INTEGRAL DOMAINS AND THE IDF-PROPERTY

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Dedeicated to the memory of Robert Gilmer,
the most influential figure in Multiplicative Ideal Theory in recent past.

Abstract. TODO...

1. Introduction

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2. Background

2.1. **General Notation.** Following common notation, we let $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, denote the sets of integers, rational numbers, and real numbers, respectively. In addition, we let $\mathbb{P}$, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of primes, positive integers, and nonnegative integers, respectively. For $a, b \in \mathbb{Z}$, we let $[\![a, b]\!]$ denote the discrete interval $\{n \in \mathbb{Z} \mid a \le n \le b\}$, allowing $[\![a, b]\!]$ to be empty when $a > b$. In addition, given $S \subseteq \mathbb{R}$ and $r \in \mathbb{R}$, we set $S_{\ge r} = \{s \in S \mid s \ge r\}$ and $S_{>r} = \{s \in S \mid s > r\}$. For $q \in \mathbb{Q} \setminus \{0\}$, we let $\mathsf{n}(q)$ and $\mathsf{d}(q)$ denote, respectively, the unique $n \in \mathbb{N}$ and $d \in \mathbb{Z}$ such that $q = n/d$ and $\gcd(n, d) = 1$. Accordingly, for any $Q \subseteq \mathbb{Q} \setminus \{0\}$, we set $\mathsf{n}(Q) = \{\mathsf{n}(q) \mid q \in Q\}$ and $\mathsf{d}(Q) = \{\mathsf{d}(q) \mid q \in Q\}$. Finally, for each $p \in \mathbb{P}$ and $n \in \mathbb{Z} \setminus \{0\}$, we let $v_p(n)$ denote the maximum $v \in \mathbb{N}_0$ such that p^v divides n, and for $q \in \mathbb{Q} \setminus \{0\}$, we set $v_p(q) = v_p(\mathsf{n}(q)) - v_p(\mathsf{d}(q))$ (in other words, v_p is the p-adic valuation map of $\mathbb{Q}$ restricted to nonzero rationals).

2.2. **Monoids.** In the scope of this paper, a *monoid* is a triple $(M, *, e)$, where M is a set, $*$ is an associative, cancellative, and commutative operation on M, and e is the identity element, that $e * x = x * e = x$ for all $x \in M$. When there seems to be no risk of ambiguity, we write M instead of $(M, *, e)$ (we shall have the occasion to study monoids for which additive notation is either standard or more convenient). For instance, we will use multiplicative notation when we deal with the multiplicative monoid of an integral domain. Let M be a monoid. We let $M^\bullet$ denote the set of nonzero elements. An element $b \in M$ is called *invertible* if there exists $c \in M$ such that $b * c = e$. We let $\mathscr{U}(M)$ denote the group of invertible elements of M. Observe that when M is the multiplicative monoid of an integral domain D, the invertible elements of M are precisely the units of D, in which case, we will use the standard notation $D^\times$ rather than $\mathscr{U}(M)$. We let M_{red} denote the quotient monoid $M/\mathscr{U}(M)$. The monoid M is called *reduced* if $\mathscr{U}(M)$ is the trivial group, in which case, M is naturally isomorphic to M_{red}. The *difference group* of M, denoted by $\text{gp}(M)$, is the unique abelian group up to isomorphism satisfying that any abelian group containing a homomorphic image of M will also contain a homomorphic image of $\text{gp}(M)$. The monoid M is *torsion-free* if $\text{gp}(M)$ is a torsion-free group (or equivalently, if for all $b, c \in M$, if $b^n = c^n$ for some $n \in \mathbb{N}$, then $b = c$).

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For a subset S of M, we let $\langle S \rangle$ denote the submonoid of M generated by S, that is, the smallest (under inclusion) submonoid of M containing S. An *ideal* of M is a subset I of M such that $I * M \subseteq I$ (or, equivalently, $I * M = I$). An ideal of M is *principal* if there exists $b \in M$ satisfying $I = b * M$. For $b_1, b_2 \in M$, we say that b_2 *divides* b_1 in M if $b_1 * M \subseteq b_2 * M$, in which case we write $b_2 \mid_M b_1$, and we say that b_1 and b_2 are *associates* if $b_1 * M = b_2 * M$. The monoid M is a *valuation monoid* if for any $b_1, b_2 \in M$ either $b_1 \mid_M b_2$ or $b_2 \mid_M b_1$. We say that M satisfies the *ascending chain condition on principal ideals* (*ACCP*) if every increasing sequence (under inclusion) of principal ideals eventually terminates. An element $a \in M \setminus \mathscr{U}(M)$ is an *atom* (or an *irreducible*) if whenever $a = u * v$ for some $u, v \in M$, then either $u \in \mathscr{U}(M)$ or $v \in \mathscr{U}(M)$. We let $\mathscr{A}(M)$ denote the set of atoms of M. The monoid M is *atomic* if every non-invertible element factors into atoms or irreducible elements. One can check that every monoid satisfying the ACCP is atomic.

2.3. **Factorizations.** Observe that the monoid M is atomic if and only if M_{red} is atomic. Let $\mathsf{Z}(M)$ denote the free (commutative) monoid on $\mathscr{A}(M_{\text{red}})$, and let $\pi\colon \mathsf{Z}(M) \to M_{\text{red}}$ be the unique monoid homomorphism fixing the set $\mathscr{A}(M_{\text{red}})$. For every $b \in M$, we set

$$\mathsf{Z}(b) = \mathsf{Z}_M(b) = \pi^{-1}(b * \mathscr{U}(M)).$$

Observe that M is atomic if and only if $\mathsf{Z}(b)$ is nonempty for any $b \in M$. The monoid M is called a *finite factorization monoid* (*FFM*) if it is atomic and $|\mathsf{Z}(b)| < \infty$ for every $b \in M$. In addition, M is called a *unique factorization monoid* (*UFM*) if $|\mathsf{Z}(b)| = 1$ for every $b \in M$. By definition, every UFM is an FFM. If $z = a_1 * \cdots * a_\ell \in \mathsf{Z}(M)$ for some $a_1, \dots, a_\ell \in \mathscr{A}(M_{\text{red}})$, then ℓ is called the *length* of z and is denoted by $|z|$. For each $b \in M$, we set

$$\mathsf{L}(b) = \mathsf{L}_M(b) = \{|z| \mid z \in \mathsf{Z}(b)\}.$$

The monoid M is called a *bounded factorization monoid* (*BFM*) if it is atomic and $|\mathsf{L}(b)| < \infty$ for all $b \in M$. Observe that if M is an FFM, then it is also a BFM. On the other hand, the reader can verify that every BFM satisfies the ACCP.

Let D be an integral domain. The set consisting of all nonzero elements of D is a monoid, which is denoted by D^* and called the *multiplicative monoid* of D. Every factorization property defined for monoids in the previous paragraph can be rephrased for integral domains. We say that D is a *unique* (resp., *finite*, *bounded*) *factorization domain* provided that D^* is a unique (resp., finite, bounded) factorization monoid, respectively. Accordingly, we use the acronyms UFD, FFD, and BFD. Observe that this new definition of a UFD coincides with the standard definition of a UFD. In order to simplify notation, we write $\mathsf{Z}(D) = \mathsf{Z}(D^*)$, and for every $x \in D^*$, we write $\mathsf{Z}(x) = \mathsf{Z}_{D^*}(x)$ and $\mathsf{Z}(x) = \mathsf{Z}_{D^*}(x)$. As for monoids, we let $\mathscr{A}(D)$ denote the set of atoms/irreducibles of D.

2.4. **Integral Domains and Monoid Domains.** Let D be an integral domain, and let M be a torsion-free monoid. Following R. Gilmer [17], we let $D[M]$ denote the monoid ring of M over D, that is, the ring consisting of all polynomial expressions with exponents in M and coefficients in D. It follows from [17, Theorem 8.1] that $D[M]$ is an integral domain (as M is torsion-free and, by definition, cancellative). Accordingly, we often call $D[M]$ a monoid domain. In addition, it follows from [17, Theorem 11.1] that $D[M]^\times = \{rx^u \mid r \in D^\times \text{ and } u \in \mathscr{U}(M)\}$. In light of [17, Corollary 3.4], we can assume that M is a totally ordered monoid. Let $f(x) = c_n x^{q_n} + \cdots + c_1 x^{q_1}$ be a nonzero element in $D[M]$ for some coefficients $c_1, \dots, c_n \in D^*$ and exponents $q_1, \dots, q_n \in M$ satisfying $q_n > \cdots > q_1$. Then we call $\deg f = \deg_{D[M]} f := q_n$ and $\operatorname{ord} f = \operatorname{ord}_{D[M]} f := q_1$ the *degree* and the *order* of f, respectively. In addition, we call the set $\operatorname{supp} f := \operatorname{supp}_{D[M]}(f(x)) := \{q_1, \dots, q_n\}$ the *support* of f.

2.5. **t-ideals.** Let D be an integral domain with quotient field K, and let $F(D)$ denote the set of fractional ideals of D. For a fractional ideal A of D, let A^{-1} denote the fractional ideal

$$(D :_K A) = \{x \in K \mid xA \subseteq D\}.$$

The function $A \mapsto A_v := (A^{-1})^{-1}$ on $F(D)$ is called the *v-operation* on D (or on $F(D)$). Associated to the v-operation is the t-operation on $F(D)$: for each fractional ideal A of D,

$$A \mapsto A_t := \bigcup \big\{H_v \mid H \in F(D) \text{ is a finitely generated and } H \subseteq A\big\}.$$

For every fraction ideal A of D, it is clear that $A \subseteq A_t \subseteq A_v$. The v-operation and the t-operation are examples of the so called star operations (see [18, Sections 32 and 34] or [14, Chapter 1]). A fractional ideal $A \in F(D)$ is called a *v-ideal* (resp., a *t-ideal*) if $A = A_v$ (resp., $A = A_t$). An integral t-ideal maximal among integral t-ideals is a prime ideal called a *maximal t-ideal*. If A is a nonzero integral ideal with $A_t \neq D$, then A is contained in at least one maximal t-ideal. A prime ideal that is also a t-ideal is called a *prime t-ideal*. We say that $A \in F(D)$ is *v-invertible* (resp., *t-invertible*) if $(AA^{-1})_v = D$ (resp., $(AA^{-1})_t = D$). A prime t-ideal that is also t-invertible is a maximal t-ideal [23, Proposition 1.3]. Two fractional ideals $A, B \in F(D)$ are called *t-comaximal* if $(A, B)_t = D$, where (A, B) denotes the fractional ideal in $F(D)$ generated by A and B. The integral domain D is called a *Prufer v-multiplication domain* (or a *PVMD*) if every nonzero finitely generated ideal of D is t-invertible. It is well known that the class of PVMDs contains relevant classes of integral domains, including GCD-domains, Krull domains, and Prüfer domains.

Some generalizations of GCD-domains will show up in coming sections. The *content* A_f of a polynomial $h \in K[X]$ is the fractional ideal of D generated by the coefficients of f. Let f be a polynomial in $D[X]$. Then f is called *primitive* (*over* D) provided that $A_f = D$, that is, no nonunit of D divides all coefficients of f. On the other hand, f is called *super-primitive* (*over* D) if $(A_f)_v = D$, that is, $A_f^{-1} = D$. By [30, Theorem C], every super-primitive polynomial is primitive. Following Arnold and Sheldon [7], we say that D is a *PSP-domain* if every primitive polynomial over D is super-primitive. Every GCD-domain is a PSP-domain [30, Theorem H]. Finally, D is called an *AP-domain* if every irreducible is prime in D. It follows from [30, Theorem F] and [7, Proposition 4.1] that every PSP-domain is an AP-domain.

