Applications of t-invertible uppers to zero

Let D be an integral domain with quotient field K and let $F(D)$ denote the set of fractional ideals of D. Denote by A^{-1} the fractional ideal $D :_K A = \{x \in K | xA \subseteq D\}$. The function $A \mapsto A_v = (A^{-1})^{-1}$ on $F(D)$ is called the v-operation on D (or on $F(D)$). Associated to the v-operation is the t-operation on $F(D)$ defined by $A \mapsto A_t = \cup\{H_v | \ H$ ranges over finitely generated subideals of $A\}$. The v and t-operations are examples of the so called star operations, well explained in sections 32 and 34 of [8]. Indeed $A \subseteq A_t \subseteq A_v$. A fractional ideal $A \in F(D)$ is called a v-ideal (resp., a t-ideal) if $A = A_v$ (resp., $A = A_t$). An integral t-ideal maximal among integral t-ideals is a prime ideal called a maximal t-ideal. If A is a nonzero integral ideal with $A_t \neq D$ then A is contained in at least one maximal t-ideal. A prime ideal that is also a t-ideal is called a prime t-ideal. Call $I \in F(FD)$ v-invertible (resp., t-invertible) if $(II^{-1})_v = D$ (resp., $(II^{-1})_t = D$). A prime t-ideal that is also t-invertible was shown to be a maximal t-ideal in Proposition 1.3 of [12, Theorem 1.4].

Let X be an indeterminate over K. Given a polynomial $g \in K[X]$, let A_g denote the fractional ideal of D generated by the coefficients of g. A prime ideal P of $D[X]$ is called a prime upper to 0 if $P \cap D = (0)$. Thus a prime ideal P of $D[X]$ is a prime upper to 0 if and only if $P = h(X)K[X] \cap D[X]$, for a prime h in $K[X]$. It follows from [12, Theorem 1.4] that P a prime upper to zero of D is a maximal t-ideal if and only if P is t-invertible if and only if P contains a polynomial f such that $(A_f)_v = D$. Based on this it was concluded in [10] that if f is a polynomial in $D[X]$ such that $(A_f)_v = D$, then $f(X)D[X]$ is a t-product of uppers to zero. Call a polynomial f super primitive if $(A_f)_v = D$ and call D a PSP domain if every primitive polynomial over D is super primitive. (In [10], using the fact that every ideal of $D[X]$ that contained a super primitive polynomial was t-invertible we concluded that $fD[X]$ was a t-product of maximal t -ideals. An element e was called a t-invertibility element if every ideal containing e was t-invertible. It was shown in Theorem 1.3 of [10] that a t-invertibility element is a t-product of maximal t-ideals.) The following result makes the above conclusion somewhat more obvious. Yet, before we state the lemma, let's note that every non-constant polynomial in $D[X]$ belongs to at most a finite number of uppers to zero, some of which may be t-invertible.

Lemma 1 . *Let $f \in D[X]$ be a non-constant polynomial and suppose that $P_1, ..., P_n$ are the only prime uppers to zero containing f that are maximal t-ideals. Then (1) for some positive integers r_i we have $f(X)D[X] = (AP_1^{r_1}...P_n^{r_n})_t$ where $(A, P_1^{r_1}...P_n^{r_n})_t = D[X]$, i.e. A is t-co-maximal with $P_1^{r_1}...P_n^{r_n}$ (2) if f is super primitive, i.e. is such that $(A_f)_v = D$, then $fD[X] = (P_1^{r_1}...P_n^{r_n})_t$, (3) Any non-constant polynomial f of $D[X]$ has at most a finite number of super primitive divisors.*

Proof. (1). The proof can be taken from the proof of Proposition 3.7 of [5]. For (2), note that if P is a maximal t-ideal containing A, then P contains f. This makes P t-invertible. But the only t-invertible maximal t-ideals containing

