Proof. If S is a URD, all the non units of R are non units of S and hence products of mutually co-prime packets and R is an HCF domain as well.

To prove the converse we first prove the

Lemma 6. Let R be an HCF domain, K its field of fractions and x an indeterminate over R then $R + xK[x]$ is an HCF domain.

Proof. A general element $y \in S$ can be written as

$$y = r_0 + \sum_{i=1}^{n} a_i x^i \quad ; r_0 \in R \text{ and } a_i \in K.$$

As we observed in Example 1, y can be of two types corresponding to $r_0 = 0$ or $r_0 \neq 0$, that is

(α) $(r_0 = 0)$; $y = bx^s(1 + \sum_{j=1}^{n-s} a_j' x^j)$; $b \in K$

(β) $(r_0 \neq 0)$; $y = r_0(1 + \sum_{i=1}^{n} (a_i/r_0)x^i)$.

The case where one of the elements of S is zero or is a unit, is obvious and so we consider a pair y_1, y_2 of arbitrary non zero non units of S. Let

$$y_1 = r_{01} + \sum_{i_1=1}^{n_1} a_{i_1} x^{i_1}, \quad y_2 = r_{02} + \sum_{i_2=1}^{n_2} a_{i_2} x^{i_2},$$ the following cases are possible:

(a) both y_1, y_2 are of type (α)

(b) y_1 is of type (α) and y is of type (β) (~~or otherwise~~ an vice versa)

(c) y_1, y_2 are both of type (β).

In case (a) holds, let

$$y_1 = b_1 x^{s_1}(1 + \sum_{j_1=1}^{n_1-s_1} a'_{j_1} x^{j_1}) , \quad y = b_2 x^{s_2}(1 + \sum_{j_2=1}^{n_2-s_2} a'_{j_2} x^{j_2})$$

the expressions in braces being elements of $K[x]$ are products of primes and so the HCF

$$d = \left((1 + \sum_{j_1=1}^{n_1-s_1} a'_{j_1} x^{j_1}),(1 + \sum_{j_2=1}^{n_2-s_2} a'_{j_2} x^{j_2}) \right)$$