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ON PRÜFER v-MULTIPLICATION DOMAINS

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Prüfer v-multiplication domains, abbreviated PVMD's, have among their special cases a variety of notions, including Krull domains, Prüfer domains, GCD domains, and unique factorization domains. The special cases of PVMD's have been the subject of many studies while PVMD's, in general, have received relatively little attention. We attempt to bridge this gap in this paper.

To give a better idea of our results, we include the following notions and notations.

Throughout, the letters R and K denote a commutative integral domain and its field of fractions respectively.

Let $F(R)$ denote the set of fractional ideals of R. Associated to each $A \in F(R)$ is the fractional ideal $(A^{-1})^{-1} = A_v$. The map $A \mapsto A_v$, on $F(R)$, is a *-operation called the v-operation. The reader is referred to sections 32 and 34 of [6] for the definition

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and properties of *-operations. However for our purposes we note the following.

Let $A, B \in F(R)$. Then

0.1 $A \subseteq A_v$, $(A_v)_v = A_v$ and $R_v = R$,

0.2 $A \subseteq B$ implies $A_v \subseteq B_v$,

0.3 if A is principal then $(AB)_v = AB_v$,

0.4 $F(R)$ is closed under the v-multiplication:

$$(AB)_v = (A_vB)_v = (A_v B_v)_v .$$

A fractional ideal A is called a v-ideal if $A = A_v$. Moreover, a v-ideal is of <u>finite type</u> if there is a finitely generated fractional ideal B such that $A = B_v$. Finally, we can take R to be the identity element of $F(R)$ whereas $F(R)$ is a semigroup under the operation defined in 0.4.

Let $H(R)$ denote the set of v-ideals of R of finite type. Then R is a PVMD if $H(R)$ is a group under the v-multiplication defined in 0.4.

We obtain a simpler definition of PVMD by introducing the notion of P-domain. Recall from [2], that a prime ideal P of R is an associated prime of a principal ideal aR, if P is minimal over $(aR: bR)$ for some $b \in R - aR$. For brevity we shall call P an <u>associated prime</u> of R. Then we say that R is a <u>P-domain</u> if R satisfies the following property:

(P). For every associated prime P of R, R_P is a valuation domain.

In section 1 we study P-domains and show that a PVMD is a P-domain. Then in section 2 we construct an example to show that a P-domain is not necessarily a

PVMD. In section 3 we show that a P-domain is a PVMD if and only if for each pair $a, b \in R$ there exists a finitely generated ideal A such that $aR \cap bR = A_v$. Also in section 3 we give other characterizations of a PVMD.

By an <u>overring</u> of R we mean a ring between R and K. A valuation overring V of R is called essential if $V = R_P$. A prime ideal P is called an <u>essential</u> or a <u>valued</u> prime if R_P is a valuation domain. In section 4 we study the valued primes of PVMD's and show that they are nothing more than the prime t-ideals. Sheldon [14] made a similar study for GCD-domains. It is well known that if R is a Krull domain then any overring T of the form $T = \cap R_{P_\alpha}$, where P_α ranges over a set of height one prime ideals of R, is also a Krull domain. In section 5 we show that if R is a PVMD and $T = \cap R_{P_\alpha}$, where P_α ranges over a set of valued primes of R, then T is a PVMD. We study the relationship between PVMD's and GCD-domains in section 6. Moreover, we derive an equivalent form of Hasse's criterion for PID's and show that there is an analogous form for UFD's. Finally in section 7 we discuss some generalizations of Krull domains.

1. INTEGRAL DOMAINS WITH PROPERTY P

Recall that an integral domain R is called <u>essential</u> if it can be expressed as an intersection of essential valuation overrings of itself.

<u>PROPOSITION 1.1.</u> <u>The following conditions are</u>

equivalent for an integral domain R .

(1) R *is an essential integral domain such that every quotient ring of* R *is also essential.*

(2) R *is a P-domain.*

PROOF. (1) $\Rightarrow$ (2) . Suppose that M is a prime ideal minimal over $aR\colon bR$. Then by (1), R_M is essential and by a theorem of Tang (Theorem E of [15] or exercise 22, p. 52 of [6]) and by Proposition 4 of [2] $aR_M\colon bR_M$ is contained in at least one valued prime PR_M of R_M . But then it follows from the minimality of M that $P = M$ and M is a valued prime of R .

(2) $\Rightarrow$ (1) . Let $\{P_\alpha\}$ be the family of associated primes of R . Then $R = \bigcap R_{P_\alpha}$ and $R_S = \bigcap\{R_{P_\alpha} \mid P_\alpha \cap S = \phi\}$ (cf. [2]). By (2), each P_α is essential and R_S is essential.

COROLLARY 1.2. *Let* S *be a multiplicitive set in* R *and let* X *be an indeterminate over* R . *If* R *has property* P *then so do* R_S *and* $R[X]$.

