ON S-GCD DOMAINS

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Abstract. In [J. Pure Appl. Alg. 221 (2017) 2869 − 2879], the authors introduced the notion of an S-GCD domain which is a generalization of GCD domains. An integral domain D is said an S-GCD domain (S a multiplicative subset of D) if each finitely generated nonzero ideal of D is S-v-principal. We give equivalent conditions for an integral domain to be an S-GCD domain. Also we study the polynomial rings and the power series rings over an S-GCD domain.

Keywords: S-GCD domains, GCD domains, formal power series ring.

Classification: 13F05, 13A15, 13F25.

1. Introduction

Let D be an integral domain with quotient field K. Let $\mathcal{F}(D)$ be the set of nonzero fractional ideals of D. For an $I \in \mathcal{F}(D)$, set $I^{-1} = \{x \in K/\ xI \subseteq A\}$. The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_v = (I^{-1})^{-1}$ is called the v-operation on D. A nonzero fractional ideal I is said to be a v-ideal or divisorial if $I = I_v$, and I is said to be of v-finite type if $I = J_v$ for some finitely generated ideal J of D. For properties of the v-operation the reader is referred to [[5], Section 34].

An integral domain D is called a GCD domain if for each pair $a, b \in D^*$, GCD(a, b)

exists. GCD domains are an important class of integral domains from classical ideal theory. In a GCD domain, every v-finite type ideal of D is principal. This property can be generalized in several different ways ([1], [6]). However, we will be mostly interested in the S-GCD property ([6]). Let S be a multiplicative subset of D and I a nonzero ideal of D. We say that I is S-v-principal if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_v$. Following [6], D is said an S-GCD domain if each finitely generated nonzero ideal of D is S-v-principal. Note that if S consists of units of D, then the properties S-GCD and GCD are equivalent.

In this paper, we continue to study the S-GCD property. We give an example of an S-GCD domain which is not a GCD domain. Also we give an equivalent conditions for an integral domain to be an S-GCD domain. We show that D is an S-GCD domain if and only if for $a,\ b \in D^*$, $aD + bD$ is S-v-principal if and only if for $a,\ b \in D^*$, $aD \cap bD$ is S-principal equivalent to any finite intersection of principal ideals of D is S-principal equivalent to for $a,\ b \in D^*$, $aD : bD$ is S-principal. Recall from [2] that a saturated multiplicatively closed subset S of an integral domain D is said to be a splitting set if for each $d \in D^*$ we can write $d = sa$ for some $s \in S$ and $a \in D$ with $s'D \cap aD = s'aD$ for all $s' \in S$. A splitting set S of D is said to be an lcm splitting set if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal. We prove that if S is a splitting set, then D is an S-GCD domain if and only if D_S is a GCD domain, and we give an example of a domain D and a multiplicative set S such that D_S is a GCD domain but D is not an S-GCD domain, in particular S is not a splitting set (Note that this example is proved by the Professor M. Zafrullah). Based on this result we link the S-GCD property withe GCD, PVMD's and UFD domains. Recall that an element p of D is said to be prime, if pD is a prime ideal of D. We give an S-version of a well known result about GCD domain where S generated by prime elements of D, we prove that, D is a UFD if and only if D is an S-GCD domain and D_S satisfy the

