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Some Quotient Based Statements in Multiplicative Ideal Theory.

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Sunto. – *Alcuni importanti domini di integrità come i domini di Dedekind, di Prüfer e di Krull sono caratterizzati in termini di proposizioni basate sui quozienti, che coinvolgono l'operazione star. Per esempio, si prova che* D *è un dominio di Krull se e soltanto se* $(AB)_t = A_t : B^{-1}$ *per tutti gli ideali frazionari non nulli* A *e* B *di* D.

1. – Introduction.

Let D be a commutative integral domain, let $F(D)$ denote the set of non-zero fractional ideals of D and let $f(D)$ be the set of finitely generated members of $F(D)$.

In [2] we proved the following two quotient based results.

THEOREM A. – *An integral domain* D *is completely integrally closed if, and only if, for all* $A, B \in F(D)$ $(AB)_v = A_v : B^{-1}$. (*Here and in the sequel* $A_v = (A^{-1})^{-1}$.)

THEOREM B. – *An integral domain* D *is a* v-domain (*for all* $A \in f(D)$, $(AA^{-1})_v = D$) *if, and only if, for all* $A \in f(D)$ *and* $B \in F(D)$, $A_v : B = (AB^{-1})_v$.

A closer look into these result revealed, later, that we had actually unearhed special cases of two schemes of theorems. To elaborate on this we state two sample variations of Theorem A.

(*a*) If in Theorem A, we remove all mention of v, we get Dedekind domains in place of completely integrally closed domains. That is: D is a Dedekind domain if, and only if, for all $A, B \in F(D)$, $A : B^{-1} = AB$.

(*b*) Defining for each $A \in F(D)$, $A_t = \cup \{F_v | F \subseteq A, F \in f(D)\}$, if we replace v by t in Theorem A we get Krull domains in place of completely integrally closed domains. That is: D is a Krull domain if, and only if for all $A, B \in F(D)$, $(AB)_t = A_t : B^{-1}$. There are other variations of Theorem B each characterizing a distinct class of integral domains. We shall give those statements in the main text after giving a proper introduction to the notions involved. Yet, to keep the reader interested we may point out that statements like Theorem A and B characterize, among themselves, a fairly large number of integral domains of interest in multiplicative ideal theory. Some of the domains to be characterized using Theorem B include Prüfer domains, Prüfer domains with all non-zero ideals divisorial and Prüfer v-multiplication domains.

This paper is split into four sections of variable sizes. In the second section we give some basic results on $*$-operations, the v-operation and the t-operations. In the third we study the schema generating results like Theorem A and in section IV we study the schema generating ones like Theorem B. The main theme of the work is that we collect a set of most general quotient based equivalent conditions for an integral domain to be a completely integrally closed domain (a v-domain) and take out subsets of statements which are say more equivalent than others—under a given set of circumstances. These subsets, under further restrictions, will give us theorems like Theorems A and B. The emphasis of this paper is on generality and thoroughness, as much as possible at this stage. An interested reader should therefore be prepared for longish list of equivalent conditions.

2. – $*$-operations.

Let, throughout this article, D be a commutative integral domain with quotient field K and let $F(D)$ and $f(D)$ be as introduced earlier. Our notation is that of [4] except that we shall denote by $(A:B)$ the ideal

$$A:_K B = \{x \in K | xB \subseteq A\} \in F(D) .$$

A mapping on $F(D)$ defined by $A \mapsto A^*$ is called a *star operation* on D if the following conditions hold for all $a \in K - \{0\}$ and $A, B \in F(D)$:

(1) $(a)^* = (a)$, $(aA)^* = aA^*$,

(2) $A \subseteq A^*$, if $A \subseteq B$ then $A^* \subseteq B^*$ and

(3) $(A^*)^* = A^*$.

For details on star operations the reader may consult sections 32 and 34 of [4]. Yet for our purposes we include some of the notions. $A \in F(D)$ is called a *$*$-ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ for some $B \in f(D)$. Further, a star operation is said to be of *finite character* if $A^* = \cup\{J^* | J \subseteq A \text{ with } J \in f(D)\}$. If $*$ is a star operation, then the function on $F(D)$ defined by $A \mapsto A^{*_s} = \cup\{J^* | J \subseteq A \text{ and } J \in f(D)\}$ is a star operation of finite character. Clearly $A^* = A^{*_s}$ for all $A \in f(D)$. Recall that the function on $F(D)$ defined by $A \mapsto (A^{-1})^{-1} = A_v$ is a star operation called the *v-operation*. Here $A^{-1} = D:A$. The *t-operation* on D is the operation defined by $A \mapsto A_{v_s} = A_t$. The identity mapping $A \mapsto A$ on $F(D)$ is obviously a star operation (and it is of finite character). Sometimes this operation is called the *d-operation*.

Given a star operation $*$ we have for all $A, B \in F(D)$, $(AB)^* = (A^*B)^* = (A^*B^*)^*$. These equations are said to define the $*$-multiplication. An $A \in F(D)$ is said to be *$*$-invertible* if there exists $B \in F(D)$ such that $(AB)^* = D$. In this case we have $B^* = A^{-1}$ and as $D = (AB)^* = (AB^*)^* = (AA^{-1})^*$ we can replace B by A^{-1}. For any star operation $*$ and for any $A \in F(D)$, we have $A \subseteq A^* \subseteq A_v$ and hence $(A^*)_v = A_v$. In particular, a v-ideal is a $*$-ideal for any $*$ and if A is $*$-invertible then A is v-invertible. A similar statement can be shown to hold for v replaced by t if $*$ has finite character. If $*$ has finite character and $A \in F(D)$ is $*$-invertible, then A^* and A^{-1} are both $*$-ideals of finite type.

An integral domain D is a *v-domain* if every $A \in f(D)$ is v-invertible. Further, D is a *Prüfer v-multiplication domain* (PVMD) if each $A \in f(D)$ is v-invertible and A^{-1} is a v-ideal of finite type. It is easy to see that D is a PVMD if and only if each $A \in f(D)$ is t-invertible. For results on v-domains and PVMD's, the reader is referred to sections 32 and 34 of [4].

3. – The schema generating Theorem A.

