Bouvier's question revisited

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Let D be an integral domain with quotient field K and let $F(D)$ denote the set of nonzero fractional ideals of D. A star operation is a function $A \mapsto A^*$ on $F(D)$ with the following properties: If $A,B \in F(D)$ and $a \in K\backslash\{0\}$, then (i) $(a)^* = (a)$ and $(aA)^* = aA^*$, (ii) $A \subseteq A^*$ and if $A \subseteq B$, then $A^* \subseteq B^*$, and (iii) $(A^*)^* = A^*$. The more well known star operations are: $A \mapsto A_v = (A^{-1})^{-1}$ called the v-operation and $A \mapsto A_t = \cup\{F_v$: where F ranges over finitely generated nonzero ideals$\}$called the t-operation. (Here $A^{-1} = \{x \in K : xA \subseteq D\}$ as usual.) It is easy to see that for an integral ideal $A \in F(D)$, we have $A \subseteq A_t \subseteq A_v \subseteq D$. Alain Bouvier once asked if there is an integral ideal A such that $A \subsetneq A_t \subsetneq A_v \subsetneq D$. An example of such an ideal A was provided in [Z]. With the discovery of a star operation $\mu : A \mapsto A^\mu$ with $A^\mu \subseteq A_t$ for all $A \in F(D)$ it is natural to ask if there is an integral ideal A with $A \subsetneq A^\mu \subsetneq A_t \subsetneq A_v \subsetneq D$. The aim of this note is to provide an example of an ideal A such that $A \subsetneq A_w \subsetneq A_t \subsetneq A_v \subsetneq D$ where w is defined by $A \mapsto A_w = \{x \in K : xF \subseteq A$ for some finitely generated $F \in F(D)$ with $F_v = D\}$.

The w-operation has been studied as a useful new star operation, in its own right, in a recent paper [MW]. To be able to explain our example better we need to review some of the notions involved. The reader may confirm the definitions and statements made here from Gilmer [Gi, sections 32 and 34]. Let $*$ be a star operation. An ideal $A \in F(D)$ is said to be a $*$ ideal if $A = A^*$ and a star ideal of finite type if $A = B^*$ for some finitely generated $B \in F(D)$. An ideal $A \in F(D)$ is $*$ invertible if $(AA^{-1})^* = D$. The star operation $*$ is said to be of finite character if for all $A \in F(D)$ $A^* = \cup\{F^*$: where F is a nonzero finitely generated subideal of $A\}$. So, the t-operation is of finite character. If $*$ is of finite character the integral ideals maximal with respect to being integral $*$ ideals exist and are prime. It was shown in [MW] that the w-operation is a star operation of finite character, that for each pair $A,B \in F(D)$, $(A \cap B)_w = A_w \cap B_w$ and that for each $A \in F(D)$, $A_w = \bigcap AD_P$ where P ranges over w-max(D) the set of maximal w-ideals of D. Because for every star operation $*$ it is true that $A^* \subseteq A_v$, for each f.g. $A \in F(D)$ we have $A_w \subseteq A_v = A_t$, we conclude that for each $A \in F(D)$ $A_w \subseteq A_t$. It was shown in [AC] that w-max$(D) =$ t-max(D). Thus for each $A \in F(D)$, $A_w = \bigcap_{P \in t-Max(D)} AD_P$. Using the properties of star operations it is easy to establish that for each $A \in F(D)$, $(A_w)_t = (A_t)_w = A_t$. Recall that an integral domain D is a Prufer v-multiplication domain (PVMD) if each finitely generated $A \in F(D)$ is t-invertible. Recall also that D is a PVMD if and only if D_P is a valuation domain for each maximal t-ideal P of D [Gr]. Now a useful result of Kang [Kan, Theorem 3.5] says that an integral domain D is a PVMD if and only if D is integrally closed and for each ideal A of D, $A_t = \bigcap_{P \in t-Max(D)} AD_P = A_w$. Using essentially the same proof we can prove the following statement.

Proposition 1. An integral domain D is a PVMD if and only if D is integrally closed and for each two generated nonzero ideal A of D, $A_t = A_w$.

