Tightly idf domains

Call a domain D a tightly idf (tidf) domain if every nonzero non unit element of D is divisible by at least one and at most a finite number of irreducible elements. I would have preferred the name strongly idf, but that name is taken [1]. They define D to be a strong idf-domain if each nonzero element of D has only finitely many divisors which are either units or atoms. The difference between the two concepts becomes apparent if we note that while a finite field may pass as a strong idf domain a tightly idf domain is not a field. It may also be noted that while tidf $\Rightarrow$ idf because every nonzero non unit being divisible by at least one and at most a finite number of atoms implies every nonzero element is at most a finite number of atoms, it is not the case the other way around. For a valuation domain (V, M) with $M^2 = M$ is an idf domain but V cannot be a tidf domain because no $m \in M\backslash\{0\}$ is divisible by any atom.

Proposition XD. Let D be a domain with quotient field K, let X be an indeterminate over L an extension field of K and let $R = D + XL[X]$ and $S = D + XL[[X]]$. Then the following hold.

(1) Given that D is not a field, then R is a tightly idf domain if and only if D has at least one and at most a finite number of atoms.

(2). Given that D is a field R is tightly idf if and only if $|L^*/D^*| < \infty$.

(3) Given that D is not a field, S has n atoms if and only if D has $n > 0$ atoms.

(4) Given that D is a field, S is tightly idf if and only if $|L^*/D^*| < \infty$.

Proof. A general element of $D + XL[X]$ is of the form $(hX^r)(1 + Xg(X))$, where $h \in L$ and $g(X) \in L[X]$. Of these $1 + Xg(X)$ is a product of powers of finitely many height one primes in $L[X]$ and hence in $D + XL[X]$. Let n_g be the number of prime divisors of $1 + Xg(X)$ and let n_D be the number of atoms in D. If $r > 0$, then the number of irreducible divisors of $(hX^r)(1 + Xg(X))$ is $n_D + n_g$. If on the other hand $r = 0$, then $h \in D$ and so the number of irreducible divisors of h is $n_h \leq n_D$ and the number of irreducible divisors of $h(1 + Xg(X))$ is m such that $1 \leq m \leq n_D + n_g$. Conversely D must have at most a finite number of irreducible elements because $X \in R$ is divisible by every element of D.

(2) Suppose that D is a field. Then a typical element of R is $(hX^r)(1 + Xg(X))$. If $r = 0$, $h \in D$ the distinct irreducible divisors is n_g and hence finite. On the other hand if $r = 1$, hX is irreducible and if $r \geq 2$ then hX^r has finitely many irreducible divisors if and only if $|L^*/D^*| < \infty$ as in [**?**].(The number of distinct irreducible divisors depends upon the distinct cosets of L^*/D^*.

(3) A typical element of S is $(hX^r)(1+Xg(X))$ where $g(X)$ is a power series in $L[[X]]$ and so $(1+Xg(X))$ is a unit in $L[[X]]$ and hence in $D = XL[[x]]$. And X being divisible by every nonzero element of D must have as many irreducible divisors as n_D. On the other hand if $(hX^r)(1+Xg(X))$ has n irreducible divisors hX^r has n irreducible divisors. Because D is not a field, X is not irreducible. So the only irreducible divisors of $(hX^r)(1 + Xg(X))$ are the irreducible elements of D. whence $n = n_D$.

(4) The proof is straightforward.

Proposition XE. The following are equivalent for an integral domain D.

(1) D is an FFD

(2) D is tidf domain and Archimean,

(3) D is a tidf domain and for each atom a of D we have $\cap a^n D = (0)$.

Proof. (1) $\Rightarrow$ (2). Being an FFD D is atomic and idf and hence tidf. Also being an FFD, D has ACCP and so must be Arcimedean [2, Theorem 2.1].

(2) $\Rightarrow$ (3). This is obvious. (3) $\Rightarrow$ (1). All we need show is that D is atomic. For this we proceed as follows. Let x be a nonzero non unit of D. Since D is tidf there is an atom $a_1|x$. Because a_1 is an atom $\cap a_1^n D = (0)$. So there is an n_1 such that $a_1^{n_1}|x$ and $a_1^{n_1+1} \nmid x$. Let $x_1 = x/a_1^{n_1}$. If x_1 is a non unit, then because of tidf x_1 is divisible by an atom a_2 and there is an n_2 such that $a_2^{n_2}|x_2$ and $a_2^{n_2+1} \nmid x_2$. This gives $x_3 = x/a_1^{n_1}a_2^{n_2}$. Continuing thus we get $x_r = x/a_1^{n_1}a_2^{n_2}...a_r^{n_r}$. Now this cannot continue indefinitely because in the presence of tidf x is divisible by at most a finite number of distinct atoms. So x is a product of atoms. Since the choice of x was arbitrary D is atomic and an atomic idf domain is an FFD. (We have already observed that tidf implies idf.)