2.6. **Irreducible-Divisor-Finite Domains.** As mentioned in the introduction, the primary purpose of this paper is to provide fundamental results on integral domains where each nonzero element has only finitely many irreducible divisors up to associates. An integral domain with this property is called an *irreducible-divisor-finite domain*, or an *IDF-domain* for short. These integral domains were introduced and first investigated by A. Grams and H. Warner in [21]. Following P. Clark [11], we say that an integral domain D is a *Furstenberg domain* if every nonunit element of D is divisible by an irreducible. Now we introduce the following definition, which is central in this paper.

Definition 2.1. We say that an integral domain is a *tight irreducible-divisor-finite domain* or a *TIDF-domain* if it is a Furstenberg IDF-domain.

As not every antimatter domain is a field, not every IDF-domain is Furstenberg. Therefore TIDF-domains form a special proper subclass of that of IDF-domains. One advantage that TIDF-domains have over IDF-domains is that the former can be linked to the study of *strongly* TIDF-domains, those with each proper nonzero ideal contained in at least one and at most finitely many non-associate irreducible elements. Though of course that does not make the domain manageable, even though every atom in such domains would have to be a prime. Example 4.3 again proves to be a killjoy example. However other examples may be constructed. Yet if we throw in conditions like (b)-(d) of Corollary 4.4, we end up with a PID.

3. t-Invertible Upper to Zero Ideals

Assume throughout this section that D is an integral domain with quotient field K. A prime ideal P of $D[X]$ is called a *prime upper to zero* if $P \cap D = (0)$. Thus, a prime ideal P of $D[X]$ is a prime upper to zero if and only if $P = h(X)K[X] \cap D[X]$ for some irreducible/prime polynomial $h \in K[X]$.

3.1. **Ascent of the IDF Property on PSP-domains.** The primary purpose of this section is to show that the IDF property ascends on PSP-domains, that is, if a PSP-domain D is an IDF-domain, then the polynomial ring $D[X]$ is also an IDF-domain. In doing so, prime upper to zero ideals play some role as combinatorial tools. We also show how they can be used to characterize UMT domains (i.e., integral domains whose uppers to zero are all maximal t-ideals). Recall that a PSP-domain is an integral domain D satisfying that every primitive polynomial f over D is super primitive, that is, $(A_f)_v = D$.

It follows from [23, Theorem 1.4] that a prime upper to zero P of D is a maximal t-ideal if and only if P is t-invertible, which happens precisely when P contains a polynomial f super-primitive over D, that is, $(A_f)_v = D$. Based on this, it was concluded in [20] that if f is a super-primitive polynomial over D, then $f(X)D[X]$ is a t-product of uppers to zero. Using the same fact, the authors in [20] also concluded that $f(X)D[X]$ is a t-product of maximal t-ideals. An element $a \in D$ is called a *t-invertibility element* if every ideal of D containing a is t-invertible. It was proved in [20, Theorem 1.3] that $a \in D$ is a t-invertibility element if and only if aD is a t-product of maximal t-ideals of D. The following result makes the above conclusion somewhat more obvious. Yet, before we state the following lemma, we note that every non-constant polynomial in $D[X]$ belongs to at most finitely many prime uppers to zero ideals, some of which may be t-invertible.

Lemma 3.1 (Upper-to-zero Representation Lemma)**.** *Let $f \in D[X]$ be a non-constant polynomial and let $P_1, \dots, P_n$ be the only prime uppers to zero ideals containing f that are maximal t-ideals. Then the following statements hold.*

1. *(1) $f(X)D[X] = (AP_1^{r_1} \cdots P_n^{r_n})_t$ for some $r_1, \dots, r_n \in \mathbb{N}$ and $A \in F(D)$ such that A and $P_1^{r_1} \cdots P_n^{r_n}$ are t-comaximal.*
2. *(2) If f is super-primitive, then $f(X)D[X] = (P_1^{r_1} \cdots P_n^{r_n})_t$.*
3. *(3) Any non-constant polynomial $f \in D[X]$ has at most finitely many super-primitive divisors.*

Proof. (1) The proof can be taken from the proof of Proposition 3.7 of [10].

(2) Let A be as in part (1). Assume, by way of contradiction, that $A_t \neq D$. Then A is contained in a maximal t-ideal M. In this case, $f \in M$ and the fact that f is super-primitive ensures that M is t-invertible. Now the fact that the only t-invertible maximal t-ideals containing f are $P_1, \dots, P_n$ implies that $M \cap D \neq (0)$. However, the inclusion $f \in A \subseteq M$ implies that $D \subseteq M$, which is a contradiction. Thus, $A_t = D$, and so

$$f(X)D[X] = (AP_1^{r_1} \cdots P_n^{r_n})_t = (A_tP_1^{r_1} \cdots P_n^{r_n})_t = (P_1^{r_1} \cdots P_n^{r_n})_t.$$

(3) Let us call an ideal I a *t-divisor* of an ideal A if there is an ideal B such that $A = (BI)_t$. Write $f(X)D[X] = (AP_1^{r_1} \cdots P_n^{r_n})_t$ as in part (1). Then proper ideals $P_1^{a_1} \cdots P_n^{a_n}$, where $0 \le a_i \le r_n$, are t-divisors of $f(X)D[X]$ and they only t-divide $P_1^{r_1} \cdots P_n^{r_n}$. Indeed, if A, B, C are ideals of D such that $(A, B)_t = D$ and $A_t \supseteq (BC)_t$, then $A_t \supseteq C_t$: this is because $A_t \supseteq (BC)_t$ if and only if

$$A_t = (A, BC)_t = (A, AC, BC)_t = (A, (A, B)C)_t = (A, (A, B)_tC)_t = (A, C)_t,$$

which implies that $A_t \supseteq C_t$. Observe that the inclusion $(P_1^{a_1} \cdots P_n^{a_n})_t \supseteq (AP_1^{r_1} \cdots P_n^{r_n})_t$, along with the fact that A and $P_1^{a_1} \cdots P_n^{a_n}$ share no maximal t-ideals, guarantees that $(P_1^{a_1} \cdots P_n^{a_n})_t \supseteq (P_1^{r_1} \cdots P_n^{r_n})_t$. Now the number of proper t-divisors of $(P_1^{r_1} \cdots P_n^{r_n})_t$ is less than $\Pi_{i=1}^n (r_i + 1)$ and hence finite. Let h be a super-primitive divisor of f in $D[X]$, and write $f(X) = g(X)h(X)$. It follows from part (2) that

$h(X)D[X] = (P_1^{a_1} \cdots P_n^{a_n})_t$ and, therefore, $(P_1^{r_1} \cdots P_n^{r_n})_t = (P_1^{a_1} \cdots P_n^{a_n})_t(g(X))$. Multiplying both sides by $(P_1^{-a_1} \cdots P_n^{-a_n})$ and applying the t-operation, we obtain $(P_1^{r_1-a_1} \cdots P_n^{r_n-a_n})_t = (g(X))$. On the other hand, $(g(X)h(X)) = (g(X)h(X))_t$ because $(g(X)h(X))$ is a principal ideal. Consequently, t-division acts like ordinary division in this case, and so if n_{sf} denotes the number of non-associate super-primitive divisors of f, then $n_{sf} < \Pi_{i=1}^n(r_i + 1) < \infty$. □

We are in a position to prove that the IDF property ascends from D to $D[X]$ when D is a PSP-domain.

Theorem 3.2. *Let D be a PSP-domain. If D is an IDF-domain, then $D[X]$ is also an IDF-domain.*

Proof. Next, verifying the IDF property entails checking that each nonzero polynomial $g \in D[X]$ is divisible by at most a finite number of irreducible divisors. If g is constant then all the divisors up to associates of g come from D alone and up to associates there are finitely many irreducible divisors for each constant g. So, let g be non-constant. Obviously each irreducible divisor of g that comes from D is a divisor of each of the coefficients of g and so g has only finitely many irreducible divisors coming from D.

According to Lemma 3.1, if $f(X) \in D[X]$ such that $(A_f)_v = D$, then $f(X)D[X] = (Q_1^{n_1} \cdots Q_m^{n_m})_t$, where Q_i are prime uppers to zero. Now let's go back to $g(X)$, that we supposed was in n uppers to zero $P_1, \ldots, P_n$ that were maximal t-ideals and hence t-invertible. As we have seen in (1) of Lemma 3.1 $g(X)D[X] = (AP_1^{r_1} \cdots P_n^{r_n})_t$ where $(A, P_1^{r_1} \cdots P_n^{r_n})_t = D[X]$. If f is an irreducible (primitive) polynomial dividing g, then $(f) = (P_1^{a_1} \cdots P_n^{a_n})_t$ where $0 \le a_i \le r_i$. (This is because if $(f) = (Q_1^{s_1} \cdots Q_n^{s_n})_t$ and say $s_i > 0$ then $g(X)D[X] = (AP_1^{r_1} \cdots P_n^{r_n})_t \subseteq (f) \subseteq Q_i$. Since A is contained in no uppers to zero, $P_1^{r_1} \cdots P_n^{r_n} \subseteq Q_i$. Because P_j are mutually t-comaximal, exactly one of the P_j is contained in Q_i. But then for a fixed j, $P_j = Q_i$ and so each of the Q s is one of the P s.) Now because A does not share a maximal t-ideal with $P_1^{a_1} \cdots P_n^{a_n}$ we have $P_1^{r_1} \cdots P_n^{r_n} \subseteq (f)$. But there can only be a finite number of such irreducible polynomials, by (3) of Lemma 3.1. This leaves the case of when $g(X)$ is not contained in any maximal uppers to zero. In this case the only irreducible divisors are divisors of coefficients of g, coming from D. □

An element $r \in D$ is called *primal* if for any $y, z \in D$, the relation $r \mid_D yz$ guarantees the existence of $r_1, r_2 \in D$ with $r = r_1 r_2$ such that $r_1 \mid_D y$ and $r_2 \mid_D z$. Following [32], we say that D is a *pre-Schreier domain* if every nonzero element $r \in D$ is *primal*, and following [12] we say that D is a *Schreier* domain if D is an integrally closed pre-Schreier domain. It is well known that every GCD-domain is a Schreier domain, and it follows from the definition that every Schreier domain is a pre-Schreier domain. In addition, every pre-Schreier domain is a PSP-domain (this is [35, Lemma 2.1], which although stated for Schreier domains is proved using properties characterizing pre-Schreier domains only). In particular, pre-Schreier domains are AP-domains. It is well known that $D[X]$ is pre-Schreier if and only if $D[X]$ is Schreier.

Let D be a pre-Schreier domain that it is not Schreier, which implies by our previous observation that $D[X]$ is not even pre-Schreier (see e.g. [32, Remark 4.6]). Since every Schreier domain is an AP-domain, some irreducible elements of $D[X]$ are not primes (why?). However, if f is an irreducible non-constant polynomial in $D[X]$, then f is primitive. Now f being a non-constant polynomial, f must belong to an upper to zero ideal P of $D[X]$ and, as $(A_f)_v = D$, every upper to zero ideal containing f must be a maximal t-ideal [23, Theorem 1.4].

We can also establish a version of Theorem 3.2 for Schreier domains as follows. It is known that if D is Schreier then so is $D[X]$, according to [12]. So the nonconstant irreducible elements of $D[X]$ are prime and generators of uppers to zero containing them. Now D being IDF the constant irreducible divisors of a general non-constant $f \in D[X]$ come from D and so are finite, up to associates, and the non-constant irreducible divisors are finite, up to associates, because they are primes and hence generators of the uppers to zero containing them.