Our ploy is to collect, in the following theorem, an extremely general set of quotient based characterizations of completely integrally closed integral domains. Once we have collected the set, we select subsets which can, under suitable circumstances, be reintepreted to give the schema.

THEOREM 3.1. – *For an integral domain D, the following are equivalent.*

(a) *For each pair, $A, B \in F(D)$ there exists a star operation $* = *(A, B)$ such that*

$$(A:B)^* = (AB^{-1})^* .$$

(b) *For each pair $A, B \in F(D)$ there exists a star operation $* = *(A, B)$ such that*

$$(A:B^{-1})^* = (AB)^* .$$

(c) *For each pair $A, B \in F(D)$ there exists a star operation $* = *(A, B)$ such that*

$$A^*:B = (AB^{-1})^* .$$

(d) *For each pair $A, B \in F(D)$ there exists a star operation $* = *(A, B)$ such that*

$$A^*:B^{-1} = (AB)^* .$$

(e) *Every member of $F(D)$ is $*$-invertible for some star operation $*$.*

(f) *Every member of $F(D)$ is v-invertible.*

(g) *D is completely integrally closed.*

(h) *For all $A, B \in F(D)$, $(A:B)_v = (AB^{-1})_v$.*

(i) *For all $A, B \in F(D)$, $(A:B^{-1})_v = (AB)_v$.*

(j) *For all $A, B \in F(D)$, $A_v:B = (AB^{-1})_v$.*

(k) *For all $A, B \in F(D)$, $A_v:B^{-1} = (AB)_v$.*

(l) *For each pair $A, B \in F(D)$ with A a v-ideal, there exists a star operation $* = *(A, B)$ such that*

$$A:B = (AB^{-1})^* .$$

(m) *For each pair $A, B \in F(D)$ with A a v-ideal, there exists a star operation $* = *(A, B)$ such that*

$$A:B^{-1} = (AB)^* .$$

(*n*) *For each pair* $A, B \in F(D)$ *of v-ideals there exists a star operation* $* = *(A, B)$ *such that*

$$A:B = (AB^{-1})^* .$$

(*o*) *For each pair* $A, B \in F(D)$ *of v-ideals there exists a star operation* $* = *(A, B)$ *such that*

$$A:B^{-1} = (AB)^* .$$

(*p*) *Every v-ideal* $A \in F(D)$ *is* $*$*-invertible for some star operation* $*$.

(*q*) *For each pair* $A, B \in F(D)$, *there exists a star operation* $* = *(A, B)$ *such that*

$$(A:B^{-1})^* = (AB_v)^* .$$

(*r*) *For each pair* $A, B \in F(D)$, *there exists a star operation* $* = *(A, B)$ *such that*

$$(A^*:B^{-1})^* = (AB_v)^* .$$

(*s*) *For each pair* $A, B \in F(D)$, *there exists a star operation* $*$ *such that*

$$A_t:B^* = (A_tB^{-1})^* .$$

(*t*) *Every t-ideal is* $*$*-invertible for some* $*$*-operation* $*$.

PROOF. – The equivalence of (*f*) and (*g*) is well known; for example see [4, Theorem 34.3]. Clearly (*f*) $\Rightarrow$ (*e*) and from the remarks in section II we have that (*e*) $\Rightarrow$ (*f*).

(*a*) $\Rightarrow$ (*e*). Let $A \in F(D)$. Take $B = A$. Then for some star operation $* = *(A, A)$ we have

$$D \subseteq (A:A)^* = (AA^{-1})^* \subseteq D .$$

Hence $(AA^{-1})^* = D$, so A is $*$-invertible for some star operation $*$.

(*b*) $\Rightarrow$ (*e*). This is similar to the proof of (*a*) $\Rightarrow$ (*e*), except that we take $A = B^{-1}$.

(*c*) $\Rightarrow$ (*e*). This is similar to (*a*) $\Rightarrow$ (*e*).

(*d*) $\Rightarrow$ (*e*). This is similar to (*b*) $\Rightarrow$ (*e*).

$(e) \Rightarrow (a)$. Suppose that for some star operation $*$, $(AA^{-1})^* = D$, for all $A \in F(D)$. Now obviously $(AB^{-1})B \subseteq A$, so

$$AB^{-1} \subseteq (A:B) \quad \text{and hence} \quad (AB^{-1})^* \subseteq (A:B)^* .$$

Conversely, let $0 \neq x \in A:B$. Then $xB \subseteq A$ and $xBB^{-1} \subseteq AB^{-1}$. But there exists a star operation $*$ such that $(BB^{-1})^* = D$. So $x \in (AB^{-1})^*$ for some star operation fixed by B. Now $(AB^{-1})^* \subseteq (A:B)^*$ for any star operation $*$ and $(A:B)^* = (AB^{-1})^*$ for some star operation $* = *(A, B)$.

The proofs that $(e) \Rightarrow (b)$, $(e) \Rightarrow (c)$ and $(e) \Rightarrow (d)$ are similar. Hence (a)-(e) are equivalent. For the equivalence of (f), (h), (i), (j) and (k), replace $*$ by v in the procedures indicated above. So (a)-(k) are all equivalent.

Now $(a) \Rightarrow (l), (n)$; $(b) \Rightarrow (m), (o)$ are obvious, so we show that $(l), (m), (n), (o) \Rightarrow (e)$. But the procedure is apparent as, in $(l) \Rightarrow (e)$ we put $A = B$, in $(m) \Rightarrow (e)$ we put $B = A^{-1}$, in $(n) \Rightarrow (e)$ we put $B = A$ and in $(o) \Rightarrow (e)$ we put $A = B^{-1}$. Further, obviously $(e) \Rightarrow (p)$. Assuming (p) we have $(A_v A^{-1})^* = D$ which gives $(A_v A^{-1})_v = D$ or $(AA^{-1})_v = D$. So for all $A \in F(D)$, $(AA^{-1})_v = D$ and this is (f) which is equivalent to (e). Next we prove the equivalence of (q), (r) with (e).