PROOF. That R_S has property P is clear from the proof of Prop. 1.1 . For the second part we show that (2) of Proposition 1.1 holds for $R[X]$. Let P be an associated prime of $R[X]$. If $P \cap R = 0$ then it is obvious that $R[X]_P$ is a valuation domain. If, on the other hand, $P \cap R \neq 0$ then, by Corollary 8 of [2], $P = p[X]$ where p is an associated prime of R and hence is a valued prime. That $R[X]_{p[X]}$ is a valuation domain is now easy to verify.

COROLLARY 1.3. *The following are equivalent for an* *integral domain* R .

(1) R *has property* P .

(2) *Every quotient ring of* R *has property* P .

(3) *For every prime ideal* P *of* R , R_P *has property* P.

(4) *For every maximal ideal* M *of* R , R_M *has property* P .

(5) *Every flat overring of* R *has property* P .

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are easy to verify. We now proceed to show that (5) is equivalent to the rest. For this we recall that S is a flat overring of R if and only if for each maximal ideal M of S , $S_M = R_{M\cap R}$. Now (5) $\Rightarrow$ (2) because every quotient ring of R is a flat overring. Further (2) $\Rightarrow$ (5) because if S is a flat overring of R then for each maximal ideal M of S , $S_M = R_{M\cap R}$. Now by (4), which is equivalent to (2) , S has property P .

According to [9] (Props. 4 and 5), a PVMD is essential and according to [11] every quotient ring of a PVMD is again a PVMD and hence is essential. Thus we have the following corollary.

COROLLARY 1.4. A PVMD has Property P .

Corollary 1.4 gives rise to the inevitable question: Is a P-domain a PVMD? The answer is no and the counter example is given in the next section.

2. EXAMPLE OF A P-DOMAIN WHICH IS NOT A PVMD

EXAMPLE 2.1. Let k be a field Y, Z, $X_1,\ldots,X_n,\ldots$ indeterminates over k and let $R = k(X_1,\ldots,X_n,\ldots)[Y, Z]_{(Y, Z)}$ be a regular local ring of dimension 2.

Let V_i denote valuation rings, containing $k(\{X_j\}_{j \neq 1})$ obtained by defining the valuation $v_i(t) = 0$ for all $t \in k(\{X_j\}_{j \neq 1})$ $v_i(X_i) = 1 = v_i(Y) = v_i(Z)$ and $v_i(f(X_i, Y, Z)) = v_i(\sum a_{i_1 i_2 i_3} X_i^{i_1} Y^{i_2} Z^{i_3}) = \inf\{i_1 + i_2 + i_3\}$.

Let $D = R \cap \{V_i\}$, $i = 1, 2, \ldots$ where each of V_i, defined above, is a discrete rank one valuation domain. According to Heinzer and Ohm [11] the integral domain D is an example of an essential domain which is not a PVMD. We now proceed to show that the domain D has the property P. According to Corollary 1.3 it is sufficient to show that for each maximal ideal M of D, D_M has the property P. In the following we proceed to study the maximal ideals of D.

$$\begin{aligned} \text{Let } D_j &= D \cap k(X_1, \ldots X_j, Y, Z) \\ &= k(X_1, \ldots, X_j)\,[Y, Z]_{(Y, Z)} \cap V_1 \cap \ldots \cap V_j . \end{aligned}$$

Now $R_j = k(X_1, \ldots, X_j)\,[Y, Z]_{(Y, Z)}$ is a UFD and, as already mentioned, each V_i is a DVR. Furthermore, D_j has exactly $j + 1$ maximal ideals. This is so because of the following two reasons.

(1) The centres $P_1', \ldots, P_j'$ of V_j respectively are maximal ideals of D_j (cf. Corollary 1.16 of [10]).

(2) R_j is a quotient ring of D_j.

Clearly, (1) above gives j maximal ideals. For the remaining one we note that R_j is a local ring and so $R_j = (D_j)_{P_0'}$ where P_0' is a prime ideal. We claim that P_0' is a maximal ideal of D_j. Clearly there are no containment relations between $P_1',\ldots,P_j'$. If $P_i' \supseteq P_0'$ then $R_j = (D_j)_{P_0'} \supseteq V_i = (D_j)_{P_i'}$. But R_j is a 2-dimensional local ring and V_i is a DVR, a contradiction. Whence there are no containment relations between $P_0', P_1',\ldots,P_j'$. Now $D_j = R_j \cap V_1 \cap V_2 \cap \ldots \cap V_j$ and by Theorem 105 of Kaplansky [13] $P_0', P_1',\ldots,P_j'$ are exactly the maximal ideals of D_j. Further since each localization of D_j is a UFD and D_j has only finitely many maximal ideals it follows that D_j is a UFD. Now it is obvious that $D = \bigcup D_j$ - an ascending union of UFD's. According to [11], $R = D_M$ and $V_i = D_{P_i}$ where M is a maximal ideal and P_i's are the centres of the V_i. It is easy to see that any prime P of D is the union of $P \cap D_j$ and M, $\{P_i\}$ are precisely the maximal ideals of D.