Question 3.3. *Does the IDF property ascend in the class consisting of AP-domains?*

3.2. **Further Applications of the Upper-to-zero Representation Lemma.** We proceed to provide further applications of the Upper-to-zero Representation Lemma. Recall that an integral domain D is said to be a Prufer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. We also recall from [34] the following result.

Proposition 3.4. *Let D be an integrally closed integral domain, let X be an indeterminate over D and let $S = \{f(X) \in D[X] \mid (A_f)_v = D\}$. Then D is a PVMD if and only if for any prime ideal P of $D[X]$ with $P \cap D = (0)$ we have $P \cap S \neq \phi$.*

In light of [23, Theorem 1.4], it has often been concluded that D is a PVMD if and only if D is integrally closed such that every upper to zero of $D[X]$ is a maximal t-ideal. In fact the above proposition and Theorem 2.6 of [22] led to the notion of a UMT domain. (A domain whose uppers to zero are maximal t-ideals.) It was stated in [23, Proposition 3.2] that D is a PVMD if and only if D is an integrally closed UMT domain.

Lemma 3.5. *Let B be a t-invertible t-ideal of $D[X]$ with $B \cap D = (0)$. Then $B = (A' P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t$, where P_i are the t-invertible prime uppers to zero of $D[X]$ containing B and $(A', P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t = D$.*

Proof. $BK[X] = f(X)K[X]$. Since, being t-invertible, B is of finite type, there is $s \in K \backslash \{0\}$ such that $B \subseteq sfD[X]$. Or $B = (A_1 sf(X)))_t$ because B is t-invertible and so is $B/sf(X)$. Now sA_1 must intersect D because $BK[X] = fK[X]$. So the only uppers to zero that contain B must contain f. Adjusting s we can assume that $f \in D[X]$. So $B = (A_1 s)_t (f(X)) = (A_1 s(A_1 P_1^{r_1} \cdots P_n^{r_n}))_t$ by Lemma 3.1. The rest is adjustments. (Alternatively, let $P_1, \ldots, P_n$ be the maximal uppers to zero and note that $D[X]_{P_i}$ are rank one DVRs. So there is r_i that $B \subseteq (P_i^{r_i})_t$ and $B \nsubseteq (P_i^{r_i+1})_t$. Now as $(P_i^{r_i})_t$ are t-invertible, $B = (B_1 P_1^{r_1})_t$, repeating with $i = 2$ we have $B = (B_2 P_1^{r_1} P_i^{r_i})_t = \cdots = (B_n P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t$. Set $B_n = A$. As $(BA^{-1})_t = (P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t \subseteq D[X]$ we have $A \subseteq D[X]$. As far as $(A, P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t = D[X]$ is concerned, it follows from the fact that A and $(P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t$ share no maximal t-ideals.) $\square$

Theorem 3.6. *An integral domain D is a PVMD if and only if for each non-constant polynomial $f(X)$ over D, there are uppers to zero $P_1, \ldots, P_n$ such that $f(X)D[X] = (AP_1^{r_1} \cdots P_n^{r_n})_t$, where $A = A_f[X]$.*

Proof. For the direct implication, assume that D be a PVMD. Let f be a nonconstant polynomial in $D[X]$. Then $fD[X] = (AP_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t$, where P_i are the maximal t-ideals containing $fD[X]$, by Lemma 3.1. Now in $K[X]$ we have $fK[X] = P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n} K[X] = P_1^{r_1} K[X] \cap P_2^{r_2} K[X] \cap \cdots \cap P_n^{r_n} K[X]$ because P_i are maximal ideals of $K[X]$. Next note that $P_i^{r_i} K[X] \cap D[X] = P_i^{r_i} K[X]_{P_i} \cap K[X] \cap D[X]$ and because $P_i \cap D = (0)$ we have $K[X]_{P_i} = D[X]_{P_i}$. Thus $P_i^{r_i} K[X]_{P_i} \cap K[X] \cap D[X] = P_i^{r_i} D[X]_{P_i} \cap D[X] = P_i^{(r_i)}$. But then $fK[X] \cap D[X] = P_1^{(r_1)} \cap \cdots \cap P_n^{(r_n)} = (P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t$ because P_i are mutually t-comaximal. On the other hand, on account of D being integrally closed, we have $fK[X] \cap D[X] = fA_f^{-1}[X]$ [29]. This gives $fA_f^{-1}[X] = (P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})$. Multiplying both sides by A_f and applying the t-operation we get $fD[X] = (A_f P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t$. Conversely, suppose that D is such that for each non-constant polynomial $f \in D[X]$ we have $fD[X] = (A_f P_1^{r_1} P_2^{r_2} \cdots P_n^{r_n})_t$. Then, by construction, A_f is t-invertible. Since for every finitely generated nonzero ideal $A = (a_0, a_1, \ldots, a_m)$ we can construct a non-constant polynomial $f = \sum_{i=0}^{m} a_i X^i$ such that $A_f = A$ we conclude that every finitely generated nonzero ideal of D is t-invertible. (Alternatively, for each pair $a, b \in D\backslash\{0\}$ we have $f = a + bX$ which gives $(f(X)) = (A_f P)_t$, forcing $A_f = (a, b)$ to be t-invertible. But this is a necessary and sufficient condition for D to be a PVMD.) $\square$

Proposition 3.7. *An integrally closed domain D is a PVMD if and only if every linear non-constant polynomial over D is contained in a t-invertible upper to zero.*

Proof. If D is a PVMD, then of course as every upper to zero is a maximal t-ideal and hence t-invertible, every linear polynomial is contained in a t-invertible upper to zero. Conversely suppose that every non-constant linear polynomial $f = a + bX$ is contained in a t-invertible upper to zero. If $f(0) = 0$, then $f = bXD[X]$ and there is nothing to be gained from this. Yet if $f(0) \neq 0$ and f is contained in a t-invertible upper P, then $(f) = (AP)_t$. Where $fK[X] = PK[x]$ and so $fK[X] \cap D = f(X)A_f^{-1}[X] = P$. Since P is t-invertible, so must be $A_f^{-1}[X]$. After multiplying both sides by A_f and taking the t-image we get $(f(X)) = (A_f[X]P)_t$. Thus for every pair of nonzero elements a, b of D, (a, b) is t-invertible. This forces D to be a PVMD. □

Proposition 3.8. *An integrally closed domain D is a PVMD if and only if every integral ideal A of $D[X]$ with $A \cap D = (0)$ is contained in a t-invertible upper to zero.*

Proof. If D is a PVMD then every upper to zero in $D[X]$ is t-invertible. Also if A is an ideal of $D[X]$ with $A \cap D = (0)$ then (because D is integrally closed) for some $s \in D\backslash\{0\}$ we have $sA = f(X)C$ for some polynomial $f \in D[X]$ and some integral ideal C with $C \cap D \neq (0)$ [4, Theorem 2.1]. Now as $fD[X]$ is contained in at least one upper to zero sA must be in an upper to zero. But s being a constant does not belong to any upper to zero. So A is contained in at least one upper to zero. Conversely let D be integrally closed and let $f(X)$ be a non-constant linear polynomial. Then $fA_f^{-1}[X] = P$, because D is integrally closed. Since P is t-invertible $A_f^{-1}[X]$ and hence A_f^{-1} is t-invertible and so is $(A_f\ )_v$. But then every two generated nonzero ideal of D is t-invertible. □

By [23, Proposition 3.2], an integral domain D is a PVMD if and only if D is an integrally closed UMT-domain. Let us drop the integrally closed part and see if we can get similar results.

Proposition 3.9. *Let D be an integral domain and X an indeterminate over D. Then D is a UMT domain if and only if for each t-invertible t-ideal A of $D[X]$ with $A \cap D = (0)$, A is contained in a t-invertible prime upper to zero.*

Proof. Since being a t-invertible t-ideal A is a v-ideal of finite type, we have $s \in D\backslash\{0\}$ such that $sA \subseteq fD[X]$ for some f where f is non-constant polynomial contained in A. (We have $A = (a_1, \ldots, a_n)_v K[X] = g(X)$. So $(s_{i1}/sa_{i2})a_i = g(X)$. Setting $s = \Pi s_{i2}$ and multiplying both sides by s we get $t_i a_i = sg(X) \in A$. Now take $sg(X) = f(X)$ we can find $s = \Pi t_i$ such that $(sa_i) \subseteq f(X)$. Now $s(a_1, \ldots, a_n) \subseteq (f)$ and so $s(a_1, \ldots, a_n)_v \subseteq (f)$. But $s(a_1, \ldots, a_n)_v = sA$.). Now f, being a nonconstant polynomial, belongs to a prime upper to zero. If D is a UMT domain, then each prime upper to zero is t-invertible. Conversely, let f be a non-constant polynomial in $D[X]$ and suppose that every t-invertible t-ideal A of $D[X]$ with $A \cap D = (0)$ is contained in a t-invertible prime upper to zero. Observe that $fD[X]$ is a t-invertible t-ideal and so, by the rule, must be contained in a t-invertible prime upper to zero say Q_1. So $fD[X] = (A_1Q_1)_t$ where $(A_1)_t$ is a t-invertible t-ideal. If $(A_1)_t \cap D \neq (0)$ we are done and if not we apply the rule again on $(A_1)_t$ to get $(A_1)_t = ((A_2)Q_2)_t$. or $fD[X] = (A_2Q_1Q_2)_t$. Continuing the recursive procedure we get at say stage $fD[X] = (A_rQ_1 \cdots Q_r)_t$ and note that as f is contained in only a finite number of uppers to zero and as $D[X]_{P_i}$ is a rank one DVR the process cannot run for ever and thus there must be a stage r when $A_r \cap D \neq (0)$. Setting $A_r = A$ and renaming and regrouping we get $fD[X] = (AP_1^{r_1} \cdots P_n^{r_n})_t$ where $A \cap D \neq (0)$. This accounts for all the prime uppers to zero containing f. Thus every prime upper containing f is a maximal t-ideal. Now let P be a prime upper to zero. Then for sone $h \in D[X]$ we have $P = hK[X] \cap D$. By the above procedure $hD[X] = (AQ)_t$ where Q is a t-invertible prime upper containing h. But then $P = hK[X] \cap D = AQK[X] \cap D = Q$, forcing the conclusion that $P = Q$ a maximal t-ideal. (This last line actually nails the proof. The earlier procedure is to indicate what goes on generally.)

Now here's something interesting! We know that a pre-Schreier PVMD is a GCD domain. What must a pre-Schreier UMT domain D be? The way I see it let $a, b \in D\backslash\{0\}$ and take $(aX + b)D[X]$.

Because D is UMT $(aX+b)D[X] = (AP)_t$ where both A and P are and $A \cap D \neq (0)$. Now we know that if D is integrally closed and A is a t-invertible t-ideal of $D[X]$ with $A \cap D \neq (0)$, then $A = (A \cap D)[X]$ and obviously $A \cap D$ is a t-invertible t-ideal [4, Corollary 3.1]. But as the tone of [4, Corollary 3.1] indicates, the jury is still out on the converse. That is the authors of [4] did not know for sure if for every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have $A = (A \cap D)[X]$, then D should be integrally closed. That is we have this question.