$(e) \Rightarrow (q)$. Since $AB_vB^{-1} \subseteq A$ we have $AB_v \subseteq A:B^{-1}$ and so for every star operation $*$ we have $(AB_v)^* \subseteq (A:B^{-1})^*$. Now for the inclusion in the opposite direction let $0 \neq x \in A:B^{-1}$. Then $xB^{-1} \subseteq A$ and $xB^{-1}B_v \subseteq AB_v$. Now by (e) there exists $* = *(B_v)$ such that $(B^{-1}B_v)^* = D$. So, $x \in (AB_v)^*$ which gives $A:B^{-1} \subseteq (AB_v)^*$ and hence $(A:B^{-1})^* \subseteq (AB_v)^*$ for $* = *(B_v)$. Hence for $*$ such that $(B^{-1}B_v)^* = D$ we have $(A:B^{-1})^* = (AB_v)^*$.

$(e) \Rightarrow (r)$. Obviously $AB^{-1}B_v \subseteq A \subseteq A^*$. So $AB_v \subseteq A^*:B^{-1}$ which, because $A^*:B^{-1}$ is a $*$-ideal [4, exercise 1, page 406], gives $(AB_v)^* \subseteq A^*:B^{-1}$. On the other hand if $x \in A^*:B^{-1}$ then $xB^{-1} \subseteq A^*$ and $xB^{-1}B_v \subseteq A^*B_v$ or, as in $(e) \Rightarrow (q)$, $x \in (A^*B_v)^* = (AB_v)^*$. So $A^*:B^{-1} \subseteq (AB_v)^*$ and the conclusion is obvious. (r), $(q) \Rightarrow (f)$. Put $A = B^{-1}$ to conclude that every v-ideal is $*$-invertible and hence v-invertible.

Finally to complete the proof we show that $(a) \Rightarrow (s) \Rightarrow (t) \Rightarrow (e)$. Obviously $(a) \Rightarrow (s)$ and $(t) \Rightarrow (e)$. For $(s) \Rightarrow (t)$ put $B = A_t$. Now A_t $*$-invertible, for some $*$, implies A_t v-invertible for each $A \in F(D)$.

The repalacement of B by B_v in some of the conditions is remarkable. Indeed it seems to be connected with the following result.

PROPOSITION 3.2. – *Let $*$ be a star operation on D. If $A \in F(D)$ is $*$-invertible then $A^* = A_v$ and so A_v is also $*$-invertible.*

PROOF. – $(AA^{-1})^* = D$ implies $A^* = (A^{-1})^{-1} = A_v$. Now

$$D = (AA^{-1})^* = (A^*A^{-1})^* = (A_v A^{-1})^* .$$

REMARK 3.3. – The converse of Proposition 3.2 is not true. For in a *GCD-domain* D (every pair of non-zero elements has a greatest common divisor), for every $A \in f(D)$, A_v is principal and hence invertible (i.e., d-invertible), but A is not necessarily invertible.

COROLLARY 3.4. – *Let $*$ be a star operation on D. Then the following are equivalent.*

(1) $(A:B)^* = (AB^{-1})^*$ *for all* $A, B \in F(D)$.

(2) $(A:B^{-1})^* = (AB)^*$ *for all* $A, B \in F(D)$.

(3) $A^*:B = (AB^{-1})^*$ *for all* $A, B \in F(D)$.

(4) $A^*:B^{-1} = (AB)^*$ *for all* $A, B \in F(D)$.

(5) *Every* $A \in F(D)$ *is* $*$*-invertible.*

(6) D *is completely integrally closed and for every* $A \in F(D)$, $A^* = A_v$.

(7) *For each pair* $A, B \in F(D)$ *such that* A *is a* v*-ideal,* $A:B^{-1} = (AB)^*$.

PROOF. – (5) $\Rightarrow$ (6). It follows from Proposition 3.2.

(6) $\Rightarrow$ (5). D is completely integrally closed implies $(AA^{-1})_v = D$ for all $A \in F(D)$. Moreover, for all $A \in F(D)$, $A^* = A_v$ implies that

$$D = (AA^{-1})_v = (A_v A^{-1})_v = (A^*A^{-1})^* = (AA^{-1})^* .$$

The remaining equivalences come from Theorem 3.1.

COROLLARY 3.5. – *Let $*$ be a star operation on D. Then following are equivalent.*

(1) *For each pair* $A, B \in F(D)$ *with* A *a* v*-ideal*, $A:B = (AB^{-1})^*$.

(2) *For each pair of* v*-ideals* $A, B \in F(D)$, $A:B = (AB^{-1})^*$.

(3) *For each pair of* v*-ideals* $A, B \in F(D)$, $A:B^{-1} = (AB)^*$.

(4) *Every* v*-ideal is* $*$*-invertible.*

(5) *For every pair* $A, B \in F(D)$, $(A:B^{-1})^* = (AB_v)^*$.

(6) *For every pair* $A, B \in F(D)$, $A^*:B^{-1} = (AB_v)^*$.

COROLLARY 3.6. – *Let* $*$ *be a* $*$*-operation on* D. *Then the following are equivalent.*

(1) *For all* $A, B \in F(D)$, $(A_t:B)^* = (A_tB^{-1})^*$.

(2) *For each* $A \in F(D)$, A_t *is* $*$*-invertible.*

If we put $* = v$ in Corollaries 3.4 and 3.5 we get a host of characterizations of v-domains which are of course contained in Theorem 3.1. The real applications of these corollaries arise when we consider $* \neq v$. The general situation seems to be beyond our reach, so we settle for $* = d$ and $* = t$.

When $* = d$, (5) of Corollary 3.4 becomes: every $A \in F(D)$ is invertible and this is equivalent to D being a Dedekind domain. So for $* = d$, Corollary 3.4 gives the following theorem.

THEOREM 3.7. – *The following are equivalent on* D.

(1) *For all* $A, B \in F(D)$, $A:B = AB^{-1}$.

(2) *For all* $A, B \in F(D)$, $A:B^{-1} = AB$.

(3) *Every* $A \in F(D)$ *is invertible.*

(4) D *is completely integrally closed and every* $A \in F(D)$ *is a* v*-ideal.*

(5) D *is a Dedekind domain.*

The proof is dealt with in Theorem 3.1 and the reader, if not satisfied with (1) and (2) of Theorem 3.6 as new results, may compare (4) with Proposition 5.5 of [6].

