The $A + XB[X]$ construction and a tale of two results

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Let D be an integral domain, S a multiplicative set in D and let X be an indeterminate over D_S. The set $D^{(S)} = D + XD_S[X] = \{f \in D_S[X] : f(0) \in D\}$ is a ring that has been a source of interesting examples since the publication of [13], by Costa, Mott and Zafrullah. The construction in the title represents a more general situation where $A \subseteq B$ is an extension of domains, X an indeterminate over B and $A + XB[X] = \{f \in B[X] : f(0) \in A\}$. Recently Chang [11] has written a brilliant article, giving a new perspective on the $D + XD_S[X]$ construction. In this note I will first, briefly, review the developments that led to [11] and got improved on by it and then, explaining the terminology used, give one aspect of the applications of the results of [11]: The state of the art has moved from immediately deciding that "if D is a UFD and S a multiplicative set of D then $D^{(S)}$ is a GCD domain" (with generalizations) to immediately concluding that "if A is a Krull domain and B is a subintersection of A, a la Fossum [15] then $A + XB[X]$ is a PVMD, with generalizations. Of course, as indicated above, there are frills and further developments to be taken care of. But we need some preparation in the form of introduction to the tools and terminology to do that.

In the beginning the emphasis was on GCD domains and questions like: When is $D^{(S)}$ a GCD domain, were asked and answered. One of the answers was: When D is a GCD domain and S is a special kind of multiplicative sets of D called a splitting set. Then there was the question: When is $A + XB[X]$ a GCD domain? And that got answered as: When A is a GCD domain $B = A_S$ for a splitting multiplicative set S of A [7]

As alluded to above the observation that led to the writing of [13] was that if D is a UFD and S a multiplicative set of D then $D+XD_S[X]$ is a GCD domain. As $D + XD_S[X] = D + XD_{S^*}[X]$ where S^* is the saturation of S, we may as well assume S to be saturated. There was a criterion for $D^{(S)}$ to be GCD when D is: GCD(d, X) exists for all $d \in D\backslash\{0\}$ in [13], but it did not fly too high as it did not provide the relationship between D and S to make $D + XD_S[X]$ a GCD domain. The relationship came to light in [25], in Corollary 1.5 as in the following statement.

Proposition 1 . *Let D be a GCD domain and let S be a multiplicative set in D. Then the following are equivalent.*

(1) $D^{(S)}$ is a GCD domain (2) For each $d \in D\backslash\{0\}$, $d = d_1 s$ where $s \in S$ and d_1 is coprime to all members of S.

(2) of Proposition 1 helped bring to the fore a concept studied by [19] and [22] as: A saturated multiplicative set S of a domain D is said to be a splitting set of D if for each $d \in D\backslash\{0\}$, $d = d_1 s$ where $s \in S$ and d_1 is such that $d_1 D \cap tD = d_1 tD$ for each $t \in S$.

Indeed this led to the extreme question: What can we say about a GCD domain D such that for every (saturated) multiplicative set S of D we have $D^{(S)}$ a GCD domain? The answer came from a study of unique factorization considering prime powers instead of primes as the building blocks of unique factorization, a part of the study that I carried out for my doctorate from the University of London before 1974. Briefly, a nonzero nonunit r in a domain D is rigid if for all $x, y \mid r$ we have $x \mid y$ or $y \mid x$. A GCD domain D is called a gneralized UFD (GUFD) if every nonzero nonunit of D is expressible as a finite product of rigid elements r with the property: For each non unit $h \mid r$ there is a positive integer $n = n(h)$ such that $r \mid h^n$. The proof of Theorem 3.1 of [25] explains how. We record that result as the following theorem.

Theorem 2 *Let D be a GCD domain. Then for $D^{(S)}$ to be a GCD domain for every saturated multiplicative set S it is necessary and sufficient that D be a GUFD.*