f are $P_1, ..., P_n$. This leaves the possibility that A is contained in a maximal t-ideal M with $M \cap D \neq (0)$. But this is impossible because $f \in A \subseteq M$, forcing $D = (f, d)_v \subseteq M$.Thus A is contained in no maximal t-ideal. Forcing $A_t = D$. But then $fD[X] = (AP_1^{r_1}...P_n^{r_n})_t = (A_t P_1^{r_1}...P_n^{r_n})_t = (P_1^{r_1}...P_n^{r_n})_t$. For (3), let's call an ideal I a t-divisor of an ideal A if there is an ideal B such that $A = (BI)_t$. If f is as in (1), i.e. f is such that $fD[X] = (AP_1^{r_1}...P_n^{r_n})_t$, then proper ideals of the kind $P_1^{a_1}...P_n^{a_n}$ $0 \leq a_i \leq r_r$ are t-divisors of $fD[X]$ and they only t-divide $P_1^{r_1}...P_n^{r_n}$. The reason is that if A, B, C are ideals such that $(A, B)_t = D$ and $A_t \supseteq (BC)_t$, then $A_t \supseteq C_t$. (This is because $A_t \supseteq (BC)_t \Leftrightarrow A_t = (A, BC)_t = (A, AC, BC)_t = (A, (A, B)C)_t = (A, (A, B)_t C)_t = (A, C)_t \Rightarrow A_t \supseteq C_t$.). Now as $(P_1^{a_1}...P_n^{a_n})_t \supseteq (AP_1^{r_1}...P_n^{r_n})_t$ and as $P_1^{a_1}...P_n^{a_n}$ and A share no maximal t-ideals, we have $(P_1^{a_1}...P_n^{a_n})_t \supseteq (P_1^{r_1}...P_n^{r_n})_t$. Now the number of proper t-divisors of $(P_1^{r_1}...P_n^{r_n})_t$ is less than $\Pi_{i=1}^n (r_i + 1)$ and hence finite. On the other hand if h is a super primitive divisor of f, then $hD[X] = (P_1^{a_1}...P_n^{a_n})_t$ by (2). Indeed if h is a super primitive divisor of f, then $f(X) = h(X)k(X)$. Or $(P_1^{r_1}...P_n^{r_n})_t = (P_1^{a_1}...P_n^{a_n})_t(k(X))$. Multiplying both sides by $(P_1^{-a_1}...P_n^{-a_n})$ and applying the t-operation, we get $(P_1^{r_1-a_1}...P_n^{r_n-a_n})_t = (k(X))$. On the other hand $(h(X)k(X)) = (h(X)k(X))_t$ because $(h(X)k(X))$ is principal. Consequently t-division acts like ordinary division in this case and so if n_{sf} denotes the number of non-associate super primitive divisors of f , then $n_{sf} < \Pi_{i=1}^n (r_i + 1) < \infty$. ∎

Call a nonzero element r in D primal if for all $x, y \in D\backslash\{0\}$, $r|xy$ implies $r = st$ where $s|x$ and $t|y$. Cohn [6] called an integrally closed integral domain D Schreier if each nonzero element of D is primal. A domain whose nonzero elements are primal was called pre-Schreier in [18]. Note that in a pre-Schreier domain every irreducible element (atom) is prime. (In fact a primal atom in any domain, is prime. For let p be an irreducible element that is also primal and let $p|ab$. So $p = rs$ where $r|a$ and $s|b$, because p is primal. But as p is also an atom, r is a unit or s is a unit. Whence $p|a$ or $p|b$.) An integral domain D is said to have the Irreducible Divisor Finite (IDF) property if every nonzero non unit of D is divisible by at most a finite number of non associated irreducible elements of D. Obviously, in a pre-Schreier domain the IDF property translates to: Every nonzero element is divisible by at most a finite number of non-associated primes. A Schreier domain has the PSP property, as a consequence of Lemma 2.1 of [19] and as in the proof of the aforementioned lemma the integrally closed property was not used one concludes that a pre-Schreier domain has the PSP property. Also it is well known that in a PSP domain, atoms are primes as well (cf [3]). Thus if D has the PSP property, the IDF property translates to: Every nonzero element is divisible by at most a finite number of non-associated primes. The point is, I will carry on with pre-Schreier and hope that the reader will draw conclusions about PSP domains.

Now if D is pre-Schreier and not Schreier, $D[X]$ is not pre-Schreier, see e.g. [18, Remark 4.6]. (It is well known that $D[X]$ being pre-Schreier if and only if $D[X]$ is Schreier.) So, some irreducible elements of $D[X]$ are not primes. However if f is an irreducible non-constant polynomial in $D[X]$ then f is primitive, i.e. the GCD of the coefficients of f is 1 and over a pre-Schreier domain a primitive polynomial is super-primitive, as we have already pointed out, meaning $(A_f)_v = D$. (As mentioned above [19], Lemma 2.1 was stated for Schreier domains but was proved using properties characterizing pre-Schreier domains only.) Now f being a non-constant polynomial, f must belong to an upper to zero P of $D[X]$ and because $(A_f)_v = D$ every upper to zero P, containing f, must be a maximal t-ideal [12, Theorem 1.4]. Thus, as mentioned above, if D is a PSP domain any prime upper to zero in $D[X]$ that contains an irreducible polynomial is a maximal t-ideal.

Next, verifying the IDF property entails checking that each nonzero polynomial $g \in D[X]$ is divisible by at most a finite number of irreducible divisors. If g is constant then all the divisors up to associates of g come from D alone and up to associates there are finitely many irreducible divisors for each constant g. So, let g be non-constant. Obviously each irreducible divisor of g that comes from D is a divisor of each of the coefficients of g and so g has only finitely many irreducible divisors coming from D.