Question 3.10. *Suppose that D is an integral domain such that for every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have $A = (A \cap D)[X]$. Must D be integrally closed?*

The answer to the above question is yes and this is how we get it. Let's say that a domain D is $**$ if for every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$ we have $A = (A \cap D)[X]$ and let's denote $(A \cap D)$ by $\mathscr{A}$. First, let us note that if $\alpha \in K$ is integral over D, then the fractional ideal $(1, \alpha)$ is invertible if and only if $\alpha \in D$, [28, Proposition 1.4]. This leads to the following lemma.

Lemma 3.11. *Suppose that $\alpha \in K$ is integral over D. If the fractional ideal $(1, \alpha)$ is t-invertible, then $\alpha \in D$.*

Proof. Suppose that $\alpha \in K$ is integral over D. Then α satisfies a monic polynomial $f = X^n + a_{n-1}X^{n-1} + \cdots + a_0$. Since $a_i = (a_i/s_i)s_i$ for s_i in any multiplicative set S, f can serve as a monic polynomial over D_S. Thus α being integral over D implies that α is integral over D_S. Consequently, α is integral over D_P each maximal t-ideal P. Now recall the easy to prove fact that a finitely generated nonzero ideal I is t-invertible if and only if ID_P is principal for each maximal t-ideal P of D. (We say that I is t-locally principal.) Thus if α is integral over D and if P that is a maximal t-ideal of D then $\alpha \in D_P$ because α is integral over D_P and $(1, \alpha)D_P$ is principal and hence invertible. Thus $\alpha \in D_P$ for each maximal t-ideal P. But then $\alpha \in D = \cap D_P$.

Proposition 3.12. *Let D be an integral domain. Then D is integrally closed if and only if D is $**$.*

Proof. If D is integrally closed, then D is $**$ by [4, Corollary 3.1]. Conversely, suppose that $\alpha = \frac{b}{a}$, where $a, b \in D \backslash \{0\}$, is integral over D. Then α satisfies a monic polynomial f . Now f splits as $(X+\alpha)g(X)$ in $K[X]$. Being linear, $(X+\alpha)$ is a prime in $K[X]$. Thus $P = (X+\alpha)K[X] \cap D[X]$ is a prime upper to zero. Obviously $f \in P$ and so P is t-invertible. Also $a(X+\alpha)D[X] = (aX+b)D[X] \subseteq P$. Since P is a t-invertible ideal we have $(aX+b)D[X] = (AP)_t$, where P and A are t-invertible. As $(aX+b)$ is linear $A \cap D \neq (0)$. Now D being $**$ forces $A = \mathscr{A}[X]$. So $(aX+b)D[X] = (AP)_t = (\mathscr{A}[X]P)_t \subseteq \mathscr{A}[X]$, forcing $aX + b$ and thus $a, b \in \mathscr{A}[X]$. Now as $(a, b)[X] \subseteq \mathscr{A}[X]$, and as $A = \mathscr{A}[X]$ is t-invertible we have $(a, b)[X](\mathscr{A}[X])^{-1} \subseteq D[X]$. On the other hand, $(\mathscr{A}[X]P)_t = (aX + b)D[X] \subseteq (a, b)[X]$. Thus $(\mathscr{A}[X]P)_t \subseteq (a, b)[X]$ and so $P \subseteq ((a, b)[X](\mathscr{A}[X])^{-1})_t \subseteq D[X]$. Or $P \subseteq ((a, b)\mathscr{A}^{-1})_t[X] \subseteq D[X]$. Since P contains f with $A_f = D$ we have $(f, a) \subseteq ((a, b)\mathscr{A}^{-1})_t[X] \subseteq D[X]$. This forces $(((a, b)\mathscr{A}^{-1})[X])_t = ((a, b)\mathscr{A}^{-1})_t[X] = D[X]$, because $(f, a)_t = D[X]$ (see [15, Proposition 3.4]). Thus $(((a, b)\mathscr{A}^{-1})_t = D$ and so (a, b) is t-invertible. But this means $(1, \frac{b}{a})$ is t-invertible. Now as $\alpha = \frac{b}{a}$ is integral over D and as $(1, \frac{b}{a})$ is t-invertible we conclude, by Lemma 3.11, that $\alpha = \frac{b}{a} \in D$.

Now [4, Corollary 3.1] can be recovered as the following statement.

Corollary 3.13. *Let D be an integral domain. Then the following are equivalent.*

(a) D is integrally closed.

(b) For every t-ideal A of $D[X]$ with $A \cap D \neq (0)$, $A = (A \cap D)[X]$.

(c) For every divisorial ideal A of $D[X]$ with $A \cap D \neq (0)$, $A = (A \cap D)[X]$,

(d) For every t-invertible t-ideal A of $D[X]$ with $A \cap D \neq (0)$, $A = (A \cap D)[X]$.

Proof. (a) $\Rightarrow$ (b): This follows from [4, Corollary 3.1].

(b) $\Rightarrow$ (c) This holds because every divisorial ideal is a t-ideal.

(c) ⇒ (d) This holds because every t-invertible t-ideal is divisorial.

(d) ⇒ (a) This is Proposition 3.12. □

4. TIDF-Domains

In this section, we consider a natural stronger notion of IDF-domains, namely, integral domains where every nonunit has a *nonempty* finite set of irreducible divisors up to associates. Recall that an integral domain R is a *Furstenberg domain* provided that every nonunit element of R is divisible by an irreducible, and also recall that a *TIDF-domain* is a Furstenberg IDF-domain. Every TIDF-domain is, in particular, an IDF-domain. The converse does not hold as fields and antimatter domains are IDF-domains but not TIDF-domains. Here is a less trivial example of an IDF-domain that is not a TIDF-domain.

Example 4.1. A valuation domain (V, M) with $M^2 = M$ is an IDF-domain but V cannot be a TIDF-domain because no $m \in M\backslash\{0\}$ is divisible by any atom.

4.1. Atomic Domains.

Every FFD is an IDF-domain, and it is well-known and proved in [1] that in the class of atomic domains the properties of being a FFD and an IDF-domain are equivalent. Moreover, given that every FFD is atomic, it follows that every FFD is an TIDF-domain. It is therefore natural to wonder under which extra condition a TIDF-domain is guaranteed to be an FFD. The following proposition gives an answer to this question.

Proposition 4.2. *The following conditions are equivalent for an integral domain D.*

(a) D is an FFD.

(b) D is an Archimedean TIDF-domain.

(c) D is a TIDF-domain and $\bigcap_{n\in\mathbb{N}} a^n D = (0)$ for each $a \in \mathscr{A}(D)$.

Proof. (a) ⇒ (b): Since D is an FFD, it is an atomic IDF-domain and, therefore, a TIDF-domain. In addition, D satisfies the ACCP because it is an FFD. Thus, it follows from [8, Theorem 2.1] that D is Archimedean.

(b) ⇒ (c): This is obvious.

(c) ⇒ (a): Since D is a TIDF-domain, it is also and IDF-domain. Thus, all we need show is that D is atomic. For this we proceed as follows. Let x be a nonzero non unit of D. Since D is a TIDF-domain there is an atom $a_1|x$. Because a_1 is an atom $\bigcap_{n\in\mathbb{N}} a_1^n D = (0)$. So there is an n_1 such that $a_1^{n_1}|x$ and $a_1^{n_1+1} \nmid x$. Let $x_1 = x/a_1^{n_1}$. If x_1 is a non unit, then because of TIDF x_1 is divisible by an atom a_2 and there is an n_2 such that $a_2^{n_2}|x_2$ and $a_2^{n_2+1} \nmid x_2$. This gives $x_3 = x/(a_1^{n_1}a_2^{n_2})$. Continuing in this fashion, we get $x_r = x/(a_1^{n_1}a_2^{n_2}\cdots a_r^{n_r})$. Now this cannot continue indefinitely because x is divisible by at most a finite number of distinct atoms up to associates. Hence x is a product of atoms. Since the choice of x was arbitrary, we conclude that D is atomic, as desired. □

Although every integral domain satisfying the ACCP is Archimedean, satisfying ACCP does not prevent an integral domain from having a nonzero nonunit with an infinite number of non-associated irreducible divisors. For example, the ring $\mathbb{Q} + X\mathbb{R}[X]$ satisfies ACCP but its element X^2 has infinitely many distinct irreducible divisors, namely, X/r for any $r \in \mathbb{R}\backslash\mathbb{Q}$. Thus, the TIDF condition is needed in the statement of Proposition 4.2. Yet the TIDF condition and atomicity together are a bit much as we already know that the IDF condition and atomicity together already guarantee the finite factorization property. On the other hand, the TIDF condition alone does not even guarantee atomicity, as the following example illustrates.

Example 4.3. Let p be a prime. Now consider the ring $R := \mathbb{Z}_{(p)} + (X,Y)Q[\![X,Y]\!]$, where X and Y are different indeterminates over $\mathbb{Q}$, and let K be the quotient field of R. In addition, let Z be an indeterminate over K and consider the subring $D := R + ZK[\![Z]\!]$ of $K[\![Z]\!]$. Then D and R are both TIDF-domains: indeed, every nonzero nonunit x of D (resp., R) is divisible by the irreducible p, and the fact that $\mathscr{A}(D) = pD^\times$ (resp., $\mathscr{A}(R) = pR^\times$) guarantees that x has only finitely many irreducible divisors up to associates. However, it is clear that neither R nor D are atomic.

We take advantage of this example to point out an interesting divisibility behavior. Consider the localization of D at the multiplicative set $S := \{p^n \mid n \in \mathbb{N}_0\}$, that is,

$$D_S = \mathbb{Q} + (X,Y)\mathbb{Q}[\![X,Y]\!] + ZK[\![Z]\!] = \mathbb{Q}[\![X,Y]\!] + ZK[\![Z]\!].$$

Observe that D_S is an integral domain where every nonzero nonunit is divisible by a prime. Still it is not a TIDF-domain because the element Z is divisible by infinitely many non-associated primes.

Recall that an integral domain is called an AP-domain provided that every irreducible is prime. It is well known that every Schreier domain is an AP domain. The second author has pointed out in [33], the ring $R = \mathbb{Z}_{(p)} + (X,Y)Q[\![X,Y]\!]$ (as in Example 4.3) is a Schreier domain. In addition, it is not too hard to show that $D = R + ZK[\![Z]\!]$ (as in Example 4.3) is also a Schreier domain. As the Schreier property implies the AP property, the previous observation thwarts all hope of using the TIDF property in tandem with the AP property to achieve the finite factorization property. However, in the class consisting of AP domains, we can refine our understanding of the TIDF property.

Corollary 4.4. *Let D be a domain with the AP property. Then the following conditions are equivalent.*

(a) D is a UFD.

(b) D is completely integrally closed TIDF-domain.

(c) D is an Archimedean TIDF-domain.

(d) D is a TIDF-domain such that $\cap(p^n) = (0)$for every prime element p.

Proof. (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d): These implications follow immediately.

(d) $\Rightarrow$ (a): Let x be a nonzero non unit of D. Since D is a TIDF-domain with every atom a prime, x is divisible by at least one and at most finitely many primes. Choose one, say p_1 dividing x. By (4) $x = x_1 p_1^{n_1}$ where $p_1 \nmid x_1$. Repeat with x_1 to choose $p_2 | x_1 | x$ to get $x = x_2 p_1^{n_1} p_2^{n_2}$. Continuing thus at stage r we get $x = x_r p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}$ where $p_r \nmid x_r$ and this can continue until x_r is a unit. Because there are only a finite number of primes dividing x we with get $x_r a$ unit for some value of r. That is exactly when we will have a canonical presentation of x as a finite product of powers of distinct (non-associate) primes. Now as x was chosen arbitrarily, D is a UFD. □

4.2. **The $D+M$ Construction.** The $D+M$ construction, which is a useful source of (counter)examples in commutative ring theory, was introduced and first studied by Gilmer in [19, Appendix II] in the context of valuation domains[1]. Let T be an integral domain, and let K and M be a subfield of T and a nonzero maximal ideal of T, respectively, such that $T = K + M$. For a subdomain D of K, set $R = D + M$. In this subsection, we consider the properties we are concerned with in this paper throughout the lenses of the $D + M$ construction. We first consider the case when D is not a field.