ACCP property. If we want to avoid condition on S (S generated by prime) we can make condition on D. Recall from [11] that a domain D is said to be a Mori domain if it satisfies the ascending chain condition on integral divisorial ideals. We show that if D is a Mori domain and S a multiplicative set. Then D is an S-GCD domain if and only if D_S is a UFD. Note that if D is an S-GCD domain, then D is not necessary a PVMD, for example if we take D is an integral domain which is not a PVMD (A classical example can be found in [8, Example 2.1]) and $S = D^*$ a multiplicative subset of D. Then D is an S-GCD domain which is not a PVMD. Let D be an integral domain, S a splitting multiplicative subset of D and T the m-complement of S. We show that if D is S-GCD as well as T-GCD, then D is at least a PVMD. We also prove that if S is an lcm splitting set of an integral domain D, then D is an S-GCD domain if and only if D is a GCD domain and consequently, D is an S-GCD domain if and only if $D[X]$ is an S-GCD domain. Note that the S-GCD property does not carry over to the power series ring. In fact, there is an example of a GCD domain D such that $D[[X]]$ is not a GCD domain [[10], Theorem 8]. (This is the case when S consists of units of D). We give with an additional condition a necessary and sufficient condition for the power series ring $D[[X]]$ to be an S-GCD domain. First, recall that from [6], the power series ring $D[[X]]$ is said to satisfy the property $(*)$, if for all integral v-invertible v-ideals I and J of $D[[X]]$ such that $(IJ)_0 \neq (0)$, we have $((IJ)_0)_v = ((IJ)_v)_0$ where $I_0 = \{f(0),\ f \in I\}$. We show that if D is a Krull domain, such that $D[[X]]$ satisfies $(*)$ and S a multiplicative subset of D. Then D is an S-GCD domain if and only if $D[[X]]$ is an S-GCD domain (Theorem 2.4). In particular, in a Krull domain D such that $D[[X]]$ satisfies $(*)$, D is a GCD domain if and only if $D[[X]]$ is a GCD domain.

2. On S-GCD domains

We begin this section by recalling the following definition in order to give an S-version of a known classical results about GCD domains.

Definition 2.1. [6] Let D be an integral domain and S a multiplicative subset of D. We say that a nonzero ideal I of D is S-v-principal if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_v$.
We also define D to be an S-GCD-domain if each finitely generated nonzero ideal of D is S-v-principal.

Remark 2.1.
(1) If S consists of units of D, then D is an S-GCD domain if and only if D is a GCD domain.
(2) In an S-GCD domain, every finitely generated divisorial ideal is S-principal.

Let D be an integral domain and S a multiplicative subset of D. The mapping on $\mathcal{F}(D)$ defined by $I \mapsto I_w = \{x \in K,\ xJ \subseteq I$ for some finitely generated ideal J of D such that $J_v = D\}$ is called the w-operation on D. Recall from [7] that, a nonzero ideal I of D is S-w-principal if there exist an $s \in S$ and $a \in D$ such that $sI \subseteq aD \subseteq I_w$. We also define D to be an S-factorial domain if each nonzero ideal of D is S-w-principal.

Example 2.1. Let S be a multiplicative subset of an integral domain D.

(1) If D is a GCD domain, then D is an S-GCD domain.
(2) Since for all fractional ideal I of D, $I_w \subseteq I_v$, then every S-w-principal ideal of D is S-v-principal. So every S-factorial domain is an S-GCD-domain.
(3) Let T be a multiplicative subset of D containing S. As every S-v-principal ideal of D is T-v-principal, then every S-GCD domain is a T-GCD domain.

The converse of (1), in the previous example is not true in general. Indeed, let $D = \mathbb{Z} + X\mathbb{Q}[i\sqrt{2}][X]$ and $S = D \setminus (0)$. Then S is a multiplicative subset of D. Let I be a nonzero ideal D. Since $I \cap S \neq \emptyset$, then I is an S-principal ideal of D. Thus D is an S-principal ideal domain, and hence D is an S-GCD domain. But by [[4], Remark 5.3(d)], D is not a GCD domain.

The following Theorem gives equivalent conditions for an integral domain to be an S-GCD domain. It is well-known that if we take S included in the set of units of D, then these conditions are all equivalent to D being a GCD domain. Note that if $a,\ b \in D^*$, then $aD : bD = \{x \in D,\ xb \in aD\}$ is an ideal of D.

Theorem 2.1. *Let D be an integral domain and S a multiplicative subset of D. Then the following assertions are equivalent.*

(1) *D is an S-GCD domain.*
(2) *Any finite intersection of principal ideals of D is S-principal.*
(3) *For $a,\ b \in D^*$, $aD \cap bD$ is S-principal.*
(4) *For $a,\ b \in D^*$, $aD + bD$ is S-v-principal.*
(5) *For $a,\ b \in D^*$, $aD : bD$ is S-principal.*

Proof: We show that (1) $\Longleftrightarrow$ (4) $\Longleftrightarrow$ (3) $\Longleftrightarrow$ (2) and (3) $\Longleftrightarrow$ (5)