According to Lemma 1, if $f(X) \in D[X]$ such that $(A_f)_v = D$, then $f(X)D[X] = (Q_1^{n_1}...Q_m^{n_m})_t$, where Q_i are prime uppers to zero. Now let's go back to $g(X)$, that we supposed was in n uppers to zero $P_1, ..., P_n$ that were maximal t-ideals and hence t-invertible. As we have seen in (1) of Lemma 1 $g(X)D[X] = (AP_1^{r_1}...P_n^{r_n})_t$ where $(A, P_1^{r_1}...P_n^{r_n})_t = D[X]$. If f is an irreducible (primitive) polynomial dividing g, then $(f) = (P_1^{a_1}...P_n^{a_n})_t$ where $0 \leq a_i \leq r_i$. (This is because if $(f) = (Q_1^{s_1}...Q_n^{s_n})_t$ and say $s_i > 0$ then $g(X)D[X] = (AP_1^{r_1}...P_n^{r_n})_t \subseteq (f) \subseteq Q_i$. Since A is contained in no uppers to zero, $P_1^{r_1}...P_n^{r_n} \subseteq Q_i$. Because P_j are mutually t-comaximal, exactly one of the P_j is contained in Q_i. But then for a fixed j, $P_j = Q_i$ and so each of the Q s is one of the P s.) Now because A does not share a maximal t-ideal with $P_1^{a_1}...P_n^{a_n}$ we have $P_1^{r_1}...P_n^{r_n} \subseteq (f)$. But there can only be a finite number of such irreducible polynomials, by (3) of Lemma 1. This leaves the case of when $g(X)$ is not contained in any maximal uppers to zero. In this case the only irreducible divisors are divisors of coefficients of g, coming from D.

Thus we have the following statement.

Theorem 2 *Let D be a domain such that for every primitive polynomial f over D we have $(A_f)_v = D$, where A_f denotes the content of f. If D is an IDF domain, then so is $D[X]$.*

The case of Schreier domains, i.e. integrally closed pre-Schreier domains, may be handled as follows: It is known that if D is Schreier then so is $D[X]$, according to [6]. So the non constant irreducible elements of $D[X]$ are prime and generators of uppers to zero containing them. Now D being IDF the constant irreducible divisors of a general non-constant $f \in D[X]$ come from D and so are finite, up to associates, and the non-constant irreducible divisors are finite, up to associates, because they are primes and hence generators of the uppers to zero containing them.

Recall that an integral domain D is said to be a Prufer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. Let's also recall from [17] the following result.

Proposition 3 *Let D be an integrally closed integral domain, let X be an indeterminate over D and let $S= \{f(X) \in D[X] | \ (A_f)_v = D\}$. Then D is a PVMD if and only if for any prime ideal P of $D[X]$ with $P \cap D = (0)$ we have $P \cap S \neq \phi$.*

In light of [12, Theorem 1.4] it has often been concluded that D is a PVMD if and only if D is integrally closed such that every upper to zero of $D[X]$ is a maximal t-ideal. In fact the above proposition and Theorem 2.6 of [11] led to the notion of a UMT domain. (A domain whose uppers to zero are maximal t-ideals.) It was stated in [12, Proposition 3.2] that D is a PVMD if and only if D is an integrally closed UMT domain.

Lemma 4 *Let B be a t-invertible t-ideal of $D[X]$ with $B \cap D = (0)$. Then $B = (A'P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t$ where P_i are the t-invertible prime uppers to 0 of $D[X]$ containing B and $(A', P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t = D$.*

Proof. $BK[X] = f(X)K[X]$. Since, being t-invertible, B is of finite type, there is $s \in K \backslash \{0\}$ such that $B \subseteq sfD[X]$. Or $B = (A_1 sf(X)))_t$ because B is t-invertible and so is $B/sf(X)$. Now sA_1 must intersect D because $BK[X] = fK[X]$. So the only uppers to zero that contain B must contain f. Adjusting s we can assume that $f \in D[X]$. So $B = (A_1 s)_t(f(X)) = (A_1 s(A_1 P_1^{r_1}...P_n^{r_n}))_t$ by Lemma 1. The rest is adjustments. (Alternatively let $P_1, ..., P_n$ be the maximal uppers to zero and note that $D[X]_{P_i}$ are rank one DVRs. So there is r_i that $B \subseteq (P_i^{r_i})_t$ and $B \nsubseteq (P_i^{r_i+1})_t$. Now as $(P_i^{r_i})_t$ are t-invertible, $B = (B_1 P_1^{r_1})_t$, repeating with $i = 2$ we have $B = (B_2 P_1^{r_1} P_i^{r_i})_t = ... = (B_n P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$. Set $B_n = A$. As $(BA^{-1})_t = (P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t \subseteq D[X]$ we have $A \subseteq D[X]$. As far as $(A, P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t = D[x]$ is concerned, it follows from the fact that A and $(P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t$ share no maximal t-ideals.) ■

Theorem 5 *An integral domain D is a PVMD if and only if for each non-constant polynomial $f(X)$ over D we have uppers to zero $P_1, ..., P_n$ such that $f(X)D[X] = (AP_1^{r_1}...P_n^{r_n})_t$ where $A = A_f[X]$.*

