Applications of t-invertible uppers to zero

Let D be an integral domain with quotient field K and let $F(D)$ denote the set of fractional ideals of D. Denote by A^{-1} the fractional ideal $D :_K A = \{x \in K | xA \subseteq D\}$. The function $A \mapsto A_v = (A^{-1})^{-1}$ on $F(D)$ is called the v-operation on D (or on $F(D)$). Associated to the v-operation is the t-operation on $F(D)$ defined by $A \mapsto A_t = \cup\{H_v |\ H$ ranges over finitely generated subideals of $A\}$. The v and t-operations are examples of the so called star operations, well explained in sections 32 and 34 of [4]. Indeed $A \subseteq A_t \subseteq A_v$. A fractional ideal $A \in F(D)$ is called a v-ideal (resp., a t-ideal) if $A = A_v$ (resp., $A = A_t$). An integral t-ideal maximal among integral t-ideals is a prime ideal called a maximal t-ideal. If A is a nonzero integral ideal with $A_t \neq D$ then A is contained in at least one maximal t-ideal. A prime ideal that is also a t-ideal is called a prime t-ideal. Call $I \in F(FD)$ v-invertible (resp., t-invertible) if $(II^{-1})_v = D$ (resp., $(II^{-1})_t = D$). A prime t-ideal that is also t-invertible was shown to be a maximal t-ideal in Proposition 1.3 of [7, Theorem 1.4].

Let X be an indeterminate over K. Given a polynomial $g \in K[X]$, let A_g denote the fractional ideal of D generated by the coefficients of g. A prime ideal P of $D[X]$ is called a prime upper to 0 if $P \cap D = (0)$. Thus a prime ideal P of $D[X]$ is a prime upper to 0 if and only if $P = h(X)K[X] \cap D[X]$, for a prime h in $K[X]$. It follows from [7, Theorem 1.4] that P a prime upper to zero of D is a maximal t-ideal if and only if P is t-invertible if and only if P contains a polynomial f such that $(A_f)_v = D$. Based on this it was concluded in [5] that if f is a polynomial in $D[X]$ such that $(A_f)_v = D$, then $f(X)D[X]$ is a t-product of uppers to zero. Call a polynomial f super primitive if $(A_f)_v = D$ and call D a PSP domain if every primitive polynomial over D is super primitive. The following result makes the above conclusion somewhat more obvious. Yet, before we state the lemma, let's note that every non-constant polynomial in $D[X]$ belongs to at most a finite number of uppers to zero, some of which may be t-invertible.

Lemma 1 . *Let $f \in D[X]$ be a non-constant polynomial and suppose that $P_1, ..., P_n$ are the only prime uppers to zero containing f that are maximal t-ideals. Then (1) for some positive integers r_i we have $f(X)D[X] = (AP_1^{r_1}...P_n^{r_n})_t$ where $(A, P_1^{r_1}...P_n^{r_n})_t = D[X]$, i.e. A is t-co-maximal with $P_1^{r_1}...P_n^{r_n}$ (2) if f is super primitive, i.e. is such that $(A_f)_v = D$, then $fD[X] = (P_1^{r_1}...P_n^{r_n})_t$, (3) Any non-constant polynomial f of $D[X]$ has at most a finite number of super primitive divisors.*