Now $R = D_M$ is a regular local ring of dimension 2, a UFD, and hence a domain with property P. Similarly D_{P_i}, being DVR's, are domains with property P. Whence it follows, in view of Corollary 1.3, that D has property P.

3. SOME CHARACTERIZATIONS OF PVMD'S

In this section we first prove the following simple result and then consider other characterizations.

THEOREM 3.1. *Let D be an integral domain with the property that for each pair $a, b \in D$; $aD \cap bD$ is of finite type. Then the following are equivalent.*

(1) *D is an essential domain.*

(2) *D has property P.*

(3) D is a PVMD.

(4) D is locally a PVMD (i.e., for each maximal ideal M, D_M is a PVMD).

PROOF. The proof is based on the fact that D in each case is an essential domain because according to Lemma 8 of [17] an essential domain is a PVMD if and only if it satisfies the hypothesis of Theorem 3.1.

Theorem 3.1 gives us the following two simple characterizations of PVMD's.

THEOREM 3.2. *An integral domain D is a PVMD if and only if D is a P-domain and for each pair $a, b \in D$; $aD \cap bD$ is of finite type.*

THEOREM 3.3. *An integral domain D is a PVMD if and only if D is essential and for each pair $a, b \in D$; $aD \cap bD$ is of finite type.*

We take 3.2 as our definition of a PVMD. We could do the same with 3.3 but a simple test for an integral domain to be essential is not known. It may be that the property P and the property of being an essential domain are the same; it seems unlikely, but at present we do not know a counterexample.

In the remaining part of this section we derive a necessary and sufficient condition for a P-domain to be a PVMD; from a known characterization of PVMD's. For this we need the notions of Kronecker functions rings and of v-domains. We also need some related definitions. For completeness we include a working introduction to these concepts (and refer the reader to sections 32 and 34 of [6] for details).

Let X be an indeterminate over R and let f be a polynomial in $R[X]$. Then the ideal A_f generated by the coefficients of f is called the <u>content</u> of f. Further let $\{V_\alpha\}_{\alpha\in I}$ be a family of valuation overrings of R such that $R = \cap V_\alpha$. Then $A \mapsto \cap AV_\alpha = A_w$ is another $*$-operation on $F(R)$ called the w-operation induced by $\{V_\alpha\}$. The ring $R^w = \cap V_\alpha(X)$ is a Bezout ring called the Kronecker function ring of R with respect to w. (Note that $V_\alpha(X)$ denotes the trivial extension of V_α to $K(X)$.) Moreover

$$R^w = \{f/g \mid f, g \in R[X]; (A_f)_w \subseteq (A_g)_w\}.$$

Finally an integral domain R is a v-domain if for any finitely generated ideals A, B, and C of R, $(AB)_v \subseteq (AC)_v$ implies that $B_v \subseteq C_v$. In particular an essential domain is a v-domain (cf. [6], 44.13).

<u>THEOREM 3.4.</u> <u>Let</u> R <u>be a P-domain and let</u> $S = \{f \in R[X] \mid (A_f)_v = R\}$. <u>Then</u> R <u>is a PVMD if and only if each associated prime</u> P <u>of</u> $R[X]$ <u>with</u> $P \cap S = \phi$ <u>is such that</u> $P \cap R \neq (0)$.

The proof of Theorem 3.4 depends upon the following theorem of Gilmer [5].

<u>THEOREM</u> (Theorem 2.5 of [5]). <u>Suppose that</u> J <u>is a v-ring and that</u> J^v <u>is the Kronecker function ring of</u> J <u>with respect to the v-operation. Then</u> J^v <u>is a quotient ring of</u> $J[X]$ <u>if and only if</u> J <u>is a PVMD. In particular if</u> J <u>is a PVMD then</u> J^v <u>is a flat J-module.</u>

<u>PROOF OF THEOREM 3.4.</u> Let R be a PVMD. Then according to the proof of Theorem 2.5 of [5],

$R^v = R[X]_S$ where S is described in the hypothesis.

Now it is well known that if R is a PVMD then for every prime ideal Q of R^v; $Q \cap R \neq (0)$. If P is an associated prime of $R[X]$ with $P \cap S = \emptyset$ then $PR[X]_S$ is a prime of R^v. Whence $PR[X]_S \cap R \neq 0$; that is $P \cap R \neq (0)$.

Conversely by Proposition 4 of [2], $R = \bigcap R_{P_\alpha}$ where $\{P_\alpha\}$ consists of all the associated primes of R. In view of the property P, R_{P_α} are valuation domains. Let w be the *-operation induced by $\{R_{P_\alpha}\}$. By 44.13 of [6], $R^w = R^v$ and we need only show that $R^w = R[X]_S$ where $S = \{f \in R[X] \mid (A_f)_v = R\}$.

