Submitted title: Examples they can play with

EXAMPLES IN MODERN ALGEBRA WITH WHICH STUDENTS CAN PLAY

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ABSTRACT: In this article, we indicate some easy to construct rings that afford examples which answer some of the questions interested undergraduate students of Modern Algebra may ask.

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Modern algebra is a beautiful branch of Mathematics, yet highly unpopular with many of our undergraduate students. This unpopularity may be related to the lack of provision of easy to construct multi-purpose examples in the beginning level modern algebra courses. So students, in the absence of simple concrete examples to experiment and play with, end up accepting modern algebra as a necessary evil and often end up memorizing their way through it. What is worse, it does not have to be this way. There is a wealth of simple multi-purpose examples, mostly hidden away as lemmas and remarks in research articles. The aim of this note is to present one such family of examples and to encourage concerned readers to look for more. This, we hope, will also encourage researchers to include in their research articles examples, results and observations of pedagogical value.

The examples we have in mind have to do with the following easy to see result: Let B be a ring with identity, I an ideal of B and A a subring of B. Then the set $A+I$ is a subring of B.

As it stands, the above statement is as abstract as it can be. So let's look at the following family of examples: If $A = \mathbf{I}$, $B = \mathbf{Q}[X]$, $I = X\mathbf{Q}[X]$, then

$$A + I = \mathbf{I} + X\mathbf{Q}[X] = \left\{ a_0 + \sum_{i=1}^{n} a_i X^i | a_0 \in \mathbf{I};\ a_i \in \mathbf{Q} \right\},$$

i.e. the set of polynomials over $\mathbf{Q}$ with integral constant terms. (Note that we use $\mathbf{N}, \mathbf{I}, \mathbf{Q}, \mathbf{R}$, and $\mathbf{C}$ to denote respectively the sets of natural numbers, integers, rationals, reals, and complex numbers.) Then of course if we let $A = \mathbf{Q}$; $B = \mathbf{R}[X]$ and $I = X\mathbf{R}[X]$, we have $Q + X\mathbf{R}[X]$. Furthermore, if students are familiar with rings of fractions we can also form for any integral domain D, and a multiplicative set S, the ring F of the form

$$D + XD_S[X] = \left\{ a_0 + \sum_{i=1}^{n} a_i X^i | a_0 \in D;\ a_i \in D_S \right\}.$$

Now the question is, "To what end can these examples be used?" We shall point out a few uses and provide the necessary references for more.

In a typical beginning modern algebra course one defines the notion of a divides b in a general integral domain, one proves that principal ideal domains have the ascending chain condition on principal ideals (ACCP), and then later one shows that unique factorization domains (UFD's) have the ACCP. It is quite natural for students to ask: Do integral domains exist that (a) do not have ACCP or (b) have ACCP but are not UFD's? In response to these questions we offer the following:

For (a) we note that $R = \mathbf{I} + X\mathbf{Q}[X]$ or $T = \mathbf{I} + X\mathbf{R}[X]$ will do the job and in general if D is and integral domain different from its field of fractions K, then $T = D + XK[X]$ would serve the same purpose. One simply notes that in the case of $R = \mathbf{I} + X\mathbf{Q}[X]$; $X, \frac{1}{2}X, \frac{1}{2^2}X, \ldots, \frac{1}{2^n}X \in R$ for all $n \in \mathbf{N}$ and that for $n > r$, $2^{n-r}(\frac{1}{2^n}X) = \frac{1}{2^r}X$, where 2^{n-r} is a non-unit in R. So there is an infinite chain of proper inclusions $(X) \subset (\frac{1}{2}X) \subset (\frac{1}{2^2}X) \subset \ldots \subset (\frac{1}{2^n}X) \subset \ldots$

For (b) we use $D = \mathbf{Q} + X\mathbf{R}[X]$. We may start by pointing out the following: (1) The only units of D are the non-zero elements of $\mathbf{Q}$. (2) That for $f, g \in D$, $\frac{f}{g} \in D$ implies that $\frac{f}{g} \in \mathbf{R}[X]$ but not conversely. For $\frac{f}{g} \in D$ implies that $\frac{f}{g} = h$,where $h \in D$, but $D \subset \mathbf{R}[X]$,so $h \in \mathbf{R}[X]$. Moreover, h is a non-unit if and only if deg $g <$deg f. In other words $(f) \subset (g)$ if and only if deg $g <$ deg f. But then for any $f \in \mathbf{Q} + X\mathbf{R}[X]$, the degree of f is finite. So $(f) \subset (f_1) \subset (f_2) \subset \ldots$ terminates. Now utilize the fact that if D has ACC on principal ideals then every non-zero non-unit of D is expressible as a product of irreducible elements. To decide that D is not a UFD note that X is not a prime in D. For example, $X \mid X^2 = (\frac{X}{\sqrt{2}})(\sqrt{2}X)$ but $X \nmid \frac{X}{\sqrt{2}}$because $\frac{1}{\sqrt{2}} \notin D$ and $X \nmid \sqrt{2}X$ because $\sqrt{2} \notin D$. Let us call

D atomic if every non-zero non-unit of D is expressible as a product of irreducible elements. So the above example is that of an atomic domain that is not a UFD.

