Non-Atomic Unique Factorization in Integral Domains

Article · March 2005

DOI: 10.1201/9781420028249.ch1

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Arithmetical Properties of Commutative Rings and Modules (ed. S. Chapman), CRC Press, Boca Raton, 2005, 1-21

Chapter 1

Non-Atomic Unique Factorization in Integral Domains

by Daniel D. Anderson

Abstract

UFDs can be characterized by the property that every nonzero nonunit is a product of principal prime elements or equivalently that every nonzero nonunit x can be written in the form $x = up_1^{a_1} \cdots p_n^{a_n}$ where u is a unit, $p_1, \ldots, p_n$ are nonassociate principal primes, and each $a_i \geq 1$. Each $p_i^{a_i}$, in addition to being a power of a prime, has a number of other properties, each of which is subject to generalization. We survey various generalizations of (unique) factorization into prime powers in integral domains.

1 Introduction

Unique factorization domains are of course the integral domains in which every nonzero nonunit element has a unique factorization (up to order and associates) into irreducible elements or atoms. Now UFDs can also be characterized by the property that every nonzero nonunit is a product of principal primes or equivalently that every nonzero nonunit has the form $up_1^{a_1} \cdots p_n^{a_n}$ where u is a unit, $p_1, \ldots, p_n$ are nonassociate principal primes, and each $a_i \geq 1$. Each of the $p_i^{a_i}$, in addition to being a power of a prime, has other properties, each of which is subject to generalization. For example, each $p_i^{a_i}$ is primary, each is contained in a unique maximal t-ideal, and the $p_i^{a_i}$ are pairwise coprime. The goal of this chapter is to survey various generalizations of (unique) factorization into prime powers in integral domains. This follows the thesis of M. Zafrullah that the $p_i^{a_i}$ are the building blocks in a UFD. The author would like to thank M. Zafrullah for a number of discussions of these topics over the past several years.

This chapter consists of four sections besides the introduction. Section 2 covers integral domains whose elements are products of primary elements, Section 3 relates locally finite intersections of localizations to factorizations, Section 4 surveys factorizations into pairwise comaximal elements, and Section 5 summarizes the various generalizations of prime powers and the integral domains whose elements are products of these various generalizations. A fairly complete list of references is also given. The next four paragraphs outline in more detail these four sections. Then a brief review of some of the terms and notation used throughout the chapter is given. In particular, star-operations, the t-operation, the group of divisibility, and splitting sets are discussed.

In Section 2 we consider *weakly factorial domains*, integral domains with the property that every nonzero nonunit is a product of primary elements. While factorization into primary elements need not be unique, it is unique (up to order and associates) once we combine primaries with the same radical, that is, it is unique when the primary elements involved are pairwise v-coprime. Now UFDs are also characterized as being the Krull domains D with trivial divisor class group $Cl(D)$ or the integral domains D whose group of divisibility $G(D)$ is a cardinal sum of copies of $\mathbb{Z}$ (with the usual order). In a similar manner, an integral domain D is weakly factorial if and only if D is *weakly Krull*, that is, $D = \cap_{P \in X^{(1)}(D)} D_P$ (here $X^{(1)}(D)$ is the set of height-one primes of D) where the intersection is locally finite and D has t-class group $Cl_t(D) = 0$ or if and only if the natural map $G(D) \to \oplus_{P \in X^{(1)}(D)} G(D_P)$ is an order isomorphism (or equivalently, is just surjective). Also, D is weakly factorial if and only if every saturated multiplicatively closed subset of D is a splitting set.

In the case of a weakly factorial domain D, the (unique) factorization of an element into v-coprime primary elements is given by the intersection $D = \cap_{P \in X^{(1)}(D)} D_P$ in the following sense. Let x be a nonzero nonunit of D that is contained in the height-one primes $P_1, \ldots, P_n$. Then each $xD_{P_i} \cap D = (x_i)$ is a principal P_i-primary ideal and $(x) = \cap_{P \in X^{(1)}(D)} xD_P = (x_1) \cap \cdots \cap (x_n) = (x_1) \cdots (x_n)$ where $(x_1) \cap \cdots \cap (x_n)$ is also the reduced primary decomposition for (x). In Section 3 we consider factorizations induced by other locally finite intersections of localizations $D = \cap_{P \in S} D_P$. For example, call a nonzero nonunit element *t-pure* if it is contained in a unique maximal t-ideal (as is the case for a nonzero primary element). Then every nonzero nonunit of D is a product of t-pure elements if and only if the intersection $D = \cap_{P \in t\text{-}Max(D)} D_P$ (where $t\text{-}Max(D)$ is the set of maximal t-ideals of D) is locally finite, is independent (distinct maximal t-ideals contain no common nonzero prime ideal) and $Cl_t(D) = 0$. Again, this factorization into t-pure elements is unique once elements contained in the same maximal t-ideal are combined, or equivalently, when the elements in the product are pairwise v-coprime. We also consider factorization into homogeneous elements and into rigid elements.

In Section 4 we consider the comaximal factorizations recently introduced by McAdam and Swan [27]. They defined a nonzero nonunit element d of an integral domain D to be *pseudo-irreducible* (*pseudo-prime*) if $d = ab$ ($d|ab$) with a and b comaximal implies that a or b is a unit ($d|a$ or $d|b$). A factorization $d = d_1 \cdots d_n$ is a (*complete*) *comaximal factorization* if each d_i is a nonzero nonunit (pseudo-irreducible) and the d_i's are pairwise comaximal. The integral domain D is a *comaximal factorization domain* (*CFD*) if each nonzero nonunit has a complete comaximal factorization and a *unique comaximal factorization domain* (*UCFD*) is a CFD in which complete comaximal factorizations of an element are unique up to order and associates. The main result is that for a CFD D the following are equivalent: (1) every two-generated invertible ideal of D is principal, (2) every nonzero nonunit of D has a comaximal factorization into pseudo-prime elements, and (3) D is a UCFD. We give a star-operation generalization, and relate these comaximal factorizations to the factorizations given in Sections 2 and 3. It is interesting to note that a one-dimensional domain is a UCFD if and only if it is weakly factorial.

In Section 5 we summarize the various prime power generalizations given in Sections 2–4 and give several additional ones. Then the integral domains whose nonzero nonunits are products of these prime power generalizations are considered. Diagrams are given to show the various implications among the prime power generalizations and the integral domains defined.

We next review some of the terms and notation used in this chapter. Throughout D is an integral domain with quotient field K. *Let* $F(D)$ denote the set of nonzero fractional ideals of D and let $f(D)$ be the set of finitely generated members of $F(D)$. A *star-operation* $*$ on D is a closure operation on $F(D)$ that further satisfies $D^* = D$ and $(xA)^* = xA^*$ for all $0 \neq x \in K$ and $A \in F(D)$. Here $A \in F(D)$ is called a *$*$-ideal* if $A^* = A$ and A has *finite type* if $A^* = B^*$ for some $B \in f(D)$. The star-operation $*$ has *finite character* if for each $A \in F(D)$, $A^* = \cup\{B^* | B \subseteq A$ with $B \in f(D)\}$. Suppose that $*$ has finite character. Then every proper $*$-ideal is contained in a maximal $*$-ideal and a maximal $*$-ideal is prime. A prime ideal minimal over a $*$-ideal is a $*$-ideal. Hence a prime ideal minimal over a principal ideal is a $*$-ideal. We use $*$-Max(D) to denote the set of maximal $*$-ideals. Here $D = \cap_{P \in *\text{-Max}(D)} D_P$. With any star-operation $*$ is associated the finite character star-operation $*_s$ where $A^{*_s} = \cup\{B^* | B \subseteq A$ where $B \in f(D)\}$. Evidently $*$ has finite character precisely when $* = *_s$.