First we note that, for $f \in R[X]$ $(A_f)_v = R$ if and only if $(A_f)_w = R$. For $(A_f)_w = \bigcap A_f R_{P_\alpha}$ and since A_f is finitely generated, by Lemma 4 of [17] $R_{P_\alpha} = ((A_f)_v R_{P_\alpha})_v = (A_f R_{P_\alpha})_v = A_f R_{P_\alpha}$ (because R_P is a valuation domain).

Clearly $R^w \supseteq R[X]$. Further, it is easy to verify that, if $f, g \in R[X]$ with $(A_f)_v = (A_g)_v = R$. Then $(A_{fg})_v = R$. Moreover, $(A_{fg})_v = R$ implies that $(A_f)_v = (A_g)_v = R$, that is, S is a saturated multiplicative set. This shows that $R^w = R^v \supseteq R[X]_S$.

Now $R[X]_S = \bigcap R[X]_{Q_\gamma}$ where Q_γ ranges over associated primes of $R[X]$ such that $Q_\gamma \cap S = \emptyset$ ([2], Proposition 4). In view of Corollary 1.2, Q_γ are valued primes. Further since $Q_\gamma \cap R \neq (0)$, $Q_\gamma = q_\gamma[X]$ where q_γ is an associated prime of R (cf. [2], Corollary 8). Now corresponding to each $R[X]_{Q_\gamma}$ there is R_{q_γ} in $\{R_{P_\alpha}\}$ such that $R_{q_\gamma}(X) = R[X]_{Q_\gamma}$. So that $\bigcap R[X]_{Q_\gamma} = \bigcap R_{q_\gamma}(X) \supseteq \bigcap R_{P_\alpha}(X) = R^W$. Now because we have already shown that $R^W \supseteq R[X]_S$, $R^W = R[X]_S$.

4. VALUED PRIMES OF PVMD'S

In the course of our study of valued primes of PVMD's we shall use the notion of t-ideals. Briefly an ideal A of an integral domain R is a t-ideal if $A = \bigcup (F)_v$ where F ranges over the finitely generated ideals contained in A. For a detailed study of t-ideals and related notions the reader is referred to Jaffard [12]. For our purposes we note the following.

A prime t-ideal is a t-ideal which is also prime while a maximal t-ideal M is a t-ideal such that if N is a t-ideal properly containing M then N equals R. It is easy to establish that a maximal t-ideal is a prime ideal. Moreover, a minimal prime of a t-ideal is a t-ideal (cf. [12], Corollary 3, p. 31). Prime t-ideals are important in the study of PVMD's because of a result of Griffin. He proves in [9], that R is a PVMD if and only if for each maximal t-ideal P of R, R_P is a valuation domain. In [9], he also shows that if R is a PVMD then $R = \bigcap R_P$ where P ranges over maximal t-ideals of R.

From the definition of a t-ideal it follows that if A is a t-ideal then for every finitely generated ideal $F \subseteq A$, $F_v \subseteq A$. We use this fact to prove the following proposition.

PROPOSITION 4.1. *Let* R *be a PVMD. Then a prime ideal* P *of* R *is a valued prime if and only if it is a t-ideal.*

PROOF. Let P be a t-ideal. Then it is contained in a maximal t-ideal M. Since R is a PVMD, R_M is a valuation domain. Consequently R_P is a valuation domain also.

Conversely let P be a valued prime in R and let F be a finitely generated ideal contained in P. We show that $F_v \subseteq P$. Clearly, since F is finitely generated, by Lemma 4 of [17] $(F_v R_P)_v = (FR_P)_v$. Further since R_P is a valuation domain and $F \subseteq P$, FR_P is principal and so $(FR_P)_v = FR_P$. Now $(F_v R_P)_v = (FR_P)_v = FR_P \neq R_P$ and this demands that $F_v \subseteq P$.

We note here that in the above proposition the only if part does not need the assumption that R should be a PVMD. We record this observation as the following corollary.

COROLLARY 4.2. *Let* R *be an integral domain and let* P *be a valued prime in* R *then* P *is a t-ideal.*

COROLLARY 4.3. *Let* R *be an integral domain. Then the following are equivalent:*

(1) R *is a* PVMD.

(2) *Every prime t-ideal of* R *is valued.*

(3) *Every maximal t-ideal of* R *is valued.*

In [14], Sheldon studied *Prime Filter (PF)* ideals in GCD domains. By a PF-prime he meant a prime ideal P such that $a, b \in P$ inplies $GCD(a, b) \in P$. We note that in a GCD domain R, $GCD(a_1, \ldots, a_n) = (a_1, \ldots, a_n)_v$. So a PF-prime is a special case of a prime t-ideal. We note that, in view of Proposition 4.1, most of the results proved in [14] for PF-primes in GCD domains can be rephrased for prime t-ideals in PVMD's. For example some of the results stated in the following proposition are restatements of Sheldon's results.