Note that ACCP implies Archimedean property in a domain but ACCP does not stop a nonzero non unit from having an infinite number of distinct, non-associated, atoms for divisors. For example the ring $Q + XR[X]$, where Q is the set of rational numbers and R that of real numbers, satisfies ACCP but say X^2 has infinitely many distinct irreducible divisors such as X/r where r is an irrational number. So, tidf is necessary. Yet tidf and atomic are a bit much, in that idf and atomic implies FFD. On the other hand tidf alone does not cut it. Here's an example to see that and much more.

Example XF. Let Z (resp., Q) be the ring (resp., field) of integers (resp., rational numbers), p be a nonzero prime number, X and Y be indeterminates over Q, $R = Z_{(p)} + (X,Y)Q[\![X,Y]\!]$, K be the quotient field of R, Z be an indeterminate over K and let $D = R + ZK[\![Z]\!]$. Then D and R are tidf domains in that every nonzero non unit of each is divisible by an atom p and as there are no other atoms present the condition of being divisible by at most a finite number of non-associate atoms is automatically satisfied.

The other interesting thing about the above example is that if we let $S = \{p^n \mid n = 0, 1, 2, \ldots\}$, then $D_S = \mathbb{Q} + (X,Y)\mathbb{Q}[\![X,Y]\!] + ZK[\![Z]\!] = \mathbb{Q}[\![X,Y]\!] + ZK[\![Z]\!]$ is a domain in which every nonzero non unit is divisible by a prime but which is not tidf because the element Z is divisible by infinitely many non-associated primes.

As pointed out in [3] the ring $R = Z_{(p)} + (X,Y)Q[\![X,Y]\!]$ is Schreier and of course it isn't too hard to show that so is $D = R + ZK[\![Z]\!]$, as given in Example XF. This thwarts all hope of using tidf property in tandem with the atoms are prime (AP) property, as the Schreier property implies the AP property. However we have the following corollary to Proposition XE.

Corollary XG. Let D be a domain with the AP property. Then the following are equivalent.

(1) D is a UFD,

(2) D is completely integrally closed with the tidf property,

(3) D is Archimedean with the tidf property.

(4) D is tidf such that for every prime element p we have $\cap(p^n) = (0)$.

Proof. Of course (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) are direct. For (4) $\Rightarrow$ (1) let x be a nonzero non unit of D. Since D is tidf with every atom a prime, x is divisible by at least one and at most finitely many primes. Choose one, say p_1 dividing x. By (4) $x = x_1 p_1^{n_1}$ where $p_1 \nmid x_1$. Repeat with x_1 to choose $p_2|x_1|x$ to get $x = x_2 p_1^{n_1} p_2^{n_2}$. Continuing thus at stage r we get $x = x_r p_1^{n_1} p_2^{n_2} ... p_r^{n_r}$ where $p_r \nmid x_r$ and this can continue until x_r is a unit. Because there are only a finite number of primes dividing x we with get $x_r a$ unit for some value of r. That is exactly when we will have a canonical presentation of x as a finite product of powers of distinct (non-asssociate) primes. Now as x was chosen arbitrarily, D is a UFD.

One reason why I like the tidf over the idf is that it (tidf) can be linked to the study of strongly tidf domains, those with each proper nonzero ideal contained in at least one and at most a finite number of non-associate irreducible elements. Though of course that does not make the domain manageable, even though every atom in such domains would have to be a prime. Example XF again proves to be a killjoy example. However other examples may be constructed. Yet throw in things like (2)-(4) of Corollary XG and you end up with a PID.

References

[1] D. D. Anderson and B. Mullins: Finite factorization domains, Proc. Amer. Math. Soc. 124 (1996) 389–396.

[2] R.A. Beauregard and D.E. Dobbs, On a class of Archimeden integral domains, Can. J. Math., Vol. XXVIII, No. 2 (1976) 365-375.

[3] Muhammad Zafrullah, On semirigid GCD domains, II, J. Algebra and Applications to appear.