Now what kind of domains are GUFDs? Recall that a domain D is called a Krull domain if D_P is a rank one discrete valuation domain for every $P \in X^1(D)$ with $D = \bigcap_{P \in X^1(D)} D_P$ and the intersection is locally finite, i.e., every nonzero element is a nonunit in only a finite number of D_P. Paulo Ribenboim [24] called a domain D a genneralized Krull domain if D_P is a rank one valuation domain for every $P \in X^1(D)$ with $D = \bigcap_{P \in X^1(D)} D_P$ and the intersection is locally finite. Using the definition of a GUFD it was easy to show that a GUFD is a GCD domain that is a generalized Krull domain. By the way, these days generalized Krull domains could also mean something different see, e.g., [8]. In 1988 Anderson and Mahaney [4] introduced domains in which every nonzero nonunit is a product of primary elements, called them weakly factorial and noted that an integrally closed weakly factorial domain is a GCD domain. In 1995 I wrote a paper [1] with Dan and David Anderson where it was shown that a GUFD is indeed a weakly factorial domain. In the mean time a study of splitting sets was taking off, in the context of factorization in integral domains and other topics but of importantance here is the fact that Dan Anderson and I studied integral domains in which every saturated multiplicative set is a splitting set and these turned out to be weakly factorial domains [6].

It's time to talk about what a weakly factorial domain looks like. But to cover a tiny detail we would need to bring in the the language of star operations. So, those readers who are familiar with the star operations may skip the following few lines and those who aren't will have to bear with me. If there's anything in the introduction that is not quite clear, the reader can consult sections 32 and 34 of Robert Gilmer's book [18].

Let R denote an integral domain with quotient field K and let $F(R)$ be the set of nonzero fractional ideals of R. A star operation $*$ on R is a function $*\colon F(R) \to F(R)$ such that for all $A, B \in F(R)$ and for all $0 \neq x \in K$

(a) $(x)^* = (x)$ and $(xA)^* = xA^*$,

(b) $A \subseteq A^*$ and $A^* \subseteq B^*$ whenever $A \subseteq B$,

(c) $(A^*)^* = A^*$.

We note that for $A, B \in F(R)$ $(AB)^* = (A^*B)^* = (A^*B^*)^*$, and call it the $*$-product. A fractional ideal $A \in F(R)$ is called a $*$-*ideal* if $A = A^*$ and a $*$-ideal of *finite type* if $A = B^*$ where B is a finitely generated fractional ideal. Any nonzero intersection of $*$-ideals for any star operation $*$ is again a star ideal. A star operation $*$ is said to be of finite character or *of* finite type if $A^* = \bigcup\{B^* \mid 0 \neq B$ is a finitely generated subideal of $A\}$. For $A \in F(R)$ define $A^{-1} = \{x \in K \mid xA \subseteq R\}$ and call $A \in F(R)$ $*$-invertible if $(AA^{-1})^* = R$. Clearly every invertible ideal is a $*$-invertible $*$-ideal for every star operation $*$. If $*$ is of finite character and A is $*$-invertible, then A^* is of finite type. The most well-known examples of star operations are the v-operation defined by $A \mapsto A_v = (A^{-1})^{-1}$, the t-operation defined by $A \mapsto A_t = \bigcup\{B_v \mid 0 \neq B$ is a finitely generated subideal of $A\}$ and the d-operation defined by $A \mapsto A$ for all $A \in F(R)$.

If $*$ is a star operation of finite character then using Zorn's Lemma we can show that an integral ideal maximal w.r.t. being a star ideal is a prime ideal and that every integral $*$-ideal is contained in a maximal $*$-ideal. Let us denote the set of all maximal $*$-ideals by $*-max(D)$. It can also be easily established that for a star operation $*$ of finite character on D we have $D = \bigcap_{M \in *-\max(D)} D_M$. A v-ideal A of finite type is t-invertible if and only if A is t-locally principal i.e. for every $M \in t-\max(D)$ we have AD_M principal. An integral domain D is called a Prüfer v-multiplication domain (PVMD) if every nonzero finitely generated ideal of D is t-invertible. A domain R is a Krull domain if and only if every nonzero ideal of R is t-invertible if and only if every t-ideal of R is t-invertible [23, Theorem 2.5].

Obviously the set of t-invertible t-ideals $Inv_t(D)$ is a group under the t-product and $P(D)$ the set of nonzero principal fractional ideals of D is a subgroup of $Inv_t(D)$. The quotient group $Inv_t(D)/P(D)$ was inroduced in Bouvier's [9] as the class group $Cl(D)$ of D. This group is also called the t-class group and is denoted by $Cl_t(D)$. $Cl_t(D)$ is the divisor class group if D is Krull and the Picard group if D is Prufer. The statement "$Cl_t(D) = 0$" means that every t-invertible t-ideal is principal. Thus if $Cl_t(D) = 0$ and D is a PVMD then D is a GCD domain.