PROPOSITION 4.4. *Let R be a PVMD then the following hold.*

(1) *The family $\{P_\alpha\}$ of prime t-ideals of R forms a tree under inclusion, that is, for any two prime t-ideals P_1, P_2 either $P_1 \subseteq P_2$, $P_2 \subseteq P_1$, or $(P_1, P_2)_t = R$.*

(2) *For every minimal prime P of R, R_P is a valuation domain.*

(3) *R is a Prüfer domain if and only if one of the following holds:*

  (a) *every prime ideal of R is a t-ideal;*

  (b) *every maximal ideal is a t-ideal;*

  (c) ~~every maximal t-ideal of R is a maximal ideal~~; or

  (d) *prime ideals of R form a tree under inclusion.*

(4) *Every prime ideal of R is a union of prime t-ideals.*

(5) *If F is a finitely generated ideal of R with $F^{-1} \neq R$ then F_v is contained in at least one valued prime.*

PROOF. (1). Let P be a prime t-ideal. Then since P is valued, any two prime ideals P_1, P_2 contained in P are (i) valued, (ii) comparable to each other under inclusion, and (iii) t-ideals (being valued).

(2). A minimal prime ideal is clearly an associated prime of each principal ideal $aR \subseteq P$. Now use the fact that a PVMD is a P-domain.

(3). It is well known that R is a Prufer domain if and only if R_M is a valuation domain for each maximal ideal M of R . The conditions (a) - (d) can be shown to be equivalent in the light of the above mentioned result.

(4). Every minimal prime of a principal ideal aR is an associated prime in the sense that it is minimal over $aR:R$. Now for each prime ideal P we have $P = \bigcup xR$ where $x \in P$. Moreover there is a prime ideal P', contained in P , minimal over xR for each $x \in P$. So that $P = \bigcup xR \subseteq \bigcup P' \subseteq P$, where P' ranges over minimal primes of principal ideals described above. Now the result follows from (2) above.

(5). Since R is a PVMD each associated prime of R is a valued prime. Now the result follows from Theorem E of Tang [15] and from the fact that in R an associated prime is valued and hence is a t-ideal.

5. AN ANALOGY BETWEEN PVMD'S AND KRULL DOMAINS

In the well studied case of Krull domains it is customary to talk of subintersections of a defining family of valuation domains in the following sense. Let R be a Krull domain and let X^1 denote the set of height one prime ideals of R . If $Y \subseteq X^1$, then the ring $T = \bigcap_{P \in Y} R_P$ is called a subintersection. Since for a Krull domain X^1 is precisely the set of prime t-ideals we extend the definition to an arbitrary domain R . If

X denotes the set of prime t-ideals of R and $Y \subseteq X$, then the ring $T = \bigcap_{P \in Y} R_P$ is a <u>subintersection</u> of R. Now one aspect of the traditional definition of a subintersection of a Krull domain has been omitted, namely, that each R_P is a valuation ring for each $P \in Y$. We take that aspect into consideration in the following definition. Let V denote the set of valued primes of R and $Y \subseteq V$ then $T = \bigcap_{P \in} R_P$ is said to be an <u>R-essential</u> overring of R.

In view of Corollary 4.2, any R-essential overring of R is a subintersection of R. Moreover, for PVMD's the two concepts are equivalent.

It is well known that a subintersection of a Krull domain is again a Krull domain, we extend this result to PVMD's.

<u>PROPOSITION 5.1</u>. <u>Let</u> R <u>be a</u> PVMD. <u>Then any subintersection of</u> R <u>is a</u> PVMD.

<u>PROOF</u>. From Theorem 2.5 of [5] it follows that if R is a PVMD, then $R[X]_S$, with $S = \{f \in R[X] \mid (A_f)_v = R\}$ is a Bezout domain. Now let $T = \bigcap_{P \in Y} R_P$, where Y is a subset of X = the set of prime t-ideals of R = the set of valued prime ideals of R. Then, if w is the *-operation induced by the valuations rings R_P for $P \in Y$, $(A_f)_w = \bigcap_{P \in Y} A_f R_P = T$ for each $f \in S$. Thus, $T^w \supseteq R[X]_S$ and since $R[X]_S$ is a Bezout domain, T^w is a quotient ring of $R[X]_S$ and hence of $R[X]$. Therefore T^w is a quotient ring of $T[X]$ and thus T is a PVMD by Theorem 2.5 of [5].

6. FROM PVMD'S TO GCD DOMAINS

It is easy to see that an integral domain R is a GCD domain if and only if for every finitely generated fractional ideal A of R, A_v is a principal fractional ideal; that is a GCD domain is a special case of PVMD's. It is, then, natural to ask, "Under what conditions is a PVMD a GCD domain?" Before we list some conditions we recall that an integrally closed integral domain R is a Schreier ring if for $a \mid bc$ in R for all appropriate $a, b, c \in R$, then $a = a_1a_2$ where $a_1 \mid b$ and $a_2 \mid c$ (cf. [4]). From [4] again, we recall that in a Schreier domain an irreducible element is prime.

