QUESTION (HD 2101) (1) Let D be a domain. How to show that $D[X^2, X^3] \cong D[Y, Z]/(Y^2 - Z^3)$?

(2) Let $R = Z_{(p)} + (X;Y)Q[[X, Y]]$ and $M = (X, Y)Q[[X, Y]]$. Why is $R[1/p] = Q[[X, Y]] = R_M$?

ANSWER: There are two answers to each of **(1)** and **(2)**. One based on a suggestion by Shiqi Xing of Sichuan Normal University, Chengdu, China and the other as a comment on these questions by Tiberiu Dumitrescu of Universitatea Bucuresti, Romania.

Xings Answer: Let $\phi : D[Y, Z] \to D[X^2, X^3]$ defined by $\phi(Y) = X^3$ and $\phi(Z) = X^2$. Then ϕ is onto. Indeed $\ker \phi \supseteq (Y^2 - Z^3)$.We need to show that every element $g = g(Y, Z)$ in $\ker \phi$ is divisible by $(Y^2 - Z^3)$. For this we first note that $g(0, 0) = (0)$. That is there is no nonzero constant term. Express g as a polynomial in Y as: $g = f_n(Z)Y^n + f_{n-1}(Z)Y^{n-1} + ... + f_0(Z)$. Suppose that g is not divisible by $(Y^2 - Z^3)$. Then, $g = q(Y, Z)(Y^2 - Z^3) + f(Y, Z)$ and degree of f is less than 2. Hence $\partial f = 1$ or 0 in Y. So $f = f_1(Z)Y + f_0(Z)$. Because $g(Y, Z) = (0)$ on setting $Y = X^3$ and $Z = X^2$, we have $f_1(X^2)X^3 + f_0(X^2) = (0)$. But this forces $f_1(X^2)X^3 = 0 = f_0(X^2)$, because one is of odd degree and the other of even. Of course as $X^3 \neq 0$ we have $f_1(X^2) = (0)$. Whence $f_1(Z) = (0)$ and $f_0(Z) = 0$. Forcing $f(Y, Z) = 0$. But then $g = q(Y, Z)(Y^2 - Z^3) \in (Y^2 - Z^3)$.

(2). Note that $\cap p^n R = \cap(p^n Z_{(p)} + (X;Y)Q[[X;Y]]) \supseteq M = (X;Y)Q[[X;Y]]$ and there is no prime between $\cap p^n R$ and $M = (X;Y)Q[[X;Y]]$ because $\cap p^n Z_{(p)} = (0)$ and indeed there is no prime between pR and M Thus $M = \cap p^n R$. But as M misses powers of p we have $R_M = R[1/p] = Q + M = Q + (X, Y)[[X, Y]] = Q[[X, Y]]$.