Examples of star-operations include (1) the *d-operation* $A_d = A$, (2) the *v-operation* $A_v = (A^{-1})^{-1} = \cap\{Dx | Dx \supseteq A$ where $0 \neq x \in K\}$, (3) the *t-operation* $A_t = A_{v_s} = \cup\{B_v | B \subseteq A$ with $B \in f(D)\}$, and (4) for a set $\{D_\alpha\}$ of overrings of D with $D = \cap D_\alpha$, the *star-operation induced by* $\cap D_\alpha$ $A^* = \cap AD_\alpha$. Here (1) and (3) have finite character while (2) generally does not. An intersection $D = \cap D_\alpha$ has *finite character* or is *locally finite* if each $0 \neq x \in D$ is a unit in almost all D_α. If the intersection $D = \cap D_\alpha$ is locally finite, then the induced star-operation has finite character, but not conversely [1, Theorem 1].

An ideal $A \in F(D)$ is *$*$-invertible* if there is a $B \in F(D)$ with $(AB)^* = D$. In this case we can take $B = A^{-1}$. The set $\text{Inv}_*(D)$ of $*$-invertible $*$-ideals forms a group under the $*$-product $A * B = (AB)^*$. The *$*$-class group* $Cl_*(D)$ of D is $\text{Inv}_*(D)/\text{Princ}(D)$ where $\text{Princ}(D)$ is the subgroup of nonzero principal fractional ideals. Of particular importance is the *t-class group* $Cl_t(D)$ (or just the *class group*) of D. For D a Krull domain $Cl_t(D)$ is the usual divisor class group while for a Prüfer domain D or one-dimensional domain D $Cl_t(D)$ is the Picard group $\text{Pic}(D) = \text{Inv}(D)/\text{Princ}(D) = Cl_d(D)$. If $*$ is a finite character star-operation, then a $*$-invertible ideal has finite type, in fact $A \in F(D)$ is $*$-invertible if and only if A has finite type and A_P is principal for each $P \in *\text{-Max}(D)$.

Two ideals A and B of D are *$*$-comaximal* if $(A, B)^* = D$. If A and B are $*$-comaximal, then it is easily proved that $A^* \cap B^* = (A \cap B)^* = (AB)^*$. In the case where $* = d$, we just say *comaximal*. Two elements $a, b \in D$ are *$*$-coprime* if

$(a,b)^* = ((a),(b))^* = D$ and a and b are *coprime* if $[a,b] = 1$ (here $[a,b]$ denotes the GCD of a and b). Thus $a,b \in D$ are *v-coprime* if $(a,b)_v = D$, or equivalently, a and b are not contained in any maximal t-ideal.

Of particular interest are star-operations induced by intersections of localizations $D = \cap_{P\in S} D_P$: $A \to A^{*_S} = \cap_{P\in S} AD_P$. For any star-operation $*$ we have the associated finite character star-operation $*_w$ defined by $A^{*_w} = \{x \in K | xI \subseteq A$ for some $I \in f(D)$ with $I^* = D\} = \cap_{P\in *_s\text{-Max}(D)} AD_P$. Here $*_s\text{-Max}(D) = *_w\text{-Max}(D)$, A is $*_s$-invertible if and only if A is $*_w$-invertible and $Cl_{*_s}(D) = Cl_{*_w}(D)$. See [7] for details.

General references for star-operations include [21], [23], and [24]. For results on $*$-invertibility, class groups, and t-ideals see [18] and [35]. For star-operations induced by overrings, see [1] and [7].

For an integral domain D with quotient field K let $K^* = K - \{0\}$ and $U(D)$ be the group of units of D. Then the multiplicative group $G(D) = K^*/U(D)$ is called the *group of divisibility of* D. Now $G(D)$ is partially ordered by $aU(D) \leq bU(D) \iff a|b$ in $D \iff Da \supseteq Db$. Note that the map $G(D) \to \text{Princ}(D)$ given by $aU(D) \to Da$ is an order isomorphism where $\text{Princ}(D)$ is ordered by reverse inclusion.

Given a family $\{(G_\lambda, \leq_\lambda)\}$ of partially ordered abelian groups, the *cardinal product* is ΠG_λ with the order $(a_\lambda) \leq (b_\lambda) \iff$ each $a_\lambda \leq b_\lambda$. The *cardinal sum* is defined in a similar manner. Suppose that $D = \cap_{P\in S} D_P$. Then we have an order preserving monomorphism φ: $G(D) \to \Pi_{P\in S} G(D_P)$ (cardinal product) given by $\varphi(xU(D)) = (xU(D_P))$. Note that if $D = \cap_{P\in S} D_P$ has finite character, then $\operatorname{im}\varphi \subseteq \oplus_{P\in S} G(D_P)$.

A subgroup H of a partially ordered abelian group $(G, \leq)$ is *convex* if whenever $0 \leq a \leq h$ with $h \in H$, then $a \in H$ and H is *directed* (or *filtered*) if given $a, b \in H$, there exists $c \in H$ with $a \leq c$ and $b \leq c$ or equivalently $H_+ = \{h \in H | h \geq 0\}$ generates H as a group. Let S be a saturated multiplicatively closed subset of D. Then $< S > = \{s_1 s_2^{-1} U(D) | s_1, s_2 \in S\}$ is a convex directed subgroup of $G(D)$. The converse is also true, see [28]. A convex directed subgroup H of $G(D)$ is a *cardinal summand* of $G(D)$ if there is a convex directed subgroup K of $G(D)$ with $G(D) = H \oplus K$ where $H \oplus K$ is the cardinal sum of H and K. A saturated multiplicatively closed subset S of D is a *splitting set* if every nonzero element x of D can be written in the form $x = st$ where $s \in S$ and t is v-coprime to each element of S. It is not hard to show that a saturated multiplicatively closed set S is a splitting set if and only if $< S >$ is a cardinal summand of $G(D)$ ([29, Proposition 4.1] and [4, Theorem 22]). Perhaps the main example of a splitting set is a multiplicatively closed subset generated by a set $\{p_\alpha\}$ of height-one principal primes satisfying $\bigcap p_{\alpha_n} D = 0$ for each countable subcollection $\{p_{\alpha_n}\}$ of $\{p_\alpha\}$. Splitting sets were introduced in [29] and studied in [4] and [17].

There are of course many other generalizations of UFDs. A UFD is characterized by the property that every nonzero prime ideal contains a nonzero principal prime (ideal). Thus for each of the generalizations of a prime power, we can ask what domains have the property that each nonzero prime ideal contains such an element. While we will not pursue that theme here, the reader is referred to [14] for various "Kaplansky-like" theorems. Recall that an integral domain D is *atomic* if every nonzero nonunit of D is a product of irreducible elements or atoms. The theme of studying integral domains with various good atomic factorization properties is put forth in [3]. For a survey of extensions of unique factorization to integral domains, rings with zerodivisors, rings without identity, or to modules, the reader is referred to [2]. In fact, this chapter is an elaboration of Section 2 of that chapter. Portions of this chapter were presented at a mini-conference on factorization organized by Professor Scott Chapman held October 23, 2003 preceding the regional American Mathematical Society meeting at the University of North Carolina.

2 Weakly Factorial Domains

Throughout D will denote an integral domain with quotient field K. Recall that D is *weakly factorial* if every (nonzero) nonunit of D is a product of primary elements or equivalently every proper principal ideal of D is a product of principal primary ideals. Weakly factorial domains were introduced by D. D. Anderson and L. A. Mahaney [9] and further studied by Anderson and Zafrullah [13]. We begin by looking at factorizations into primary elements in an integral domain.