PROPOSITION 6.1. *Let R be a PVMD. Then the following statements are equivalent.*

(1) *R is a GCD domain.*

(2) *For any two fractional ideals A, B of R such that A_v and B_v are of finite type, $(AB)_v = R$ implies that A_v and B_v are principal.*

(3) *For any two integral ideals A, B of R with A_v, B_v of finite type $(AB)_v = dR$ implies that A_v and B_v are principal.*

(4) *R is a Schreier ring.*

(5) *If X is an indeterminate over R then every irreducible element in $R[X]$ is a prime.*

PROOF. It is clear that (2) and (3) are equivalent. For the rest we show that (1) $\Leftrightarrow$ (3) and (1) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (3). Obvious from the definition of a GCD domain given in the beginning of this section.

(3) $\Rightarrow$ (1). We note that R is a PVMD. So according to Griffin [9], for all a, b in R $((aR \cap bR)(aR + bR))_v = abR$. Clearly $aR \cap bR$ is of finite type and so, by (3), $(aR \cap bR)_v = aR \cap bR$ is principal. Now recall that for R to be a GCD domain it is necessary and sufficient that $aR \cap bR$ is principal for all a, b in R.

(1). $\Rightarrow$ (4). Recall, from [4], that a GCD domain is a Schreier ring.

(4) $\Rightarrow$ (5). Recall, again from [5], that if R is a Schreier ring then so is $R[X]$. Now according to remarks prior to this proposition every irreducible element in a Schreier domain is a prime.

(5) $\Rightarrow$ (1). To show that R is a GCD domain it is sufficient to prove that for every prime P of $R[X]$ such that $P \cap R = 0$, P is principal (cf. IV of Theorem I of Tang [15]). Now since $P \cap R = (0)$, P is a rank one prime of $R[X]$ and hence an associated prime of $R[X]$. Further since R is a PVMD we use Theorem 3.4 to show that the prime ideal P described above intersects the set $S = \{f \in R[X]\ (A_f)_v = R\}$. For this we note that if $P \cap S = \emptyset$ then $P \cap R \neq 0$ and this contradiction gives us the required result that $P \cap S \neq \emptyset$. Consequently P contains a primitive polynomial of $R[X]$. Now according to (5) a primitive polynomial of $R[X]$ is a product of primes. So that P contains a principal prime. But since P is of height one, it is principal.

Since a Prüfer domain is a PVMD we have the following corollary which, incidentally, is Theorem 2.8 of [4].

COROLLARY 6.2. A Prüfer domain is a Bezout domain *if and only if it is Schreier.*

The proof of (5) $\Rightarrow$ (1) gives rise to the following rather interesting result.

COROLLARY 6.3. (to the proof of (5) $\Rightarrow$ (1)). *Let R be a Schreier ring. Then R is a GCD domain if and only if for each prime ideal P of $R[X]$ with $P \cap R = (0)$, P contains a primitive polynomial.*

Recall that an integral domain R is said to be *atomic* if every non-zero non-unit of R is expressible as a finite product of *atoms* (irreducible elements). In case of atomic integral domains the GCD property follows from a very weak condition.

PROPOSITION 6.4. *An atomic integral domain R is a UFD if and only if for any two coprime elements a, b in R, $(a, b)_v = R$.*

PROOF. Clearly if R is a GCD domain and a, b in R are coprime then $(a, b)_v = R$. So if R is a UFD the condition holds. Conversely suppose that R is atomic and for a, b coprime in R, $(a, b)_v = R$. All we need to show is that every atom in R is a prime. Let a be an atom in R and suppose that $a \mid bc$. If $a \nmid b$ then a, b are coprime and so $(a, b)_v = R$. Now $(ac, bc)_v = c(a, b)_v = cR$. If we let $bc = ad$ we have $(ac, bd)_v = a(c, d)_v = cR$. Finally since $(c, d)_v$ is an integral ideal, $a \mid c$.

We say that an integral domain R satisfies the property (λ) if for any two coprime elements a, b of R $(a, b)_v = R$. The following result is an obvious consequence of the proof of Proposition 6.4.

COROLLARY 6.5. _If an integral domain R satisfies the property (λ), then every atom in R is a prime._

A number of integral domains satisfy the property (λ) . For example it is easy to verify that a Schreier domain satisfies this property. Then there is the obvious case of the pre-Bezout domains, of Cohn, in which any two co-prime elements a, b are comaximal i.e., $ua + vb = 1$ for some u, b in the same integral domain. Finally there are integral domains which satisfy the so called PSP-property (cf. Arnold and Sheldon [1]). Here an integral domain R is said to have the PSP property if every primitive polynomial over R is superprimitive; that is, if $f \in R[X]$ is primitive then $(A_f)_v = R$. Clearly the property (λ) is a generalization of the PSP-property and in fact it can be termed as the _linear PSP-property_. For details on PSP-property the reader is referred to [1]; what interests us here is that in certain cases the linear PSP-property (or (λ)) is equivalent to the PSP-property.