If we put $* = d$ in Corollary 3.5 then (5) of this corollary becomes: every v-ideal $A \in F(D)$ is invertible. The integral domains characterized by this property are the *generalized Dedekind (G-Dedekind) domains* of [10] and the *pseudo-Dedekind domains* of [3]. This leads to the following theorem.

THEOREM 3.8. – *The following are equivalent for an integral domain D.*

(1) *For each pair $A, B \in F(D)$ with A a v-ideal, $A:B = AB^{-1}$.*

(2) *For each pair of v-ideals $A, B \in F(D)$,*

$$A:B = AB^{-1}.$$

(3) *For each pair of v-ideals $A, B \in F(D)$,*

$$A:B^{-1} = AB.$$

(4) *Every v-ideal is invertible.*

(5) *For all $A, B \in F(D)$, $A:B^{-1} = AB_v$.*

It is shown in [9] that D is a Krull domain if and only if all $A \in F(D)$ are t-invertible. According to [11], D is a Krull domain if and only if there exists a star operation $*$ of finite character such that each $A \in F(D)$ is $*$-invertible. So, fixing $*$ to be of finite character we can give Corollary 3.4 the following form.

THEOREM 3.9. – *Let $*$ be a star operation of finite character defined on D. Then the following are equivalent.*

(1) $(A:B)^* = (AB^{-1})^*$ *for all $A, B \in F(D)$.*

(2) $(A:B^{-1})^* = (AB)^*$ *for all $A, B \in F(D)$.*

(3) $A^*:B = (AB^{-1})^*$ *for all $A, B \in F(D)$.*

(4) $A^*:B^{-1} = (AB)^*$ *for all $A, B \in F(D)$.*

(5) *Every $A \in F(D)$ is $*$-invertible.*

(6) *D is completely integrally closed and for every $A \in F(D)$, $A^* = A_v$.*

(7) *For each pair $A, B \in F(D)$ with A a v-ideal $A:B^{-1} = (AB)^*$.*

(8) *D is a Krull domain.*

The proof is indeed unnecessary. Of the above equivalent conditions, (6) is rather interesting. Let us call D a *TV-domain* if every t-ideal of D is a v-ideal. Obviously D is a TV-domain if and only if for all $A \in F(D)$, $A_t = A_v$. Now (6) above compares with the following result from [7]: If D is a TV-domain, then for D to be a Krull domain it is necessary and sufficient that D should be completely integrally closed. Also, see [8].

Indeed, it goes without saying that if we put $* = t$ in Theorem 3.8 we have another set of equivalent conditions.

In [11] D is called a *pre-Krull domain* if for all $A \in F(D)$, A_v is t-invertible. Putting $* = t$ in Corollary 3.5 we get the following theorem.

THEOREM 3.10. – *For an integral domain D the following are equivalent.*

(1) *For each pair $A, B \in F(D)$ with A a v-ideal, $A:B = (AB^{-1})_t$.*

(2) *For each pair of v-ideals $A, B \in F(D)$,*

$$A:B = (AB^{-1})_t .$$

(3) *For each pair of v-ideals $A, B \in F(D)$,*

$$A:B^{-1} = (AB)_t .$$

(4) *Every v-ideal is t-invertible.*

(5) *For every pair $A, B \in F(D)$,*

$$(A:B^{-1})_t = (AB_v)_t .$$

(6) *For every pair $A, B \in F(D)$,*

$$A_t:B^{-1} = (AB_v)_t .$$

(7) *D is a pre-Krull domain.*

We know that D is a Krull domain if and only if every $A \in F(D)$ is t-invertible. So, if for each $A \in F(D)$, A_t is invertible, then D is a Krull domain with the property that for all $A \in F(D)$, A_v is invertible. This makes D a locally factorial Krull domain. It can be easily shown that if D is a locally factorial Krull domain then for each $A \in F(D)$, A_t is invertible.

Now if we put $* = d$ in Corollary 3.6 we get the following result.

PROPOSITION 3.11. – *For an integral domain D the following are equivalent.*

(1) *For all $A \in F(D)$,*

$$A_t : B = A_t B^{-1}.$$

(2) *For all $A \in F(D)$, A_t is invertible.*

(3) *D is a locally factorial Krull domain.*

REMARK 3.12. – Putting $* = t$ in Corollary 3.6 characterizes Krull domains and putting $* = v$ characterizes completely integrally closed domains.

4. – The schema generating Theorem B.

In this section we also collect a rather long list of statements equivalent to « D is a v-domain ». Then we take up special groups of these statements to characterize various special classes of v-domains.

THEOREM 4.1. – *For an integral domain D the following are equivalent.*

(1) *There exists a star operation $*$ on D such that for all $A \in f(D)$, $B \in F(D)$, $A^* : B^{-1} = (AB)^*$.*

(2) *There exists a star operation $*$ on D, such that for all $A \in f(D)$, $B \in F(D)$, $A^* : B^{-1} = (A_v B)^*$.*

(3) *For each pair $A \in f(D)$, $B \in F(D)$ there exists a star operation $* = *(A, B)$ such that $A_v : B^{-1} = (AB)^*$.*

(4) *Every $A \in f(D)$ is $*$-invertible for some star operation $* = *(A)$.*

(5) *Every $A \in f(D)$ is v-invertible.*

(6) *D is a v-domain.*

(7) *For each pair $A \in f(D)$, $B \in F(D)$ there exists a star operation $* = *(A, B)$ such that $A^* : B = (AB^{-1})^*$.*

(8) *For each pair $A \in f(D)$, $B \in F(D)$ there exists a star operation $* = *(A, B)$ such that $(A : B)^* = (AB^{-1})^*$.*

(9) *For each pair* $A \in f(D)$, $B \in F(D)$ *there exists a star operation* $* = *(A, B)$ *such that* $A_v : B^{-1} = (AB_v)^*$.

(10) *For each pair* $A \in f(D)$, $B \in F(D)$ *there exists* $* = *(A, B)$ *such that* $A_v : B = (AB^{-1})^*$.

(11) *For each pair* $A \in f(D)$, $B \in F(D)$ *there exists* $* = *(A, B)$ *such that* $(A : B)_v = (AB^{-1})^*$.