Proof. Let D be a PVMD and let f be a non-constant polynomial in $D[X]$. Then $fD[X] = (AP_1^{r_1}P_2^{r_2}...P_n^{r_n})_t$, where P_i are the maximal t-ideals containing $fD[X]$, by Lemma 1. Now in $K[X]$ we have $fK[X] = P_1^{r_1}P_2^{r_2}...P_n^{r_n}K[X] = P_1^{r_1}K[X] \cap P_2^{r_2}K[X] \cap ... \cap P_n^{r_n}K[X]$ because P_i are maximal ideals of $K[X]$. Next note that $P_i^{r_i}K[X] \cap D[X] = P_i^{r_i}K[X]_{P_i} \cap K[X] \cap D[X]$ and because $P_i \cap D = (0)$ we have $K[X]_{P_i} = D[X]_{P_i}$. Thus $P_i^{r_i}K[X]_{P_i} \cap K[X] \cap D[X] = P_i^{r_i}D[X]_{P_i} \cap D[X] = P_i^{(r_i)}$. But then $fK[X] \cap D[X] = P_1^{(r_1)} \cap ... \cap P_n^{(r_n)} = (P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t$ because P_i are mutually t-comaximal. On the other hand, on account of D being integrally closed, we have $fK[X] \cap D[X] = fA_f^{-1}[X]$ [16]. This gives $fA_f^{-1}[X] = (P_1^{r_1}P_2^{r_2}...P_n^{r_n})$. Multiplying both sides by A_f and applying the t-operation we get $fD[X] = (A_f P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t$. Conversely suppose that D is such that for each non-constant polynomial $f \in D[X]$ we have $fD[X] = (A_f P_1^{r_1}P_2^{r_2}...P_n^{r_n})_t$. Then, by construction, A_f is t-invertible. Since for every finitely generated nonzero ideal $A = (a_0, a_1, ..., a_m)$ we can construct a non-constant polynomial

$f = \sum_{i=0}^{m} a_i X^i$ such that $A_f = A$ we conclude that every finitely generated nonzero ideal of D is t-invertible. (Alternatively for each pair $a, b \in D\backslash\{0\}$ we have $f = a + bX$ which gives $(f(X)) = (A_f P)_t$, forcing $A_f = (a, b)$ to be t-invertible. But this is a necessary and sufficient condition for D to be a PVMD.) ■

Proposition 6 *An integrally closed domain D is a PVMD if and only if every linear non-constant polynomial over D is contained in a t-invertible upper to zero.*

Proof. If D is a PVMD, then of course as every upper to zero is a maximal t-ideal and hence t-invertible, every linear polynomial is contained in a t-invertible upper to zero. Conversely suppose that every non-constant linear polynomial $f = a + bX$ is contained in a t-invertible upper to zero. If $f(0) = 0$, then $f = bXD[X]$ and there is nothing to be gained from this. Yet if $f(0) \neq 0$ and f is contained in a t-invertible upper P, then $(f) = (AP)_t$. Where $fK[X] = PK[x]$ and so $fK[X] \cap D = f(X)A_f^{-1}[X] = P$. Since P is t-invertible, so must be $A_f^{-1}[X]$. multiplying both sides by A_f and taking the t-image we get $(f(X)) = (A_f[X]P)_t =$. Thus for every pair of nonzero elements a, b of D, (a, b) is t-invertible. This forces D to be a PVMD. ■

Proposition 7 *An integrally closed domain D is a PVMD if and only if every integral ideal A of $D[X]$ with $A \cap D = (0)$ is contained in a t-invertible upper to zero.*

Proof. If D is a PVMD then every upper to zero in $D[X]$ is t-invertible. Also if A is an ideal of $D[X]$ with $A \cap D = (0)$ then (because D is integrally closed) for some $s \in D\backslash\{0\}$ we have $sA = f(X)C$ for some polynomial $f \in D[X]$ and some integral ideal C with $C \cap D \neq (0)$ [2, Theorem 2.1]. Now as $fD[X]$ is contained in at least one upper to zero sA must be in an upper to zero. But s being a constant does not belong to any upper to zero. So A is contained in at least one upper to zero. Conversely let D be integrally closed and let $f(X)$ be a non-constant linear polynomial. Then $fA_f^{-1}[X] = P$, because D is integrally closed. Since P is t-invertible $A_f^{-1}[X]$ and hence A_f^{-1} is t-invertible and so is $(A_f\ )_v$. But then every two generated nonzero ideal of D is t-invertible. ■

By Proposition 3.2 of [12] D is a PVMD if and only if D is an integrally closed UMT domain. Let's drop the integrally closed part and see if we can get similar results.