Although, like Schreier rings, the rings with property (λ) have the property that if a, b have a common factor they have a higher common factor; the property (λ) does not imply the Schreier property. This follows from an example, in [4], provided by G.M. Bergmann, of a pre-Bezout ring which is not Schreier. This observation indicates that the property (λ) is more general than the Schreier property. It is not clear whether a PVMD with the property (λ) should be a GCD domain, as in the case of Schreier property (cf. Proposition 6.1). A detailed study of this property is not our purpose, so we postpone it to some future publication and consider a closely related special case of the property (λ) . The pre-Bezout property, which is a special case of the property (λ), gives us a rather interesting criterion for an integral domain R to be a PID.

COROLLARY 6.6. *An integral domain R is a PID if and only if (1) R is atomic and (2) each pair of coprime elements of R are comaximal.*

PROOF. Clearly a PID is atomic and its coprime elements are comaximal. To prove the converse we show that the conditions (1) and (2) imply the celebrated *Hasse Criterion* for PID's. For the sake of completeness we include the statement of the criterion.

Hasse Criterion. *An integral domain R is a PID if and only if there exists a function f from $R - \{0\}$ to the set of natural numbers such that*

(H_1) $a \mid b$ *implies* $f(a) \leq f(b)$ *with equality if* $b \mid a$ *also.*

(H_2) *if* $a \nmid b$ *and* $b \nmid a$ *then there exist* u, v, d *in* R *such that* $d = ua + vb$ *and* $f(d) < \min(f(a), f(b))$.

Now according to Proposition 6.4 R is a UFD. So we can define a function f from $R - \{0\}$ to the set of natural numbers such that (H_1) is satisfied. Now if $a \nmid b$ and $b \nmid a$, $GCD(a, b) = d$ divides a and b properly and thus $f(d) < \min(f(a), f(b))$. Now we can write $a = a_1 d$ and $b = b_1 d$ where a_1 and b_1 are coprime. By condition (2) above there exist u, v in R such that $ua_1 + vb_1 = 1$, that is $ua_1 d + vb_1 d = d$. Thus Hasse criterion is satisfied and consequently R is a PID.

7. SOME GENERALIZATIONS OF KRULL DOMAINS

Let $\{V_\alpha\}_{\alpha\in I}$ be a family of valuation overrings of R and consider the following conditions (on R).

(1) $R = \bigcap V_\alpha$.

(2) For every non-zero non-unit x in R, x is a non-unit in only a finite number of V_α.

(3) Each V_α is essential.

(4) Each V_α is of rank one.

(5) Each V_α is rank one discrete.

Integral domains which satisfy (1) and (2) are called rings of finite character. Griffin [8] gave the name, '<u>Rings of Krull type</u>' to integral domains satisfying (1), (2), and (3). The integral domains which satisfy (1), (2), (3), and (4) are called <u>generalized Krull domains</u>. This name is again due to Griffin [8]. Finally the integral domains satisfying (1), (2), (3), and (5) are the well known Krull domains. Rings of finite character are not necessarily PVMD's but rings of Krull type, as shown by Griffin in [9], are PVMD's. In this section we show that, as the study of Krull domains was made in the light of results on UFD's, we can study rings of Krull type with a reference to GCD rings of finite character. Following this line we introduce a further generalization of Krull domains with reference to a generalization of UFD's discussed in [18].

By Lemma 13 of [3], an integrally closed integral domain is a ring of finite character if and only if it has a Kronecker function ring of finite character. For PVMD's of finite character we have the following Proposition.

PROPOSITION 7.1. *The following are equivalent for an integral domain* R .

(1) R *is a PVMD and a ring of finite character.*

(2) R *is a PVMD and* R^V *is a ring of finite character.*

(3) R *is a PVMD and every non-zero non-unit of* R^V *is divisible by at most a finite number of mutually coprime non-units.*

(4) R *is a ring of Krull type.*

PROOF. We show that (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) .

(1) $\Rightarrow$ (2). If R is a ring of finite character then so are $R[X]$ and R_S where X is an indeterminate over R and S is a multiplicative set of R (cf. [9]). Now, since R is a PVMD, by Theorem 2.5 of Gilmer [6] R^V is a quotient ring of $R[X]$ and hence a ring of finite character.

(2) $\Rightarrow$ (3). This equivalence follows from the fact that a Bezout ring of finite character is a ring of Krull type and from the results of [16].

(3) $\Rightarrow$ (4). We note that $R = R^V \cap K$ where K is the quotient field of R . But since R is a PVMD, $R^V = \cap R_{P_\alpha}(X)$ where P_α ranges over the maximal t-ideals of R . So that $R = R^V \cap K = \cap R_{P_\alpha}$ where each non-zero non-unit of R is a non-unit in only a finite number of R_{P_α} .

(4) $\Rightarrow$ (1). Obvious.