Proposition 4.5. *Let T be an integral domain, and let K and M be a subfield of T and a nonzero maximal ideal of T, respectively, such that $T = K + M$. For a subring D of K, set $R = D + M$. If D is not a field, then the following statements hold.*

(1) If R is an IDF-domain, then D has only finitely many nonassociate irreducibles.

[1]The $D + M$ construction in the general context of integral domains was first investigated in 1976 by J. Brewer and E. Rutter [9].

(2) If R is a TIDF-domain, then D has a finite set of nonassociate irreducibles and $\mathscr{A}(D)$ is nonempty provided that D is a divisor-closed subring of R.

In addition, if T is a local domain, then the following stronger statements hold.

(3) R is an IDF-domain if and only if D has only finitely many nonassociate irreducibles.

(4) R is a TIDF-domain if and only if D is a TIDF-domain with a nonempty finite set of nonassociate irreducibles.

Proof. (1) Every nonzero $d \in D$ divides any $m \in M$; indeed, $r = d(d^{-1})m$ and it is clear that $d^{-1}m \in KM \subseteq M$. Therefore, since M is a nonzero ideal, the fact that R is an IDF-domain, immediately implies that R contains only finitely many nonassociate irreducibles.

(2) Assume that R is a TIDF-domain. It follows by the previous part that R has only finitely many nonassociate irreducibles. Since D is not a field, it must contain a nonzero nonunit d, which must remain a nonunit in R. As R is a TIDF-domain, there is an $a \in \mathscr{A}(R)$ such that $a \mid_R x$. Then $a \in \mathscr{A}(D)$ because D is a divisor-closed subring of R. Thus, $\mathscr{A}(D)$ is not empty.

(3) This is [1, Proposition 4.3].

(4) For the direct implication, suppose first that R is a TIDF-domain. In light of part (2) and the fact that D is not a field, it suffices to argue that every nonzero nonunit in D has an irreducible divisor in D. Take a nonzero nonunit $d \in D$. Since d is a nonunit of R, we can take $a_1 + m_1 \in \mathscr{A}(R)$ with $a_1 \in D$ and $m_1 \in M$ such that $a_1 + m_1 \mid_R d$. As $a_1(1 + a_1^{-1}m_1) \in \mathscr{A}(R)$ either $a_1 \in \mathscr{A}(R)$ or $1 + a_1^{-1}m_1 \in \mathscr{A}(R)$. However, the fact that T is local ensures that $1 + M \subseteq R^\times$. Hence $a_1 \in \mathscr{A}(R)$. Since $D^\times = R^\times \cap D$, it follows that $a_1 \in \mathscr{A}(D)$.

For the reverse implication, suppose that D is a TIDF-domain with a (nonempty) finite maximal set of nonassociate irreducibles. Because of part (3), it is enough to verify that every nonzero nonunit $x \in R$ has an irreducible divisor in R. This is clear if $x \in M$ because we have already observed that every nonzero element of D divides every element of M. Assume, therefore, that $x \notin M$. Then the fact that T is a local domain ensures that x is associate in R with an element of D. As D is a TIDF-domain, x must be divisible by an irreducible in D and so by an irreducible in R. Hence R is also a TIDF-domain. ☐

Corollary 4.6. *Let D be an integral domain with quotient field K, and let L be a field extension of K. Consider the subrings $R = D + XL[X]$ and $S = D + XL[\![X]\!]$ of $L[X]$ and $L[\![X]\!]$, respectively. If D is not a field, then the following statements hold.*

(1) If R is a TIDF-domain, then $\mathscr{A}(D)$ is nonempty and finite (up to associates).

(2) S is a TIDF-domain, then $\mathscr{A}(D)$ is nonempty and finite (up to associates).

Proof. (1) The direct implication follows from Proposition 4.5 because D is a divisor-closed subring of both R and S. For the reverse implication, consider the general element $(hX^r)(1 + Xg(X))$ of $D + XL[X]$, where $h \in L$ and $g(X) \in L[X]$. The element $1 + Xg(X)$ is a product of powers of finitely many height-one primes in $L[X]$ and hence in $D + XL[X]$. Let n_g be the number of prime divisors of $1 + Xg(X)$ and let n_D be the number of atoms in D (up to associates). If $r > 0$, then the number of irreducible divisors of $(hX^r)(1 + Xg(X))$ is $n_D + n_g$. On the other hand, if $r = 0$, then $h \in D$ and so the number of irreducible divisors of h is $n_h \leq n_D$, and so the number of irreducible divisors of $h(1 + Xg(X))$ is m such that $1 \leq m \leq n_D + n_g$. Conversely D must have at most a finite number of irreducible elements because $X \in R$ is divisible by every element of D.

(2) This is similar to part (1). ☐

It follows from [6, Lemma 4.18] that $\mathscr{A}(R) \subseteq T^\times \cup \mathscr{A}(T)$ and, if D is a field, $\mathscr{A}(R) \subseteq \mathscr{A}(T)$.

Proposition 4.7. *Let T be an integral domain, and let K and M be a subfield of T and a nonzero maximal ideal of T, respectively, such that $T = K + M$. For a subfield F of K, set $R = F + M$.*

- *When $M \cap \mathscr{A}(R) \neq \emptyset$, then the following statements hold.*
  (1) R is an IDF-domain if and only if T is an IDF-domain and $|K^\times/F^\times| < \infty$.
  (2) R is an TIDF-domain if and only if T is an TIDF-domain and $|K^\times/F^\times| < \infty$.
- *When $M \cap \mathscr{A}(R) = \emptyset$, then the following statements hold.*
  (4) R is an IDF-domain if and only if T is an IDF-domain.
  (5) R is an TIDF-domain if and only if T is an TIDF-domain.

Proof. (1) This is [1, Proposition 4.2(a)].

(2) For the direct implication, suppose that R is an TIDF-domain. In light of part (1), we only need to argue that $D_T(t)$ is nonempty for all nonzero nonunit $t \in T$. Let t be a nonzero nonunit of T. After replacing t by one of its associate elements in T, we can assume that $t \in R$. Since t is a nonunit of T, then it must be a nonunit of R. This, along with the fact that R is an TIDF-domain, ensures the existence of $a \in \mathscr{A}(R)$ such that $a \mid_R t$. Because F is a field, it follows from [6, Lemma 4.18] that $\mathscr{A}(R) \subseteq \mathscr{A}(T)$. Thus, $a \in \mathscr{A}(T)$, and so $a \mid_T t$. Hence $D_T(t)$ is nonempty.

Conversely, suppose that T is an TIDF-domain and $|K^\times/F^\times| < \infty$. Because of part (1), it suffices to show that $D_R(r)$ is nonempty for all nonzero nonunit $r \in R$. Fix a nonzero nonunit $r \in R$. From the fact that F is a field, one can easily argue that $R^\times = T^\times \cap R$ (see [6, Lemma 4.7]) and, therefore, we obtain that r is a nonunit in T. We split the rest of the proof in the following two cases.

Case 1: $r \notin M$. In this case, after replacing r for one of its associates in R, we can assume that $r = 1+m$ for some nonzero $m \in M$. Because T is an TIDF-domain, $1+m$ has an irreducible divisor in T, and we can assume that such an irreducible divisor of $1+m$ has the form $1+m'$ for some $m' \in M$ (note that no element of M can divide $1+m$ in T). From $1+m' \in \mathscr{A}(T) \setminus M$ and $R^\times = T^\times \cap R$, one infers that $1+m' \in \mathscr{A}(R)$. Moreover, it is clear that $1+m'$ divides r in R. Thus, $D_R(r)$ is nonempty.

Case 2: $r \in M$. As T is an TIDF-domain, we can pick $a \in \mathscr{A}(T)$ such that $a \mid_T r$. If $a \notin M$, then a is associate in T with an irreducible element of the form $1+m$ for some nonzero $m \in M$ and, proceeding as we did in the previous case, we can conclude that $1+m \in \mathscr{A}(R)$ and $1+m \mid_R r$. Suppose, on the other hand, that $a \in M$. Since every element in T is associate in T with an element of R, after replacing a by one of its associates in T, we can assume that $a \mid_R r$. Since $R^\times = T^\times \cap R$, it follows that $a \in \mathscr{A}(R)$. Hence $D_R(r)$ is nonempty.

(3) This is [1, Proposition 4.2(a)].

(4) This follows the lines of parts (2) as the hypothesis $M \cap \mathscr{A}(R) \neq \emptyset$ was only needed to transfer the IDF property. □

As in the case when D is not a field, we highlight the following special cases.

Corollary 4.8. *Let D be an integral domain with quotient field K, and let L be a field extension of K. For the subrings $R = D + XL[X]$ and $S = D + XL[\![X]\!]$ of $L[X]$ and $L[\![X]\!]$, respectively, the following statements hold.*

- *If D is not a field, then R is a TIDF-domain if and only if $\mathscr{A}(D)$ is nonempty and finite (up to associates).*
- *If D is a field, then the following statements hold*
  (1) R is a TIDF-domain if and only if $|L^*/D^*| < \infty$.
  (2) S has n atoms if and only if D has $n > 0$ atoms.
  (3) S is a TIDF-domain if and only if $|L^*/D^*| < \infty$.

We end this subsection by briefly discussing the $D + M$ construction on another natural class of domains strictly contained in that of IDF-domains. Following [26], we say that an integral domain R is a *powerful IDF-domain* (or an *PIDF-domain*) if for every nonzero $x \in R$ the set $\bigcup_{n\in\mathbb{N}} D_n(x)$ is finite up to associates, where $D_n(x)$ denote the set of divisors of x^n in R. Examples of PIDF-domains include Krull domains and, in particular, Dedekind domains and rings of integers of algebraic number fields (see [26, Corollary 3.3]). In addition, every valuation domain is a PIDF: this follows immediately from the fact that a valuation domain can contain only one principal ideal generated by an irreducible element. Observe that R is an IDF-domain if and only if $D_1(x)$ is finite for every nonzero $x \in R$ and, therefore, every PIDF-domain is an IDF-domain. Not every IDF-domain, however, is a PIDF-domain.

Example 4.9. [26,] Let F be a field of characteristic zero, and consider the subring $F[X^2, X^3]$ of $F[x]$. It is not hard to verify that R is an FFD and, therefore, it is an IDF-domain. On the other hand, it follows from [26, Proposition 4.1] that R is not a PIDF-domain.

Proposition 4.10. *Let T be an integral domain, and let K and M be a subfield of T and a nonzero maximal ideal of T, respectively, such that $T = K + M$. For a subfield F of K, set $R = F + M$. Then the following statements hold.*

(1) When $M \cap \mathscr{A}(R) \neq \emptyset$, then R is a PIDF-domain if and only if T is a PIDF-domain and $|K^\times/F^\times| < \infty$.

(2) When $M \cap \mathscr{A}(R) = \emptyset$, then R is a PIDF-domain if and only if T is a PIDF-domain.

