $R_{P(r_i)}$ $(i = 1,2,\ldots,n)$ while u is a unit in $R_{P(r_i)}$ (because $(u,v) = 1$). But according to our assumption

$u/v \in R_{P(r_\alpha)}$ for all r_α, which implies that $u/v \in R_{P(r_i)}$ – an absurd conclusion. Thus we conclude that $R' - R = \phi$ that is IKT_4 also holds.

Remark 2.

(i) We observe that integral domains satisfying IKT_{1-4} are a special case of rings of Krull type and are called Independent rings of Krull type (IKT), (see Griffin [5] section 9 and [1] footnote at p.40.)

(ii) The converse of Theorem 5, that is, "If R is a GCD domain and is IKT then R is Semirigid." is also true and is included in a discussion of GCD rings of Krull type in the author's doctoral thesis.

(iii) Let R be a Semirigid GCD domain such that every rigid element of R satisfies the property:

(λ) for each non unit h dividing r, there exists a positive integer n such that $r|h^n$.

It can be verified that R is a generalized Krull domain (GKD) (cf. [1] for definition of GKD's).

Theorem 5 suggests that without going deeply into the theory of the rings of finite character, one can determine whether or not a given GCD domain is an IKT. As an application of this theorem we give a simple example of a Bezout domain which is an IKT.

EXAMPLE 4. Let R be a valuation domain with field of fractions K then

$$T = \left\{a_0 + \sum_{i=1}^{n} a_i x^i \mid a_0 \in R, a_i \in K\right\} = R + x K[x]$$

is a Bezout IKT .

<u>ILLUSTRATION</u>:- According to [4] T is a Bezout domain and hence is a GCD domain.

To prove that T is an IKT we only have to show that T is semirigid (cf. Theorem 5).

Let $t = a_o + \sum_{i=1}^{n} a_i x^i$ be an arbitrary element of T then

$$t = a_o(1 + \sum_{i=1}^{n} a_i/a_o\, x^i) \quad \text{if} \quad a_o \neq 0$$

or

$$t = \frac{b}{c} x^r (1 + \sum b_j x^j) \quad \text{if} \quad a_o = 0 .$$

It is easy to verify that in both cases the expressions in brackets are products of atoms and thus are semirigid. Moreover if $a_o \neq 0$ then a_o is either rigid or a unit, that is if $a_o \neq 0$, $a_o + \sum a_i x^i$ is semirigid. To show that the same holds when $a_o = 0$ we prove that $b/c\, x^r$ is rigid. Let $u/v\, x^\alpha$, $t/w\, x^\beta$ be any two factors of $b/c\, x^r$. If $\alpha < \beta$ then obviously $u/v\, x^\alpha \mid t/w\, x^\beta$, so let $\alpha = \beta$. But since R is a valuation domain, either $(u/v)^{-1}(t/w) \in R$ or $(u/v)(t/w)^{-1} \in R$,which implies that $u/v\, x^\alpha \mid t/w\, x^\alpha$ or $t/w\, x^\alpha \mid u/v\, x^\alpha$. Thus we have shown that one out of every two factors of $b/c\, x^r$ divides the other. We conclude that $\frac{b}{c} x^r(1 + \sum b_j x^j)$ is semirigid. Finally since t is arbitrary, the integral domain T is Semirigid.

ACKNOWLEDGEMENT:-

Thanks are due to Dr. John Shepperd who made a critical study of the manuscript.

BIBLIOGRAPHY

[1] BREWER, J.W. and MOTT, J.L., "Integral domains of finite character." J. Reine Angew. Math. 241, 34-41 (1970).

[2] COHN, P.M., "Free rings and their relations." Academic Press, London and New York (1971).

[3] COHN, P.M., "Bezout rings and their sub-rings." Proc. Camb. Phil. Soc., 64, 251-264 (1968).

[4] COSTA, D., MOTT, J., ZAFRULLAH, M., "The construction $R + x R_S[x]$, to appear.

[5] GRIFFIN, M., "Rings of Krull type." J. Reine Angew. Math., 229, 1-27 (1968).

[6] KAPLANSKY, I., "Commutative rings." Allyn and Bacon, (1970).

[7] SHELDON, P.B., "Prime Ideals in GCD domains." Can. J. Math., Vol XXVI, No.1, 98-107 (1974).

Muhammad Zafrullah
Department of Mathematics
UMIST, P.O. Box 88
MANCHESTER M60 1QD. U.K.

(Received May 29, 1975)