(12) *For each pair* $A \in f(D)$, $B \in F(D)$ *there exists* $* = *(A, B)$ *such that* $A_v : B = (A_v B^{-1})^*$.

(13) *For each* $A \in f(D)$, A_v *is* $*$*-invertible for some star operation* $* = *(A_v)$.

(14) *For each pair* $A \in f(D)$, $B \in F(D)$ *there exists* $* = *(A, B)$ *such that* $A_v : B^{-1} = (A_v B)^*$.

(15) *For each pair* $A \in f(D)$, $B \in F(D)$, $A_v : B = (AB^{-1})_v$.

(16) *For each pair* $A \in f(D)$, $B \in F(D)$, $A_v : B^{-1} = (AB)_v$.

(17) *For each pair* $A \in f(D)$, $B \in F(D)$, $(A : B)_v = (AB^{-1})_v$.

(18) *For each pair* $A \in F(D)$, $B \in f(D)$ *there exists* $* = *(A, B)$ *such that* $A^* : B = (AB^{-1})^*$.

(19) *For each pair* $A \in F(D)$, $B \in f(D)$ *there exists* $* = *(A, B)$ *such that* $A^* : B^{-1} = (AB)^*$.

(20) *For each pair* $A \in F(D)$, $B \in f(D)$ *there exists* $* = *(A, B)$ *such that* $(A : B)^* = (AB^{-1})^*$.

(21) *For each pair* $A \in F(D)$, $B \in f(D)$ *there exists* $* = *(A, B)$ *such that* $(A : B^{-1})^* = (AB)^*$.

(22) *For each pair* $A \in F(D)$, $B \in f(D)$, $A_v : B = (AB^{-1})_v$.

(23) *For each pair* $A \in F(D)$, $B \in f(D)$, $A_v : B^{-1} = (AB)_v$.

(24) *For each pair* $A \in F(D)$, $B \in f(D)$, $(A : B)_v = (AB^{-1})_v$.

(25) *For each pair* $A \in F(D)$, $B \in F(D)$, $(A : B^{-1})_v = (AB)_v$.

PROOF. – (5) $\Leftrightarrow$ (6). This equivalence is well known, (see [4, 34.6]).

(4) $\Rightarrow$ (5). If A is $*$-invertible, for some star operation $*$, then $(AA^{-1})^* = D$. But as $(AA^{-1})_v = ((AA^{-1})^*)_v = D_v = D$ we have the conclusion.

(5) $\Rightarrow$ (4). This is obvious because the v-operation is a $*$-operation.

(3) $\Rightarrow$ (4). Put $B = A^{-1}$. Then $A_v : A_v = (AA^{-1})^*$ leads to $(AA^{-1})^* = D$.

(4) $\Rightarrow$ (3). Obviously $AB \subseteq A : B^{-1} \subseteq A_v : B^{-1}$ and as $A_v : B^{-1}$ is a v-ideal we have $(AB)^* \subseteq A_v : B^{-1}$ for any star operation $*$. Now let $*$ be such that $(AA^{-1})^* = D$. Then $(AA^{-1})_v = D$ and if $x \in A_v : B^{-1}$ then $xB^{-1} \subseteq A_v$ and $(1/x)B_v \supseteq A^{-1}$ or $xA^{-1} \subseteq B_v$. Multiplying by A on both sides of the previous inclusion we get $xA^{-1}A \subseteq AB_v$ which gives $x \in (AB)_v$ and consequently $A_v : B^{-1} \subseteq (AB)_v$.

(2) $\Rightarrow$ (5). Put $A = D$. Then $B_v = B^*$ for all $B \in F(D)$. Now put $B = A^{-1}$ and this gives $A_v : A_v = (AA^{-1})_v$; which leads in turn to $(AA^{-1})_v = D$.

(5) $\Rightarrow$ (2). Put $*(A, B) = v$ and use (4) $\Rightarrow$ (3).

(5) $\Rightarrow$ (1). Put $*(A, B) = v$ and use the proof of (4) $\Rightarrow$ (3).

(1) $\Rightarrow$ (4). Put $A = D$. Then $B^* = B_v$ for each $B \in F(D)$. Now putting $B = A^{-1}$ we get $(AA^{-1})^* = D$ from (1).

(5) $\Rightarrow$ (7), (9), (10), (12) and (14). Put $v = *(A, B)$ and use the proof of (4) $\Rightarrow$ (3).

(5) $\Rightarrow$ (8). We prove the implication for $* = *(A, B) = v$. Obviously $AB^{-1} \subseteq A : B$ and so $(AB^{-1})_v \subseteq (A : B)_v$. Now let $x \in A : B$. Then $xB \subseteq A$ and $(1/x)B^{-1} \supseteq A^{-1}$ or $xA^{-1} \subseteq B^{-1}$. Multiplying both sides by A gives $xA^{-1}A \subseteq AB^{-1}$ or on applying the v-operation, we get $x \in (AB^{-1})_v$. This gives $A : B \subseteq (AB^{-1})_v$ and hence $(A : B)_v \subseteq$ $\subseteq (AB^{-1})_v$.

(5) $\Rightarrow$ (11) This is identical to (5) $\Rightarrow$ (8).

(7) $\Rightarrow$ (5). Put $B = A^*$. This gives $A^* : A^* = (AA^{-1})^*$ which leads to $(AA^{-1})^* = D$ for some $*$ and hence for $* = v$.

(8) $\Rightarrow$ (5). Put $A = B$. This gives $(A : A)^* = (AA^{-1})^*$ and hence $(AA^{-1})^* = D$.

(9) $\Rightarrow$ (5). Put $B = A^{-1}$.

(10) $\Rightarrow$ (5). Put $B = A_v$.

(11) $\Rightarrow$ (5). Put $B = A$.

(12) $\Rightarrow$ (5). Put $B = A_v$.

(5) $\Rightarrow$ (13). Put $*(A_v) = v$.

$(13) \Rightarrow (5)$. If A_v is $*$-invertible for some star operation $*$, then $(A_v A^{-1})^* = D$ which gives $(A_v A^{-1})_v = D$ or $(AA^{-1})_v = D$.