Proof. (1). The proof can be taken from the proof of Proposition 3.7 of [2]. For (2), note that if P is a maximal t-ideal containing A, then P contains f. This makes P t-invertible. But the only t-invertible maximal t-ideals containing f are $P_1, ..., P_n$. This leave the possibility that A is contained in a maximal t-ideal M with $M \cap D \neq (0)$. But this is impossible because $f \in A \subseteq M$, forcing $D = (f, d)_v \subseteq M$.Thus A is contained in no maximal t-ideal. Forcing $A_t = D$. But then $fD[X] = (AP_1^{r_1}...P_n^{r_n})_t = (A_tP_1^{r_1}...P_n^{r_n})_t = (P_1^{r_1}...P_n^{r_n})_t$. For (3), let's call an ideal I a t-divisor of an ideal A if there is an ideal B such that $A = (BI)_t$. If f is as in (1), i.e. f is such that $fD[X] = (AP_1^{r_1}...P_n^{r_n})_t$, then proper ideals of the kind $P_1^{a_1}...P_n^{a_n}$ $0 \leq a_i \leq r_r$ are t-divisors of $fD[X]$ and they only t-divide $P_1^{r_1}...P_n^{r_n}$. The reason is that if A, B, C are ideals such that $(A, B)_t = D$ and $A_t \supseteq (BC)_t$, then $A_t \supseteq C_t$. (This is because $A_t \supseteq (BC)_t \Leftrightarrow A_t = (A, BC)_t = (A, AC, BC)_t = (A, (A, B)C)_t = (A, (A, B)_t C)_t = (A, C)_t \Rightarrow A_t \supseteq C_t$.). Now as $(P_1^{a_1}...P_n^{a_n})_t \supseteq (AP_1^{r_1}...P_n^{r_n})_t$ and as $P_1^{a_1}...P_n^{a_n}$ and A share no maximal t-ideals. Thus we have $(P_1^{a_1}...P_n^{a_n})_t \supseteq (P_1^{r_1}...P_n^{r_n})_t$, alone. Now the number of proper t-divisors of $(P_1^{r_1}...P_n^{r_n})_t$ is less than $\Pi_{i=1}^n (r_i + 1)$ and hence finite. On the other hand if h is a super primitive divisor of f, then $hD[X] = (P_1^{a_1}...P_n^{a_n})_t$ by (2). Indeed if h is a super primitive divisor of f, then $f(X) = h(X)k(X)$. Or $(P_1^{r_1}...P_n^{r_n})_t = (P_1^{a_1}...P_n^{a_n})_t(k(X))$. Multiplying both sides by $(P_1^{-a_1}...P_n^{-a_n})$ and applying the t-operation, we get $(P_1^{r_1-a_1}...P_n^{r_n-a_n})_t = (k(X))$. On the other hand $(h(X)k(X)) = (h(X)k(X))_t$ because $(h(X)k(X))$ is principal. Consequently t-division acts like ordinary division in this case and so if n_{sf} denotes the number of non-associate super primitive divisors of f , then $n_{sf} < \Pi_{i=1}^n (r_i + 1) < \infty$. ∎

Call a nonzero element r in D primal if for all $x, y \in D\backslash\{0\}$, $r|xy$ implies $r = st$ where $s|x$ and $t|y$. Cohn [3] called an integrally closed integral domain D Schreier if each nonzero element of D is primal. A domain whose nonzero elements are primal was called pre-Schreier in [10]. Note that in a pre-Schreier domain every irreducible element (atom) is prime. (In fact a primal atom in any domain, is prime. For let p be an irreducible element that is also primal and let $p|ab$. So $p = rs$ where $r|a$ and $s|b$, because p is primal. But as p is also an atom, r is a unit or s is a unit. Whence $p|a$ or $p|b$.) An integral domain D is said to have the Irreducible Divisor Finite (IDF) property if every nonzero non unit of D is divisible by at most a finite number of non associated irreducible elements of D. Obviously, in a pre-Schreier domain the IDF property translates to: Every nonzero element is divisible by at most a finite number of non-associated primes. A Schreier domain has the PSP property, as a consequence of Lemma 2.1 of [11] and as in the proof of the aforementioned lemma the integrally closed property was not used one concludes that a pre-Schreier domain has the PSP property. Also it is well known that in a PSP domain, atoms are prime as well (cf [1]). Thus if D has the PSP property, the IDF property translates to: Every nonzero element is divisible by at most a finite number of non-associated primes. The point is, I will carry on with pre-Schreier and hope that the reader will draw conclusions about PSP domains.

Now if D is pre-Schreier, $D[X]$ may not be pre-Schreier, see e.g. [10, Remark 4.6]. So, some irreducible elements of $D[X]$ may not be prime. However if f is an irreducible non-constant polynomial in $D[X]$ then f is primitive, i.e. the GCD of the coefficients of f is 1 and over a pre-Schreier domain a primitive polynomial is super-primitive, as we have already pointed out, meaning $(A_f)_v = D$. (As mentioned above [11], Lemma 2.1 was stated for Schreier domains but was proved using properties characterizing pre-Schreier domains only.) Now f being a non-constant polynomial, f must belong to an upper to zero P of $D[X]$ and because $(A_f)_v = D$ every upper to zero P, containing f, must be a maximal

t-ideal [7, Theorem 1.4]. Thus, as mentioned above, if D is a PSP domain any prime upper to zero in $D[X]$ that contains an irreducible polynomial is a maximal t-ideal.

Next, verifying the IDF property entails checking that each nonzero polynomial $g \in D[X]$ is divisible by at most a finite number of irreducible divisors. If g is constant then all the divisors of g come from D alone and there are finitely many irreducible divisors for each constant g. So, let g be non-constant. Obviously each irreducible divisor of g that comes from D is a divisor of each of the coefficients of g and so g has only finitely many irreducible divisors coming from D.