Suppose that (q) is a nonzero P-primary ideal of D. Then P is the unique prime t-ideal containing (q) [5, Lemma 1]. Thus nonzero primary elements with distinct radicals are v-coprime. If (q_1) and (q_2) are P-primary, then so is $(q_1)(q_2) = (q_1q_2)$ [9, Corollary 2]. Thus if a nonzero nonunit $x \in D$ is a product of primary elements, then by combining primary elements with the same radical, we can write $x = q_1 \cdots q_n$ where (q_i) is P_i-primary and the P_i's are distinct, or equivalently, the q_i's are pairwise v-coprime. In this case, $(x) = (q_1)\cdots(q_n) = (q_1) \cap \cdots \cap (q_n)$ where $(q_1) \cap \cdots \cap (q_n)$ is the unique normal decomposition for (x) [9, Corollary 5]. It follows that the factorization of an element into a product of pairwise v-coprime primary elements is unique up to order and associates.

We next give a number of conditions equivalent to D being weakly factorial and then sketch the proof of several of the implications.

Theorem 2.1. *For an integral domain D the following conditions are equivalent.*

(1) *D is weakly factorial.*

(2) *Every nonzero nonunit of D has a unique factorization (up to order and associates) into a product of pairwise v-coprime primary elements.*

(3) *$D = \cap_{P \in X^{(1)}(D)} D_P$ is locally finite and $Cl_t(D) = 0$.*

(4) *For every nonzero nonunit $x \in D$, if P is a prime ideal minimal over (x), then $\operatorname{ht} P = 1$ and $xD_P \cap D$ is principal.*

(5) *$D = \cap_{P \in X^{(1)}(D)} D_P$ is locally finite and the natural map*

$$G(D) \to \oplus_{P \in X^{(1)}(D)} G(D_P)$$

is surjective (and hence an order isomorphism).

(6) *Every convex directed subgroup of $G(D)$ is a cardinal summand of $G(D)$.*

(7) *For every (nonzero) prime ideal P of D, $\langle D - P \rangle = \{s_1 s_2^{-1} U(D) | s_1, s_2 \in D - P\}$ is a cardinal summand of $G(D)$.*

(8) *Every saturated multiplicatively closed subset of D is a splitting set.*

(9) *For every (nonzero) prime ideal P of D, $D - P$ is a splitting set.*

Proof. We have already remarked the equivalence of (1) and (2). The equivalence of (1) and (3)-(7) is [13, Theorem]. Since every convex directed subgroup of $G(D)$ is of the form $\langle S \rangle = \{s_1 s_2^{-1} U(D) \,|\, s_1, s_2 \in S\}$ for some saturated multiplicatively closed subset S of D and conversely and since $\langle S \rangle$ is a cardinal summand of $G(D)$ if and only if S is a splitting set, we get the equivalence of (6) and (8) and of (7) and (9). ∎

We next sketch the proof of several of the implications of the previous theorem. Suppose that D is weakly factorial. Since each nonzero prime ideal contains a nonzero primary element, each nonzero prime ideal contains the radical of a nonzero principal primary ideal which is a maximal t-ideal. Hence $X^{(1)}(D) = t\text{-Max}\,(D)$. As we always have $D = \cap_{P \in t\text{-Max}(D)} D_P$, we have $D = \cap_{P \in X^{(1)}(D)} D_P$. Moreover, each proper principal ideal having a primary decomposition involving only height-one primes gives that each nonzero principal ideal is contained in only finitely many height-one primes and hence the intersection is locally finite. (In fact, $D = \cap_{P \in X^{(1)}(D)} D_P$ with the intersection being locally finite if and only if every nonzero principal ideal has a primary decomposition involving only height-one primes [9, Theorem 13].) So D is weakly Krull. Let A be an integral t-invertible ideal of D. Let $P_1, \cdots, P_n$ be the maximal t-ideals (=height-one primes) containing A. Now A_{P_i} is principal, so $A_{P_i} = q_i D_{P_i}$ where $q_i D$ is P_i-primary. Then $A \subseteq (A_{P_1} \cap D) \cap \cdots \cap (A_{P_n} \cap D) = q_1 D \cap \cdots \cap q_n D = q_1 \cdots q_n D$. So $A = q_1 \cdots q_n A'$ where A' is contained in no height-one prime. Hence $A'_t = D$, so $A_t = q_1 \cdots q_n D$ is principal. Thus $Cl_t(D) = 0$.

Conversely, suppose that $D = \cap_{P \in X^{(1)}(D)} D_P$ where the intersection is locally finite and that $Cl_t(D) = 0$. Let x be a nonzero nonunit of D. Then $xD = (xD_{P_1} \cap D) \cap \cdots \cap (xD_{P_n} \cap D)$ where $P_1, \cdots, P_n$ are the height-one primes containing xD. But each $xD_{P_i} \cap D$ is P_i-primary and $xD_{P_1} \cap D, \cdots, xD_{P_n} \cap D$ are v-coprime. Hence

$$xD = (xD_{P_1} \cap D) \cap \cdots \cap (xD_{P_n} \cap D) = ((xD_{P_1} \cap D) \cdots (xD_{P_n} \cap D))_t\,;$$

so each $xD_{P_i} \cap D$ is t-invertible. Hence $Cl_t(D) = 0$ gives $(xD_{P_i} \cap D)_t = q_i D$ where $q_i D$ is P_i-primary. So $xD = q_1 D \cdots q_n D$. Thus D is weakly factorial.

We next show that (1) and (4) are equivalent. Suppose that D is weakly factorial. If P is minimal over (x), then $\operatorname{ht} P = 1$ and $(x)_P \cap D$ is principal being the P-primary component in the reduced primary decomposition of (x). Conversely, suppose that (4) holds. Let S be the multiplicatively closed subset of D consisting of all products of primary elements and units. Then S is saturated. Let y be a nonunit factor of an element $x \in S$. Then (x) being a product of principal primary ideals has only finitely many minimal primes each of which by hypothesis has height-one. Since y is a factor of x, it too is contained in only finitely many height-one primes $P_1, \cdots, P_n$. Now $y \in (y)_{P_1} \cap D = (q_1)$ where (q_1) is P_1-primary and $y = q_1 t_1$ where $t_1 \in P_2 \cap \cdots \cap P_n - P_1$. Continuing in this manner we get $y = q_1 \cdots q_n t_n$ where (q_i) is P_i-primary and t_n is contained in no height-one prime, hence a unit. So $y \in S$. Suppose that some nonzero nonunit of D is not in S. Then $(a) \cap S = \varnothing$ since S is saturated. Hence (a) can be enlarged to a prime ideal $P \supseteq (a)$ maximal with respect to $P \cap S = \varnothing$. Shrinking P to a prime ideal Q minimal over (a), we have a height-one prime ideal Q containing no primary elements. But this is absurd, for $(a)_Q \cap D$ is itself a principal primary ideal contained in Q.

We next give some examples of weakly factorial domains.

Example 2.2.

(1) A one-dimensional quasilocal domain has every nonunit element primary and hence is weakly factorial. Conversely, a domain in which every nonunit is primary is either a field or one-dimensional quasilocal.

(2) [9, page 149] A one-dimensional domain D is weakly factorial if and only if each nonunit of D is contained in only finitely many maximal ideals and $Pic(D) = 0$.

(3) [9, Theorem 21] Given any collection $\{H_\lambda\}$ of rank-one totally ordered abelian groups, there is a weakly factorial Bezout domain D with $G(D)$ order-isomorphic to the cardinal direct sum $\oplus H_\lambda$.