COROLLARY 7.2. *Let* R *be a ring of Krull type. Then every subintersection of* R *(e.g. a flat overring of* R*) is a ring of Krull type.*

PROOF. Let R be a ring of Krull type and let T be a subintersection. By Proposition 5.1, T is a PVMD.

Now T^v being an overring of R^v, and hence being a quotient ring of R^v, T^v is a ring of finite character.

According to results of section 4, if R is a PVMD then every minimal prime of a principal ideal of R is a valued prime. Clearly if R is a ring of Krull type then every principal ideal of R has only a finite number of minimal primes. We generalize rings of Krull type to those PVMD's whose principal ideals have finitely many minimal primes and call them pre-Krull rings. Pre-Krull GCD domains have some interesting factorization properties which can be described as follows. (For details the reader is referred to [18]).

Let R be a GCD domain. An element x in R is called a packet if for any factorization $x = x_1 x_2$ of x, $x_1 \mid x^2$ or $x_2 \mid x_1^2$. A product of finitely many packets, in a GCD domain, is expressible as a product of mutually coprime packets. Finally it is easy to see that a non-zero non-unit x of a GCD domain R is a finite product of packets if and only if xR has finitely many minimal primes. A pre-Krull GCD domain is called a URD (unique representation domain) in [18]. With reference to these observations we state the following proposition.

PROPOSITION 7.3. *A PVMD R is a pre-Krull domain if and only if R^v is a URD.*

PROOF. It is easy to see that if every principal ideal of R has finitely many minimal primes then the quotient rings of R, and the polynomial rings over R have the same property. As a result every principal ideal of R^v, which is a quotient ring of $R[X]$, has finitely many primes. Now R^v being Bezout, and hence a GCD domain, is a URD. The converse is obvious.

Methods of constructing new URD's (or GCD pre-Krull domains) have been discussed in [18]. With some effort those methods can be modified to produce examples of pre-Krull domains which are not necessarily GCD domains. But the process involved in making the adaptation is lengthy and demands a separate treatment. The point that we want to make in this section is that rings of Krull type (and other known generalizations of Krull domains) can be studied as generalizations of GCD rings with special factorization properties. For example, according to [16] a ring of Krull type is a generalization of GCD domains whose non-zero non-units are divisible by at most a finite number of mutually coprime non-units.

REMARK 7.4. We notice that, once we show that the quotient ring of a PVMD is again a PVMD, the theory of PVMD's runs along lines seemingly parallel to those of GCD domains. This state of affairs leads one to look for points of difference. One difference comes to light when we consider atomicity. We know that an atomic GCD domain is a UFD. In view of this analogy we can ask if an atomic PVMD is a Krull domain or not. The answer, according to Anne Grams [7], is no. In this paper she provides an example of a Prüfer generalized Krull domain which is atomic but not a Dedekind domain.

8. REFERENCES

[1] Arnold, J. and Sheldon, P., Integral domains that satisfy Gauss' Lemma, Michigan Math. J. 22(1975) 39 - 51

[2] Brewer, J. and Heinzer, W., Associated primes of principal ideals, Duke Math. J. 41(1974) 1 - 7

[3] Brewer, J. and Mott, J., Integral domains of finite character, J. Reine Angew. Math. 241(1970) 34-41

[4] Cohn, P., Bezout rings and their subrings, Proc. Cambridge Phil. Soc. 64(1968) 251 - 264

[5] Gilmer, R., An embedding theorem for HFC rings, Proc. Cambridge Phil. Soc. 68(1970) 583 - 587

[6] Gilmer, R., Multiplicative ideal theory, Marcel Dekker, New York (1972)

[7] Grams, A., Atomic rings and the ascending chain condition for principal ideals, Proc. Camb. Phil. Soc. 75(1974) 321 - 329

[8] Griffin, M., Rings of Krull type, J. Reine Angew. Math. 229(1968) 1 - 27

[9] __________, Some results on v-multiplication rings, Canad. J. Math. 19(1967) 710 - 722

[10] Heinzer, W., Noetherian intersections of integral domains, Kansas Comm. Alge. Conference Notes, Springer Notes 311

[11] Heinzer, W. and Ohm, J., An essential ring which is not a v-multiplication ring, Can. J. Math. 25(1973) 856 - 861

[12] Jaffard, P., Les Systemes d'Ideaux, Dunod, Paris (1960)

[13] Kaplansky, I., Commutative Rings, Allyn & Bacon, Boston (1970)

[14] Sheldon, P., Prime ideals in GCD domains, Can. J. Math. 26(1974) 98 - 107

[15] Tang, H., Gauss' Lemma, Proc. Amer. Math. Soc. 35(1972) 372 - 376

[16] Zafrullah, M., Rigid elements in GCD domains, J. Natur. Sco. and Math. Vol. 17, No. 2(1977)

[17] ________, On finite conductor domains, Manuscripta Math. 24(1978) 191 - 204

[18] ________, Unique representation domains, to appear in J. Natur. Sci. and Math.

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