According to Lemma 1, if $f(X) \in D[X]$ such that $(A_f)_v = D$, then $f(X)D[X] = (Q_1^{n_1}...Q_m^{n_m})_t$, where Q_i are prime uppers to zero. Now let's go back to $g(X)$, that we supposed was in n uppers to zero $P_1, ..., P_n$ that were maximal t-ideals and hence t-invertible. As we have seen $g(X)D[X] = (AP_1^{r_1}...P_n^{r_n})_t$ where $(A, P_1^{r_1}...P_n^{r_n})_t = D[X]$. Thus if f is an irreducible (primitive) polynomial dividing g, then $(f) = (P_1^{a_1}...P_n^{a_n})_t$ where $0 \leq a_i \leq r_i$, because A does not share a maximal t-ideal with $P_1^{a_1}...P_n^{a_n}$.But there can only be a finite number of such irreducible polynomials, by (3) of Lemma 1. This leaves the case of when $g(X)$ is not contained in any maximal uppers to zero. In this case the only irreducible divisors are divisors of coefficients of g, coming from D.

Thus we have the following statement.

Theorem 2 *Let D be a domain such that for every primitive polynomial f over D we have $(A_f)_v = D$, where A_f denotes the content of f. If D is an IDF domain, then so is $D[X]$.*

The case of Schreier domains, i.e. integrally closed pre-Schreier domains, may be handled as follows: It is known that if D is Schreier then so is $D[X]$, according to [3]. So the non constant irreducible elements of $D[X]$ are prime and generators of uppers to zero containing them. Now D being IDF the constant irreducible divisors of a general non-constant $f \in D[X]$ come from D and so are finite, up to associates, and the non-constant irreducible divisor are finite, up to associates, because they are primes and hence generators of the uppers to zero containing them.

Recall that an integral domain D is said to be a Prufer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. Let's also recall from [9] the following result.

Proposition 3 *Let D be an integrally closed integral domain, let X be an indeterminate over D and let $S = \{f(X) \in D[X] | \ (A_f),= D\}$. Then D is a PVMD if and only if for any prime ideal P of $D[X]$ with $P \cap D = (0)$ we have $P \cap S \neq \phi$.*

In light of [7, Theorem 1.4] it has often been concluded that D is a PVMD if and only if D is integrally closed such that every upper to zero of $D[X]$ is a maximal t-ideal. In fact the above proposition and Theorem 2.6 of [6] led to the notion of a UMT domain. (A domain whose uppers to zero are maximal t-ideals.) It was stated in [7, Proposition 3.2] that D is a PVMD if and only if D is an integrally closed UMT domain.

Lemma 4 *Let B be a t-invertible t-ideal of $D[X]$ with $B \cap D = (0)$. Then $B = (A' P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$ where P_i are the t-invertible prime uppers to 0 of $D[X]$ containing B and $(A', P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t = D$.*

Proof. $BK[X] = f(X)K[X]$. Since, being t-invertible, B is of finite type, there is $s \in K \backslash \{0\}$ such that $B \subseteq sfD[X]$. Or $B = (A_1 sf(X)))_t$ because B is t-invertible and so is $B/sf(X)$. Now sA_1 must intersect D because $BK[X] = fK[X]$. So the only uppers to zero that contain B must contain f. Adjusting s we can assume that $f \in D[X]$. So $B = (A_1 s)_t(f(X)) = (A_1 s(A_1 P_1^{r_1} ... P_n^{r_n}))_t$ by Lemma 1. The rest is adjustments. (Alternatively let $P_1, ..., P_n$ be the maximal uppers to zero and note that $D[X]_{P_i}$ are rank one DVRs. So there is r_i that $B \subseteq (P_i^{r_i})_t$ and $B \nsubseteq (P_i^{r_i+1})_t$. Now as $(P_i^{r_i})_t$ are t-invertible, $B = (B_1 P_1^{r_1})_t$, repeating with $i = 2$ we have $B = (B_2 P_1^{r_1} P_i^{r_i})_t = ... = (B_n P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$. Set $B_n = A$. As $(BA^{-1})_t = (P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t \subseteq D[X]$ we have $A \subseteq D[X]$. As far as $(A, P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t = D[x]$ is concerned, it follows from the fact that A and $(P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$ share no maximal t-ideals.) ∎

Theorem 5 *An integral domain D is a PVMD if and only if for each non-constant polynomial $f(X)$ over D we have uppers to zero $P_1, ... P_n$ such that $f(X)D[X] = (AP_1^{r_1} ... P_n^{r_n})_t$ where $A = A_f[X]$.*