(4) [13, Example] Let $K \subseteq L$ be fields and $D = K + XL[X] = \{f(X) \in L[X] | f(0) \in K\}$. Then D is an atomic weakly factorial domain. It is easily seen that every nonzero nonunit of D may be written in one of the following two forms: $p_1 \cdots p_n$ or $p_1 \cdots p_n (aX^r)$ where $p_1, \cdots, p_n$ are principal primes and $aX^r (r \geq 1,\ a \in L)$ is primary.

(5) [5, Lemma 3] An atomic domain in which almost all atoms are prime is weakly factorial.

(6) [5, Corollary 2] An atomic domain is weakly factorial $\iff$ every atom is primary $\iff$ each atom is contained in a unique maximal t-ideal.

While $D[X]$ is a UFD whenever D is, this need not be the case for a weakly factorial domain. We have the following result. Recall that D is a *generalized Krull domain* if D is weakly Krull and for each $P \in X^{(1)}(D)$, D_P is a valuation domain.

Theorem 2.3. *For an integral domain D the following conditions are equivalent.*

(1) $D[X]$ is weakly factorial.

(2) D is a weakly factorial GCD domain.

(3) D is a weakly factorial generalized Krull domain.

(4) D is a generalized Krull domain and a GCD domain.

(5) D is weakly factorial and if p and q are non v-coprime primary elements then $p | q$ or $q | p$.

Proof. (1)$\iff$(2) [9, Theorem 17]. The equivalence of (2)–(4) is [9, Theorem 20]. (2)$\iff$(5) [9, Theorem 18] and its proof. ∎

It is interesting to observe that $D[X]$ is an atomic weakly factorial domain if and only if $D[X]$ and hence D is a UFD. This follows from the previous theorem and the well-known fact that an atomic GCD domain is a UFD.

3 Independent Locally-Finite Intersections of Localizations

In Section 2 we saw that factorization into primary elements was intimately connected to the locally finite representation $D = \cap D_{P \in X^{(1)}(D)} D_P$. Now of course an integral domain need not have such a representation. In this section we extend the results of Section 2 to more general intersections of localizations. Much of this comes from Anderson and Zafrullah [16]. We begin with some definitions given there.

Let D be an integral domain with quotient field K. A family $\mathcal{F} = \{P_i\}_{i \in I}$ of nonzero prime ideals of D is called a *defining family of primes for D* if $D = \cap_{i \in I} D_{P_i}$. If, further, every nonzero nonunit of D belongs to at most finitely many members of $\mathcal{F}$, $\mathcal{F}$ is of *finite character*, and if no two members of $\mathcal{F}$ contain a common nonzero prime ideal, $\mathcal{F}$ is *independent*. An integral domain is *independent of finite character $\mathcal{F}$* (or an *$\mathcal{F}$-IFC domain*) if it has a defining family $\mathcal{F}$ of prime ideals that is independent and of finite character. Let us note that the h-local domains of Matlis [26] (where $\mathcal{F} = \mathrm{Max}(D)$), Noetherian domains whose grade-one primes are of height one, Krull domains, generalized Krull domains of Ribenboim [30], independent rings of Krull type [22], and weakly Krull domains [10] are all $\mathcal{F}$-IFC domains with well-defined family of primes.

Let $\mathcal{F}$ be a defining family of primes for D. Then $\mathcal{F}$ induces the star-operation $*_{\mathcal{F}}$ on D by $A^{*_{\mathcal{F}}} = \cap_{P \in \mathcal{F}} AD_P$ for $A \in F(D)$. Moreover, if $\mathcal{F}$ has finite character, so does $*_{\mathcal{F}}$ [1, Theorem 1]. We call an integral ideal A of D *unidirectional* if A is contained in a unique member of $\mathcal{F}$. The following four theorems come from [16].

Theorem 3.1. *([16, Theorem 2.1]) Let D be an integral domain and let $\mathcal{F}$ be a defining family of primes for D. Then the following are equivalent.*

(1) D is an $\mathcal{F}$-IFC domain.

(2) For every nonzero nonunit x of D, (x) is a $*_{\mathcal{F}}$-product of finitely many unidirectional $*_{\mathcal{F}}$-ideals of D.

(3) Every nonzero prime ideal of D contains a nonzero element x such that (x) is a $*_{\mathcal{F}}$-product of finitely many unidirectional $*_{\mathcal{F}}$-ideals.

If $P_1, \cdots, P_n$ are the members of $\mathcal{F}$ containing (x) in (2), then the factorization alluded to is

$$(x) = (xD_{P_1} \cap D) \cap \cdots \cap (xD_{P_n} \cap D) = ((xD_{P_1} \cap D) \cdots (xD_{P_n} \cap D))^{*_{\mathcal{F}}}.$$

Theorem 3.2. *([16, Theorem 3.3]) Let $\mathcal{F}$ be a defining family of mutually incomparable primes for D such that $*_{\mathcal{F}}$ is of finite character. Then the following are equivalent.*

(1) $\mathcal{F}$ is independent of finite character.

(2) Every nonzero prime ideal of D contains an element x such that (x) is a $*_{\mathcal{F}}$-product of unidirectional $*_{\mathcal{F}}$-ideals.

(3) Every nonzero prime ideal of D contains a unidirectional $*_{\mathcal{F}}$-invertible $*_{\mathcal{F}}$-ideal.

(4) For $P \in \mathcal{F}$ and $0 \neq x \in P$, $xD_P \cap D$ is $*_{\mathcal{F}}$-invertible and unidirectional.

(5) $\mathcal{F}$ is independent and for any nonzero ideal A of D, $A^{*_{\mathcal{F}}}$ is of finite type whenever AD_P is finitely generated for all P in $\mathcal{F}$.

Theorem 3.3. *([16, Corollary 3.5]) Let $\mathcal{F}$ be a defining family for D consisting of incomparable primes. Then the following are equivalent.*

(1) $\mathcal{F}$ is independent of finite character and every $*_{\mathcal{F}}$-invertible $*_{\mathcal{F}}$-ideal is principal.

(2) Every nonzero nonunit x of D may be written in the form $x = x_1 \cdots x_n$ where each x_i generates a unidirectional ideal.

(3) For each nonzero nonunit x in D and for each P in $\mathcal{F}$ containing x, $xD_P \cap D$ is principal and unidirectional.

(4) The natural map $G(D) \to \Pi_{P \in \mathcal{F}} G(D_P)$ has image $\oplus_{P \in \mathcal{F}} G(D_P)$.

The factorization in (2) becomes unique (up to order and associates) once the x_i's belonging to the same unidirectional ideal are combined. If $P_1, \cdots, P_n$ are the primes of $\mathcal{F}$ containing x and if $xD_{P_i} \cap D = (x_i)$, then this factorization is $x = ux_1 \cdots x_n$ for some unit u.

Now suppose that $*$ is a star-operation on D. Now $D = \cap_{P \in *_s\text{-Max}(D)} D_P$, so $\mathcal{F} = *_s\text{-Max}(D)$ is a defining family for D. Here the star-operation $*_{s\mathcal{F}}$ is given by $A^{*_{s\mathcal{F}}} = \cap_{P \in \mathcal{F}} AD_P$. But by [7], $\cap_{P \in \mathcal{F}} AD_P = A^{*_w} = \{x \in K | xI \subseteq A$ for some $I \in f(D)$ with $I^* = D\}$. Also, $*_{s\mathcal{F}} = *_w$ has finite character, A is $*_{s\mathcal{F}}$-invertible if and only if A is $*_s$-invertible and $Cl_{*_{s\mathcal{F}}}(D) = Cl_{*_s}(D)$. So we have the following result.