Proposition 8 *Let D be an integral domain and X an indeterminate over D. Then D is a UMT domain if and only if for each t-invertible t-ideal A of $D[X]$ with $A \cap D = (0)$, A is contained in a t-invertible prime upper to zero.*

Proof. Since being a t-invertible t-ideal A is a v-ideal of finite type, we have $s \in D\backslash\{0\}$ such that $sA \subseteq fD[X]$ for some f where f is non-constant polynomial contained in A. (We have $A = (a_1, ..., a_n)_v K[X] = g(X)$. So $(s_{i1}/sa_{i2})a_i = g(X)$. Setting $s = \Pi s_{i2}$ and multiplying both sides by s we get $t_i a_i = sg(X) \in A$.

Now take $sg(X) = f(X)$ we can find $s = \Pi t_i$ such that $(sa_i) \subseteq f(X)$. Now $s(a_1, ..., a_n) \subseteq (f)$ and so $s(a_1, ..., a_n)_v \subseteq (f)$. But $s(a_1, ..., a_n)_v = sA$.). Now f, being a nonconstant polynomial, belongs to a prime upper to zero. If D is a UMT domain, then each prime upper to zero is t-invertible. Conversely let f be a non-constant polynomial in $D[X]$ and suppose that every t-invertible t-ideal A of $D[X]$ with $A \cap D = (0)$ is contained in a t-invertible prime upper to zero. Observe that $fD[X]$ is a t-invertible t-ideal and so, by the rule, must be contained in a t-invertible prime upper to zero say Q_1. So $fD[X] = (A_1Q_1)_t$ where $(A_1)_t$ is a t-invertible t-ideal. If $(A_1)_t \cap D \neq (0)$ we are done and if not we apply the rule again on $(A_1)_t$ to get $(A_1)_t = ((A_2)Q_2)_t$. or $fD[X] = (A_2Q_1Q_2)_t$. Continuting the recursive procedure we get at say stage $fD[X] = (A_rQ_1...Q_r)_t$ and note that as f is contained in only a finite number of uppers to zero and as $D[X]_{P_i}$ is a rank one DVR the process cannot run for ever and thus there'd be a stage r when $A_r \cap D \neq (0)$. Setting $A_r = A$ and renaming and regrouping we get $fD[X] = (AP_1^{r_1}...P_n^{r_n})_t$ where $A \cap D \neq (0)$. This accounts for all the prime uppers to zero containing f. Thus every prime upper containing f is a maximal t-ideal. Now let P be a prime upper to zero. Then for sone $h \in D[X]$ we have $P = hK[X] \cap D$. By the above procedure $hD[X] = (AQ)_t$ where Q is a t-invertible prime upper containing h. But then $P = hK[X] \cap D = AQK[X] \cap D = Q$, forcing the conclusion that $P = Q$ a maximal t-ideal. (This last line actually nails the proof. The earlier procedure is to indicate what goes on generally.)

Now here's something interesting! We know that a pre-Schreier PVMD is a GCD domain. What must a pre-Schreier UMT domain D be? The way I see it let $a, b \in D \backslash \{0\}$ and take $(aX + b)D[X]$. Because D is UMT $(aX + b)D[X] = (AP)_t$ where both A and P are and $A \cap D \neq (0)$. Now we know that if D is integrally closed and A is a t-invertible t-ideal of $D[X]$ with $A \cap D \neq (0)$, then $A = (A \cap D)[X]$ and obviously $A \cap D$ is a t-invertible t-ideal [2, Corollary 3.1]. But as the tone of [2, Corollary 3.1] indicates, the jury is still out on the converse. That is the authors of [2] did not know for sure if for every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have $A = (A \cap D)[X]$, then D should be integrally closed. That is we have this question.

Question Suppose that D is an integral domain such that for every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have $A = (A \cap D)[X]$. Must D be integrally closed?

The answer to the above question is yes and this is how we get it. Let's say that a domain D is $**$ if for every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have $A = (A \cap D)[X]$ and let's denote $(A \cap D)$ by $\mathcal{A}$. First let us note that if $\alpha \in K$ is integral over D then the fractional ideal $(1, \alpha)$ is invertible if and only if $\alpha \in D$, [15, Proposition 1.4] This leads to the following lemma.

Lemma 9 *Suppose that $\alpha \in K$ is integral over D. If the fractional ideal $(1, \alpha)$ is t-invertible, then $\alpha \in D$.*

Proof. Suppose that $\alpha \in K$ is integral over D. Then α satisfies a monic polynomial $f = X^n + a_{n-1}X^{n-1} + ... + a_0$. Since $a_i = (a_i/s_i)s_i$ for s_i in any multiplicative set S, f can serve as a monic polynomial over D_S. Thus α being integral over D implies that α is integral over D_S. Consequently α is integral over D_P each maximal t-ideal P. Now recall the easy to prove fact that a finitely generated nonzero ideal I is t-invertible if and only if ID_P is principal for each maximal t-ideal P of D. (We say that I is t-locally principal.) Thus if α is integral over D and if P that is a maximal t-ideal of D then $\alpha \in D_P$ because α is integral over D_P and $(1,\alpha)D_P$ is principal and hence invertible. Thus $\alpha \in D_P$ for each maximal t-ideal P. But then $\alpha \in D = \cap D_P$.