From this result, it follows that if D is integrally closed then D is a PVMD if and only if for every pair of elements $a,b \in D\backslash\{0\}$ $(a,b)_t = (a,b)_w$. Thus if D is an integrally closed non-PVMD then there must be a pair of elements $a,b \in D\backslash\{0\}$ such that $(a,b)_w \subsetneq (a,b)_t$. We claim that this (a,b) is not t-invertible. For, if $A = (a,b)$ is t-invertible then $(AA^{-1})_t = D$ and so $(AA^{-1}) \nsubseteq P$ for every maximal t-ideal P and so AD_P is principal, but then $AD_P = (AD_P)_t = A_t D_P$. This means that $A_w = \bigcap_{P \in t-max(D)} AD_P = \bigcap_{P \in t-max(D)} A_t D_P = (A_t)_w = A_t$. For the same reasons $A = (a,b)$ is not w-invertible. Now if we recall that D is essential if there is a set of primes $\{P_\alpha\}$ such that D_{P_α} is a valuation domain for each α and $D = \bigcap D_{P_\alpha}$, we are in a position to state the following result.

Proposition 2. Let D be a locally GCD non PVMD with at least one pair of v-coprime elements that are not comaximal then there exists an ideal $A \in F(D)$ such that $A \subsetneq A_w \subsetneq A_t \subsetneq A_v \subsetneq D$.

Proof. Take R any locally GCD non PVMD then $R[X] = D$ has the property required in the hypothesis. Being a locally GCD domain, D is, essential and so, integrally closed and being a non-PVMD D contains a pair of nonzero elements a, b such that $(a,b)_w \subsetneq (a,b)_t$ as we have already shown. From the above discussion it also follows that $((a,b)((a) \cap (b)))_w \neq (ab) \neq ((a,b)((a) \cap (b)))_t$. However as D is an essential domain we have $((a,b)((a) \cap (b)))_v = (ab)$ [Z] Let us put $\mathsf{A} =((a,b)((a) \cap (b)))$ and show that $\mathsf{A}_w \subsetneq \mathsf{A}_t$. Suppose that this is not the case then $\mathsf{A}_w = \mathsf{A}_t$. So for each maximal t-ideal P, $\mathsf{A}D_P = \mathsf{A}_w D_P = \mathsf{A}_t D_P$ which gives $((a,b)((a) \cap (b)))D_P = ((a,b)((a) \cap (b)))_t D_P \supseteq ((a,b)_t((a) \cap (b)))D_P$. Since D_P is a GCD domain we have $(a,b)D_P \supseteq (a,b)_t D_P$ which means that $(a,b)D_P = (a,b)_t D_P$ which in turn leads to $(a,b)_w = (a,b)_t$. So $\mathsf{A}_w \subsetneq \mathsf{A}_t \subsetneq (ab) = \mathsf{A}_v \subsetneq D$. Now as D has a pair of v-coprime elements that are not comaximal, not every maximal ideal of D is a t-ideal. So there is at least one maximal ideal M such that $M_t = D$. But this means that there is a finitely generated ideal $F \subseteq M$ such that $F_t = D = F_w = F_v$. We claim that $A = F\mathsf{A}$ is the required ideal. This is because $A_* = (F\mathsf{A})_* = (F_*\mathsf{A})_* = \mathsf{A}_*$ for $* =$ w, t or v. Clearly $A \subsetneq \mathsf{A} \subseteq \mathsf{A}_w \subsetneq \mathsf{A}_t = A_t \subsetneq \mathsf{A}_v = A_v = (ab) \subsetneq D$. Noting that $\mathsf{A}_w = A_w$ we have our example of an ideal A such that $A \subsetneq A_w \subsetneq A_t \subsetneq A_v \subsetneq D$.

Acknowledgement *I came up with this form of Bouvier's question at the Fez Conference* (2001). *I am thankful to the organizers for inviting me to the conference.*

References:

[AC] D. Anderson and S. Cook, Two star-operations and their induced lattices. Comm. Algebra 28 (2000), no. 5, 2461–2475.

[Gi] R. Gilmer, Multiplicative Ideal Theory, Marcel-Dekker, New York, 1972.

[Gr] M. Griffin, Some results on v-multiplication rings. Canad. J. Math. 19 1967 710–722.

[Kan] B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. J. Algebra 123 (1989), no. 1, 151–170.

[MW] R. McCasland and F. Wang, On w-modules over strong Mori domains. Comm. Algebra 25 (1997), no. 4, 1285–1306.

[Z] M. Zafrullah, The $D + XD_S[X]$ construction from GCD-domains, J. Pure Appl. Algebra 50(1988), 93-107.