Theorem 3.4. *Let $*$ be a finite character star-operation on the integral domain D. Then the following conditions are equivalent.*

(1) *$*$-Max(D) is independent of finite character and $Cl_*(D) = 0$.*

(2) *Every nonzero nonunit x of D may be written in form $x = x_1 \cdots x_n$ where each x_i is contained in a unique member of $*$-Max(D).*

(3) *Every nonzero nonunit x of D may be written uniquely (up to order and associates) in the form $x = x_1 \cdots x_n$ where each x_i is contained in a unique $P_i \in *$-Max(D) and for $i \neq j$, $P_i \neq P_j$, or equivalently, x_i and x_j are $*$-coprime.*

(4) *For each nonzero nonunit x in D and for each $P \in *$-Max(D), $xD_P \cap D$ is principal and unidirectional.*

(5) *The natural map $G(D) \to \Pi_{P \in *\text{-Max}(D)} G(D_P)$ has image $\oplus_{P \in *\text{-Max}(D)} G(D_P)$ (and hence is an order isomorphism where $\oplus G(D_P)$ has the cardinal order).*

(6) *$*$-Max(D) is independent, $Cl_*(D) = 0$, and for each nonzero ideal A of D, A^{*_w} is of finite type whenever AD_P is finitely generated for all $P \in *$-Max(D).*

Of special interest is the case where $*$ is the t-operation. Here $*_w = t_w$ is the w-operation introduced by Wang and McCasland [20]: $A_w = \{x \in K | xI \subseteq D$ for some finitely generated ideal I with $I^{-1} = D\}$. Let P be a prime ideal of a domain D. Then an ideal A of D is *P-pure* if $A_P \cap D = A$ (we usually assume $A \subseteq P$) and A is *t-pure* if A is P-pure for some $P \in t$-Max(D) [10]. For A t-invertible (with $A_t \neq D$), A_t is t-pure if and only if A is contained in a unique maximal t-ideal [11, Lemma 2.1]. A nonzero nonunit $x \in D$ is *t-pure* if (x) is t-pure, or equivalently, if x is contained in a unique maximal t-ideal. Following [11] we call an integral domain D *t-pure* if every nonzero nonunit of D is t-pure and we call D *semi-t-pure* if every nonzero nonunit of D is a finite product of t-pure elements. Evidently D is t-pure if and only if D is quasilocal with maximal ideal a t-ideal. Observe that if $0 \neq (q)$ is P-primary, then (q) is P-pure and t-pure and q is t-pure. We have the following t-version of Theorem 3.4.

Theorem 3.5. *For an integral domain D, the following conditions are equivalent.*

(1) *t-Max(D) is independent of finite character and $Cl_t(D) = 0$.*

(2) *D is semi-t-pure, i.e., every nonzero nonunit of D is a finite product of t-pure elements.*

(3) *Every nonzero nonunit x of D may be written uniquely in the form $x = x_1 \cdots x_n$ where each x_i is t-pure and for $i \neq j$, x_i and x_j are v-coprime.*

(4) *For $0 \neq x \in D$ and $x \in P \in t$-Max(D), $xD_P \cap D$ is principal and t-pure.*

(5) *The natural map $G(D) \to \Pi_{P \in t\text{-Max}(D)} G(D_P)$ has image $\oplus_{P \in t\text{-Max}(D)} G(D_P)$ (and hence is an order ismorphism where $\oplus G(D_P)$ has the cardinal order).*

(6) t-Max(D) is independent, $Cl_t(D) = 0$, and for each nonzero ideal A of D, A_w is of finite type whenever AD_P is finitely generated for all $P \in t$-Max(D).

(7) For each $P \in t$-Max(D), $D - P$ is a splitting set, or equivalently, $< D - P > = \{s_1 s_2^{-1} U(D) | s_1 s_2 \in D - P\}$ is a cardinal summand of $G(D)$.

(8) For each nonempty subset $F \subseteq t$-Max(D), $S = D - U_{P \in F} P$ is a splitting set, or equivalently $< S >$ is a cardinal summand of $G(D)$.

(9) D is an $\mathcal{F}$-IFC domain, for each $Q \in \mathcal{F}$, D_Q is t-pure, and $Cl_t(D) = 0$.

Proof. The equivalence (1)–(6) follows from Theorem 3.4. The equivalence of (1), (7) and (8) follows from [16, Theorem 2.14]. And the equivalence of (1)–(3) and (8) is given in [11, Theorem 3.4]. ∎

Observe that Theorem 2.1 (D weakly factorial) is just the case $X^{(1)}(D) = t$-Max(D).

We next give three more generalizations of prime powers that turn out to be equivalent to being t-pure in GCD domains. Let h be a nonzero nonunit of an integral domain D. Then h is *rigid* [31] (*homogeneous, strongly homogeneous* [11]) if for $x, y \in D$, $x, y | h$ (with $[x, y] = 1$, $(x, y)_v = D$) implies $x | y$ or $y | x$ (x or y is a unit). And D is (*semi-*) *rigid* if every nonzero nonunit of D is (a product of) rigid (elements). Similar definitions for (*semi-*) *homogeneous domains* and (*semi-*) *strongly homogeneous domains* may be given.

It is easy to see that an atom or prime power is rigid, that a rigid element is strongly homogeneous, and that a strongly homogeneous or t-pure (and hence primary) element is homogeneous. Evidently an integral domain is rigid if and only if it is a valuation domain. Thus any one-dimensional quasilocal domain that is not a valuation domain will have a nonzero primary element that is not rigid.

To study factorization in a nonatomic setting, P.M. Cohn [19] defined a nonzero nonunit $h \in D$ to be *primal* if $h | xy$ implies $h = h_1 h_2$ where $h_1 | x$ and $h_2 | y$ and to be *completely primal* if each of its nonunit factors is primal. (See [15] for a survey of completely primal elements.) Thus an atom is (completely) primal if and only if it is prime. An integral domain in which every nonzero nonunit is (completely) primal is called a *pre-Schreier domain* [34] and an integrally closed pre-Schreier domain is a *Schreier domain*. A GCD domain is a Schreier domain [19]. We have the following result.

Theorem 3.6. *([11, Theorem 2.3, Corollary 2.4]) Let h be a completely primal element of an integral domain D. Then the following are equivalent.*

(1) h is homogeneous.

(2) h is strongly homogeneous.

(3) h is t-pure.

If further D is a GCD domain, we have (1)–(3) equivalent to

(4) h is rigid.

It is well-known that $Cl_t(D) = 0$ for a pre-Schreier domain D [34, Corollary 3.7]. Combining this with Theorem 3.5 and Theorem 3.6 we get our next result.

Theorem 3.7. *([11, Corollary 3.7]) For an integral domain D, the following are equivalent.*

(1) D is a pre-Schreier semi-homogeneous integral domain.

(2) *Every nonzero nonunit of D may be written uniquely as a product of mutually coprime completely primal homogeneous elements.*
(3) *D is pre-Schreier and t-Max(D) is independent of finite character.*
(4) *D is a $\mathcal{F}$-IFC domain, for each $Q \in \mathcal{F}$, D_Q is a homogeneous pre-Schreier domain, and $Cl_t(D) = 0$.*

Combining Theorem 3.5 and the second part of Theorem 3.6 we get the following result which should be compared to Theorem 2.3. Recall that an integral domain D is an *independent ring of Krull type* if t-Max(D) is independent of finite character and D_P is a valuation domain for each $P \in t$-Max(D). The implication (1)$\Longrightarrow$(3) was first given in [31] while the implication (3)$\Longrightarrow$(1) was given in [32].