$(14) \Rightarrow (5)$. Put $B = A^{-1}$.

$(5) \Rightarrow (15)$, (16), (17). Special cases of $(5) \Rightarrow (7)$, (8), (9), (10), (11), (12), and (14).

$(15) \Rightarrow (5)$. Put $B = A_v$.

$(16) \Rightarrow (5)$. Put $B = A^{-1}$.

$(17) \Rightarrow (5)$. Put $B = A$.

$(4) \Rightarrow (18)$. Obviously $AB^{-1} \subseteq A:B \subseteq A^*:B$ and so $(AB^{-1})^* \subseteq$ $\subseteq A^*:B$ for any star operation $*$. Now let $*$ be such that $(BB^{-1})^* = D$ and let $x \in A^*:B$. Then $xB \subseteq A^*$. Multiplying both sides by B^{-1} and applying $* = *(B)$ we get

$$x \in (A^*B^{-1})^* = (AB^{-1})^* .$$

$(4) \Rightarrow (19)$. Obviously, now, $(AB)^* \subseteq A^*:B^{-1}$ for any star operation $*$. Let $*$ be the star operation such that $(BB^{-1})^* = D$ and let $x \in A^*:B^{-1}$. Then $xB^{-1} \subseteq A^*$. Multiplying both sides by B and applying the star operation $*$ we get $x \in (AB)^*$.

$(4) \Rightarrow (20)$. $AB^{-1} \subseteq A:B$ and so for every star operation $*$, $(AB^{-1})^* \subseteq (A:B)^*$. Now for $* = *(B)$ such that $(BB^{-1})^* = D$ let $x \in A:B$. Then $xB \subseteq A$. Multiplying both sides by B^{-1} and applying $* = *(B)$ we get $x \in (AB^{-1})^*$ and so $A:B \subseteq (AB^{-1})^*$ which gives $(A:B)^* \subseteq (AB^{-1})^*$.

$(4) \Rightarrow (21)$. The proof is similar to that of $(4) \Rightarrow (20)$.

$(5) \Rightarrow (22)$. Use the proof of $(4) \Rightarrow (18)$.

$(5) \Rightarrow (23)$. Use the proof of $(4) \Rightarrow (19)$.

$(5) \Rightarrow (24)$. Use the proof of $(4) \Rightarrow (20)$.

$(5) \Rightarrow (25)$. Use the proof of $(4) \Rightarrow (20)$.

$(18) \Rightarrow (4)$. Put $A = B$. Then $B^*:B \supseteq D$ and $BB^{-1} \subseteq D$ and this forces $(BB^{-1})^* = D$ for the star operation of (18).

$(19) \Rightarrow (4)$. Put $A = B^{-1}$.

$(20) \Rightarrow (4)$. Put $A = B$.

$(21) \Rightarrow (4)$. Put $A = B^{-1}$.

(22) $\Rightarrow$ (5). Put $A = B$.

(23) $\Rightarrow$ (5). Put $A = B^{-1}$.

(24) $\Rightarrow$ (5). Put $A = B$.

(25) $\Rightarrow$ (5). Put $A = B^{-1}$.

COROLLARY 4.2. – *Let* $*$ *be a star operation on* D. *Then the following are equivalent.*

(1) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A^* : B^{-1} = (AB)^*$.

(2) *For all* $A \in f(D)$, $B \in F(D)$, $A^* : B^{-1} = (A_v B)^*$.

(3) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v : B^{-1} = (AB)^*$.

(4) *Every* $A \in f(D)$ *is* $*$*-invertible and for all* $X \in F(D)$, $X^* = X_v$.

(5) D *is a v-domain and for every* $X \in F(D)$, $X^* = X_v$.

(6) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v : B^{-1} = (A_v B)^*$.

PROOF. – In view of Theorem 4.1 it is sufficient to show the equivalence of (1), (4), (5) and (6).

(1) $\Rightarrow$ (4). Putting $A = D$ we find that $B^* = B_v$ for all $B \in F(D)$. Now put $B = A^{-1}$ to complete the proof.

(4) $\Rightarrow$ (5). Obvious in the light of Theorem 4.1.

(5) $\Rightarrow$ (6). $X^* = X_v$ for all $X \in F(D)$ gives $A_v : B^{-1} = (A_v B)_v = (AB)_v$ and this equation can be shown to hold for a v-domain.

(6) $\Rightarrow$ (1). For $A = D$ we have $B^* = B_v$ for all $B \in F(D)$ and the rest follows once we replace $*$ by v.

COROLLARY 4.3. – *Let* $*$ *be a star operation on* D. *Then the following are equivalent.*

(1) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A^* : B = (AB^{-1})^*$.

(2) *For all* $A \in f(D)$ *and* $B \in F(D)$, $(A : B)^* = (AB^{-1})^*$.

(3) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v : B^{-1} = (AB_v)^*$.

(4) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v : B = (AB^{-1})^*$.

(5) *For all* $A \in f(D)$ *and* $B \in F(D)$, $(A : B)_v = (AB^{-1})^*$.

(6) *For all* $A \in F(D)$ *and* $B \in f(D)$, $A^* : B = (AB^{-1})^*$.

(7) *For all* $A \in F(D)$ *and* $B \in f(D)$, $A^*:B^{-1} = (AB)^*$.

(8) *For all* $A \in F(D)$ *and* $B \in f(D)$, $(A:B)^* = (AB^{-1})^*$.

(9) *For all* $A \in F(D)$ *and* $B \in f(D)$, $(A:B^{-1})^* = (AB)^*$.

(10) *Every* $A \in f(D)$ *is* $*$*-invertible.*

PROOF. – The proofs given in Theorem 4.1 suffice.

COROLLARY 4.4. – *Let* $*$ *be a star operation on* D. *Then the following are equivalent.*

(1) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v:B = (A_vB^{-1})^*$.

(2) *For each* $A \in f(D)$, A_v *is* $*$*-invertible.*

Let us put $* = d$. Then (4) of Corollary 4.2 becomes: Every $A \in f(D)$ is invertible and for all $X \in F(D)$, $X = X_v$. But if every $X \in f(D)$ is invertible then D is a Prüfer domain. So (4) of Corollary 4.2, for $* = d$, determines a Prüfer domain whose non-zero ideals are all divisorial (v-ideals). Thus we have the following theorem.