Proposition 10 *Let D be an integral domain. Then D is integrally closed if and only if D is $**$.*

Proof. If D is integrally closed, then D is $**$ by [2, Corollary 3.1]. Conversely, suppose that $\alpha = \frac{b}{a}$, where $a, b \in D\backslash\{0\}$, is integral over D. Then α satisfies a monic polynomial f. Now f splits as $(X+\alpha)g(X)$ in $K[X]$. Being linear, $(X+\alpha)$ is a prime in $K[X]$. Thus $P = (X+\alpha)K[X] \cap D[X]$ is a prime upper to zero. Obviously $f \in P$ and so P is t-invertible. Also $a(X+\alpha)D[X] = (aX+b)D[X] \subseteq P$. Since P is a t-invertible ideal we have $(aX+b)D[X] = (AP)_t$, where P and A are t-invertible. As $(aX+b)$ is linear $A \cap D \neq (0)$. Now D being $**$ forces $A = \mathcal{A}[X]$. So $(aX+b)D[X] = (AP)_t = (\mathcal{A}[X]P)_t \subseteq \mathcal{A}[X]$, forcing $aX+b$ and thus $a, b \in \mathcal{A}[X]$. Now as $(a,b)[X] \subseteq \mathcal{A}[X]$, and as $A = \mathcal{A}[X]$ is t-invertible we have $(a,b)[X](\mathcal{A}[X])^{-1} \subseteq D[X]$. On the other hand $(\mathcal{A}[X]P)_t = (aX+b)D[X] \subseteq (a,b)[X]$. Thus $(\mathcal{A}[X]P)_t \subseteq (a,b)[X]$ and so $P \subseteq ((a,b)[X](\mathcal{A}[X])^{-1})_t \subseteq D[X]$. Or $P \subseteq ((a,b)\mathcal{A}^{-1})_t[X] \subseteq D[X]$. Since P contains f with $A_f = D$ we have $(f,a) \subseteq ((a,b)\mathcal{A}^{-1})_t[X] \subseteq D[X]$. This forces $(((a,b)\mathcal{A}^{-1})[X])_t = ((a,b)\mathcal{A}^{-1})_t[X] = D[X]$, because $(f,a)_t = D[X]$ (see [7, Proposition 3.4]). Thus $(((a,b)\mathcal{A}^{-1})_t = D$ and so (a,b) is t-invertible. But this means $(1, \frac{b}{a})$ is t-invertible. Now as $\alpha = \frac{b}{a}$ is integral over D and as $(1, \frac{b}{a})$ is t-invertible we conclude, by Lemma 9, that $\alpha = \frac{b}{a} \in D$.

Now [2, Corollary 3.1] can be recovered as the following statement.

Corollary M. Let D be an integral domain. Then the following are equivalent.

(1) D is integrally closed,

(2) For every t-ideal A of $D[X]$ with $A \cap D \neq (0)$, $A = (A \cap D)[X]$,

(3) For every divisorial ideal A of $D[X]$ with $A \cap D \neq (0)$, $A = (A \cap D)[X]$,

(4) For every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$, $A = (A \cap D)[X]$.

Proof. (1) $\Rightarrow$ (2) follows from [2, Corollary 3.1], (2) $\Rightarrow$ (3) because every divisorial ideal is a t-ideal and (3) $\Rightarrow$ (4) because every t-invertible t-ideal is divisorial. Finally (4) $\Rightarrow$ (1) is Proposition 10.

Notes.

(1). On the idf front the following Q/A often goes unnoticed:

Let D be an the idf domain, let L be a field extension of $K = qf(D)$ and let X be an indeterminate over L. Under what conditions must $D + XL[X]$ (resp., $D + XL[[X]]$) be an the idf domain?

Answer: Not generally, yet if D has only finitely many, or no, irreducible elements. Thus if D is a Cohen-Kaplansky or an antimatter domain with quotient field $K \neq D$, then $D + XK[X]$ is an the idf domain.

Suppose that D has only finitely many or no atoms and $D \neq K$. Let $f \in D + XL[X]$. If $f(0) = d \neq 0$, $f = d(1 + Xg(X))$ where $d \in D$ and $1 + Xg(X)$ is a product of primes. As D has only a finite number of atoms, d is divisible by only finitely many atoms and so f is divisible by only finitely many atoms. If $f(0) = 0$, $f = (X^r/s)(s + Xg(X))$. Notice that if $D \neq K$, X is not irreducible. so the only atomic factors of f in this case are atoms of D or primes of the form $1 + Xh(X)$. Next if $D + XL[X]$ is an the idf domain and $D \neq K$, then D must have only finitely many atoms or no atoms because each of X/s, for $s \in D \backslash \{0\}$, is divisible by all the nonzero elements of D.