Proof. (1) Assume that R is a PIDF-domain. In particular, R is an IDF-domain, and it follows from part (1) that T is an IDF-domain and $|K^\times/F^\times| < \infty$. Suppose, towards a contradiction, that T is not a PIDF-domain. As T is an IDF-domain, for some nonzero $t \in T$ we can choose a strictly increasing sequence $(k_n)_{n\imath\mathbb{N}}$ of positive integers and a sequence $(a_n)_{n\in\mathbb{N}}$ of pairwise non-associate irreducibles of T with $a_n \mid_T t^{k_n}$ for every $n \in \mathbb{N}$. After replacing t by one of its associates in T, we can assume that $t \in R$. Similarly, we can assume that $a_n \in \mathscr{A}(R)$ and $a_n \mid_R t^{k_n}$ for every $n \in \mathbb{N}$ (here we are using that $R^\times = T^\times \cap R$). Therefore $(a_n)_{n\in\mathbb{N}}$ is a sequence of non-associate irreducibles in R with $a_nR^\times \in \bigcup_{n\in\mathbb{N}} D_R(t^n)$. Thus, $|\bigcup_{n\in\mathbb{N}} D_R(t^n)| = \infty$, contradicting that R is a PIDF-domain.

Conversely, suppose that T is a PIDF-domain and $|K^\times/F^\times| < \infty$, and take $\alpha_1, \dots, \alpha_m \in K^\times$ such that $K^\times/F^\times = \{\alpha_1F^\times, \dots, \alpha_mF^\times\}$. Fix a nonzero nonunit $r \in R$. Since F is a field, it follows from [6, Lemma 4.18] that $\mathscr{A}(R) \subseteq \mathscr{A}(T)$, and so $aT^\times \in \bigcup_{n\in\mathbb{N}} D_T(r^n)$ for every $a \in \mathscr{A}(R)$ with $aR^\times \in \bigcup_{n\in\mathbb{N}} D_R(r^n)$. Hence we can define $\varphi\colon \bigcup_{n\in\mathbb{N}} D_R(r^n) \to \bigcup_{n\in\mathbb{N}} D_T(r^n)$ by $\varphi(aR^\times) = aT^\times$. Now for $bT^\times \in \bigcup_{n\in\mathbb{N}} D_T(r^n)$, we see that $\varphi^{-1}(bT^\times) \subseteq \{\alpha_1bR^\times, \dots, \alpha_mbR^\times\}$. This, along with the fact that $\bigcup_{n\in\mathbb{N}} D_T(r^n)$ is finite, guarantees that $\bigcup_{n\in\mathbb{N}} D_R(r^n)$ is a finite set. As a result, R is a PIDF-domain.

(2) These follow the lines of part (1) as the hypothesis $M \cap \mathscr{A}(R) \neq \emptyset$ was only needed to transfer the IDF property. □

4.3. **Localization.** It is well known that the property of being atomic is not preserved under localization. Indeed, there are atomic domains that have localizations that are antimatter domains (i.e., do not contain any irreducibles) but not fields. The following example sheds some light upon this observation.

Example 4.11. Consider the monoid ring $F[M]$, where F is the field of p elements and M is the monoid $\{0\} \cup \mathbb{Q}_{\geq 1}$. Observe that the localization of $F[M]$ at the multiplicative set $S = \{X^m \mid m \in M\}$ is the group ring $F[\mathbb{Q}]$. It follows from [17, Theorem 14.17] that $F[\mathbb{Q}]$ does not satisfy ACCP. Moreover, since F is a perfect field of characteristic p, every nonunit element of $F[\mathbb{Q}]$ is the p-th power of a nonunit element, and so $F[\mathbb{Q}]$ is an antimatter domain.

However, under some special conditions on the multiplicative set, the property of being atomic is preserved under localization. Let $A \subseteq B$ be a ring extension. Following Cohn [12], we call $A \subseteq B$ an *inert extension* if $xy \in A$ for $x, y \in B^*$ implies that $ux, u^{-1}y \in A$ for some $u \in B^\times$. Let $A \subseteq B$ be an inert extension of integral domains, then one can readily verify that $\mathscr{A}(A) \subseteq B^\times \cup \mathscr{A}(B)$. Therefore if $A \subseteq B$ is inert and $A^\times = B^\times \cap A$, then $\mathscr{A}(A) = \mathscr{A}(B) \cap A$.

Example 4.12.

(1) If R is an integral domain, then the extension $R \subseteq R[X]$ is inert.

(2) Furthermore, under the usual notation of the $D+M$ construction, it is not hard to argue that both extensions $D \subseteq R$ and $R \subseteq T$ are inert.

(3) Fix $n \in \mathbb{Z}$ with $n \ge 2$, and then consider the extension $R[X^n] \subseteq R[X]$. It is clear that $R[X^n]^\times = R^\times$. Observe, on the other hand, that although $X^n \in R[X^n]$ there is no $u \in R^\times$ such that $uX \notin R[X^n]$. As a result, $R[X^n] \subseteq R[X]$ is not an inert extension.

Let R be an integral domain, and let S be a multiplicative set of R. If S is generated by primes, then the localization extension $R \subseteq R_S$ is inert by [2, Proposition 1.9]. Now if $R \subseteq R_S$ is an inert extension, then we know by [2, Theorem 2.1] that R_S is atomic provided that R is atomic. Unfortunately, the same statement is not longer true if atomicity is replaced by the IDF property [2, Example 2.3] or the TIDF property (Example 4.13). The ring in the following example is basically the same ring used in [2, Example 2.3].

Example 4.13. Let D denote the localization of $\mathbb{Z}$ at a prime p, and then set $R = D + X\mathbb{R}[\![X]\!]$. Observe that D has only one irreducible up to associates, namely p, and this irreducible element divides every nonunit of D. In particular, D is a TIDF-domain with finitely many irreducibles up to associates. Since $\mathbb{R}[\![X]\!]$ is a local domain, it follows from part (4) of Proposition 4.5 that R is also a TIDF-domain. One can easily check that p is also prime in R. Thus, if S is the multiplicative set $pR^\times$, then the extension $R \subseteq R_S$ is inert. However, as argued in [2, Example 2.3], the ring R_S is not even an IDF-domain.

A saturated multiplicative subset S of an integral domain R is called *splitting* if every $r \in R$ can be written as $r = as$ for some $a \in R$ and $s \in S$ such that $aR \cap s'R = as'R$ for all $s' \in S$. If S is a splitting multiplicative set, then the extension $R \subseteq R_S$ is inert by [2, Proposition 1.5]. In addition, if R is atomic, then every multiplicative subset of R generated by primes is a splitting multiplicative set. In the next proposition we establish the statements of both [2, Theroem 3.1] and [2, Theorem 2.4(a)] to the case of TIDF-domains.

Proposition 4.14. *Let R be an integral domain, and let S be a splitting multiplicative subset of R generated by primes. Then R is a TIDF-domain if and only if R_S is a TIDF-domain.*

Proof. Let B be the subset of R consisting of all elements that are not divisible in R by any prime contained in S. Write $r = bs$ for some $b \in R$ and $s \in S$. If p is a prime in S and $r = bs$ for some $b \in R$ and $s \in S$ such that $bR \cap s'R = bs'R$ for all $s' \in S$, then [2, Proposition 1.6] ensures the existence of a maximum $m \in \mathbb{N}_0$ such that $p^m \mid_R b$, and so $b \in bR \cap p^m R = bp^m R$, which implies that $m = 0$. Therefore every element of $r \in R$ can be written as $r = bs$ for some $b \in B$ and $s \in S$.

By virtue of [2, Theroem 3.1] and [2, Theorem 2.4(a)], the ring R is an IDF-domain if and only if R_S is an IDF-domain. Therefore it suffices to show that R is a Furstenberg domain if and only if R_S is a Furstenberg domain. For the direct implication, assume that R is a Furstenberg domain. It suffices to argue that every nonzero nonunit $r \in R$ has an irreducible divisor in R_S. Since $R \subseteq R_S$ is an inert extension, $\mathscr{A}(R) \subseteq R_S^\times \cup \mathscr{A}(R_S)$ by [2, Lemma 1.1]. Write $r = bs$ for some $b \in B$ and $s \in S$. Since R is Furstenberg, there is an $a \in \mathscr{A}(R)$ such that $b = ar'$ for some $r' \in R$. Since none of the primes contained in S can divide a in R, it follows that $a \notin S = R_S^\times$. Hence $a \in \mathscr{A}(R_S)$. Thus, R_S is Furstenberg.

Conversely, suppose that R_S is a Furstenberg domain. We will argue that every nonzero nonunit $r \in R$ has an irreducible divisor. We assume that $p \nmid_R r$ for any $p \in S$ as otherwise we are done. Since R_S is Furstenberg, there exists $a' \in \mathscr{A}(R_S)$ dividing r in R_S. As $R_S^\times = S$, we can actually assume that $a' \in R$. As S is splitting, $a' = as$ for some $a \in R$ and $s \in S$ such that $aR \cap s'R = as'R$ for all $s' \in S$.

It follows from [2, Corollary 1.4] that $a \in \mathscr{A}(R)$. Write $r = a\frac{b}{s}$ for some $b \in R$ and $s \in S$. Now the fact that $s \mid_R ab$ implies that $s \mid_R b$. As a result, a is an irreducible divisor of r in R. Thus, R is a Furstenberg domain. $\square$

TODO: Maybe consider PIDF-domains under localization (for the sake of completion)...

4.4. **Direct Limits.** A ring homomorphism $f\colon R \to S$ is called a *divisor homomorphism* if $x \mid_R y$ provided that $f(x) \mid_S f(y)$ for all $x, y \in R$. Observe that if f is a divisor homomorphism, then $f^{-1}(S^\times) = R^\times$, and so $f(R^\times) = S^\times \cap f(R)$, which implies that $f^{-1}(\mathscr{A}(S)) \subseteq \mathscr{A}(R)$. However, divisor homomorphisms are not, in general, atomically inert. We say that f is *divisor-closed homomorphism* if f is a divisor homomorphism such that $f(R)$ is a divisor-closed subring of S.

Lemma 4.15. *For a divisor-closed homomorphism $f\colon R \to S$, the following statements hold.*

(1) $f(R^\times) = S^\times$, and so $f^{-1}(S^\times) = R^\times$.

(2) $f^{-1}(\mathscr{A}(S)) = \mathscr{A}(R)$, and so $f(\mathscr{A}(R)) = \mathscr{A}(S) \cap f(S)$. In particular, f is atomically inert.

Proof. (1) It is clear that $f(R^\times) \subseteq S^\times$. For the reverse implication, take $y \in S^\times$. Since $y \mid_S 1 = f(1)$, the fact that $f(R)$ is a divisor-closed subring of S allows us to write $y = f(x)$ for some $x \in R$ and the fact that f is a divisor homomorphism now implies that $x \mid_R 1$. Thus, $x \in R^\times$ and so $y \in f(R^\times)$. For the second equality, first observe that if $f(x) \in S^\times$ for some $x \in R$, then $f(x) \mid_S f(1)$ and, as f is a divisor homomorphism, $x \in R^\times$. Therefore $R^\times = f^{-1}(f(R^\times)) = f^{-1}(S^\times)$.