THEOREM 4.5. – *The following are equivalent for an integral domain* D.

(1) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A:B^{-1} = AB$.

(2) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A:B^{-1} = A_vB$.

(3) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v:B^{-1} = AB$.

(4) D *is a Prüfer domain in which every non-zero ideal is divisorial.*

(5) D *is a v-domain in which every non-zero ideal is divisorial.*

(6) *For all* $A \in f(D)$, $B \in F(D)$, $A_v:B^{-1} = A_vB$.

Putting $* = d$ in Corollary 4.3 we get the following characterizations of ordinary Prüfer domains.

THEOREM 4.6. – *The following are equivalent for* D.

(1) *For all* $A \in f(D)$, $B \in F(D)$, $A:B = AB^{-1}$.

(2) *For all* $A \in f(D)$, $B \in F(D)$, $A_v:B^{-1} = AB_v$.

(3) *For all* $A \in f(D)$, $B \in F(D)$, $A_v:B = AB^{-1}$.

(4) *For all* $A \in f(D)$, $B \in F(D)$, $(A:B)_v = AB^{-1}$.

(5) *For all* $A \in F(D)$, $B \in f(D)$, $A:B = AB^{-1}$.

(6) *For all* $A \in F(D)$, $B \in f(D)$, $A:B^{-1} = AB$.

(7) *Every* $A \in f(D)$ *is invertible.*

(8) D *is a Prüfer domain.*

Again, putting $* = d$ makes (2) of Corollary 4.4: *For each* $A \in f(D)$, A_v *is invertible.* Now, D such that for all $A \in f(D)$, A_v is invertible is a generalized GCD-domain (G-GCD-domain) of [1]. So we have the following quotient bsed characterization of G-GCD-domains.

THEOREM 4.7. – D *is a G-GCD-domain if and only if for all* $A \in f(D)$ *and* $B \in F(D)$,

$$A_v:B = A_vB^{-1}.$$

Now we put $* = t$ in Corollaries 4.2 to 4.4. We recall that D is a PVMD if every $A \in f(D)$ is t-invertible. Now (4) of Corollary 4.2 becomes: *Every* $A \in f(D)$ *is t-invertible and every t-ideal of* D *is a v-ideal.* So Corollary 4.2 characterizes TV-PVMD's. For a study of TV-PVMD's the reader may consult [7] and [8].

THEOREM 4.8. – *The following are equivalent for an integral domain* D.

(1) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_t:B^{-1} = (AB)_t$.

(2) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_t:B^{-1} = (A_vB)_t$.

(3) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v:B^{-1} = (AB)_t$.

(4) *Every* $A \in f(D)$ *is t-invertible and for every* $X \in F(D)$, $X_t = X_v$.

(5) D *is a v-domain and for every* $X \in F(D)$, $X_t = X_v$.

(6) *For all* $A \in f(D)$, $B \in F(D)$, $A_v:B^{-1} = (A_vB)_t$.

(7) D *is a TV-PVMD.*

Now for a characterization of ordinary PVMD's we put $* = t$ in Corollary 4.3.

THEOREM 4.9. – *The following are equivalent for an integral domain* D.

(1) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_t : B = (AB^{-1})_t$.

(2) *For all* $A \in f(D)$ *and* $B \in F(D)$, $(A : B)_t = (AB^{-1})_t$.

(3) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v : B^{-1} = (AB_v)_t$.

(4) *For all* $A \in f(D)$ *and* $B \in F(D)$, $A_v : B = (AB^{-1})_t$.

(5) *For all* $A \in f(D)$ *and* $B \in F(D)$, $(A : B)_v = (AB^{-1})_t$.

(6) *For all* $A \in F(D)$ *and* $B \in f(D)$, $A_t : B = (AB^{-1})_t$.

(7) *For all* $A \in F(D)$ *and* $B \in f(D)$, $A_t : B^{-1} = (AB)_t$.

(8) *For all* $A \in F(D)$ *and* $B \in f(D)$, $(A : B)_t = (AB^{-1})_t$.

(9) *For all* $A \in F(D)$ *and* $B \in f(D)$, $(A : B^{-1})_t = (AB)_t$.

(10) *Every* $A \in f(D)$ *is t-invertible.*

(11) *D is a PVMD.*

Finally, for $* = t$, Corollary 4.4 does not give us anything new and in fact provides another characterization of a PVMD. This follows from the fact that if $A \in f(D)$ then $A_v = A_t$.

In Theorem 4.1 the condition that $(A : B^{-1})^* = (AB)^*$ for all $A \in f(D)$, $B \in F(D)$ and for $* = *(A, B)$, has been noticeably absent. The reason for excluding this condition is that it does not generally give a v-domain. In fact, there are integral domains D, which are not v-domains, not even integrally closed domains, but which satisfy for all $A \in f(D)$, $B \in F(D)$, $(A : B^{-1})_v = (AB)_v$ [2]. On the other hand, if D is integrally closed, then this condition holds [2]. Now as the condition in question fails to deliver the property of being integrally closed we may ask, « Under what conditions does the condition in question provide the property of being integrally closed? ». Our next result answers this question in a restricted manner. Yet, for this we have to prepare a little. Recall that an integral domain D is called a *finite conductor domain* if for all $a, b \in K - \{0\}$, $aD \cap bD$ is finitely generated.

THEOREM 4.10. – *Let D be a finite conductor domain. Then D is a PVMD if, and only if, for all $A \in f(D)$ and $B \in F(D)$, $(A : B^{-1})_v = (AB)_v$.*

PROOF. – It is well known that if D is an integrally closed domain then for all $A \in f(D)$ and $B \in F(D)$, $(A : B^{-1})_v = (AB)_v$ (see [2] and the references there) and that a PVMD is integrally closed. So the condition is necessary. Now for the sufficiency, all we have to show is that for every two generated $X \in F(D)$, X is v-invertible and X^{-1} is a v-ideal of finite type (i.e., every two generated ideal is t-invertible) [5]. Of course the finite type bit is taken care of by the finite conductor property. For the v-invertibility, let $X = (a, b)$, $a, b \in K - \{0\}$. Then $X^{-1} = A = (a_1, \ldots, a_n)$. So for all $B \in F(D)$ $(A : B^{-1})_v = (AB)_v$. But if $B = X$ then

$$(A : A)_v = (X^{-1}X)_v = D .$$

This leads to the following well known result.