However if $D = K$, $K + XL[X]$ is the idf if and only if $|L^*/K^*| < \infty$. For if $K + XL[X]$ is the idf and $f(0) \neq (0)$ we have $f = (lX^r)(1 + Xg(X))$ and elements of the form lX are irreducible. Yet if $|L^*/K^*| < \infty$ there are only finitely many non-associate atoms l_iX, where l_i represents the coset l_i/K^*. Of course elements of the form $1 + Xg(X)$ are products of primes. If on the other hand $|L^*/K^*| = \infty$ we have infinitely many lX that are not associated to each other. so if $r > 1$, $(lX^r)(1 + Xg(X))$ has infinitely many non-associate irreducible factors.

(2). Also goes missing a mention of a PSP the idf domain that is Prufer due to Loper [13]. If we call a PSP domain described in [13] a loper domain, then $D[X]$ is a PVMD that is not a PSP domain (use Corollaries 3.5 and 3.6 of [4]).

(3). It appears, no one has considered "strongly" idf domains: Every nonzero non unit is divisible by at least one and at most a finite number of irreducible elements, up to associates. Examples abound.

(4). The question of why $D[X]$ is Schreier when it is pre-Schreier, has baffled quite a few people. A somewhat convoluted proof was provided in [1, Corollary 7]. Given below is a direct proof, in the hope that you can convert it into a result on monoid algebras.

Let R be an integral domain, with quotient field K. Cohn [6] called an element x of R primal if for all $y, z \in R$, $x|yz$ implies $x = rs$, where $r|y$ and $s|z$. He called an integrally closed integral domain whose elements were all primal a Schreier ring and proved that if R is a Schreier ring and X an indeterminate over R, then so is $R[X]$. Later McAdam and Rush [14] proved that if every element of $R[X]$ is primal then, R is Schreier (Theorem 3 of [14]).

Theorem 11 *Let R be an integral and let X be an indeterimate over R. If every element of R is primal in the polynomial ring $R[X]$, then R is a Schreier ring.*

In the course of his study of Bezout rings and their subrings, P.M. Cohn [6] introduced the notion of a primal element in the manner already mentioned. Also, he called an element $c \in R$ completely primal if all factors of c are primal. He then proved that in an integral domain any product of (completely) primal elements is (completely) primal. From this it is clear that if S is generated by completely primal elements then the saturation $\overline{S}$ of S consists of completely primal elements. He then goes on to state what may be called "Nagata like theorem".

Theorem 12 *[cf. [6] Theorem 2.6]. Let R be an integrally closed integral domain and S a multiplicative subset of R. Then (i) if R is a Schreier ring, so is R_S, (ii) (Nagata like theorem) if R_S is a Schreier ring and S is generated by completely primal elements of R, then R is a Schreier ring.*

Using his Nagata type theorem he goes on to prove the following result as Theorem 2.7 of [6].

Theorem 13 *Let R be an integral domain and X an indterminate over R. If R is a Schreier ring then so is $R[X]$.*

In his proof he noted that since R is integrally closed, so is $R[X]$. He then shows that elements of R are primal in $R[X]$ and then uses his Nagata like theorem in the following way: Since $S = R\backslash\{0\}$ consists of completely primal elements of $R[X]$ and since $R[X]_S = K[X]$ is a Schreier ring, $R[X]$ is a Schreier ring. Our Theorem 11 says that if we must assume or prove that elements of R are primal in $R[X]$, then the integrally closed assumption is unnecessary.
Proof. (Proof of Theorem 11) Note that every element of R being primal in $R[X]$ entails every element of R being primal in R. For when y, z are in R and $x|yz$, the elements y, z are in $R[X]$ as well. So $x|yz$ implies $x = r(X)s(X)$ where $r(X)|y$ and $s(X)|z$ and the degree considerations put $r(X)$ and $s(X)$ in R. So all we are needing to show is that R is integrally closed. For this let $\alpha = \frac{a}{b}$ be integral over R, where $a, b \in R\backslash\{0\}$. Then α satisfies a monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + ... + a_0$. ■

Now, in $K[X]$, we have $f(X) = (X - \frac{a}{b})g(X)$. Let $d \in R$ such that $dg(X) \in R[X]$. Then, $bdf(X) = (bX - a)dg(X)$ where the expressions on both sides are in $R[X]$. Since b is primal in $R[X]$, we have $b = pq$ where $p|(bX - a)$ in $R[X]$ and $q|dg(X)$, in $R[X]$. But this means that $df = \frac{(bX-a)}{p}\frac{dg(X)}{q}$, where $\frac{(bX-a)}{p} = \frac{b}{p}X - \frac{a}{p}$, $\frac{dg(X)}{q} = h(X) \in R[X]$.