(2) Take $a \in \mathscr{A}(R)$ and write $f(a) = y_1 y_2$ for some $y_1, y_2 \in N$. Since $f(R)$ is a divisor-closed subring of S, then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in R$, and so $f(a) = f(x_1 x_2)$. As f is a divisor homomorphism, we can take $u \in R^\times$ with $a = u x_1 x_2$, which implies that either $x_1 \in R^\times$ or $x_2 \in R^\times$. Hence $y_1 \in S^\times$ or $y_2 \in S^\times$, and so $f(a) \in \mathscr{A}(S)$. Thus, f is atomically inert, and so $\mathscr{A}(R) \subseteq f^{-1}(\mathscr{A}(S))$. For the reverse inclusion, take $x \in R$ with $f(x) \in \mathscr{A}(S)$, and then write $x = x_1' x_2'$ for $x_1', x_2' \in R$. Therefore $f(x_1') \in S^\times$ or $f(x_2') \in S^\times$, and so $x_1' \in f^{-1}(S^\times) = R^\times$ or $x_2' \in f^{-1}(S^\times) = R^\times$ by part (1), and we obtain that $x \in \mathscr{A}(R)$. Hence $f^{-1}(\mathscr{A}(S)) = \mathscr{A}(R)$. The second equality is an immediate implication of the first one. $\square$

Being a divisor(-closed) homomorphism transfers from the homomorphism of a directed system of integral domains to the homomorphisms of its direct limit.

Proposition 4.16. *Let I be a directed set, and let $(f_{ij}\colon R_i \to R_j)_{i,j \in I}$ be a directed system of homomorphisms with colimit $(\varinjlim R_i, (f_i)_{i \in I})$. Then the following statements hold.*

(1) f_{ij} is a divisor homomorphism for all $i \in I$ if and only if f_i is a divisor homomorphism for all $i \in I$.

(2) If f_{ij} is a divisor-closed homomorphism for all $i \in I$, then f_i is a divisor-closed homomorphism for all $i \in I$.

Proof. (1) Let R denote the colimit of the system $(R_i, (f_i)_{i \in I})$. For the direct implication, suppose that f_{ij} is a divisor homomorphism for all $i, j \in I$. Now fix $i \in I$, and take $x_i, y_i \in R_i$ such that $f_i(x_i) \mid_R f_i(y_i)$. Pick $j \in I$ and $x_j \in R_j$ such that $[x_i] \cdot [x_j] = [y_i]$. Take $k \in I$ with $k \ge i, j$ such that $[x_i] \cdot [x_j] = [f_{ik}(x_i) f_{jk}(x_j)]$, and then $\ell \in I$ with $\ell \ge k$ such that $f_{i\ell}(x_i) f_{j\ell}(x_j) = f_{i\ell}(y_i)$ in R_ℓ. Since $f_{i\ell}$ is a divisor homomorphism and $f_{i\ell}(x_i) \mid_{R_\ell} f_{i\ell}(y_i)$, it follows that $x_i \mid_{R_i} y_i$. Hence f_i is a divisor homomorphism.

For the reverse implication, assume that f_i is a divisor homomorphism for every $i \in I$. Fix $i, j \in I$ with $i \le j$, and let us verify that f_{ij} is a divisor homomorphism. To do this, take $x_i, y_i \in M_i$ such that $f_{ij}(x_i) \mid_{R_j} f_{ij}(y_i)$. This implies that $f_j(f_{ij}(x_i))$ divides $f_j(f_{ij}(y_i))$ in R, that is, $f_i(x_i) \mid_R f_i(y_i)$. As f_i is a divisor homomorphism, $x_i \mid_R y_i$. Thus, we conclude that f_{ij} is a divisor homomorphism.

(2) Assume that f_{ij} is a divisor-closed homomorphism for all $i \in I$. Fix $i \in I$ and suppose that $[x_j]$ divides $f_i(x_i)$ in R for some $j \in I$ and $x_j \in R_j$. Take $k \in I$ with $k \geq i, j$. Since f_k is a divisor homomorphism by the previous statement, the fact that $f_k(f_{jk}(x_j))$ divides $f_k(f_{ik}(x_i))$ in R ensures that $f_{jk}(x_j)$ divides $f_{ik}(x_i)$ in R_k. As f_{ik} is divisor-closed, there exists $x_i' \in R_i$ such that $f_{ik}(x_i') = f_{jk}(x_j)$, and so $[x_j] = f_k(f_{jk}(x_j)) = f_k(f_{ik}(x_i')) = f_i(x_i') \in f_i(R_i)$. Thus, f_i is divisor-closed for every $i \in I$. $\square$

In the following theorem we give some conditions under which the IDF properties we study in this section are preserved under direct limits.

Theorem 4.17. *Let I be a directed set, and let $(f_{ij} \colon R_i \to R_j)_{i \in I}$ be a directed system of monoid homomorphisms with direct limit $(\varinjlim R_i, (f_i)_{i\in I})$. If f_{ij} is a divisor-closed homomorphism for all $i, j \in I$ with $i \leq j$, then the following statements hold.*

(1) If R_i is an IDF-domain for each $i \in I$, then $\varinjlim R_i$ is an IDF-domain.

(2) If R_i is a TIDF-domain for each $i \in I$, then $\varinjlim R_i$ is a TIDF-domain.

Proof. Set $R := \varinjlim R_n$. Since f_{ij} is a divisor-closed homomorphism for all $i, j \in I$ with $i \leq j$, it follows from part (2) of Proposition 4.16 that f_i is a divisor-closed homomorphism for each $i \in I$. Therefore, by part (2) of Lemma 4.15, the homomorphism f_i is atomically inert and satisfies $f_i^{-1}(\mathscr{A}(R)) = \mathscr{A}(R_i)$ for each $i \in I$.

(1) Assume that R_i is an IDF-domain for all $i \in I$. For all $i, j \in I$ with $i \leq j$, the homomorphism f_{ij} is atomically inert by Lemma 4.15. Now take $[x] \in R$ and $i \in I$ with $x \in R_i$. We claim that $|D_R([x])| \leq |D_{R_i}(x)|$. Suppose that $[a]$ and $[b]$ are two non-associate irreducible divisors of $[x]$ in R with $a \in R_j$ and $b \in R_k$ for some $j, k \in I$. Take $\ell \in I$ large enough so that $i, j, k \leq \ell$. Since f_j and f_k are divisor-closed homomorphisms and $[a], [b] \in \mathscr{A}(R)$, we see that $a \in \mathscr{A}(R_j)$ and $b \in \mathscr{A}(R_k)$. Now, as f_ℓ is a divisor homomorphism, the fact that $[a] \mid_R [x]$ implies that $f_{j\ell}(a) \mid_{R_\ell} f_{i\ell}(x)$. Then we see that $f_{j\ell}(a) \in f_{i\ell}(R_i)$ because the latter is a divisor-closed subring of R_ℓ. Take $a' \in R_i$ such that $f_{j\ell}(a) = f_{i\ell}(a')$. Since $f_{j\ell}$ is atomically inert by Lemma 4.15 and $f_{i\ell}$ is a divisor-closed homomorphism, $a' \in \mathscr{A}(R_i)$. Similarly, we can choose $b' \in \mathscr{A}(R_i)$ with $f_{j\ell}(b) = f_{i\ell}(b')$. Then the fact that f_i is a divisor homomorphism guarantees that both a' and b' divide x in R_i. Observe in addition that a' and b' are non-associate elements of R_i because their images $[a]$ and $[b]$ are not associate elements in R. Hence $|D_R([x])| \leq |D_{R_i}(x)|$, as claimed. Then the fact that each R_i is an IDF-domain implies that R is an IDF-domain.

(2) Suppose that R_i is a TIDF-domain for every $i \in I$. Given part (1), it suffices to prove that $D_1([x])$ is nonempty for every nonunit $[x] \in R$. To do so, take a nonunit $[x] \in R$, and then take $i \in I$ such that $x \in R_i$. Since R_i is an TIDF-domain, there is an $a \in \mathscr{A}(R_i)$ such that $a \mid_{R_i} x$. This, along with the fact that f_i is atomically inert, implies that $[a]$ is an irreducible in R dividing $[x]$, and so $D_R([x])$ is nonempty. Hence we can conclude that R is an TIDF-domain. $\square$

As we did with the $D + M$ construction, we end this subsection by briefly discussing direct limits of PIDF-domains.

Proposition 4.18. *Let I be a directed set, and let $(f_{ij} \colon R_i \to R_j)_{i \in I}$ be a directed system of monoid homomorphisms with direct limit $(\varinjlim R_i, (f_i)_{i\in I})$. Let f_{ij} be a divisor-closed homomorphism for all $i, j \in I$ with $i \leq j$. If R_i is a PIDF-domain for each $i \in I$, then $\varinjlim R_i$ is a PIDF-domain.*

Proof. Set $R := \varinjlim R_n$. Since f_{ij} is a divisor-closed homomorphism for all $i, j \in I$ with $i \leq j$, it follows from part (2) of Proposition 4.16 that f_i is a divisor-closed homomorphism for each $i \in I$. Therefore, by part (2) of Lemma 4.15, the homomorphism f_i is atomically inert and satisfies $f_i^{-1}(\mathscr{A}(R)) = \mathscr{A}(R_i)$ for each $i \in I$.

Assume that R_i is a PIDF-domain for each $i \in I$. We proceed by contradiction: suppose that there exists a nonzero $[x] \in R$ such that $\bigcup_{n\in\mathbb{N}} D_R([x]^n)$ is an infinite set, and fix $i \in I$ such that $x \in R_i$. Since R_i is a PIDF-domain, we can pick $N \in \mathbb{N}$ with $|\bigcup_{n\in\mathbb{N}} D_{R_i}(x^n)| < N$. It follows from part (1) that $D_R([x]^n)$ is finite for every $n \in \mathbb{N}$. Then we can take a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ of positive integers and a sequence $([a_n])_{n\in\mathbb{N}}$ of pairwise non-associate irreducibles of R such that $[a_n] \mid_R [x^{k_n}]$. For each $n \in [\![1, N]\!]$, choose $i_n \in I$ such that $a_n \in R_{i_n}$. Now take $\ell \in I$ with $\ell \geq i_1, \dots, i_N$. Observe that $f_{n\ell}(a_n) \mid_{R_\ell} f_{i\ell}(x^{k_n})$ for every $n \in [\![1, N]\!]$ because f_ℓ is a divisor homomorphism. Proceeding as we did in part (1), for each $n \in [\![1, N]\!]$ we can produce $a'_n \in \mathscr{A}(R_i)$ with $f_i(a'_n) = [a_n]$. Since $[a_1], \dots, [a_N]$ are pairwise non-associates in R, it follows that $a'_1, \dots, a'_N$ are pairwise non-associate irreducibles of R_i, each of them dividing some power of x in R_i. However, this contradicts that $|\bigcup_{n\in\mathbb{N}} D_{R_i}(x^n)| < N$. Thus, R is a PIDF-domain. □

5. Monoid Domains and the IDF Property

Throughout the remainder of paper, we tacitly assume that every monoid is cancellative and commutative. For a torsion-free monoid M and an integral domain R, we can construct the monoid ring $R[M]$, which is also an integral domain. This document offers a brief discussion of the IDF property on monoid rings. To begin with, we show that, for every $n \in \mathbb{N}$, there exists an n-dimensional monoid ring that is not an IDF domain.

Proposition 5.1. *For every field F and $d \in \mathbb{N}$, there exists a monoid ring over F with Krull dimension d that is a BFD but no an FFD.*

Proof. First, suppose that $d = 1$. Consider the monoid $M = \{0\} \cup \mathbb{Q}_{\geq 1}$. It is clear that the difference group of M is $\mathbb{Q}$. We can argue, following the lines of [6, Example 4.7], that $F[M]$ is a BFD that is not an FFD. On the other hand, $\dim F[M] = \dim F[\mathbb{Q}] = \dim F[x] = 1$, where the first equality follows from [17, Theorem 21.4] and the second equality follows from [17, Theorem 17.1].