Proof. Let D be a PVMD and let f be a non-constant polynomial in $D[X]$. Then $fD[X] = (AP_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$, where P_i are the maximal t-ideals containing $fD[X]$, by Lemma 1. Now in $K[X]$ we have $fK[X] = P_1^{r_1} P_2^{r_2} ... P_n^{r_n} K[X] = P_1^{r_1} K[X] \cap P_2^{r_2} K[X] \cap ... \cap P_n^{r_n} K[X]$ because P_i are maximal ideals of $K[X]$. Next note that $P_i^{r_i} K[X] \cap D[X] = P_i^{r_i} K[X]_{P_i} \cap K[X] \cap D[X]$ and because $P_i \cap D = (0)$ we have $K[X]_{P_i} = D[X]_{P_i}$. Thus $P_i^{r_i} K[X]_{P_i} \cap K[X] \cap D[X] = P_i^{r_i} D[X]_{P_i} \cap D[X] = P_i^{(r_i)}$. But then $fK[X] \cap D[X] = P_1^{(r_1)} \cap ... \cap P_n^{(r_n)} = (P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$ because P_i are mutually t-comaximal. On the other hand, on account of D being integrally closed, we have $fK[X] \cap D[X] = fA_f^{-1}[X]$ [8].This gives $fA_f^{-1}[X] = (P_1^{r_1} P_2^{r_2} ... P_n^{r_n})$. Multiplying both sides by A_f and applying the t-operation we get $fD[X] = (A_f P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$. Conversely suppose that D is such that for each non-constant polynomial $f \in D[X]$ we have $fD[X] = (A_f P_1^{r_1} P_2^{r_2} ... P_n^{r_n})_t$. Then, by construction, A_f is t-invertible. Since for every finitely generated nonzero ideal $A = (a_0, a_1, ..., a_m)$ we can construct a non-constant polynomial $f = \sum_{i=0}^{m} a_i X^i$ such that $A_f = A$ we conclude that every finitely generated nonzero ideal of D is t-invertible. (Alternatively for each pair $a, b \in D \backslash \{0\}$ we have $f = a + bX$ which gives $(f(X)) = (A_f P)_t$, forcing $A_f = (a, b)$ to be t-invertible. But this is a necessary and sufficient condition for D to be a PVMD.) ∎

Proposition 6 *An integrally closed domain D is a PVMD if and only if every linear non-constant polynomial over D is contained in a t-invertible upper to zero.*

Proof. If D is a PVMD, then of course as every upper to zero is a maximal t-ideal and hence t-invertible, every linear polynomial is contained in a t-invertible upper to zero. Conversely suppose that every non-constant linear polynomial $f = a + bX$ is contained in a t-invertible upper to zero. If $f(0) = 0$, then $f = bXD[X]$ and there is nothing to be gained from this. Yet if $f(0) \neq 0$ and f is contained in a t-invertible upper P, then $(f) = (AP)_t$. Where $fK[X] = PK[x]$ and so $fK[X] \cap D = f(X)A_f^{-1}[X] = P$. Since P is t-invertible, so must be $A_f^{-1}[X]$. multiplying both sides by A_f and taking the t-image we get $(f(X)) = (A_f[X]P)_t =$. Thus for every pair of nonzero elements a, b of D, (a, b) is t-invertible. This forces D to be a PVMD. ■

Proposition 7 *An integrally closed domain D is a PVMD if and only if every integral ideal A of $D[X]$ with $A \cap D = (0)$ is contained in a t-invertible upper to zero.*

Proof. If D is a PVMD then every upper to zero in $D[X]$ is t-invertible. Also if A is an ideal of $D[X]$ with $A \cap D = (0)$ then for some $s \in D\backslash\{0\}$ we have $sA = f(X)C$ for some polynomial $f \in D[X]$ and some integral ideal C with $C \cap D \neq (0)$ [**?**, Theorem 2.1]. Now as $fD[X]$ is contained in at least one upper to zero sA must be in an upper to zero. But s being a constant does not belong to any upper to zero. So A is contained in at least one upper to zero. Conversely let D be integrally closed and let $f(X)$ be a non-constant linear polynomial. Then $fA_f^{-1}[X] = P$, because D is integrally closed. Since P is t-invertible $A_f^{-1}[X]$ and hence A_f^{-1} is t-invertible and so is $(A_f\ )_v$. But then every two generated nonzero ideal of D is t-invertible. ■

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