Next note that in $df = (\frac{b}{p}X - \frac{a}{p})h(X)$, d is primal in $R[X]$ and so $d = rs$ where $r|_{R[X]}(\frac{b}{p}X - \frac{a}{p})$ and $s|_{R[X]}h(X)$. But then $f = (\frac{b}{pr}X - \frac{a}{pr})h_1(X)$, where all the expressions involved are in $R[X]$.

Now f is monic and the expressions on the right are in $R[X]$. So the leading coefficients of $(\frac{b}{pr}X - \frac{a}{pr})$ and $h_1(X)$ must be units. Thus b is an associate of pr, making r an associate of q and thus proving that $\frac{a}{pr}$ is an associate of $\frac{a}{pq} = \frac{a}{b}$. But $\frac{a}{pr} \in R$, which leads to the conclusion that $\alpha = \frac{a}{b} \in R$.

Corollary 14 *Let R be an integral domain and X an indeterminate over R. Then the following are equivalent for R. (1) Every element of $R[X]$ is primal in $R[X]$, (2) Every element of R is primal in $R[X]$ (3) R is a Schreier ring.*

Proof. (1) $\Rightarrow$ (2) is obvious, (2) $\Rightarrow$ (3) follows from Theorem 11 and (3) $\Rightarrow$ (1) is Theorem 13. ■

We note that the above results hinge on the fact that $K[X]$ is a Schreier ring. This allows us to extend the above results to a generalization, called a monoid ring, of polynomial rings, in a limited way. To see that we prepare as follows. Let $S =< S, +, 0 >$ be a commutative monoid and let R be a ring. The monoid ring of S over R, denoted by $R[X; S]$ or $R[S]$, is the set of finite sums of the form $\sum a_s X^s$, where $s \in S$ and $a_s \in R \backslash \{0\}$, with addition and multiplication defined as for polynomials. According to Theorem 8.1 of [9] $R[X; S]$ is an integral domain if and only if R is an integral domain and S is torsion free and cancellative. Here S is torsion free if $ms = ns$ implies $m = n$ for any $m, n \in N$ and any $s \in S$.

References

[1] D.D. Anderson, T. Dumitrescu and M. Zafrullah, Quasi-Schreier domains II, Comm. Algebra 35 (2007), 2096-2104

[2] D.D. Anderson, D.J. Kwak and M. Zafrullah, On agreeable domains, Comm. Algebra 23 (13) (1995), 4861-4883.

[3] D.D. Anderson and M. Zafrullah, The Schreier Property and Gauss' Lemma, Bollettino U. M. I.(8) 10-B (2007), 43-62

[4] J. T. Arnold and P.B Sheldon, Integral domains that satisfy Gauss Lemma, Michigan Math. J. 22 (1975) 39-51.

[5] G.W. Chang, T. Dumitrescu and M. Zafrullah, t-Splitting sets in integral domains, J. Pure Appl. Algebra 187 (2004) 71–86.

[6] P. Cohn, Bezout rings and their subrings, Proc. Cambridge Phil. Soc.64 (1968), 251-264.

[7] S. Gabelli, On divisorial ideals in polynomial rings over Mori domains, Comm. Algebra 15(11)(1987) 2349-2370.

[8] R. Gilmer, Multiplicative Ideal Theory, Marcel-Dekker, New York, 1972.

[9] R. Gilmer, Commutative Semigroup Rings, The University of Chicago Press, 1984.

[10] R. Gilmer, J. Mott and M. Zafrullah, On t-invertibility and comparability, Commutative Ring Theory (eds. P.-J. Cahen, D. Costa, M. Fontana and S.-E. Kabbaj), Marcel Dekker, New York, 1994, 141-150.

[11] E. Houston, S. Malik and J. Mott, Characterizations of $*$-multiplication domains, Canad. Math. Bull. 27(1)(1984) 48-52.

[12] E. Houston and M. Zafrullah,"On t-invertibility II" Cornm. Algebra 17(8)(1989) 1955-1969.

[13] K. Alan Loper, Two Prufer domain counterexamples, J. Algebra 221 (1999), 630–643.

[14] S. McAdam and D. E. Rush, Schreier rings, Bull. London Math. Soc, 10 (1978), 77-80.

[15] J. Mott, B. Nashier and M. Zafrullah, Contents of polynomials and invertibility. Comm. Algebra18 (1990), 1569–1583.

[16] J. Querre, Ideaux divisoriels d'un anneau de polynomes, J. Algebra 64 (1980) 270-284.

[17] M. Zafrullah, Some polynomial characterizations of Prüfer v-multiplication domains, J. Pure Appl. Algebra 32 (1984), 231-237.

[18] M. Zafrullah, On a property of pre-Schreier domains, Comm. Algebra 15 (1987), 1895-1920.

[19] M. Zafrullah, "Well behaved prime t-ideals" J . Pure Appl. Algebra 65(1990) 199-207.