(1) $\Longleftrightarrow$ (4) It is obvious. Conversely, let $I = b_1D + \cdots + b_nD$ be a nonzero finitely generated ideal of D. By hypothesis, there exist an $s_1 \in S$ and $a_1 \in D$ such that $s_1(b_1D + b_2D) \subseteq a_1D \subseteq (b_1D + b_2D)_v$. Then $s_1I \subseteq a_1D + b_3D + \cdots + b_nD \subseteq I_v$. Again by hypothesis, there exist an $s_2 \in S$ and $a_2 \in D$ such that $s_2(a_1D + b_3D) \subseteq a_2D \subseteq (a_1D + b_3D)_v$. Then $s_1s_2I \subseteq a_2D + b_4D + \cdots + b_nD \subseteq I_v$. By induction, there exist an $s_{n-1} \in S$ and $a_{n-1} \in D$ such that $s_{n-1}(a_{n-2}D + b_nD) \subseteq a_{n-1}D \subseteq (a_{n-2}D + b_nD)_v$. Let $t = s_1 \cdots s_{n-1} \in S$. Then $tI \subseteq a_{n-1}D \subseteq I_v$, and hence I is S-v-principal.

(4) $\Longrightarrow$ (3) Let $a,\ b \in D^*$. Since $I = aD + bD$ is S-v-principal, then there exist an $s \in S$ and $d \in D \setminus (0)$ such that $sI \subseteq dD \subseteq I_v$. Thus $I^{-1} \subseteq \frac{1}{d}D \subseteq \frac{1}{s}I^{-1}$. Therefore $sI^{-1} \subseteq \frac{s}{d}D \subseteq I^{-1}$. But $I^{-1} = \frac{1}{a}D \cap \frac{1}{b}D = \frac{1}{ab}(aD \cap bD)$. So $s(aD \cap bD) \subseteq \frac{sab}{d}D \subseteq aD \cap bD$, and hence $aD \cap bD$ is S-principal.

(3) $\Longrightarrow$ (4) Let $a,\ b \in D^*$, and let $I = aD + bD$. By hypothesis $aD \cap bD$ is S-principal, then there exist an $s \in S$ and $d \in D \setminus (0)$ such that $s(aD \cap bD) \subseteq dD \subseteq aD \cap bD$. Since $I^{-1} = \frac{1}{a}D \cap \frac{1}{b}D = \frac{1}{ab}(aD \cap bD)$, then $sI^{-1} \subseteq \frac{d}{ab}D \subseteq I^{-1}$. Which implies that $sI \subseteq \frac{sab}{d}D \subseteq I_v$. Hence I is S-v-principal.

(2) $\Longleftrightarrow$ (3) It is obvious. Conversely, let $a_1, ..., a_n \in D$. We show that $I = a_1D \cap \cdots \cap a_nD$ is S-principal. By hypothesis there exist an $s_1 \in S$ and an $\alpha_1 \in D$ such that $s_1(a_1D \cap a_2D) \subseteq \alpha_1 D \subseteq a_1D \cap a_2D$. Then $s_1 I \subseteq (s_1(a_1D \cap a_2D)) \cap a_3D \cap \cdots \cap a_nD \subseteq \alpha_1 D \cap a_3D \cap \cdots \cap a_nD \subseteq I$. By induction, there exist an $s_{n-1} \in S$ and an $\alpha_{n-1} \in D$ such that $s_1 \cdots s_{n-1} I \subseteq s_{n-1}(\alpha_{n-2}D \cap a_nD) \subseteq \alpha_{n-1}D \subseteq I$. Let $t = s_1 \cdots s_{n-1} \in S$. Then $tI \subseteq \alpha_{n-1}D \subseteq I$, and hence I is S-principal.

(3) $\Longleftrightarrow$ (5) It is sufficient to remark that, for each $a,\ b \in D$, $aD \cap bD = (aD : bD)(bD)$.