Theorem 3.8. *([11, Corollary 3.8]) For an integral domain D, the following conditions are equivalent.*
(1) *D is a semi-rigid GCD domain.*
(2) *D is a semi-t-pure GCD domain.*
(3) *D is a GCD domain that is an independent ring of Krull type.*
(4) *D is an independent ring of Krull type with $Cl_t(D) = 0$.*

We end this section with the following result. Recall that two ideals I and J of a ring are *condensed* if $IJ = \{ij | i \in I, j \in J\}$.

Theorem 3.9. *([8, Theorem 2.1]) Let D be an atomic domain. Then the following statements are equivalent.*
(1) *Every pair of comaximal ideals of D is condensed.*
(2) *Every pair of distinct maximal ideals of D is condensed.*
(3) *Each atom of D is unidirectional.*
(4) *D is h-local with* Pic(D) $= 0$.
(5) *Every nonzero nonunit of D is a product of unidirectional elements.*
(6) *For each nonzero nonunit $x \in D$ and each maximal ideal M containing x, $xD_M \cap D$ is a principal unidirectional ideal.*

As before, the factorization into unidirectional elements given in (5) is unique up to units and associates once elements contained in the same maximal ideal are combined. Note that (4)–(6) are still equivalent if we drop the hypothesis that D is atomic. Indeed these three statements are just three of the equivalences of Theorem 3.4 where $*$ is the d-operation.

4 Comaximal Factorizations and Generalizations

Recently McAdam and Swan [27] studied comaximal factorizations of elements and ideals in integral domains and rings. In this section we discuss their comaximal factorizations of elements in integral domains, give a generalization to star-operations, and compare it to results from Sections 2 and 3. We begin with some definitions from their chapter.

Let D be an integral domain. A nonzero nonunit b of D is *pseudo-irreducible* (resp. *pseudo-prime*) if $b = cd$ with c and d comaximal (i.e., $(c, d) = D$) implies c or d is a unit (resp. $b|cd$ with c and d comaximal implies that $b|c$ or $b|d$). Clearly b pseudo-prime implies that b is pseudo-irreducible. Note that b is pseudo-prime if and only if $D/(b)$ is indecomposable [27, Lemma 3.1]. For a nonzero nonunit b of D we call $b = b_1 \cdots b_m$ a *(complete) comaximal factorization of* b if the b_i are pairwise comaximal nonunit (pseudo-irreducible) elements. Evidently a comaximal factorization $b = b_1 \cdots b_m$ is complete if and only if it has no proper refinements that are also comaximal factorizations of b. Then D is a *comaximal factorization domain* (*CFD*) if every nonzero nonunit of D has a complete comaximal factorization and D is a *unique comaximal factorization domain* (*UCFD*) if D is a CFD in which complete comaximal factorizations are unique up to order and associates.

They showed [27, Lemma 1.1] that an integral domain D is a CFD if either (i) each nonzero nonunit of D has only finitely many minimal primes or (ii) each nonzero nonunit of D is contained in only finitely many maximal ideals. Here (i) insures that a Noetherian domain is a CFD. To this list of CFDs we can add: D has a defining family of primes of finite character. The proof is similar to the proof given for (i).

A key concept in their work is the notion of an S-ideal. A nonzero ideal I of D is called an *S-ideal* if $I = (a,c) = (a^2, c)$ for some $a, c \in I$. The relation to comaximal factorizations is the observation that $(a,c) = (a^2,c)$ if and only if there is an element $b \in D$ with $(a,b) = D$ and $c|ab$ [27, Lemma 1.2]. They proved that S-ideals are invertible and that any two-generated invertible ideal is isomorphic to an S-ideal [27, Lemma 1.5]. They gave the following characterization of UCFDs (we have added (3)).

Theorem 4.1. *([27, Theorem 1.7]) For an integral domain D the following conditions are equivalent.*

(1) D is a UCFD.

(2) D is a CFD and every pseudo-irreducible element of D is pseudo-prime.

(3) Every nonzero nonunit of D has a comaximal factorization into pseudo-prime elements.

(4) Every two-generated invertible ideal of D is principal.

(5) Every S-ideal of D is principal.

It should be noted that in the previous theorem we can not add $Pic(D) = 0$ as the following example from Section 4 of [27] shows (but see Theorem 4.2 below). Let A_n be the subring of $B_n = \mathbb{R}[x_0, \cdots, x_n]/\left(x_0^2 + \cdots + x_n^2 - 1\right)$, the ring of real-valued polynomial functions on the n-sphere S^n, consisting of all even functions. Then for $n \geq 2$, A_n is a regular domain that is a UCFD, but $Pic(A_n) \neq 0$.

Now a UFD D is a UCFD, indeed every nonzero nonunit of D is contained in only finitely many minimal prime ideals and $Pic(D) = 0$. For exactly the same reason, a weakly factorial domain is a UCFD. Note that the domains given by Theorem 3.4 for the case where $*$ is the d-operation are UCFDs. Indeed, these are precisely the UCFDs in which every pseudo-irreducible element is contained in a unique maximal ideal. In the case of a one-dimensional UFD (that is, a PID) a pseudo-prime element is just a primary element and a complete comaximal factorization is the same thing as a factorization into v-coprime prime powers or primary elements. Actually this result extends to any one-dimensional domain.

Theorem 4.2. *For a one-dimensional integral domain D the following are equivalent.*

(1) D is weakly factorial.

(2) *Every nonzero nonunit of D is contained in only finitely many maximal ideals and $Pic(D) = 0$.*

(3) *D is a UCFD.*

Proof. (1)$\Longleftrightarrow$(2) Example 2.2 (2). (2)$\Longrightarrow$(3) Since every nonzero nonunit is contained in only finitely many minimal primes, D is a CFD. By Theorem 4.1 D is a UCFD. (3)$\Longrightarrow$(1) Let $d \in D$ be pseudo-prime. Hence $D/(d)$ is indecomposable and thus $D/\sqrt{(d)}$ is indecomposable. Since $D/\sqrt{(d)}$ is zero-dimensional and reduced, $D/\sqrt{(d)}$ is von Neumann regular and hence a field since it is indecomposable. Thus $\sqrt{(d)}$ is a maximal ideal and hence (d) is a primary ideal. Thus every nonzero nonunit of D is a product of pseudo-prime elements or equivalently a product of primary elements. So D is weakly factorial. ∎

Also, a quasilocal domain (D, M) is a UCFD. Indeed, for $0 \neq a \in M$, $a = a$ is the unique comaximal factorization of a ! One should note that while a UFD is a UCFD, the complete comaximal factorization of an element need not be the prime-power factorization. Indeed in $K[X,Y]$, K a field, XY is pseudo-irreducible.

We next generalize comaximal factorizations to coprime and $*$-comaximal factorizations where $*$ is a finite character star operation on D. A nonzero nonunit $a \in D$ is *$*$-pseudo-irreducible* (resp. *$*$-pseudo-prime*) if for $b, c \in D$ with $(b,c)^* = D$, $a = bc$ (resp. $a|bc$) implies b or c is a unit (resp. $a|b$ or $a|c$). Certainly a $*$-pseudo-prime element is $*$-pseudo-irreducible. A factorization $b = b_1 \cdots b_m$ into nonunits is a (*complete*) *$*$-comaximal factorization* if for $i \neq j$ $(b_i, b_j)^* = D$ (and each b_i is $*$-pseudo-irreducible). Evidently $b = b_1 \cdots b_m$ is a complete $*$-comaximal factorization if and only if it has no proper refinements that are $*$-comaximal factorizations. Finally, D is a *$*$-CFD* if each nonzero nonunit of D has a complete $*$-comaximal factorization and a $*$-CFD is a *$*$-UCFD* if each complete $*$-comaximal factorization is unique up to units and associates.