Some readers may wish to point out that we already have examples of rings, due to Eisenstein, for this purpose. These rings, such as the ring $\mathbf{I}[\sqrt{-5}]$, afford examples like $6 = (2)(3) = (1+\sqrt{-5})(1-\sqrt{-5})$. But to be able to conclude that $\mathbf{I}[\sqrt{-5}]$ is an atomic domain that is not a UFD we have either to wait until we can introduce the ideas involved or risk bringing into the course ideas which need a non-trivial amount of effort to understand. Besides $\mathbf{I}[\sqrt{-5}]$ does not afford the variety of examples that $\mathbf{Q}+X\mathbf{R}[X]$ offers. For example, X^2 can be expressed as a product of two distinct non-associated atoms in infinitely many ways as $X^2 = (\alpha X)(\frac{1}{\alpha}X)$ where $\alpha \in \{\sqrt{P_n} \mid P_n \text{is the n}^{th} \text{ prime}\}$. Moreover, $\mathbf{Q}+X\mathbf{R}[X]$ is not Noetherian [1, pp. 109-129] and is probably the only accessible example of a non-Noetherian domain that has ACC on principal ideals. Besides, depending upon the course level, $\mathbf{Q}+X\mathbf{R}[X]$ can be replaced by $F_1 + XF_2[X]$ where F_2 is a field and F_1 is a subfield of F_2 [1, pp.109-129] and [3, p.353]. Then $\mathbf{R}+X\mathbf{C}[X]$ can be offered as a Noetherian domain that is not a UFD, [1, pp.109-129]. With further sophistication some students may construct examples like $\mathbf{Q}+X\mathbf{Q}\sqrt{2}[X]$. $\to \mathbf{Q}+X\,\mathbf{Q}(\sqrt{2})[X]$.

Another set of uses of $\mathbf{Q}+X\mathbf{R}[X]$ comes from the abundance of non-principal ideals in $\mathbf{Q}+X\mathbf{R}[X]$.
Examples:

1. $(X)\cap(\sqrt{2}X) = X^2\mathbf{R}[X] = \{f(X) = \sum a_iX^i \mid a_0 = a_1 = 0, a_i \in \mathbf{R}\}$ For if $(X)\cap(\sqrt{2}X) = (d)$ then $d = Xf(X)$ where $f(X) \in D$ and $d = \sqrt{2}Xg(X)$ where $g(X) \in D$. Equating $Xf(X) = \sqrt{2}Xg(X)$ in $\mathbf{R}[X]$ we get $\frac{1}{\sqrt{2}}f(X) \in D$ which is possible only if the constant term of $f(X)$ is zero which means that $d = X(\frac{1}{\sqrt{2}})Xh(X)$ where $h(X) \in \mathbf{R}[X]$. But then $X, \sqrt{2}X$ both divide X^2. Now $X^2 \in (d)$ is absurd even if $h(X) = 1$ for $\frac{1}{\sqrt{2}}X^2 \nmid X^2$.

2. The above example can be used to demonstrate the existence of two elements which have a GCD but not a LCM. For the GCD $(X, \sqrt{2}X) = 1$ exists but the LCM $[X, \sqrt{2}X]$ does not because $(X)\cap(\sqrt{2}X)$ is not principal.

3. The general expression for each $f \in D = \mathbf{Q}+X\mathbf{R}[X]$, which is $f(X) = \alpha X^r g(X)$; where $g(X) \in D$ and $\alpha \in \mathbf{Q}$ if $r = 0$,* can be a great source of problems on divisibility. At the somewhat advanced level the polynomial constructions of the type $D + XD_S[X]$ can be used

* and $g(0) \neq 0$

(see [4, pp. 423-439]) to give examples of (1) integral domains that are GCD domains but are not UFD's ($\mathbf{I} + X\mathbf{Q}[X]$ is one such example) and (2) integral domains that are not GCD domains [6, pp. 93-107].

Finally we are pleased to see that recently there have been some advances in the direction we suggest. For example Hungerford [5] uses $\mathbf{I} + X\mathbf{Q}[X]$ in several places and Scott Chapman [2] has recently written an article giving examples, of irreducible elements that are not primes, to serve as alternatives to the classical example $\mathbf{I}[\sqrt{-5}]$ mentioned earlier. But this is not enough...hence this note.

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BIOGRAPHICAL SKETCHES

Tess Jackson is an assistant professor of Mathematics and Secondary Education at Winthrop University. At Winthrop University she teaches undergraduate and graduate courses in the areas of mathematics and mathematics education. She is interested in pedagogy issues relating to the teaching of mathematics at the secondary and post-secondary levels.

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