Corollary 2.1. *For an integral domain D, the following statements are equivalent.*

1. *D is a GCD domain.*
2. *Any finite intersection of principal ideals of D is principal.*
3. *For $a,\ b \in D \setminus (0)$, $(aD + bD)_v$ is principal.*
4. *For $a,\ b \in D \setminus (0)$, $aD : bD$ is principal.*

Our next result gives with an additional condition a necessary and sufficient condition for an integral domain D to be an S-GCD domain.

Theorem 2.2. *Let D be an integral domain and S a multiplicative subset of D such that for each $d \in D \setminus \{0\}$, there is an $s \in S$ such that $s(dD_S \cap D) \subseteq dD$. Then the following assertions are equivalent.*

(1) *D is an S-GCD domain.*

(2) *D_S is a GCD domain.*

Proof: (1) $\Longrightarrow$ (2) This application is always true and does not need the assumption : for each $d \in D \setminus \{0\}$, there is an $s \in S$ such that $s(dD_S \cap D) \subseteq dD$. Indeed, Assume that D is an S-GCD domain. We show that D_S is a GCD domain. Let (a) and (b) be principal ideals of D_S. Then $(a) = I_S$ and $(b) = J_S$ for some principal ideals I and J of D. Then by Theorem 2.2, $I \cap J$ is an S-principal ideal of D. Hence $(a) \cap (b) = I_S \cap J_S = (I \cap J)_S$ is a principal ideal of D_S.
(2) $\Longrightarrow$ (1) Let aD and bD be principal ideals of D. Since $(aD \cap bD)_S = aD_S \cap bD_S$ is a principal ideal of D_S, then there exists a $d \in (aD \cap bD)$ such that $(aD \cap bD)_S = dD_S$. Thus $(aD \cap bD)_S \cap D = (dD_S) \cap D$. But by assumption $s(dD_S \cap D) \subseteq dD$ for some $s \in S$. Then $aD \cap bD \subseteq (aD \cap bD)_S \cap D = (dD_S) \cap D = d : s$, and hence $s(aD \cap bD) \subseteq dD \subseteq aD \cap bD$.

We next give an example of a domain D and a multiplicative set S such that D_S is a GCD domain but D is not an S-GCD domain. Note that this example is proved by the Professor M. Zafrullah.

Example 2.2. Let $R = \mathbb{Z}_{(p)} + Y\mathbb{Q}[[Y]]$ where $\mathbb{Z}$ is the ring of integers, p a prime number, $\mathbb{Q}$ the field of rational numbers and Y an indeterminate over $\mathbb{Q}$. It is easy to see that R is a discrete rank two valuation domain. Let $S = \{p^n,\ 0 \leq n \in \mathbb{Z}\}$ and note that S is a multiplicative set of R such that $R_S = \mathbb{Q}[[Y]]$.
Now let $D = R + XR_S[X] = \mathbb{Z}_{(p)} + Y\mathbb{Q}[[Y]] + X\mathbb{Q}[[Y]][X]$.
Indeed as $D_S = R_S[X] = \mathbb{Q}[[Y]][X]$, a polynomial ring over a valuation domain and so we conclude that D_S is a GCD domain.

Now consider the ideal $(Y)\cap(X)$ in D. Indeed X and Y are non units, $X\nmid Y$ and $Y\nmid X$ and every power of p divides both X and Y. So if s is a power of p, then $\frac{XY}{s}\in(Y)\cap(X)$, as $\frac{XY}{s}\in(X)$ because $\frac{XY}{s}=X\frac{Y}{s}$ and $\frac{XY}{s}\in(Y)$ because $\frac{XY}{s}=\frac{X}{s}Y$.

Now let $a\in(Y)\cap(X)$. Then $a=Yf=Xg$ where $f,g\in D$. Taking f as a function of X over R_S, we note $f=Xh(X)$ where $h(X)\in R_S[X]$. So $a=YXh(X)$. Indeed for some $s\in S$, $sh(X)\in D$, so $a=(YX/s)k(X)$ where $k(X)=sh(X)\in D$. As for any $t\in S$, $a/t=(YX/st)k(X)$ we conclude that for any $t\in S$ and any $a\in(Y)\cap(X)$ we have $a/t\in(Y)\cap(X)$.