Note that if we take $*$ to be the d-operation ($A_d = A$) we get the comaximal factorizations of McAdam and Swan. We can also define coprime factorizations by replacing $(a,b)^* = D$ by $[a,b] = 1$, but we must be careful (see next paragraph). Define a nonzero nonunit $a \in D$ to be *[ ]-pseudo-irreducible* (resp. *[ ]-pseudo-prime*) if $a = a_1 \cdots a_n$ (resp. $a|a_1 \cdots a_n$) where $[a_i, a_j] = 1$ for $i \neq j$ implies some a_i is a unit ($a|a_i$ for some i). A (*complete*) *coprime factorization* is a factorization $b = b_1 \cdots b_m$ where each b_i is a nonunit and $[b_i, b_j] = 1$ for $i \neq j$ (with each b_i [ ]-pseudo-irreducible). Evidently a coprime factorization is complete if and only if it has no proper refinement which is again a coprime factorization. The domain D is a *[ ]-CFD* if each nonzero nonunit has a complete coprime factorization and a [ ]-CFD is a *[ ]-UCFD* if the complete coprime factorization of an element is unique up to units and associates.

Note that if a is $*$-pseudo-irreducible and $a = a_1 \cdots a_n$ where $(a_i, a_j)^* = D$ for $i \neq j$, then some a_i is a unit. Indeed, this follows from the fact that if $(a,b)^* = (a,c)^* = D$, then $(a,bc)^* = D$. For we can write $a = a_1(a_2 \cdots a_n)$ where $(a_1, a_2 \cdots a_n)^* = D$. However, $[a,b] = [a,c] = 1$ does not imply $[a,bc] = 1$. (In fact, a domain D satisfies this property (called the PP-property) if and only if D satisfies GL: the product of two primitive polynomials is primitive. And it is well-known that an atomic domain satisfying GL is a UFD. See [12] for details.) Consider the element X^3 of $\mathbb{Q}+X\mathbb{R}[X]$. Then X^3 is not [ ]-pseudo-irreducible since

$X^3 = X(\sqrt[3]{2}X)(\frac{1}{\sqrt[3]{2}}X)$ is a coprime factorization. However, X^3 does *not* have a coprime factorization into two elements. For if $X^3 = ab$, then we can take $a = uX$ and $b = u^{-1}X^2$ where $u \in \mathbb{R}$. But $[uX, u^{-1}X^2] = uX$.

Note that if $*_1$ and $*_2$ are finite character star-operations on D with $*_1 \leq *_2$ (i.e., $A^{*_1} \subseteq A^{*_2}$ for each $A \in F(D)$), then $a \in D$ $*_2$-pseudo-irreducible implies a is $*_1$-pseudo-irreducible and that a [ ]-pseudo-irreducible implies that a is t-pseudo-irreducible. Similar statements hold for the corresponding types of pseudo-prime elements. However, for comaximal factorizations we have the reverse situation. If $*_1 \leq *_2$, then a $*_1$-comaximal factorization is a $*_2$-comaximal factorization. Thus the relationship between $*_1$-UCFDs and $*_2$-UCFDs is somewhat murky.

As in the case of CFDs, a domain D is a $*$-CFD if each nonzero nonunit is either (i) contained in only finitely many minimal primes or (ii) contained in only finitely many maximal $*$-ideals. Also, it is not hard to show that a domain satisfying ACCP is a $*$-CFD. We have the following partial analog of Theorem 4.1.

Theorem 4.3. *Let $*$ be a finite character star-operation on an integral domain D. Consider the following four conditions.*

(1) D is a $*$-CFD and each two-generated $*$-invertible $*$-ideal is principal.

(2) D is a $*$-CFD and each $*$-pseudo-irreducible element is $*$-pseudo-prime.

(3) Every nonzero nonunit of D has a $*$-comaximal factorization into $*$-pseudo-prime elements.

(4) D is a $*$-UCFD.

Then (1)$\Longrightarrow$(2)$\Longleftrightarrow$(3)$\Longrightarrow$(4).

Proof. The proof of (1)$\Longrightarrow$(2)$\Longrightarrow$(4) is the star operation version of [27, Lemma 1.3]. We give the details for the readers not familiar with star-operations. (1)$\Longrightarrow$(2) Let a be $*$-pseudo-irreducible. Suppose that $a|bc$ where $(b,c)^* = D$. Then $((a,b),(a,c))^* = D$, so $(a,b)^* \cap (a,c)^* = ((a,b)(a,c))^*$. Hence $(a) \subseteq (a,b)^* \cap (a,c)^* = ((a,b)(a,c))^* \subseteq (a)$; so $(a) = ((a,b)(a,c))^*$. So (a,b) and (a,c) are $*$-invertible. Thus $(a,b)^* = (d)$ and $(a,c)^* = (e)$ for some $d, e \in D$. So $(a) = (d)(e)$ and hence $a = ude$ for some unit u. But then a $*$-pseudo-irreducible and $(d,c)^* = D$ gives ud or e is a unit. If ud is a unit, $(a) = (e)$ and $c \in (a,c)^* = (e) = (a)$. Likewise e a unit gives $c \in (a)$. So a is $*$-pseudo-prime. (2)$\Longrightarrow$(3) Clear. (3)$\Longrightarrow$(2) Let a be $*$-pseudo-irreducible. Then a has a $*$-comaximal factorization $a = a_1 \cdots a_n$ where each a_i is $*$-pseudo-prime. Since a is $*$-pseudo-irreducible, $n = 1$, that is, $a = a_1$ is $*$-pseudo-prime. (2)$\Longrightarrow$(4) Suppose that $x_1 \cdots x_n = y_1 \cdots y_m$ are two complete $*$-comaximal factorizations. By hypothesis, each x_i, y_j is $*$-pseudo-prime. Now x_1 $*$-pseudo-prime and $x_1|y_1 \cdots y_m$ implies $x_1|y_i$ for some i which we can take to be 1. Likewise $y_i|x_j$ for some j. Then $x_1|x_j$ and since $(x_1,x_j)^* = D$ for $j \neq 1$, we must have $j = 1$. Thus x_1 and y_1 are associates. Cancelling x_1 gives $x_2 \cdots x_n = (uy_2) \cdots y_m$ for some unit u. By induction $n = m$ and after re-ordering x_i and y_i are associates. ∎

Note that if D is a domain satisfying the equivalent conditions of Theorem 3.4, then D is a $*$-UCFD. Indeed, these are precisely the $*$-UCFDs in which each $*$-pseudo-irreducible element is contained in a unique maximal $*$-ideal. We do not know if (4)$\Longrightarrow$(1), even in the case of the t-operation. However, in the one-dimensional case we have the following result.

Theorem 4.4. *For a one-dimensional domain D and a finite character star-operation $*$ on D the following are equivalent.*
(1) D is weakly factorial.
(2) D is a $*$-UCFD.