Now we are able to say what a weakly factorial domain is. It is an integral domain D such that $D = \bigcap_{P \in X^1(D)} D_P$ and the intersection is locally finite and $Cl_t(D) = 0$. Other characterizing properties of weakly Krull domains are given in [6]. What is interesting here is the fact that minus the condition that $Cl_t(D) = 0$ from "weakly factorial domain" we get a weakly Krull domain as a domain D such that $D = \bigcap_{P \in X^1(D)} D_P$ and the intersection is locally finite. Weakly Krull domains were studied in [5] and were often used in the study of factorization in integral domains, and so were weakly Krull monoids [17]. These domains were characterized in [5] as given in the following Proposition.

Proposition 3 *([5, Theorem 3.1]) For an integral domain D, the following conditions are equivalent. (1) $D = \bigcap_{P \in X^1(D)} D_P$ and the intersection is of finite character. (2) Every proper principal ideal of D has a primary decomposition where all the associated prime ideals have height one. (3) Every principal ideal of D is a t-product of primary (t-) ideals. (4) Every proper t-ideal is a t-product of primary (t-) ideals. (5) Every nonzero prime ideal contains a t-invertible primary t-ideal. (6) Every proper principal ideal of D is a finite intersection of t-invertible primary t-ideals. (7) If P is a prime ideal minimal over a proper principal ideal (x) then $xD_P \cap D$ is t-invertible.*

We thought those were all the characterizations there were of weakly Krull domains. But later developments showed that there was more in store for us. But for now we'll have to digress a little to talk about what caused those later developments.

With the popularity of Prufer v-Multiplication domains (PVMDs) came the question: When is $D^{(S)}$ a PVMD? It was answered in [2] by Anderson, Anderson and Zafrullah, using a somewhat tentative approach with: When D is a PVMD and S is a "t-splitting set" of D.

A multiplicative set S of an integral domain D is said to be a t-splitting set if for each $d \in D\backslash\{0\}$ we can write $d = (AB)_t$ for integral ideals A, B such that $(A, s)_t = D$ for all $s \in S$ and $B_t \cap S \neq \phi$. Note that a multiplicative set S is a t-splitting set if and only if for each $d \in D\backslash\{0\}$ we have $dD_S \cap D$ t-invertible. All this resulted from the thought that if, for a multiplicative set S, $D^{(S)}$ were a PVMD (a GCD domain) then (d, X) should be t-invertible (respectively, $GCD(d, X)$ should exist) for all $d \in D\backslash\{0\}$.

It was established in [2, p. 8], using a characterizing property of weakly Krull domains, that every saturated multiplicative set in a weakly Krull domain is a t-splittiing set. Thus if D is a weakly Krull PVMD (i.e. a generalized Krull domain) and S a multiplicative set of D then $D^{(S)}$ is a PVMD [2, Corollary 2.7].

While the t-splitting sets of elements worked, as we have seen above, they led to the study of t-splitting sets of ideals. These were studied in [12], but as they come very chose to what we are going to prove we need to put in a bit more detail.

Let S be a multiplicative set of ideals of D and let $D_S = \{x \in K \mid xA \subseteq D$ for some $A \in S\}$ be the S-transform of D (see [3] for basic properties of this construction). If I is an ideal of D , then $I_S = \{x \in K \mid xA \subseteq I$ for some $A \in S\}$ is an ideal of D_S containing I. Denote by $S^\perp$ the set of all ideals B of D with $(A+B)_t = D$ for all $A \in S$. Note that $S^\perp$ is also a multiplicative set of ideals. We call $S^\perp$ the t-complement of S. Consider also, the multiplicative set of ideals $sp(S) \supseteq S$ consisting of all ideals C of D with $C_t \supseteq A$ for some $A \in S$. It is easy to see that $sp(sp(S)) = sp(S)$, $sp(S)^\perp = S^\perp$ and $D_S = D_{sp(S)}$.

Call a multiplicative set of ideals S a t-splitting set of ideals if every nonzero principal ideal dD of D can be written as $dD = (AB)_t$ with $A \in sp(S)$ and $B \in S^\perp$.

Indeed it was shown in [12, Proposition 2] that if S is a t-splitting set of ideals of D then so is $S^{\perp}$ and for every $C \in S$, C_t contains some t-invertible ideal of $sp(S)$. Moreover the set S_i of all t-invertible ideals in $sp(S)$ is a t-splitting set with t-complement $S^{\perp}$ and $sp(S_i) = sp(S)$. A multiplicative set S of ideals of D is called v-finite if for each $A \in S$, A_t is contained in a v-finite ideal $J \in sp(S)$ (see Gabelli [16]). The above comment shows that a t-splitting set of ideals is v-finite.