Suppose now that $d \geq 2$. Let M be the additive submonoid $\{\mathbf{0}\} \cup (\mathbb{Z}^{d-1} \times \mathbb{N})$ of the free abelian group $\mathbb{Z}^d$. Clealry, M is torsion-free. Consider the monoid domain $F[M]$. One can easily see that $\mathscr{A}(M) = \mathbb{Z}^{d-1} \times \{1\}$. This immediately implies that M is a BFM (indeed, an HFM). Since M is a reduced torsion-free BFM, it follows from [3, Theorem 13] that $F[M]$ is a BFD. On the other hand, $x_d^2 = (x_1^{-k} x_d)(x_1^k x_d)$ for every $k \in \mathbb{N}$ (after identifying $F[M]$ with the obvious subring of the Laurent polynomial ring $F[x_1^{\pm 1}, \dots, x_d^{\pm 1}]$), and it is clear that the monomials $x_1^k x_d$ (for all $k \in \mathbb{Z}$) are pairwise non-associate irreducible elements in $F[M]$. Hence $F[M]$ is not an FFD. Finally, $\dim F[M] = \dim F[\mathbb{Z}^d] = d$ by virtue of [17, Theorem 21.4] and [17, Theorem 17.1]. □

As an integral domain is an FFD if and only if it is an atomic IDF-domain [1, Theorem 5.1], we obtain the following corollary.

Corollary 5.2. *For every $d \in \mathbb{N}$, there exists an atomic monoid ring with Krull dimension d that is not an IDF-domain.*

Provided that the monoid M is chosen as simple as it can possibly be, meaning $M = \mathbb{N}_0$, the monoid ring $R[M]$ becomes a ring of polynomials, and it is known that the property of being irreducible-divisor-finite does not ascend, in general, from R to $R[M]$ (see [27, Theorem 2.5]). Following the same methodology, one can choose R as simple as it can possibly be, namely $R \in \{\mathbb{Q}, \mathbb{F}_p\}$, and wonder whether the property of being irreducible-divisor-finite ascends from M to $R[M]$. As for the case of polynomial rings, this question has a negative answer, and our next goal is to provide explicit construction of counterexamples. Observe that the monoids used to establish Corollary 5.2 are not IDF-monoids.

Let us introduce another family of monoids, which will be more suitable for our purposes. For $p \in \mathbb{P}$, set $M_p := \langle 1/p^n \mid n \in \mathbb{N} \rangle$, and for a nonempty set $\mathscr{P}$ consisting of primes, we let $M_{\mathscr{P}}$ denote the additive submonoid of $\mathbb{Q}_{\ge 0}$ generated by the set $\bigcup_{p \in \mathscr{P}} M_p$, that is,

$$M_{\mathscr{P}} := \langle p^{-n} \mid p \in \mathscr{P} \text{ and } n \in \mathbb{N} \rangle.$$

One can easily see that none of the defining generators of $M_{\mathscr{P}}$ is an atom. This implies that $M_{\mathscr{P}}$ is an antimatter monoid. In addition, it follows from [16, Proposition 3.1] that $M_{\mathscr{P}}$ is not root-closed. The most useful fact about the monoid $M_{\mathscr{P}}$ is that its subset $M_p \cap [0,1)$ is divisor-closed for every $p \in \mathscr{P}$: we say that a subset S of a monoid M is *divisor-closed* if for all $s \in S$ and $t \in M$ the divisibility relation $t \mid_M s$ implies that $t \in S$. Let us prove this.

Lemma 5.3. *Let $\mathscr{P}$ be a set of primes with $|\mathscr{P}| \ge 2$. Then $M_p \cap [0,1)$ is a divisor-closed subset of $M_{\mathscr{P}}$ for every $p \in \mathscr{P}$.*

Proof. Fix $p \in \mathscr{P}$, and take an element $c/p^k \in M_p \cap (0,1)$ for some $k \in \mathbb{N}$ and $c \in [\![1, p^k - 1]\!]$ such that $\gcd(p,c) = 1$. Suppose, towards a contradiction, that $1/q^n \mid_{M_{\mathscr{P}}} c/p^k$ for some $q \in \mathscr{P} \setminus \{p\}$ and $n \in \mathbb{N}$. Then we can write

$$\frac{c}{p^k} = \frac{c_1}{q^\ell} + \frac{c_2}{d} \tag{5.1}$$

for some $c_1, c_2, d, \ell \in \mathbb{N}$ such that $c_2/d \in M_{\mathscr{P}}$ and $\gcd(q,d) = 1$. After multiplying both sides of (5.1) by $p^k q^\ell d$, we see that q^ℓ divides c_1. However, this contradicts the fact that $c/p^k < 1$. Hence $1/q^n \nmid_{M_{\mathscr{P}}} c/p^k$ for any $q \in \mathscr{P} \setminus \{p\}$ and $n \in \mathbb{N}$, which means that every nonzero divisor of c/p^n in $M_{\mathscr{P}}$ belongs to $M_p \cap (0,1)$. Hence $M_p \cap [0,1)$ is a divisor-closed subset of $M_{\mathscr{P}}$. □

Let R be an integral domain, and let M be a totally ordered monoid. If $f(x) = c_n x^{q_n} + \dots + c_1 x^{q_1} \in R[M]$ for some nonzero coefficients $c_1, \dots, c_n \in R$ and $q_1 < \dots < q_n$, then we call $\operatorname{supp}(f(x)) := \{q_1, \dots, q_n\}$ the *support* of f. If $S \subseteq R[M] \setminus \{0\}$, then we set $\operatorname{supp}(S) := \bigcup_{s(x) \in S} \operatorname{supp}(s(x))$. We are in a position to prove that $\mathbb{Q}[M_{\mathscr{P}}]$ is not an IDF-domain provided that $|\mathscr{P}| \ge 2$.

Proposition 5.4. *Let $\mathscr{P}$ be a set of primes with $|\mathscr{P}| \ge 2$. Then $M_{\mathscr{P}}$ is an IDF-monoid while $\mathbb{Q}[M_{\mathscr{P}}]$ is not an IDF-domain.*

Proof. We have observed before that $M_{\mathscr{P}}$ is antimatter; therefore it is an IDF-monoid. In order to prove that $\mathbb{Q}[M_{\mathscr{P}}]$ is not an IDF-domain, we will argue that $x - 1 \in \mathbb{Q}[M_{\mathscr{P}}]$ has infinitely many non-associate irreducible divisors. For each $m \in \mathbb{N}$, let $\Phi_m(x)$ denote the m-th cyclotomic polynomial. Since

$$x - 1 = \left(x^{1/p^n}\right)^{p^n} - 1 = \prod_{j=0}^{n} \Phi_{p^j}\left(x^{1/p^n}\right)$$

and $\Phi_{p^j}\left(x^{1/p^n}\right)$ belongs to $\mathbb{Q}[M_{\mathscr{P}}]$ for all $p \in \mathscr{P}$ and $j, n \in \mathbb{N}$, it suffices to show that $\Phi_{p^j}(x^{1/p^k})$ is irreducible in $\mathbb{Q}[M_{\mathscr{P}}]$ provided that $p \in \mathscr{P}$ and $1 \le j < k$. Fix $p \in \mathscr{P}$ and $j, k \in \mathbb{N}$ with $1 \le j < k$, and then write $\Phi_{p^j}(x^{1/p^k}) = a(x)b(x)$ for some $a(x), b(x) \in \mathbb{Q}[M_{\mathscr{P}}]$. Now write

$$a(x) = a_p(x) + a'(x) \quad \text{ and } \quad b(x) = b_p(x) + b'(x)$$

for some $a_p(x), a'(x), b_p(x), b'(x) \in \mathbb{Q}[M_{\mathscr{P}}]$ such that $\operatorname{supp}(a_p(x))$ and $\operatorname{supp}(b_p(x))$ are contained in M_p while $\operatorname{supp}(a'(x))$ and $\operatorname{supp}(b'(x))$ are disjoint from M_p, that is, $a_p(x)$ (resp., $b_p(x)$) is the addition of all monomials in $a(x)$ (resp., $b(x)$) whose exponents belong to M_p. Now set $f(x) := a_p(x)b'(x) + a'(x)b_p(x) + a'(x)b'(x)$ (note that $f(x) = a(x)b(x) - a_p(x)b_p(x)$). Since $\deg a(x)b(x) = \deg(\Phi_{p^j}(x^{1/p^k})) < 1$, it follows that

$$\operatorname{supp}(f(x)) = \operatorname{supp}\left(\Phi_{p^j}(x^{1/p^k}) - a_p(x)b_p(x)\right) \subseteq M_p \cap (0,1). \tag{5.2}$$

As the support of both $a'(x)$ and $b'(x)$ is disjoint from M_p, then every element in the support of $f(x)$, which belongs to $M_p \cap (0,1)$ by (5.2), must be divisible by an element of $M_{\mathscr{P}} \setminus M_p$. Since $M_p \cap (0,1)$ is a divisor-closed subset of $M_{\mathscr{P}}$ by Lemma 5.3, it follows that $\text{supp}(f(x))$ is empty, that is, $f(x) = 0$. Hence $\Phi_{p^j}(x^{1/p^k}) = a(x)b(x) = a_p(x)b_p(x)$.

We proceed to show that either $a_p(x)$ or $b_p(x)$ has degree zero. Take the minimum $\ell \in \mathbb{N}$ such that $p^\ell \text{supp}(a_p(x))$ and $p^\ell \text{supp}(b_p(x))$ are subsets of $\mathbb{N}$. Because

$$p^{j+\ell-k-1}(p-1) = \varphi(p^j)p^{\ell-k} = \deg \Phi_{p^j}\left(\left(x^{p^\ell}\right)^{1/p^k}\right) = \deg a_p(x^{p^\ell})b_p(x^{p^\ell}) \in \mathbb{N},$$

we see that $j + \ell - k - 1 \geq 0$. Therefore $\Phi_{p^j}\left(x^{\ell-k}\right) = \Phi_p\left(x^{p^{\ell-k+j-1}}\right) = \Phi_{j+\ell-k}$ is irreducible in $\mathbb{Q}[x]$. Thus, the equality $\Phi_{p^j}(x^{\ell-k}) = a_p(x^{p^\ell})b_p(x^{p^\ell})$ implies that one of the polynomial expressions $a_p(x)$ or $b_p(x)$ is constant, as desired.

Suppose, without loss of generality, that $a_p(x) \in \mathbb{Q} \setminus \{0\}$. Since $\deg a(x)$ divides $\deg \Phi_{p^j}(x^{1/p^k}) = (p-1)/p^{k-j+1} \in M_p \cap (0,1)$, the fact that $M_p \cap (0,1)$ is divisor-closed in $M_{\mathscr{P}}$ ensures that $\deg a(x) \in M_p$. Therefore $\deg a(x) = \deg a_p(x) = 0$, which implies that $a(x) \in \mathbb{Q}$. Hence $\Phi_{p^j}(x^{1/p^k})$ is irreducible in $\mathbb{Q}[M_{\mathscr{P}}]$ for all $j,k \in \mathbb{N}$ with $1 \leq j < k$, and it is clear that none of these polynomial expressions are associates in $\mathbb{Q}[M_{\mathscr{P}}]$. Hence $x-1$ has infinitely many non-associates irreducible divisors, and we can conclude that $\mathbb{Q}[M_{\mathscr{P}}]$ is not an IDF-domain. $\square$

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