Now if D were a S-GCD, then $(Y)\cap(X)$ would be S-principal, that is for some $a\in(Y)\cap(X)$ and $s\in S$ we would have $s((Y)\cap(X))\subseteq aD\subseteq(Y)\cap(X)$. But then $(Y)\cap(X)\subseteq(a/s)D\subseteq(Y)\cap(X)$ implying that $(Y)\cap(X)=(a/s)D$. This leads to (a) $(a/sp)|(a/s)$ (obviously) and (b) $(a/s)|(a/sp)$ (because $a/sp\in(Y)\cap(X)$). But then $(a/s)D=(a/sp)D$ leading to $pD=D$ a contradiction.

Recall from [2] that a saturated multiplicatively closed subset S of an integral domain D is said to be a splitting set if for each $d\in D^*$ we can write $d=sa$ for some $s\in S$ and $a\in D$ with $s'D\cap aD=s'aD$ for all $s'\in S$.

Corollary 2.2. *Let D be an integral domain and S a splitting set in D. Then D is an S-GCD domain if and only if D_S is a GCD domain.*

Proof: Since S is a splitting set in D, then by [2, Theorem 2.2], there exists a multiplicatively closed subset T of D such that for each $d\in D\setminus\{0\}$ we can write $d=st$ for some $s\in S$ and $t\in T$ with $dD_S\cap D=tD$.

Let $d\in D\setminus\{0\}$. Then $d=st$ for some $s\in S$ and $t\in T$ with $dD_S\cap D=tD$. So $s(dD_S\cap D)=stD=dD$. Hence by Theorem 2.2, D is an S-GCD domain if and only if D_S is a GCD domain.

Recall from [11] that a domain D is said to be a Mori domain if it satisfies the ascending chain condition on integral divisorial ideals.

Proposition 2.1. *Let D be a Mori domain and let S be a multiplicative set. Then D is an S-GCD domain if and only if D_S is a UFD.*

Proof: If D is S-GCD, then D_S is a GCD Mori domain and so a UFD. Conversely, if D_S is a UFD then for each $a \in D \setminus \{0\}$, $aD_S \cap D$ is a t-ideal and hence a v-ideal of finite type ([[11], Théorème 1]). Say $aD_S \cap D = (a_1, ..., a_n)_v$ and let $s_i \in S$ be such that $s_i a_i \in (a)$. Then $s = s_1 s_2 .... s_n$ is such that $s(aD_S \cap D) \subseteq (a)$. Hence by Theorem 2.2, D is an S-GCD domain.

The following result is an immediate consequence of the previous Proposition. Note that in [12], the author gave an example of Mori domain D such that $D[X]$ is not a Mori domain.

Corollary 2.3. *Let D be a Mori domain such that $D[X]$ is Mori and let S be a multiplicative set of D. Then D is S-GCD if and only if $D[X]$ is an S-GCD domain.*

Recall that an element p of D is said to be prime, if pD is a prime ideal of D. Our next Theorem give an S-version of a well known result about GCD domain where S generated by prime elements of D, that is, an integral domain D is UFD if and only if D is a GCD domain satisfying ACCP.

Theorem 2.3. *Let D be an integral domain and S a multiplicative subset generated by prime elements of D. Then D is a UFD if and only if D is an S-GCD domain and D_S satisfy the ACCP property.*

Proof: If D is an S-GCD domain such that D_S is an ACCP domain, then D_S is a GCD domain satisfy the ACCP property. So D_S is a UFD and since

S is generated by prime elements of D, then by [9, Lemma 2.1], D is a UFD. Conversely, if D is a UFD, then D is a GCD domain in particular an S-GCD domain. As D is a UFD, then D_S is a UFD which implies that D_S is an ACCP domain.

Note that if D is an S-GCD domain, then D is not necessary a PVMD. Indeed, Let D be an integral domain which is not a PVMD (A classical example can be found in [8, Example 2.1]) and let $S = D^*$ a multiplicative subset of D. It is easy to show that D is an S-principal ideal domain. So D is an S-GCD domain which is not a PVMD. The next Proposition link the S-GCD property with PVMDs. First, let us recall that a splitting set S of D is said to be an lcm splitting set if for each $s \in S$ and $d \in D$, $sD \cap dD$ is principal [2].

