Theorem 12. An integral domain R is a GUFD iff it is an HCF-GKD.

Proof. Let R be a GUFD then

(1) every non zero non unit of R is contained in a finite number of minimal prime ideals (Cor. 3 and the fact that every non zero non unit of R is the product of a finite number of prime quanta)

(2) for every minimal prime P, R_P is a valuation domain(Prop. 4 and Cor. 6)

(3) $R = \cap R_P$, where P ranges over all minimal prime ideals of R.

Proof of(3). Obviously $R \subseteq \cap R_P$ where P ranges over minimal primes. Let $x \in \cap R_P$, then since R is an HCF domain, we can write $x = r/s$ where $(r,s) = 1$. Now $r/s \in R_P$ for every minimal prime P implies that s is a unit in each R_P, consequently s is in no minimal prime ideal and so has no prime quantum as a factor which in a GUFD is possible only if s is a unit and hence $x \in R$.

The properties (1),(2) and (3) as we have mentioned at the beginning of this section, show that R is a GKD and with the help of Prop. 4 we have proved that a GUFD is an HCF-GKD.

Conversely let R be an HCF-GKD. Let x be a non zero non unit element of R, then by the definition of a GKD, x is contained in a finite number of minimal prime ideals $P_1, P_2, . P_n$ say. We may assume that there is no other minimal prime which contains x. Now since P_i are distinct there exists an element $y \in P_1$ such that $y \notin P_2$. We claim that $(x,y) \neq 1$, for otherwise $(x,y) = 1$ in R implies that $xR \cap yR = xyR$ in R and so $xR_{P_1} \cap yR_{P_1} = xyR_{P_1}$ in R_{P_1} (cf Proof of Lemma 9) which further implies that $(x,y) = 1$ in R_{P_1}. But R_{P_1} being a valuation domain