COROLLARY 4.11. – *An integrally closed finite conductor domain is a PVMD.*

In the spirit of this article one may ask what happens if in Theorem 4.10, we replace v by d or by t. For the d-operation we have nothing new. That is, we get: *A finite conductor domain is a Prüfer domain with the TV-property if and only if for all $A \in f(D)$ and $B \in F(D)$, $A : B^{-1} = AB$.* But (1) of Theorem 4.5 is much more than this statement. On the other hand, if we replace v by t we do get a new result.

THEOREM 4.12. – *A finite conductor domain D is a TV-PVMD if and only if for all $A \in f(D)$ and $B \in F(D)$, $(A : B^{-1})_t = (AB)_t$.*

PROOF. – (*Sufficiency*). The TV-property comes from putting $A = D$. So we are left with $(A : B^{-1})_v = (AB)_v$ and Theorem 4.10 applies.

(*Necessity*). If D is a TV-PVMD, then as D is integrally closed we have, for all $A \in f(D)$ and $B \in F(D)$, $(A : B^{-1})_v = (AB)_v$. But then, by the TV-property,

$$(A : B^{-1})_t = (AB)_t .$$

On the more familiar grounds we can prove, with the help of quotient based statements, the following extremely well known result.

THEOREM 4.13. – *For a Noetherian domain D the following are equivalent.*

(1) *D is integrally closed.*

(2) *D is completely integrally closed.*

(3) *For every pair $A, B \in F(D)$ there exists a star operation $* = *(A, B)$ such that $(A : B^{-1})^* = (AB)^*$.*

(4) *D is a v-domain.*

(5) *D is a Krull domain.*

PROOF. – (1) ⇒ (2). Because D is integrally closed for all $A \in f(D) = F(D)$; $B \in F(D)$; $(A : B^{-1})_v = (AB)_v$. So for all $A, B \in F(D)$ we have $(A : B^{-1})_v = (AB)_v$ and by Theorem 3.1, D is completely integrally closed.

(2) ⇔ (3). This is obvious by Theorem 3.1.

(3) ⇒ (4). Obvious because (2) ⇒ (4).

(4) ⇒ (5). Since D is a v-domain we have for all $A \in f(D) = F(D)$, and $B \in F(D)$

$$A_v : B^{-1} = (AB)_v .$$

But as a Noetherian domain is obviously a TV-domain [7] the above equation becomes $A_t : B^{-1} = (AB)_t$ for all $A, B \in F(D)$. But this, by Theorem 3.9, is equivalent to D being Krull.

(5) ⇒ (4). This is because a Krull domain is integrally closed.

The above study includes notions whose characterizations generalize

(1) $\forall A \in F(D)$; A is invertible (Dedekind domains),

(2) $\forall A \in f(D)$; A is invertible (Prüfer domains).

To complete the picture we also include generalizations of

(3) $\forall A \in F(D)$; A is principal (PID)

(4) $\forall A \in f(D)$; A is principal (Bezout domains).

For this we note that for all $A \in f(D)$ we have $A_t = A_v$, that D is a GCD-domain if and only if for all $A \in f(D)$; A_v is principal and that a GCD-Krull domain is a UFD. As a result of these observations we have the following theorem.

THEOREM 4.14. – *An integral domain D is a UFD if and only if for each $A \in F(D)$, A_t is principal.*

PROOF. – Obviously if for all $A \in F(D)$, A_t is principal, then D is a GCD-domain. Now for all $A \in F(D)$, A_t principal implies A_t is invertible, hence t-invertible. That is, each $A \in F(D)$ is t-invertible. But then according to [9] D is Krull. However, a Krull GCD-domain is a UFD.

Conversely, if D is a UFD, it is a Krull GCD-domain. Now D Krull implies that for all $A \in F(D)$, A_t is t-invertible and hence

	Principal then D is a	Invertible then D is a	t-invertible then D is a	v-invertible then D is a
If $\forall A \in F(D)$, A is	PID	Dedekind domain	Krull domain	Completely integrally closed domain CIC-domain
If $\forall A \in f(D)$, A is	Bezout domain	Prüfer domain	Prüfer v-multiplication domain PVMD	v-domain
If $\forall A \in f(D)$, $A_t = A_v$ and $\forall A \in f(D)$, A is	TV-Bezout domain *1	TV-Prüfer domain *2	TV-PVMD *3	TV-PVMD
If $\forall A \in F(D)$, A_t is	UFD	locally factorial Krull domain	Krull domain	CIC-domain
If $\forall A \in f(D)$, A_t is	GCD-domain	Generalized GCD-domain	PVMD	v-domain
If $\forall A \in F(D)$, $A_t = A_v$ and $\forall A \in f(D)$, A_t is	TV-GCD-domain	locally GCD TV-domain	TV-PVMD	TV-PVMD
If $\forall A \in F(D)$, A_v is	GCD-domain with complete group of divisibility	Generalized Dedekind domain	Pre-Krull domain *4	CIC-domain
If $\forall A \in f(D)$, A_v is	GCD-domain	Generalized GCD-domain	PVMD	v-domain
If $\forall A \in F(D)$, $A_t = A_v$ and $\forall A \in f(D)$, A_v is	TV-GCD-domain	TV-Generalized GCD-domain	TV-PVMD	TV-PVMD

a v-ideal of finite type. But then as D is a GCD-domain, every v-ideal of finite type is principal.

Finally, if for all $A \in F(D)$, A_v is principal, then D is a GCD-domain with the property that every set of non-zero elements of D has a GCD. This is obviously a GCD-domain with the group of divisibility a complete (lattice ordered) group. The rather obvious examples are a rank one valuation domain with value group the set of reals, and the ring of entire functions.

The following diagram indicates the notions that arise from the various generalizations of statements (1)-(4). For example, the first statement is «If $\forall A \in F(D)$, A is principal then D is a PID».

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