Proposition 2.2. *Let D be an integral domain, S a splitting multiplicative subset of D, and T the m-complement of S. If D is S-GCD as well as T-GCD, then D is at least a PVMD.*

Proof: Since D is a T-GCD domain, then by Theorem 2.2, D_T is a GCD domain. So by [2, Proposition 2.4], S is an lcm splitting set. On the other hand, as D is an S-GCD domain, then D_S is GCD which implies that D_S is a PVMD. So by [2, Theorem 4.3], D is a PVMD.

Remark 2.2. Note that if S is an lcm splitting set of an integral domain D, then D is an S-GCD domain if and only if D is a GCD domain. Indeed, if D is S-GCD, then D_S is a GCD domain. So by [2, Theorem 4.3], D is a GCD domain. The other implication is obvious.

Corollary 2.4. *Let D be an integral domain and S be an lcm splitting set of D. Then D is an S-GCD domain if and only if $D[X]$ is an S-GCD domain.*

Proof: By [3, Theorem 2.2], S is an lcm splitting set in $D[X]$. So by the previous Remark, D is an S-GCD domain if and only if D is a GCD domain which equivalent to $D[X]$ is a GCD domain if and only if $D[X]$ is an S-GCD domain.

Let D be an integral domain. A splitting set S of D is called a t-lcm splitting set if $sD \cap dD$ is t-invertible for all $s \in S$ and $0 \neq d \in D$. This concept was introduced by the third author at the conference held in Incheon, Korea (May, 2001). He showed that if S is a t-lcm splitting set, then D is a PVMD if and only if D_S is a PVMD.

Proposition 2.3. *Let D be an integral domain and S be a t-lcm splitting set of D. If D is an S-GCD domain, then D is a PVMD.*

Proof: If D is S-GCD, then D_S is a GCD domain. So D_S is a PVMD and hence D is a PVMD.

Proposition 2.4. *Let D be an integral domain with quotient field K and S a multiplicative subset of D. If D is an S-GCD domain, then for each $x \in \overline{D}$ (the integral closure of D), there exists an $s \in S$ such that $sx \in D$.*

Proof: Since D is an S-GCD domain, then by Theorem 2.2, D_S is a GCD domain. Thus D_S is integrally closed. Which implies that $D_S = \overline{D_S} = \overline{D}_S$, and hence for each $x \in \overline{D}$, there exists an $s \in S$ such that $sx \in D$.

Corollary 2.5. *If D is a GCD domain, then D is an integrally closed domain.*

Proof: In the previous Proposition it suffices to tack $S = \{1\}$.

Remark 2.3. The S-GCD property does not carry over to the power series ring. In fact, there is an example of a GCD domain D such that $D[[X]]$ is not a GCD domain [[10], Theorem 8]. (This is the case when S consists of units of D).

Let D be an integral domain with quotient field K. Recall from [6] that, the power series ring $D[[X]]$ is said to satisfy the property $(*)$, if for all integral v-invertible v-ideals I and J of $D[[X]]$ such that $(IJ)_0 \neq (0)$, we have $((IJ)_0)_v = ((IJ)_v)_0$ where $I_0 = \{f(0),\ f \in I\}$. For example, $\mathbb{Z}[i\sqrt{5}][[X]]$ satisfies the property $(*)$ [[6], Example 3.1]. We are closing this section with the following two results.

Theorem 2.4. *Let D be a Krull domain, such that $D[[X]]$ satisfies $(*)$ and S a multiplicative subset of D. Then D is an S-GCD domain if and only if $D[[X]]$ is an S-GCD domain.*

Proof: By [[6], Theorem 4.4], $S\text{-}Cl_t(D) \simeq S\text{-}Cl_t(D[[X]])$. So by [[6], Theorem 4.2], D is an S-GCD domain if and only if $S\text{-}Cl_t(D) = 0$ if and only if $S\text{-}Cl_t(D[[X]]) = 0$ which equivalent to $D[[X]]$ is an S-GCD domain.

Corollary 2.6. *Let D be a Krull domain, such that $D[[X]]$ satisfies $(*)$. Then D is a GCD domain if and only if $D[[X]]$ is a GCD domain.*

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