Proof. This follows from Theorem 4.2 and the fact that for D one-dimensional, $*$-Max$(D) = \text{Max}(D)$. ∎

However, we can not add D is a [ ]-UCFD to Theorem 4.4. For let $D = K + XL[[X]]$ where $K \subseteq L$ are fields. Now D is always a [ ]-CFD, but D is a [ ]-UCFD if and only if $K = L$, i.e., D is a UFD. Now up to associates the nonunits (atoms) of D have the form uX^n where $u \in L^*$ and $n \geq 1$ ($n = 1$). Let $u, v \in L^*$. Then uX and vX are associates $\Longleftrightarrow uv^{-1} \in K$ and for $n > 1$, $uX^n = (vX)(uv^{-1}X^{n-1})$. Hence $[uX^{n_1}, vX^{n_2}] = 1 \Longleftrightarrow n_1 = n_2 = 1$ and $uv^{-1} \notin K$. Let a be a nonzero nonunit of D. Then either a is [ ]-pseudo-irreducible or $a = a_1 \cdots a_n$ where each a_i is a nonunit and $[a_i, a_j] = 1$ for $i \neq j$. In the latter case $[a_i, a_j] = 1$ gives that each a_i has order one and hence is irreducible and thus [ ]-pseudo-irreducible. Thus D is a [ ]-CFD. Suppose that $K \subsetneq L$ and let $u \in L - K$. Then $uX^2 = uX \cdot X = u^2X \cdot u^{-1}X = u(u+1)X \cdot (u+1)^{-1}X$. Now the first two complete coprime factorizations are distinct unless $u^2 \in K$ while the first and the third complete coprime factorizations are distinct unless $u(u+1) \in K$. But then uX^2 having a unique complete coprime factorization gives that $u = u(u+1) - u^2 \in K$, a contradiction.

5 Generalizations of Prime Powers

In this section we summarize the various generalizations of prime powers given in Sections 2–4 and give several new ones (see Figure 1). For each of these prime power generalizations, we consider the domains whose nonzero nonunits are a product of such elements (see Figure 2).

The pattern runs as follows. Suppose that D is an integral domain and that $\mathcal{P}$ is a property generalizing prime power. Then D could be called a *$\mathcal{P}$-domain* if each nonzero nonunit of D has a property $\mathcal{P}$ and a *semi-$\mathcal{P}$-domain* if each nonzero nonunit of D is a finite product of elements with property $\mathcal{P}$. In several cases we have seen that D is a semi-$\mathcal{P}$-domain if and only if D is an $\mathcal{F}$-IFC domain where for each $P \in \mathcal{F}$, D_P is a $\mathcal{P}$-domain. For example, this is the case where property $\mathcal{P}$ is "prime power", "primary", or "t-pure".

In [6] a nonzero nonunit q was defined to be a *prime quantum* if q satisfies Q_1: For every nonunit $r|q$, there exists a natural number n with $q|r^n$, Q_2: For every natural number n, if $r|q^n$ and $s|q^n$, then $r|s$ or $s|r$, and Q_3: For every natural number n, each element t with $t|q$ has the property that if $t|ab$, then $t = t_1t_2$ where $t_1|a$ and $t_2|b$. Thus Q_2 says that each power of q is rigid and Q_3 says that q is completely primal (since a product of completly primal elements is completely primal). A prime power p^n is a prime quantum and a prime quantum q is primary. An integral domain D is a *generalized unique factorization domain* (*GUFD*) if every nonzero nonunit of D is a product of prime quanta. Clearly a UFD is a GUFD and a GUFD is weakly factorial and a GCD domain [6].

M. Zafrullah [33] considered yet another generalization of a prime power. He defined a nonzero nonunit $x \in D$ to be a *packet* if $\sqrt{(x)}$ is prime, that is, there is a unique minimal prime over (x). He studied GCD domains, called *unique representations domains* (*URD*), with the property that every nonzero nonunit is a product of packets. In a URD, each nonzero nonunit has only finitely many primes minimal over it. Thus by Theorem 4.1, a URD is a UCFD. For a study of domains with this and related properties, see [25].

The various generalizations of a prime element are indicated in Figure 1.

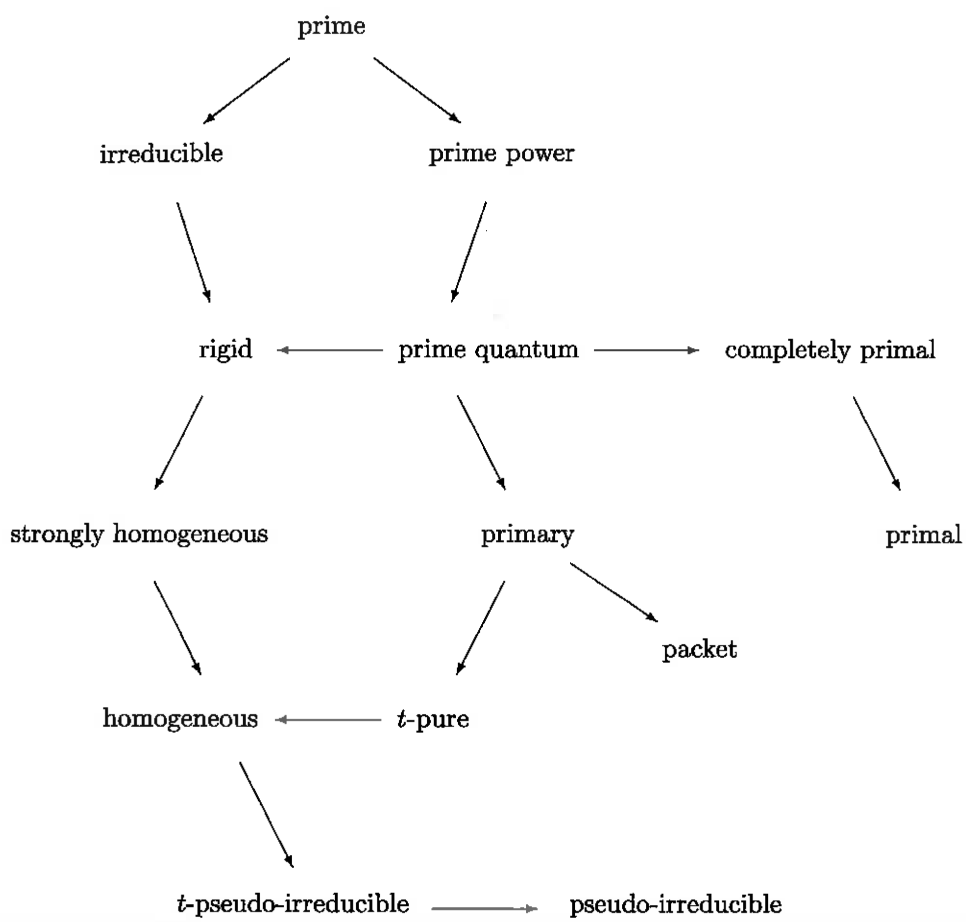
prime
irreducible
prime power
rigid
prime quantum
completely primal
strongly homogeneous
primary
primal
packet
homogeneous
t-pure
t-pseudo-irreducible
pseudo-irreducible

Figure 1.

Corresponding to Figure 1, we have Figure 2 showing the various generalizations of UFDs. Again, the only implications are the obvious ones.

$$
\begin{tikzcd}
\text{atomic domain} \arrow[d] & \text{UFD} \arrow[l] \arrow[d] & \\
\text{semi-rigid domain} \arrow[d] & \text{GUFD} \arrow[l] \arrow[r] \arrow[d] & \text{GCD domain} \arrow[d] \\
\text{semi-strongly-homogeneous domain} \arrow[d] & \text{weakly factorial domain} \arrow[d] & \text{Schreier domain} \arrow[d] \\
\text{semi-homogeneous domain} & \text{semi-}t\text{-pure domain} \arrow[l] \arrow[d] \arrow[dr] & \text{pre-Schreier domain} \\
 & t\text{-UCFD} & \text{UCFD}
\end{tikzcd}
$$

Figure 2.

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