Next if Λ is a nonempty set of nonzero prime ideals of D, we define $F(\Lambda) = \{A \subseteq D : A \text{ is an ideal of } D \text{ such that } A \ncong P \text{ for all } P \in \Lambda\}$. Thus defined $F(\Lambda)$, called a spectral localizing system, is a saturated multiplicative set of ideals of D and $D_{F(\Lambda)} = \bigcap\limits_{P \in \Lambda} D_P$ [14, Proposition 5.1.4]. By Corollary 2.6 of Chang [10], D is a weakly Krull domain if and only if $F(\Lambda)$ is a t-splitting set (of ideals) for every non-empty set Λ of prime t-ideals. So if D is weakly Krull and Λ is a nonempty subset of $X^1(D)$ we can call $D_{F(\Lambda)} = \bigcap\limits_{P \in \Lambda} D_P$ a subintersection, in the same manner as we call a subintersection of a Krull domain. Indeed if D is weakly Krull then by Theorem 2.3 of [10] if $R = \bigcap\limits_{P \in \Lambda} D_P$ then $t\text{-}max(R) = \{P_{F(\Lambda)} : P \in \Lambda\}$. Thus, in this case, for each $Q \in t\text{-}max(R)$, $R_Q = D_{Q \cap D}$.

Recall that an overring R of an integral domain D is called flat if for each maximal ideal M of R we have $R_M = D_{M \cap D}$. The reader may look up [21], for introduction and equivalent conditions for flatness of an overring. Based on this an overring R of D was called t-flat, in [20] if for each $M \in t\text{-}max(R)$ we have $R_M = D_{M \cap D}$. Thus, by the above comment, if R is a subintersection of a weakly Krull domain D then R is t-flat over D. Indeed, as shown in [20] "t-flat" runs almost parallel to "flat". For our purposes it suffices to quote Chang's result [11, Theorem 1.7].

Theorem 4 *(cf [11, Theorem 1.7]) If T is an overring of D, then the following statements are equivalent.*

(1) T is t-flat over D.

(2) There is a multiplicative set S of ideals of D such that $T = D_S$ and $(AT)_t = T$ for all $A \in S$.

(3) $((D : \alpha)T)_t = T$ for all $\alpha \in T \backslash \{0\}$. (Here $(D : \alpha) = \{x \in D : x\alpha \in D\}$.)

Now let us start collecting the results that would help us bring this tale of two results to an end. First we shall have a simple result

Theorem 5 *([11, Theorem 2.2]) If D is a PVMD and S is a t-splitting set of ideals of D, then*

$R = D + XD_S[X]$ is a PVMD.

Corollary 6 *Let A be a generalized Krull domain and let B be a subintersection of A then $A + XB[X]$ is a PVMD.*

Proof. We have already established in a discussion prior to Theorem 4 that being a subintersection of a weakly Krull domain, B is t-flat and so $B = A_S$ where S is a t-splitting set of ideals of A.

Of course the above corollary applies in particular when A is Krull.

Theorem [11, Theorem 2.5]. Let $R = A + XB[X]$, and assume that B is t-flat over A. Then R is a PVMD if and only if A is a PVMD and $B = A_S$ for S a t-splitting set of ideals of A.

The following corollary uses the fact that if D is a PVMD then each t-linked overring is t-flat, as mentioned in [11, Theorem 1.9], and a subintersection is t-linked.

Corollary H Let $R = A + XB[X]$. Then A is a generalized Krull domain if and only if for each B that is t-flat over A the $A + XB[X]$ construction is a PVMD.

Proof. Note that if A is a generalized Krull domain then every t-flat B is a subintersection and hence $B = A_{F(\Lambda)}$ where $F(\Lambda)$ is a t-splitting set of ideals of A by [10, Theorem 2.3]. Thus by , , $A + XB[X]$ is a PVMD. Conversely suppose that for every t-flat B, $A + XB[X]$ is a PVMD. But then A is a PVMD and so, being t-flat, $B = A_{F(\Lambda)}$ where Λ is a nonempty set of prime t-ideals of A and $F(\Lambda)$ is a t-splitting set of ideals. But then by [10, Corollary 2.6], A is a weakly Krull domain and hence a generalized Krull domain.

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